THE GEOMETRICAL INVARIANTS OF ALGEBRAIC LOCI

(*Second Paper*)

By J. A. TODD.

[Received 10 June, 1938.—Read 16 June, 1938.—Revised 10 October, 1938.]

Introduction.

In a previous paper† with the above title I defined certain systems of equivalence $\{X_h\}$ for each dimension h from 0 to $d-1$ on an algebraic V_d, which I there termed *canonical systems* of dimension h. The proof of the invariance of these systems which I gave in that paper depended, for $h > 0$, on the assumption of an unproved criterion of equivalence for varieties of dimension h on V_d. After that paper was written, but before it appeared in print, Eger published two notes‡ in which he defined canonical systems from a different point of view. These notes merely give an account of Eger's results, and his proofs have not yet been published, but it would appear from his account that these proofs are of a function-theoretic character, and moreover require the existence of certain simple integrals attached to the V_d in question. It therefore seems desirable to give a purely geometric discussion of the matter which will establish the existence of the canonical systems without making use of any unproved assumption. This is the object of the present paper, which essentially replaces the earlier one with the same title. I should like, however, to make my obligations to Eger quite clear, since it seems likely that the geometrical reasoning given below is closely related to the function-theoretic considerations employed by Eger.

† J. A. Todd, *Proc. London Math. Soc.* (2), 43 (1937), 127.

‡ M. Eger, *Comptes rendus*, 204 (1937), 92, 217.

1. As a starting point we recall that on a non-singular variety V_d of d dimensions the canonical system of dimension $d-1$ may be defined by taking an irreducible linear system $|S|$ of V_{d-1}'s of freedom d and constructing its Jacobian variety S_J. If $|S|$ is free from base points it is then easily proved that the linear system $|X_{d-1}|$ defined by

$$|X_{d-1}| \equiv |S_J-(d+1)\,S| \tag{1}$$

is independent of $|S|$, the proof being a simple generalization of the corresponding result for $d=2$. The system $|X_{d-1}|$ is easily shown to have the property expressed by the relation of (linear) equivalence on S

$$X_{d-2}[S] \equiv (S\,.\,X_{d-1}+S) \tag{2}$$

which we call the *relation of adjunction.*

Suppose now that we have, on V_d, d irreducible linear pencils $|S_i|$ ($i=1, \ldots, d$), and seek the variety $J_{d-1}[S_1, S_2, \ldots, S_d]$ which is the locus of points P such that the d tangent $[d-1]$'s to the varieties of the pencils $|S_i|$ which pass through P have a common line. We shall prove that

$$J_{d-1}[S_1, S_2, \ldots, S_d] \equiv X_{d-1}+2(S_1+S_2+\ldots+S_d). \tag{3}$$

The proof of (3) is by induction on d. When $d=1$, (3) is equivalent to (1). We therefore assume that (3) holds on varieties of dimension $d-1$.

Let S be a particular variety of the pencil $|S_1|$ and let $|\Sigma_j|$ denote the linear system cut by $|S_j|$ on S for $j=2, 3, \ldots, d$. The variety

$$J_{d-1}[S_1, S_2, \ldots, S_d]$$

contains the base V_{d-2}, (S_1^2), of the pencil $|S_1|$. For the tangent $[d-1]$'s at a point P of (S_1^2) to the varieties of the pencils $|S_2|, \ldots, |S_d|$ which pass through P meet in a line, and just one member of the pencil $|S_1|$ can be made to touch this line at P. At a point Q common to

$$J_{d-1}[S_1, S_2, \ldots, S_d]$$

and S which does not lie on (S_1^2) the tangent $[d-1]$'s at Q to the varieties of $|S_2|, \ldots, |S_d|$ which pass through Q meet the tangent $[d-1]$ to S at Q in the tangent $[d-2]$'s to $|\Sigma_2|, \ldots, |\Sigma_d|$ at Q, and these $[d-2]$'s meet in the common tangent line to the varieties of $|S_1|, |S_2|, \ldots, |S_d|$ which pass through Q. The locus of Q is therefore the Jacobian variety

$$J_{d-2}[\Sigma_2, \Sigma_3, \ldots, \Sigma_d]$$

of the pencils $|\Sigma_j|$. But, by the inductive hypothesis,

$$J_{d-2}[\Sigma_2, \Sigma_3, \ldots, \Sigma_d] \equiv X_{d-2}[S]+2(\Sigma_2+\Sigma_3+\ldots+\Sigma_d),$$

and hence

$$(S \,.\, J_{d-1}[S_1, S_2, \ldots, S_d]) \equiv (S_1^2)+X_{d-2}[S]+2(S \,.\, S_2+S_3+\ldots+S_d)$$
$$\equiv (S \,.\, X_{d-1}+2[S_1+S_2+\ldots+S_d])$$

by (2), the equivalences being on S.

The right and left sides of (3) thus cut out equivalent V_{d-2}'s on each variety S of the pencil $|S_1|$, and so by a well-known criterion of linear equivalence† the two sides of (3) are linearly equivalent on V_d, which proves the theorem.

2. In seeking to extend these results, and to define invariant systems of lower dimension h on V_d I considered, in the previous paper, the Jacobian variety of a linear system of freedom $h+1$, when $h+1<d$. Eger, on the other hand, considers the Jacobian variety of $h+1$ linear pencils, and in the present paper we follow Eger. Our proof is inductive; we assume the existence of the system of equivalence $\{X_h\}$, for a fixed value of h, to be established on varieties of dimension less than d, and deduce the existence of the corresponding system on V_d. The "first case" of the induction is that in which $d = h+1$, and the existence of the canonical system in this case has just been established above.

It is worthy of remark that the geometrical argument which we use is exactly parallel to a familiar argument‡ which leads to the existence of the Zeuthen-Segre invariant of a surface—which is indeed merely the particular case $h = 0$, $d = 2$. This point seems to be worth stressing, since in most of the current accounts available of the algebro-geometric treatment of the theory of algebraic surfaces this invariant appears as an isolated phenomenon with no very clear relation to the general theory. The present paper shows, I think, that it has a very real place in the organic scheme of development of the theory of invariants.

It will be convenient for formal purposes to introduce a set of linear operators K_n, due essentially to Eger, whose field of operation consists of all proper algebraic subvarieties of V_d. These operators are defined by the

† See, *e.g.*, O. Zariski, *Algebraic surfaces* (Berlin, 1935), 89. The proof given there for the case $d = 2$ extends immediately to varieties of higher dimension.

‡ See H. F. Baker, *Principles of geometry*, 6 (Cambridge, 1933), 207.

condition of linearity, and by their effects on varieties lying on V_d which are as follows:

$$\begin{aligned} K_n[V_t] &= X_n[V_t] \quad (t > n), \\ &= V_t \qquad\quad (t = n), \\ &= 0 \qquad\quad\; (t < n). \end{aligned} \tag{4}$$

3. We suppose henceforward that $h < d-1$. We consider $h+1$ irreducible linear pencils $|S_i|$ $(i = 0, 1, \ldots, h)$ on V_d, and define their Jacobian variety $J_h[S_0, S_1, \ldots, S_h]$ of h dimensions as follows. When $h = 0$ the variety consists of the set of double points of members of the pencil $|S_0|$; when $h > 0$ it consists of the points P of V_d such that the $h+1$ tangent $[d-1]$'s to the members of the pencils which pass through P have a common $[d-h]$.

Let S be a particular variety of $|S_0|$, and let (S_0^2) be the base V_{d-2} of the pencil. Denote by $|\Sigma_j|$ and $|\sigma_j|$ the pencils cut out by $|S_j|$ on S and (S_0^2) respectively, for $j = 1, 2, \ldots, h$. We prove that

$$(S \,.\, J_h[S_0, S_1, \ldots, S_h]) = J_{h-1}[\Sigma_1, \ldots, \Sigma_h] + J_{h-1}[\sigma_1, \ldots, \sigma_h]. \tag{5}$$

The sign of equality is here intentional, implying that the intersection does in fact break up into the two varieties figuring on the right of (5).

The proof of (5) is immediate. Let P be a point common to (S_0^2) and $J_h[S_0, S_1, \ldots, S_h]$. The tangent $[d-1]$'s at P to the varieties of

$$|S_1|, \ldots, |S_h|$$

which pass through P meet in a $[d-h]$. Since P lies on $J_h[S_0, S_1, \ldots, S_h]$ this $[d-h]$ lies in the tangent $[d-1]$ to some member of $|S_0|$ at P and therefore meets in a $[d-h-1]$ the $[d-2]$ common to the ∞^1 tangent $[d-1]$'s to the varieties of $|S_0|$ at P (all of which lie in the tangent $[d]$ to V_d at P). Thus the tangent $[d-3]$'s to the varieties of $|\sigma_1|, \ldots, |\sigma_h|$ which pass through P meet in a $[d-h-1]$, and the locus of P is $J_{h-1}[\sigma_1, \ldots, \sigma_h]$. On the other hand, at a point Q common to S and $J_h[S_0, S_1, \ldots, S_h]$ which does not lie on (S_0^2), the tangent $[d-1]$'s to the varieties of $|S_1|, \ldots, |S_h|$ which pass through Q meet in a $[d-h]$ lying in the tangent $[d-1]$ to S at Q, and hence the tangent $[d-2]$'s to the members of $|\Sigma_1|, \ldots, |\Sigma_h|$ which pass through Q meet in a $[d-h]$. Thus the locus of Q is $J_{h-1}[\Sigma_1, \ldots, \Sigma_h]$, and (5) follows at once.

4. We can, however, go further than this. At a point P of $J_{h-1}[\sigma_1, \ldots, \sigma_h]$ there is just one variety, S_P say, of the pencil $|S_0|$ whose tangent $[d-1]$

at P contains the $[d-h]$ common to the tangent $[d-1]$'s at P of the varieties of the pencils $|S_1|, \ldots, |S_h|$ which pass through P. We prove that this $[d-1]$ contains the tangent $[h]$ to $J_h[S_0, S_1, \ldots, S_h]$ at P.

This may be seen by the following differential argument, though a direct synthetic proof is doubtless possible. Since V_d is an analytic non-singular variety we can represent the points of V_d in a suitable neighbourhood of P by the values of d parameters $u_1, u_2, \ldots, u_d$ in such a way that P corresponds to the set of values $(0, 0, \ldots, 0)$ and that the coordinates of all points of V_d in the neighbourhood are analytic functions of the u's. Since P is supposed not to belong to the base-locus of any of the pencils $|S_1|, \ldots, |S_h|$, these parameters can be chosen so that the varieties of these pencils are given respectively, near P, by $u_1=c_1, \ldots, u_h=c_h$, where the c's are constants vanishing for the particular varieties passing through P. Since P lies on (S_0^2) the varieties of $|S_0|$ will be given near P by $\phi(u_1, \ldots, u_d) = c$ where c is a constant and ϕ is a function analytic near P which is indeterminate at the origin. Since $h < d$ the choice of u_d is certainly at our disposal, and there is therefore no loss of generality in supposing that this equation takes the form $cu_d = f(u_1, \ldots, u_d)$, where f is a power series in the u's vanishing at the origin.

The components du_i of any vector common to the tangent spaces at P to the varieties $u_1 = 0, \ldots, u_h = 0$ satisfy

$$du_1 = du_2 = \ldots = du_h = 0, \tag{6}$$

while the components of a vector lying in the tangent space at P to

$$cu_d = f(u_1, \ldots, u_d)$$

satisfy the single relation

$$\sum_{i=1}^{d-1} \left(\frac{\partial f}{\partial u_i}\right)_0 du_i + \left[\left(\frac{\partial f}{\partial u_d}\right)_0 - c\right] du_d = 0, \tag{7}$$

the suffix indicating that the derivatives are evaluated at the origin.

Now the Jacobian of the $h+1$ pencils is given by the vanishing of all $h+1$ rowed determinants of the Jacobian matrix

$$\left\| \frac{\partial(u_1, u_2, \ldots, u_h, f/u_d)}{\partial(u_1, u_2, \ldots, u_d)} \right\|,$$

a matrix which reduces to

$$\left\| \begin{matrix} I_h & 0_{d-h} & & \\ v_1 & v_2 & \ldots & v_d \end{matrix} \right\|.$$

where I_h is the unit matrix of h rows and columns, 0_{d-h} is a block of zeros of h rows and $d-h$ columns, and

$$v_i = \frac{\partial}{\partial u_i}\left(\frac{f}{u_d}\right) \quad (i = 1, \ldots, d).$$

Thus the equations of the Jacobian are given, near P, by

$$v_{h+1} = v_{h+2} = \ldots = v_d = 0,$$

i.e. (neglecting the inessential factor u_d) by

$$\left.\begin{aligned} \frac{\partial f}{\partial u_i} &= 0 \quad (i = h+1, h+2, \ldots, d-1), \\ u_d \frac{\partial f}{\partial u_d} - f &= 0. \end{aligned}\right\} \tag{8}$$

Since P lies on this Jacobian it follows that

$$\left(\frac{\partial f}{\partial u_i}\right)_0 = 0 \quad (i = h+1, h+2, \ldots, d-1). \tag{9}$$

Hence, if we choose $c = \left(\frac{\partial f}{\partial u_d}\right)_0$, the left-hand side of (7) reduces to

$$\sum_{i=1}^{h} \left(\frac{\partial f}{\partial u_i}\right)_0 du_i = 0, \tag{10}$$

which is satisfied by any vector satisfying (6). Thus this value of c gives the variety S_P, and (10) is the equation satisfied by all vectors in the tangent $[d-1]$ to S_P at P.

On the other hand, a vector du_i lying in the tangent $[h]$ to

$$J_h[S_0, S_1, \ldots, S_h]$$

at P, whose equations are given by (8), satisfies

$$\sum_{j=1}^{d} \left(\frac{\partial^2 f}{\partial u_i \partial u_j}\right)_0 du_j = 0 \quad (i = h+1, \ldots, d-1)$$

and

$$\left(\frac{\partial f}{\partial u_d}\right)_0 du_d - \sum_{i=1}^{d} \left(\frac{\partial f}{\partial u_i}\right)_0 du_i = 0,$$

that is, when (9) is taken into account,

$$\sum_{i=1}^{h} \left(\frac{\partial f}{\partial u_i}\right)_0 du_i = 0,$$

which is just (10). Thus all such vectors lie in the tangent space to S_P at P, which is our theorem.

5. With these preliminaries we can now proceed to our main result. Let X_h denote the variety (possibly virtual) defined by

$$X_h \equiv J_h[S_0, S_1, \ldots, S_h] - K_h\left[\prod_{i=0}^{h}(1+S_i)^2 - 1\right], \tag{11}$$

where K_h is the operator defined in § 2, and where the expression operated on by K_h stands for its formal expansion, in which products are interpreted as intersections and meaningless symbols (intersections whose virtual dimension is negative) are replaced by zero. We shall prove that X_h defines a system of equivalence $\{X_h\}$ on V_d which is independent of the pencils $|S_i|$.

The proof is by induction on d. When $d = h+1$ (11) reduces to (3), as is easily seen by noticing that in $\left[\prod_{i=0}^{h}(1+S_i)^2 - 1\right]$ the only varieties of dimension h are $2(S_0 + \ldots + S_h)$ and all varieties of lower dimension are annihilated by the operator K_h. We therefore assume our theorem to be true for varieties of dimension less than d, and prove it for varieties of dimension d. The proof falls into two parts.

6. In the first place we must prove that (11) does actually define a system of equivalence on V_d. To prove this it is clearly sufficient to prove that, if $T_1, \ldots, T_r$ are r V_{d-1}'s on V which can vary in irreducible linear systems $|T_1|, |T_2|, \ldots, |T_r|$ (not necessarily all different), then, as they vary, the variety $X_h[T_1 T_2 \ldots T_r]$ moves in a system of equivalence.

The theorem is true trivially when $d = h+1$, since the only case which arises then is that in which $r = 1$, and by definition, if T is a V_h on V_{h+1}, $X_h[T] = T$. We assume as an inductive hypothesis that the theorem is valid on varieties of dimension less than d, and prove it for varieties of dimension d.

Consider first the case in which $r = 1$, and let T, T' denote two members of the linear system $|T_1|$. We consider the $h+1$ pencils $|S_i|$ on V_d and let

$$|\Sigma_i|, \ |\Sigma_i'|$$

denote the pencils cut out by $|S_i|$ on T and T' respectively. By (11), applied to T and T' (of dimension $d-1$),

$$X_h[T] \equiv J_h[\Sigma_0, \Sigma_1, \ldots, \Sigma_h] - K_h\left[\prod_{i=0}^{h}(1+\Sigma_i)^2 - 1\right] \qquad (\text{on } T),$$

$$X_h[T'] \equiv J_h[\Sigma_0', \Sigma_1', \ldots, \Sigma_h'] - K_h\left[\prod_{i=0}^{h}(1+\Sigma_i')^2 - 1\right] \quad (\text{on } T'),$$

where, since we are assuming our main theorem for varieties of dimension less than d, the expressions in the second term on the right of these equivalences are independent of the particular varieties of the pencils $|\Sigma_i|$ or $|\Sigma_i'|$ which we use. We can thus suppose that Σ_i' is cut out on T' by the same member of $|S_i|$ which cuts out Σ_i on T. But then, if $\Sigma_a \Sigma_b \ldots \Sigma_c$ is any term appearing in the expansion,

$$X_h[\Sigma_a \Sigma_b \ldots \Sigma_c] = X_h[S_a S_b \ldots S_c T],$$

$$X_h[\Sigma_a' \Sigma_b' \ldots \Sigma_c'] = X_h[S_a S_b \ldots S_c T'],$$

and $(S_a S_b \ldots S_c T)$ and $(S_a S_b \ldots S_c T')$ are equivalent varieties on $(S_a S_b \ldots S_c)$. Hence, applying our inductive hypothesis, since $(S_a S_b \ldots S_c)$ has dimension less than or equal to $d-1$,

$$X_h[\Sigma_a \Sigma_b \ldots \Sigma_c] \equiv X_h[\Sigma_a' \Sigma_b' \ldots \Sigma_c']$$

on $(S_a S_b \ldots S_c)$ and therefore on V_d. So

$$K_h\left[\prod_{i=0}^{h} (1+\Sigma_i)^2 - 1\right] \equiv K_h\left[\sum_{i=0}^{h} (1+\Sigma_i')^2 - 1\right] \quad \text{on} \quad V_d.$$

To prove our result when $r = 1$ it is therefore necessary only to show that

$$J_h[\Sigma_0, \Sigma_1, \ldots, \Sigma_h] \equiv J_h[\Sigma_0', \Sigma_1', \ldots, \Sigma_h'].$$

Now, since $T \equiv T'$, there is a linear pencil $|T^*|$ contained in $|T_1|$ to which T and T' belong. If $|\sigma_i|$ is the pencil cut out by $|S_i|$ on the base of T^*, then, by (5),

$$J_h[\Sigma_0, \Sigma_1, \ldots, \Sigma_h] \equiv (T \,.\, J_{h+1}[T^*, S_0, S_1, \ldots, S_h]) - J_h[\sigma_0, \sigma_1, \ldots, \sigma_h].$$

As T varies in $|T^*|$ the second variety on the right remains fixed while the first describes a linear pencil on $J_{h+1}[T^*, S_0, S_1, \ldots, S_h]$. Hence

$$J_h[\Sigma_0, \Sigma_1, \ldots, \Sigma_h]$$

varies in a system of equivalence, which proves the result.

The extension to the case $r > 1$ is trivial. For instance, if $T_1 \equiv T_1'$ and $T_2 \equiv T_2'$, then

$$X_h[T_1 T_2] \equiv X_h[T_1 T_2'] \qquad (\text{on } T_1)$$

and

$$X_h[T_1 T_2'] \equiv X_h[T_1' T_2'] \qquad (\text{on } T_2'),$$

so that $$X_h[T_1 T_2] \equiv X_h[T_1' T_2'] \qquad (\text{on } V_d),$$

since equivalence on a subvariety implies equivalence on V_d, and the equivalence relation is transitive. The procedure when $r > 2$ is clear.

We have thus shown that (11) does define a system of equivalence. The next, and more important step, is to prove that this system is invariant.

7. The proof of the invariance of the system defined in (11) follows very closely that of the existence of the Zeuthen-Segre invariant on a surface. We introduce a further linear pencil S_{h+1}, and for convenience make the following definitions:

$$J = J_{h+1}[S_0, S_1, \ldots, S_h, S_{h+1}];$$

$$\Sigma_{ij} = (S_i S_j), \quad \sigma_{ij} = (S_i^2 S_j) \quad (i \neq j);$$

$$\Delta_i = (J S_i);$$

j_i and γ_i are the Jacobian varieties of the pencils $|\Sigma_{ij}|$, $|\sigma_{ij}|$ $(j \neq i)$ cut by $|S_j|$ on a particular variety S_i of $|S_i|$ and on (S_i^2) respectively; they are each of dimension h.

Since (11) is assumed for varieties of dimension less than d, it holds for S_i and (S_i^2). Hence

$$j_i \equiv X_h[S_i] + K_h \left[\prod_{j=0}^{h+1}{}' (1+\Sigma_{ij})^2 - 1 \right],$$

the intersections being evaluated on S_i and the dash indicating that the factor for which $j = i$ does not occur in the product. Thus

$$j_i \equiv K_h \left[S_i + S_i \left\{ \prod_{j=0}^{h+1}{}' (1+S_j)^2 - 1 \right\} \right]$$

$$\equiv K_h \left[S_i \prod_{j=0}^{h+1}{}' (1+S_j)^2 \right]. \tag{12}$$

Similarly $$\gamma_i \equiv K_h \left[S_i^2 \prod_{j=0}^{h+1}{}' (1+S_j)^2 \right]. \tag{13}$$

By (5), $$\Delta_i = (J S_i) \equiv j_i + \gamma_i. \tag{14}$$

Consider now the Jacobian locus Γ of the $h+1$ pencils Δ_i $(i = 0, 1, \ldots, h)$ on J. By (3),

$$\Gamma \equiv X_h[J] + 2 \sum_{i=0}^{h} \Delta_i. \tag{15}$$

Points of this locus arise in four different ways.

(i) Each point of γ_i (for $i = 0, 1, \ldots, h$) is a fixed part of the corresponding pencil Δ_i. It appears clearly from (3) that every such fixed part separates out twice from the Jacobian locus.

(ii) At any point $J_h[S_0, S_1, \ldots, S_h]$ the sections of the tangent $[d-1]$'s to the corresponding varieties of the pencils $|S_0|, |S_1|, \ldots, |S_h|$ by the tangent $[h+1]$ to J are the tangent $[h]$'s to the varieties of the pencils

$$|\Delta_0|, \quad |\Delta_1|, \quad \ldots, \quad |\Delta_h|$$

which pass through the point, and these $[h]$'s meet in a line, the section by the tangent $[h+1]$ to J of the $[d-h]$ in which the $h+1$ $[d-1]$'s meet. Thus $J_h[S_0, S_1, \ldots, S_h]$ belongs to Γ.

(iii) The variety Γ also contains γ_{h+1}. For the reasoning of § 4 shows that at any point P of γ_{h+1} the $[d-h-1]$ of intersection of the tangent spaces to the varieties of $|S_0|, |S_1|, \ldots, |S_h|$ which pass through P lies in the tangent $[d-1]$ at P to a definite member of the pencil $|S_{h+1}|$, and that the tangent $[h+1]$ to J at P lies in this $[d-1]$. It thus meets the $[d-h-1]$ in a line common to the tangent spaces of the varieties of $|\Delta_0|, |\Delta_1|, \ldots, |\Delta_h|$ passing through P, and hence P is a point of Γ. (The reader may be reminded that only $h+1$ pencils were involved in § 4 as compared with $h+2$ here: this accounts for the difference in the dimensions of the intersections involved.)

(iv) At any point of Γ distinct from the foregoing there is a distinct tangent space to each of the varieties of $|S_0|, |S_1|, \ldots, |S_{h+1}|$ which pass through the point, these spaces having a common $[d-h-1]$. The tangent $[d-1]$'s to the varieties of $|S_0|, |S_1|, \ldots, |S_h|$ through the point have this $[d-h-1]$ as their complete intersection, but since the point lies on Γ the section of this $[d-h-1]$ by the tangent $[h+1]$ to J is a line. The locus Λ of such points may thus be defined as the locus of those points of J at which the $[d-h-1]$ common to the tangent spaces of the $h+2$ pencils meets the tangent $[h+1]$ of J in a line. It is a locus of contacts of the pencils of some special nature, but its only importance from our point of view is that *it is symmetrically related to the $h+2$ pencils $|S_0|, |S_1|, \ldots, |S_{h+1}|$*. We have, accordingly,

$$\Gamma = \Lambda + J_h[S_0, S_1, \ldots, S_h] + 2\sum_{i=1}^{h} \gamma_i + \gamma_{h+1}. \tag{16}$$

From (12), (13), (14), (15), and (16) it now follows that

$$J_h[S_0, S_1, \ldots, S_h]$$
$$\equiv \Gamma - 2\sum_{i=0}^{h} \gamma_i - \gamma_{h+1} - \Lambda$$
$$\equiv X_h[J] + 2\sum_{i=0}^{h} j_i - \gamma_{h+1} - \Lambda$$
$$\equiv X_h[J] - \Lambda + K_h\left[2\sum_{i=0}^{h} S_i \prod_{j=0}^{h+1}{}' (1+S_j)^2 - S_{h+1}^2 \prod_{j=0}^{h} (1+S_j)^2\right],$$

and so

$$J_h[S_0, S_1, \ldots, S_h] - K_h\left[\prod_{i=0}^{h} (1+S_i)^2 - 1\right]$$
$$\equiv X_h[J] - \Lambda + K_h\left[2\sum_{i=0}^{h} S_i \prod_{j=0}^{h+1}{}' (1+S_j)^2 - S_{h+1}^2 \prod_{j=0}^{h} (1+S_j)^2 - \prod_{j=0}^{h} (1+S_j)^2 + 1\right]$$
$$\equiv X_h[J] - \Lambda + K_h\left[2\sum_{i=0}^{h+1} S_i \prod_{j=0}^{h+1}{}' (1+S_j)^2 - \prod_{j=0}^{h+1} (1+S_j)^2 + 1\right]. \tag{17}$$

The right-hand side of (17) is symmetrical in $S_0, S_1, \ldots, S_{h+1}$. But the left-hand side does not involve the arbitrary auxiliary pencil $|S_{h+1}|$. Hence the left-hand side of (17) is independent of the pencils $|S_0|, |S_1|, \ldots, |S_h|$, *i.e.* the system $\{X_h\}$ given by (11) is an invariant system on V_d. This completes the proof of the theorem.

We may note that if we define

$$K_h[1] = X_h[V_d], \tag{18}$$

then (11) can be written more concisely

$$J_h[S_0, S_1, \ldots, S_d] \equiv K_h\left[\prod_{i=0}^{h} (1+S_i)^2\right], \tag{19}$$

which is the form obtained by Eger in the first of the notes already cited.

8. We can now show that the invariant systems thus defined possess the *general property of adjunction* expressed by the relation

$$X_{h-1}[S] \equiv (S \,.\, X_h[S] + X_h[V_d]) \quad (\text{on } S), \tag{20}$$

where $h > 0$ and where S is an irreducible V_{d-1}. We suppose first that S belongs to an irreducible linear pencil $|S_0|$. Since (20) reduces to (2) when $d = h+1$ we assume the result to hold on varieties of dimension less than d, and prove it for varieties of dimension d.

We take h additional irreducible linear pencils $|S_i|$ on V_d, $(i = 1, 2, \ldots, h)$ and denote by $|\Sigma_i|$, $|\sigma_i|$ the pencils cut out by $|S_i|$ on S and on the base (S_0^2) of $|S_0|$ respectively. Then, by the inductive hypothesis, if T is an irreducible algebraic subvariety of $k(\leqslant d-1)$ dimensions lying on V_d,

$$X_{h-1}[ST] \equiv (S \,.\, X_h[ST] + X_h[T]),$$

that is,

$$K_{h-1}[ST] \equiv (S \,.\, K_h[(1+S)T]). \tag{21}$$

Now, from (13),

$$J_{h-1}[\sigma_1, \sigma_2, \ldots, \sigma_h] \equiv K_{h-1}\left[\prod_{i=1}^{h} (1+\sigma_i)^2\right] \equiv K_{h-1}\left[S^2 \prod_{i=1}^{h} (1+S_i)^2\right]. \tag{22}$$

Hence, from (11),

$$\begin{aligned} X_{h-1}[S] &\equiv J_{h-1}[\Sigma_1, \Sigma_2, \ldots, \Sigma_h] - K_{h-1}\left[\prod_{i=1}^{h} (1+\Sigma_i)^2 - 1\right] \\ &\equiv (S \,.\, J_h[S_0, S_1, \ldots, S_h]) - J_{h-1}[\sigma_1, \sigma_2, \ldots, \sigma_h] \\ &\qquad\qquad - K_{h-1}\left[S \prod_{i=1}^{h} (1+S_i)^2 - S\right] \quad \text{by (5)} \\ &\equiv (S \,.\, J_h[S_0, S_1, \ldots, S_h]) - K_{h-1}\left[S(1+S) \prod_{i=1}^{h} (1+S_i)^2 - S\right] \quad \text{by (22)} \\ &\equiv (S \,.\, J_h[S_0, S_1, \ldots, S_h]) \\ &\qquad - \left(S \,.\, K_h\left[(1+S) \,.\, \left\{(1+S) \prod_{i=1}^{h} (1+S_i)^2 - 1\right\}\right]\right) \quad \text{by (21)} \\ &\equiv \left(S \,.\, J_h[S_0, S_1, \ldots, S_h] - K_h\left[\prod_{i=0}^{h} (1+S_i)^2 - 1 - S\right]\right) \\ &\equiv (S \,.\, X_h[S] + X_h[V_d]) \quad \text{by (11)}. \end{aligned}$$

This proves the theorem stated when S belongs to an irreducible linear pencil.

Now let S and T be two V_{d-1}'s on V_d belonging to irreducible linear systems $|S|$ and $|T|$ of freedom greater than zero satisfying the following conditions:

(i) The linear system $|S+T|$ is irreducible,

(ii) The linear systems cut by $|T|$ on a general S, and by $|S|$ on a general T, are irreducible and have freedom greater than zero,

(iii) The linear system cut by $|S+T|$ on the intersection (ST) of a general S and a general T is irreducible and has freedom greater than zero.

We prove that

$$X_h[S+T] \equiv X_h[S]+X_h[T]+2X_h[ST]+X_h[ST(S+T)]. \qquad (23)$$

This is true trivially when $h=d-1$, since then the last two terms on the right are zero. We assume the theorem to hold for canonical systems of dimension $h+1$. Conditions (i), (ii), and (iii) are sufficient to ensure that (20) can be applied to S, T, $S+T$, on V_d, to (ST) considered as a locus lying on S or T, and to $ST(S+T)$ considered as a locus lying on (ST). Hence, using the inductive hypothesis,

$$\begin{aligned} X_h[S+T] &\equiv (S+T\,.\,X_{h+1}[S+T]+X_{h+1}[V_d]) \\ &\equiv (S+T\,.\,X_{h+1}[S]+X_{h+1}[T]+2X_{h+1}[ST] \\ &\qquad\qquad +X_{h+1}[ST(S+T)]+X_{h+1}[V_d]) \\ &\equiv (S\,.\,X_{h+1}[S]+X_{h+1}[V_d])+(T\,.\,X_{h+1}[T]+X_{h+1}[V_d]) \\ &\qquad +(S\,.\,X_{h+1}[T]+X_{h+1}[ST])+(T\,.\,X_{h+1}[S]+X_{h+1}[ST]) \\ &\qquad +\big(S+T\,.\,X_{h+1}[ST]+X_{h+1}[ST(S+T)]\big) \\ &\equiv X_h[S]+X_h[T]+(ST\,.\,X_{h+1}[T]+X_{h+1}[ST])_T \\ &\qquad +(ST\,.\,X_{h+1}[S]+X_{h+1}[ST])_S \\ &\qquad +\big(ST(S+T)\,.\,X_{h+1}[ST]+X_{h+1}[ST(S+T)]\big)_{(ST)} \end{aligned}$$

[the suffixes indicating that the intersections are evaluated on S, T, (ST) respectively], and the right-hand side is consequently

$$X_h[S]+X_h[T]+2X_h[ST]+X_h[ST(S+T)],$$

which proves (23).

We now use (23) to *define* $X_h[S+T]$ even when $|S+T|$ is reducible. We may also use (23) to define $X_h[S]$ on a *virtual* variety S of $d-1$ dimensions. To do this we take an effective variety U such that $T\equiv U-S$ is effective, and define

$$X_h[S] \equiv X_h[U]-X_h[T]-2X_h[ST]-X_h[STU].$$

The varieties (ST) and (STU) may be virtual, but their dimension is less than $d-1$, and they lie on the effective varieties T, (TU) respectively. If they are in fact virtual their canonical systems may be defined in the same manner. In any event, after a finite number of steps only canonical systems of effective varieties appear on the right, since when the dimension of a variety is less than h its canonical V_h does not exist†.

It follows from the proof of (23), and from Macpherson's result just cited, that the system $X_h(S)$ defined on *any* S, reducible or irreducible, effective or virtual, satisfies the relation (20). The system

$$A_h[S] \equiv X_h[S] + X_h[V_d] \tag{24}$$

we call the *adjoint* system to S, of dimension h, and we clearly have

$$X_{h-1}[S] \equiv (S \,.\, A_h[S]). \tag{25}$$

9. We have finally to identify the invariant system $\{X_h\}$ just considered with that defined in the previous paper in terms of the Jacobian of a linear ∞^{h+1} system. To do this it is sufficient to show‡ that the Jacobian $J_h[S]$ of a sufficiently general linear ∞^{h+1} system $|S|$ is given by

$$J_h[S] \equiv \sum_{i=0}^{h+2} \binom{h+2}{i} X_h[S^i]$$

that is, symbolically,

$$J_h[S] \equiv K_h[(1+S)^{h+2}]. \tag{26}$$

As Eger remarks, this may be deduced by limiting considerations from (19). It is convenient to establish the more general result§ that the Jacobian of a linear ∞^{r+1} system $|S|$ and of $h-r$ other linear pencils $|S_1|, \ldots, |S_{h-r}|$ is given by

$$J_h[S;\ S_1, \ldots, S_{h-r}] \equiv K_h\left[(1+S)^{r+2} \prod_{i=1}^{h-r} (1+S_i)^2\right]. \tag{27}$$

The proof of (27) proceeds by induction on r, since it reduces to (19) when $r=0$. When $h=d-1$, (27) is easily proved by considerations similar to those employed in the proof of (3). We suppose then that (27) has been proved for canonical systems of dimension greater than h on V_d, and prove it for the system $\{X_h\}$.

We consider a linear ∞^{r+2} system $|T|$ containing $|S|$, and a linear ∞^r system $|U|$ contained in $|S|$, and consider a general pencil $|S_0|$ lying in

† In a note appearing in *Proc. Cambridge Phil. Soc.*, 35 (1939), 389–393, R. E. Macpherson has extended (23) to the case in which S is an isolated variety, and has shown directly that the relation of adjunction (20) is valid in this case also.

‡ *C.f.* equation (5.1) of the previous paper.

§ Eger, *loc. cit. ante*.

$|T|$. The Jacobian J_{h+1} of T, S_1, ..., S_{h-r} is of $h+1$ dimensions and contains $J_h[U; S_0, S_1, ..., S_{h-r}]$, and as $|S_0|$ varies in T this locus describes a linear system on J_{h+1}. When $|S_0|$ lies in $|S|$ it has a member S^* in common with $|U|$. As $|S_0|$ tends to the limiting position in which it lies in $|S|$ the variety $J_h[U; S_0, S_1, ..., S_{h-1}]$ breaks up into $J_h[S; S_1, ..., S_{h-r}]$ and a residual variety which is easily seen to be the intersection of J_{h+1} and S^*. Thus

$$J_h[U; S_0, S_1, ..., S_{h-r}] \equiv J_h[S; S_1, ..., S_{h-r}] + (S \,.\, J_{h+1}[T; S_1 ..., S_{h-r}]).$$

Hence by the inductive hypotheses, on r and h,

$$J_h[S; S_1, ..., S_{h-r}]$$

$$\equiv K_h\left[(1+S)^{r+1}(1+S)^2 \prod_{i=1}^{h-r}(1+S_i)^2\right] - \left(S \,.\, K_{h+1}\left[(1+S)^{r+3}\prod_{i=1}^{h-r}(1+S_i)^2\right]\right)$$

$$\equiv K_h\left[(1+S)^{r+3}\prod_{i=1}^{h-r}(1+S_i)^2 - S(1+S)^{r+2}\prod_{i=1}^{h-r}(1+S_i)^2\right] \quad \text{by} \quad (21)$$

$$\equiv K_h\left[(1+S)^{r+2}\prod_{i=1}^{h-r}(1+S_i)^2\right],$$

which is (27). Putting $h = r$ in (27) we obtain (26). The identity of the systems considered here with those previously obtained is therefore established.

Trinity College,
Cambridge.