THE GEOMETRICAL INVARIANTS OF ALGEBRAIC LOCI

By J. A. Todd.

[Received 21 October, 1936.—Read 12 November, 1936.]

Until quite recently the theory of the invariants of an algebraic variety V_d of d dimensions has centred almost entirely round the canonical system of V_{d-1}'s lying on it. The definition of this system follows closely that of the canonical system of curves on a surface, and dates back to Noether†. If $|C|$ is a linear system of ∞^d V_{d-1}'s on V_d, and if C_J is its Jacobian V_{d-1}, then the system $|C_J-(d+1)C|$ is independent of $|C|$ and is the canonical system in question. An alternative expression for the system is $|C'-C|$, where $|C'|$ is the linear system adjoint to $|C|$, cutting each C in a V_{d-2} belonging to the canonical system on C.

It is only within the last few years that any attention has been drawn to other geometrical invariants of algebraic varieties. In 1932 Severi‡ introduced an invariant series of sets of points on an algebraic surface, and still more recently Segre§ has considered a similar series on a V_3, and an invariant system of curves on V_3.

We show here how to define on a given V_d an invariant system of varieties of each dimension from 0 up to $d-1$; the system of dimension $d-1$ being the canonical system already mentioned, and the systems of sets of points and curves being (for $d=2, 3$) the invariant systems of Severi and Segre. We shall refer to these invariant systems as the *canonical systems* of appropriate dimension, and shall denote the canonical system of dimension k by the symbol $\{X_k(V_d)\}$.

† For references see the article in *Encyk. Mat. Wiss.*, III C 6b, § 47, or the later reports by Lefschetz, *Mémorial des Sciences Mathématiques*, 40 (1929), and Rosenblatt, *Atti del Congresso Internationale dei Matematici* (Bologna, 1928), 4, 93.

‡ Severi, *Commentarii Matematici Helvetici*, 4 (1932), 268.

§ B. Segre, *Mem. R. Accademia d'Italia*, 5 (1934), 479.

In order to simplify the formal work somewhat, we shall attach the following conventional meaning to the symbol $X_k(V_d)$ when $k \geqslant d$. The symbol $X_d(V_d)$ shall mean V_d itself; the symbol $X_k(V_d)$, for $k > d$, shall represent zero.

In the case of the canonical system $\{X_{d-1}(V_d)\}$ it is well known that the arithmetical invariants of a general member of the system give invariants of V_d. For the canonical systems of lower dimensions, such invariants do not necessarily arise. For example, on a V_3 the virtual genus of a curve is not necessarily constant as the curve moves in a system of equivalence, as is easily seen by taking V_3 to be a linear space, in which the aggregate of curves of given order forms a single connected system of equivalence†. We do, however, obtain invariants by considering the virtual number of intersections of a set of canonical varieties of such number and dimension that their virtual intersection consists of a finite set of points. In this way the results obtained here lead to significant results in the theory of the *arithmetical* invariants of V_d, which it is hoped to consider in a subsequent paper.

The canonical system of dimension zero on V_d is easily defined in terms of a pencil $|C|$ of V_{d-1}'s with an irreducible base V_{d-2}, (C^2). In fact, if δ denotes the set of double points of the pencil, and $X_0(C)$, $X_0(C^2)$ denote canonical sets on C and (C^2) respectively, then we show in § 2 below that the series of equivalence (possibly virtual) defined by the set

$$\delta - 2X_0(C) - X_0(C^2)$$

is independent of $|C|$; this is the system $\{X_0(V_d)\}$.

To define the canonical varieties of higher dimension we introduce the idea of the *adjoints of various dimensions* of a linear system $|C|$ of V_{d-1}'s on V_d, supposing (in the first place) that $|C|$ is a sufficiently ample linear system and is free from base-points. The definition of these systems is similar to that of the adjoint system of V_{d-1}'s; the adjoint system of dimension k being the set of V_k's on V_d which cut out on a general C a V_{k-1} belonging to $\{X_{k-1}(C)\}$. There is no difficulty in defining (possibly virtual) varieties with this property, and we assume that the aggregate of adjoint V_k's to $|C|$ forms on V_d a system of equivalence $\{A_k(C)\}$. It then appears that the (possibly virtual) system of equivalence $\{A_k(C) - X_k(C)\}$ is independent of $|C|$; this is the canonical system $\{X_k(V_d)\}$ of dimension k on V_d, and may be used to define the system adjoint to V_{d-1}'s on V_d which are not members of sufficiently general linear systems for the preceding definition to apply.

† See, *e.g.*, Severi, *Mem. R. Accademia d'Italia*, 5 (1934), 242.

It may be remarked that, as defined, these systems are *a priori* virtual, but there certainly exist V_d's on which they may be realized effectively. Further, their invariance under birational transformation of the V_d is relative, in general, since the introduction of fundamental varieties of various dimensions in the transformation will in general affect the canonical systems.

1. *Equivalence of V_k's on a V_d.*

We recall the notion of *equivalence* between V_k's on V_d, a concept introduced by Severi†. The ideas underlying the present formulation are due to the author‡. We consider on V_d $d-k$ linear systems $|C_1|, |C_2|, \ldots, |C_{d-k}|$ of V_{d-1}'s and consider the aggregate of V_k's obtained as the complete intersection of $d-k$ V_{d-1}'s, chosen one from each of the $d-k$ systems. This aggregate of V_k's we call a *system of intersection.* Consider now the aggregate of cycles of $2k$ dimensions on the Riemannian R_{2d} of V_d which represent virtual algebraic V_k's on V_d, *i.e.* which are homologous on R_{2d} to the difference of two cycles representing effective V_k's. We call these the *algebraic* $2k$-cycles of R_{2d}. They form an abelian group with addition as the law of composition; let this group be denoted by $\mathfrak{G}$. We define a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ as follows: $\mathfrak{H}$ is generated by all the cycles of the form $\Gamma'-\Gamma''$, where Γ', Γ'' are cycles representing two effective V_k's, A', A'', with the property that for some effective V_k, A say, $A+A'$ and $A+A''$ belong to the same system of intersection. Two V_k's A, B will be called *equivalent* if $A-B$ is represented by a cycle of $\mathfrak{H}$, *i.e.* if the cycles representing A and B lie in the same coset of $\mathfrak{G}$ with respect to $\mathfrak{H}$. When this happens we write $A \equiv B$. The relation of equivalence is symmetrical, reflexive, and transitive; and it is invariant under the operation of group-addition, *i.e.* from $A \equiv B$ and $C \equiv D$ follows $A+C \equiv B+D$. Further, if two V_k's A, B are equivalent on V_d, and a V_{d-l} meets them in V_{k-l}'s A_1, B_1, then $A_1 \equiv B_1$ on V_{d-l}.

A *system of equivalence* is essentially a connected algebraic system of mutually equivalent effective V_k. In the present work we stress the notion of equivalence rather than that of systems of equivalence, since the former is better adapted to deal with varieties which may in general be virtual.

It is of importance to have a criterion for deciding whether two varieties A and B on a V_d are equivalent or not. If A and B are of dimension $d-1$, then it is known that a sufficient condition for the equivalence of A and B

† Severi, *Mem. R. Accademia d'Italia*, 4 (1933), 71.

‡ Todd, *Annals of Mathematics* (2), 35 (1934), 702.

is that they should cut equivalent V_{d-2}'s on the V_{d-1}'s of a sufficiently general linear pencil. It is natural to seek an extension of this result to the case in which A and B are of dimension smaller than $d-1$. The result that suggests itself is that, if A and B are of k dimensions, they are equivalent if they meet the V_{d-1}'s of a sufficiently general linear system $|S|$, free from fundamental varieties, in equivalent V_{k-1}'s. In this paper we shall assume that this is so. That the hypothesis made is probable is suggested from the considerations which follow, though these scarcely amount to a proof. The justification for making the assumption lies in the consequences that follow from it. Algebraic geometry, especially when concerned with loci of three or more dimensions, is still an experimental science, and the results obtained here, by using this hypothesis, seem to be very suggestive.

In order to examine more clearly the nature of the assumption made, we consider in a little more detail the case in which A and B are curves on a V_3; the general case presents no serious new complication. Let us suppose, in the first instance, that A and B cut out on a general surface S_0 of $|S|$ sets of points A_1 and B_1 which belong to a series of intersection. This means that we can find on S_0 two pencils of curves $|f|$, $|g|$, with the property that the complete intersection of one pair of curves f_1, g_1 of the pencils is the set A_1, while the complete intersection of another pair f_2, g_2 is the set B_1. It may be possible to select such a pair of pencils $|f|$ and $|g|$ in more than one way. Suppose that $|f'|$, $|g'|$ are another pair of pencils with the same property. Then, if the curves of $|f'|$ are members of the same complete continuous system as the curves of $|f|$, they must belong to the same linear system. For f_1 and f_1', being algebraically equivalent, cut any g in the same number of points, hence the set A_1 is the complete intersection of f_1' and g_1, and it then follows from the criterion of linear equivalence for curves on a surface that f_1 and f_1' are linearly equivalent. It may happen, however, that $|f'|$ and $|g'|$ consist of curves respectively of the same orders as, but not equivalent to, the curves $|f|$ and $|g|$, *i.e.* that the surface possesses several descriptively similar linear systems which are non-equivalent (*e.g.* the cubic surface, with twenty-seven pencils of conics). It seems likely, however, that this can happen only for a limited type of surface on V_3, and that, if $|S|$ is sufficiently general, this possibility cannot arise.

This being the case, we can now isolate the complete linear systems on S_0 to which $|f|$ and $|g|$ belong and determine *uniquely* on the other surfaces of the pencil the corresponding linear systems. Then, by assigning fixed base points to these systems, lying on the base curve of the pencil, we can reduce the freedom of the systems passing through the sets A_1 and B_1 to zero. If now S_0 is allowed to vary in the pencil, the curves f_1 and g_1 describe surfaces F_1, G_1 which meet in the curve A, and f_2, g_2 describe surfaces F_2, G_2 meeting in the curve B, these being the complete intersections of the respective pairs of surfaces. Since the surfaces F_1, F_2 are clearly equivalent on V_3, as are G_1, G_2, it follows that the curves A and B belong to a system of intersection.

The extension to the case when the sets A_1 and B_1 do not belong to a series of intersection is simple, since in the most general case in which A_1 and B_1 are equivalent there exists a set D_1 such that A_1+D_1, B_1+D_1 belong to the sum or difference of a finite number of series of intersection. These being, as we have seen, determinable rationally on each S if $|S|$ is sufficiently general, it follows by an obvious process that A and B are such that there exists a curve D such that $A+D$ and $B+D$ belong to the sum or difference of a finite number of systems of intersection on V_3, and are hence equivalent.

This outline may serve to explain how the hypothesis mentioned above is suggested, and may form the lines on which a complete proof can be based.

2. *The canonical series* $\{X_0(V_d)\}$ *of sets of points on* V_d.

We shall now prove, by induction on d, that if δ denotes the set of double points of the members of a pencil $|C|$ of V_{d-1}'s, with irreducible base (C^2), on V_d (assumed to be free from singularities), and if $X_0(C)$, $X_0(C^2)$ are canonical sets of C and (C^2), then the series of equivalence defined by

$$\delta-2X_0(C)-X_0(C^2) \tag{2.1}$$

is independent of $|C|$. The proof is very similar to that of Segre in the case $d=3$, to which we have already referred, and is essentially the same as the argument establishing the existence of the Zeuthen-Segre invariant I_d of V_d. In fact, the number of points in the set $X_0(V_d)$ is $I_d+(-1)^d.2d$.

We consider two pencils, $|C|$, $|D|$, of V_{d-1}'s with irreducible base-loci (C^2), (D^2). Let ϕ_1 be the curve of contact of a C and a D, and τ the group of points of stationary contact (lying on ϕ_1). Let δ, δ' be the sets of double points of members of the pencils $|C|$, $|D|$, respectively, and let ρ, ρ' be the sets of double points of the pencils $|C^*|$, $|D^*|$, cut out respectively by $|D|$ on (C^2) and by $|C|$ on (D^2).

Consider a fixed V_{d-1}, C_0, of the pencil $|C|$, and the pencil of V_{d-2}'s cut out on C_0 by $|D|$. Assuming (2.1) to define the canonical series on varieties of dimension less than d, we have

$$X_0(C)\equiv\delta_1-2X_0(CD)-X_0(CD^2), \tag{2.2}$$

where δ_1 is the set of double points of the pencil $|(C_0 D)|$. Again, considering the pencil $|C^*|$ on (C^2), we have

$$X_0(C^2) \equiv \rho - 2X_0(C^2 D) - X_0(C^2 D^2). \tag{2.3}$$

The points δ_1 and ρ make up the intersection of C_0 and ϕ_1; hence

$$(\phi_1 C) \equiv \delta_1 + \rho. \tag{2.4}$$

The set of double points of the linear series cut on ϕ_1 by $|C|$ consists of the set δ of double points of $|C|$, the set ρ of double points of $|(C^2 D)|$, each counted twice since they are base-points of the series in question, the set ρ' of double points of $|(CD^2)|$, and the set τ of stationary points. Hence

$$\delta + 2\rho + \rho' + \tau \equiv 2(\phi_1 C) + X_0(\phi_1). \tag{2.5}$$

From (2.2), (2.3), (2.4), (2.5) we deduce that

$$\delta - 2X_0(C) - X_0(C^2) \equiv X_0(\phi_1) - \tau - \rho - \rho' + 4X_0(CD)$$
$$+ 2X_0(C^2 D) + 2X_0(CD^2) + X_0(C^2 D^2). \tag{2.6}$$

Since the right-hand member of (2.6) is symmetrical in C and D, it follows that

$$\delta - 2X_0(C) - X_0(C^2) \equiv \delta' - 2X_0(D) - X_0(D^2),$$

which establishes the existence of the canonical series $\{X_0(V_d)\}$ on V_d, since (2.1) reduces, when $d = 1$, to the familiar expression for the canonical series on a curve†.

3. *The canonical system* $\{X_1(V_3)\}$ *of curves on a* V_3.

Having thus established the existence of the system $\{X_0(V_d)\}$ for all values of d, we proceed to consider the canonical curves. The existence of these on a surface is known from quite elementary considerations. We shall establish the existence of a canonical system of curves, first on a V_3, and then on a V_d of any dimension. The proof given here for the V_3 is different from, and somewhat simpler than, that of Segre.

We consider two pencils $|C|$, $|D|$ of surfaces on V_3, and retain the notation of the last section. The equation (2.2) becomes

$$X_0(C) \equiv \delta_1 - 2X_0(CD) - (CD^2), \tag{3.1}$$

† When $d = 2$ the proof requires verbal modification, since the base (C^2) consists of a set of points and is thus "reducible."

since (CD^2) is a set of points, and so $X_0(CD^2) \equiv (CD^2)$. Now $X_0(CD)$ is the canonical series on the curve (CD), and is therefore cut out on (CD) by that curve on the surface D which is adjoint to (CD), namely $(CD)+X_1(D)$, where $X_1(D)$ is a canonical curve of D. Hence

$$X_0(CD) \equiv \big((CD).(CD)+X_1(D)\big)_D \equiv (C^2D)+\big(C.X_1(D)\big). \quad (3.2)$$

In the same way we find that

$$X_0(C^2) \equiv \big((C^2).(C^2)+X_1(C)\big)_C \equiv (C^3)+\big(C.X_1(C)\big). \quad (3.3)$$

The equation (2.3) reduces, since $(C^2D^2)=0$, to

$$X_0(C^2) \equiv \rho-2(C^2D),$$

whence, by (3.3),

$$\rho \equiv \big(C.(C^2)+X_1(C)+2(CD)\big).$$

Hence, from (3.1), (3.2), (2.4),

$$X_0(C) \equiv \big(C.\phi_1-(C^2)-4(CD)-(D^2)-X_1(C)-2X_1(D)\big). \quad (3.4)$$

Thus the curves

$$A_1(C) \equiv \phi_1-(C^2)-4(CD)-(D^2)-X_1(C)-2X_1(D) \quad (3.5)$$

cut out on C sets of the canonical series $\{X_0(C)\}$. The system $\{A_1(C)\}$ is called the *adjoint system of curves*† of $|C|$. We see from (3.5) that $A_1(C)-X_1(C)$ is symmetrical in C and D. Hence the system

$$\{X_1(V_3)\} \equiv \{A_1(C)-X_1(C)\}$$

is independent of C. This is the *canonical system of curves* on V_3.

Now let $|S|$ be a net of surfaces on V_3, and let J_1 be its Jacobian curve. On a fixed surface S_0 of $|S|$ the other surfaces cut out a pencil of curves, and the double points of these curves lie at the intersections of S_0 and J_1. Hence

$$X_0(S) \equiv (S.J_1)-2X_0(S^2)-(S^3)$$
$$\equiv \big(S.J_1-2X_1(S)-3(S^2)\big)$$

by virtue of (3.3). Thus

$$J_1-2X_1(S)-3(S^2) \equiv A_1(S) \equiv X_1(S)+X_1(V_3),$$

† The assumption made in §1 is, of course, used here (and in similar places throughout the paper) to deduce the uniqueness of the system $\{A_1(C)\}$.

that is to say

$$X_1(V_3) \equiv J_1 - 3X_1(S) - 3(S^2). \tag{3.6}$$

Since $X_1(S) \equiv (S^2) + (SX)$, where X is a canonical surface of V_3, we deduce from (3.6) the relation

$$X_1(V_3) \equiv J_1 - 3(SX) - 6(S^2)$$

which is used by Segre to define the system $\{X_1(V_3)\}$.

4. *Canonical curves on* V_d.

The extension of the considerations of the preceding section to varieties of any dimension is now quite simple. We assume the existence of the canonical system $\{X_1(V_l)\}$ for $l < d$, and of the adjoint system $\{A_1(V_{l-1})\}$ for a linear system of V_{l-1}'s lying on V_l, and deduce the existence of the system $\{X_1(V_d)\}$.

Considering once again the two pencils $|C|$ and $|D|$ of V_{d-1}'s, we have the relations (2.2), (2.3), (2.4). By the hypothesis of the induction, if α and β are not both zero, we have

$$X_0(C^{1+\alpha} D^\beta) \equiv \left(C^{1+\alpha} D^\beta . A_1(C^{1+\alpha} D^\beta) \right)_{(C^\alpha D^\beta)},$$

the notation implying that the adjoint curves lie on $(C^\alpha D^\beta)$. Hence

$$X_0(C^{1+\alpha} D^\beta) \equiv \left(C^{1+\alpha} D^\beta . X_1(C^{1+\alpha} D^\beta) + X_1(C^\alpha D^\beta) \right)_{(C^\alpha D^\beta)}$$

$$\equiv \left(C . X_1(C^{1+\alpha} D^\beta) + X_1(C^\alpha D^\beta) \right).$$

The equations (2.2), (2.3), (2.4) now give

$$X_0(C) \equiv \left(C . \phi_1 - X_1(C) - 2X_1(D) - X_1(C^2) - 4X_1(CD) - X_1(D^2) - 2X_1(C^2 D) - 2X_1(CD^2) - X_1(C^2 D^2) \right),$$

showing that the system of curves

$$\{A_1(C)\} \equiv \{\phi_1 - X_1(C) - 2X_1(D) - X_1(C^2) - 4X_1(CD) - X_1(D^2) - 2X_1(C^2 D) - 2X_1(CD^2) - X_1(C^2 D^2)\}$$

is adjoint to C. Further, the system

$$X_1(V_d) \equiv \{A_1(C) - X_1(C)\},$$

being symmetrical in C and D, is independent of C and defines the canonical system of curves on V_d. It also follows that

$$\phi_1 \equiv \big(X_1(V_d)+2X_1(C)+X_1(C^2)\big)+2\big(X_1(D)+2X_1(DC)+X_1(DC^2)\big)$$
$$+\big(X_1(D^2)+2X_1(D^2C)+X_1(D^2C^2)\big). \quad (4.1)$$

It is a simple matter to deduce from this an expression for the Jacobian curve of a net $|S|$ of V_{d-1}'s on V_d. If J_1 is the Jacobian curve in question, a repetition of the argument of the last section shows that

$$X_0(S) \equiv (S\,.\,J_1)-2X_0(S^2)-X_0(S^3)$$
$$\equiv \big(S\,.\,J_1-2X_1(S)-3X_1(S^2)-X_1(S^3)\big),$$

so that $$A_1(S) \equiv J_1-2X_1(S)-3X_1(S^2)-X_1(S^3).$$

But $A_1(S) \equiv X_1(S)+X_1(V_d)$, and so

$$J_1 \equiv X_1(V_d)+3X_1(S)+3X_1(S^2)+X_1(S^3). \quad (4.2)$$

5. *Canonical V_k's on a V_d.*

The extension of the considerations of the last two sections to prove the existence of an invariant system of V_k's on a V_d for any value of k (less than d) presents no essentially new difficulty. Let us suppose that the canonical system $\{X_{k-1}(V_d)\}$ is defined for all d. On a V_{k+1} the existence of the canonical V_k follows by elementary methods. We assume that the system $\{X_k(V_l)\}$ is defined on varieties of dimension l less than d, and proceed to establish their existence on V_d. As part of the inductive hypothesis we assume the following generalizations of (4.1) and (4.2): (i) that on a V_l (of any dimension l) the Jacobian J_{k-1} of a linear ∞^k system of V_{l-1}'s, $|S|$, is given by

$$J_{k-1} \equiv \sum_{r=0}^{k+1} \binom{k+1}{r} X_{k-1}(S^r), \quad (5.1)\dagger$$

and (ii) that the V_{k-1} of contact of a pencil $|T|$ of V_{l-1}'s and a linear ∞^{k-1} system $|U|$ of V_{l-1}'s is given by

$$\phi_{k-1} \equiv \sum_{r=0}^{k} \binom{k}{r} \big(X_{k-1}(U^r)+2X_{k-1}(U^rT)+X_{k-1}(U^rT^2)\big). \quad (5.2)$$

These reduce respectively to (4.2) and (4.1) when $k=2$.

† With the convention that $S^0 = V_l$.

We take on V_d a pencil $|C|$ of V_{d-1}'s and a linear system $|D|$ of freedom k. Let ϕ_k be the locus of contacts of a C and a D. The system $|D|$ cuts out on a fixed C_0 of $|C|$ a linear ∞^k system of V_{d-2}'s whose Jacobian J_{k-1} is, by (5.1), given by

$$J_{k-1} \equiv \sum_{r=0}^{k+1} \binom{k+1}{r} X_{k-1}(D^r C)$$

$$\equiv X_{k-1}(C) + \sum_{r=1}^{k+1} \binom{k+1}{r} X_{k-1}(D^r C). \tag{5.3}$$

The system $|D|$ cuts on the base (C^2) of $|C|$ a linear ∞^k system of V_{d-3}'s whose Jacobian J'_{k-1} is given by

$$J'_{k-1} \equiv \sum_{r=0}^{k+1} \binom{k+1}{r} X_{k-1}(D^r C^2). \tag{5.4}$$

It is easily seen that

$$(\phi_k \,.\, C) \equiv J_{k-1} + J'_{k-1}. \tag{5.5}$$

Now, by the definition of $X_k(V_l)$ for $l < d$,

$$X_{k-1}(CD^r) \equiv \Big((CD^r) \,.\, A_k(CD^r)\Big)_{(D^r)} \equiv \Big((CD^r) \,.\, X_k(CD^r) + X_k(D^r)\Big)_{(D^r)}$$

$$\equiv \Big(C \,.\, X_k(CD^r) + X_k(D^r)\Big),$$

and, in the same way,

$$X_{k-1}(C^2 D^r) \equiv \Big(C \,.\, X_k(C^2 D^r) + X_k(CD^r)\Big).$$

Hence, from (5.3), (5.4), and (5.5),

$$X_{k-1}(C) \equiv \Bigg(C \,.\, \phi_k - \sum_{r=1}^{k+1} \binom{k+1}{r} \Big(X_k(CD^r) + X_k(D^r)\Big) - \sum_{r=0}^{k+1} \binom{k+1}{r} \Big(X_k(C^2 D^r) + X_k(CD^r)\Big)\Bigg),$$

and so the system

$$\{A_k(C)\} \equiv \Big\{\phi_k - X_k(C) - X_k(C^2) - \sum_{r=1}^{k+1} \binom{k+1}{r} \Big(X_k(D^r) + 2X_k(D^r C) + X_k(D^r C^2)\Big)\Big\} \tag{5.6}$$

is the adjoint system of $|C|$ of dimension k.

Consider now the characteristic system cut by $|D|$ on a particular V_{d-1}, D_0, of $|D|$, and the pencil cut by $|C|$ on D_0. The locus of contacts of these systems is $(\phi_k . D_0)$, so that, from (5.2),

$$(\phi_k . D) \equiv \sum_{r=0}^{k} \binom{k}{r} \Big(X_{k-1}(D^{r+1}) + 2X_{k-1}(D^{r+1}C) + X_{k-1}(D^{r+1}C^2) \Big)$$

$$\equiv X_{k-1}(D) + 2X_{k-1}(DC) + X_{k-1}(DC^2)$$

$$+ \sum_{r=1}^{k} \binom{k}{r} \Big(X_{k-1}(D^{r+1}) + 2X_{k-1}(D^{r+1}C) + X_{k-1}(D^{r+1}C^2) \Big).$$

Hence, since

$$X_{k-1}(D^{\alpha+1}C^{\beta}) \equiv \Big(D . X_k(D^{\alpha}C^{\beta}) + X_k(D^{\alpha+1}C^{\beta}) \Big)$$

if α, β are not both zero, we find that

$$X_{k-1}(D) \equiv \Big(D . \phi_k - 2X_k(C) - X_k(C^2) + X_k(D) - \sum_{r=1}^{k+1} \binom{k+1}{r} \Big(X_k(D^r) + 2X_k(D^r C) + X_k(D^r C^2) \Big) \Big).$$

Hence the system

$$\{A_k(D)\} \equiv \Big\{ \phi_k - 2X_k(C) - X_k(C^2) + X_k(D) - \sum_{r=1}^{k+1} \binom{k+1}{r} \Big(X_k(D^r) + 2X_k(D^r C) + X_k(D^r C^2) \Big) \Big\} \quad (5.7)$$

is adjoint to $|D|$. From (5.6) and (5.7) it follows that

$$A_k(C) - X_k(C) \equiv A_k(D) - X_k(D),$$

which establishes the existence of the invariant system $X_k(V_d)$. Also, from (5.6) or (5.7) we find

$$\phi_k \equiv \sum_{r=0}^{k+1} \binom{k+1}{r} \Big(X_k(D^r) + 2X_k(D^r C) + X_k(D^r C^2) \Big),$$

which is (5.2) with $k-1$ replaced by k.

To complete the induction it is only necessary to extend (5.1). For this we consider a linear ∞^{k+1} system $|S|$ and its Jacobian J_k. The Jacobian of the characteristic system cut by $|S|$ on a fixed S_0 is $(S_0 J_k)$, so that, by (5.1),

$$(J_k . S) \equiv \sum_{r=0}^{k+1} \binom{k+1}{r} X_{k-1}(S^{r+1}).$$

Now $$X_{k-1}(S^{r+1}) \equiv \Big(S . X_k(S^{r+1}) + X_k(S^r) \Big) \quad (r > 0),$$

so that

$$(J_k . S) \equiv X_{k-1}(S) + \left(S . \sum_{r=1}^{k+1} \binom{k+1}{r} \left(X_k(S^{r+1}) + X_k(S^r) \right) \right)$$

$$\equiv X_{k-1}(S) + \left(S . \sum_{r=1}^{k+2} \binom{k+2}{r} X_k(S^r) - X_k(S) \right).$$

Hence
$$A_k(S) \equiv J_k + X_k(S) - \sum_{r=1}^{k+2} \binom{k+2}{r} X_k(S^r)$$

and
$$J_k \equiv X_k(V_d) + \sum_{r=1}^{k+2} \binom{k+2}{r} X_k(S^r)$$

$$\equiv \sum_{r=0}^{k+2} \binom{k+2}{r} X_k(S^r).$$

This is (5.1) with k instead of $k-1$. The induction is thus complete, and the canonical system $\{X_k(V_d)\}$ is defined for all values of k and d.

[*Note*—Added 25 February, 1937.] Since the above paper was written two notes on the same subject have appeared by M. Eger [*Comptes rendus*, 204 (1937), 92, 217]. In the second of these papers the canonical system of dimension k on a V_d is defined in terms of the Jacobian of $k+1$ pencils of V_{d-1}'s, and a formula equivalent to our (5.1) is obtained (equation γ of Eger's note).

University of Manchester.