FIRST QUADRANT SPECTRAL SEQUENCES IN ALGEBRAIC K-THEORY

by

R.W. Thomason*

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139
U.S.A.

By algebraic K-theory I understand the study of the following process: one takes a small category $\underline{\underline{S}}$ provided with a "direct sum" operation, $\oplus$; "group completes" the monoid structure induced by $\oplus$ on the classifying space $B\underline{\underline{S}}$; and then takes the homotopy groups of the resulting space. For R a ring, this process applied to the category of finitely generated projective R-modules yields Quillen's $K_*(R)$. Karoubi's L-theory is also a special case of this generalized algebraic K-theory.

My aim in this paper is to show how K-theory may be axiomatized as a generalized homology theory on the category of such categories $\underline{\underline{S}}$, and to give a construction that yields the K-theoretic analogues of mapping cones, mapping telescopes and the like. Even if one is interested only in the K-theory of rings, these results for generalized K-theory should be useful technical tools. In particular the mapping cone construction of §6 may be useful in situations where appeal to

* Author partially supported by NSF Grant MCS77-04148.

Quillen's Theorem B fails. Two examples are given in §6 to illustrate this point.

§1. Definitions of some algebraic structures on categories.

K-theory assigns a graded abelian group to each small *symmetric monoidal category*. Recall this is a category $\underline{\underline{S}}$ together with a selected object $0 \in \underline{\underline{S}}$, a bifunctor $\oplus : \underline{\underline{S}} \times \underline{\underline{S}} \longrightarrow \underline{\underline{S}}$, and natural isomorphisms

$$\alpha : (A \oplus B) \oplus C \xrightarrow{\sim} B \oplus A$$

$$\gamma : A \oplus B \xrightarrow{\sim} B \oplus A$$

$$\lambda : A \xrightarrow{\sim} 0 \oplus A$$

These natural isomorphisms are subject to "coherence conditions;" we require a certain five diagrams to commute, and these imply all (generic) diagrams made up of α's, γ's, and λ's commute. Check [2], II, §1, III, §1; or [6], 3.3; or [9], VII, §1, VII, §7; for details. The word "small" means only that $\underline{\underline{S}}$ has a set of objects, rather than a proper class of them; this is a technicality required so the classifying space [14] $B\underline{\underline{S}}$ exists.

The standard example of a symmetric monoidal category is any additive category with $\oplus$ given by direct sum. The subcategory of isomorphisms in an additive category has an induced symmetric monoidal structure.

A *permutative category* is a symmetric monoidal one where the α's and λ's are required to be identity natural transformations: thus $\oplus$ is strictly associative and unital. Every symmetric monoidal category is equivalent to a permutative one ([7], 1.2; [11], 4.2), so we may assume things are permutative whenever convenient.

Let $\underline{\underline{S}}$, $\underline{\underline{T}}$ be symmetric monoidal categories. A <u>lax symmetric monoidal functor</u> $F : \underline{\underline{S}} \longrightarrow \underline{\underline{T}}$ is a functor together with natural transformations

$$\tilde{f} : O_T \longrightarrow FO_S \qquad\qquad \bar{f} : FA \oplus FB \longrightarrow F(A \oplus B)$$

such that the following diagrams commute

$$\begin{array}{ccc}
FA \oplus FB & \xrightarrow{\bar{f}} & F(A \oplus B) \\
\gamma \downarrow & & \downarrow F\gamma \\
FB \oplus FA & \xrightarrow[\bar{f}]{} & F(B \oplus A)
\end{array}$$

$$\begin{array}{ccccc}
(FA \oplus FB) \oplus FC & \xrightarrow{\bar{f} \oplus 1} & F(A \oplus B) \oplus FC & \xrightarrow{\bar{f}} & F((A \oplus B) \oplus C) \\
\alpha \downarrow & & & & \downarrow F\alpha \\
FA \oplus (FB \oplus FC) & \xrightarrow{1 \oplus \bar{f}} & FA \oplus F(B \oplus C) & \xrightarrow{\bar{f}} & F(A \oplus (B \oplus C))
\end{array}$$

$$\begin{array}{ccc}
FA & \xrightarrow{F\lambda} & F(O_S \oplus A) \\
\lambda \downarrow & & \uparrow \bar{f} \\
O_T \oplus FA & \xrightarrow{\tilde{f} \oplus 1} & FO_S \oplus FA
\end{array}$$

F is a <u>strong</u> symmetric monoidal functor (the usual notion in infinite loop space theory) if in addition $\tilde{f}$ and $\bar{f}$ are isomorphisms. F is a <u>strict</u> symmetric monoidal functor if $\tilde{f}$ and $\bar{f}$ are the identity natural transformation. F is a lax (strong, strict) permutative functor if its source and target are permutative.

If F, G are lax symmetric monoidal functors, a symmetric monoidal natural transformation $\eta : F \Rightarrow G$ is a natural transformation making the following diagrams commute:

$$\begin{array}{ccc} & & FO_S \\ & \overset{\tilde{f}}{\nearrow} & \\ O_T & & \Big\downarrow \eta \\ & \underset{\tilde{g}}{\searrow} & \\ & & GO_S \end{array} \qquad\qquad \begin{array}{ccc} FA \oplus FB & \xrightarrow{\bar{f}} & F(A \oplus B) \\ {\scriptstyle \eta \oplus \eta} \Big\downarrow & & \Big\downarrow {\scriptstyle \eta} \\ GA \oplus GB & \xrightarrow{\bar{g}} & G(A \oplus B) \end{array}$$

As an example, any additive functor between two additive categories is a strong symmetric monoidal functor. Any natural transformation between two additive functors is a symmetric monoidal natural transformation.

For any of the above concepts, we have a corresponding nonunital version obtained by dropping any condition relating to the unit 0.

§2. Passage to the associated spectrum and definition of algebraic K-theory.

Let Sym Mon be the category of small symmetric monoidal categories and lax symmetric monoidal functors. By [17], 4.2.1, there is a functor, $\mathrm{Spt} : \underline{\mathrm{Sym}}\ \underline{\mathrm{Mon}} \longrightarrow \underline{\mathrm{Spectra}}$, into the category of (connective) spectra, and a natural transformation $B\underline{\underline{S}} \longrightarrow \mathrm{Spt}_0(\underline{\underline{S}})$, which exhibits the zeroth space of the spectrum $\mathrm{Spt}(\underline{\underline{S}})$ as the group completion of the classifying space of the category $\underline{\underline{S}}$. This extends the usual functors defined on the subcategory of strong symmetric monoidal functors by [11], [15]. The proof of [13] is easily adapted to show Spt is uniquely determined up to natural equivalence by the above properties.

A symmetric monoidal natural transformation $\eta : F \Rightarrow G$ induces a homotopy of maps of spectra from Spt F to Spt G. Thus one may construct a homotopy theory of symmetric monoidal categories which is related via Spt to the homotopy theory of spectra, just as in [14] a homotopy theory of categories is constructed which is related by the classifying space functor B to the homotopy theory of spaces. The key thing to keep in mind is that a symmetric monoidal natural transform is like a homotopy; this motivates the construction of §3 and its relation to the homotopy colimit constructions of §4.

I define K-theory as a functor from Sym Mon to graded abelian groups by $K_*(\underline{\underline{S}}) = \pi_*^S(\mathrm{Spt}\ \underline{\underline{S}})$, the reduced stable homotopy groups of the spectrum Spt $\underline{\underline{S}}$. As by spectrum I mean what used to be called an Ω-spectrum, this is equivalent to $\pi_*(\mathrm{Spt}_0\ \underline{\underline{S}})$, the homotopy groups of the group completion of $B\underline{\underline{S}}$. Thus if $\underline{\underline{S}}$ is the subcategory of isomorphisms in an additive category $\underline{\underline{\mathcal{A}}}$, my $K_*(\underline{\underline{S}})$ is $K_*(\underline{\underline{\mathcal{A}}})$ as defined by Quillen's plus construction or group completion method.

I prefer to think of K_* as π_*^S Spt rather than in terms of homotopy groups, as the former is more "homological."

§3. The fundamental construction and its first quadrant K-theory spectral sequence.

I will now present a construction on diagrams in Sym Mon, show it has a reasonable universal mapping property (so it is not an ad hoc construction), and then give a spectral sequence for its K-groups.

Let $\underline{\underline{L}}$ be a small category. For simplicity, consider a diagram of permutative categories; i.e., a functor $F : \underline{\underline{L}} \longrightarrow \underline{\text{Perm}}$ into the category of permutative categories.

Let Perm-hocolim F be the permutative category with objects $n[(L_1,X_1),\ldots,(L_n,X_n)]$ where $n = 0,1,2,\ldots$; L_i is an object of $\underline{\underline{L}}$, and X_i is an object of $F(L_i)$. A morphism $n[(L_1,X_1),\ldots,(L_n,X_n)] \longrightarrow k[(L_1',X_1'),\ldots,(L_k',X_k')]$ consists of data $(\psi;\ell_i;x_i)$

1) a surjection of sets $\psi : \{1,\ldots,n\} \twoheadrightarrow \{1,\ldots,k\}$
2) maps $\ell_i : L_i \longrightarrow L'_{\psi(i)}$ in $\underline{\underline{L}}$
3) maps $x_j : \bigoplus_{i\in\psi^{-1}(j)} F(\ell_i)(X_i) \longrightarrow X_j'$ in L_j'

There is a straightforward rule giving the composition of morphisms. I will not explicitly give it, but it is implicitly determined by the universal mapping property below.

The unit of Perm-hocolim F is $0[\ ]$, and $\oplus$ is given by $n[(L_1,X_1),\ldots,(L_n,X_n)] \oplus m[(L_{n+1},X_{n+1}),\ldots,(L_{n+m},X_{n+m})] = n+m[(L_1,X_1),\ldots,(L_{n+m},X_{n+m})]$.

The universal mapping property is given by

Lemma: Strict permutative functors G : Perm hocolim $F \longrightarrow \underline{\underline{T}}$ correspond bijectively with systems consisting of non-unital lax permutative functors $G_L : F(L) \longrightarrow \underline{\underline{T}}$ for each $L \in \underline{\underline{L}}$, and non-unital permutative natural transformations $G_\ell : G_L \Rightarrow G_{L'} \cdot F(\ell)$ for each $\ell : L \longrightarrow L'$ in $\underline{\underline{L}}$; which must satisfy the conditions $G_1 = \text{id}$ and $G_\ell \cdot G_{\ell'} = G_{\ell\ell'}$.

Proof: Let $J_L : F(L) \longrightarrow$ Perm hocolim F be given by $J_L(X) = 1[(L,X)]$. This J_L is a non-unital lax permutative functor in an obvious way, and for $\ell : L \longrightarrow L'$ there is an obvious choice of $J_\ell : J_L \Rightarrow J_{L'} \cdot F(\ell)$.

Given this system, the bijective correspondence sends G to the system $G_L = G \cdot J_L$, $G_\ell = G \cdot J_\ell$. It is tedious but easy to see this works.

Given a diagram $F : \underline{\underline{L}} \longrightarrow \underline{\text{Sym}}\ \underline{\text{Mon}}$, there is an analogous Sym-Mon-hocolim F with the corresponding universal mapping property. All I say below about Perm-hocolim F applies to it as well. The explicit description of the objects and morphisms of Sym-Mon-hocolim F differs slightly from that given above.

This construction turns out to have good properties with respect to K-theory.

Theorem: Let $F : \underline{\underline{L}} \longrightarrow \underline{\text{Perm}}$ be a diagram. Then there is a natural first quadrant spectral sequence

$$E^2_{p,q} = H_p(\underline{\underline{L}};K_qF) \Rightarrow K_{p+q}(\text{Perm-hocolim } F)$$

Here $H_*(\underline{\underline{L}};K_qF)$ is the homology of the category $\underline{\underline{L}}$ with coefficients in the functor $L \longmapsto K_qF(L)$. A convenient source for information on this is [5], IX §6 or [14], §1. I'll identify the E^2 term with more familiar objects for the examples of §4.

This theorem is an immediate corollary of the theorem of §5 and the proposition of §4.

§4. Facts about and examples of homotopy colimits.

To prepare the way for the statement of the fundamental theorem of §5, and to explain the strange-looking name Perm-hocolim, I will review the homotopy colimit (homotopy direct limit) construction of Bousfield and Kan [1]. Some version of this exists for every category admitting a reasonable homotopy theory, e.g., *Sym Mon* and *Spectra*; but [1] concentrates on the category of simplicial sets. I'll give some of their results translated for the category of topological spaces, *Top*. One can also read Vogt [20] for this material.

Let $F : \underline{\underline{L}} \longrightarrow \underline{Top}$ be a diagram. Associated naturally to F is a space Top-hocolim F, the homotopy colimit of F. It is characterized by a universal mapping property ([1], XII, 2.3) establishing a bijective correspondence between maps g : Top-hocolim $F \longrightarrow X$ and a system of maps $g_L : F(L) \longrightarrow X$ and homotopies relating them. With the philosophy of §2 that symmetric monoidal natural transforms are like homotopies, this universal mapping property of Top-hocolim F is much like that of Perm-hocolim F given in the lemma of §3 (cf. [18], 1.3.2).

For any generalized homology theory E_* on *Top*, there is a first quadrant spectral sequence [1], XII, 5.7

$$E^2_{p,q} = H_p(\underline{\underline{L}}; E_q F) \Rightarrow E_{p+q}(\text{Top-hocolim } F)$$

This construction subsumes many well-known constructions.

Example 1: Let $F : \underline{\underline{L}} \longrightarrow \underline{Top}$ be the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

Then Top-hocolim F is the double mapping cyclinder on $A \longrightarrow B$ and $A \longrightarrow C$. In this case the spectral sequence collapses to the long exact Mayer-Vietoris sequence

$$\cdots \xrightarrow{\partial} E_q(A) \longrightarrow E_q(B) \oplus E_q(C) \longrightarrow E_q(\text{double mapping cylinder}) \xrightarrow{\partial} \cdots$$

For C a point, the Top-hocolim is the mapping cone of $A \longrightarrow B$; and for B and C points, it is the suspension of A.

Example 2: Let $\underline{\underline{L}}$ be the category of the positive integers as a partially ordered set. Then $F : \underline{\underline{L}} \longrightarrow \underline{\text{Top}}$ is a diagram: $F(1) \longrightarrow F(2) \longrightarrow F(3) \longrightarrow \cdots$, and Top-hocolim F is the mapping telescope. In the spectral sequence $H_p(\underline{\underline{L}};E_qF) = 0$ if $p > 0$, and $H_0(\underline{\underline{L}};E_qF) = \varinjlim_n E_qF(n)$.

Example 3: Let $\underline{\underline{L}}$ be a group G considered as a category with one object $*$, and morphisms being the elements of G. A functor $F : \underline{\underline{L}} \longrightarrow \underline{\text{Top}}$ is a homomorphism $G \longrightarrow \text{Aut}(F(*))$; that is, an action of G on $F(*)$. If EG is a free acyclic G-complex, Top-hocolim F is $EG \times_G F(*)$. The spectral sequence is identified to the usual one

$$H_p(G,E_qF(*)) \Rightarrow E_{p+q}(EG \times_G F(*)) .$$

Example 4: Let $\underline{\underline{L}}$ be Δ^{op}. Then $F : \Delta^{op} \longrightarrow \underline{\text{Top}}$ is just a simplicial space. It follows from [1], XII, 3.4 that Top-hocolim F is the "thickened" geometric realization "$\|\ \|$" of Segal [15], which is homotopy equivalent to the geometric realization of F for "good" F. To interpret the E^2 term of the spectral sequence, recall that for any functor E from Δ^{op} into the category of abelian groups, i.e., for E a simplicial abelian group, $H_*(\Delta^{op};E)$ is the homology of the chain complex which in degree p is E_p, and has differential $\partial = \sum(-1)^i d_i$. This follows from [1], XII, 5.6 and [12], 22.1.

One has analogous results in many categories admitting a homotopy theory. In particular, consider a functor $F : \underline{\underline{L}} \longrightarrow \underline{\text{Spectra}}$. One may define Spectra-hocolim F as follows. Let $F_n : \underline{\underline{L}} \longrightarrow \underline{\text{Top}}$ be the diagram of $n^{\underline{th}}$ spaces of the spectra. Form Top-hocolim F_n. As homotopy colimits in $\underline{\text{Top}}$ commute with suspensions, we get maps

$$\Sigma\ \text{Top-hocolim}\ F_n \xrightarrow{\cong} \text{Top-hocolim}\ \Sigma F_n \longrightarrow \text{Top-hocolim}\ F_{n+1}$$

induced by the maps $\Sigma F_n \longrightarrow F_{n+1}$ adjoint to the structure maps $F_n \longrightarrow \Omega F_{n+1}$. Passing to the adjoints again, we get maps Top-hocolim $F_n \longrightarrow \Omega$Top-hocolim F_{n+1}. These maps are not in general equivalences; so the sequence of spaces Top-hocolim F_n is not a spectrum, but only a prespectrum. To this prespectrum one canonically associates an equivalent spectrum [10]; this spectrum is our Spectra-hocolim F. As above, we have

Proposition: For any connective generalized homology theory E_* on Spectra, there is a first quadrant spectral sequence natural in $F : \underline{\underline{L}} \longrightarrow \underline{\text{Spectra}}$

$$E^2_{p,q} = H_p(\underline{\underline{L}}; E_q F) \Rightarrow E_{p+q}(\text{Spectra-hocolim } F).$$

Proof: Use the fact $E_*(\text{Spectra-hocolim } F) = \varinjlim_n E_{k+n}(\text{Top-hocolim } F_n)$ and the spectral sequences for Top-hocolim F_n. Here one regards E_* as a generalized homology theory on spaces in the usual way, via the suspension spectrum functor.

For special diagrams, we may identify the E^2 term as in the examples above. In particular, for a diagram of spectra

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ * & & \end{array}$$

the Spectra-hocolim is the mapping cone or cofibre spectrum of $A \longrightarrow B$, and the spectral sequence degenerates into the long exact cofibre sequence.

One may also consider homotopy colimits in *Cat*, the category of small categories. This is treated in [17], [18]. It is shown there that the classifying space functor $B : \underline{Cat} \longrightarrow \underline{Top}$ commutes with homotopy colimits up to homotopy equivalence. This is an essential ingredient at several points in the proof of the theorem of §5.

§5. *Homotopy colimits are preserved by* Spt.

Theorem: Let $F : \underline{\underline{L}} \longrightarrow \underline{Perm}$ be a functor. There is a natural equivalence of spectra

$$\text{Spectra-hocolim (Spt } F) \simeq \text{Spt (Perm-hocolim } F)$$

Sketch of proof: The universal mapping property of a homotopy colimit gives a natural map Spectra-hocolim (Spt F) $\longrightarrow$ Spt (Perm-hocolim F). This map will be the equivalence. The proof uses the resolution technique of [19].

Use the monad T on *Cat* which sends a category to the free permutative category over it to construct a Kliesli standard simplicial resolution of F. This is a simplicial object in the category of diagrams of permutative categories, $n \longmapsto T^{n+1}F$, with face and degeneracy operators induced by the action of T on F, the multiplication of T, and the unit of T. This simplicial object is augmented to F via the action $TF \longrightarrow F$.

Applying Spectra-hocolim (Spt ?) $\longrightarrow$ Spt (Perm-hocolim ?), one gets a map of simplicial objects in Spectra. One may "geometrically realize" such simplicial spectra. The augmentation induces a map from the realization of $n \longmapsto$ Spectra-hocolim (Spt $T^{n+1}F$) to Spectra-hocolim (Spt F), and one from the realization of $n \longmapsto$ Spt (Perm-hocolim $T^{n+1}F$) to Spt (Perm-hocolim F). The first map is a homotopy equivalence by general nonsense; the second, by a calculation given below. Granted this, one is reduced to showing that Spectra-hocolim (Spt $T^{n+1}F$) $\longrightarrow$ Spt (Perm-hocolim $T^{n+1}F$) is an equivalence, using the usual fact that a simplicial map which is an equivalence in each degree has a geometric realization which is an equivalence.

One next reduces to the theorem [18], 1.2 relating homotopy colimits in Top and Cat. Recall if $T\underline{\underline{C}}$ is the free permutative category on $\underline{\underline{C}}$, Spt $T\underline{\underline{C}}$ is equivalent to $\Sigma^{\infty}B\underline{\underline{C}}$, the suspension spectrum on the classifying space of the category $\underline{\underline{C}}$. Thus for $G : \underline{\underline{L}} \longrightarrow$ Cat, one has equivalences: Spectra-hocolim (Spt TG) $\simeq$ Spectra-hocolim $\Sigma^{\infty}BG \simeq \Sigma^{\infty}$ Top-hocolim BG. On the other hand, recall from [18] that Cat has homotopy colimits given by the Grothendieck construction $\underline{\underline{L}}\int G$ on $G : \underline{\underline{L}} \longrightarrow$ Cat, and that there is a natural equivalence Top-hocolim BG $\simeq B(\underline{\underline{L}}\int G)$. One finds functors and natural transforms giving inverse homotopy equivalences between Perm-hocolim TG and $T(\underline{\underline{L}}\int G)$, so there are equivalences Spt (Perm-hocolim TG) $\simeq$ Spt $(T(\underline{\underline{L}}\int G) \simeq \Sigma^{\infty}(\underline{\underline{L}}\int G) \simeq \Sigma^{\infty}$(Top-hocolim BG). Combining the two series of equivalences, we get Spectra-hocolim (Spt $T^{n+1}F$) $\simeq$ Spt (Perm-hocolim $T^{n+1}F$) as required.

It remains only to indicate why the map of the realization of $n \longmapsto$ Spt (Perm-hocolim $T^{n+1}F$) to Spt (Perm-hocolim F) is an equivalence. The functor Spt factors as the composite of an infinite loop space machine and a functor which regards the classifying space of a permutative category as an E_∞-space, as in [11]. As the machine commutes with geometric realization, it suffices to show the realization of the simplicial E^∞-space $n \longmapsto B(\text{Perm-hocolim } T^{n+1}F)$ is equivalent as a space to $B(\text{Perm-hocolim } F)$. But by the preceding paragraph, this simplicial space is equivalent to $n \longmapsto B(T(\underline{\underline{L}} \int T^n F))$, and so by [18] its realization is equivalent to the classifying space of $\Delta^{op} \int n \longrightarrow T(\underline{\underline{L}} \int T^n F)$, or even $[\Delta^{op} \int n \longrightarrow [T(\underline{\underline{L}} \int T^n F)]^{op}]^{op}$. There is a functor from this last category to Perm-hocolim F which induces a homotopy equivalence of classifying spaces by an argument similar to the proofs of [17], VI, 2.3, VI, 3.4. This completes the sketch.

To produce an honest proof, a few technical tricks must be applied. For example, at various points there are problems with basepoints of spaces and with units of permutative categories. One deals with this by noting that if one adds a new disjoint unit 0 to a permutative category, the associated spectrum doesn't change up to homotopy. Also, the above proof works only in the case where $F : \underline{\underline{L}} \longrightarrow \underline{\text{Perm}}$ is such that for each morphism ℓ in $\underline{\underline{L}}$, $F(\ell)$ is a strict permutative functor. The general case is deduced from this special case. Finally, to avoid trouble with the universal mapping property of Spectra-hocolim, part of the argument must be done in the category of prespectra. A fully detailed honest proof will appear elsewhere, someday.

§6. A simplified mapping cone and how to use it.

The category Perm-hocolim F is generally somewhat complicated, although not impossibly so. In certain situations of interest it may be replaced by a simpler homotopy equivalent construction. I will indicate how to do this in the case of mapping cones. This construction should be useful for producing exact sequences in the K-theory of rings. It can be employed in place of Quillen's Theorem B ([14], §1) for this purpose. It has the advantage that its hypotheses are easier to satisfy in practice than those of Theorem B. As with Quillen's theorem, it leaves one with the problem of identifying what it gives as the third terms in a long exact sequence of K-groups with what one wants there. This problem is generally that of showing some functor induces a homotopy equivalence of classifying spaces; and may be attacked by the methods of [14], essentially by cleverness and the use of Quillen's Theorem A. These points should become clear in the two examples below.

To make the simplified double mapping cylinder consider a diagram of symmetric monoidal categories and strong symmetric monoidal functors

$$\begin{array}{ccc} \underline{\underline{A}} & \xrightarrow{V} & \underline{\underline{B}} \\ {\scriptstyle U}\downarrow & & \\ \underline{\underline{C}} & & \end{array}$$

Suppose every morphism of $\underline{\underline{A}}$ is an isomorphism. Let $\underline{\underline{P}}$ be the category with objects (C,A,B) where C is an object of $\underline{\underline{C}}$; A, of $\underline{\underline{A}}$; and B, of $\underline{\underline{B}}$. A morphism in $\underline{\underline{P}}$, $(C,A,B) \to (C',A',B')$ is given by an equivalence class of data:

1) $\psi : A \longrightarrow A_1 \oplus A' \oplus A_2$ an isomorphism in $\underline{\underline{A}}$;

2) $\psi_1 : C \oplus UA_1 \longrightarrow C'$ a morphism in $\underline{\underline{C}}$

3) $\psi_2 : VA_2 \oplus B \longrightarrow B'$ a morphism in $\underline{\underline{B}}$.

Equivalent data are obtained by changing A_1 and A_2 up to isomorphism; thus if $a : A_1 \xrightarrow{\cong} A_1'$, $b : A_2 \xrightarrow{\cong} A_2'$ are isomorphisms, the above data (ψ, ψ_1, ψ_2) is equivalent to $(a \oplus A' \oplus b \cdot \psi,\ \psi_1 \cdot C \oplus Ua^{-1},\ \psi_2 \cdot Vb^{-1} \oplus B)$.

This $\underline{\underline{P}}$ is the simplified double mapping cylinder of the diagram. It has the obvious symmetric monoidal structure with $(C,A,B) \oplus (C',A',B') = (C \oplus C', A \oplus A', B \oplus B')$. There are strong symmetric monoidal functors $\underline{\underline{B}} \longrightarrow \underline{\underline{P}}$, $\underline{\underline{C}} \longrightarrow \underline{\underline{P}}$, and $\underline{\underline{P}}$ has a simple universal mapping property. In the special case where $\underline{\underline{C}} = \underline{\underline{0}}$ is a point, the construction of $\underline{\underline{P}}$ yields the simplified mapping cone on $\underline{\underline{A}} \longrightarrow \underline{\underline{B}}$.

One uses Theorem A of [14] to show the canonical map from the double mapping cylinder in the sense of §3 to the simplified version $\underline{\underline{P}}$ is a homotopy equivalence. This justifies the name "double mapping cylinder" for $\underline{\underline{P}}$ and yields:

<u>Proposition</u>: In the situation above, there is a long exact Mayer-Vietoris sequence

$$\longrightarrow K_{i+1}(\underline{\underline{P}}) \xrightarrow{\partial} K_i(\underline{\underline{A}}) \longrightarrow K_i(\underline{\underline{B}}) \oplus K_i(\underline{\underline{C}}) \longrightarrow K_i(\underline{\underline{P}}) \xrightarrow{\partial} \cdots$$

As an example of how to use this construction, I will give a quick proof of Quillen's theorem that his two definitions of K-theory agree [3]. Let $\underline{\underline{\mathcal{A}}}$ be an additive category, and $\underline{\underline{A}} = \text{Iso}\,\underline{\underline{\mathcal{A}}}$ the category of isomorphisms in $\underline{\underline{\mathcal{A}}}$. As remarked above, $\underline{\underline{A}}$ is symmetric monoidal with $\oplus$ given by direct sum. For $\underline{\underline{\mathcal{A}}}$ the category of finitely generated projective R-modules, $\coprod_n GL_n(R)$ is cofinal in $\underline{\underline{A}}$, so $K_i(\underline{\underline{A}}) = \pi_i^S \operatorname{Spt}(\underline{\underline{A}}) = \pi_i \operatorname{Spt}_0(\underline{\underline{A}}) \cong \pi_i(BGL(R)^+)$ for $i > 0$. Thus $K_*(\underline{\underline{A}})$ coincides with Quillen's "plus construction" or group completion definition of K-theory.

Consider now the suspension of $\underline{\underline{A}}$, $\Sigma\underline{\underline{A}} = \underline{\underline{P}}$ obtained as the simplified double mapping cylinder from the diagram where both $\underline{\underline{B}}$ and $\underline{\underline{C}}$ are points. By the Proposition above we have an isomorphism $\partial : K_{i+1}(\Sigma\underline{\underline{A}}) \cong K_i(\underline{\underline{A}})$, and $K_{i+1}(\Sigma\underline{\underline{A}}) = \pi_{i+1}^S \operatorname{Spt}(\Sigma\underline{\underline{A}}) \cong \pi_{i+1} \operatorname{Spt}_0(\Sigma\underline{\underline{A}}) \cong \pi_{i+1} B\Sigma\underline{\underline{A}}$, the last isomorphism being due to the fact $B\Sigma\underline{\underline{A}}$ is connected, hence group complete.

Now $\Sigma\underline{\underline{A}}$ has objects $(0,A,0)$, which I abbreviate to A. A morphism $A \rightarrow A'$ in $\Sigma\underline{\underline{A}}$ consists of an equivalence class of data, which reduces to giving an isomorphism $\psi : A \xrightarrow{\cong} A_1 \oplus A' \oplus A_2$, up to isomorphism in A_1 and A_2. From ψ, one constructs a diagram of epimorphisms and monomorphisms in $\underline{\underline{\mathcal{A}}}$, with choices of splittings $A \cong A_1 \oplus A' \oplus A_2 \twoheadrightarrow A' \oplus A_2 \twoheadleftarrow A'$. In fact, $\Sigma\underline{\underline{A}}$ is easily seen to be isomorphic to the category $Q^S\mathcal{A}$ ([16], §3) whose objects are those of $\mathcal{A}$; and whose morphisms $A \rightarrow A'$ are equivalence classes of data $A \twoheadrightarrow E \leftarrowtail A'$, where the indicated arrows are splittable mono- and epimorphisms, together with a choice of splitting (shown as dotted arrows). Changing E and all arrows by the same isomorphism $E \cong E'$ gives the equivalence relation. Composition is induced by taking pullbacks as in Quillen's $Q\mathcal{A}$ [14], and there is a functor $\Sigma\underline{\underline{A}} \cong Q^S\mathcal{A} \rightarrow Q\mathcal{A}$ that forgets the choice of splittings. (Actually, this $Q\mathcal{A}$ is the opposite category of the one in [14].) As $K_i(\underline{\underline{A}}) \cong \pi_{i+1}BQ^S\mathcal{A}$ by the above, to show $K_i(A) = \pi_{i+1}BQ\mathcal{A}$; i.e., that the group completion definition of K-theory agrees with the Q-construction definition, it remains only to show $Q^S\mathcal{A} \rightarrow Q\mathcal{A}$ is a homotopy equivalence.

To see this, consider the category $Q^{se}\mathcal{A}$ defined like $Q^S\mathcal{A}$, but where the morphisms are classes of $A \leftarrowtail E \twoheadleftarrow A'$, with a choice of splitting made only for the epimorphism. The functor

$Q^s\mathcal{a} \longrightarrow Q\mathcal{a}$ factors as $Q^s\mathcal{a} \longrightarrow Q^{se}\mathcal{a} \longrightarrow Q\mathcal{a}$. I'll show $\rho : Q^{se}\mathcal{a} \longrightarrow Q\mathcal{a}$ is a homotopy equivalence; the proof that $Q^s\mathcal{a} \longrightarrow Q^{se}\mathcal{a}$ is, is similar. To show ρ is a homotopy equivalence, by Theorem A I need only show the categories ρ/A are contractible for each A in $Q\mathcal{a}$ [14]. But ρ/A contains as a reflexive subcategory, and so as a deformation retract, the full subcategory whose objects are splittable epimorphisms $j : B \twoheadrightarrow A$, and whose morphisms from $j : B \twoheadrightarrow A$ to $j' : B' \twoheadrightarrow A$ are epimorphisms $g : B \underset{\leftarrow --}{\twoheadrightarrow} B'$ with a choice of splitting, and such that $j'g = j$. This subcategory has a symmetric monoidal structure induced by pullback over A, so $(B \twoheadrightarrow A) \otimes (B' \twoheadrightarrow A) = B \times_A B \twoheadrightarrow A$. Using this structure, the proof of [3], p. 227 that "For any C in P, $\langle S, E_C \rangle$ is contractible" applies to show this subcategory, and so ρ/A, is contractible, as required. This argument is the clever part of Quillen's proof given in [3]; the virtue of my machinery here is merely that it gets one down to this crux very quickly.

As a second example, I will indicate how one can approach the Lichtenbaum conjecture. I will reduce the problem to showing that a certain functor is a homotopy equivalence, but I have been unable to complete the proof by showing this is so.

Recall that the conjecture states that if $\bar{k}$ is an algebraically closed field in characteristic p, and $\bar{\mathbb{F}}_p$ is the algebraic closure of the prime field, there is a short exact sequence

$$0 \longrightarrow K_*(\bar{\mathbb{F}}_p) \longrightarrow K_*(\bar{k}) \longrightarrow K_*(\bar{k}) \otimes \mathbb{Q} \longrightarrow 0$$

Proof of this is crucial to understanding K-theory in characteristic p. By the work of Howard Hiller [4], this conjecture is equivalent to the conjecture of Quillen: Let $\phi^q : \bar{k} \longrightarrow \bar{k}$ be the Frobenius map $x \longmapsto x^q$, then $BGL(\mathbb{F}_q)^+$ is the homotopy fibre of $1-\phi^q : BGL(\bar{k})^+ \longrightarrow BGL(\bar{k})^+$. Equivalently, there should be a fibre sequence of infinite loop spaces, or cofibre sequence of spectra. I'll produce the cofibre of $BGL(\mathbb{F}_q)^+ \to BGL(\bar{k})^+$; to prove the conjecture one then wants to show it's $BGL(\bar{k})^+$.

Let $\underline{\underline{A}} = \coprod_n GL_n(\mathbb{F}_q)$, $\underline{\underline{B}} = \coprod_n GL_n(\bar{k})$, with symmetric monoidal structure induced by direct sum. Thus $Spt_0(\underline{\underline{A}}) \simeq \mathbb{Z} \times BGL(\mathbb{F}_q)^+$, $Spt_0(\underline{\underline{B}}) \simeq \mathbb{Z} \times BGL(\bar{k})^+$. Consider the simplified mapping cone $\underline{\underline{P}}$ on the functor $\bar{k} \otimes_{\mathbb{F}_q} : \underline{\underline{A}} \to \underline{\underline{B}}$. This $\underline{\underline{P}}$ has objects (A,B), with A a vector space over $\mathbb{F}_q$ and B a vector space over $\bar{k}$. A morphism $(A,B) \to (A',B')$ is a class of data: $\psi : A \cong A_1 \oplus A' \oplus A_2$, $\psi_2 : B \oplus (\bar{k} \otimes A_2) \xrightarrow{\cong} B'$.

Let $\underline{\underline{B}}^{-1}\underline{\underline{B}}$ be Quillen's group completed category, as described in [3]. Then $Spt_0(\underline{\underline{B}}^{-1}\underline{\underline{B}}) \simeq \mathbb{Z} \times BGL(\bar{k})^+$, and as $\pi_0(\underline{\underline{B}}^{-1}\underline{\underline{B}})$ is a group, $Spt_0(\underline{\underline{B}}^{-1}\underline{\underline{B}}) \simeq B(\underline{\underline{B}}^{-1}\underline{\underline{B}})$. Let $(\underline{\underline{B}}^{-1}\underline{\underline{B}})_0$ be the connected component of $(0,0)$. Then $B(\underline{\underline{B}}^{-1}\underline{\underline{B}})_0 \simeq BGL(\bar{k})^+$.

There is a symmetric monoidal functor $\rho : \underline{\underline{P}} \to (\underline{\underline{B}}^{-1}\underline{\underline{B}})_0$ given on objects by $\rho(A,B) = (B, \phi^q B)$, where $\phi^q B$ is the $\bar{k}$ vector space B with the $\bar{k}$-module structure changed by $\phi^q : \bar{k} \to \bar{k}$. On a morphism $(A,B) \to (A',B')$ of $\underline{\underline{P}}$ given by data as above, ρ is given by the morphism in $(\underline{\underline{B}}^{-1}\underline{\underline{B}})_0$ determined ([3]) by the object $\bar{k} \otimes A_2$, and the pair of isomorphisms

$$B \oplus (\bar{k} \otimes_{\mathbb{F}_q} A_2) \xrightarrow{\psi_2} B'$$

$$\phi^q B \oplus (\bar{k} \otimes_{\mathbb{F}_q} A_2) \cong \phi^q B \oplus \phi^q(\bar{k} \otimes_{\mathbb{F}_q} A_2) \xrightarrow{\phi^q(\psi_2)} \phi^q B'$$

where one uses the canonical isomorphism $\bar{k} \otimes_{\mathbb{F}_q} A_2 \cong \phi^q \bar{k} \otimes_{\mathbb{F}_q} A_2$ which is the identity on the subgroup $\mathbb{F}_q \otimes_{\mathbb{F}_q} A_2$.

The diagram

$$
\begin{array}{ccccc}
\mathbb{Z} \times BGL(\mathbb{F}_q)^+ & \longrightarrow & \mathbb{Z} \times BGL(\bar{k})^+ & \longrightarrow & BGL(\bar{k})^+ \\
\wr\| & & \wr\| & & \wr\| \\
Spt_0(\underline{\underline{A}}) & \longrightarrow & Spt_0(\underline{\underline{B}}) & \xrightarrow{1-\phi^q} & Spt_0((\underline{\underline{B}}^{-1}\underline{\underline{B}})_0) \\
\| & & \| & & \uparrow{\scriptstyle Spt_0(\rho)} \\
Spt_0(\underline{\underline{A}}) & \longrightarrow & Spt_0(\underline{\underline{B}}) & \longrightarrow & Spt_0(\underline{\underline{P}})
\end{array}
$$

commutes, and the bottom row is a fibre sequence, as the sequence of zeroth spaces of a cofibre sequence of spectra. Thus Quillen's conjecture is equivalent to the statement that $Spt_0(\rho)$ is an equivalence. As both $\underline{\underline{P}}$ and $(\underline{\underline{B}}^{-1}\underline{\underline{B}})_0$ are connected and so group complete, this is in fact equivalent to the functor $\rho : \underline{\underline{P}} \longrightarrow (\underline{\underline{B}}^{-1}\underline{\underline{B}})_0$ being a homotopy equivalence.

So far, I have been unable to show this, but there are signs that it is true. One could try to appeal to Quillen's Theorem A. I know the fibre $(0,0)/\rho$ is contractible by Lang's Theorem [8] that all torsors for the Frobenius action on $GL_n(\bar{k})$ are trivial with trivialization unique up to $GL_n(\mathbb{F}_q)$. Unfortunately, $(B,B)/\rho$ is in general disconnected, although I suspect each component is contractible. One could hope to show $H^*(\rho)$ is an isomorphism by considering the Grothendieck spectral sequence $H^p((\underline{\underline{B}}^{-1}\underline{\underline{B}})_0,\ H^q(\ /\rho)) \Rightarrow H^{p+q}(\underline{\underline{P}})$ and proving it collapses by analysis of $H^*(\ /\rho)$ and the action of the $GL_n(\bar{k})$ on it. Something like Tits buildings seems to play a role here. The interested reader is invited to try to make sense of this.

§7. Axiomitization of K-theory as a generalized homology theory on Sym Mon.

I will give an axiomatiziation characterizing the functor K_*. The axioms are reminiscent of the usual axioms for a generalized homology theory on Top if one accepts the idea that the appropriate Sym-Mon-hocolim is the analogue of the mapping cone. This gives a reassuring picture of K-theory as a generalized homology theory on Sym Mon, and suggests that analogues of the usual theorems about homology theories on Spectra ought to be true for K-theory.

I do not see how to characterize K-theory restricted to rings or exact categories in any similar fashion.

Consider the following four axioms on a functor K_* from Sym Mon to the category of non-negatively graded abelian groups.

I. (Homotopy axiom). If $F : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ is a morphism that induces an isomorphism on homology with $\mathbb{Z}$-coefficients after group completion, $\pi_0^{-1}H_*(F) : \pi_0^{-1}H_*(\underline{\underline{A}}) \xrightarrow{\cong} \pi_0^{-1}H_*(\underline{\underline{B}})$; then $K_*(F)$ is an isomorphism.

Here $H_*(\underline{\underline{A}})$ is homology of the category $\underline{\underline{A}}$, as in [5], IX, §6, or [14]. By [14], $H_*(\underline{\underline{A}})$ is isomorphic to $H_*(B\underline{\underline{A}})$, the homology of the classifying space. By $\pi_0^{-1}H_*(\underline{\underline{A}})$, I mean the homology localized with respect to the multiplicative subset $\pi_0\underline{\underline{A}} \subseteq H_0(\underline{\underline{A}})$. As is well known, $\pi_0^{-1}H_*(\underline{\underline{A}})$ is isomorphic to $H_*(\mathrm{Spt}_0\ \underline{\underline{A}})$, so this axiom is equivalent to the statement that $K_*(F)$ is an isomorphism if $\mathrm{Spt}_0(F)$ is a homotopy equivalence.

II. (Cofibre sequence axiom). For $F : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ a morphism, let

$\underline{\underline{B}} \longrightarrow \underline{\underline{P}}$ be the mapping cone on F. Then there is a long exact sequence

$$\longrightarrow K_{i+1}(\underline{\underline{P}}) \xrightarrow{\partial} K_i(\underline{\underline{A}}) \longrightarrow K_i(\underline{\underline{B}}) \longrightarrow K_i'(\underline{\underline{P}}) \xrightarrow{\partial} \cdots \longrightarrow K_0(\underline{\underline{P}}) \longrightarrow 0$$

Here the mapping cone $\underline{\underline{P}}$ is the Sym Mon-hocolim of a diagram as in §4, Example 1.

III (Continuity axiom). If $\underline{\underline{A}}_i$, $i \in I$ is a directed system of symmetric monoidal categories, $\varinjlim K_*(\underline{\underline{A}}_i) \cong K_*(\varinjlim \underline{\underline{A}}_i)$.

IV (Normalization axiom). Let $\underline{\underline{\mathbb{Z}}}$ be the category whose objects are integers $\underline{n}$, and whose morphisms are all identity morphisms. Let $\underline{\underline{\mathbb{Z}}}$ have the symmetric monoidal structure $\underline{n} \oplus \underline{m} = \underline{n+m}$. Then $K_0(\underline{\underline{\mathbb{Z}}}) = \mathbb{Z}$, $K_i(\underline{\underline{\mathbb{Z}}}) = 0$ for $i > 0$.

Theorem: If K_* is any functor from Sym Mon to non-negatively graded abelian groups satisfying the above four axioms, then K_* is naturally isomorphic to algebraic K-theory, π_*^S Spt.

Idea of Proof: Because of the homotopy axiom, K_* induces a functor out of the homotopy category of Sym Mon (obtained by formally inverting all $F : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ such that Spt(F) is an equivalence) into the homotopy category of Spectra. Suppose first one knew the induced map of homotopy categories was an equivalence. Then one shows K_* is stable homotopy. Using axiom II and the tower of higher connected coverings of a system, which is a sort of upside-down Postnikov tower, one can reduce to checking K_* is π_*^S on Eilenberg-MacLane spectra. Using axiom II again, one can shift dimensions until one is dealing with

$K(\pi,0)$-spectra. By axiom III, reduce to the case π is finitely generated; by axiom II, to the case π is cyclic; and by axiom II again to the case $\pi = \mathbb{Z}$; which holds by axiom IV.

While I do not know the two homotopy categories are the same, I can show the homotopy category of Sym Mon is a retract of that of Spectra, and that the retraction does not change the homotopy type of Spt_0. Proof of this involves 2-category theory, and a generalization of the theorem of §5, so I'll say no more about it. This relation between the homotopy categories is strong enough to make possible an argument along the lines of the first paragraph.

Bibliography

[1] Bousfield, A.K., and Kan, D.M.: Homotopy Limits, Completions, and Localizations, Springer Lecture Notes in Math., Vol. 304, (1972).

[2] Eilenberg, S., and Kelly, G.M.: "Closed categories"; in Proceedings of the Conference on Categorical Algebra: La Jolla 1965, pp. 421-562 (1966).

[3] Grayson, D.: "Higher algebraic K-theory: II (after Quillen)," in Algebraic K-Theory: Evanston 1976, Springer Lecture Notes in Math., Vol. 551, pp. 217-240 (1976).

[4] Hiller, H.: "Fixed points of Adams operations," thesis, MIT (1978).

[5] Hilton, P., and Stammbach, U.: A Course in Homological Algebra, Springer Graduate Texts in Math. Vol. 4 (1971).

[6] Kelly, G.M.: "An abstract approach to coherence," in Coherence in Categories, Springer Lecture Notes in Math., Vol. 281, pp. 106-147 (1972).

[7] Kelly, G.M.: "Coherence theorems for lax algebras and for distributive laws," in Category Seminar, Springer-Lecture Notes in Math., Vol. 420, pp. 281-375 (1974).

[8] Lang, S.: "Algebraic groups over finite fields," Amer. J. Math. Vol. 78, no. 3, pp. 555-563, (1956).

[9] MacLane, S.: Categories for the Working Mathematician, Springer Graduate Texts in Math., vol. 5, (1971).

[10] May, J.P.: "Categories of spectra and infinite loop spaces," in Category Theory, Homology Theory, and Their Applications III, Springer Lecture Notes in Math., vol. 99, pp. 448-479 (1969).

[11] May, J.P.: "E_∞ spaces, group completions, and permutative categories," in New Developments in Topology, London Math. Soc. Lecture Notes, no. 11, pp. 61-94 (1974).

[12] May, J.P.: "Simplicial Objects in Algebraic Topology," D. VanNostrand Co., (1967).

[13] May, J.P.: "The spectra associated to permutative categories," preprint.

[14] Quillen, D.: "Higher algebraic K-theory: I," in Higher K-Theories, Springer Lecture Notes in Math., vol. 341, pp. 85-147 (1973).

[15] Segal, G.: "Categories and cohomology theories," Topology, vol. 13, pp. 293-312 (1974).

[16] Segal, G.: "K-homology theory and algebraic K-theory," in K-Theory and Operator Algebras: Athens 1975, Springer Lecture Notes in Math., vol. 575, pp. 113-127 (1977).

[17] Thomason, R.W., "Homotopy colimits in Cat, with applications to algebraic K-theory and loop space theory," thesis, Princeton, (1977).

[18] Thomason, R.W.: "Homotopy colimits in the category of small categories," to appear, Math. Proc. Cambridge Phil. Soc.

[19] Thomason, R.W.: "Uniqueness of delooping machines," preprint.

[20] Vogt, R.M.: "Homotopy limits and colimits," Math. Z., vol. 134, no. 1, pp. 11-52 (1973).