

Shorter Notes: Some Simple Examples of Symplectic Manifolds Author(s): W. P. Thurston Source: Proceedings of the American Mathematical Society, Vol. 55, No. 2 (Mar., 1976), pp. 467-468 Published by: <u>American Mathematical Society</u> Stable URL: <u>http://www.jstor.org/stable/2041749</u> Accessed: 03/04/2013 07:27

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## SOME SIMPLE EXAMPLES OF SYMPLECTIC MANIFOLDS

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ABSTRACT. This is a construction of closed symplectic manifolds with no Kaehler structure.

A symplectic manifold is a manifold of dimension 2k with a closed 2-form  $\alpha$  such that  $\alpha^k$  is nonsingular. If  $M^{2k}$  is a closed symplectic manifold, then the cohomology class of  $\alpha$  is nontrivial, and all its powers through k are nontrivial. M also has an almost complex structure associated with  $\alpha$ , up to homotopy.

It has been asked whether every closed symplectic manifold has also a Kaehler structure (the converse is immediate). A Kaehler manifold has the property that its odd dimensional Betti numbers are even. H. Guggenheimer claimed [1], [2] that a symplectic manifold also has even odd Betti numbers. In the review [3] of [1], Liberman noted that the proof was incomplete. We produce elementary examples of symplectic manifolds which are not Kaehler by constructing counterexamples to Guggenheimer's assertion.

There is a representation  $\rho$  of  $Z \oplus Z$  in the group of diffeomorphisms of  $T^2$  defined by

$$(1,0) \xrightarrow{\rho} \mathrm{id}, \qquad (0,1) \xrightarrow{\rho} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

where " $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ " denotes the transformation of  $T^2$  covered by the linear transformation of  $\mathbb{R}^2$ . This representation determines a bundle  $M^4$  over  $T^2$ , with fiber  $T^2$ :  $M^4 = \tilde{T}^2 \times_{Z \oplus Z} T^2$ , where  $Z \oplus Z$  acts on  $\tilde{T}^2$  by covering transformations, and on  $T^2$  by  $\rho$  ( $M^4$  can also be seen as  $\mathbb{R}^4$  modulo a group of affine transformations). Let  $\Omega_1$  be the standard volume form for  $T^2$ . Since  $\rho$  preserves  $\Omega_1$ , this defines a closed 2-form  $\Omega'_1$  on  $M^4$  which is nonsingular on each fiber. Let  $\rho$  be projection to the base: then it can be checked that  $\Omega'_1 + p^* \Omega_1$  is a symplectic form. (It is, in general, true that  $\Omega''_1 + K\rho^* \Omega_1$  is a symplectic form, for any closed  $\Omega''_1$  which is a volume form for each fiber, and K sufficiently large.) But  $H_1(M^4) = Z \oplus Z \oplus Z$ , so  $M^4$  is not a Kaehler manifold.

Many more examples can be constructed. In the same vein, if  $M^{2k}$  is a closed symplectic manifold, and if  $N^{2k+2}$  fibers over  $M^{2k}$  with the fundamental class of the fiber not homologous to zero in N, then N is also a symplectic manifold. If, for instance, the Euler characteristic of the fiber is not zero, this

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Received by the editors July 31, 1974.

AMS (MOS) subject classifications (1970). Primary 57D15, 58H05.

hypothesis is satisfied. To do this, one must see that if there is a closed 2-form  $\alpha_1$  whose integral on a fiber is nonzero, then  $\alpha_1$  is cohomologous to a 2-form  $\alpha$  which is nonsingular on each fiber. To find  $\alpha$ , first find a 2-form  $\beta$ , not necessarily closed, which is nonsingular on each fiber, and whose integral on each fiber agrees with that of  $\alpha_1$ : this exists by convexity considerations. On each fiber, F, there is a form  $\gamma_F$  such that  $\beta_F - (\alpha_1)_f = d(\gamma_F)$ . This equation can also be solved differentiably in a small neighborhood of the base, so, by convexity considerations, there is a global 1-form  $\gamma$  such that on each fiber,  $\beta_F - (\alpha_1)_F = d(\gamma_F)$ . Let  $\alpha = \alpha_1 + d(\gamma)$ . If  $\Omega_1$  is a symplectic form for  $M^{2k}$ , then  $\Omega = \alpha + K(\rho^*\Omega_1)$  is a symplectic form for  $N^{2k+2}$ , K is sufficiently large. This construction, although it applies only to a narrow range of examples,

nonetheless has a certain amount of flexibility. This leads me to make the

CONJECTURE. Every closed 2k-manifold which has an almost complex structure  $\tau$  and a real cohomology class  $\alpha$  such that  $\alpha^k \neq 0$  has a symplectic structure realizing  $\tau$  and  $\alpha$ .

I would like to thank Alan Weinstein for pointing out this question and for helpful discussions.

## References

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