Lecture Notes in Mathematics

An informal series of special lectures, seminars and reports on mathematical topics

Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

13

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Seminar on Fiber Spaces

Lectures delivered in 1964 in Berkeley
and 1965 in Zürich
Berkeley notes by J. F. Mc Clendon

1966

Springer-Verlag · Berlin · Heidelberg · New York

Research supported in part by the U. S. Air Force Office of Scientific Research.

Library of Congress Catalog Card Number 66–19122. Printed in Germany. Title No. 7333.

Foreword

These notes are divided into seven sections. The material in the first five was given as a series of informal lectures at Berkeley in the Summer of 1964. The material in the last two was discussed in Professor Eckmann's seminar in Zurich during the Spring of 1965.

Mr. J.F. Mc Clendon took my lecture notes from the Berkeley lectures and wrote them up in a presentable form. In particular the formulation of Theorem 5 in section five is due to him.

The first five lectures develop the theory of the Postnikov resolution of a map. In particular the main results of Mahowald's paper [3] are covered. Section six applies the theory to a specific example - the universal fibration with fiber the Stiefel manifold $V_{n,2}$. The last section deals with the problem of computing Postnikov invariants. A method for doing this is outlined and a theorem proved that implies that every (4k+3)-spin manifold has a tangent 2-field [12].

I would like to take this opportunity to thank Professor Eckmann for making possible my visit to the Mathematics Research Institute, E.T.H., Zurich.

14 September, 1965

E. Thomas

Table of Contents

I. Introduction

We consider three classical problems concerning the following diagram of spaces and maps:

(1)
$$\begin{array}{ccc} & & E \\ & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

i) (Existence problem) Does there exist a map $g: X \to E$ such that $pg = f$?

ii) (Enumeration problem) How many homotopy classes* of such g's are there ?

iii) (Classification problem) Given two such g's, can we distinguish between them by algebraic invariants ?

An example of (1) is:

$$\begin{array}{ccc} & & BO(n-k) \\ & & \downarrow i \\ M^n & \xrightarrow{f} & BO(n) \end{array}$$

where M^n is a smooth n-dimensional manifold, BO(n) is the classifying space for the orthogonal group, $i: BO(n-k) \to BO(n)$, $1<k<n$, is the natural inclusion, and f is the map inducing the tangent bundle over M. In this example a lifting of f corresponds to a field of tangent orthonormal k-frames and questions (i), (ii), and (iii), can be reformulated as familiar questions about such fields.

Other problems in differential topology can be stated in a similar fashion.

One method of getting a <u>negative</u> answer to the Existence

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* There are two possible meanings for "homotopy" in question (ii): "free homotopy" or "homotopy relative to f."

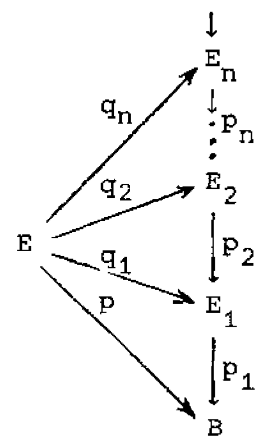
question (i) is to derive an algebraic diagram from (1) and show that it cannot be commutative. Affirmative answers to question (i) have been obtained via obstruction theory. That is, if $E \to B$ is a fiber bundle and $q: Y \to X$ is the bundle induced by f, then a lifting of f corresponds to a cross-section of q. Obstruction theory for bundles and the theory of characteristic classes are then applicable. For example, if $X(M)$ is the Euler class of the manifold M, then $X(M) = 0$ if and only if M has an everywhere continuous non-zero vector field. (i.e. the Euler class is the only obstruction to lifting f to $BO(n-1)$.)

These lectures will discuss the elaboration of this positive method due initially to Postnikov, with important contributions by Moore, Hermann, and Mahowald. Mahowald's paper [3] will be discussed in some detail.

General description of the method

We hope to factor the map p of (1) into a diagram

(2)



where $(p_1 p_2 \dots p_n)q_n = p$ $(n \geqslant 1)$ and:

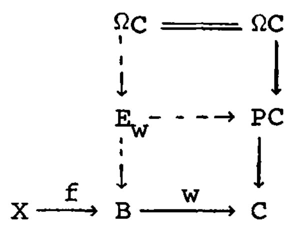
A) At each stage, the obstruction to lifting a map from E_q to E_{q+1} is given in terms of "computable" algebraic invariants (e.g., a finite set of cohomology classes.)

B) There exists a sequence $\{r_n\}$ of integers such that $1<r_1<r_2<\ldots<r_n \ldots$ and such that the morphism $q_{r_n *}: \pi_i(E) \to \pi_i(E_{r_n})$ is bijective for $0<i<n$ and surjective for $i = n$.

Let X be a complex and consider the morphism[1] $q_{r_n *}: [X,E] \to [X,E_{r_n}]$. It follows from B) that if dim $X<n$, then $q_{r_n *}$ is bijective, while if dim $X = n$, then $q_{r_n *}$ is surjective. Thus if X has dim $< n$, then a map $f: X \to B$ lifts to E iff it lifts to E_{r_n}.

In order to achieve the factorization given in (2) we will use the following key construction:

(3)



Here PC is the space of paths of C beginning at $*$ and ΩC is the space of loops of C at $*$. Thus $PC \to C$, given by $\alpha \to \alpha(1)$, is a Hurewicz fibration with fiber ΩC [8] and PC is contractible. $E_w \to B$ is the fibration induced by w, so $E_w = \{(b,\alpha) \in B \times PC: w(b) = \alpha(1)\}$. It is easily seen that f lifts to E_w iff $wf \simeq *$.

An important special case is when $C = K(\pi,n)$, an Eilenberg-MacLane space of type (π,n). Since $H^n(B;\pi) = [B,C]$ and $w \in H^n(B,\pi)$,

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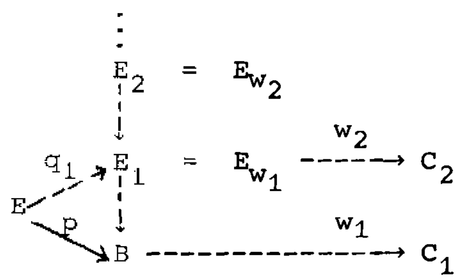
1) We work in the category of spaces with base point (which is always denoted by $*$). $[X,Y]$ denotes the set of homotopy classes of base point preserving maps from X to Y.

it follows that $wf \simeq *$ iff $f^*w = 0$ (provided that B is a complex).

We now apply (3) to (1) taking $C_i = K(\pi_i, n_i)$, $i \geq 1$, and obtain the following diagram

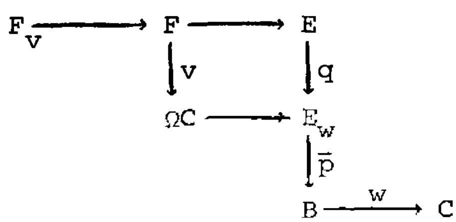
(4)



The w_i's could be chosen as follows. First, consider ker p^*, where $p^*: H^{n_1}(B;\pi_1) \to H^{n_1}(E;\pi_1)$, choose a w_1 there and use it to construct the fibration $E_1 \to B$. Since $p^*w_1 = 0$ we can lift p to q_1. Then we look in ker q_1^* for a w_2, etc.

We have constructed a diagram (2) satisfying condition A. Our plan is to re-examine, under certain restrictions, construction (4) with condition B in mind. Assume that $p: E \to B$ is a fibration with an $(n-1)$-connected fiber F. It follows that $p_*: \pi_r(E) \to \pi_r(B)$ is bijective for $0 < r < n-1$ and surjective for $r = n$ and that $p^*: H^r(B,\pi) \to H^r(E,\pi)$ is bijective for $1 < r < n-1$ and injective for $r = n$. Now assume that $C = K(\pi, n+1)$ and choose $w \in H^{n+1}(B,\pi) \cap$ ker p^*. The situation after rearranging (4) slightly is:

(5)



where $p = \bar{p}q$ and $p^*w = 0$. Note that $q(F) \subset \Omega C$; set $v = q|F$. Both the maps v and q are homotopically equivalent to fiber maps [7], and moreover the fibers themselves are homotopically equivalent [2]. Let F_v denote the fiber of v. Then one obtains the commutative diagram given in (5), such that the triples

$$F_v \longrightarrow F \xrightarrow{v} \Omega C ,$$

$$F_v \longrightarrow E \xrightarrow{q} E_w ,$$

are homotopically equivalent to fibrations.

If F_v were n-connected it would follow from the exact sequence for $q: E \to E_w$ that $q_*: \pi_i(E) \to \pi_i(E_w)$ is bijective for $0 < i < n$, and surjective for $i = n+1$. Thus we would have gained a dimension in the passage to the second stage of the construction. This is a step in the direction of (B) - so we seek conditions under which F_v is n-connected.

From the exact sequence for $v: F \to \Omega C$ we see that $\pi_i(F_v) \to \pi_i(F)$ is bijective for $i \neq n, n-1$ and that

$$\pi_n(F_v) = 0 \iff v_* \text{ is injective in dimension } n,$$

$$\pi_{n-1}(F_v) = 0 \iff v_* \text{ is surjective in dimension } n.$$

The question as to whether or not the construction (4) will satisfy B) now can be rephrased as:

1) Take $\pi = \pi_n(F)$. Can we choose $w \in H^{n+1}(B,\pi)$ such that the resulting v gives $v_*: \pi_n(F) \approx \pi$?

2) If so, can we repeat the process - i.e., can we (a) compute $H^*(E_w)$ and $\ker q^*$; (b) determine which classes in $\ker q^*$

"kill" the higher homotopy groups of F (in the sense that in going from F to F_v we kill the n'th homotopy group of F)?

In order to gain some insight into the problem let us consider part of (5) separately

$$
(6) \qquad \begin{array}{ccc} F_v & \longrightarrow & F \\ & & \big\downarrow v \\ & & \Omega C = K(\pi,n) \end{array} \qquad \begin{array}{l} F \text{ is } (n-1)\text{-connected,} \\ \pi_n(F) = \pi. \end{array}
$$

Since $[F,\Omega C] = H^n(F;\pi) = \mathrm{Hom}(\pi,\pi)$, taking $v = 1 \in \mathrm{Hom}(\pi,\pi)$ gives v_* an isomorphism. The question is whether or not (6) can be fitted into (5) - that is, can we choose w to produce such a v.

The answers to these questions are, more or less, yes, as we shall see later.

II. Principal Fiber Spaces

If $w\colon (B,B_0) \to (C,C_0)$ is a (base point preserving) map then we have the diagram

$$\begin{array}{ccc} (\Omega C,\Omega C_0) & = & (\Omega C,\Omega C_0) \\ \downarrow & & \downarrow \\ (E,E_0) & \xrightarrow{\bar{w}} & (PC,PC_0) \\ \downarrow & & \downarrow \\ (B,B_0) & \xrightarrow{w} & (C,C_0) \end{array}$$

where $E = \{(b,\alpha) \in B \times PC\colon w(b) = \alpha(1)\}$

$E_0 = \{(b,\alpha) \in B_0 \times PC_0\colon w(b) = \alpha(1)\}$

<u>Definition</u>: $(\Omega C,\Omega C_0) \xrightarrow{i} (E,E_0) \xrightarrow{p} (B,B_0)$ is the <u>principal fibration induced by w</u>, with principal fibre space (E,E_0).

<u>Lemma</u> 1. Let $g\colon (X,X_0) \to (B,B_0)$ be a map. Then g lifts to (E,E_0) iff $wg \simeq *$.

<u>Lemma</u> 2. Let (X,X_0) be a pair of spaces. Then the following sequence of sets is exact:

$$\begin{aligned} &[(X,X_0);(\Omega B,\Omega B_0)] \xrightarrow{(\Omega w)_*} [(X,X_0);(\Omega C,\Omega C_0)] \xrightarrow{i_*} [(X,X_0);(E,E_0)] \\ (7)\quad &\xrightarrow{p_*} [(X,X_0);(B,B_0)] \xrightarrow{w_*} [(X,X_0);(C,C_0)]. \end{aligned}$$

<u>Proof</u>: Exactness at $[(X,X_0);(B,B_0)]$ follows from the first lemma. The rest is not hard to verify (see [5]).

If α and β are paths such that $\alpha(1) = \beta(0)$, denote their product by $\alpha \vee \beta$. There is a natural product

$$(\Omega C, \Omega C_0) \times (\Omega C, \Omega C_0) \longrightarrow (\Omega C, \Omega C_0)$$
$$(\alpha, \beta) \longrightarrow \alpha \vee \beta$$

and a natural action

$$(\Omega C, \Omega C_0) \times (E, E_0) \longrightarrow (E, E_0)$$
$$(\alpha, (b, \beta)) \longrightarrow (b, \alpha \vee \beta) .$$

They induce a product (denoted by $\vee$) and an action (denoted by $\cdot$)

$$[(X,X_0);(\Omega C,\Omega C_0)] \times [(X,X_0);(\Omega C,\Omega C_0)] \longrightarrow [(X,X_0);(\Omega C,\Omega C_0)]$$
$$[(X,X_0);(\Omega C,\Omega C_0)] \times [(X,X_0);(E,E_0)] \longrightarrow [(X,X_0);(E,E_0)].$$

For any pairs (with base point) (X,X_0), (Y,Y_0) let $0 \in [(X,X_0),(Y,Y_0)]$ denote the class of the constant map.

Lemma 3: (a) $0 \cdot q = q$ for $q \in [(X,X_0);(E,E_0)]$

(b) $u \cdot 0 = i_* u$ for $u \in [(X,X_0);(\Omega C,\Omega C_0)]$

(c) Let $q,q' \in [(X,X_0);(E,E_0)]$. Then $p_* q = p_* q'$ iff there is a $u \in [(X,X_0);(\Omega C,\Omega C_0)]$ such that $q' = u \cdot q$.

(d) The sequence (7) and the operations defined above are natural in the obvious ways.

Proof: For the proof in the absolute case see [6].

Transgression in Fiber Spaces

Suppose that $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration and B is path connected. Take cohomology with coefficients in a fixed group G. Denote reduced homology by $\bar{H}(\ )$. $\bar{H}^*(B)$ will sometimes be identified with $H^*(B,*)$ and sometimes with a subgroup of $H^*(B)$. Denote by $\bar{p}$: $(E,F) \to (B,*)$ the map defined by p.

Define $T^*(F) \subset H^*(F)$, $S^*(B) \subset \bar{H}^*(B)$ by

$$T^*(F) = \delta^{-1}\bar{p}^*\bar{H}^*(B)$$

$$S^*(B) = \bar{p}^{*-1}\delta H^*(F)$$

where δ: $H^*(F) \to H^*(E,F)$. Note that $S^*(B)$ = kernel p^*, $T^*(F) \supset$ image i^*, $S^*(B) \supset$ ker $\bar{p}^*$.

Define τ: $T^*(F) \to S^*(B)/\ker \bar{p}^*$

σ: $S^*(B) \to T^*(F)/ \operatorname{im} i^*$

by $\tau(u) = [u']$, where $\delta u = \bar{p}^*u'$

$\sigma(v) = [v']$, where $\bar{p}^*v = \delta v'$.

Clearly, kernel τ = image i^*, kernel σ = kernel $\bar{p}^*$ so τ, σ induce inverse isomorphisms

$$T^*(F)/\operatorname{im} i^* = S^*(B)/\ker \bar{p}^*.$$

τ is called the transgression in the fiber space, σ, the suspension. (We can define σ: $S^*(B) \to H^*(F)$, and then its values are cosets of $i^*H^*(E)$; similarly for τ.)

<u>The Lifting Problem</u>

Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration, B path connected, and E, B have the homotopy type of complexes. We form the diagram

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & & \\ \downarrow{\scriptstyle v_q} & & \downarrow{\scriptstyle q} & & \\ \Omega C & \longrightarrow & E_w & & \\ & & \downarrow{\scriptstyle p_1} & & \\ & & B & \xrightarrow{w} & C \end{array} \qquad (p_1 \circ q = p)$$

Suppose that $wp \simeq *$. By lemma 1, p lifts to $q: (E,F) \to (E_w,\Omega C)$. Let $v_q = q|F: F \to \Omega C$.

Definition: v_q is (geometrically) realized by the pair (w,q). A reasonable question is: which maps v_q can be realized?

To answer this question we consider the sequence

$$(\Omega C,\Omega C) \xrightarrow{j} (E_w,\Omega C) \xrightarrow{p_1} (B,*) \xrightarrow{w} (C,C),$$

where p_1 is the principal fibration induced by w (see page 7). Mapping the pair (E,F) into the sequence we obtain the commutative diagram shown below:

$$\begin{array}{ccccccc}
[E,\Omega C] & \xrightarrow{j_*} & [(E,F),(E_w,\Omega C)] & \xrightarrow{p_{1*}} & [(E,F),(B,*)] & \xrightarrow{w_*} & [E,C] \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & & & \\
[F,\Omega C] & = & [F,\Omega C] & . & & &
\end{array}$$

Define $\Sigma w = i^* p_{1*}^{-1}[\bar{p}] \subset [F,\Omega C]$. Using Lemma 3(d), one easily shows that Σw is a coset of $i^*[E,\Omega C]$. Furthermore, Σw = all homotopy classes that can be geometrically realized by (w,q) for some lifting q of p.

If $C = K(G,n)$, $n>0$, this can be translated into cohomology. We have $w \in H^n(B;G)$ and Σw is a coset of $i^*H^{n-1}(E,G)$ in $H^{n-1}(F;G)$. Now $p^*w = 0$, so for the fibration $F \xrightarrow{i} E \xrightarrow{p} B$, $\sigma(w)$ is defined (= the suspension of w, see page 9) and is a coset of $i^*H^{n-1}(E;G)$ in $H^{n-1}(F;G)$

Theorem 1. $-\sigma w = \Sigma w$.

Proof: It suffices to prove that $-\sigma w$ and Σw have a common representative. Suppose that $v \in \Sigma w$, so that $v: F \to \Omega C$ is the restriction of some $q: (E,F) \to (E_w, \Omega C)$ and $p_1 q = \bar{p}$.

If σ^1 denotes suspension in $\Omega C \to E_w \to B$ then it is easily checked that $\sigma^1(-w)$ is represented by ι_{n-1} = the characteristic class of $H^{n-1}(\Omega C;G)$. By the naturality of the suspension for the commutative diagram

$$\begin{array}{ccccc} \Omega C & \longrightarrow & E_w & \longrightarrow & B \\ \uparrow v & & \uparrow q & & \| \\ F & \longrightarrow & E & \longrightarrow & B \end{array}$$

we have

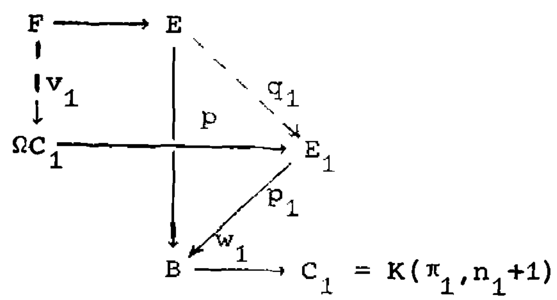
$$v = v^* \iota_{n-1} \in v^* \sigma^1(-w) \subset \sigma(-w) ,$$

which completes the proof.

We now can carry out the program of the first lecture.

III. The "classical" (Moore-Postnikov) method of decomposing a fibration

Suppose that $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration, that $\pi_1(B)$ acts trivially on $H_*(F;G)$[2], and that F has non-zero homotopy groups $\pi_i = \pi_{n_i}(F)$ in dimension $n_1, n_2, \ldots$ with $0 < n_1 < n_2 < \ldots$ (if $n_1 = 1$ assume π_1 is abelian). Let $v_1 \in H^{n_1}(F, \pi_1)$ be the fundamental class of F. That is, v_1 corresponds to the Hurewicz map under $H^{n_1}(F, \pi_1) = \mathrm{Hom}\,(H_{n_1}(F), \pi_1)$. It follows from the spectral sequence of the fibration that v_1 is transgressive. Let $w_1 = -\tau(v_1)$. We have, as before, the diagram



By Theorem 1 we can choose q_1: $(E,F) \to (E_1, \Omega C_1)$ such that $v_{q_1} = v_1$.

Let F_1 = the fiber of v_1. Then in the homotopy sequence for the fibration $F_1 \to F \to \Omega C_1$, v_{1*}: $\pi_n(F) \to \pi_n(\Omega C_1) = \pi_1$ is an isomorphism, implying $\pi_r(F_1) = 0$, $r < n$, $\pi_r(F_1) \approx \pi_r(F)$, $r > n$.
Now let $v_2 \in H^{n_2}(F_1; \pi_2)$ be the fundamental class of F_1. Again it is transgressive (in the fibration $F_1 \to E \to E_1$).

Let $w_2 = -\tau(v_2) \in H^{n_2+1}(E_1, \pi_2)$ induce E_2 over E_1, etc.

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[2] Henceforth, "$F \to E \to B$ is a fibration" will include this condition.

Some questions about this construction are:

1) How can we compute the transgression?

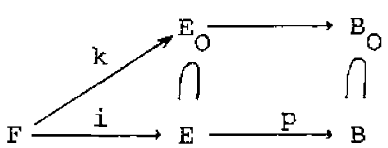
2) Can we relate the classes $w_2, w_3, \ldots$ to the cohomology of F, E, B ? In particular, how do the Steenrod operations behave on the w's ?

3) If $f: X \to B$, $f^*w_1 = 0$, and $f_1, f_2: X \to E_1$, are two liftings of f, what is $f_1^*w_2 - f_2^*w_2$?

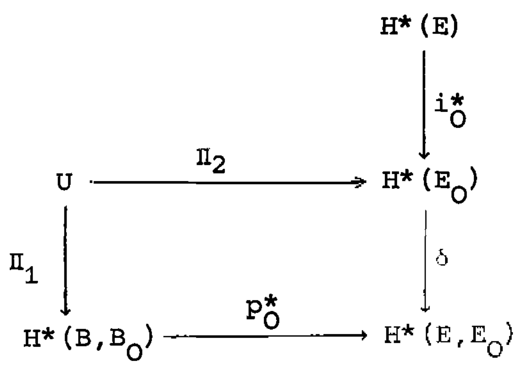
The answers to these questions will depend on 1) knowing the cohomology of the E_i's and 2) knowing the action of ΩC_i on E_i vis-a-vis cohomology.

<u>Relative Transgression</u>

Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ a fiber space, $* \in B_0 \subset B$, $E_0 = p^{-1}(B_0) \subset E$, $p_0: (E, E_0) \to (B, B_0)$. In short:



Define $U = U(B, B_0, E_0)$ = all pairs $(u,v) \in H^*(B, B_0) \oplus H^*(E_0)$ such that $\delta v = p_0^* u$.



Π_1, Π_2 are the projections

Define $S^*(B,B_0) = \Pi_1 U$, $T^*(E_0) = \Pi_2 U$,

$$\sigma_0 : S^*(B,B_0) \longrightarrow T^*(E_0)/\mathrm{Im}\, i_0^* \text{ induced by } \Pi_2\Pi_1^{-1}$$

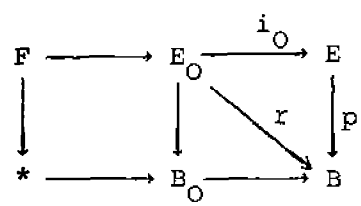
$$\tau_0 : T^*(E_0) \longrightarrow S^*(B,B_0)/\ker p_0^* \text{ induced by } \Pi_1\Pi_2^{-1}$$

σ_0 is the <u>relative suspension</u>, τ_0 the <u>relative transgression</u> (We can think of τ_0 as mapping $T^*(E_0)$ to $H^*(B,B_0)$; its values are then cosets of $\ker p_0^*$; similarly for σ_0.)

<u>Properties of the relative transgression τ_0</u>

<u>Property</u> 1. Let $j: B \to (B,B_0)$, $k: F \to E_0$. Then $\tau k^* = j^*\tau_0$ where τ is the absolute transgression of the last lecture. (Proof obvious.)

Consider:



where $r = pi_0$. Using r^* we can make $H^*(E_0)$ into an algebra over $H^*(B)$. That is, given $u \in H^*(E_0)$, $v \in H^*(B)$, define $u \cdot v = u \cup r^*v$. Also, $H^*(B,B_0)$ is an algebra over $H^*(B)$ and $\ker p_0^*$ is stable under $H^*(B)$.

<u>Property</u> 2. τ_0 is an $H^*(B)$-morphism. That is, if $u \in T^*(E_0)$, $v \in H^*(B)$, then $u \cdot v \in T^*(E_0)$ and $\tau_0(u \cdot v) = \tau_0(u) \cdot v$.

<u>Proof</u>: $u \in T^*(E_0)$ implies there is a $u' \in H^*(B,B_0)$ such that $\delta u = p_0^* u'$. Then $\delta(u \cdot v) = \delta(u \cup r^*v) = \delta(u \cup i_0^* p^* v) = \delta u \cup p^*v = p_0^* u' \cup p^*v = p_0^*(u' \cup v)$.

Next, suppose that φ is a primary cohomology operation, Ψ its suspension, then:

Property 3: $\varphi\tau_0 = \tau_0\Psi$

Proof: Use the fact: $\varphi\delta = \delta\Psi$.

For the next property we will need the following theorem of Serre [9]:

Theorem Let $p: E \to B$ be a fiber space with arcwise connected fiber F and base B. Suppose $\pi_1(B)$ acts trivially on $H_*(F;Z)$.Let $* \in B_0 \subset B$ and set $E_0 = p^{-1}(B_0)$. Assume that $H_i(B,B_0;Z) = 0$ for $0 < i < a$, $H_j(F;Z) = 0$, $0 < j < b$. Then the homomorphism $p^*: H^k(B,B_0;G) \to H^k(E,E_0;G)$ is injective for $k < a+b$ and surjective for $k < a+b-1$, where G is any abelian group.

Proof: Apply the universal coefficient theorem to the homology version [9, p.268].

Property 4. Under the hypotheses of the above theorem, the following sequence is exact:

$$\cdots \longrightarrow H^i(E_0) \xrightarrow{\tau_0} H^{i+1}(B,B_0) \xrightarrow{1^*} H^{i+1}(E) \xrightarrow{i_0^*} H^{i+1}(E_0)$$

$$\cdots \longrightarrow H^{a+b-1}(E_0), \quad \text{where } 1 = jp.$$

Proof: Start with the sequence for (E,E_0) and insert (B,B_0) by the theorem. Notice that 1^* can be obtained from the commutative diagram

$$\begin{array}{ccc} H^{i+1}(B,B_0) & \longrightarrow & H^{i+1}(B) \\ \Big\downarrow p_0^* \approx & \searrow l^* & \Big\downarrow p^* \\ H^{i+1}(E,E_0) & \longrightarrow & H^{i+1}(E) \end{array} .$$

Define $\tau_1 : T^*(E_0) \to H^*(B)/j^* \operatorname{Ker} p_0^*$ to be the composite homomorphism shown below:

$$T^*(E_0) \xrightarrow{\tau_0} H^*(B,B_0)/\operatorname{Ker} p_0^* \xrightarrow{j^*} H^*(B)/j^* \operatorname{Ker} p_0^* .$$

Notice that Property 1 can now be written $\tau k^* = \tau_1$, and that τ_1 continues to enjoy Properties 2 and 3.

<u>Property</u> 5. Let $k_0 : B_0 \subset B$ denote the inclusion. Let t be an integer such that $0 < t < a+b-1$, and suppose that

$$\text{Kernel } p^* \supset \text{Kernel } k_0^* \text{ in dim } t,$$

$$k_0^* \text{ is surjective in dim } t.$$

Then the following sequence is exact:

$$0 \longrightarrow H^t(E) \xrightarrow{i_0^*} H^t(E_0) \xrightarrow{l^*} H^{t+1}(B) .$$

<u>Proof</u>: By exactness, Image j^* = Kernel k_0^* . Thus,

$$\text{Image } l^* = p^*(\text{Image } j^*) = p^*(\text{Kernel } k_0^*) = 0 ,$$

in dim t. Hence by the exactness of the sequence given in Property 4,

$$\text{Kernel } i_0^* = \text{Image } l^* = 0 \text{ in dim } t ,$$

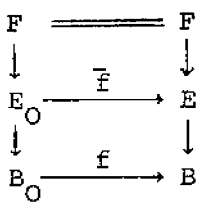
as claimed.

If k_0^* is surjective in dim t, then j^* is injective in dim t+1 (using the cohomology sequence of the pair (B,B_0)), and so

Kernel τ_0 = Kernel τ_1 in dim t,

which completes the proof.

<u>Property</u> 6: Given any fiber space commutative diagram:

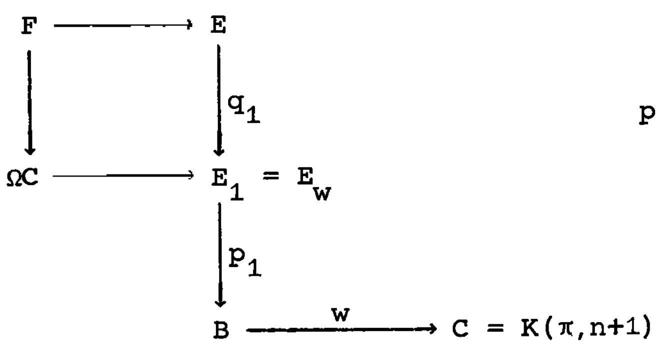


such that $H_j(F;Z) = 0$, $0<j<b$, and $f_*: H_r(B_0;Z) \to H_r(B;Z)$ is isomorphic for $0 < r < a-1$, and epic for $r = a-1$, then the sequence in (4) is still defined and exact with $i_0^* = \bar{f}^*$ and (B,B_0) thought of as (M_f,B_0) where M_f is the mapping cylinder of f.

The proof follows from usual argument using the mapping cylinder.

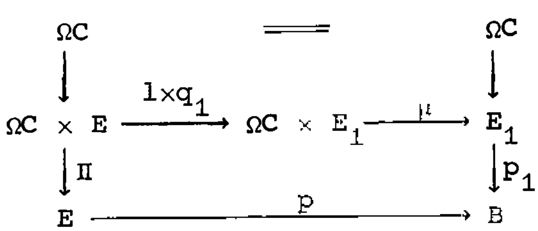
<u>Application</u>

Consider the usual diagram with $w \in \ker p^*$.



$p = p_1q_1$

<u>Lemma</u> 4: There is a commutative diagram of fiber spaces



where Π is the projection and μ is the action of ΩC in the principal fiber space E_1.

Proof: Recall that q_1 is defined by $q_1(e) = (p(e),\alpha_e)$ where α_e is a path from $*$ to $wp(e)$. If $\lambda \in \Omega C$ we have

$$p_1\mu(1 \times q_1)(\lambda,e) = p_1\mu(\lambda,(p(e),\alpha_e)) = p_1(p(e),\lambda \vee \alpha_e) = p(e)$$
$$= p\Pi(\lambda,e).$$

Corollary 1: Let $F \to E \overset{p}{\to} B$ be a fibration with (n-1)-connected fiber and let C and E_1 be as above. Then there is an exact sequence

$$\cdots H^i(\Omega C \times E) \xrightarrow{\tau_0} H^{i+1}(B,E) \longrightarrow H^{i+1}(E_1) \xrightarrow{\nu^*} H^{i+1}(\Omega C \times E)$$
$$\cdots \longrightarrow H^{2n}(\Omega C \times E)$$ where $\nu = \mu\cdot(1 \times q_1)$. ((B,E) should be thought of as (M_p,E)).

For future use define $s\colon E \to \Omega C \times E$ by $s(e) = (*,e)$ and note that

$$\nu \cdot s = \mu \cdot (1 \times q_1)\cdot s \simeq q_1\colon E \to E_1$$

since $\mu(*,e_1) = e_1, e_1 \in E_1$.

IV. We will now illustrate the method by considering the classifying space for oriented (n-1)-sphere bundles $S^{n-1} \xrightarrow{i} BSO(n-1) \xrightarrow{p} BSO(n)$. Abbreviate $BSO(q) = B_q$. Note that p is homotopically equivalent to the natural inclusion $B_{n-1} \subset B_n$; upon occasion it will be thought of as that inclusion.

We will use the scheme described in the last lecture in order to factor p. Let $0 < n_1 < n_2 < \cdots$ be the dimensions in which S^{n-1} has non-zero homotopy groups and let $\Pi_i = \pi_{n_i}(S^{n-1})$. So $n_1 = n-1$, $n_2 = n$, $n_3 = n+1$, $\Pi_1 = Z$, $\Pi_2 = \Pi_3 = Z_2$. Let s_{n-1} be a generator for the group $H^{n-1}(S^{n-1};Z)$. We know that s_{n-1} is transgressive and we are interested in $\tau(s_{n-1})$. The Serre exact sequence [8;p.468] in this case is:

$$\cdots H^{n-1}(B_{n-1}) \longrightarrow H^{n-1}(S^{n-1}) \xrightarrow{\tau} H^n(B_n) \xrightarrow{p^*} H^n(B_{n-1}) \longrightarrow H^n(S^{n-1}),$$

so that in dimension n, $\text{im}\ \tau = \ker p^*$. Now it follows from the Gysin sequence for the oriented sphere bundle $S^{n-1} \subset B_{n-1} \xrightarrow{p} B_n$ that $\ker p^* \cap H^n(B_n)$ is cyclic infinite generated by the Euler class X_n. Hence we can choose s_{n-1} so that $\tau(s_{n-1}) = -X_n$. As in the general situation (see p.12) we can find q_{n-1} such that the following diagram is commutative

$$\begin{array}{ccccc}
F_n & \xrightarrow{i_n} & S_{n-1} & \longrightarrow & B_{n-1} & & \\
 & & \downarrow{\scriptstyle s_{n-1}} & & \downarrow{\scriptstyle q_{n-1}} & & \\
 & & K(Z,n-1) & \longrightarrow & E_{n-1} & & \\
 & & & & \downarrow{\scriptstyle p_{n-1}} & & \\
 & & & & B_n & \xrightarrow{X_n} & K(Z,n)
\end{array}$$

where F_n is the fiber of q_{n-1}, F_n is (n-1)-connected, and the morphism $q_{n-1*}:\pi_r(B_{n-1}) \to \pi_r(E_{n-1})$ is bijective for $0<r<n-1$, and surjective for $r = n$. Moreover, $i_{n_*}: \pi_r(F_n) \approx \pi_r(S^{n-1})$, $r \geq n$.

Let v_2 = characteristic class of $F_n \in H^n(F_n, \Pi_2) = H^n(F_n, Z_2)$. We know that v_2 is transgressive and we are interested in $\tau(v_2) \in H^{n+1}(E_{n-1}, Z_2) \cap \ker q_{n-1}{}^*$.

We digress for a while to discuss $H^*(E_{n-1}, Z_2)$. Now for $q \geq 2$, $H^*(B_q; Z_2)$ is a polynomial algebra on the Stiefel-Whitney classes $w_2, \ldots, w_q$. Consequently,

i) The morphism $p^*: H^*(B_n; Z_2) \to H^*(B_{n-1}; Z_2)$ is surjective.

ii) The kernel of p^* is the ideal in $H^*(B_n; Z_2)$ generated by w_n.

Since $w_n = X_n$ mod 2, it follows that Kernel p^*_{n-1} = Kernel p^* in all dimensions. Therefore, by Property 5, we have the following exact sequence (with mod 2 coefficients) for $0<r<2n-2$.

$$0 \longrightarrow H^r(E_{n-1}) \xrightarrow{\nu^*} H^r\big((Z, n-1) \times B_{n-1}\big) \xrightarrow{\tau_1} H^{r+1}(B_n).$$

Let ι denote the mod 2 reduction of the fundamental class of $(Z, n-1)$; thus $\iota \in H^{n-1}(Z, n-1; Z_2)$ and $\tau(\iota) = w_n$. Therefore by Property 2 we have

iii) $\tau_1(\iota \otimes b) = w_n \cdot b$,

where $b \in H^i(B_n)$, $i<n-2$. Furthermore, by exactness of the above sequence we have

iv) $H^q(E_{n-1}) \approx$ Kernel τ_1, for $q<2n-2$.

Let $s: B_{n-1} \to (Z,n-1)\times B_{n-1}$ denote the inclusion. Since $q_{n-1} \simeq \nu\circ s$ (see page 18), our attention is shifted from

$$H^*(E_{n-1},Z_2)\cap \ker q_{n-1}{}^* \text{ to } \ker \tau_1 \cap \ker s^*.$$

Three facts of note are

<u>Fact</u> 1. Let $1 \in H^0(Z,(n-1))$. Then $\tau(1) = 0$ and thus in $H^*((Z,n-1)\times B_{n-1})$ any term of the form $1 \otimes b$, $b \in H^*(B_{n-1})$, is in $\ker \tau_1$.

<u>Fact</u> 2. By a formula of Wu [14], $Sq^i w_n = w_n \cdot w_i$, $0<i<n$. Thus by Property 3, and iii) above,

$$\tau_1(Sq^i\iota \otimes 1) = Sq^i\tau_1(\iota \otimes 1) = Sq^i w_n = w_n\cdot w_i = \tau_1(\iota \otimes w_i),$$

and so

$$\tau_1(Sq^i \otimes 1 + \iota \otimes w_i) = 0 \qquad \text{for } i < n-2$$

<u>Fact</u> 3. Let $u \in H^*(E_{n-1})$, set $\nu^*(u) = 1 \otimes b + \sum_j a_j \otimes c_j$ where $a_j \in \bar{H}^*(Z,n-1)$, $b,c_j \in H^*(B_{n-1})$. Then $q^*_{n-1}u = s^*\nu^*u = s^*(1 \otimes b + \Sigma a_j \otimes c_j) = 1 \otimes b$ (since $s^*\bar{H}^*((Z,n-1)) = 0$). So $u \in \ker q^*_{n-1} \Longleftrightarrow \nu^*u \in \ker s^* \Longleftrightarrow \nu^*u = \Sigma a_j \otimes c_j$, $\deg a_j > 0$.

Using the above facts one can easily calculate $H^q(E_{n-1})$ for $q < 2n-2$. For example, $\ker \tau_1 \cap \ker s^*$ is 0 in dimension $< n$ while in dimension $n+1$ it is $Sq^2\iota \otimes 1 + 1 \otimes w_2$.

Let $k^{n+1} \in H^{n+1}(E_{n-1})$ be the unique element such that $\nu^*(k^{n+1}) = Sq^2\iota \otimes 1 + 1 \otimes w_2$.

We now can proceed with the factorization of p. Since $\tau(v_2) = k_{n+1}$ we have

$$
\begin{array}{ccccc}
F_{n+1} & \longrightarrow & F_n & \longrightarrow & B_{n-1} & & \\
 & & \downarrow{\scriptstyle v_2} & & \downarrow{\scriptstyle q_2} & & \\
 & & K(Z_2,n) & \longrightarrow & E_n & & \\
 & & & & \downarrow{\scriptstyle p_2} & & \\
 & & & & E_{n-1} & \xrightarrow{k_{n+1}} & K(Z_2,n+1)
\end{array}
$$

where F_{n+1} is the fiber of q_2, F_{n+1} is n-connected, the morphism q_2^*: $\pi_r(B_{n-1}) \to \pi_r(E_n)$ is bijective for $0<r<n$ and surjective for $r = n+1$, and i_{n+1_*}: $\pi_r(F_{n+1}) \approx \pi_r(S^{n-1})$ $r \geq n+1$. We have gained one more stage.

An unresolved question is: by altering the method somewhat, can we kill several homotopy groups of F_n at once instead of only one?

V. Killing Homotopy Groups

Suppose that $v \in H^n(Y,J)$ where $J = Z$ or Z_p, p prime, and Y is a complex.

We have

$$\begin{array}{ccc} \Omega K \longrightarrow & E_v & \\ & \downarrow & \\ & Y \xrightarrow{\ v\ } & K(J,n) = K \end{array}$$

and we can ask: when does v kill a factor J of $\pi_n(Y)$ - i.e. when does $\pi_n(E_v)$ have one less factor J than $\pi_n(Y)$?

We know [7] that any map is homotopically equivalent to a fibration and it is easy to check that if v is changed to a fibration the fiber will be E_v. Hence we may consider $E_v \to Y \to K$ to be a fibration. The homotopy sequence of that fibration shows that if v_* is surjective in dim n then $\pi_i(E_v) \approx \pi_i(Y)$, $i \neq n$, and the sequence $0 \to \pi_n(E_v) \to \pi_n(Y) \to J \to 0$ is exact. In particular, if $\pi_n(Y)$ is finite and $J = Z_p$ we have: order $\pi_n(E_v)$ = order $\pi_n(Y)/p$.

Theorem 2. v_* is surjective in dim n if and only if there exists a map $f: S^n \to Y$ such that $f^*(v)$ = a generator of $H^n(S^n;J)$.

Proof: We have $S^n \xrightarrow{f} Y \xrightarrow{v} K$ and $[Y,K] \xrightarrow{f^*} [S^n,K]$, $\pi_n(Y) = [S^n,Y] \xrightarrow{v_*} [S^n,K] = \pi_n(K)$, so that $f^*v = vf = v_*f$. Also

$$\begin{array}{ccccc} H^n(Y;J) & = & [Y,K] & & \\ \downarrow f^* & & & & \\ H^n(S^n;J) & = & [S^n,K] & = & \pi_n(K) \end{array}$$

so $f^*(v)$ = generator $H^n(S^n, J) = \Pi_n(K)$, iff $v_*(f)$ = generator $\Pi_n(K)$, iff v_* is surjective.

Definition A class $v \in H^n(Y;J)$ is spherical if there exists $f: S^n \to Y$ such that $f^*(v)$ = generator of $H^n(S^n;J)$.[1)]

We are interested in finding spherical classes. In particular we are interested in spherical classes of F where $F \to E \overset{p}{\to} B$ is a fibration and in finding spherical classes of the spaces $F_1, F_2, \cdots$ etc. which arise in the factorization of p.

Irreducible Cohomology Operations

Suppose that φ is a cohomology operation of type $(J, Z_p, n, n+q+1)$ with $q \geq 0$, i.e., for each space X, $\varphi: H^n(X;J) \to H^{n+q+1}(X;Z_p)$.

Definition [see 3.13 of 3] φ is irreducible relative to (X, w, α) if X is a space, $w \in H^n(X,J)$, $\alpha: S^{n+q} \to X$ and

(1) $\alpha^*(w) = 0$, $\varphi(w) = 0$, $\alpha^* H^{n+q}(X;Z_p) = 0$

(2) $\varphi_\alpha(w)$ = generator $H^{n+q}(S^{n+q};Z_p)$ (i.e. $\varphi_\alpha(w) \neq 0$) where φ_α is the functional cohomology operation (see [10]).

Theorem 3. [see 3.14 of 3]. Suppose φ is irreducible relative to (X, w, α). Consider the fiber space $E_w \to X \overset{w}{\to} K(J,n) = K$ and think of φ as an element of $H^{n+q+1}(K;Z_p)$. Then

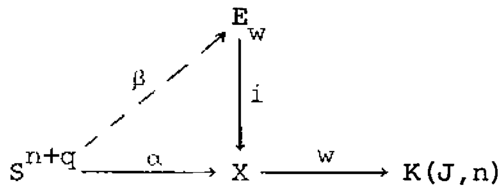
(1) $w^*(\varphi) = 0$, i.e. $\varphi \in S^*(K)$ (see p.9, $w = p$)

(2) If $y \in H^{n+q}(E_w, Z_p) \cap \sigma(\varphi)$, then y is spherical.

---

1) Notice that if v is spherical, then so is $-v$.

<u>Proof</u>:



Since $\alpha^*(w) = 0$, α lifts to β. Since $wi \simeq *$ and $\varphi w \simeq *$, the functional operation $\varphi_i(w)$ is defined as a coset in $H^{n+q}(E_w, Z_p)$. Moreover, $\beta^*\varphi_i(w) = \varphi_\alpha(w)$ (naturality of the functional operation) = generator of $H^{n+q}(S^{n+q};Z_p)$. Hence each element of $\varphi_i(w)$ is spherical and (since $\sigma(\varphi) \subset \varphi_i(w)$) the proof is complete.

This property of irreducibility is natural in the following sense. Suppose that φ is irreducible relative to (X,w,α) and let $f: X \to Y$, $v \in H^n(Y,J)$, $f^*v = w$, and $\varphi(v) = 0$. Then φ is irreducible relative to $(Y,v,f\alpha)$.

This naturality is particularly useful when, for a given φ, a simple X (i.e., having few cells) can be found. For example, if φ is irreducible relative to (S^n,s,α), where $s \in H^n(S^n,J)$ is the generator, and if $v \in H^n(Y,J)$ is spherical, then the map f is available and we need only check the condition: $\varphi(v) = 0$, to insure that φ is irreducible relative to $(Y,v,f\alpha)$. If φ is irreducible relative to $(S^n \cup_\beta e^q, s, \alpha)$ and v is spherical via $\gamma: S^n \to Y$ then if (1) $\gamma\beta \simeq *$, we can find an f, and if (2) $\varphi(v) = 0$ we then have that φ is irreducible relative to $(Y,v,f\alpha)$.

Suppose a cohomology operation φ is irreducible relative to (X,w,α) and X is a CW-complex formed by attaching i-cells to a sphere. Call φ irreducible of type $(i+1)$.

<u>Theorem</u> 4. The following operations are irreducible of type 1:

Sq^{2^i}, $i = 0,1,2,3$, β_p, P_p^1 p = prime > 2. The following are irreducible of type 2: $Sq^{2^i}Sq^{2^{i+1}}, Sq^{2^{i+1}}Sq^{2^i}$, $i = 0,1,2$, Sq^{16}. The attaching maps for the type 2 operations are Hopf maps - except for Sq^2Sq^1 where it is a map of degree 2.

<u>Proof</u>: The type 1 results are well known. For example, if $\varphi = Sq^2$ use $X = S^n$, w = generator of $H^n(S^n;J)$, $\alpha: S^{n+1} \to S^n$ the suspension of the Hopf map. For Sq^2Sq^1 see [3, p.323]. The others can be done similarly or by using a lemma of Toda [13; pp.84,190]. For Sq^{16} see [13; p.86].

<u>Definition</u> [cf 2.1.1 of 3] Let $w_i \in H^{n_i}(X;J_i)$, J_i cyclic, $i = 1, \cdot\cdot, l$. $\{w_i\}$ is a <u>spherical set</u> if there exist $\alpha_i: S^{n_i} \to X$, $i = 1, \cdot\cdot, l$, such that $\alpha_i^* w_i$ = generator of $H^{n_i}(S^{n_i};J_i)$ and $\alpha_j^* w_i = 0$ if $i \neq j$.

Let $w_i \in H^{n_i}(X;J_i)$, J_i cyclic, $i = 1, \cdot\cdot, l$, and let $C = \mathop{X}\limits_{i=1}^{l} K(J_i, n_i)$, $w = (w_1, \cdot\cdot, w_l): X \to C$. It is easy to check that $w_*: \pi_m(X) \to \pi_m(C)$ is surjective for each m iff $\{w_i\}$ is a spherical set.

Let each $J_i = Z$ or Z_p, and suppose that $\Psi \in H^m(C;Z_p)$, where $m < 2 \min \{n_i\}$. Then we may write

(*) $\Psi = \Sigma\, \varphi_i(\iota_i)$ where φ_i is a primary operation of type (n_i, J_i, m, Z_p). Suppose also that for some t, $0 < t < l$, φ_t (in the representation (*) of Ψ) is either:

(a) irreducible of type 1

or

(b) irreducible of type 2 relative to $M = S^{n_t} \cup_\gamma e^n$ such that $\alpha_t \gamma \simeq *$. (Here n is an integer with $n_t < n < m$).

<u>Theorem</u> 5. [see 3.1.4 and 3.1.7 of 3] Let $\{w_i\}$ be a spherical set and let Ψ be given as above. If $w^*(\Psi) = 0$, then $\sigma(\Psi) \subset H^{m-1}(E_w;Z_p)$ consists of spherical classes, where σ is calculated in the fibration $E_w \to X \xrightarrow{w} C$.

<u>Proof</u>: (a) The proof of (a) is similar to, and easier than that of (b)

(b) Since $\alpha_t \gamma \simeq *$ we can extend α_t to $\alpha\colon M \to X$ so that $\alpha^* w_t = s$ where $s \in H^{n_t}(M;Z_p)$ corresponds to the generator of $H^{n_t}(S^{n_t};Z_p)$. It can be checked that α can be chosen so that $\alpha^* w_i = 0$, $i \neq t$. (Using a Puppe sequence, alter α by means of $\alpha_1,\ldots,\alpha_{t-1},\alpha_{t+1},\ldots,\alpha_l$, if necessary). We have a commutative diagram

$$\begin{array}{ccccc} E_s & \longrightarrow & M & \xrightarrow{s} & K(J_t,n_t) \\ \downarrow r & & \downarrow \alpha & & \downarrow p \\ E_w & \longrightarrow & X & \xrightarrow{w} & C \end{array}$$

where p is the natural inclusion. Denote by σ' suspension in $E_s \to M \to K(J_t,n_t)$. Then $r^*\sigma(\Psi) \subset \sigma'(p^*\Psi) = \sigma'(\varphi_t) \Rightarrow r^*\sigma(\Psi)$ consists of spherical classes $\Rightarrow \sigma(\Psi)$ consists of spherical classes.

The above theorem gives a method of finding spherical classes. We are interested in spherical sets. Using the above notation, call $\Psi \in H^n(C;Z_p)$ <u>allowable</u> if $\sigma(\Psi)$ consists of spherical classes. Suppose $\{\Psi_1,\cdots,\Psi_a\} \subset H^n(C;Z_p)$ and $v_i \in \sigma(\Psi_i)$. Then we have the following criterion [see 3.4.2 of 3]: $\{v_i\}$ is a spherical set if every non-trivial linear combination of $\Psi_1,\cdots,\Psi_a$ is allowable. This is just a statement in terms of the Ψ's of a fact about a collection of mod p cohomology classes of any space - namely, the collection forms a spherical set iff every non-trivial linear combination of its elements is a spherical class. Or, to put it somewhat differently, let $\Sigma_p \subset H_*(X;Z_p)$ denote the mod p reduction of the image of the Hurewicz homomorphism, and let $T \subset H^*(X;Z_p)$ denote any subspace such that

$$T \cap \text{Annihilator } \Sigma_p = 0.$$

Then any finite set of linearly independent elements in T is a spherical set.

VI. An illustration

We give an example that will illustrate the use of Theorem 5, page 27. Again let $B_q = BSO(q)$, $q \geqslant 1$, and consider the fibration

$$V_{n+2,2} \xrightarrow{i} B_n \xrightarrow{p} B_{n+2} , \qquad n \geqslant 2 ,$$

where $V_{n+2,2}$ denotes the Stiefel manifold of orthonormal 2 frames in R^{n+2}. Given a complex A, a map $\xi: A \to B_{n+2}$ can be regarded as an orientable $(n+2)$-plane bundle over A. Moreover, ξ lifts to B_n iff ξ has two linearly independent (global) cross sections. In particular if A is an $(n+2)$-dimensional orientable manifold M and if ξ = tangent bundle of M, then ξ lifts to B_n iff M has 2 linearly independent tangent vector fields.

We take the case $n = 4s+1$, $s \geqslant 1$, and construct the first three stages in a Postnikov resolution of the map p.

Set $V = V_{4s+3,2}$. By [4], V is 4s-connected and

$$\pi_{4s+1}(V) \approx \pi_{4s+2}(V) \approx Z_2 , \quad \pi_{4s+3}(V) \approx Z_4 .$$

Take cohomology with mod 2 coefficients and consider the homomorphism

$$p^*: H^*(B_{4s+3}) \longrightarrow H^*(B_{4s+1}) .$$

Now Kernel p^* is the ideal in $H^*(B_{4s+3})$ generated by w_{4s+2} and w_{4s+3}. Thus if $u \in H^{4s+1}(V)$ denotes the fundamental class, it follows by the Serre exact sequence [8] that $\tau u = w_{4s+2}$.

Following the method given in lecture III we construct the diagram shown below.

$$\begin{array}{ccccc}
V_1 \longrightarrow & V & \xrightarrow{\ i\ } & B_{4s+1} & \\
 & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle q_1} & \\
 & K(Z_2,4s+1) & \xrightarrow{\ j\ } & E_1 & \\
 & & & \downarrow{\scriptstyle p_1} & \\
 & & & B_{4s+3} & \xrightarrow[w_{4s+2}]{} K(Z_2,4s+2)
\end{array}$$

Here $p_1q_1 = p$,q_1 is chosen so that $v_{q_1} = u$ (see pages 10,12), and V_1 is the fiber of the map q_1 ($\equiv$ fiber of the map u; see page 5). Because u is spherical, the space V_1 is (4s+1)-connected. (See Theorem 2, page 23). Since $p_1^* w_{4s+2} = 0$ and $p_1^* w_{4s+3} = p_1^* Sq^1 w_{4s+2} = 0$, it follows that Kernel p^* = Kernel p_1^* in all dimensions. Also, p^* is surjective in all dimensions. Thus by Property 5 and Corollary 1 we have an exact sequence

$$(*) \qquad 0 \to H^r(E_1) \xrightarrow{\nu^*} H^r(F\times B_{4s+1}) \xrightarrow{\tau_1} H^{r+1}(B_{4s+3}),$$

for all $0<r<8s-1$, where $F = K(Z_2,4s+1)$. Moreover, by Property 1,

$$\tau_1(\iota\otimes 1) = w_{4s+2} ,$$

where ι denotes the fundamental class of F. Recall (see page 18) that if $s: B_{4s+1} \to F\times B_{4s+1}$ denotes the canonical injection, then $\nu\circ s \simeq q_1$. Thus to compute Kernel $q_1^* \subset H^r(E_1)$ it suffices to compute Kernel $s^* \cap$ Kernel τ_1 in $H^r(F\times B_{4s+1})$. Using Properties

2 and 3 (as they apply to τ_1) we obtain the following chart:

dimension	Kernel $s^* \cap$ Kernel τ_1 spanned by:
$\leqslant 4s+2$	0
$4s+3$	$(Sq^2\iota\otimes 1 + \iota\otimes w_2) = A$
$4s+4$	$Sq^1 A$, $(Sq^2 Sq^1\iota\otimes 1 + Sq^1\iota\otimes w_2) = B$.

Moreover, one easily calculates that

$$Sq^1 B = Sq^2 A + A\cdot w_2 \ .$$

By sequence (*) there are classes $k_i \in H^{4s+i}(E_1)$, $i = 3,4$, such that

$$\nu^* k_3 = A, \quad \nu^* k_4 = B \ ,$$
$$Sq^1 k_4 + Sq^2 k_3 + k_3\cdot w_2 = 0.$$

Since $q_1^* = s^*\nu^*$, it follows that Kernel q_1^* in dimension $\leqslant 4s+4$ looks as follows.

dimension	Kernel q_1^* spanned by:
$\leqslant 4s+2$	0
$4s+3$	k_3 ,
$4s+4$	$Sq^1 k_3$, k_4 .

Let σ denote th suspension in the fibration given by q_1. Since $k_i \in$ Kernel q_1^* each k_i is in domain σ. Choose any non-zero classes $\alpha_i \in \sigma(k_{i+1}) \subset H^{4s+i}(V_1)$, $i = 2,3$. By naturality, $\alpha_i \in \sigma_1(j^*k_{i+1})$, where σ_1 denotes the suspension in the fibration

$$(**) \qquad V_1 \to V \xrightarrow{u} K(Z_2, 4s+1).$$

By definition of k_i we have

$$j^*k_3 = Sq^2\iota, \quad j^*k_4 = Sq^2Sq^1\iota$$

Now apply Theorem 5 to (**), taking $C = K(Z_2,4s+1)$, $X = V_1$, $w = u$, $E_u = V$. By the theorem it follows that $\{\alpha_2,\alpha_3\}$ is a spherical set. (Since Sq^2Sq^1 is an irreducible operation of type 2, we need to remark that $2\pi_{4s+1}(V) = 0$.) Therefore by Theorem 1 we can construct the following diagram:

$$\begin{array}{ccccc}
V_2 & \longrightarrow & V_1 & \longrightarrow & B_{4s+1} \\
 & & \Big\downarrow{\scriptstyle (\alpha_2,\alpha_3)} & & \Big\downarrow{\scriptstyle q_2} \\
 & & K(Z_2,4s+2) \times K(Z_2,4s+3) & \longrightarrow & E_2 \\
 & & & & \Big\downarrow{\scriptstyle p_2} \\
 & & & & E_1 \xrightarrow{k_3 \times k_4} K(Z_2,4s+3) \times K(Z_2,4s+4).
\end{array}$$

Here $q_1 = p_2q_2$, q_2 is chosen so that $v_{q_2} = (\alpha_2,\alpha_3)$ and V_2 denotes the fiber of q_2. Since α_2 and α_3 are spherical, and since V_2 can also be regarded as the fiber of the map (α_2,α_3), it follows by Theorem 2 (page 23) that V_2 is (4s+2)-connected and that $\pi_{4s+3}(V_2) \cong Z_2$. We seek an invariant in $H^{4s+4}(E_2)$ to kill this group. Notice that Kernel $p_2^* =$ Kernel q_1^* through dimension $4s+4$, and that q_1^* is surjective in all dimensions since p^* is. Thus by Property 5 and Corollary 1 we have the following exact sequence for $0 \leq r \leq 4s+4$.

$$(***) \qquad 0 \to H^r(E_2) \xrightarrow{\nu} H^r(F_1 \times B_{4s+1}) \xrightarrow{\tau_1} H^{r+1}(E_1).$$

where $F_1 = K(Z_2,4s+2) \times K(Z_2,4s+3)$. Let γ_i, $i = 2,3$, denote the fundamental class of the factor $K(Z_2,4s+i)$ in the fiber F_1. Then by construction of the fibration p_2 it follows that

$$\tau\gamma_i = k_{i+1} \qquad (i = 2,3)$$

and so by Property 1,

$$\tau_1(\gamma_i \otimes 1) = k_{i+1} \quad , \quad i = 2,3.$$

Let $s_1: B_{4s+1} \to F_1 \times B_{4s+1}$ denote the injection. Using sequence (***) and the calculations given above for k_3, k_4 one finds that

Kernel $s_1^* \cap$ Kernel $\tau_1 = 0$ in dim. $\leq 4s+3$, while in dim. $4s+4$, Kernel $s_1^* \cap$ Kernel τ_1 is spanned by

$$(Sq^2\gamma_2 \otimes 1 + \gamma_2 \otimes w_2 + Sq^1\gamma_3 \otimes 1) = c$$

Let $\ell \in H^{4s+4}(E_2)$ be the class such that $\nu^*(\ell) = C$. Then in the fibration

$$V_2 \rightarrow B_{4s+1} \xrightarrow{q_2} E_2 ,$$

$q_2^* \ell = 0$ and $\sigma(\ell)$ consists of spherical classes, as is seen by applying Theorem 5 to the fibration

$$V_2 \rightarrow V_1 \xrightarrow{\alpha_2 \times \alpha_3} K(Z_2, 4s+2) \times K(Z_2, 4s+3) .$$

Thus ℓ can be used as the invariant for constructing the next fibration $p_3 : E_3 \rightarrow E_2$, and since $\sigma(\ell)$ consists of spherical classes there will be a map $q_3 : B_{4s+1} \rightarrow E_3$ such that $p_3 q_3 = q_2$ and such that the fiber of q_3 will be (4s+3)-connected.

VII. Computing Postnikov invariants

Theorem 5 provides a satisfactory criterion for deciding at each stage which classes in $H^*(E_i)$ can be used as invariants for constructing the next fibration $p_i: E_{i+1} \to E_i$. However, we are still left with the problem of "computing" these invariants. For example, take the case $i = 1$ and let $k \in H^*(E_1)$ be such an invariant. Let A be a complex and let $\xi: A \to B$ be a map that lifts to E_1. Define

$$k(\xi) = \bigcup_{\eta} \eta^* k ,$$

where the union is taken over all maps $\eta: A \to E_1$ such that $p_1 \eta = \xi$, where $p_1: E_1 \to B$. Thus if $\deg k = t$, then $k(\xi) \subset H^t(A)$. We consider the problem: compute the set $k(\xi)$. Only if a satisfactory method for solving this problem can be found is the theory of Postnikov invariants of more than limited use. In this section we consider a method that works in some situations.

We modify slightly the example given in lecture VI. Let Spin (n), $n \geq 2$, denote the universal covering group for $SO(n)$. Since Z_2 is the fiber of the covering homomorphism Spin (n) $\to$ $SO(n)$, there is a principal fibration

$$K(Z_2, 1) \to B\,\mathrm{Spin}\,(n) \xrightarrow{\Pi_n} BSO(n) , \qquad n \geq 3 ,$$

induced by $w_2 \in H^2(BSO(n))$, where we regard w_2 as a map $BSO(n) \to K(Z_2, 2)$. Thus given a complex A and an orientable bundle $\xi: A \to BSO(n)$, ξ lifts to B Spin(n) iff $w_2(\xi) = 0$.

Since the inclusion $SO(n) \subset SO(n+1)$ induces an isomorphism on the fundamental group, it follows that

$$\frac{SO(n+k)}{SO(n)} = \frac{Spin(n+k)}{Spin(n)} = V_{n+k,k}$$

for $n \geq 3$, $k \geq 1$. In particular taking $k = 2$, we obtain the following commutative diagram of fiber spaces:

$$\begin{array}{ccc} V_{n+2,2} & = & V_{n+2,2} \\ \downarrow & & \downarrow \\ B\,Spin(n) & \xrightarrow[\Pi_n]{} & BSO(n) \\ \tilde{p}\downarrow & & \downarrow p \\ B\,Spin(n+2) & \xrightarrow[\Pi_{n+2}]{} & BSO(n+2) \end{array} .$$

(Here $\tilde{p}$ and p are induced by inclusions). Thus if we have a Spin(n+2)-bundle on A (i.e., a map $\xi: A \to B\,Spin(n+2)$), then ξ has 2 linearly independent cross-sections iff there exists a map $\eta: A \to B\,Spin(n)$ such that $\tilde{p}\eta = \xi$.

We take the case $n = 4s+1$, $s > 0$, $\dim A \leq 4s+3$. Set $\tilde{B}_q =$ $B\,Spin(q)$. Since the fiber map $\tilde{p}$ can be regarded as the fibration induced from p by Π_{n+2}, so can the Postnikov resolution for $\tilde{p}$ be induced by Π_{n+2} from the resolution for p, constructed in lecture VI. In particular we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\tilde{B}_{4s+1} & \xrightarrow{\Pi_{4s+1}} & B_{4s+1} & & \\
\downarrow \tilde{q}_1 & & \downarrow q_1 & & \\
\tilde{E}_1 & \xrightarrow{\zeta} & E_1 & & \\
\downarrow \tilde{p}_1 & & \downarrow p_1 & & \\
\tilde{B}_{4s+3} & \xrightarrow[\Pi_{4s+3}]{} & B_{4s+3} & \xrightarrow[w_{4s+2}]{} & K(Z_2, 4s+2)
\end{array}$$

Here p_1, q_1 have the same meaning as in VI (so $p = p_1 q_1$), and $\tilde{p}_1$ is induced from p_1 by Π_{4s+3}. Since points in $\tilde{E}_1$ are pairs (b,e), with $b \in \tilde{B}_{4s+3}$, $e \in E_1$, such that $p_1(e) = \Pi_{4s+3}(b)$, we define

$$\zeta \colon \tilde{E}_1 \to E_1 \ , \quad \tilde{q}_1 \colon \tilde{B}_{4s+1} \to \tilde{E}_1$$

by

$$\zeta(b,e) = e \ , \quad \tilde{q}_1(x) = (\tilde{p}x,\ q_1 \Pi_{4s+1} x) \, ,$$

for $x \in \tilde{B}_{4s+1}$. Then the diagram is commutative, $\tilde{p} = \tilde{p}_1 \tilde{q}_1$, and the fundamental class of $V_{4s+3,2}$ is geometrically realized by $\tilde{q}_1$. (See pages 10, 11).

Set $\tilde{k} = \zeta^* k_3 \in H^{4s+3}(\tilde{E}_1)$. Then $\tilde{k}$ is the only obstruction to lifting a map $\eta \colon A \to \tilde{E}_1$ into $\tilde{B}_{4s+1}$, provided that $\dim A < 4s+3$.

Recall that k_3 is characterized by the fact that

$$\nu^*(k_3) = Sq^2 \iota \otimes 1 + \iota \otimes w_2 \in H^{4s+3}(K(Z_2, 4s+1) \times B_{4s+1}) \, .$$

(See sequence (*) in VI.) Since $\Pi^*_{4s+1} w_2 = 0$, it follows by naturality that

$$\tilde{\nu}^*(\tilde{k}) = Sq^2\iota\otimes 1 \in H^{4s+3}(K(Z_2,4s+1)\times\tilde{B}_{4s+1}),$$

where $\tilde{\nu}^*$ denotes the operator ν^* in the fibration $\tilde{p}_1$. Consequently, the invariant $\tilde{k}$ is characterized uniquely by the properties

$$(1) \qquad \tilde{q}_1^* \tilde{k} = 0, \quad \tilde{i}^* \tilde{k} = Sq^2\iota,$$

where $\tilde{i}\colon K(Z_2,4s+1) \subset \tilde{E}_1$ denotes the inclusion of the fiber.

Now $\tilde{p}_1$ is a principal fibration with fiber $K(Z_2,4s+1)$. Let

$$m\colon K(Z_2,4s+1)\times\tilde{E}_1 \to \tilde{E}_1$$

denote the action in this fibration (see lecture II). Then by Lemma 3(a), (b), and (1) above, since $\tilde{E}_1$ is 3-connected, we have

$$(2) \qquad m^*\tilde{k} = Sq^2\iota\otimes 1 + 1\otimes\tilde{k}.$$

Given $\xi\colon A \to \tilde{B}_{4s+3}$, define as above,

$$\tilde{k}(\xi) = \bigcup_{\eta} \eta^*\tilde{k} \subset H^{4s+3}(A),$$

where the union is over all maps $\eta\colon A \to \tilde{E}_1$ such that $\tilde{p}_1\eta = \xi$. By Lemma 3(c) and (2) above we see that $\tilde{k}(\xi)$ is a coset of the subgroup

$$Sq^2 H^{4s+1}(A) \subset H^{4s+3}(A).$$

Summarizing, we have proved:

(3) Let $\xi: A \to \tilde{B}_{4s+3}$, where A is a complex of dim $\leq 4s+3$. Suppose that $w_{4s+2}(\xi) = 0$. Then ξ lifts to $\tilde{B}_{4s+1}$, iff $0 \in \tilde{k}(\xi) \subset H^{4s+3}(A)$. Moreover, $\tilde{k}(\xi)$ is a coset of the subgroup $Sq^2 H^{4s+1}(A)$.

We seek a way to compute $\tilde{k}(\xi)$. Our method uses the secondary cohomology operations of Adams. (See Chapter 3 of [1]). Let Φ denote the operation associated with the Adem relation

(4) $$Sq^2 Sq^2 + Sq^1(Sq^2 Sq^1) = 0.$$

Thus Φ is defined on those classes $u \in H^*(A)$ such that

$$Sq^2(u) = Sq^2 Sq^1(u) = 0.$$

And $\Phi(u)$ is then a coset of the subgroup

$$Sq^2 H^{n+1}(A) + Sq^1 H^{n+2}(A)$$

in $H^{n+3}(A)$, assuming dim $u = n$.

We prove

Theorem 6. Let A and ξ be as in (3), with $s>0$. If $Sq^1 H^{4s+2}(A) \subset Sq^2 H^{4s+1}(A)$, then

$$\tilde{k}(\xi) = \Phi(w_{4s}(\xi)),$$

as cosets of $Sq^2H^{4s+1}(A)$ in $H^{4s+3}(A)$. Thus, by (3), ξ has 2 linearly independent cross-sections, iff $0 \in \Phi(w_{4s}(\xi))$.

Notice that Φ is indeed defined on $w_{4s}(\xi)$. For by the formulae of Wu,

$$Sq^2 w_{4s}(\xi) = w_2(\xi) \cdot w_{4s}(\xi) + w_{4s+2}(\xi) = 0,$$

since $w_{4s+2}(\xi) = 0$ by hypothesis and $w_2(\xi) = 0$ since ξ is a Spin (4s+3)-bundle. Similarly,

$$Sq^2\ Sq^1\ w_{4s}(\xi) = w_2(\xi) \cdot w_{4s+1}(\xi) = 0.$$

The point of the theorem is that there are several good techniques for computing secondary operations, especially if A is a manifold. As an example we state (without proof) an important consequence of the theorem.

Theorem 7. Let M be a closed , connected, smooth manifold of dim 4s+3, $s \geq 0$. Suppose that $w_1(M) = w_2(M) = 0$. Then M has 2 linearly independent tangent vector fields.

Here $w_i(M) \in H^i(M)$, $i \geq 0$, denotes the i^{th} Stiefel-Whitney class of the tangent bundle of M. Recall that by the classical theorem of H.Hopf, one knows that M has at least one non-zero vector field since dim M is odd. The proof of Theorem 7 consists in applying 6 to the tangent bundle of M and showing that $0 \in \Phi(w_{4s}(M))$. For details see [12].

Proof of Theorem 6. The case s = 1 is somewhat anomalous, and so we assume that $s > 1$. We begin by constructing the universal example for the operation Φ. Consider the following (commutative) diagram of spaces and maps:

$$
\begin{array}{ccccccc}
K(Z_2,4s+1) & \xrightarrow{i} & K(Z_2,4s+1)\times K(Z_2,4s+2) & \xrightarrow{j} & Z & & \\
 & & \Big\downarrow{\scriptstyle \varrho} & & \Big\downarrow{\scriptstyle g} & & \\
 & & K(Z_2,4s+2) & \longrightarrow & Y & \xrightarrow{Sq^2\circ f} & K(Z_2,4s+2) \\
 & & & & \Big\downarrow{\scriptstyle f} & & \\
 & & & & K(Z_2,4s) & \xrightarrow{Sq^2Sq^1} & K(Z_2,4s+3)\,.
\end{array}
$$

Here f is the principal fibration induced by Sq^2Sq^1, and g is the principal fibration induced by $Sq^2\circ f$. Thus the composite fibration $f\circ g\colon Z \to K(Z_2,4s)$ has $K(Z_2,4s+1)\times K(Z_2,4s+2)$ as fiber. The map j denotes the inclusion of this fiber into the total space Z. The maps ϱ and i denote, respectively, the projection and inclusion. By construction the fiber of g is $K(Z_2,4s+1)$, imbedded by the composite map $j\circ i$. Because of the Adem relation (4) and the Serre exact sequence, there is a (unique) class $\phi\in H^{4s+3}(Z)$ such that

$$j^*\phi = Sq^2\iota_1\otimes 1 + 1\otimes Sq^1\iota_2,$$

where ι_i denotes the fundamental class of the factor $K(Z_2,4s+i)$, $i = 1,2$.

Let A be a complex and $u\in H^{4s}(A)$ a class such that $Sq^2(u) = Sq^2Sq^1(u) = 0$. If we regard u as a map $A \to K(Z_2,4s)$, then u lifts to Z and by definition

$$\Phi(u) = \bigcup_{v} v^* \emptyset,$$

where the union is taken over all maps $v: A \to Z$ such that $fgv = u$.

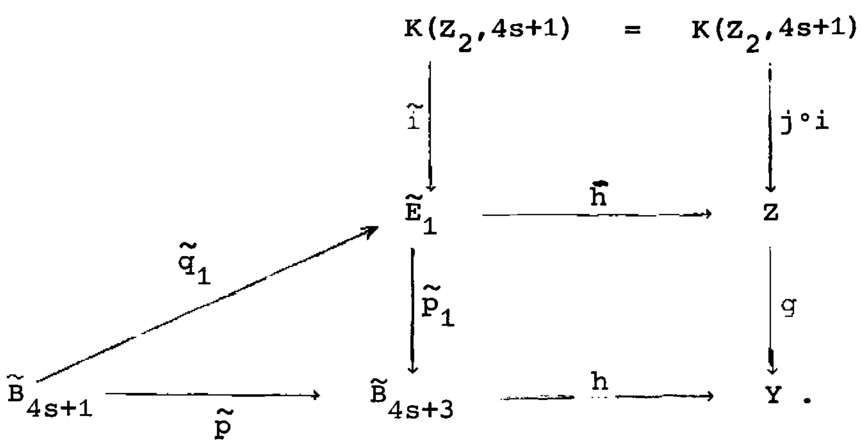
We consider again the Postnikov resolution for the Spin-fibration $\tilde{p}$. Regard $w_{4s} \in H^{4s}(\tilde{B}_{4s+3})$ as a map $\tilde{B}_{4s+3} \to K(Z_2, 4s)$. Since

$$Sq^2 Sq^1 w_{4s} = Sq^2 w_{4s+1} = w_2 \cdot w_{4s+1} = 0 ,$$

there is a map $h: \tilde{B}_{4s+3} \to Y$ such that $fh = w_{4s}$. (Here Y and f refer to the above diagram). But

$$Sq^2 \circ f \circ h = Sq^2 w_{4s} = w_{4s+2} .$$

Since $\tilde{p}_1: \tilde{E}_1 \to \tilde{B}_{4s+3}$ was defined as the fibration induced by w_{4s+2}, we can regard $\tilde{p}_1$ as the fibration induced by h from the fibration $g: Z \to Y$, since g is induced by the map $Sq^2 \circ f$ and $Sq^2 \circ f \circ h = w_{4s+2}$. Consequently, we obtain the commutative diagram below:



Set $k_1 = \bar{h}^*\phi \in H^{4s+3}(\tilde{E}_1)$. Then,

$$\tilde{i}^* k_1 = \tilde{i}^*\bar{h}^*\phi = j^* i^*\phi = Sq^2 \iota_1 = \tilde{i}^*\tilde{k} .$$

Furthermore, by definition,

$$k_1 \in \Phi(\tilde{p}_1^* w_{4s}) ,$$

since $fh\tilde{p}_1 = w_{4s}\tilde{p}_1$, and so by naturality

$$(5) \qquad \tilde{q}_1^* k_1 \in \Phi(\tilde{p}^* w_{4s}) ,$$

since $\tilde{p} = \tilde{p}_1 \tilde{q}_1$.

For convenience we define for any space X

$$I^q(X) = Sq^2 H^{q-2}(X) + Sq^1 H^{q-1}(X), \qquad q \geqslant 3.$$

Thus $I^q(X)$ is the indeterminacy subgroup of the operation Φ, defined on classes of degree $q-3$.

We now need a fact whose proof is given in [12].

$$\Phi(\tilde{p}^* w_{4s}) = \tilde{p}^* I^{4s+3}(\tilde{B}_{4s+3}).$$

Assuming this we see by (5) that there is a class $\alpha \in I^{4s+3}(\tilde{B}_{4s+3})$ such that

$$\tilde{q}_1^* k_1 = \tilde{p}^* \alpha .$$

Set

$$k_2 = k_1 - \tilde{p}_1^* \alpha \in H^{4s+3}(\tilde{E}_1).$$

Then,

$$\tilde{i}^*k_2 = \tilde{i}^*k_1 - \tilde{i}^*\tilde{p}_1^*\alpha = \tilde{i}^*k_1 = \tilde{i}^*k ,$$
$$\tilde{q}_1^*k_2 = \tilde{q}_1^*k_1 - \tilde{q}_1^*\tilde{p}_1^*\alpha = \tilde{q}_1^*k_1 - \tilde{p}^*\alpha = 0 .$$

Therefore by (1), $k_2 = \tilde{k}$, and so we have shown that

$$\tilde{k} \in \Phi(\tilde{p}_1^* \, w_{4s}) ,$$

since $\tilde{k}$ differs from k_1 by an element of $I^{4s+3}(\tilde{E}_1)$.

Now let $\xi\colon A \to \tilde{B}_{4s+3}$ be a bundle as in (3) and let $\eta\colon A \to \tilde{E}_1$ be a map such that $\tilde{p}_1\eta = \xi$. Then, by definition,

$$\eta^*\tilde{k} \in \tilde{k}(\xi) .$$

But by naturality,

$$\eta^*\tilde{k} \in \Phi(\eta^*\tilde{p}_1^*w_{4s}) = \Phi(w_{4s}(\xi)) ,$$

which completes the proof, since by hypothesis $\tilde{k}(\xi)$ and $\Phi(w_{4s}(\xi))$ are cosets of the same subgroup.

The method of proof given for Theorem 6 is generalized in [12] to a method that handles other sorts of fibrations. In particular the method applies to give results on cross-sections of complex vector bundles, on immersions of certain manifolds in Euclidean space, and on the existence of almost-complex structures on certain 8-manifolds. Also, it is shown in [12] that by using "twisted" cohomology operations, one can remove the hypothesis $w_2(\xi) = 0$ in Theorem 6 (and hence remove the hypothesis $w_2(M) = 0$ in Theorem 7). For a resumé of these results, see [11].

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