THE GENERALIZED PONTRJAGIN COHOMOLOGY OPERATIONS AND RINGS WITH DIVIDED POWERS.

by
Emery Thomas*
University of California (Berkeley)

INTRODUCTION

This paper concerns a generalization of the Pontrjagin square cohomology operation. This cohomology operation was defined originally by Pontrjagin in 1942 (see [5]), and was then studied in detail by J. H. C. Whitehead [17], [19], [20]. Additional research on the operation has been done by Eilenberg and MacLane [2], [3]; Wu [21]; Nakaoka [4]; and Yamanoshita [22].

In its simplest form the Pontrjagin square is a function

$$\mathfrak{P}_2\colon\ H^{2n}(K;Z/2^rZ) \longrightarrow H^{4n}(K;Z/2^{r+1}Z),$$

where $H^q(K;G)$ denotes the q^{th} cohomology group of a complex K with coefficients in a group G, and Z is the integers. The cohomology operations defined in this paper generalize the Pontrjagin square in two ways: first, an entire sequence of operations will be defined, $\mathfrak{P}_t$ $(t = 0, 1, \ldots)$, such that when t is a prime number p,

$$\mathfrak{P}_p\colon\ H^{2n}(K;Z/p^rZ) \longrightarrow H^{2pn}(K;Z/p^{r+1}Z).$$

In particular, the function $\mathfrak{P}_2$ is the Pontrjagin square. Secondly, the coefficients will consist of groups which are summands of a certain type of graded ring. This special kind of ring is a ring with divided powers, a concept recently defined by H. Cartan (see [1]).

Rings with divided powers may be viewed as generalizations of the following example: consider a commutative, graded algebra over the field of the rationals. For each integer t denote by g_t the mapping $x \longrightarrow x^t/t!$ It is precisely the properties of the functions g_t which we take as a definition for a ring with divided powers (see §1).

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We will consider as coefficient domain only those rings with divided powers where each summand of the graded ring is a cyclic group of infinite or prime power order; such rings will be called *p-cyclic*. The main result of the paper may then be stated roughly (see theorem I):

> *let X be a space and consider its cohomology with coefficients in a p-cyclic ring with divided powers. Then, the cohomology ring of the space is itself a (bi-graded) ring with divided powers.*

In fact, the generalized Pontrjagin operations, $\mathfrak{P}_t$, will play the role of the divided power functions, g_t, mentioned above.

The significance of rings with divided powers is underlined by the fact that the integral homology ring of an Eilenberg - MacLane space is a ring with divided powers (see [1]). An important class of these rings is provided by a construction of Eilenberg - MacLane ([3]): for each abelian group π they construct a ring with divided powers, $\Gamma(\pi)$ (see §1). Of particular importance is the fact that $\Gamma(\pi)$ appears as a direct summand of the homology ring $H_*(\pi,n)$ (n even).

The present definition of the functions $\mathfrak{P}_t$ is based upon an earlier definition, given without details in [15]. In that paper the ring $\Gamma(\pi)$ was used as coëfficients, but it was not regarded as a ring with divided powers. I would like to thank Professors H. Cartan and S. Eilenberg for suggesting to me the idea of using rings with divided powers. This generalization seems to indicate the underlying algebraic nature of the cohomology operations $\mathfrak{P}_t$. It should be emphasized that these are cohomology operations whose definition requires the specifying of an entire graded *ring* as a coefficient domain, and not just a pair of groups.

In as much as these operations generalize the Pontrjagin square, it is reasonable to expect that applications of the new operations will arise in situations which generalize the applications of the Pontrjagin square. At present the applications of that operation are: computing the second obstruction in the low dimensional case, expressing the Pontrjagin characteristic classes (mod 4) in terms of the Stiefel - Whitney classes, and calculating certain cohomology classes in the symmetric products of complexes. These topics indicate the direction in which application of the operations $\mathfrak{P}_t$ should be sought.

The plan of the paper is this: in section 1 the two main results are stated. The first of these is the one given above; the second gives some special properties of the operations $\mathfrak{P}_t$ when we restrict the coefficients to a certain category. The two main theorems are proved by first defining a set of subsidiary functions P_t (called "model operations"), and obtaining properties of these. The operations $\mathfrak{P}_t$ are then defined using the functions P_t, and their properties follow from the properties of the functions P_t. Thus, section 2 states the properties of these model operations, section 3 defines the operations $\mathfrak{P}_t$, and section 4 gives the proofs of the two main theorems. The remainder of the paper is then devoted to proving the properties of the model operations P_t. In the appendix an example is given of a computation of the operations $\mathfrak{P}_t$: namely, for the infinite complex projective space. Furthermore, a theorem is stated there to the effect that the operations $\mathfrak{P}_t$ give information not obtained by other known cohomology invarients.

The method used here to define the operations $\mathfrak{P}_t$ is that developed by Professor N. Steenrod (see [10], [11], [13]). I would like to express to him my sincere gratitude and warm thanks for the advice and encouragement he has extended to me throughout the preparation of this paper.

1. THE MAIN THEOREMS

We denote by $\mathcal{R}$ the category whose objects are commutative, graded rings with unit, and whose maps are all functions between such rings. If R is a ring in $\mathcal{R}$ and $R = \sum_{k=0}^{\infty} R_k$, we will in general assign degree k to the elements of the subgroup R_k (1).

Let R be a ring in $\mathcal{R}$. We will say that R is a *ring with divided powers* (see H. Cartan [1; chapter 7]), if to each element $x \in R$ of even degree ≥ 2, there is assigned a sequence of elements $g_r(x) \in R$ $(r = 0, 1, 2, \ldots)$, such that the functions g_r have the following properties:

(1.1) $g_r\colon R_{2k} \longrightarrow R_{2rk}$,

(1.2) $g_0(x) =$ unit of R; $g_1(x) = x$,

(1.3) $g_r(x)g_s(x) = (r,s)g_{r+s}(x)$,

$$(1.4)\qquad g_t(x+y) = \Sigma_{r+s=t}\, g_r(x)g_s(y),$$

$$(1.5)\qquad g_s(g_r(x)) = \epsilon_{s,r}\, g_{rs}(x), \qquad (r > 0)$$

$$(1.6)\qquad g_r(xy) = r!g_r(x)g_r(y) \qquad (r \geq 2)$$
if x and y have even degree ≥ 2,

$$(1.7)\qquad g_r(xy) = 0 \qquad (r \geq 2)$$
if x and y have odd degree,

where (r,s) denotes the binomial coefficient $(r+s)!/(r!)(s!)$, and $\epsilon_{s,r} = (r,r-1)(2r,r-1) \ldots ((s-1)r,r-1)$.

Let R and R' be rings with divided powers. By a *map* α from R to R' we will mean the following:

(1.8) (i) α is a ring homomorphism of degree zero; that is, if $x \in R_i$, then $\alpha(x) \in R_i'$, $(i = 0, 1, \ldots)$.

(ii) If g_i, g_i' $(i = 0, 1, \ldots)$ are the divided power functions for R and R' respectively, then

$$\alpha g_i = g_i'\alpha.$$

We denote by $\Gamma(\mathcal{R})$ the category whose objects are rings with divided powers and whose maps are the functions defined in 1.8. A ring R in $\Gamma(\mathcal{R})$ will be called a Γ-*ring*, and the divided power functions g_i $(i = 0, 1, \ldots)$ will be called the γ-functions for R.

Let π be an abelian group; following Eilenberg-MacLane we will say that π is *p-cyclic* if it is a cyclic group whose order is infinite or a power of some prime. Let R be a ring in $\mathcal{R}$. We will say that R is a *p-cyclic ring* if each summand R_k of R is a p-cyclic group, where $R = \Sigma R_k$. We denote by ${}_p\Gamma(\mathcal{R})$ the subcategory of $\Gamma(\mathcal{R})$ which consists of all p-cyclic Γ-rings. It is this category which is the basic coefficient domain for the remainder of the paper. Examples of such rings will be given later in this section.

Suppose that X is a topological space and that G is an abelian group. As usual we denote the q^{th} (singular) cohomology group of X with coefficients in G by $H^q(X;G)$. If we are given a ring $R \in \mathcal{R}$, then we define the *cohomology ring* of X with coefficients in R, written $H^*(X;R)$, to be the bi-graded ring $\Sigma_{m,n} H^m(X;R_n)$. The multiplication in $H^*(X;R)$ is the cup-product defined using the natural pairing given by multiplication in the

ring R. That is, if $u \in H^m(X;R_i)$ and $v \in H^n(X;R_j)$, then

$$u \smile v \in H^{m+n}(X;R_{i+j}).$$

If $u \in H^m(X;R_n)$ we define the *base degree* of u to be m, the *coefficient degree* of u to be n, and the *total degree* of u to be $m + n$. If m, $n \geq 1$ and at least one of them is odd, we will say that u has *semi-odd degree*. We now can state the first main result of the paper.

THEOREM I: *Let X be a space, and let A be a p-cyclic Γ-ring. Then, the cohomology ring $H^*(X;A)$ is a bi-graded ring with divided powers. That is, to each element $u \in H^*(X;A)$ with base degree and coefficient degree even ≥ 2, there is assigned a sequence of elements $\not{p}_r(u)$ $(r = 0, 1, 2, \ldots)$, such that the functions $\not{p}_r$ have the following properties:*

$$(1)\qquad \not{p}_r\colon H^{2m}(X;A_{2n}) \longrightarrow H^{2rm}(X;A_{2rn}),$$

$$(2)\qquad \not{p}_0(u) = \text{unit of } H^*(X;A);\quad \not{p}_1(u) = u,$$

$$(3)\qquad \not{p}_r(u) \smile \not{p}_s(u) = (r,s)\not{p}_{r+s}(u),$$

$$(4)\qquad \not{p}_t(u + v) = \Sigma_{r+s=t}\, \not{p}_r(u) \smile \not{p}_s(v),$$

$$(5)\qquad \not{p}_s(\not{p}_r(u)) = \epsilon_{s,r}\, \not{p}_{rs}(u), \qquad (r > 0)$$

$$(6)\qquad \not{p}_r(u \smile u') = r!\, \not{p}_r(u) \smile \not{p}_r(u') \qquad (r \geq 2)$$
if u and u' have even base and coefficient degree ≥ 2,

$$(7)\qquad \not{p}_r(u \smile u') = 0 \qquad (r \geq 2)$$
if u and u' each have semi-odd degree,

$$(8)\qquad \not{p}_r\, f^*(u) = f^*\, \not{p}_r(u),$$

$$(9)\qquad \not{p}_r\, \alpha_*(u) = \alpha_*\, \not{p}_r(u),$$

where f^* is induced by a map f of a space Y to X, and α_* is induced by a homomorphism α of A to a p-cyclic Γ-ring A'.

The proof of theorem I is given in section 4. We now construct two examples of p-cyclic Γ-rings, which will play important roles in what is to follow.

Let Π be an abelian group. We define a commutative, graded ring $\Gamma(\Pi)$ as follows (see Eilenberg-MacLane [3; p. 107]): $\Gamma(\Pi)$ has as generators the elements $\gamma_t(x)$ for each $x \in \Pi$ and non-negative integer t. These generators have the following relations (2):

(1.9) $\gamma_o(x) = 1,$

(1.10) $\gamma_r(x)\ \gamma_s(x) = (r,s)\ \gamma_{r+s}(x),$

(1.11) $\gamma_t(x + y) = \Sigma_{r+s=t}\ \gamma_r(x)\ \gamma_s(y).$

We assign degree 2t to the generator $\gamma_t(x)$, and set (1),

(1.12) $\Gamma_t(\Pi)$ = subgroup of $\Gamma(\Pi)$ of elements of degree 2t.

Then, $\Gamma(\Pi) = \Gamma_o(\Pi) + \Gamma_1(\Pi) + \dots$ (direct sum). Eilenberg-MacLane show that $\Gamma_o(\Pi) = Z$, and that $\Gamma_1(\Pi)$ is naturally isomorphic to Π by the map $\gamma_1(x) \longrightarrow x$. We make $\Gamma(\Pi)$ into a Γ-ring by defining the γ-functions g_s ($s = 0, 1, \dots$) as (see [1]):

(1.13) $g_s\ \gamma_t(x) = \epsilon_{s,t}\ \gamma_{st}(x)$ $(t > 0)$

where $\epsilon_{s,t} = (t,t-1)(2t,t-1)\ \dots\ ((s-1)t,t-1)$. The functions are defined on a product by means of 1.6 and 1.7. Notice that

$$g_s\ \gamma_1(x) = \gamma_s(x).$$

Let Π' be a second abelian group and α a homomorphism from Π to Π'. We define a ring homomorphism $\Gamma(\alpha)$ mapping $\Gamma(\Pi)$ to $\Gamma(\Pi')$ by

(1.14) $\Gamma(\alpha)\ \gamma_k(x) = \gamma_k'(\alpha x),$

$\Gamma(\alpha)\ [\gamma_r(x)\ \gamma_s(y)] = \gamma_r'(\alpha x)\ \gamma_s'(\alpha y),$

where $\gamma_t'(x')$ denotes a generator of $\Gamma(\Pi')$ (3). The function $\Gamma(\alpha)$ is

in fact a <u>map</u> in the category $\Gamma(\mathcal{R})$; for,

$$\Gamma(\alpha)g_s(\gamma_t(x)) = \Gamma(\alpha)\epsilon_{s,t}\ \gamma_{st}(x) = \epsilon_{s,t}\ \gamma_{st}'(\alpha x)$$

$$= g_s'\ \Gamma(\alpha)\ \gamma_t(x),$$

which satisfies 1.8. Thus, assigning $\Gamma(\Pi)$ to Π and $\Gamma(\alpha)$ to α gives us a functor from the category of abelian groups to the category $\Gamma(\mathcal{R})$.

Suppose now that Π is a cyclic group of order θ (where $\theta = 0$ means that Π is infinite cyclic). Let r be any non-negative integer, and set

(1.15) $$[r,\theta^\infty] = \begin{cases} \text{common value of G. C. D. } [r,\theta^e] \text{ for large e,} & r > 0 \\ 0, & r = 0 \end{cases}$$

Then, Eilenberg-MacLane show that

(1.16) $\Gamma_r(\Pi)$ = cyclic group of order $\theta[r,\theta^\infty]$. $(r \geq 0)$

For example,

$$\Gamma_{p^r}(Z/p^sZ) \approx Z/p^{r+s}Z, \qquad (p \text{ prime})$$

$$\Gamma_t(Z) \approx Z \qquad (t \geq 0)$$

where Z denotes the integers.

It follows that if Π is a p-cyclic group, then $\Gamma(\Pi)$ is a p-cyclic Γ-ring. Thus, as an immediate corollary to theorem I we have:

COROLLARY A: <u>Let</u> X <u>be a space, and let</u> Π <u>be a p-cyclic group. Then,</u> $H^*(X; \Gamma(\Pi))$ <u>is a bi-graded ring with divided powers.</u>

To construct a second example of Γ-rings, we begin with a particular category of cyclic groups. Let r be any non-negative integer, and set Z_r = integers mod r, where $Z_o = Z$. Define a category $\mathcal{G}$ as follows: an object of $\mathcal{G}$ is a group Z_θ, where θ is zero or a power of a prime. The maps of $\mathcal{G}$ are all homomorphisms between such groups. Thus, each group of $\mathcal{G}$ is a p-cyclic group. We define a functor G from $\mathcal{G}$ to the category

$_p\Gamma(\mathcal{R})$ by setting

$$G(Z_\theta) = G_o(Z_\theta) + G_1(Z_\theta) + \ldots \text{ (direct sum)},$$

where

(1.17) $$G_r(Z_\theta) = Z_{\theta[r,\theta^\infty]} \qquad (r = 0,1,2,\ldots)$$

and $\theta[r,\theta^\infty]$ is the integer defined in 1.15. We assign degree 2k to the elements of $G_k(Z_\theta)$. Thus, $G(Z_\theta)$ is a graded group in which $G_1(Z_\theta) = Z_\theta$; unless stated otherwise, we will always regard the group Z_θ as the summand $G_1(Z_\theta)$ of $G(Z_\theta)$. We introduce a multiplication into the group $G(Z_\theta)$ by observing first that

$$\text{order } G_r(Z_\theta) = \text{order } \Gamma_r(Z_\theta). \qquad (r = 0,1,\ldots)$$

Thus, we set up a canonical isomorphism

(1.18) $$\rho_k = \rho\colon\ G_k(Z_\theta) \approx \Gamma_k(Z_\theta)$$

by defining

$$\rho(1_{\theta[k,\theta^\infty]}) = \gamma_k(1_\theta),$$

where $1_r = 1 \bmod r \in Z_r$ $(r = 0, 1, \ldots)$. If $c_r \in G_r(Z_\theta)$ and $c_s \in G_s(Z_\theta)$, we define the product c_rc_s in $G_{r+s}(Z_\theta)$ by

(1.19) $$c_rc_s = \rho^{-1}[\rho(c_r)\rho(c_s)].$$

Hence, $G(Z_\theta)$ is a graded ring. To make $G(Z_\theta)$ a Γ-ring we define γ-functions g_r $(r = 0, 1, \ldots)$ by

(1.20) $$g_r = \rho^{-1}g_r'\rho,$$

where g_r' are the γ-functions in $\Gamma(Z_\theta)$. In particular, 1.18 and 1.20 imply:

(1.21) $$g_r(a) = \rho^{-1}\gamma_r(a),\ g_r(na) = n^rg_r(a),$$

$$g_r(1_\theta) = 1 \bmod \theta[r,\theta^\infty],$$

where $a \in Z_\theta$ $(= g_1(a) \in G_1(Z_\theta))$ and n is any integer. It is easily verified that the functions g_r defined in 1.20 satisfy the properties 1.1 through 1.7, since the γ-functions g_r' do. Thus, $G(Z_\theta)$ is a p-cyclic Γ-ring.

Let Z_θ be a group in $\mathcal{G}$; then, for each integer $r \geq 0$, $G_r(Z_\theta)$ is itself a group in $\mathcal{G}$. Hence, we can apply the functor G to the group $G_r(Z_\theta)$. From definition 1.17 one has:

(1.22) $$G_s(G_r(Z_\theta)) = G_{sr}(Z_\theta),$$

that is, from the standpoint of groups. Of course, considered as summands of graded rings the two groups in general have different degrees.

Finally, suppose that Z_θ and Z_τ are two groups in $\mathcal{G}$, and that α is a homomorphism of Z_θ to Z_τ. With α we associate a map $G(\alpha)$ of $G(Z_\theta)$ to $G(Z_\tau)$ by

(1.23) $$G(\alpha) = \rho^{-1}\Gamma(\alpha)\rho,$$

where ρ is defined in 1.18 and $\Gamma(\alpha)$ in 1.14. In particular,

(1.24) $$G(\alpha)g_r = g_r\alpha \qquad (r = 0,1,\ldots)$$

as mappings of Z_θ to $G_r(Z\ )$. Thus, assigning $G(Z_\theta)$ to Z_θ and $G(\alpha)$ to α defines a covariant functor from the category $\mathcal{G}$ to the category $_p\Gamma(\mathcal{R})$. Therefore, as a second corollary to theorem I we have:

COROLLARY B: <u>Let</u> X <u>be a space, and let</u> Z_θ <u>be a group in the category</u> $\mathcal{G}$. <u>Then</u>, $H^{\#}(X;G(Z_\theta))$ <u>is a bi-graded ring with divided powers</u>.

If we restrict ourselves to the ring $G(Z_\theta)$ as coefficients, then the functions $\mathfrak{p}_t$ have certain special properties. Suppose that Z_θ is any group in $\mathcal{G}$. Then there is a canonical way of introducing a multiplication into the group, since Z_θ is the factor group of the ring Z by the ideal θZ. Using this multiplication as a pairing we define a second cup-product, written $\smile'$. That is, if $u \in H^m(X;Z_\theta)$ and $v \in H^n(X;Z_\theta)$, then

$$u \smile' v \in H^{m+n}(X;Z_\theta).$$

Since $G_r(Z_\theta) \in \mathcal{G}$ for $r = 0, 1, \ldots$, we clearly have the same pairing defined in each such group.

Let us compare the groups Z_θ and $G_r(Z_\theta)$, using 1.15 and 1.17: namely,

(i) if $\theta = 0$, then $Z_o = Z = G_r(Z_o)$;

(ii) if $r = 0$, then $G_o(Z_\theta) = Z$,

(iii) if $r, \theta > 0$, then the order of $G_r(Z_\theta)$ is a multiple of the order of Z_θ.

Thus, in all cases a well-defined (group) homomorphism η_r mapping $G_r(Z_\theta)$ to Z_θ is given by:

(1.25) $$\eta_r g_r(1_\theta) = 1_\theta.$$

Now, $G_r(Z_\theta)$ is itself a group in $\mathcal{G}$. Hence, using the functor G we have:

$$G(\eta_r): G(G_r(Z_\theta)) \longrightarrow G(Z_\theta).$$

In particular, for any integer $s \geq 0$, $G_s(\eta_r)$ maps $G_s(G_r(Z_\theta))$ to $G_s(Z_\theta)$. But from the group standpoint, $G_s(G_r(Z_\theta)) = G_{sr}(Z_\theta)$, as remarked in 1.22. Thus, if we continue to denote $G_s(\eta_r)$ by η_r, we have for $r,s \geq 0$ a well-defined (group) homomorphism η_r mapping $G_{sr}(Z_\theta)$ to $G_s(Z_\theta)$ given by

(1.26) $$\eta_r\, g_{sr}(1_\theta) = g_s(1_\theta).$$

Using the $\smile'$-product defined above and the functions η_r given by 1.26, we have the following result.

THEOREM II: *Let X be a space, and let Z_θ be a group in $\mathcal{G}$. Let $G(Z_\theta)$ be the p-cyclic Γ-ring defined in 1.17. Then, for*

$u \in H^{2n}(X;Z_\theta)$ and integers r, s, $t \geq 1$, we have:

(i) $\eta_{r*}\mathfrak{P}_{rs}(u) = [\mathfrak{P}_s(u)]^r$, (r-fold $\smile'$-product)

(ii) $\mathfrak{P}_t(u) = u^t$, *if θ is zero or is prime to t.*

Suppose further that $u_i \in H^{2n_i}(X;Z_\theta)$ $(i = 1, \ldots, r)$, and that t is a positive integer. Then,

(iii) *if t is odd,*

$$\mathfrak{P}_t(u_1 \smile' \ldots \smile' u_r) = \mathfrak{P}_t(u_1) \smile' \ldots \smile' \mathfrak{P}_t(u_r),$$

(iv) *if t is even*

$$\mathfrak{P}_t(u_1 \smile' \ldots \smile' u_r) = \mathfrak{P}_t(u_1) \smile' \ldots \smile' \mathfrak{P}_t(u_r) + \Psi_t(u_1, \ldots, u_r)$$

where

(a) $\Psi_t(u_1, \ldots, u_r) \in H^{2tn}(X;G_t(Z_\theta))$, $(n = n_1 + \ldots + n_r)$

(b) $\Psi_t(u_1, \ldots, u_r) = 0$, *if θ is odd or equals zero,*

(c) $\Psi_t(u, \ldots, u) = 0$, $(u_1 = \ldots = u_r = u)$

(d) $2\Psi_t(u_1, \ldots, u_r) = 0$.

The proof of this theorem is given in section 4.

2. THE MODEL OPERATIONS, P_t $(t = 0, 1, \ldots)$

Suppose that we have proved theorems I and II for the category of regular cell complexes (see [9; §2]). We indicate here how to extend the proofs of the theorems to arbitrary topological spaces.

Let A be a p-cyclic ring with divided powers (see §1), and let A_{2k} be any summand of A of even degree $(k > 0)$. Let n be any positive integer. Since A_{2k} is a finitely generated abelian group, there exists a regular cell complex K_n with the following properties:

(i) K_n is an Eilenberg-MacLane space of type $(A_{2k}, 2n)$; i.e.,
$\pi_{2n}(K_n) \approx A_{2k}$, $\pi_r(K_n) = 0$, $r \neq 2n$;

(ii) the r-skeleton of K_n is a finite regular cell complex for each $r = 0, 1, \ldots$.

The existence of such a complex is remarked by Thom in [14; §6]. Let us assume for the moment that we have proved theorems I and II for the category of regular cell complexes; in particular, the functions $\wp_t$ are then defined for the cohomology ring $H^*(K_n;A)$. Now, let M be any CW-complex (see [18; §5]). We define the functions $\wp_t$ for the complex M in the usual way: that is, if $u \in H^{2n}(M;A_{2k})$, we define

$$\wp_t(u) = f_u^* \wp_t(\iota_n),$$

where ι_n is the fundamental class of $H^{2n}(K_n;A_{2k})$, and f_u^* is induced by a map f_u: $M \longrightarrow K_n$ such that $f_u^*(\iota_n) = u$. One easily verifies that the functions defined in this way continue to satisfy theorems I and II.

Finally, let X be an arbitrary space; denote by K(X) the geometric realization of the singular complex of X (see [19; §19, 20]). Then, K(X) is a CW-complex and there is a natural map $k : K(X) \longrightarrow X$ such that k^*: $H^*(X;G) \approx H^*(K(X);G)$, for any coefficient group G. Let $u \in H^{2n}(X;A_{2k})$; we set

$$\wp_t(u) = k^{*-1} \wp_t k^*(u).$$

Once again the functions so defined satisfy theorems I and II, as is easily verified. Thus, we need prove I and II only for the category of regular cell complexes. For the remainder of the paper, "complex" will mean "regular cell complex" unless otherwise noted.

Theorems I and II are proved for the category of complexes in the following way. We define a set of cohomology operations P_r $(r = 0, 1, \ldots)$ which take coefficients only in the category $\mathcal{G}$ (see §1). Then the γ-functions $\wp_r$ (see theorem I) are defined by means of the functions P_r, and theorems I and II are proved using the properties of the functions P_r stated in this section.

(2.1) PROPOSITION: Let K be a complex, and let Z_θ be a group in the category $\mathcal{G}$. Then, there is a sequence of functions P_r $(r = 0, 1, \ldots)$ with the following properties:

(i) P_r: $H^{2n}(K;Z_\theta) \longrightarrow H^{2rn}(K;G_r(Z_\theta))$, $(n > 0)$

(ii) $P_0(u) = 1 \in H^0(K;G_0(Z_\theta))$,
$P_1(u) = u$,

(iii) $P_r(u) \smile P_s(u) = (r,s)\, P_{r+s}(u)$,

(iv) $P_t(u + v) = \Sigma_{r+s=t}\, P_r(u) \smile P_s(v)$,

(v) $P_s(P_r(u)) = P_{rs}(u)$, $(r > 0)$

(vi) $P_r f^* = f^* P_r$,

(vii) $P_r \alpha_* = G(\alpha)_* P_r$,

where α_* is induced by a homomorphism α of Z_θ to Z_τ $(Z_\tau \in \mathcal{G})$, and $G(\alpha)$ is the homomorphism defined in 1.23.

The proof of this theorem is given in sections 10 through 12. We call the functions P_t model operations, because their only use will be to define the functions $\wp_t$.

Let $Z_\theta \in \mathcal{G}$, and let η_r be the homomorphism from $G_{sr}(Z_\theta)$ to $G_s(Z_\theta)$ defined in 1.26. We have

(2.2) PROPOSITION: Let $u \in H^{2n}(K;Z_\theta)$. Then,

(i) $\eta_{r*} P_{rs}(u) = [P_s(u)]^r$, (r-fold $\smile'$-product)

(ii) $P_t(u) = u^t$, if θ is zero or is prime to t.

(2.3) PROPOSITION: Let $u_i \in H^{2n_i}(K;Z_\theta)$ $(i = 1, \ldots, r)$, and let t be a positive integer. Then

(i) if t is odd,
$P_t(u_1 \smile' \ldots \smile' u_r) = P_t(u_1) \smile' \ldots \smile' P_t(u_r)$

(ii) <u>if</u> t <u>is even</u>,

$$P_t(u_1 \smile' \cdots \smile' u_r) = P_t(u_1) \smile' \cdots \smile' P_t(u_r) + \Psi_t(u_1, \ldots, u_r),$$

<u>where</u>

(a) $\Psi_t(u_1, \ldots, u_r) \in H^{2tn}(K; G_t(Z_\theta))$, $\quad (n = n_1 + \cdots + n_r)$

(b) $\Psi_t(u_1, \ldots, u_r) = 0$, $\quad$ <u>if</u> θ <u>is odd or equals zero</u>

(c) $\Psi_t(u, \ldots, u) = 0$, $\quad (u_1 = \cdots = u_r = u)$

(d) $2\Psi_t(u_1, \ldots, u_r) = 0$.

(iii) <u>Let</u> $v_i \in H^{2n_i+1}(K; Z_\theta)$ $(i = 1, 2)$. <u>Then</u>, <u>for</u> $t \geq 2$,

$$2P_t(v_1 \smile' v_2) = 0.$$

The proofs for these propositions are given in section 13.

3. THE DEFINITION OF THE OPERATIONS $\wp_t$

We use the model operations P_t $(t = 0, 1, \ldots)$ defined in the preceding section to define the γ-functions $\wp_t$ described in theorem I. Throughout this section let A be a fixed p-cyclic Γ-ring, where $A = \Sigma A_i$. Let A_{2k} $(1 \leq k < \infty)$ be a fixed summand of A. For a given complex K we wish to define the functions $\wp_r$ mapping $H^{2n}(K; A_{2k})$ to $H^{2rn}(K; A_{2rk})$ $(n > 0)$. Since each summand of A is a p-cyclic group (see §1), there exists an integer θ, either zero or a power of a prime, such that

$$\text{order } A_{2k} = \text{order } Z_\theta.$$

Let μ be an isomorphism from A_{2k} to Z_θ. We define first a function $\wp_{r,\mu}$, which depends upon the isomorphism μ; and we then show that in fact our definition is independent of the particular choice of μ.

Let g_r': $A \longrightarrow A$, g_r: $G(Z_\theta) \longrightarrow G(Z_\theta)$ $(r = 0, 1, \ldots)$ be the γ-functions in A and $G(Z_\theta)$.

(3.1) LEMMA: <u>For each integer</u> r, <u>there is a homomorphism</u> ζ_r: $G_r(Z_\theta) \longrightarrow A_{2rk}$ <u>such that</u>

$$g_r' \mu^{-1} = \zeta_r g_r$$

<u>in the following diagram</u>:

$$\begin{array}{ccc} G_r(Z_\theta) & \xrightarrow{\ \zeta_r\ } & A_{2rk} \\ {\scriptstyle g_r}\uparrow & & \uparrow{\scriptstyle g_r'} \\ Z_\theta & \xrightarrow{\ \mu^{-1}\ } & A_{2k} \end{array}$$

PROOF: Since A is a Γ-ring, we can define homomorphisms

(3.2) $$f_r\colon \Gamma_r(A_{2k}) \longrightarrow A_{2rk} \qquad (r = 0, 1, \ldots)$$

by setting

$$f_r \gamma_r(a) = g_r'(a)$$

$$f_{r+s}[\gamma_r(a) \ \gamma_s(b)] = g_r'(a) g_s'(b),$$

for $a, b \in A_{2k}$. Relations 1.1 through 1.7 and 1.9 through 1.11 imply that each function f_r is well-defined. We define the homomorphism ζ_r by

(3.3) $$\zeta_r = f_r \ \Gamma_r(\mu^{-1}) \rho;$$

that is, as the following sequence of homomorphisms:

$$G_r(Z_\theta) \xrightarrow{\ \rho\ } \Gamma_r(Z_\theta) \xrightarrow{\ \Gamma_r(\mu^{-1})\ } \Gamma_r(A_{2k}) \xrightarrow{\ f_r\ } A_{2rk}.$$

Here, ρ is the isomorphism defined in 1.18; and $\Gamma_r(\mu^{-1})$ is the isomorphism corresponding to μ^{-1} using the functor Γ_r (see 1.14).

To prove 3.1, let $a \in Z_\theta$. Then, by 3.3 and 1.21,

$$\zeta_r g_r(a) = f_r \ \Gamma_r(\mu^{-1}) \ \rho \ g_r(a) = f_r \ \Gamma_r(\mu^{-1}) \ \gamma_r(a).$$

But, from 1.14 and 3.2 we have

$$f_r\, \Gamma_r(\mu^{-1})\, \gamma_r(a) = f_r\, \gamma_r(\mu^{-1}(a)) = g_r'\mu^{-1}(a).$$

Thus, $\zeta_r g_r(a) = g_r'\mu^{-1}(a)$, which completes the proof.

We define the function $\not\wp_{r,\mu}$ mapping $H^{2n}(K;A_{2k})$ to $H^{2rn}(K;A_{2rk})$ by

(3.4) $$\not\wp_{r,\mu} = \zeta_{r*}\, P_r\, \mu_*.$$

That is, $\not\wp_{r,\mu}$ is the composition of the following functions:

$$H^{2n}(K;A_{2k}) \xrightarrow{\mu_*} H^{2n}(K;Z_\theta) \xrightarrow{P_r} H^{2rn}(K;G_r(Z_\theta)) \xrightarrow{\zeta_{r*}} H^{2rn}(K;A_{2rk}).$$

We proceed to show that the function $\not\wp_{r,\mu}$ is independent of the choice of μ. Let B be a second p-cyclic Γ-ring, where $B = \Sigma_i B_i$. Then, order B_{2k} = order Z_τ for some integer τ. Choose an isomorphism ν from B_{2k} to Z_τ. We have:

(3.5) <u>Let α be any (group) homomorphism from</u> A_{2k} to B_{2k}. <u>Then,</u>

$$\Gamma_r(\nu^{-1})_*\rho_*\, P_r\, \nu_*\alpha_* = \Gamma_r(\alpha)_*\, \Gamma_r(\mu^{-1})_*\rho_*\, P_r\, \mu_*.$$

PROOF: Define a homomorphism β of Z_θ to Z_τ by

$$\beta = \nu\alpha\,\mu^{-1}.$$

Then, $\nu\,\alpha = \beta\,\mu$. Since μ, ν are isomorphisms and Γ_r is a functor, we have:

$$\Gamma_r(\alpha)\, \Gamma_r(\mu^{-1}) = \Gamma_r(\nu^{-1})\, \Gamma_r(\beta).$$

From 2.1 (vii) we have:

$$P_r\beta_* = G_r(\beta)_*\, P_r,$$

when $G_r(\beta) = \rho^{-1}\, \Gamma_r(\beta)\, \rho$. Hence,

$$\begin{aligned}\Gamma_r(\nu^{-1})_*\rho_*\, P_r\nu_*\, \alpha_* &= \Gamma_r(\nu^{-1})_*\, \rho_*\, P_r\beta_*\mu_* \\ &= \Gamma_r(\nu^{-1})_*\, \rho_*\, G_r(\beta)_* P_r\mu_* \\ &= \Gamma_r(\nu^{-1})_*\, \Gamma_r(\beta)_*\, \rho_*\, P_r\mu_* \\ &= \Gamma_r(\alpha)_*\, \Gamma_r(\mu^{-1})_*\, \rho_*\, P_r\mu_*,\end{aligned}$$

as was to be proved.

Now, let ω and σ be any two isomorphisms from A_{2k} to Z_θ.

(3.6) PROPOSITION: $\not\wp_{r,\omega} = \not\wp_{r,\sigma}$.

PROOF: Define an isomorphism α from A_{2k} to A_{2k} by $\alpha = \sigma^{-1}\omega$. This implies

(3.7) $$\Gamma_r(\omega^{-1}) = \Gamma_r(\alpha^{-1})\, \Gamma_r(\sigma^{-1}).$$

Using 3.4, 3.7, and the definition of α we have

$$\begin{aligned}\not\wp_{r,\omega} = \zeta_{r*}\, P_r\omega_* &= f_{r*}\, \Gamma_r(\omega^{-1})_*\, \rho_*\, P_r\, \sigma_*\alpha_* \\ &= f_{r*}\, \Gamma_r(\alpha^{-1})_*\, \Gamma_r(\sigma^{-1})_*\, \rho_*\, P_r\sigma_*\alpha_*\, .\end{aligned}$$

Let us now apply 3.5 by setting $B_{2k} = A_{2k}$, $\mu = \sigma$, $\nu = \sigma$, and $Z_\theta = Z_\tau$. Then,

$$\Gamma_r(\sigma^{-1})_*\rho_*\, P_r\sigma_*\alpha_* = \Gamma_r(\alpha)_*\, \Gamma_r(\sigma^{-1})_*\, \rho_*\, P_r\, \sigma_*.$$

Hence, combining this with the previous equation we have

$$\begin{aligned}\not\wp_{r,\omega} &= f_{r*}\, \Gamma_r(\alpha^{-1})_*\, \Gamma_r(\alpha)_*\, \Gamma_r(\sigma^{-1})_*\, \rho_*\, P_r\, \sigma_* \\ &= f_{r*}\, \Gamma_r(\sigma^{-1})_*\, \rho_*\, P_r\sigma_* \\ &= \zeta_{r*}\, P_r\, \sigma_* = \not\wp_{r,\sigma}\, .\end{aligned}$$

This proves the proposition and shows that we can now uniquely define the functions $\wp_r$ mapping $H^{2n}(K;A_{2k})$ to $H^{2rn}(K;A_{2rk})$ by

(3.8) DEFINITION: $\wp_r = \wp_{r,\nu}$ $(r = 0, 1, \ldots)$ <u>where</u> ν <u>is any isomorphism of</u> A_{2k} <u>to</u> Z_θ.

To emphasize the difference between the γ-functions $\wp_t$ and the model operations P_t, we show the relation between the two in the case where the coefficient ring is $G(Z_\theta)$, for $Z_\theta \in \mathcal{C}$.

(3.9) THEOREM: <u>Let</u> Z_θ <u>be a group in the category</u> $\mathcal{C}$, <u>and let</u> $G(Z_\theta)$ <u>be the</u> Γ<u>-ring defined in</u> 1.17. <u>Let</u> u <u>be an element of</u> $H^{2n}(K;G_r(Z_\theta))$ $(r \geq 1)$, <u>and set</u> $\epsilon_{s,r} = (r,r-1)(2r,r-1)\ldots((s-1)r,r-1)$. <u>Then,</u>

$$\wp_s(u) = \epsilon_{s,r}\, P_s(u), \qquad (s = 0, 1, \ldots)$$

<u>in</u> $H^{2sn}(K;G_{sr}(Z_\theta))$.

(3.10) COROLLARY: <u>Let</u> $u \in H^{2n}(K;Z_\theta)$. <u>Then,</u>

$$\wp_s(u) = P_s(u), \qquad (s = 0, 1, \ldots)$$

<u>in</u> $H^{2sn}(K;G_s(Z_\theta))$.

PROOF: In order to use definition 3.4 we must find a group in $\mathcal{C}$ whose order is equal to the order of $G_r(Z_\theta)$. But $G_r(Z_\theta)$ is itself a cyclic group of order $\theta[r,\theta^\infty]$ (see 1.15) generated by $g_r(1_\theta) = 1 \bmod \theta[r,\theta^\infty]$. Set $\tau = \theta[r,\theta^\infty]$. Then, $G_r(Z_\theta) = Z_\tau \in \mathcal{C}$. In addition we must choose an isomorphism ν from $G_r(Z_\theta)$ to Z_τ. Set $\nu g_r(1_\theta) = 1_\tau$. Then, as a mapping of <u>groups</u>, ν = identity; but as a mapping from a summand of one Γ-ring into a summand of another, ν maps elements of degree 2r in $G(Z_\theta)$ into elements of degree 2 in $G(Z_\tau)$. From 3.4 we have

$$\wp_s = \zeta_{s*} P_s \nu_* = \zeta_{s*} P_s,$$

since ν_* = identity, as a coefficient group homomorphism. Recall that ζ_s is the composition of the following functions:

$$G_s(Z_\tau) \xrightarrow{\ \rho\ } \Gamma_s(Z_\tau) \xrightarrow{\ \Gamma_s(\nu^{-1})\ } \Gamma_s(G_r(Z_\theta)) \xrightarrow{\ f_s\ } G_{sr}(Z_\theta).$$

Thus, by 3.3, 1.21, 1.14, 3.2, and 1.5 we have

$$\begin{aligned}\zeta_s g_s(1_\tau) &= f_s\ \Gamma_s(\nu^{-1})\rho\, g_s(1_\tau)\\ &= f_s\ \Gamma_s(\nu^{-1})\, \gamma_s(1_\tau)\\ &= f_s\, \gamma_s g_r(1_\theta)\\ &= g_s(g_r(1_\theta)) = \epsilon_{s,r} g_{rs}(1_\theta).\end{aligned}$$

But considered simply as cyclic groups with generators, we know that

$$G_s(Z_\tau) = G_{sr}(Z_\theta),\quad g_s(1_\tau) = g_{sr}(1_\theta).$$

Thus, $\zeta_s(a) = \epsilon_{s,r}(a)$, for each element a of $G_{sr}(Z_\theta)$. Hence,

$$\wp_s(u) = \zeta_{s*} P_s(u) = \epsilon_{s,r} P_s(u),$$

which completes the proof. The corollary follows immediately from the fact that $G_1(Z_\theta) = Z_\theta$, and $\epsilon_{s,1} = 1$.

4. THE PROOF OF THE MAIN THEOREMS

With the γ-functions $\wp_r$ $(r = 0, 1, \ldots)$ now defined (see 3.4 and 3.8), we show that these functions satisfy theorems I and II. Throughout the section let A denote a fixed p-cyclic Γ-ring, and A_{2k} a fixed summand of even degree. Suppose that A_{2k} has order θ, and let ν be a fixed isomorphism from A_{2k} to Z_θ.

PROOF OF I(1): This follows at once from definitions 3.4 and 3.8.

PROOF OF I(2): In 3.3 we defined a homomorphism ζ_r mapping $G_r(Z_\theta)$ to A_{2rk} such that $g_r'\nu^{-1} = \zeta_r g_r$, where g_r, g_r' denote the γ-functions in $G(Z_\theta)$ and A respectively. Let $\iota \in A_o$ denote the unit of A. Then, using 1.2 and 3.3 it is easily verified that $\zeta_o(1) = \iota$, where $1 \in G_o(Z_\theta) = Z$. This implies that if 1', 1" are the units of $H^*(K;G(Z_\theta))$, $H^*(K;A)$ respectively, then

$$\zeta_{o*}(1') = 1".$$

But from 3.4 and 2.1(ii) we have

$$\mathscr{P}_o(u) = \zeta_{o*}P_o\nu_*(u) = \zeta_{o*}(1') = 1",$$

for any class $u \in H^{2n}(K;A_{2k})$. This proves part one of I(2).

For the second part of I(2), we observe that the function $\zeta_1 = \nu^{-1}$, by 1.2, 3.2, and 3.3. Hence,

$$\mathscr{P}_1(u) = \zeta_{1*}P_1\nu_*(u) = (\nu^{-1})_*\nu_*(u) = u,$$

by 2.1(ii). This completes the proof.

PROOF OF I(3): We need a preliminary result. Let λ denote the multiplication in both the ring A and $G(Z_\theta)$. Then

(4.1) <u>In the following diagram,</u>

$$\lambda(\zeta_r \otimes \zeta_s) = \zeta_{r+s}\lambda.$$

$$\begin{array}{ccc} G_r(Z_\theta) \otimes G_s(Z_\theta) & \xrightarrow{\lambda} & G_{r+s}(Z_\theta) \\ \downarrow{\scriptstyle \zeta_r \otimes \zeta_s} & & \downarrow{\scriptstyle \zeta_{r+s}} \\ A_{2rk} \otimes A_{2sk} & \xrightarrow{\lambda} & A_{2(r+s)k}. \end{array}$$

PROOF: It is sufficient to verify commutativity on a generator of $G_r(Z_\theta) \otimes G_s(Z_\theta)$; namely, on the element $g_r(1_\theta) \otimes g_s(1_\theta)$. Then, by 1.3 and 3.1, we have

$$\begin{aligned} \zeta_{r+s}\,\lambda(g_r(1_\theta) \otimes g_s(1_\theta)) &= (r,s)\ \zeta_{r+s}g_{r+s}(1_\theta) \\ &= (r,s)\ g'_{r+s}\ \nu^{-1}(1_\theta) \\ &= g'_r(\nu^{-1}1_\theta)g'_s(\nu^{-1}1_\theta) \end{aligned}$$

$$= \lambda(\zeta_r \otimes \zeta_s)(g_r(1_\theta) \otimes g_s(1_\theta)),$$

which completes the proof.

We use the lemma to prove I(3):

$$\begin{aligned} \mathscr{P}_r(u) \smile \mathscr{P}_s(u) &= \zeta_{r*}P_r\nu_*(u) \smile \zeta_{s*}P_s\nu_*(u) \\ &= \lambda_*(\zeta_r \otimes \zeta_s)_*[P_r\nu_*(u) \otimes P_s\nu_*(u)], \end{aligned}$$

by 3.4 and the definition of the $\smile$-product. However, using 4.1, 2.1(iii), and 3.4 we have

$$\begin{aligned} \lambda_*(\zeta_r \otimes \zeta_s)_*[P_r\nu_*(u) \otimes P_s\nu_*(u)] &= (\zeta_{r+s})_*\ \lambda_*[P_r\nu_*(u) \otimes P_s\nu_*(u)] \\ &= (r,s)\ \zeta_{r+s*}\ P_{r+s}\nu_*(u) \\ &= (r,s)\ \mathscr{P}_{r+s}(u), \end{aligned}$$

completing the proof.

PROOF OF I(4): This follows at once from 4.1 and 2.1(iv), by a proof entirely similar to the one used for I(3).

PROOF OF I(5): Set τ = order A_{2rk}, and choose isomorphisms ν from A_{2k} to Z_θ and μ from A_{2rk} to Z_τ. Let ζ_r be the function defined in 3.3 mapping $G_r(Z_\theta)$ to A_{2rk}, and let $G(\mu\ \zeta_r)$ be the mapping from $G(G_r(Z_\theta))$ to $G(Z_\tau)$, defined using the functor G (see 1.23). Then,

(4.2) LEMMA: $\zeta_s G_s(\mu\ \zeta_r) = \epsilon_{s,r}\ \zeta_{sr}$, <u>in the following diagram.</u>

$$\begin{array}{ccc} G_s(G_r(Z_\theta)) & \xrightarrow{G_s(\mu\ \zeta_r)} & G_s(Z_\tau) \\ \| & & \downarrow{\scriptstyle \zeta_s} \\ G_{sr}(Z_\theta) & \xrightarrow{\zeta_{sr}} & A_{2srk} \end{array}$$

PROOF: Let $\omega = \theta[r,\theta^{\infty}]$ (see 1.15). Then,

$$G_r(Z_\theta) = Z_\omega, \quad \text{and} \quad g_r(1_\theta) = 1_\omega .$$

Since all the functions involved in 4.2 are homomorphisms, we may verify commutativity by applying the functions to a generator for $G_s(Z_\omega)$, say $g_s(1_\omega)$. Then, by 1.24 and 3.1,

$$\begin{aligned} \zeta_s G_s(\mu\,\zeta_r) g_s(1_\omega) &= \zeta_s g_s(\mu\,\zeta_r\,1_\omega) \\ &= g_s'\,\mu^{-1}\,\mu\,\zeta_r(1_\omega) \\ &= g_s'\,\zeta_r(1_\omega). \end{aligned}$$

But, $1_\omega = g_r(1_\theta)$; and $\zeta_r g_r(1_\theta) = g_r'\,\mu^{-1}(1_\theta)$, by 3.1. Hence,

$$\begin{aligned} g_s'\,\zeta_r(1_\omega) &= g_s'\,g_r'\,(\mu^{-1}1_\theta) = \epsilon_{s,r}\,g_{sr}'\,\mu^{-1}(1_\theta) \\ &= \epsilon_{s,r}\,\zeta_{sr}g_{sr}(1_\theta), \end{aligned}$$

by 1.6 and 3.1. This proves the lemma.

We use 4.2 to prove I(5):

$$\begin{aligned} \not{P}_s(\not{P}_r(u)) &= \zeta_{s*}P_s\mu_*(\zeta_{r*}P_r\nu_*(u)) \\ &= \zeta_{s*}G_s(\mu\,\zeta_r)_*\,P_sP_r\nu_*(u), \end{aligned}$$

by definition 3.4 and 2.1(vii). But $P_sP_r = P_{sr}$, by 2.1(v). Hence, from 4.2, we have

$$\zeta_{s*}G_s(\mu\,\zeta_r)_*P_sP_r\nu_*(u) = \epsilon_{s,r}\,\zeta_{sr*}P_{sr}\nu_*(u) = \epsilon_{s,r}\not{P}_{sr}(u),$$

as was to be proved.

We consider together the proofs of I(6) and I(7). Suppose that $u \in H^m(K;A_{2r})$, $u' \in H^n(K;A_{2s})$ where $m + n$ is even. If A_{2r}, A_{2s} are both finite groups with relatively prime orders, then $A_{2r} \otimes A_{2s} = 0$. Therefore, $u \smile u' = 0$, $\not{P}_t(u) \smile \not{P}_t(u') = 0$, and I(6), I(7) are trivial in this instance. Thus, from now on we assume the contrary; that is, that there

is an integer θ, either zero or a power of a prime, such that $A_{2r} \otimes A_{2s} \approx Z_\theta$.

Let us denote by θ_i the order of the summand A_i of A $(i = 0, 1, \ldots)$. Set $Z^i = Z_{\theta_{2i}}$, and choose isomorphisms ν_i: $A_{2i} \approx Z^i$. Set $a_{2i} = \nu_i^{-1}(1_{\theta_{2i}})$. Then, a_{2i} is a generator for A_{2i}.

Let θ be the integer referred to above. If $\theta = 0$, then Z^r, Z^s and Z_θ are all equal to Z. If $\theta > 0$, then the orders of Z^r and Z^s are multiples of the order of Z_θ, since $A_{2r} \otimes A_{2s} \approx Z_\theta$. In all cases if we denote by ξ_r and ξ_s the natural homomorphisms

$$Z_r \xrightarrow{\ \xi_r\ } Z_\theta, \quad Z_s \xrightarrow{\ \xi_s\ } Z_\theta,$$

we see that $\xi_r \otimes \xi_s$ is an isomorphism of $Z^r \otimes Z^s$ onto $Z_\theta \otimes Z_\theta$ $(\approx Z_\theta)$. Let λ: $A_{2r} \otimes A_{2s} \longrightarrow A_{2(r+s)}$ and ω: $Z_\theta \otimes Z_\theta \longrightarrow Z_\theta$ be the multiplication in the rings A and Z_θ respectively (see §1). We then have:

(4.3) LEMMA: _In the following diagram,_

$$\Psi\omega(\xi_r \otimes \xi_s)(\nu_r \otimes \nu_s) = \nu_{r+s}\lambda,$$

where

$$\Psi(1_\theta) = \nu_{r+s}(a_{2r}a_{2s}).$$

$$\begin{array}{ccccc} A_{2r}\otimes A_{2s} & \xrightarrow{\ \lambda\ } & A_{2(r+s)} & \xrightarrow{\ \nu_{r+s}\ } & Z^{r+s} \\ \Big\downarrow \nu_r\otimes\nu_s & & & & \Big\uparrow \Psi \\ Z^r\otimes Z^s & \xrightarrow[\ \xi_r\otimes\xi_s\ ]{} & Z_\theta\otimes Z_\theta & \xrightarrow[\ \omega\ ]{} & Z_\theta \end{array}$$

The proof is immediate, since ν_{r+s}, $\nu_r \otimes \nu_s$, $\xi_r \otimes \xi_s$, and ω are all isomorphisms.

Now, let $\zeta'_t\colon\ G_t(Z^r) \longrightarrow A_{2rt}$, $\zeta''_t\colon\ G_t(Z^s) \longrightarrow A_{2st}$ and $\zeta_t\colon\ G_t(Z^{r+s}) \longrightarrow A_{2(r+s)t}$ be the maps defined in 3.3. Let $\omega'\colon\ G_t(Z_\theta) \otimes G_t(Z_\theta) \longrightarrow G_t(Z_\theta)$ be the multiplication in the ring $G_t(Z_\theta)$ (see §1). We then have:

(4.4) LEMMA: In the following diagram,

$$(t!)\,\lambda(\zeta'_t \otimes \zeta''_t) = \zeta_t G_t(\Psi)\omega'[G_t(\xi_r) \otimes G_t(\xi_s)].$$

$$\begin{array}{ccccc}
G_t(Z^r)\otimes G_t(Z^s) & \xrightarrow{\zeta'_t\otimes\zeta''_t} & A_{2rt}\otimes A_{2st} & \xrightarrow{\lambda} & A_{2(r+s)t} \\
\Big\downarrow{\scriptstyle G_t(\xi_r)\otimes G_t(\xi_s)} & & & & \Big\uparrow{\scriptstyle \zeta_t} \\
G_t(Z_\theta)\otimes G_t(Z_\theta) & \xrightarrow[\omega']{} & G_t(Z_\theta) & \xrightarrow[G_t(\Psi)]{} & G_t(Z^{r+s})
\end{array}$$

The proof is purely mechanical, and is omitted.

Since ω' and $G_t(\xi_r) \otimes G_t(\xi_s)$ are both isomorphisms, as an immediate consequence of 4.4 we have:

(4.5) $$t!\,\Lambda = \zeta_t G_t(\Psi),$$

where

$$\Lambda = \lambda(\zeta'_t \otimes \zeta''_t)[G_t(\xi_r) \otimes G_t(\xi_s)]^{-1}\,\omega'^{-1}.$$

PROOF OF I(6): Let $u \in H^{2m}(K;A_{2r})$, $u' \in H^{2n}(K;A_{2s})$ $(r, s \geq 1)$. We first need a lemma.

(4.6) $$\not{P}_t(u \smile u') = t!\,\Lambda_* P_t[\xi_{r*}\nu_{r*}(u) \smile' \xi_{s*}\nu_{s*}(u)],$$

where Λ is the homomorphism defined in 4.5.

PROOF: From 3.4 and the definition of the $\smile$-product, we have

$$\not{P}_t(u \smile u') = \zeta_{t*}P_t\nu_{(r+s)*}\lambda_*(u \otimes u').$$

But by 4.3, 2.1(vii), and 4.5, it follows that

$$\begin{aligned}
\zeta_{t*}P_t\nu_{(r+s)*}\lambda_*(u \otimes u') &= \zeta_{t*}P_t\Psi_*\omega'_*[\xi_{r*}\nu_*(u) \otimes \xi_{s*}\nu_{s*}(u')] \\
&= \zeta_{t*}G_t(\Psi)_*P_t[\xi_{r*}\nu_*(u) \smile' \xi_{s*}\nu_{s*}(u')] \\
&= t!\,\Lambda_*P_t[\xi_{r*}\nu_{r*}(u) \smile' \xi_{s*}\nu_{s*}(u')].
\end{aligned}$$

Thus, the lemma follows from these two equations.

In order to prove I(6), set

$$\begin{aligned}
\Omega_t &= \Psi_t[\xi_{r*}\nu_{r*}(u),\ \xi_{s*}\nu_{s*}(u')], && (t \text{ even}) \\
&= 0, && (t \text{ odd})
\end{aligned}$$

where Ψ_t is the function described in 2.3(ii). Then $2\Omega_t = 0$, using 2.3(ii d) in case t is even. Hence,

$$\begin{aligned}
\not{P}_t(u \smile u') &= t!\,\Lambda_*[P_t(\xi_{r*}\nu_{r*}u) \smile' P_t(\xi_{s*}\nu_{s*}u') + \Omega_t] \\
&= t!\,\Lambda_*\omega'_*[P_t(\xi_{r*}\nu_{r*}(u)) \otimes P_t(\xi_{s*}\nu_{s*}(u'))] \\
&= t!\,\Lambda_*\omega'_*[G_t(\xi_r) \otimes G_t(\xi_s)]_*[P_t\nu_{r*}(u) \otimes P_t\nu_{s*}(u')] \\
&= t!\,\lambda_*(\zeta'_t \otimes \zeta''_t)_*[P_t\nu_{r*}(u) \otimes P_t\nu_{s*}(u')] \\
&= t!\,\lambda_*[\zeta'_{t*}P_t\nu_{r*}(u) \otimes \zeta''_{t*}P_t\nu_{s*}(u')] \\
&= t!\ \not{P}_t(u) \smile \not{P}_t(u'),
\end{aligned}$$

which completes the proof. Here we have used 4.6, 2.3(ii), 2.1(vii), 4.5, 3.4, and the fact that $t! \equiv 0 \bmod 2$, when $t \geq 2$.

PROOF OF I(7): Let $u \in H^m(K;A_i)$, $u' \in H^n(K;A_j)$, where u and u' have semi-odd degree. That is, either m and n are odd, or i and j are odd, or both pair are odd. Let us suppose first that i, j are even and m, n are odd; set $i = 2r$, $j = 2s$ $(r, s \geq 1)$. Then, from 4.6 and 2.3(iii), we have:

$$\mathfrak{P}_t(u \smile u') = t!\Lambda_* P_t[\xi_{r*}\nu_{r*}(u) \smile' \xi_{s*}\nu_{s*}(u')] = 0,$$

since the base degrees of $\xi_{r*}\nu_{r*}(u)$ and $\xi_{s*}\nu_{s*}(u')$ are odd, and $t! \equiv 0$ mod 2, for $t \geq 2$.

Suppose now that i and j are both odd, say $i = 2r+1$, $j = 2s+1$. For this case we need a preliminary lemma: let G be any finitely generated abelian group and K a complex. Then, there is a natural identification of $K^{\#} \otimes G$ and Hom (K,G), where $K^{\#} = \text{Hom}\,(K,Z)$ (see §5). In particular, suppose that G is a cyclic group, and that $\overline{u} \in H^m(K;G)$. Then, a representative cocycle can be chosen for $\overline{u}$ of the form $\overline{u} = \{u \otimes n\}$, where $u \in C^m(K)$, $n \in G$, and $\{\ \}$ denotes cohomology class.

(4.7) LEMMA: <u>Let</u> $\overline{u} \in H^m(K;Z_\theta)$, <u>where</u> $Z_\theta \in \mathcal{G}$. <u>Suppose that</u> $\overline{u} = \{u \otimes n\}$, <u>where</u> $u \in C^m(K)$, <u>and</u> $n \in Z_\theta$. <u>Denote the</u> γ<u>-functions in</u> $G(Z_\theta)$ <u>by</u> g_i $(i = 0, 1, \ldots)$. <u>Then, there is a cochain</u> $z \in C^{tm}(K)$ <u>such that</u>

$$P_t(\overline{u}) = \{z \otimes g_t(n)\}.$$

The proof of the lemma depends only upon the properties of the functions P_r and will be given in section 8.

Using 4.7 we complete the proof of I(7). By hypothesis, we have $u \in H^m(K;A_{2r+1})$, $u' \in H^n(K;A_{2s+1})$. Since A_{2r+1} and A_{2s+1} are cyclic groups generated respectively by a_{2r+1} and a_{2s+1}, there are integer cochains u_1, u_1' such that

$$u = \{u_1 \otimes a_{2r+1}\},\ u' = \{u_1' \otimes a_{2s+1}\}.$$

Then, $u \smile u' = \{w \otimes (a_{2r+1}a_{2s+1})\}$, where $w = d^{\#}(u_1 \otimes u_1')$, and $d^{\#}$ is induced by the diagonal map d of K into $K \times K$. Let ν be the isomorphism $\nu_{2(r+s+1)}$ from $A_{2(r+s+1)}$ to Z^{r+s+1}, defined at the beginning of the proof of I(6). Then, by 4.7 we have,

$$P_t\nu_*(u \smile u') = P_t\{w \otimes \nu(a_{2r+1}a_{2s+1})\} = \{z \otimes g_t\nu(a_{2r+1}a_{2s+1})\}$$

for some cochain $z \in C^{2t(r+s)+2}(K)$. Thus

$$\begin{aligned}\mathfrak{P}_t(u \smile u') &= \zeta_{t*}P_t\nu_*(u \smile u')\\ &= \zeta_{t*}\{z \otimes g_t\nu(a_{2r+1}a_{2s+1})\}\\ &= \{z \otimes \zeta_t g_t\nu(a_{2r+1}a_{2s+1})\}\\ &= \{z \otimes g_t'\nu^{-1}\nu(a_{2r+1}a_{2s+1})\}\\ &= \{z \otimes g_t'(a_{2r+1}a_{2s+1})\} = 0,\end{aligned}$$

by 3.1 and 1.7. This completes the proof of I(7).

PROOF OF I(8): This follows at once from 3.4, 2.1(vi), and the fact that $f^{\#}$ commutes with all cohomology homomorphisms induced by coefficient group homomorphisms.

PROOF OF I(9): Since α is a ring homomorphism of A to A', it induces a group homomorphism α_i of A_i to A_i' for each $i = 0, 1, \ldots$. Let $f_t\colon \Gamma_t(A_{2k}) \longrightarrow A_{2kt}$, $f_t'\colon \Gamma_t(A_{2k}') \longrightarrow A_{2kt}'$ be the functions defined in 3.2, and let g_i, g_i' be the γ-functions in A and A' respectively. We then have:

(4.8) LEMMA: <u>In the following diagram</u>:

$$f_t'\ \Gamma_t(\alpha_{2k}) = \alpha_{2kt}f_t.$$

$$\begin{array}{ccc} \Gamma_t(A_{2k}) & \xrightarrow{\Gamma_t(\alpha_{2k})} & \Gamma_t(A_{2k}') \\ \downarrow{\scriptstyle f_t} & & \downarrow{\scriptstyle f_t'} \\ A_{2kt} & \xrightarrow{\alpha_{2kt}} & A_{2kt}' \end{array}$$

PROOF: By 1.14, 3.2, and 1.8,

$$f'_t \Gamma_t(\alpha_{2k}) \gamma_t(a) = f'_t \gamma_t(\alpha_{2k} a)$$
$$= g'_t(\alpha_{2k} a)$$
$$= \alpha_{2kt} g_t(a)$$
$$= \alpha_{2kt} f_t \gamma_t(a),$$

where $\gamma_t(a)$ is any element of $\Gamma_t(A_{2k})$. Now, let $\nu\colon A_{2k} \longrightarrow Z_\theta$, $\mu\colon A'_{2k} \longrightarrow Z_\tau$ be isomorphisms. We then are in a position to use 3.5. Namely,

$$\Gamma_t(\mu^{-1})_* \rho_* P_t \mu_* \alpha_{2k*} = \Gamma_t(\alpha_{2k})_* \Gamma_t(\nu^{-1})_* \rho_* P_t \nu_*. \tag{4.9}$$

Using this we have at once the proof of I(9): for

$$\not\mathcal{P}_t \alpha_*(u) = \zeta_{t*} P_t \mu_* \alpha_{2k*}(u) \qquad (3.4)$$
$$= f'_{t*} \Gamma_t(\mu^{-1})_* \rho_* P_t \mu_* \alpha_{2k*}(u) \qquad (3.3)$$
$$= f'_{t*} \Gamma_t(\alpha_{2k})_* \Gamma_t(\nu^{-1})_* \rho_* P_t \nu_*(u) \qquad (4.9)$$
$$= \alpha_{2kt*} f_{t*} \Gamma_t(\nu^{-1})_* \rho_* P_t \nu_*(u) \qquad (4.8)$$
$$= \alpha_* \zeta_{t*} P_t \nu_*(u) \qquad (3.3)$$
$$= \alpha_* \not\mathcal{P}_t(u), \qquad (3.4)$$

which gives the desired result.

This concludes the proof of theorem I. The proof of theorem II follows at once from 2.2 and 2.3, using 3.10. Thus, the two main theorems are proved, and we are left now with proving 2.1, 2.2, and 2.3.

5. DEFINITION OF THE MODEL OPERATIONS P_p, (p prime).

We define the model operations P_p (p a prime number) using the method developed by N. E. Steenrod (see [11]). Since we will make constant use of the techniques developed in [11], in the following paragraphs we summarize briefly the contents of §2 of that paper.

Let K be a regular cell complex, and let K^* denote the associated cochain complex with integer coefficients: i.e., $K^* = \text{Hom}(K,Z)$. If u is a q-cochain of K, and c is a (finite) q-chain of K, then $u \cdot c \in Z$ denotes the value of u on c.

Let $\theta \geq 0$ be an integer, and suppose that $\bar{u} \in H^q(K;Z_\theta)$ is a cohomology class mod θ of K. In order to define a cochain representative for $\bar{u}$, we first define an elementary cochain complex $M = M(\theta,q)$ as follows: the cochain groups $C^r(M) = 0$ if $r \neq q$ or $q + 1$; $C^q(M)$ is an infinite cyclic group with generator u; $C^{q+1}(M)$ is zero if $\theta = 0$, and otherwise is infinite cyclic with generator v. The coboundary in M is defined by $\delta u = \theta v$. Suppose now that $f\colon M \longrightarrow K^*$ is a cochain mapping (i.e., $f\delta = \delta f$); then fu is a cocycle mod θ and determines a cohomology class $\bar{u} \in H^q(K;Z_\theta)$. Conversely, starting with $\bar{u}$, there is an integral cochain u_1 which is a cocycle mod θ (i.e., $\delta u_1 = \theta v_1$) and whose cohomology class is $\bar{u}$; hence, setting $fu = u_1$, $fv = v_1$ we define a cochain map $M \longrightarrow K^*$. Such a map f we will call a cochain representation of $\bar{u}$. It is easily shown that homotopic cochain maps $M \longrightarrow K^*$ represent the same $\bar{u}$; and hence the cohomology class $\bar{u}$ may be regarded as a homotopy class of cochain maps $M \longrightarrow K^*$, any one of which is a representation of $\bar{u}$.

Let π denote a permutation group of degree n. We shall regard π as a group of permutations of the factors of any n-fold tensor product such as $K^{*n} = K^* \otimes \dots \otimes K^*$ (n factors). Let W be an acyclic complex on which π operates freely. Also denote by W the chain complex it determines; its chain groups are free abelian.

The construction of cohomology operations applied to $\bar{u}$ (called the π-reduced powers of $\bar{u}$ in [11]) is based on the following diagram:

$$W \otimes_\pi M^n \xrightarrow{\ \Psi\ } W \otimes_\pi K^{*n} \xrightarrow{\ \zeta\ } W \otimes_\pi K^{n*} \xrightarrow{\ \phi\ } K^*. \tag{5.1}$$

The undefined terms in 5.1 are explained as follows:

If W is a chain complex, and A is a cochain complex, their tensor product $W \otimes A$ is the cochain complex whose cochain groups are

$$C^r(W \otimes A) = \Sigma_{i=0}^{\infty} C_i(W) \otimes C^{r+i}(A), \tag{5.2}$$

and whose coboundary operator is defined by

(5.3) $\delta(w \otimes a) = \partial w \otimes a + (-1)^i w \otimes \delta a,$

where $i = \dim w$. In case π operates on both W and A, we define operations in $W \otimes A$ by

(5.4) $x(w \otimes a) = xw \otimes xa, \ x \in \pi.$

Then, $W \otimes_\pi A$ is the factor complex by the subcomplex generated by cochains of the form $x(w \otimes a) - (w \otimes a)$.

The map Ψ of 5.1 is induced by a map f: $M \longrightarrow K^*$ representing $\bar{u}$; i.e., $\Psi = 1 \otimes_\pi f^n$.

The map ζ of 5.1 is induced by a natural map ζ': $K^{*n} \longrightarrow K^{n*}$ (see 2.5 in [11]). In case K is a finite complex, ζ' is an isomorphism. Furthermore, the action of π in K^n yields a dual action in K^{n*} with respect to which ζ' is equivariant. Thus, $\zeta = 1 \otimes_\pi \zeta'$.

The map ϕ is defined as the dual of an equivariant chain map ϕ' called a <u>diagonal approximation</u>:

(5.5) $\phi'\colon \ W \otimes K \longrightarrow K^n.$

For the details of ϕ', see 2.7 in [11]. Once we are given ϕ', the map ϕ dual to ϕ' is defined by

(5.6) $\phi(w \otimes y)\cdot\sigma = (-1)^{\frac{1}{2}i(i-1)} \ y\cdot\phi'(w \otimes \sigma),$

where $i = \dim w$, y is a cochain of K^*, and σ is an oriented cell of K with $\dim \sigma = \dim y - i$. From the equivariance of ϕ' we deduce that $\phi x = \phi$ for every $x \in \pi$; hence, ϕ is defined on $W \otimes_\pi K^{n*}$.

Having defined completely the terms of 5.1, we come to the final step in defining cohomology operations. Let G be an abelian group of coefficients. There is a natural transformation

$$\omega\colon \ K^* \otimes G \longrightarrow \mathrm{Hom}\ (K,G)$$

given by $\omega(y \otimes g)\cdot\sigma = (y\cdot\sigma)g$, where $y \in K^*$, $g \in G$, and σ is an oriented cell of K. In case K is finitely generated in each dimension, ω is an isomorphism. In any case $\delta\omega = \omega\delta$, so that ω induces a homomorphism

(5.7) $\omega\colon \ H^r(K^* \otimes G) \longrightarrow H^r(K;G).$

Now tensor the diagram 5.1 with G and pass to the derived diagram of cohomology groups and induced homomorphisms. The composition of the three induced homomorphisms and the homomorphism ω of 5.7 is a homomorphism denoted by

(5.8) $\Phi\colon \ H^r(W \otimes_\pi M^n \otimes G) \longrightarrow H^r(K;G).$

The image of Φ for all dimensions r is called the <u>set of π-reduced powers</u> of the cohomology class $\bar{u}$ of K.

To make the notation somewhat less unwieldy, in practise we will suppress the homomorphisms ζ and ω; namely, we let ϕ denote henceforth the composition $\phi\zeta$ of 5.1, and ignore ω. Thus, on the level of cochains we have mappings

$$W \otimes_\pi M^n \otimes G \xrightarrow{\ \Psi\ } W \otimes_\pi K^{*n} \otimes G \xrightarrow{\ \phi\ } K^* \otimes G,$$

which induce cohomology homomorphisms

(5.9) $H^r(W \otimes_\pi M^n \otimes G) \xrightarrow{\Psi^*} H^r(W \otimes_\pi K^{*n} \otimes G) \xrightarrow{\phi^*} H^r(K;G),$

whose composition is Φ.

It should be emphasized that the map Ψ^* is induced by the representation $M \longrightarrow K^*$ of $\bar{u}$. The other homomorphism of 5.9, ϕ^*, is induced by the diagonal approximation ϕ' of 5.5 and is independent of $\bar{u}$.

This, then, is a summary of Steenrod's general method of obtaining cohomology operations. We proceed to specialize this method to obtain the function P_r mapping $H^{2n}(K;Z_\theta)$ to $H^{2rn}(K;G_r(Z_\theta))$.

Let p be a prime number, and let π be the cyclic group of permutations of K^{*p} with generator T, where T moves each factor of K^{*p} one place to the right and moves the last factor to the first position.

In the group ring $Z(\pi)$ set

(5.10) $\Delta = T - 1, \ \Sigma = \Sigma_{k=0}^{p-1} T^k.$

Construct a π-complex W having one cell e_1 and its transforms $T^k e_1$ in each dimension $i \geq 0$. Define a boundary homomorphism, ∂, by

(5.11) $$\partial e_{2i+1} = \Delta e_{2i}, \quad \partial e_{2i} = \Sigma\, e_{2i-1}.$$

Then, W is a π-free, acyclic complex (see [10; §4]).

Suppose that C is a group in the category $\mathcal{C}$ (see §1); i.e., $C = Z_\theta$ where $\theta = 0$ or a power of a prime. Let K be a fixed complex, and let $\bar{u} \in H^{2n}(K;Z_\theta)$. Construct an elementary cochain complex $M = M(\theta,2n)$, as described earlier in this section: in order to define the map P_p, we first define a cochain function:

$$R = R_p\colon\ C^{2n}(M) \longrightarrow C^{2pn}(W \otimes_\pi M^p).$$

Namely,

(5.12) $$R(u) = e_o \otimes_\pi u^p + (\Sigma^* e_1) \otimes_\pi u^{p-1}\,\delta u,$$

where u is the generator for $C^{2n}(M)$, and

$$\Sigma^* = \Sigma_{k=o}^{p-1}\ k\cdot T^k \in Z(\pi).$$

For any integer $r \geq 0$, let $\theta[r,\theta^\infty]$ be the integer defined in 1.15. We then have

(5.13) PROPOSITION: $\delta R_p(u) \equiv 0 \bmod \theta[p,\theta^\infty]$.

For proof we need the following easily verified facts:

Let Δ, Σ, and Σ^* be the elements of $Z(\pi)$ defined in 5.10 and 5.12. Then

(5.14) $$\Delta\Sigma^* = p\cdot 1 - \Sigma = \Sigma^*\,\Delta,$$

(5.15) $$\delta(u^p) = \Sigma\,(u^{p-1}\delta u),$$

where $u_1 u_2 \ldots u_m$ denotes the tensor product $u_1 \otimes \ldots \otimes u_m \in K^{*m}$ $(m \geq 2)$.

Using (5.14) and (5.15), we have

$$\begin{aligned}\delta R(u) &= \delta(e_o \otimes_\pi u^p) + \delta[(\Sigma^* e_1) \otimes_\pi u^{p-1}\,\delta u]\\ &= e_o \otimes_\pi \delta(u^p) + (\Sigma^*\partial e_1) \otimes_\pi u^{p-1}\,\delta u - (\Sigma^* e_1) \otimes_\pi \delta(u^{p-1}\delta u)\\ &= e_o \otimes_\pi \Sigma\,(u^{p-1}\,\delta u) + (\Delta\Sigma^* e_1) \otimes_\pi u^{p-1}\,\delta u\\ &\qquad - (\Sigma^* e_1) \otimes_\pi \delta(u^{p-1}\,\delta u)\\ &= p\theta e_1 \otimes_\pi u^{p-1}\,v - \theta^2(\Sigma^* e_1) \otimes_\pi z,\end{aligned}$$

where $\delta u = \theta v$, and $\delta(u^{p-1}\,\delta u) = \theta^2 z$. We have used here the coboundary formula 5.3 and the fact that $(\Sigma e) \otimes_\pi y = e \otimes_\pi \Sigma\, y$, for $y \in M^p$, $e \in W$.

Now, if $\theta = 0$, then $p\theta = \theta^2 = 0 = \theta[p,\theta^\infty]$. If $\theta > 0$, it is clear from 1.15 that $\theta[p,\theta^\infty]$ divides both $p\theta$ and θ^2. Hence, in either case $\delta R_p(u) \equiv 0 \bmod \theta[p,\theta^\infty]$, proving the proposition.

From 1.17 we see that $G_p(Z_\theta) = Z_{\theta[p,\theta^\infty]}$. Thus, we can define a cohomology class ξ in $H^{2pn}(W \otimes_\pi M^p \otimes G_p(Z_\theta))$ by setting

(5.16) $$\xi = \{R_p(u) \otimes \iota_p\},$$

where { } denotes cohomology class and

$$\iota_r = 1 \bmod \theta[r,\theta^\infty] = g_r(1_\theta), \qquad (r = 0, 1, \ldots).$$

We define the function P_p by

(5.17) $$P_p(\bar{u}) = \Phi(\xi) \in H^{2pn}(K;G_p(Z_\theta)),$$

where $\bar{u} \in H^{2n}(K;Z_\theta)$ and Φ is the function defined in 6.8. Since Φ is in fact the composition $\phi^*\Psi^*$ (see 5.9), we have

(5.18) $$P_p(\bar{u}) = \phi^*\Psi^*(\xi) = \phi^*\{\Psi R_p(u) \otimes \iota_p\}.$$

(5.19) PROPOSITION: The cohomology class $P_p(\bar{u})$ is independent of the choice of cochain representation $f\colon\ M \longrightarrow K^*$.

PROOF: This is simply a particular case of theorem 3.1 in [11].

In the introduction we remarked that the functions $\not{p}_r$ generalize the Pontrjagin square cohomology operation: this is justified by the following:

(5.20) PROPOSITION: The class $P_2(\bar{u})$ coincides with the Pontrjagin square of $\bar{u}$.

PROOF: The Pontrjagin square of $\bar{u}$ is represented by the following cochain in $W \otimes_\pi M^2$:

$$e_o \otimes_\pi u^2 + e_1 \otimes_\pi u\, \delta u.$$

Since $p = 2$, in this case $\Sigma^* = T$. Also, since dim u is even,

$$\Delta(\delta uu) = u\delta u - \delta uu.$$

Hence, it is easily verified that the cochain $e_o \otimes_\pi u^2 + (\Sigma^* e_1) \otimes_\pi u\delta u$ (i.e., $R_2(u)$) is cohomologous mod $\theta[2,\theta^\infty]$ to $e_o \otimes_\pi u^2 + e_1 \otimes_\pi u\, \delta u$. Thus, the class $P_2(\bar{u})$ defined in 5.18 coincides with the Pontrjagin square of $\bar{u}$.

We state the naturality properties of the functions P_p before proceeding.

(5.21) PROPOSITION: Let L be a second complex, and f: L ⟶ K a map. Then,

$$P_p f^*(\bar{u}) = f^* P_p(\bar{u}),$$

where $\bar{u} \in H^{2n}(K;Z_\theta)$.

PROOF: This is an immediate consequence of theorem 3.6 of [11].

(5.22) Let $Z_\tau \in \mathcal{G}$ and let α be a homomorphism from Z_θ to Z_τ. Let $G_p(\alpha)$: $G_p(Z_\theta) \longrightarrow G_p(Z_\tau)$ be the homomorphism defined in 1.23. Then,

$$P_p \alpha_*(\bar{u}) = G_p(\alpha)_* P_p(\bar{u}),$$

where $\bar{u} \in H^{2n}(K;Z_\theta)$.

PROOF: Let $M = M(\theta,2n)$, $N = N(\tau,2n)$ be elementary cochain complexes, where u generates $C^{2n}(M)$ and w generates $C^{2n}(N)$. Let f: $M \longrightarrow K^*$ be a cochain representation for $\bar{u}$. We define a cochain representation for $\alpha_*(\bar{u})$, say g: $N \longrightarrow K^*$, as follows: suppose that the homomorphism α of Z_θ to Z_τ is given by

$$\alpha(1_\theta) = s(1_\tau). \qquad \text{(s an integer)}$$

Define the representation g by

$$g(w) = sf(u).$$

Let Ψ_f, Ψ_g be the homomorphisms in 5.1 induced respectively by f and g. Set $k_p = \tau[p,\tau^\infty] = g_p(1_\tau)$. Then, from 5.18 and 5.12 we have:

$$\begin{aligned} P_p\alpha_*(\bar{u}) &= \not{p}^*\{\Psi_g(e_o \otimes_\pi w^p + (\Sigma^* e_1) \otimes_\pi w^{p-1}\, \delta w) \otimes k_p\} \\ &= \not{p}^*\{[e_o \otimes_\pi g(w)^p + (\Sigma^* e_1) \otimes_\pi g(w)^{p-1}\, \delta g(w)] \otimes k_p\} \\ &= \not{p}^*\{s^p[e_o \otimes_\pi f(u)^p + (\Sigma^* e_1) \otimes_\pi f(u)^{p-1}\, \delta f(u)] \otimes k_p\} \\ &= \not{p}^*\{\Psi_f R_p(u) \otimes s^p\, k_p\}. \end{aligned}$$

But it is clear that $s^p\, k_p = G_p(\alpha)(\iota_p)$, where $\iota_p = g_p(1_\theta)$; for,

$$G_p(\alpha)(\iota_p) = G_p(\alpha)g_p(1_\theta) = g_p(\alpha 1_\theta) = g_p(s1_\tau) = s^p g_p(1_\tau) = s^p\, k_p.$$

Thus,

$$\begin{aligned} \not{p}^*\{\Psi_f\, R_p(u) \otimes s^p\, k_p\} &= \not{p}^*\{\Psi_f\, R_p(u) \otimes G_p(\alpha)(\iota_p)\} \\ &= G_p(\alpha)_*\, \not{p}^*\{\Psi_f\, R_p(u) \otimes \iota_p\} \\ &= G_p(\alpha)_* P_p(\bar{u}), \end{aligned}$$

where we use the fact that $\not{p}^*$ commutes with all coefficient group homomorphisms. This completes the proof.

6. REMARKS ON CUP-PRODUCTS

We are left with two things to do: first, define the functions P_r for any non-negative integer r; and then prove theorems 2.1, 2.2, and 2.3. The definition of the functions P_r involves the behaviour of P_p (p prime) on cup-products. Hence, in this section we digress to make some general remarks on cup-products.

Let $A_1, \ldots, A_r$, and A be abelian groups, and λ a homomorphism from $A_1 \otimes \ldots \otimes A_r$ to A. We call λ a pairing, and define a cup-product (written $\smile_\lambda$) relative to λ as follows: suppose that $u_i \in H^{n_i}(K;A_i)$ ($i = 1, \ldots, r$). Let d: $K \longrightarrow K^r$ be the diagonal map $x \longrightarrow (x,\ldots,x)$. We define

$$(6.1) \qquad u_1 \smile_\lambda \ldots \smile_\lambda u_r = \lambda_{\#} d^*(u_1 \times \ldots \times u_r),$$

where d^* is induced by d, $\lambda_{\#}$ by λ, and $u_1 \times \ldots \times u_r \in H^n(K^r; A_1 \otimes \ldots \otimes A_r)$ ($n = n_1 + \ldots + n_r$).

Our first concern is to compare the cup-products given by different pairings. Suppose we also have abelian groups $B_1, \ldots, B_r$, B, and a homomorphism ω of $B_1 \otimes \ldots \otimes B_r$ to B. Then, by 6.1 we have a cup-product, $\smile_\omega$, relative to the pairing ω. We compare $\smile_\lambda$ and $\smile_\omega$ as follows:

(6.2) LEMMA: Let η_i: $A_i \longrightarrow B_i$ ($i = 1, \ldots, r$) and ρ: $B \longrightarrow A$ be homomorphisms such that $\rho\omega(\eta_1 \otimes \ldots \otimes \eta_r) = \lambda$ in the following diagram:

$$\begin{array}{ccc} A_1 \otimes \ldots \otimes A_r & \xrightarrow{\lambda} & A \\ \Big\downarrow \eta_1 \otimes \ldots \otimes \eta_r & & \Big\uparrow \rho \\ B_1 \otimes \ldots \otimes B_r & \xrightarrow[\omega]{} & B \end{array}$$

Then,

$$\rho_*(\eta_{1*}u_1 \smile_\omega \eta_{2*}u_2 \smile_\omega \ldots \smile_\omega \eta_{r*}u_r) = u_1 \smile_\lambda \ldots \smile_\lambda u_r,$$

where $u_i \in H^{n_i}(K;A_i)$.

The proof follows at once from 6.1 and the fact that d^* commutes with coefficient homomorphisms.

As an application of 6.2, suppose that (r, s, ..., t) is any finite set of non-negative integers (say q in number) whose sum is positive. Let Z_θ be a group in $\mathcal{G}$ (see §1), and let $G(Z_\theta)$ be the Γ-ring defined in 1.17. Denote by λ the ring multiplication in $G(Z_\theta)$; that is, the homomorphism from $G_r(Z_\theta) \otimes G_s(Z_\theta) \otimes \ldots \otimes G_t(Z_\theta)$ to $G_{(r+s+\ldots+t)}(Z_\theta)$. In particular we have

$$\lambda[g_r(1_\theta) \otimes \ldots \otimes g_t(1_\theta)] = (\alpha_{r,s,\ldots,t})g_{r+s+\ldots+t}(1_\theta)$$

where $\alpha_{r,s,\ldots,t}$ denotes the multinomial coefficient $(r+s\ldots+t)!/(r!)\ldots(t!)$. We factor the homomorphism λ as follows: set

$$(6.3) \qquad b = \begin{cases} 1, & \text{if } \theta = 0 \\ [r, s, \ldots, t, \theta^\infty], & \text{if } \theta > 0 \end{cases}$$

where $[r, s, \ldots, \theta^\infty]$ = common value of G. C. D. $(r, s, t, \ldots, \theta^e)$ for large e (see 1.15). Then, from 1.17 one readily concludes that

$$G_r(Z_\theta) \otimes G_s(Z_\theta) \otimes \ldots \otimes G_t(Z_\theta) \approx G_b(Z_\theta).$$

From 6.3 we see that there are non-negative integers c, d, ..., e such that

$$r = bc, \ s = bd, \ \ldots, \ t = be.$$

Hence, using 1.26, we define an isomorphism $\eta = \eta_c \otimes \eta_d \otimes \ldots \otimes \eta_e$ mapping $G_r(Z_\theta) \otimes \ldots \otimes G_t(Z_\theta)$ to $G_b(Z_\theta) \otimes \ldots \otimes G_b(Z_\theta)$ (q factors). Now, let ω be the isomorphism of the q-fold tensor product $G_b(Z_\theta) \otimes \ldots \otimes G_b(Z_\theta)$ to $G_b(Z_\theta)$ given by the natural multiplication in the ring $G_b(Z_\theta)$ (see §1). Then, the following diagram is commutative

$$\begin{array}{ccc} G_r(Z_\theta) \otimes \cdots \otimes G_t(Z_\theta) & \xrightarrow{\ \lambda\ } & G_{r+\cdots+t}(Z_\theta) \\ \Big\downarrow \eta & & \Big\uparrow \mu \\ G_b(Z_\theta) \otimes \cdots \otimes G_b(Z_\theta) & \xrightarrow{\ \omega\ } & G_b(Z_\theta), \end{array}$$

where $\mu g_b(1_\theta) = [(r + s + \cdots + t)!/(r!) \cdots (t!)] g_{r+\cdots+t}(1_\theta)$.

Thus, as a special case of 6.2 we have:

(6.4) LEMMA: Let $u_i \in H^{n_i}(K;G_i(Z_\theta))$ $(i = r, s, \ldots, t)$. Then

$$u_r \smile u_s \smile \cdots \smile u_t = \mu_*[\eta_{c*}u_r \smile' \cdots \smile' \eta_{e*}u_t].$$

In 6.1 we gave a general definition of the cup-product. We now show that in certain cases the cup-product can be obtained using the complex $W \otimes_\pi K^{\#n}$ (see §5). Let $\phi^\#$: $H^*(W \otimes_\pi K^{\#n} \otimes G) \longrightarrow H^*(K;G)$ be the homomorphism described in 5.6. Let $\smile'$ denote the cup-product defined in §1; we then have:

(6.5) PROPOSITION: Let $Z_\theta \in \mathcal{C}$, and let $\bar{u}_i \in H^{n_i}(K;Z_\theta)$ $(i = 1, \ldots, q)$. Choose cochains $u_i \in C^{n_i}(K)$ which are integral cochains representing each $\bar{u}_i$. Then

$$\bar{u}_1 \smile' \cdots \smile' \bar{u}_q = \phi^\#\{e_o \otimes_\pi u_1 \otimes \cdots \otimes u_q \otimes 1_\theta\},$$

where $1_\theta = 1 \bmod \theta$.

PROOF: Define a chain map β of K into $W \otimes K$ by $\beta(\sigma) = e_o \otimes \sigma$, where $\sigma \in K$ and e_o is the basic vertex of W. Let ϕ': $W \otimes K \longrightarrow K^q$ be a diagonal approximation chain map (see 5.5). Set $d = \phi'\beta$ mapping K to K^n. Clearly, d is a chain map carried by the diagonal carrier. Thus, by 6.1,

$$\bar{u}_1 \smile' \cdots \smile' \bar{u}_q = \omega_* d^*\{(u_1 \otimes \cdots \otimes u_q) \otimes (1_\theta \otimes \cdots \otimes 1_\theta)\}$$
$$= d^*\{(u_1 \otimes \cdots \otimes u_q) \otimes 1_\theta\},$$

where ω is the multiplication of $Z_\theta \otimes \cdots \otimes Z_\theta$ (q factors) to Z_θ. Define a π-equivariant cochain map $\beta^\#$ mapping $K^{\#q}$ into $W \otimes_\pi K^{\#q}$ by $\beta^\#(v_1 \otimes \cdots \otimes v_q) = e_o \otimes_\pi v_1 \otimes \cdots \otimes v_q$, where $v_i \in K^\#$. If we denote the cochain dual to ϕ' by ϕ (see 5.6), it is easily verified that

$$\phi\beta^\# = d^\#,$$

where $d^\#$ is the cochain map dual to d. Hence

$$\bar{u}_1 \smile' \cdots \smile' \bar{u}_q = d^*\{(u_1 \otimes \cdots \otimes u_q) \otimes 1_\theta\}$$
$$= \{d^\#(u_1 \otimes \cdots \otimes u_q) \otimes 1_\theta\}$$
$$= \{\phi\beta^\#(u_1 \otimes \cdots \otimes u_q) \otimes 1_\theta\}$$
$$= \phi^\#\{e_o \otimes_\pi (u_1 \otimes \cdots \otimes u_q) \otimes 1_\theta\},$$

which proves the lemma.

Finally, we use 6.5 to prove a special case of 2.2.

(6.6) PROPOSITION: Let $\bar{u} \in H^{2n}(K;Z_\theta)$. Let η_p: $G_p(Z_\theta) \longrightarrow Z_\theta$ be the homomorphism defined in 1.25. Then,

$$\eta_{p*}P_p(\bar{u}) = \bar{u}^p. \qquad \text{(p-fold } \smile'\text{-product)}$$

PROOF: Let F: $M \longrightarrow K^*$ be a cochain representative for $\bar{u}$. Set $\Psi = 1 \otimes_\pi f^p$, as in 5.1. Then, from 5.12 and 5.18 we have:

$$P_p(\bar{u}) = \phi^\#\{\Psi[e_o \otimes_\pi u^p + \theta(\Sigma^* e_1) \otimes_\pi u^{p-1}v] \otimes g_p(1_\theta)\}.$$

Now, η_{p*} and $\phi^\#$ commute. Hence,

$$\eta_{p*}P_p(\bar{u}) = \phi^\#\{\Psi[e_o \otimes_\pi u^p + \theta(\Sigma^* e_1) \otimes_\pi u^{p-1}v) \otimes 1_\theta\},$$
$$= \phi^\#\{\Psi(e_o \otimes_\pi u^p) \otimes 1_\theta\},$$

by 1.25 and the fact that $\theta\, 1_\theta = 0$. But from 6.5,

$$\phi^*\{\Psi(e_o \otimes_\pi u^p) \otimes 1_\theta\} = \phi^*\{e_o \otimes_\pi f(u)^p \otimes 1_\theta\} = \bar{u}^p,$$

since f(u) is an integral cochain representing $\bar{u}$. Thus, $\eta_{p*}P_p(\bar{u}) = \bar{u}^p$, which completes the proof.

7. THE CASE OF DIMENSION $\bar{u}$ ODD

Before defining the functions P_r $(r \geq 0)$, we must consider the case of a cohomology class $\bar{u}$ whose dimension is odd. Let $M = M(\theta,2n+1)$ be an elementary cochain complex (see §5). Then, equation 5.12 still defines a cochain function R_p mapping $C^{2n+1}(M)$ to $C^{(2n+1)p}(W \otimes_\pi M^p)$, and proposition 5.13 is still valid; i.e., $\delta R_p(u) \equiv 0 \bmod \theta[p,\theta^\infty]$. If p is an odd prime, the proof given for 5.13 is still valid. If p = 2, the proof must be changed slightly, as 5.15 is not true in this case. However, we easily verify that

(7.1) $$\delta R_2(u) = -\theta^2(e_1 \otimes_\pi v^2),$$

where $R_2(u) = e_o \otimes_\pi u^2 + e_1 \otimes_\pi \delta uu$. But $\theta^2 \equiv 0 \bmod \theta[2,\theta^\infty]$; hence, 5.13 still holds.

Thus, we can continue to use 5.18 to define $P_p(\bar{u}) = \phi^*\{\Psi R_p(u) \otimes g_p(1_\theta)\}$, where $\bar{u} \in H^{2n+1}(K;Z_\theta)$ and Ψ is defined using a cochain representation for $\bar{u}$, say f: $M \longrightarrow K^*$.

(7.2) THEOREM: <u>Let</u> $Z_\theta \in \mathcal{G}$, <u>where</u> $\theta = p^i$, p <u>an odd prime</u> $(i > 0)$. <u>Then,</u>

$$P_p(\bar{u}) = 0,$$

<u>for</u> $\bar{u} \in H^{2n+1}(K;Z_\theta)$.(4)

The proof of this theorem will be given in a forthcoming paper by N. E. Steenrod and the author [13].

Let p now be any prime number, and let Z_τ be a group in $\mathcal{G}$. Suppose that $\tau = 0$ or q^j, where q is a prime different from p. Then, from 1.15 and 1.17 one has that $G_p(Z_\tau) = Z_\tau$; and from 1.25 it follows that

(7.3) $$\eta_p = \text{identity}: \ G_p(Z_\tau) \longrightarrow Z_\tau.$$

Now 6.6 is still valid even if dim $\bar{u}$ is odd; thus, we have at once:

(7.4) PROPOSITION: <u>Let</u> $\bar{u} \in H^m(K;Z_\tau)$. <u>Then,</u>

$$P_p(\bar{u}) = \bar{u}^p. \qquad \text{(p-fold } \smile'\text{-product)}$$

However, the $\smile'$-product is anti-commutative. Thus, if dim $\bar{u}$ is odd, $2\bar{u}^2 = 0$. Since the $\smile'$-product is also associative, this implies that $2\bar{u}^n = 0$ $(n \geq 2)$.

Combining this with 7.2 and 7.4 we obtain:

(7.5) COROLLARY: <u>Let</u> Z_σ <u>be any group in</u> $\mathcal{G}$, <u>and let</u> $\bar{u} \in H^{2n+1}(K;Z_\sigma)$. <u>Then,</u>

(i) $2P_p(\bar{u}) = 0$, if p odd

(ii) $2P_2(\bar{u}) = 0$, if σ is odd or zero.

We now look at the function P_2 defined on an odd-dimensional class $\bar{u}$, where the coefficients are the integers mod a power of 2. Consider the following exact coefficient sequences $(\theta = 2^k)$,

(*) $$0 \longrightarrow G_2(Z_\theta) \xrightarrow{\tau} G_2(Z_{2\theta}) \xrightarrow{\alpha} Z_2 \longrightarrow 0,$$

(**) $$0 \longrightarrow Z_\theta \longrightarrow Z_{\theta^2} \longrightarrow Z_\theta \longrightarrow 0,$$

where $\tau g_2(1_\theta) = 2g_2(1_{2\theta})$, Z_2 is identified with the factor group $G_2(Z_{2\theta})/\text{image } \tau$, and α is the factor map. Let δ_*, δ_{**} be the Bockstein coboundary operators associated with (*) and (**) (see [12; 38.5]). Let $\bar{u} \in H^q(K;Z_\theta)$; we define a cohomology operation(5)

$$Sq_i: \ H^q(K;Z_\theta) \longrightarrow H^{2q-i}(K;Z_2) \quad (i = 0, 1, \ldots)$$

by

(7.6) $$Sq_i(\bar{u}) = \phi^*\{\Psi(e_i \otimes_\pi u^2) \otimes 1_2\},$$

where u is a generator for the elementary cochain complex $M = M(\theta,q)$ and Ψ is induced by a cochain map $M \longrightarrow K^*$ representing $\bar{u}$ (see §5). Finally, let β be the homomorphism of Z_2 to $G_2(Z_\theta)$ given by $\beta(1_2) = (\theta^2/2)g_2(1_\theta)$, where $\theta = 2^k$. Notice that if $k > 1$, then $\beta = 0$. In any case, $2\beta = 0$. The operation P_2 can now be characterized as follows:

(7.7) PROPOSITION: Let $Z_\theta \in \mathcal{C}$ $(\theta = 2^k)$, and let $\bar{u} \in H^{2q+1}(K;Z_\theta)$. Then,

$$P_2(\bar{u}) = \delta_* Sq_1(\bar{u}) + \beta_* Sq_2\, \delta_{**}(\bar{u}),$$

in $H^{4q+2}(K;G_2(Z_\theta))$.

COROLLARY: If $Z_\theta = Z_{2^k}$, where $k > 1$, then

$$P_2(\bar{u}) = \delta_* Sq_1(\bar{u}).$$

PROOF: Now $2\, Sq_1(\bar{u}) = 0$. Hence,

$$(7.8)\qquad Sq_1(\bar{u}) = -Sq_1(\bar{u}) = \phi^*\Psi^*\{-(e_1 \otimes_\pi u^2) \otimes 1_2\}$$

$$= \phi^*\Psi^*\{(-e_1 \otimes_\pi u^2 + \theta e_2 \otimes_\pi uv) \otimes 1_2\},$$

since $\theta 1_2 = 0$. However

$$\delta(-e_1 \otimes_\pi u^2 + \theta e_2 \otimes_\pi uv) = 2R_2(u) + \theta^2 e_2 \otimes_\pi v^2,$$

where R_2 is the function defined in 5.12. This implies that

$$\delta_*\{(-e_1 \otimes_\pi u^2 + \theta e_2 \otimes_\pi uv) \otimes 1_2\} = \{R_2(u) \otimes g_2(1_\theta)\} + \beta_*\{e_2 \otimes_\pi v^2 \otimes 1_2\}.$$

Now δ_* commutes with ϕ^* and Ψ^*. Therefore, applying δ_* to both sides of 7.8 we have

$$\delta_* Sq_1(\bar{u}) = P_2(\bar{u}) + \beta_*\phi^*\Psi^*\{e_2 \otimes_\pi v^2 \otimes 1_2\}$$

$$= P_2(\bar{u}) + \beta_* Sq_2(\bar{v})$$

$$= P_2(\bar{u}) + \beta_* Sq_2\, \delta_{**}(\bar{u}).$$

The theorem then follows from the fact that $2\beta_* = 0$.

As an immediate consequence of 7.7 we have:

(7.9) PROPOSITION: Let $Z_\theta \in \mathcal{C}$ $(\theta = 2^k)$, and let $\bar{u} \in H^{2q+1}(K;Z_\theta)$. Then,

$$2P_2(\bar{u}) = 0.$$

PROOF: The result follows at once from 7.7 when we observe that $2\, Sq_1 = 0 = 2\beta_*$.

Notice from 7.2, 7.3, and 7.7 the fact that $P_p(u)$ is always given in terms of other known operations, when dim u is odd. Thus, no new information is obtained by considering the functions P_p in the odd-dimensional case. In the following section, however, we will need to make use of the results of this section; for we will study the function P_p applied to the cup-product of two odd-dimensional classes.

8. THE DEFINITION OF THE OPERATIONS P_r

We define the functions P_r $(r \geq 0)$ as follows: first, suppose that $r = 0$ or 1. Now for any group Z_θ in $\mathcal{C}$, we know that $G_0(Z_\theta) = Z_1$ and $G_1(Z_\theta) = Z_\theta$. Thus, the coefficient groups are correct if we simply use 2.1(ii) as a definition: that is, for $u \in H^{2n}(K;Z_\theta)$, set

$$(8.1)\qquad P_0(u) = \text{unit of } H^*(K;G(Z_\theta)),$$

$$P_1(u) = u.$$

Suppose now that $r \geq 2$. Then, $r = p_1 \dots p_k$, where each p_i is a prime number. From 1.22 we know that $G_i(G_j(Z_\theta)) = G_{ij}(Z_\theta)$, for any pair of integers i, j. Thus, $G_r(Z_\theta) = G_{p_k}(G_{p_{k-1}}(\dots(G_{p_1}(Z_\theta)\dots)$. Again our coefficient group is correct if we define

$$(8.2)\qquad P_r(u) = P_{p_k}(P_{p_{k-1}}(\dots(P_{p_1}(u)\dots)),$$

in $H^{2rn}(K;G_r(Z_\theta))$.

We must show that definition 8.2 is independent of the order of the primes $p_1, \ldots, p_k$. This follows at once from the following lemma:

(8.3) LEMMA: Let p and q be any two primes, and let $u \in H^{2n}(K;Z_\theta)$. Then,

$$P_p(P_q(u)) = P_q(P_p(u))$$

in $H^{2pqn}(K;G_{pq}(Z_\theta))$.

The proof of 8.3 will occupy the remainder of this section. It depends upon analysing the behaviour of the function P_p on the $\smile'$-product of classes. We turn first to a study of these. The results needed are:

(8.4) THEOREM: Let $\bar{u}_i \in H^{2n_i}(K;Z_\theta)$ $(i = 1, \ldots, r)$, and let p be an odd prime. Then,

$$P_p(\bar{u}_1 \smile' \ldots \smile' \bar{u}_r) = P_p(\bar{u}_1) \smile' \ldots \smile' P_p(\bar{u}_r).$$

(8.5) THEOREM: Let $u_i \in H^{2n_i}(K;Z_\theta)$ $(i = 1, \ldots, r)$. Then,

$$P_2(\bar{u}_1 \smile' \ldots \smile' \bar{u}_r) = P_2(\bar{u}_1) \smile' \ldots \smile' P_2(\bar{u}_r) + \Psi_2(\bar{u}_1, \ldots, \bar{u}_r),$$

where

(a) $\Psi_2(\bar{u}_1, \ldots, \bar{u}_r) \in H^{4n}(K;G_2(Z_\theta))$, $(n = n_1 + \ldots + n_r)$

(b) $2\Psi_2(\bar{u}_1, \ldots, \bar{u}_r) = 0$,

(c) $\Psi_2(\bar{u}_1, \ldots, \bar{u}_r) = 0$, if θ is odd or is zero

(d) $\Psi_2(\bar{u}, \ldots, \bar{u}) = 0$, $(\bar{u}_1 = \ldots = \bar{u}_r = \bar{u})$.

We consider first the proof of 8.4. Suppose we have proved 8.4 for the case of 2 factors; that is, when r = 2. Then, the theorem for more than 2 factors is a simple induction on r, using the fact that the cup-product is associative. Hence, we assume r = 2. Also, the cup-product is defined in terms of the $\times$-product (see 6.1). Thus we prove 8.4 first for the case of the $\smile'$-product replaced by the $\times$-product.

We recall that the cohomology homomorphism $\phi^*\colon H^r(W \otimes_\pi K^{*p} \otimes G) \longrightarrow H^r(K;G)$ is independent of the particular choice of π-equivariant chain map, ϕ', used to define it (see §5). Let K_1, K_2 be any two complexes, and form the product complex $K_1 \times K_2$. Define a chain map ϕ' mapping $W \otimes (K_1 \otimes K_2)$ to $(K_1 \otimes K_2)^p$ as the composition of the chain maps in the diagram below:

(8.6) $$\phi' = \lambda(\phi_1' \otimes \phi_2')\mu\, d_\#',$$

$$W \otimes (K_1 \otimes K_2) \xrightarrow{d_\#'} (W \otimes W) \otimes (K_1 \otimes K_2) \xrightarrow{\mu}$$
$$(W \otimes K_1) \otimes (W \otimes K_2) \xrightarrow{\phi_1' \otimes \phi_2'} K_1^p \otimes K_2^p \xrightarrow{\lambda} (K_1 \otimes K_2)^p.$$

Here, $d_\#' = d_\# \otimes 1 \otimes 1$, where $d_\#$ is a chain map representing the diagonal homomorphism $d\colon \pi \longrightarrow \pi \times \pi$, μ and λ are the natural chain maps permuting factors, and ϕ_i' is a π-equivariant chain map for the diagonal map $d_i\colon W \times K_i \longrightarrow K_i^p$ $(i = 1, 2)$. Then, as remarked in [10; §5], ϕ' is carried by the diagonal carrier and is π-equivariant. Hence, we can define the dual to ϕ',

$$\phi\colon W \otimes_\pi (K_1 \otimes K_2)^{p*} \longrightarrow (K_1 \otimes K_2)^*,$$

by 5.6. Combining ϕ with the map ζ of 5.1, we obtain a map, which we still denote by ϕ, mapping $W \otimes_\pi (K_1 \otimes K_2)^{*p}$ to $(K_1 \otimes K_2)^*$.

Now let ρ be the cochain map defined as the composition of the following homomorphisms:

(8.7) $$\rho = (\phi_1 \otimes \phi_2)\mu^\# (d_\# \otimes \lambda^\#),$$

$$W \otimes_\pi (K_1 \otimes K_2)^{*p} \xrightarrow{d_\# \otimes \lambda^\#} (W \otimes W) \otimes_{\pi \times \pi} (K_1^{*p} \otimes K_2^{*p})$$
$$\xrightarrow{\mu^\#} (W \otimes_\pi K_1^{*p}) \otimes (W \otimes_\pi K_2^{*p}) \xrightarrow{\phi_1 \otimes \phi_2} (K_1 \otimes K_2)^*.$$

Here, $\lambda^\#$ is the natural shuffle cochain map permuting the factors, $d_\#$ is the map defined above, ϕ_1, ϕ_2 are cochain duals to ϕ_1', ϕ_2' (see 5.6), and

$\mu^{\#}$ is the shuffle cochain map given by

$$(e_1 \otimes e_2) \otimes_{\pi \times \pi} (u \otimes v) \longrightarrow (-1)^{km}(e_1 \otimes_\pi u) \otimes (e_2 \otimes_\pi v),$$

where $e_1, e_2 \in W$, $u \in K_1^{*p}$, $v \in K_2^{*p}$, $\dim e_2 = k$, $\dim u = m$.

(8.8) PROPOSITION: Let ϕ be the cochain dual to ϕ', and let ρ be the cochain map defined in 8.7. Then,

$$\phi = \rho: \quad W \otimes_\pi (K_1 \otimes K_2)^{*p} \longrightarrow (K_1 \otimes K_2)^*.$$

The proof is a lengthy and mechanical verification of the fact that ϕ and ρ have the same value on a cell $\sigma \otimes \tau \in K_1 \otimes K_2$. It is omitted here.

Now, let $M_i = M_i(\theta, 2n_i)$ be an elementary cochain complex with generators u_i, v_i in dimensions $2n_i$, $2n_i+1$ respectively $(i = 1, 2)$. For shortened notation set $S_i = W \otimes_\pi M_i^p$. Let

$$a = (\Sigma^* e_o \otimes_\pi u_1^{p-1} v_1) \otimes (e_1 \otimes_\pi u_2^p),$$

$$d = (e_1 \otimes_\pi u_1^p) \otimes (T \Sigma^* e_o \otimes_\pi u_2^{p-1} v_2),$$

be elements in $S_1 \otimes S_2$. For future use we prove:

(8.9) LEMMA: There exist elements $A, B, C, D, E, F \in S_1 \otimes S_2$ such that

$$a = pA + \theta B + \delta C,$$

$$d = pD + \theta E + \delta F.$$

PROOF: Observe that $\Sigma(\Sigma^* - [p(p-1)/2])u_1^{p-1}v_1 = 0$ in M_1^p. But π operates freely on M_1^p in dimensions r where $2pn+1 \leq r \leq (2n+1)p-1$. Thus, if x is any element in M_1^p in this range of dimensions, then $\Sigma x = 0$ implies there exists an element $y \in M_1^p$ such that $x = \Delta y$. Set $\Delta^* = T^{p-1} - 1 \in Z(\pi)$. Then, Δ^* and Σ still generate the annihilator of each other in $Z(\pi)$. Thus, $\Sigma x = 0$ also implies that there exists $y' \in M_1^p$ such that $x = \Delta^* y'$.

In particular, then, there is an element $y_1 \in C^{2pn+1}(M_1^p)$ such that

$$\Delta^* y_1 = (\Sigma^* - [p(p-1)/2])u_1^{p-1}v_1.$$

Set $\delta y_1 = \theta z_1$. Then, since $\Delta^* u_2^p = 0$, we obtain the expression for a by setting

$$A = [(p-1)/2](e_o \otimes_\pi u_1^{p-1} v_1) \otimes (e_1 \otimes_\pi u_2^p),$$

$$B = (e_1 \otimes_\pi z_1) \otimes (e_1 \otimes_\pi u_2) + (e_1 \otimes_\pi y_1) \otimes (\Sigma e_1 \otimes_\pi u_2^{p-1} v_2),$$

$$C = (e_1 \otimes_\pi y_1) \otimes (e_1 \otimes_\pi u_2^p).$$

The coefficient $(p-1)/2$ is an integer, since p is an odd prime by hypothesis. The expression for d is obtained in an entirely similar way.

We now are in a position to prove 8.4, making use of the complexes $M_i = M_i(\theta, 2n_i)$ $(i = 1, 2)$ defined above. Form the product complex $M_1 \otimes M_2$, and consider the subcomplex $M \subset M_1 \otimes M_2$ defined by

$C^{2(n_1+n_2)}(M)$ generated by $u_1 \otimes u_2 = u$,

$C^{2(n_1+n_2)+1}(M)$ generated by $v_1 \otimes u_2 + u_1 \otimes v_2 = v$.

Then, $\delta u = \theta v$, and M is an elementary cochain complex of type $M(\theta, 2(n_1+n_2))$. Let f_i: $M_i \longrightarrow K_i^*$ be a cochain representation for $\bar{u}_i$ $(i = 1, 2)$. Then, $f = f_1 \otimes f_2 | M$: $M \longrightarrow K_1^* \otimes K_2^*$ is a cochain representation for $\bar{u}_1 \times \bar{u}_2 \in H^{2(n_1+n_2)}(K_1^* \otimes K_2^*; Z_\theta)$. Hence, from 5.18 and 8.8,

$$\text{(8.10)} \qquad P_p(\bar{u}_1 \times \bar{u}_2) = \rho^*\{\Psi R_p(u_1 \otimes u_2) \otimes \iota_p\} = \{\rho \Psi R_p(u_1 \otimes u_2) \otimes \iota_p\},$$

where $\Psi = 1 \otimes_\pi f^p$, and $\iota_p = g_p(1_\theta)$. Set $x_i = f_i(u_i)$, $y_i = f_i(v_i)$ in K_i^* $(i = 1, 2)$, and $x = x_1 \otimes x_2$ in $K_1^* \otimes K_2^*$. Then, from 8.7,

$$\text{(8.11)} \qquad \rho \Psi R_p(u_1 \otimes u_2) = (\phi \otimes \phi_2)\mu^{\#}(d_{\#} \otimes \lambda^{\#})[e_o \otimes_\pi x^p + \theta(\Sigma^* e_1) \otimes_\pi x^{p-1}(y_1 \otimes x_2 + x_1 \otimes y_2)].$$

Now, $d_{\#}e_o = e_o \otimes e_o$, $d_{\#}e_1 = e_o \otimes e_1 + e_1 \otimes Te_o$, (see [10; 5.2]). Thus, one has

$$(8.12)\quad \begin{aligned} \mu^{\#}(d_{\#} \otimes \lambda^{\#})\Psi R_p(u_1 \otimes u_2) &= (e_o \otimes_\pi x_1^p) \otimes (e_o \otimes_\pi x_2^p) \\ &\quad - \theta[(\Sigma^* e_o) \otimes_\pi x_1^{p-1} y_1) \otimes (e_1 \otimes_\pi x_2^p)] \\ &\quad + \theta[(e_o \otimes_\pi x_1^p) \otimes (\Sigma^* e_1 \otimes_\pi x_2^{p-1} y_2)] \\ &\quad + \theta[(\Sigma^* e_1 \otimes_\pi x_1^{p-1} y_1) \otimes (e_o \otimes_\pi x_2^p) \\ &\quad + (e_1 \otimes_\pi x_1^p) \otimes (\Sigma^* Te_o \otimes_\pi x_2^{p-1} y_2)] \\ &= \Psi_1 R_p(u_1) \otimes \Psi_2 R_p(u_2) \\ &\quad + \theta[-(\Sigma^* e_o \otimes_\pi x_1^{p-1} y_1) \otimes (e_1 \otimes_\pi x_2^p) \\ &\quad + (e_1 \otimes_\pi x_1^p) \otimes (\Sigma^* Te_o \otimes_\pi x_2^{p-1} y_2)] \\ &\quad - \theta^2(\Sigma^* e_1 \otimes_\pi x_1^{p-1} y_1) \otimes (\Sigma^* e_1 \otimes_\pi x_2^{p-1} y_2) \\ &= \Psi_1 R_p(u_1) \otimes \Psi_2 R_p(u_2) \\ &\quad + \theta[-(\Psi_1 \otimes \Psi_2)a + (\Psi_1 \otimes \Psi_2)d] - \theta^2 G, \end{aligned}$$

where a and d are defined in 8.9, $\Psi_i = 1 \otimes_\pi f_i^p$, and $G = (\Sigma^* e_1 \otimes_\pi x_1^{p-1} y_1) \otimes (\Sigma^* e_1 \otimes_\pi x_2^{p-1} y_2)$. Now, let A, B, ..., F be the elements defined in 8.10, and set

$$H = \Psi_1 \otimes \Psi_2(-A + D),\quad I = \Psi_1 \otimes \Psi_2(-B + E) - G,$$

$$J = \Psi_1 \otimes \Psi_2(-\theta C + \theta F).$$

Then,

$$\mu^{\#}(d_{\#} \otimes \lambda^{\#})\Psi R_p(u_1 \otimes u_2) = \Psi_1 R_p(u_1) \otimes \Psi_2 R_p(u_2) + p\,\theta\, H + \theta^2 I + \delta J\,.$$

Therefore, from 8.11,

$$\rho\Psi R_p(u_1 \otimes u_2) = (\phi_1 \otimes \phi_2)[\Psi_1 R_p(u_1) \otimes \Psi_2 R_p(u_2) + p\,\theta\, H + \theta^2 I + \delta J],$$

and by 8.10,

$$\begin{aligned} P_p(\bar{u} \times \bar{u}_2) &= \{\phi_1 \Psi_1 R_p(u_1) \otimes \phi_2 \Psi_2 R_p(u_2) \otimes \iota_p\} \\ &= \{\phi_1 \Psi_1 R_p(u_1) \otimes \iota_p\} \times \{\phi_2 \Psi_2 R_p(u_2) \otimes \iota_p\} \\ &= P_p(\bar{u}_1) \times P_p(\bar{u}_2). \end{aligned}$$

This completes the proof of 8.4 for the case of the $\times$-product. The theorem for the $\cup'$-product follows at once from the fact that $P_p d^* = d^* P_p$, where d^* is the homomorphism induced by the diagonal map (see 6.1).

In §7 we defined the functions P_p on odd dimensional cohomology classes. Let $\bar{u}_i \in H^{2n_i+1}(K;Z_\theta)$ $(i = 1, 2)$. Applying the preceding proof to the element $P_p(\bar{u}_1 \cup' \bar{u}_2)$ we have, in an entirely similar way, $P_p(\bar{u}_1 \cup' \bar{u}_2) = \pm P_p(\bar{u}_1) \cup' P_p(\bar{u}_2)$. But from 7.5 and the fact that the $\cup'$-product is bilinear, it follows that $2P_p(\bar{u}_1 \cup' \bar{u}_2) = 0$. Therefore,

$$(8.13)\qquad P_p(\bar{u}_1 \cup' \bar{u}_2) = P_p(\bar{u}_1) \cup' P_p(\bar{u}_2),$$

a result we will need in the next section.

We turn now to the proof of 8.5. In order to study P_2 on the $\cup'$-product, we need to introduce a new cohomology operation. Let $\bar{u} \in H^q(K;Z_\theta)$ $(\theta = 0$ or $2^j)$, and suppose that f: $M \longrightarrow K^*$ is a cochain representation for $\bar{u}$, where $M = M(\theta, q)$. Let $1_r = 1 \bmod r$ $(r = 0, 1, \ldots)$, and let $g_2(1_\theta)$ be a generator for $G_2(Z_\theta)$. Let Φ: $H^*(W \otimes_\pi M^2 \otimes G) \longrightarrow H^*(K;G)$ be the homomorphism defined in 5.8 (G any coefficient group). Define

$$(8.14)\qquad w(\bar{u}) = \Phi\{e_o \otimes_\pi uv \otimes 1_\theta\} \;\in\; H^{2q+1}(K;Z_\theta).$$

Denote by ν the homomorphism of Z_θ to $G_2(Z_\theta)$ given by $\nu(1_\theta) = \theta g_2(1_\theta)$. Define

(8.15) $$\mathfrak{P}(\bar{u}) = \nu_* w(\bar{u}) \in H^{2q+1}(K;G_2(Z_\theta)).$$

The operation $\mathfrak{P}$ is the Postnikov square; we note here some well-known properties of this operation (see [7], [20]).

(8.16)
(i) $\mathfrak{P}(u_1 + u_2) = \mathfrak{P}(u_1) + \mathfrak{P}(u_2)$

(ii) $2\mathfrak{P}(u) = 0$

(iii) $\mathfrak{P}(u) = 0,$ if θ is zero.

Consider now the exact coefficient sequence

(8.17) $$0 \longrightarrow G_2(Z_\theta) \xrightarrow{\tau} G_2(Z_{2\theta}) \xrightarrow{\alpha} Z_2 \longrightarrow 0,$$

which was denoted by (*) in §7; again let δ_* be the coboundary associated with the exact sequence. Define an operation (see [4;§§4, 5]),

$$\Delta: H^q(K;Z_\theta) \longrightarrow H^{2q-1}(K;G_2(Z_\theta))$$

by the composition

(8.18) $$H^q(K;Z_\theta) \xrightarrow{Sq_2} H^{2q-2}(K;Z_2) \xrightarrow{\delta_*} H^{2q-1}(K;G_2(Z_\theta)),$$

where Sq_2 is defined in 7.6. The following result is due to Nakaoka, [4; §5]:

(8.19) THEOREM: Let $u_i \in H^{2n_i}(K;Z_\theta)$ $(i = 1, 2)$, where $Z_\theta \in \mathcal{G}$. Then,

$$P_2(u_1 \smile' u_2) = P_2(u_1) \smile' P_2(u_2) + \mathfrak{P}(u_1) \smile' \Delta(u_2) + \Delta(u_1) \smile' \mathfrak{P}(u_2); \quad \text{if } \theta = 2^j;$$

$$= P_2(u_1) \smile' P_2(u_2), \quad \text{if } \theta \text{ is zero or odd.}$$

The result can be obtained by applying the method of 8.12 to the case p = 2. Using 8.19 we now can prove 8.5. We give a proof by induction: to

simplify the notation, in 8.5 replace $\bar{u}_i$ by u_i $(i = 1, \ldots, r)$. Suppose first that r = 2. Define

(8.20) $$\Psi_2(u_1,u_2) = \mathfrak{P}(u_1) \smile' \Delta(u_2) + \Delta(u_1) \smile' \mathfrak{P}(u_2) \quad (\theta = 2^j)$$

$$= 0, \quad \text{if } \theta \text{ is zero or odd.}$$

Then, by 8.19

$$P_2(u_1 \smile' u_2) = P_2(u_1) \smile' P_2(u_2) + \Psi_2(u_1,u_2).$$

Properties (a) and (b) of 8.5 follow at once from 8.16; 8.5(c) follows from 8.20; and 8.5(d) follows from 8.20 and the fact that the $\smile'$-product is anti-commutative.

Thus, 8.5 is proved for the case r = 2. Let r be an integer > 2, and suppose by induction that we have proved 8.5 for all integers k < r. We prove it now for k = r, and hence for all integers.

Because the cup-product is associative, $u_1 \smile' \ldots \smile' u_r = (u_1 \smile' u_2) \smile' (u_3 \smile' \ldots \smile' u_r)$. Therefore,

$$\begin{aligned} P_2(u_1 \smile' \ldots \smile' u_r) &= P_2[(u_1 \smile' u_2) \smile' (u_3 \smile' \ldots \smile' u_r)] \\ &= P_2(u_1 \smile' u_2) \smile' P_2(u_3 \smile' \ldots \smile' u_r) \\ &\quad + \Psi_2(u_1 \smile' u_2, (u_3 \smile' \ldots \smile' u_r)) \\ &= P_2(u_1) \smile' \ldots \smile' P_2(u_r) + \Psi_2(u_1, \ldots, u_r), \end{aligned}$$

where,

$$\begin{aligned} \Psi_2(u_1, \ldots, u_r) = {} & \Psi_2(u_1,u_2) \smile' P_2(u_3 \smile' \ldots \smile' u_r) \\ & + P_2(u_1) \smile' P_2(u_2) \smile' \Psi_2(u_3, \ldots, u_r) \\ & + \Psi_2(u_1,u_2) \smile' \Psi_2(u_3, \ldots, u_r) \\ & + \Psi_2(u_1 \smile' u_2, u_3 \smile' \ldots \smile' u_r). \end{aligned}$$

From the inductive hypothesis and the fact that the cup-product is bilinear, we see that 8.5(a), (b), and (c) are all verified for this definition of $\Psi_2(u_1, \ldots, u_r)$. When $u_1 = u_2 = \ldots = u_r = u$, we have by induction that $\Psi_2(u,u) = 0 = \Psi_2(u, \ldots, u)$. Thus, to prove 8.5(d), it remains to show that $\Psi_2(u \smile u, u \smile' \ldots \smile' u) = 0$. But this is simply a special case of the following lemma:

(8.21) Let $u \in H^{2n}(K;Z_\theta)$, $v \in H^{2m}(K;Z_\theta)$ $(\theta = 2^j)$. Then

$$\Psi_2(u \smile' u, v) = 0.$$

PROOF: From 8.20 we see that

$$\Psi_2(u \smile' u, v) = \mathfrak{P}(u \smile' u) \smile' \Delta(v) + \Delta(u \smile' u) \smile' \mathfrak{P}(v).$$

Now in [4; §5], it is shown that

$$\mathfrak{P}(u_1 \smile' u_2) = \mathfrak{P}(u_1) \smile' P_2(u_2) + P_2(u_1) \smile' \mathfrak{P}(u_2),$$

for cohomology classes u_1, u_2. Thus, $\mathfrak{P}(u \smile' u) = 0$, since $2\mathfrak{P}(u) = 0$, and the cup-product is anti-commutative. Therefore, 8.21 follows at once when we show

(8.22) Let $u \in H^{2n}(K;Z_{2^k})$; then,

$$\Delta(u \smile' u) = 0.$$

PROOF: By definition 8.18, $\Delta(u \smile' u) = \delta_* Sq_2(u \smile' u)$. However, from [10; §5], we have

$$Sq_2(u \smile' u) = Sq_0(u) \smile' Sq_2(u) + Sq_1(u) \smile' Sq_1(u) + Sq_2(u) \smile' Sq_0(u).$$

Now the $\smile'$-product is anti-commutative, and $2\,Sq_2 = 0$. Therefore, $Sq_2(u \smile' u) = Sq_1(u) \smile' Sq_1(u)$. However, it is known that

$$Sq_1(u) = \delta_{**}\, Sq_2(u),$$

where δ_{**} is the coboundary associated with the exact sequence

$$(**) \qquad 0 \longrightarrow Z_2 \xrightarrow{2} Z_4 \longrightarrow Z_2 \longrightarrow 0.$$

Thus,

$$Sq_2(u \smile' u) = \delta_{**}\, Sq_2(u) \smile' \delta_{**} Sq_2(u) = \delta_{**}(Sq_2(u) \smile' \delta_{**}\, Sq_2(u)).$$

But $\delta_{**} = \alpha_* \delta'$, where δ' is the coboundary associated with the exact sequence

$$0 \longrightarrow G_2(Z_{2\theta}) \longrightarrow G_2(Z_{4\theta}) \longrightarrow Z_2 \longrightarrow 0,$$

and α is the natural map $G_2(Z_{2\theta}) \longrightarrow Z_2$. Hence,

$$\Delta(u \smile' u) = \delta_* Sq_2(u \smile' u) = \delta_* \delta_{**}[Sq_2(u) \smile' \delta_{**} Sq_2(u)]$$

$$= \delta_* \alpha_* \delta'[Sq_2(u) \smile' \delta_{**} Sq_2(u)] = 0,$$

since $\delta_* \alpha_* = 0$ by exactness of the sequence 8.17. This proves 8.22 and hence 8.21.

We turn now to the proof of 8.3. That is, we are to show that $P_p(P_q(u)) = P_q(P_p(u))$, where p, q are primes and $u \in H^{2n}(K;Z_\theta)$. If $p = q$, the statement is trivially true. Also, if $\theta = 0$, then

$$P_p(P_q(u)) = u^{pq} = P_q(P_p(u)),$$

by 7.4, since $G_p(G_q(Z_o)) = Z = G_q(G_p(Z_o))$. Thus, we assume that $p \neq q$, and $\theta = t^k$ where t is a prime. Since $p \neq q$, one of these is prime to t; say $p \neq t$. Then, by 7.4,

$$P_p(u) = u^p.$$

Thus, by 8.4 or 8.5(d), and 7.4,

$$P_q(P_p(u)) = P_q(u^p) = [P_q(u)]^p = P_p(P_q(u)).$$

This completes the proof and shows that definition 8.2 is independent of the order in which we take the primes. Notice that we have used here the hypothesis that $Z_\theta \in \mathcal{C}$; for if θ were a composite integer, say pq, then in general

$$P_p(u) \neq u^p, \quad P_q(u) \neq u^q,$$

and the proof of 8.3 would break down.

As an immediate consequence of 8.2 and 8.3 we have:

(8.23) PROPOSITION: Let $u \in H^{2n}(K;Z_\theta)$; then,

$$P_r(P_s(u)) = P_s(P_r(u)) = P_{rs}(u). \qquad (r, s > 0)$$

Combining 8.23 with 6.6 we have,

(8.24) PROPOSITION: Let $u \in H^{2n}(K;Z_\theta)$, and let r, s be two non-negative integers. Let η_r be the homomorphism from $G_{rs}(Z_\theta)$ to $G_s(Z_\theta)$ defined in 1.26. Then,

$$\eta_{r\#}P_{rs}(u) = [P_s(u)]^r \qquad \text{(r-fold } \smile' \text{-product)}.$$

We conclude this section with the proof of 4.7. Let us first suppose that t = p, p a prime. Since $\bar{u} = \{u \otimes n\} = \{nu \otimes 1_\theta\}$, we have by 5.18,

$$P_p(\bar{u}) = \{\phi\Psi R_p(nu) \otimes \iota_p\} = \{\phi\Psi R_p(nu) \otimes g_p(1_\theta)\}.$$

From 5.12 one easily verifies that $R_p(nu) = n^p R_p(u)$. Hence,

$$P_p(\bar{u}) = \{n^p\phi\Psi R_p(u) \otimes g_p(1_\theta)\} = \{\phi\Psi R_p(u) \otimes g_p(n)\},$$

by 1.21. Setting $z = \phi\Psi R_p(u)$, 4.7 follows. Now let $t = p_1 \ldots p_k$ (p_i prime), and suppose that we have proved 4.7 for all integers t such that k, the number of primes, is < r. Let $t' = p_1 \ldots p_r$; we prove 4.7 for t', and hence for all integers. Set $t = p_2 \ldots p_r$. Then, 4.7 holds for t, by the inductive hypothesis. That is,

$$P_t(\bar{u}) = \{z_t \otimes g_t(n)\},$$

for some cochain $z_t \in C^{tm}(K)$. But, $g_t(n) = n^t g_t(1_\theta)$. Hence,

$$P_t(\bar{u}) = \{n^t z_t \otimes g_t(1_\theta)\} = \{n^t z_t \otimes 1_\tau\},$$

where $\tau = \theta[t,\theta^\infty]$. Set $p = p_1$; then, by 8.3 and 5.18,

$$P_{t'}(\bar{u}) = P_p(P_t(\bar{u})) = \{\phi\Psi R_p(n^t z_t) \otimes g_p(1_\tau)\}.$$

But $R_p(n^t z_t) = n^{pt}R_p(z_t)$, and $g_p(1_\tau) = g_{pt}(1_\theta) = g_{t'}(1_\theta)$. Thus,

$$P_{t'}(\bar{u}) = \{\phi\Psi R_p(z_t)\otimes n^{pt}g_{pt}(1_\theta)\} = \{\phi\Psi R_p(z_t) \otimes g_{t'}(n)\}.$$

Setting $z = \phi\Psi R_p(z_t)$, completes the proof of 4.7.

9. THE OPERATION P_p ON A SUM

The functions P_r are now defined for all integers $r \geq 0$, and we are ready to begin the proofs of theorems 2.1, 2.2, and 2.3. These proofs are based entirely on the formal properties of the functions P_r. However, there remains one such formal property which must be obtained directly from the definition of the functions P_p, p a prime. Namely, a special case of 2.1(iv):

(9.1) THEOREM: Let $\bar{u}_1, \bar{u}_2 \in H^{2n}(K;Z_\theta)$, where $Z_\theta \in \mathcal{C}$. Let p be a prime number. Then,

$$P_p(\bar{u}_1 + \bar{u}_2) = \Sigma_{i+j=p}\, P_i(\bar{u}_1) \smile P_j(\bar{u}_2).$$

(9.2) COROLLARY: Let $\bar{u}_o, \ldots, \bar{u}_m \in H^{2n}(K;Z_\theta)$. Then,

$$P_p(\bar{u}_o + \ldots + \bar{u}_m) = \Sigma\, P_{t_o}(\bar{u}_o) \smile \ldots \smile P_{t_m}(\bar{u}_m),$$

where the summation is taken over all distinct sets of non-negative integers $(t_o, \ldots, t_m)$ such that $t_o + \ldots + t_m = p$.

We begin the proof of 9.1 by analysing the behaviour of the cochain function R_p (see 5.12) on the sum of two cochains.

Let $M_i = M_i(\theta,2n)$ $(i = 1, 2)$ be an elementary cochain complex with generators u_i and v_i, and $\delta u_i = \theta v_i$ (see §5). Define the cochain complex $M_1 + M_2$ (direct sum) in the natural way.

(9.3) LEMMA:[6] There is a cochain $Q = Q(u_1,u_2)$ in $(M_1 + M_2)^p$ such that,

$$(u_1 + u_2)^p = u_1^p + u_2^p + \Sigma Q,$$

where $\Sigma = \Sigma_{k=0}^{p-1} T^k \in Z(\pi)$.

To see this, note that each p-fold product of u_1 and u_2 occurs just once in the expansion of $(u_1 + u_2)^p$. Since p is a prime, the periodicity of such a product under cyclic permutations is either p or 1. But u_1^p and u_2^p are the only products of period 1. Hence, Q is formed by choosing one product from each equivalence class of products under cyclic permutations. Also, since u_1 and u_2 have even dimension, no change of sign occurs from permutations by T. This completes the proof.

Now in the cochain complex $M_1 + M_2$ set $u = u_1 + u_2$, $v = v_1 + v_2$, and define an elementary cochain complex $M = M(\theta,2n)$ as the subcomplex of $M_1 + M_2$ with cochain groups generated respectively by u and v. Let $\bar{u}_i \in H^{2n}(K;Z_\theta)$ $(i = 1, 2)$ be the classes given in 9.1. Choose a cochain representation f_i: $M_i \longrightarrow K^*$ for each $\bar{u}_i$ (see §5). Then, $f = f_1 + f_2$ is a cochain map of $M_1 + M_2$ to K^*, and $f|M$ is a cochain representation for $\bar{u} = \bar{u}_1 + \bar{u}_2$. Set

$$\Psi_i = 1 \otimes_\pi f_i^p: \ W \otimes_\pi M_i^p \longrightarrow K^* \qquad (i = 1, 2),$$

$$\Psi_{12} = 1 \otimes_\pi f^p: \ W \otimes_\pi (M_1 + M_2)^p \longrightarrow K^*.$$

Using the function Ψ_{12} and the cochain Q defined in 9.3, we have:

(9.4) $$P_p(\bar{u}_1 + \bar{u}_2) = P_p(\bar{u}_1) + P_p(\bar{u}_2) + \phi^*\{\Psi_{12}(e_o \otimes_\pi Q) \otimes pg_p(1_\theta)\},$$

where ϕ is the cochain map defined in 5.6.

Thus, theorem 9.1 follows at once from 9.4 when we show:

(9.5) $$\Sigma_{k=1}^{p-1} P_k(\bar{u}_1) \smile P_{p-k}(\bar{u}_2) = \phi^*\{\Psi_{12}(e_o \otimes_\pi Q) \otimes pg_p(1_\theta)\}.$$

The remainder of this section is devoted to the proofs of 9.4 and 9.5.

The proof of 9.4 is based on the following two lemmas: set $\Delta^* = T^{p-1} - 1 \in Z(\pi)$ (see 8.9). It is clear that Δ^* and Σ generate the annihilator of each other in $Z(\pi)$. We use this fact to prove the following lemma:

(9.6) LEMMA: Let Q be the cochain defined in 9.3, and let y be a cochain in $(M_1 + M_2)^p$ such that $\delta Q = \theta y$. Then, there is a cochain z in $(M_1 + M_2)^p$ such that

$$(u_1 + u_2)^{p-1}(v_1 + v_2) - (u_1^{p-1} v_1 + u_2^{p-1} v_2 + y) = \Delta^* z.$$

PROOF: From 5.15 we see that

$$\delta[(u_1+u_2)^p] = \Sigma[(u_1+u_2)^{p-1}\delta(u_1+u_2)] = \theta\, \Sigma\, [(u_1+u_2)^{p-1}(v_1+v_2)].$$

But from 9.3 and 5.15 we have:

$$\delta[(u_1+ u_2)^p] = \delta(u_1^p + u_2^p + \Sigma Q) = \theta\, \Sigma\, (u_1^{p-1} v_1 + u_2^{p-1} v_2 + y).$$

Hence,

$$\Sigma\, [(u_1 + u_2)^{p-1}(v_1 + v_2) - (u_1^{p-1} v_1 + u_2^{p-1} v_2 + y)] = 0.$$

The lemma then follows from the same argument as that used in proving 8.9.

(9.7) LEMMA: Let Q and z be the cochains given respectively in 9.3 and 9.6. Suppose that $\delta z = \theta w$, for some cochain $w \in (M_1 + M_2)^p$. Let R_p be the cochain function defined in 5.12. Then,

$$R_p(u_1 + u_2) = R_p(u_1) + R_p(u_2) + pe_o \otimes_\pi Q + p\theta e_1 \otimes_\pi z + \theta^2 e_2 \otimes_\pi w$$
$$- \delta[(\Sigma^* e_1) \otimes_\pi Q + \theta e_2 \otimes_\pi z)].$$

The proof is a mechanical verification using 5.11, 5.12, 5.14, 9.3, 9.6

and the fact that

$$e_o \otimes_\pi \Sigma Q = (\Sigma e_o) \otimes_\pi Q; \; e_1 \otimes_\pi \Delta^* z = (\Delta e_1) \otimes_\pi z.$$

The result of 9.7 enables us to prove 9.4. For let Ψ_i $(i = 1, 2)$ and Ψ_{12} be the cochain representation maps defined prior to 9.4. Then, $\Psi_{12} R_p(u_i) = \Psi_i R_p(u_i)$ $(i = 1, 2)$, where u_i is the generator of $C^{2n}(M_i)$. Hence, by 5.12 and 9.7,

$$\begin{aligned} P_p(\bar{u}_1 + \bar{u}_2) &= \phi^*\{\Psi_{12} R_p(u_1 + u_2) \otimes g_p(1_\theta)\} \\ &= \phi^*\{\Psi_{12}[R_p(u_1) + R_p(u_2) + pe_o \otimes_\pi Q] \otimes g_p(1_\theta)\} \\ &= \phi^*\{\Psi_1 R_p(u_1) \otimes g_p(1_\theta)\} + \phi^*\{\Psi_2 R_p(u_2) \otimes g_p(1_\theta)\} \\ &\quad + \phi^*\{\Psi_{12}(pe_o \otimes_\pi Q) \otimes g_p(1_\theta)\} \\ &= P_p(\bar{u}_1) + P_p(\bar{u}_2) + \phi^*\{\Psi_{12}(pe_o \otimes_\pi Q) \otimes g_p(1_\theta)\} \end{aligned}$$

which completes the proof of 9.4.

We turn to the proof of 9.5. Define a homomorphism β_i of Z_θ to $G_p(Z_\theta)$ by

(9.8) $$\beta_i(1_\theta) = (i, p-i) g_p(1_\theta), \qquad (i = 1, \ldots, p-1)$$

where (r,s) is the binomial coefficient $(r+s)!/(r)!(s)!$, and $g_p(1_\theta) \equiv 1 \bmod \theta[p,\theta^\infty]$ (see 1.15). Then, 9.5 follows at once from the following two results:

(9.9) $$P_i(\bar{u}_1) \smile P_{p-i}(\bar{u}_2) = \beta_{i*}(\bar{u}_1^i \smile \bar{u}_2^{p-i}) \qquad (i = 1, \ldots, p-1)$$

(9.10) $$\phi^*\{\Psi_{12}(e_o \otimes_\pi Q) \otimes pg_p(1_\theta)\} = \sum_{i=1}^{p-1} \beta_{i*}(\bar{u}_1^i \smile \bar{u}_2^{p-i}),$$

<u>where</u> Q <u>is the cochain defined in 9.3</u>.

PROOF OF 9.9: Let i be any positive integer $< p$. Then, i and $p-i$ are relatively prime. Thus, 9.9 follows at once from 6.3, 6.4, and 6.6,

where we take $r = i$, $s = p-i$, $b = 1$ and $\beta_1 = \mu$ in 6.4.(7)

PROOF OF 9.10: We begin by analysing the cochain element $Q = Q(u_1, u_2)$. From the proof of 9.3 we see that Q may be written as the sum of generators of $C^{2pn}((M_1 + M_2)^p)$, one generator for each equivalence class under cyclic permutations. A typical generator w of Q will have the form

$$w = u_1^{m_1} \otimes u_2^{n_1} \otimes \ldots \otimes u_1^{m_r} \otimes u_2^{n_r},$$

where $m_1 \geq 0$, $n_r \geq 0$, and $n_1, m_2, \ldots, n_{r-1}, m_r > 0$. Define the <u>weight</u> of the generator w to be the integer $m = m_1 + m_2 + \ldots + m_r$. From the definition of Q we see that $1 \leq m \leq p-1$, and $n_1 + \ldots + n_r = p-m$. Using the generator w we have:

(9.11) LEMMA: $\phi^*\{\Psi_{12}(e_o \otimes_\pi w) \otimes 1_\theta\} = \bar{u}_1^m \smile \bar{u}_2^{p-m}$.

PROOF: Set $x_i = f_i(u_i)$, where f_i: $M \longrightarrow K^*$ is a cochain representation of $\bar{u}_i$ $(i = 1, 2)$. Then, x_i is an integral cochain in K^* representing $\bar{u}_i$. Therefore, from 6.5, it follows that

$$\begin{aligned} \phi^*\{\Psi_{12}(e_o \otimes_\pi w) \otimes 1_\theta\} &= \phi^*\{(e_o \otimes_\pi x_1^{m_1} \otimes \ldots \otimes x_2^{n_r}) \otimes 1_\theta\} \\ &= \bar{u}_1^{m_1} \smile \bar{u}_2^{n_1} \smile \ldots \smile \bar{u}_1^{m_r} \smile \bar{u}_2^{n_r}. \end{aligned}$$

But $\bar{u}_1$, $\bar{u}_2$ have even dimension and the cup-product is anti-commutative. Hence

$$\bar{u}_1^{m_1} \smile \bar{u}_2^{n_1} \smile \ldots \smile \bar{u}_1^{m_r} \smile \bar{u}_2^{n_r} = \bar{u}_1^m \smile \bar{u}_2^{p-m},$$

since $m = m_1 + \ldots + m_r$, and $p - m = n_1 + \ldots + n_r$. This proves the lemma.

From the definition of Q we see that $Q = w_1 + \ldots + w_{p-1}$, where w_i is a sum of generators of $(M_1 + M_2)^p$ each one of which has weight i $(i = 1, 2, \ldots, p-1)$. But since each generator in Q is a representative of an equivalence class under cyclic permutations, there are precisely $(i,p-i)/p$ generators in the term w_i. Thus, as a consequence of 9.11, we have at once

(9.12) $$\phi^*\{\Psi_{12}(e_o \otimes_\pi w_i) \otimes 1_\theta\} = [(i,p-i)/p]\bar{u}_1^i \smile \bar{u}_2^{p-i},$$

for $(i = 1, 2, \ldots, p-1)$.

Let β_i be the homomorphism defined in 9.8. Notice that $\beta_1(1_\theta) = pg_p(1_\theta)$, and $\beta_i(1_\theta) = (i,p-i)g_p(1_\theta) = [(i,p-i)/p]\beta_1(1_\theta)$. Using this fact and 9.12, we have

$$\begin{aligned} \phi^*\{\Psi_{12}(e_o \otimes_\pi Q) \otimes pg_p(1_\theta)\} &= \phi^*\beta_{1*}\{\Psi_{12}(e_o \otimes_\pi (w_1+\ldots+w_{p-1})) \otimes 1_\theta\} \\ &= \Sigma_{i=1}^{p-1} \beta_{1*}\phi^*\{\Psi_{12}(e_o \otimes_\pi w_i) \otimes 1_\theta\} \\ &= \Sigma_{i=1}^{p-1} [(i,p-i)/p]\beta_{1*}(\bar{u}_1^i \smile' \bar{u}_2^{p-i}) \\ &= \Sigma_{i=1}^{p-1} \beta_{i*}(\bar{u}_1^i \smile' \bar{u}_2^{p-i}). \end{aligned}$$

This completes the proof of 9.10, and hence of theorem 9.1.

10. PROOF OF THEOREM 2.1(i), (ii), AND (iii).

The first two parts of theorem 2.1 are concerned with the definition of the functions P_t. These definitions have already been given: namely. in 5.17, 8.1, and 8.2.

The proof of 2.1(iii) is based upon the following lemma, which also will be used in a later section. Let r, s be non-negative integers such that $r + s > 0$; let $Z_\theta \in \mathcal{C}$. Denote by b the integer defined in 6.3; from the definition it is clear that there are integers c and d such that $r = bc$, $s = bd$. Let $\mu = \mu_{r,s}$ be the homomorphism from $G_b(Z_\theta)$ to $G_{r+s}(Z_\theta)$ defined in 6.4; i.e.,

$$\mu g_b(1_\theta) = (r,s)g_{r+s}(1_\theta), \tag{10.1}$$

where (r,s) is the binomial coefficient. Let $G_p(\mu)$: $G_{pb}(Z_\theta) \longrightarrow G_{p(r+s)}(Z_\theta)$ be the function obtained from μ by using the functor G (see 1.23). We then have:

(10.2) LEMMA: *Let* $u, v \in H^{2n}(K;Z_\theta)$. *Let* p *be a prime, and let* r, s *be non-negative integers such that* $r + s > 0$. *If* $p = 2$, *assume that* $(r,s) \equiv 0 \bmod 2$. *Let* b, c, d, *be the integers defined above. Then,*

$$P_p(P_r(u) \smile P_s(u)) = G_p(\mu)_*\{[P_{pb}(u)]^c \smile' [P_{pb}(v)]^d\},$$

where $[\ ]^m$ *denotes the* m-*fold* $\smile'$-*product* $(m \geq 1)$.

PROOF: From 6.4 and 8.24 we have, for any value of p,

$$\begin{aligned} P_r(u) \smile P_s(v) &= \mu_*[\eta_{c*}P_r(u) \smile' \eta_{d*}P_s(v)] \\ &= \mu_*\{[P_b(u)]^c \smile' [P_b(v)]^d\}. \end{aligned} \tag{10.3}$$

Suppose that $p \neq 2$. Then

$$\begin{aligned} P_p(P_r(u) \smile P_s(v)) &= G_p(\mu)_*P_p\{[P_b(u)]^c \smile' [P_b(v)]^d\} \\ &= G_p(\mu)_*\{[P_pP_b(u)]^c \smile' [P_pP_b(v)]^d\} \\ &= G_p(\mu)_*\{[P_{pb}(u)]^c \smile' [P_{pb}(v)]^d\}, \end{aligned}$$

by 10.3, 8.4, and 8.23. This proves 10.2 for the case of an odd prime p.

Suppose now that $p = 2$. In this case we need a short lemma to complete the proof. Let $Z_\theta, Z_\tau \in \mathcal{C}$, where $\theta = 2^i$, $\tau = 2^j$. Let λ: $Z_\theta \longrightarrow Z_\tau$ be a homomorphism of the form

$$\lambda(1_\theta) = 2m\ 1_\tau, \tag{10.4}$$

for some integer m. Of course, $2m\theta \equiv 0 \bmod \tau$, since λ is assumed to be well-defined. Let $G_2(\lambda)$: $G_2(Z_\theta) \longrightarrow G_2(Z_\tau)$ be the homomorphism defined using the functor G. We then have:

(10.5) *Let* u *be an element of* $H^q(K;G_2(Z_\theta))$ *such that* $2u = 0$. *Then,*

$$G_2(\lambda)_*(u) = 0,$$

in $H^q(K;G_2(Z_\tau))$.

To prove 10.5 we define homomorphisms

$$\alpha: G_2(Z_\theta) \longrightarrow G_2(Z_\tau), \quad \beta: G_2(Z_\theta) \longrightarrow G_2(Z_\theta),$$

by

$$\alpha g_2(1_\theta) = 2m g_2(1_\tau), \quad \beta g_2(1_\theta) = 2m g_2(1_\theta).$$

It is clear that $G_2(\lambda) = \alpha\beta: G_2(Z_\theta) \longrightarrow G_2(Z_\tau)$, since

$$G_2(\lambda) g_2(1_\theta) = g_2(\lambda 1_\theta) = g_2(2m1_\theta) = 4m^2 g_2(1_\theta) = \alpha\beta g_2(1_\theta).$$

Hence, $G_2(\lambda)_* = \alpha_*\beta_*$. But, $\beta_*(u) = 2mu = 0$, by the hypothesis on u. Thus, $G_2(\lambda)_*(u) = \alpha_*\beta_*(u) = 0$, and the lemma is proved.

We now can prove 10.2 for the case p = 2. From 10.3 and 8.5 we have,

$$P_2(P_r(u) \smile P_s(u)) = G_2(\mu)_* P_2\{[P_b(u)]^c \smile' [P_b(v)]^d\}$$

$$= G_2(\mu)_*\{[P_{2b}(u)]^c \smile' [P_{2b}(v)]^d + \Psi_2\}$$

where $\Psi_2 = \Psi_2(P_b(u), \ldots, P_b(v))$. If θ is zero or is odd, $\Psi_2 = 0$ and the lemma is proved. Thus, suppose that $\theta = 2^k$. Then, there are integers i, $j \geq k$ such that

$$G_b(Z_\theta) = Z_{2^i}, \quad G_{r+s}(Z_\theta) = Z_{2^j}.$$

Also,

$$\mu g_b(1_\theta) = (r,s) g_{r+s}(1_\theta) = 2m g_{r+s}(1_\theta),$$

for some integer m, since $(r,s) \equiv 0 \bmod 2$ by hypothesis. Now by 8.5(b), $2\Psi_2 = 0$. Therefore, from 10.5, $G_2(\mu)_* \Psi_2 = 0$. Hence,

$$P_2(P_r(u) \smile P_s(v)) = G_2(\mu)_*\{[P_{2b}(u)]^c \smile' [P_{2b}(v)]^d\},$$

in all cases, and the lemma is proved.

We now turn to the proof of 2.1(iii). Suppose first that r + s > 0.

Then, from 10.3, we have

$$P_r(u) \smile P_s(u) = \mu_*\{[P_b(u)]^c \smile' [P_b(u)]^d\} = \mu_*\{[P_b(u)]^{c+d}\}.$$

But from 8.23 and 8.24,

$$[P_b(u)]^{c+d} = \eta_{c+d*} P_{c+d}[P_b(u)] = \eta_{c+d*} P_{r+s}(u),$$

since (c+d)b = r+s. Also,

$$\mu\eta_{c+d}\, g_{r+s}(1_\theta) = (r,s) g_{r+s}(1_\theta),$$

as is easily verified from the definitions of μ and η. Thus,

$$P_r(u) \smile P_s(u) = \mu_* \eta_{c+d*} P_{r+s}(u) = (r,s) P_{r+s}(u),$$

as was to be proved.

If r = s = 0, then 2.1(iii) is simply a consequence of 2.1(ii) and the fact that $1 \smile 1 = 1$, where 1 denotes the unit of the ring $H^*(K;G(Z_\theta))$. Thus, 2.1(iii) is proved in all cases.

11. PROOF OF THEOREM 2.1(iv).

We are to prove that

(*) $$P_t(u + v) = \Sigma_{r+s=t} P_r(u) \smile P_s(v),$$

where u, v $\in H^{2n}(K;Z_\theta)$, and $Z_\theta \in \mathcal{G}$. We prove this by induction on the number of primes which occur in the integer t. By 9.1, we have proved (*) for the case t = p, p a prime. Suppose that we have proved (*) for all integers t which are a product of less than q primes (q > 1). Let k be an integer which is the product of q primes; say $k = p_q \cdots p_1$, where $p_{j+1} \geq p_j$ (j = 1, ..., q-1). We now prove (*) for the function P_k, and hence by induction, for the function P_t for all integers t.

Set $p = p_q$ and $m = p_{q-1} \cdots p_1$. Notice that p is the largest prime which occurs in k. By 8.23 and the inductive hypothesis, we have

(11.1) $P_k(u + v) = P_p[P_m(u + v)] = P_p[\Sigma_{i+j=m} P_i(u) \smile P_j(v)]$.

Set $c_i = P_i(u) \smile P_{m-i}(v)$ $(i = 0, 1, \ldots, m)$. Then, by 9.2 we have

(11.2) $$P_k(u + v) = \Sigma P_{t_o}(c_o) \smile \ldots \smile P_{t_m}(c_m),$$

where the summation is taken over all distinct sets of non-negative integers $(t_o, \ldots, t_m)$ such that $t_o + \ldots + t_m = p$.

We prove (*) by examining in more detail the sum on the right hand side of 11.2. To condense the notation, define the following sets of integers:

(11.3) $$T^* \equiv T^*_p = \{t | t = (t_o, \ldots, t_m), t_i \geq 0, \Sigma^m_{i=o} t_i = p\}$$

$$T(r) \equiv T_p(r) = \{t | t \in T_p, \Sigma^m_{j=o} jt_j = r\},$$

<u>for</u> $r = 0, 1, \ldots, k$.

Thus, (11.2) can be stated more succinctly:

(11.4) $$P_k(u + v) = \Sigma^k_{r=o} \Sigma_{t \in T(r)} P_{t_o}(c_o) \smile \ldots \smile P_{t_m}(c_m).$$

The proof of this follows at once from the two easily verified facts:

(a) $T^*_p = \bigcup^k_{r=o} T_p(r)$

(b) $T_p(r) \cap T_p(s) = \emptyset$ $(r \neq s)$

Thus, we will have proved (*) when we show:

(11.5) $$P_r(u) \smile P_{k-r}(u) = \Sigma_{t \in T(r)} P_{t_o}(c_o) \smile \ldots \smile P_{t_m}(c_m),$$

<u>for</u> $r = 0, 1, \ldots, k$.

But 11.5 is itself an immediate consequence of the following lemma:

(11.6) LEMMA: <u>Let</u> r <u>be a fixed integer</u>, $0 \leq r \leq k$. <u>For each</u> $t \in T(r)$, <u>let</u> δ_t, ϵ_t <u>be the integers defined in 11.11 and 11.12</u>, ϕ_t <u>the homomorphism defined in 11.14, and</u> u_t, v_t <u>the cohomology classes defined in 11.16. Then,</u>

(a) $P_{t_o}(c_o) \smile \ldots \smile P_{t_m}(c_m) = \phi^*_t [(u_t)^{\delta_t} \smile (v_t)^{\epsilon_t}]$

(b) $P_r(u) \smile P_{k-r}(v) = \Sigma_{t \in T(r)} \phi^*_t [(u_t)^{\delta_t} \smile (v_t)^{\epsilon_t}]$,

<u>where</u> $(\ )^q$ <u>denotes the</u> q-<u>fold</u> $\smile$-<u>product</u> $(q \geq 1)$.

Thus, to prove (*) we need only prove 11.6. We begin by defining the elements δ_t, ϵ_t, etc. involved in the statement of the lemma. Let $t = (t_o, \ldots, t_m) \in T_p(r)$. We call the integers $t_o, \ldots, t_m$ the <u>factors</u> of t, and distinguish between two types of elements in $T_p(r)$ as follows: define

(11.7) $$T'(r) = \{t | t \in T_p(r), \text{G.C.D.} (t_o, \ldots, t_m, \theta) = 1\},$$

$$T''(r) = \{t | t \in T_p(r), \text{G.C.D.} (t_o, \ldots, t_m, \theta) = p\},$$

where θ is the order of the coefficient group of the cohomology classes u and v. From 11.3 we know that $t_o + \ldots + t_m = p$. Therefore, since p is a prime it follows that

$$T_p(r) = T'(r) \cup T''(r).$$

Suppose that $t \in T''(r)$. Then each factor t_i of t $(i = 0, \ldots, m)$ is divisible by p, and the integer θ is divisible by p. But $t_o + \ldots + t_m = p$; hence, in this case, t has only one non-zero factor, say t_i, and $t_i = p$. Also, since $Z_\theta \in \mathcal{C}$, θ either equals zero or is a power of p.

Let $t \in T(r)$, and suppose that $t_{j_o}, \ldots, t_{j_a}$ are the non-zero factors of t $(0 \leq a \leq m;\ j_c < j_d$ if $c < d)$. Define

(11.8) $$\tau_i = t_{j_i};\quad x_i = j_i;\quad y_i = m-j_i. \qquad (i = 0, \ldots, a)$$

Set

(11.9) $$b_t = \begin{cases} 1, & \text{if } \theta = 0 \\ \text{G.C.D.}(x_o, \ldots, x_a, y_o, \ldots, y_a, \theta^\infty), & \text{if } \theta > 0. \end{cases}$$

Then, there are integers d_i, e_i $(i = 0, \ldots, a)$ such that

(11.10) $$x_i = d_i b_t, \quad y_i = e_i b_t.$$

We define the non-negative integers δ_t, ϵ_t by

(11.11) $$\delta_t = \begin{cases} \Sigma_{i=0}^{a} \tau_i d_i, & \text{if } t \in T'(r) \\ d_o & \text{if } t \in T''(r). \end{cases}$$

(11.12) $$\epsilon_t = \begin{cases} \Sigma_{i=0}^{a} \tau_i e_i, & \text{if } t \in T'(r) \\ e_o, & \text{if } t \in T''(r). \end{cases}$$

Define a cyclic group H_t with generator h_t by

(11.13) $$H_t = \begin{cases} G_{b_t}(Z_\theta),\ h_t = g_{b_t}(1_\theta), & \text{if } t \in T'(r) \\ G_{pb_t}(Z_\theta),\ h_t = g_{pb_t}(1_\theta), & \text{if } t \in T''(r) \end{cases}$$

and define the homomorphism ϕ_t mapping H_t to $G_k(Z_\theta)$ by

(11.14) $$\phi_t(h_t) = \rho_t g_k(1_\theta),$$

where

(11.15) $$\rho_t = (\tau_o, \ldots, \tau_a)(x_o, y_o)^{\tau_o} \ldots (x_a, y_a)^{\tau_a},$$

and $(a_1, \ldots, a_n)$ denotes the multinomial coefficient $(a_1 + \ldots + a_n)!/a_1! \ldots a_n!$. The fact that ϕ_t is well-defined is shown in 11.19. Finally, set

(11.16) $$w_t = \begin{cases} P_{b_t}(w), & \text{if } t \in T'(r) \\ P_{pb_t}(w), & \text{if } t \in T''(r) \end{cases}$$

where w is either u or v in $H^{2n}(K;Z_\theta)$.

We proceed to prove 11.6(a). Since $P_o(w) = 1$, and $1 \smile z = z$, for any cohomology classes w and z, it follows that

$$P_{t_o}(c_o) \smile \ldots \smile P_{t_m}(c_m) = P_{\tau_o}(c_{x_o}) \smile \ldots \smile P_{\tau_a}(c_{x_a}),$$

where $\tau_o, \ldots, \tau_a$ are the non-zero factors of t (see 11.8). Again let r be a fixed integer, $0 \leq r \leq k$; and let $t \in T(r)$. Suppose that $t \in T''(r)$. Then, as remarked after 11.7, t has only one non-zero factor, $\tau_o = p$. Thus, to prove 11.6(a) for this case we need only show that

(11.17) $$P_p[P_{x_o}(u) \smile P_{y_o}(v)] = \phi_{t*}[(u_t)^{\delta_t} \smile' (v_t)^{\epsilon_t}].$$

To prove 11.17 notice first that $\phi_t = G_p(\mu)$ mapping $G_{pb_t}(Z_\theta)$ to $G_{pm}(Z_\theta)$, where μ $(= \mu_{x_o,y_o})$ is the homomorphism of $G_{b_t}(Z_\theta)$ to $G_m(Z_\theta)$ defined in 6.4. This follows from the fact that $\mu g_{b_t}(1_\theta) = (x_o,y_o)g_m(1_\theta)$; hence, $G_p(\mu)g_{pb_t}(1_\theta) = (x_o,y_o)^p g_{pm}(1_\theta)$. But, $(x_o,y_o)^p = (x_o,y_o)^{\tau_o} = \rho_t$, by 11.12; and $g_{pb_t}(1_\theta) = h_t$, by 11.13. Therefore, $\phi_t g_{pb_t}(1_\theta) = G_p(\mu)g_{pb_t}(1_\theta)$, as was asserted. Since $t \in T''(r)$, we have from 11.11 and 11.12 that $\delta_t = d_o$, $\epsilon_t = e_o$, where $x_o = d_o b_t$, $y_o = e_o b_t$. Finally, $u_t = P_{pb_t}(u)$, $v_t = P_{pb_t}(v)$, from 11.16. Thus, 11.17 is simply a special case of 10.2. In particular, the case p = 2 is also covered. For suppose p = 2; then k = 2m. But by definition, $p = p_q = 2$ is the largest prime occuring in k. Hence, $m = 2^i$ for $i \geq 1$. Since $x_o + y_o = m = 2^i$, it is clear that $(x_o,y_o) \equiv 0 \bmod 2$, thus satisfying the hypotheses of 10.2. If either of x_o, y_o is zero, the proof follows at once from the definitions.

Thus, we have proved 11.6(a) in the case $t \in T''(r)$. Now suppose that $t \in T'(r)$. Then, by definition 11.7, G.C.D.$(\tau_o, \ldots, \tau_a, \theta) = 1$. Hence, from 6.4 we have

(11.18) $$P_{\tau_o}(c_{x_o}) \smile \ldots \smile P_{\tau_a}(c_{x_a}) = \mu_*[(c_{x_o})^{\tau_o} \smile' \ldots \smile' (c_{x_a})^{\tau_a}],$$

where μ is the homomorphism of $G_m(Z_\theta)$ to $G_k(Z_\theta)$ given in 6.4. We now use lemma 6.2 as follows: define integers α_i $(i = 0, \ldots, a)$ by $\alpha_o = 0$, $\alpha_j = \tau_o + \ldots + \tau_{j-1}$ $(j = 1, \ldots, a)$. In lemma 6.2 set r = 2p, and

$$A_{2j-1} = G_{x_i}(Z_\theta), \quad A_{2j} = G_{y_i}(Z_\theta),$$

$$\eta_{2j-1} = \eta_{d_i}: \ G_{x_i}(Z_\theta) \longrightarrow G_{b_t}(Z_\theta),$$

$$\eta_{2j} = \eta_{e_i}: \ G_{y_i}(Z_\theta) \longrightarrow G_{b_t}(Z_\theta),$$

for $2\alpha_i+1 \leq 2j-1 \leq 2\alpha_{i+1}$ $(i = 0, \ldots, a-1)$. Define the remaining groups in 6.2 by

$$B_1 = \cdots = B_{2p} = B = G_{pb_t}(Z_\theta), \ A = G_m(Z_\theta).$$

Let $\lambda = \omega'[(\lambda_{x_o,y_o})^{\tau_o} \otimes \cdots \otimes (\lambda_{x_a,y_a})^{\tau_a}]$, where λ_{x_i,y_i} maps $G_{x_i}(Z_\theta) \otimes G_{y_i}(Z_\theta)$ to $G_m(Z_\theta)$ by multiplication in the ring $G(Z_\theta)$, and ω' maps $G_m(Z_\theta) \otimes \cdots \otimes G_m(Z_\theta)$ (p factors) to $G_m(Z_\theta)$ by multiplication in the ring $G_m(Z_\theta)$ (see §1). Similarly, let ω be the mapping from $B_1 \otimes \cdots \otimes B_{2p}$ to B given by the multiplication in the ring $G_{b_t}(Z_\theta)$. Finally, define a map ρ from B $(= G_{b_t}(Z_\theta))$ to A $(= G_m(Z_\theta))$ by

$$\rho g_{b_t}(1_\theta) = [(x_o,y_o)^{\tau_o} \ldots (x_a,y_a)^{\tau_a}] \, g_m(1_\theta).$$

One easily verifies that

$$\rho\,\omega(\eta_1 \otimes \cdots \otimes \eta_{2p}) = \lambda\,;$$

hence, from 6.2 we have

$$\begin{aligned} &(c_{x_o})^{\tau_o} \smile' \cdots \smile' (c_{x_a})^{\tau_a} \\ &\quad = \rho_*\{[(u_t)^{d_o} \smile' (v_t)^{e_o}]^{\tau_o} \smile' \cdots \smile' [(u_t)^{d_a} \smile' (v_t)^{e_a}]^{\tau_a}\}, \\ &\quad = \rho_*[(u_t)^{\delta_t} \smile' (v_t)^{\epsilon_t}], \end{aligned}$$

since u_t, v_t have even dimension and the cup-product is anti-commutative. If we now observe that

(11.19) $$\phi_t = \mu\rho,$$

then 11.6(a) follows from 11.18.

Before considering the proof of 11.6(b), we must first obtain several lemmas. Throughout the remainder of the section, let r be a fixed integer such that $0 \leq r \leq k$.

(11.20) LEMMA: Let $t \in T_p(r)$, and let b_t be the integer defined in 11.9:

(i) if $t \in T'(r)$, then b_t divides r and k-r;

(ii) if $t \in T''(r)$, then pb_t divides r and k-r.

PROOF: Let $t = (t_o, \ldots, t_m) \in T_p(r)$. Suppose that $t \in T'(r)$. Then, by 11.3 and 11.7, we have

$$r = \Sigma_{j=o}^{m} jt_j = \Sigma_{i=o}^{a} x_i\tau_i = \Sigma_{i=o}^{a} b_t d_i \tau_i = b_t(\Sigma_{i=o}^{a} d_i\tau_i).$$

Similarly, $k - r = b_t(\Sigma_{i=o}^{a} e_i\tau_i)$, which proves (i). Now let $t \in T''(r)$. Then, τ_o is the only non-zero factor of t and $\tau_o = p$. Hence, by 11.3 and 11.10,

$$r = x_o\tau_o = b_t d_o \tau_o = b_t d_o p$$

$$k-r = y_o\tau_o = b_t e_o \tau_o = b_t e_o p;$$

which completes the proof of the lemma.

Thus, by 11.20 we know that for every $t \in T(r)$, there are non-negative integers f_t and g_t such that

(11.21) $$r = b_t f_t, \quad k-r = b_t g_t \qquad t \in T'(r)$$

$$r = pb_t f_t, \quad k-r = pb_t g_t \qquad t \in T''(r).$$

Hence, for each $t \in T(r)$ we have homomorphisms (see 1.26)

$$\eta_t = \eta_{f_t} \otimes \eta_{g_t}: \ G_r(Z_\theta) \otimes G_{k-r}(Z_\theta) \longrightarrow H_t \otimes H_t,$$

where the group H_t is defined in 11.13. Let ω_t mapping $H_t \otimes H_t$ to H_t be the multiplication in the ring H_t, and let $\lambda_{r,k-r}$ mapping $G_r(Z_\theta) \otimes G_{k-r}(Z_\theta)$ to $G_k(Z_\theta)$ be the multiplication in the ring $G(Z_\theta)$. Finally, let ϕ_t be the map of H_t to $G_k(Z_\theta)$ defined in 11.14. Then,

(11.22) In the following diagram,

$$(\Sigma\, \phi_t) \circ (\Sigma\, \omega_t) \circ (\Sigma\, \bar{\eta}_t) = \lambda_{r,k-r},$$

where the summation sign indicates direct sum of groups and homomorphisms and the summation is over all $t \in T(r)$.

$$\begin{array}{ccc} G_r(Z_\theta) \otimes G_{k-r}(Z_\theta) & \xrightarrow{\lambda_{r,k-r}} & G_k(Z_\theta) \\ \downarrow \Sigma\, \eta_t & & \uparrow \Sigma\, \phi_t \\ H_t \otimes H_t & \xrightarrow[\Sigma\, \omega_t]{} & H_t \end{array}$$

PROOF: Let $(r,k-r)$ be the binomial coefficient, and let ρ_t be the integer defined in 11.15. Then,

$$(r,k-r) = \Sigma_{t \in T(r)}\, \rho_t,$$

as may be seen by comparing coefficients in $(x+y)^k = [(x+y)^p]^m$. The lemma then follows immediately from the definitions of the groups and homomorphisms involved.

Using 11.22 and a mild extension of 6.2, we have at once:

(11.23) $$P_r(u) \smile P_{k-r}(u) = \Sigma_{t \in T(r)}\, \phi_*[(u_t)^{f_t} \smile (v_t)^{g_t}].$$

Involved in the proof is the fact that $H^*(K;G_1+G_2)$ is isomorphic in a natural way to $H^*(K;G_1) + H^*(K;G_2)$, for any coefficient groups G_1 and G_2. Thus, 11.6(b) will follow from 11.23 when we show that

(11.24) $$f_t = \delta_t, \quad g_t = \epsilon_t.$$

PROOF: Let $t \in T'(r)$. Then,

$$r = \Sigma_{j=o}^{a}\, x_j\tau_j = b_t(\Sigma_{j=o}^{a}\, \tau_j d_j) = b_t\delta_t.$$

But by 11.21, $r = b_t f_t$. Hence, since $b_t > 0$, $\delta_t = f_t$. On the other hand, if $t \in T''(r)$, then

$$r = x_o\tau_o = b_t d_o p = b_t\delta_t.$$

But, by 11.21, $r = pb_t f_t$, and again $\delta_t = f_t$. Similarly, $\epsilon_t = g_t$ and the lemma is proved. Thus, we have proved 11.6(b); and in fact 2.1(iv).

12. PROOF OF THEOREM 2.1(v), (vi), AND (vii).

The proof of theorem 2.1(v) has already been given in 8.23.

PROOF OF 2.1(vi): $P_r f^* = f^* P_r$. The case $r = p$, p a prime, has already been proved in 5.21. Assume then that $r = p_k, \ldots, p_1$, p_i prime. From 8.2 we have

$$P_r = P_{p_k} \circ P_{p_{k-1}} \circ \cdots \circ P_{p_1}.$$

Then, 2.1(vi) follows by induction on k, using the fact that $P_{p_i} f^* = f^* P_{p_i}$ for each $i = 1, 2, \ldots, k$.

PROOF OF 2.1(vii): $P_r\alpha_* = G_r(\alpha)_* P_r$. Again the case $r = p$, p a prime, has been covered in 5.22. Let p be a prime, and q be any positive integer; let α be a homomorphism from Z_θ to Z_τ ($Z_\theta, Z_\tau \in \mathcal{C}$). Let $G_r(\alpha)$ be the homomorphism of $G_r(Z_\theta)$ to $G_r(Z_\tau)$ defined in 1.23 ($r = 0, 1, \ldots$).

(12.1) LEMMA: Let $\alpha_q = G_q(\alpha)$, mapping $G_q(Z_\theta)$ to $G_q(Z_\tau)$. Then,

$$G_{pq}(\alpha) = G_p(\alpha_q),$$

mapping $G_{pq}(Z_\theta)$ to $G_{pq}(Z_\tau)$, where we identify $G_p(G_q(Z_\theta)) = G_{pq}(Z_\theta)$.

PROOF: Set $\iota_r = 1 \bmod \theta[r,\theta^\infty]$, $k_r = 1 \bmod \tau[r,\tau^\infty]$ $(r = 0, 1, \ldots)$. Then, ι_r and k_r are generators respectively for $G_r(Z_\theta)$ and $G_r(Z_\tau)$. Also, set $1_\theta = 1 \bmod \theta$, $1_\tau = 1 \bmod \tau$. Then, $\iota_r = g_r(1_\theta)$, $k_r = g_r(1_\tau)$. Suppose that the homomorphism α from Z_θ to Z_τ is given by

$$\alpha 1_\theta = a 1_\tau,$$

for some integer a. Let us compute the homomorphism $G_q(\alpha) = \alpha_q$, mapping $G_q(Z_\theta)$ to $G_q(Z_\tau)$:

$$G_q(\alpha)(\iota_q) = G_q(\alpha)g_q(1_\theta) = g_q(\alpha 1_\theta) = a^q g_q(1_\tau) = a^q k_q.$$

Now regard the group $G_q(Z_\theta)$, generated by ι_q, as itself a group in $\mathcal{G}$; and apply the functor G_p to the homomorphism α_q. Then,

$$G_p(\alpha_q)g_p(\iota_q) = g_p(\alpha_q \iota_q) = g_p(a^q k_q) = a^{pq} g_p(k_q) = a^{pq} k_{pq}.$$

Since $k_q = 1 \bmod \tau[q,\tau^\infty]$, it is clear from 1.15 and 1.17 that

$$g_p(k_q) = 1 \bmod \tau[pq,\tau^\infty] = k_{pq}.$$

On the other hand,

$$G_{pq}(\alpha)(\iota_{pq}) = G_{pq}(\alpha)g_{pq}(1_\theta) = g_{pq}(\alpha 1_\theta) = a^{pq} k_{pq}.$$

Thus, since $\iota_{pq} = 1 \bmod \theta[pq,\theta^\infty] = g_p(\iota_q)$, the homomorphisms $G_{pq}(\alpha)$ and $G_p(\alpha_q)$ are equal, completing the proof of 12.1.

Then, from 13.1, 8.23, and 5.22, we have:

$$P_{pq}\alpha_* = P_p(P_q\alpha_*) = P_pG_q(\alpha)_*P_q = G_p(G_q(\alpha))_*P_p \circ P_q$$
$$= G_{pq}(\alpha)_* P_{pq}.$$

Thus, the proof of 2.1(vii) is now completed by a simple induction based on the number of primes which occur in the integer r.

13. PROOF OF THEOREMS 2.2 AND 2.3.

The first part of proposition 2.2 has already been proved (see 8.24), as we have needed to use the result in previous sections. The second part of 2.2 follows at once from the first part, for if θ is zero or is prime to t, then

$$G_t(Z_\theta) = Z_\theta, \text{ and } \eta_t = \text{identity}.$$

PROOF OF 2.3: Part (i) of 2.3 follows at once from 8.4 and 8.2, using induction on the number of odd primes which occur in the odd integer t.

To prove part (ii), suppose first that $t = 2^s$ $(s \geq 1)$. If $s = 1$, then part (ii) is simply a restatement of 8.5. Suppose we have proved 2.3(ii) for all integers $t = 2^r$ where $1 \leq r < s$. We now prove it for $t = 2^s$. Set $k = 2^{s-1}$. From 8.5, 8.23, and 2.1(iv) we have:

$$P_{2^s}(u_1 \smile' \cdots \smile' u_r) = P_2(P_k(u_1 \smile' \cdots \smile' u_r))$$
$$= P_2[P_k(u_1) \smile' \cdots \smile' P_k(u_r) + \Psi_k(u_1, \ldots, u_r)]$$
$$= P_2[P_k(u_1) \smile' \cdots \smile' P_k(u_r)]$$
$$+ [P_k(u_1) \smile' \cdots \smile' P_k(u_r)] \smile \Psi_k(u_1, \ldots, u_r)$$
$$+ P_2[\Psi_k(u_1, \ldots, u_r)].$$

But,

$$P_2[P_k(u_1 \smile' \cdots \smile' P_k(u_r)] = P_{2^s}(u_1) \smile' \cdots \smile' P_{2^s}(u_r)$$
$$+ \Psi_2(P_k(u_1), \ldots, P_k(u_r)).$$

Set,

$$\Psi_{2^s}(u_1, \ldots, u_r) = \Psi_2(P_k(u_1), \ldots, P_k(u_r))$$
$$+ [P_k(u_1) \smile' \cdots \smile' P_k(u_r)] \smile \Psi_k(u_1,\ldots,u_r) + P_2[\Psi_k(u_1,\ldots,u_r)].$$

We must show that the function Ψ_{2^s} satisfies properties (a), (b), (c), and (d) of 2.3(ii). Parts (a), (b), and (c) are immediate from 8.5 and the inductive hypothesis. Thus, to prove 2.3(ii) for the case $t = 2^s$, we need only show that

(13.1) $$2\Psi_{2^s}(u_1, \ldots, u_r) = 0.$$

But from the inductive hypothesis we have at once:

$$2\Psi_2(P_k(u_1), \ldots, P_k(u_r)) = 0, \quad 2\Psi_k(u_1, \ldots, u_r) = 0;$$

therefore, since the cup-product is bilinear, to prove 13.1 we need only show

(13.2) $$2P_2(\Psi_k(u_1, \ldots, u_r)) = 0.$$

From 2.3(ii) (b) we know that $\Psi_k(u_1, \ldots, u_r) = 0$ unless θ is a power of 2. Suppose, therefore, that $\theta = 2^j$. Then, $\Psi_k(u_1, \ldots, u_r) \in H^{4n}(K;Z_{2^{j+s-1}})$, where $n = n_1 + \ldots + n_r$. Since $j \geq 1$ and $s-1 \geq 1$, it follows that $j+s-1 \geq 2$. Thus, 13.2 follows at once from the following lemma, whose proof we give at the end of this section.

(13.3) LEMMA: Let $u \in H^{2q}(K;Z_{2^r})$, where $r \geq 2$. Suppose that $2u = 0$. Then,

$$2P_2(u) = 0.$$

Having proved 2.3(ii) for integers of the form $t = 2^s$, we now let t be any even integer; say, $t = k2^s$, where k is odd. Then, by 2.3(i), 8.23, and what we have just proved, it follows that

$$\begin{aligned} P_t(u_1 \smile' \ldots \smile' u_r) &= P_{2^s}[P_k(u_1 \smile' \ldots \smile' u_r)] \\ &= P_{2^s}[P_k(u_1) \smile' \ldots \smile' P_k(u_r)] \\ &= P_t(u_1) \smile' \ldots \smile' P_t(u_r) + \Psi_{2^s}[P_k(u_1), \ldots, P_k(u_r)]. \end{aligned}$$

If we set

$$\Psi_t(u_1, \ldots, u_r) = \Psi_{2^s}(P_k(u_1), \ldots, P_k(u_r)),$$

the function Ψ_t clearly satisfies parts (a), (b), (c), and (d) of 2.3(ii). This completes the proof.

PROOF OF 2.3(iii): We begin by proving a special case:

(13.4) $$2P_2(v_1 \smile' v_2) = 0,$$

where $v_i \in H^{2n_i+1}(K;Z_\theta)$.

There are two cases to consider: first, suppose that θ is zero or odd. Then, from 7.4, it follows that

$$P_2(v_1 \smile' v_2) = (v_1 \smile' v_2) \smile' (v_1 \smile' v_2) = -(v_1^2 \smile' v_2^2),$$

since the cup-product is associative and anti-commutative. Hence,

$$2P_2(v_1 \smile' v_2) = -2(v_1^2 \smile' v_2^2) = -(2v_1^2 \smile' v_2^2) = 0,$$

again, because the cup-product is bilinear and anti-commutative.

Now, suppose that θ is even. Then, from equation 8.12 one obtains in this case (see also Wu [21; theorem 2]):

$$\begin{aligned} P_2(v_1 \smile' v_2) = P_2(v_1) \smile' P_2(v_2) + \nu_\#[&Sq_1(v_1) \smile' \mu_\# w(v_2) \\ &+ \mu_\# w(v_1) \smile' Sq_1(v_2)]. \end{aligned}$$

Here, Sq_1 and w are the operations defined respectively in 7.6 and 8.14, μ is the natural homomorphism of Z_θ to Z_2, and ν is the homomorphism of Z_2 to $G_2(Z_\theta)$ given by $\nu(1_2) = \theta g_2(1_\theta)$. Clearly, $2\nu_\# = 0$; also, $2P_2(v_1) = 2P_2(v_2) = 0$, from 7.9. Therefore, $2P_2(v_1 \smile' v_2) = 0$, completing the proof of 13.4.

Suppose now that $t = 2^s$ $(s \geq 1)$. It then follows at once from 8.2, 13.3 and 13.4 by induction on s that

(13.5) $$2P_{2^s}(v_1 \smile' v_2) = 0.$$

Consider next the case of an odd integer t. Then,

$$P_t(v_1 \smile' v_2) = P_t(v_1) \smile' P_t(v_2),$$

by 8.2 and 8.13. Therefore,

$$2P_t(v_1 \smile' v_2) = [2P_t(v_1)] \smile' P_t(v_2) = 0,$$

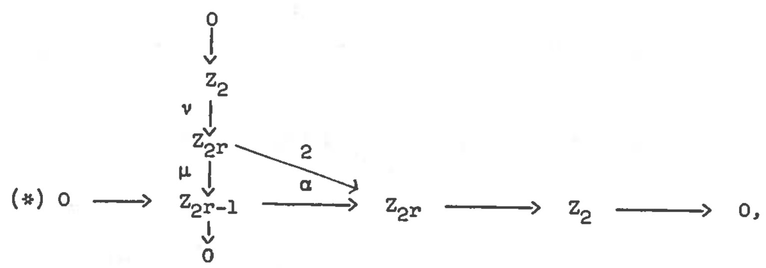
by 7.5 and induction on the number of odd primes in t.

Finally, suppose that $t = 2^s m$, where m is odd and $s \geq 1$. Then, by 8.23 and 8.13, we have

$$P_t(v_1 \smile' v_2) = P_{2^s}(P_m(v_1 \smile' v_2)) = P_{2^s}(P_m(v_1) \smile' P_m(v_2)).$$

Thus, $2P_t(v_1 \smile' v_2) = 2P_{2^s}(P_m(v_1) \smile' P_m(v_2)) = 0$, by 13.5, since $P_m(v_1)$ and $P_m(v_2)$ still have odd dimension. This completes the proof of 2.3(iii), and hence of 2.3.

PROOF OF 13.3: Consider the following commutative diagram:



where μ is the factor map, $\nu(1_2) = 2^{r-1}1_{2^r}$, and $\alpha(1_{2^r-1}) = 21_{2^r}$. Then, both the vertical and horizontal sequences are exact. Let δ_* denote the coboundary operator associated with the horizontal sequence (*). Since $2u = 0$ by hypothesis, we have $\alpha_*\mu_*(u) = 0$, by commutativity. Therefore, because the sequence (*) is exact, there exists a class $v \in H^{2q-1}(K;Z_2)$ such that

$$\mu_*(u) = \delta_*(v).$$

But, $\delta_* = \mu_*\lambda_*\delta_{**}$, where δ_{**} is the coboundary associated with the exact sequence

$$(**) \qquad 0 \longrightarrow Z \xrightarrow{2} Z \longrightarrow Z_2 \longrightarrow 0,$$

and λ is the factor map $Z \longrightarrow Z_{2^r}$. Hence,

$$\mu_*(u - \lambda_*\delta_{**}v) = 0.$$

Consequently, since the vertical sequence is also exact, there is a class $w \in H^{2q}(K;Z_2)$ such that

$$u = \lambda_*\delta_{**}(v) + \nu_*(w).$$

Therefore, it follows from 2.1(iv) that

$$P_2(u) = \Sigma_{r=0}^{2} P_r(\lambda_*\delta_{**}v) \smile P_{2-r}(\nu_*w).$$

We will show that $2P_2(u) = 0$, by showing this for each term in the above sum.

First, let r = 1 or 2. Then,

$$\begin{aligned} 2P_r(\lambda_*\delta_{**}v) &= 2G_r(\lambda)_*P_r(\delta_{**}v) \\ &= 2G_r(\lambda)_*[\delta_{**}v]^r \\ &= G_r(\lambda)_*[(2\delta_{**}v) \smile' (\delta_{**}v)^{r-1}] \\ &= 0, \end{aligned}$$

since $2\delta_{**} = 0$ by exactness of the sequence (**). Here we have used 2.1(vii) and the fact that $P_r(\delta_{**}v) = (\delta_{**}v)^r$, since $\delta_{**}v$ has integer coefficients. Thus, by bilinearity,

$$2[P_r(\lambda_*\delta_{**}v) \smile' P_{2-r}(\nu_*w)] = 0,$$

for $r = 1, 2$.

We now show that $2P_2(v_{\#}w) = 0$. This follows at once from the observation that $2G_2(v) = 0$. To show this, notice that $g_2(1_2)$ is a generator for $G_2(Z_2)$, and $g_2(1_{2r}) = 1_{2r+1}$ is a generator for $G_2(Z_{2r}) = Z_{2r+1}$. Hence,

$$\begin{aligned} 2G_2(v)g_2(1_2) &= 2g_2(v1_2) \\ &= 2g_2(2^{r-1}1_{2r}) \\ &= 2\cdot(2^{r-1})^2\, g_2(1_{2r}) \\ &= 2^{2r-1}\, 1_{2r+1} = 0, \end{aligned}$$

since $r \geq 2$, and $2r-1 \geq r+1$ when $r \geq 2$. This completes the proof of 13.3.

APPENDIX: COMPUTATION OF THE OPERATIONS $\not{p}_t$

We will give an example of a complex in which the operations $\not{p}_t$ are non-trivial. Let M_{2n} be the complex projective space of 2n real dimensions ($n \geq 0$). We regard M_{2n} as a subcomplex of M_{2n+2}, and define

$$M_\infty = \bigcup_{n=0}^{\infty} M_{2n};$$

that is, M_∞ is the infinite complex projective space. It is well-known that the cohomology ring of M_∞ with integer coefficients is a polynomial ring in a single generator u, where u is a generator for the cyclic infinite group $H^2(M_\infty;Z)$.

In order to compute the operations $\not{p}_t$ in the cohomology ring of M_∞, we first compute the model operations P_t. For each $r \geq 0$ let η_r: $Z \longrightarrow Z_r$ be reduction mod r. If K is any complex, and $v \in H^q(K;Z)$, let

$$v \bmod r = \eta_{r\#}(v),$$

in $H^q(K;Z_r)$.

(1) LEMMA: <u>Let Z_θ be a group in the category $\mathcal{C}$ (see §1), and let</u> u <u>be a generator for $H^2(M_\infty;Z)$. Then, for</u> $t \geq 0$,

$$P_t(u \bmod \theta) = u^t \bmod \theta[t,\theta^\infty],$$

<u>where $\theta[t,\theta^\infty]$ is the integer defined by 1.15</u>.

PROOF: Using 2.1(vii) and 2.2(ii) we have:

$$\begin{aligned} P_t(u \bmod \theta) &= P_t\,\eta_{\theta\#}(u) \\ &= G_t(\eta_\theta)_{\#} P_t(u) \\ &= G_t(\eta_\theta)_{\#}(u^t). \end{aligned}$$

But, $G_t(Z_\theta) = Z_\tau$, and $G_t(\eta_\theta) = \eta_\tau$: $Z \longrightarrow Z_\tau$, where $\tau = \theta[t,\theta^\infty]$. Thus, the lemma is proved.

We use (1) together with 2.3 to compute P_t on any summand of $H^*(M_\infty;Z_\theta)$.

(2) LEMMA: <u>Let u^r be a generator for $H^{2r}(M_\infty;Z)$</u> ($r \geq 1$). <u>Then, for</u> $t \geq 0$,

$$P_t(u^r \bmod \theta) = u^{rt} \bmod \theta[t,\theta^\infty].$$

PROOF:

$$\begin{aligned} P_t(u^r \bmod \theta) &= P_t[(u \bmod \theta)^r] \\ &= [P_t(u \bmod \theta)]^r \\ &= (u^t \bmod \theta[t,\theta^\infty])^r \\ &= u^{rt} \bmod \theta[t,\theta^\infty]. \end{aligned}$$

Here we have used either 2.3(i) or 2.3(ii c), depending on whether t is odd or even.

Lemma (2) enables us to compute the operation $\wp_t$ in $H^*(M_\infty;G(Z_\theta))$.

(3) THEOREM: Let Z_θ be a group in the category $\mathcal{C}$, and let $G(Z_\theta)$ be the p-cyclic Γ-ring defined in 1.17. Let $u^r \bmod \theta[s,\theta^\infty]$ be a generator for $H^{2r}(M_\infty;G_s(Z_\theta))$ $(r, s > 0)$. Then, for $t \geq 0$,

$$\wp_t(u^r \bmod \theta[s,\theta^\infty]) = \epsilon_{t,s}u^{rt} \bmod \theta[ts,\theta^\infty],$$

where $\epsilon_{t,s} = (s,s-1)(2s,s-1) \ldots ((t-1)s,s-1)$.

PROOF: The theorem follows at once from 3.9 and lemma (2) above, since $G_t(G_s(Z_\theta)) = G_{ts}(Z_\theta)$ = cyclic group of order $\theta[ts,\theta^\infty]$.

(4) COROLLARY: Let $Z_\theta \in \mathcal{C}$, where $\theta = p^i$, p a prime number. Let $u^r \bmod p^i$ generate $H^{2r}(M_\infty;Z_\theta)$. Then,

$$\wp_{p^j}(u^r \bmod p^i) = u^{rp^j} \bmod p^{i+j}. \qquad (i, j \geq 1)$$

Hence, $\wp_{p^j}(u^r \bmod p^i)$ is a generator for $H^{2rp^j}(M_\infty;G_{p^j}(Z_\theta))$, which is a cyclic group of order p^{i+j}.

We are interested not only in showing that the functions $\wp_t$ are non-trivial, but also in finding examples where these operations give new information. We indicate here such an example.

(5) DEFINITION: Let C be a cohomology operation relative to $[n_1, n_2, A_1, A_2]^{(8)}$, and let K and L be two complexes. We say that C distinguishes K and L if the following conditions obtain:

(a) $H^{n_i}(K;A_i) \approx H^{n_i}(L;A_i)$; $\qquad (i = 1, 2)$

(b) for any pair of isomorphisms $\lambda = (\lambda_1, \lambda_2)$ where

$$\lambda_i\colon \ H^{n_i}(K;A_i) \approx H^{n_i}(L;A_i),$$

there exists an element $u_\lambda \in H^{n_1}(K;A_1)$ such that

$$c\lambda_1(u_\lambda) \neq \lambda_2 c(u_\lambda).$$

An immediate consequence of (5) is:

(6) Let K and L be two complexes. If there is a cohomology operation C, which distinguishes K and L, then the two complexes do not have the same homotopy type.

Our result is this:

(7) THEOREM: There exist finite complexes K_t and L_t which the operation $\wp_t$ distinguishes $(t \geq 2)$, but which other known cohomology invariants fail to distinguish.

We omit the proof; a sketch is given in [16].

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FOOTNOTES

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(1) In some later, specific cases (see 1.12, 1.17) the ring R will have only elements of even degree. In this case R_k will denote the subgroup of elements of degree 2k.

(2) It is easily shown that relation 18.2 in [3; p. 107], namely $\gamma_t(rx) = r^t \gamma_t(x)$, is a consequence of the other three relations.

(3) See J. H. C. Whitehead [19].

(4) This corrects an omission in [15]; namely, equation (2.5) of that paper is valid only if the integer m is odd.

(5) See N. E. Steenrod [8], [13].

(6) This lemma is due to N. E. Steenrod.

(7) If p = 2, then i = p - i = 1; hence, r = s = b = 1.

(8) We define C to be a cohomology operation relative to $[n_1, n_2, A_1, A_2]$ if, for any complex K, C is a natural function mapping $H^{n_1}(K;A_1)$ to $H^{n_2}(K;A_2)$ (see Steenrod [9; §1]).

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