Noncommutative localization in topology

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Introduction

The topological applications of the Cohn noncommutative localization considered in this paper deal with spaces (especially manifolds) with infinite fundamental group, and involve localizations of infinite group rings and related triangular matrix rings. Algebraists have usually considered noncommutative localization of rather better behaved rings, so the topological applications require new algebraic techniques.

Part 1 is a brief survey of the applications of noncommutative localization to topology: finitely dominated spaces, codimension 1 and 2 embeddings (knots and links), homology surgery theory, open book decompositions and circle-valued Morse theory. These applications involve chain complexes and the algebraic K- and L-theory of the noncommutative localization of group rings.

Part 2 is a report on work on chain complexes over generalized free products and the related algebraic K- and L-theory, from the point of view of noncommutative localization of triangular matrix rings. Following Bergman and Schofield, a generalized free product of rings can be constructed as a noncommutative localization of a triangular matrix ring. The novelty here is the explicit connection to the algebraic topology of manifolds with a generalized free product structure realized by a codimension 1 submanifold, leading to noncommutative localization proofs of the results of Waldhausen and Cappell on the algebraic K- and L-theory of generalized free products. In a sense, this is more in the nature of an application of topology to noncommutative localization! But this algebra has in turn topological applications, since in dimensions $\geqslant 5$ the surgery classification of manifolds within a homotopy type reduces to algebra.

Part 1. A survey of applications

We start by recalling the universal noncommutative localization of P.M.Cohn [5]. Let A be a ring, and let $\Sigma = \{s : P \to Q\}$ be a set of morphism of f.g. projective A-modules. A ring morphism $A \to R$ is Σ-*inverting* if for every $s \in \Sigma$ the induced morphism of f.g. projective R-modules $1 \otimes s : R \otimes_A P \to R \otimes_A Q$ is an isomorphism. The noncommutative localization $A \to \Sigma^{-1}A$ is Σ-inverting, and has the universal property that any Σ-inverting ring morphism $A \to R$ has a unique factorization $A \to \Sigma^{-1}A \to R$. The applications to topology involve homology with coefficients in a noncommutative localization $\Sigma^{-1}A$.

Homology with coefficients is defined as follows. Let X be a connected topological space with universal cover $\widetilde{X}$, and let the fundamental group $\pi_1(X)$ act on the left of $\widetilde{X}$, so that the (singular) chain complex $S(\widetilde{X})$ is a free left $\mathbb{Z}[\pi_1(X)]$-module complex. Given a morphism of rings $\mathcal{F} : \mathbb{Z}[\pi_1(X)] \to \Lambda$ define the Λ-*coefficient homology of* X to be

$$H_*(X;\Lambda) \;=\; H_*(\Lambda \otimes_{\mathbb{Z}[\pi_1(X)]} S(\widetilde{X}))\ .$$

If X is a CW complex then $S(\widetilde{X})$ is chain equivalent to the cellular free $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\widetilde{X})$ with one generator in degree r for each r-cell of X, and

$$H_*(X;\Lambda) \;=\; H_*(\Lambda \otimes_{\mathbb{Z}[\pi_1(X)]} C(\widetilde{X}))\ .$$

1.1 Finite domination

A topological space X is *finitely dominated* if there exist a finite CW complex K, maps $f : X \to K$, $g : K \to X$ and a homotopy $gf \simeq 1 : X \to X$. The finiteness obstruction of Wall [31] is a reduced projective class $[X] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ such that $[X] = 0$ if and only if X is homotopy equivalent to a finite CW complex.

In the applications of the finiteness obstruction to manifold topology $X = \overline{M}$ is an infinite cyclic cover of a compact manifold M – see Chapter 17 of Hughes and Ranicki [13] for the geometric wrapping up procedure which shows that in dimension $\geqslant 5$ every tame manifold end has a neighbourhood which is a finitely dominated infinite cyclic cover $\overline{M}$ of a compact manifold M. Let $f : M \to S^1$ be a classifying map, so that $\overline{M} = f^*\mathbb{R}$, and let $\overline{M}^+ = f^*\mathbb{R}^+$. The finiteness obstruction $[\overline{M}^+] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\overline{M})])$ is the end

obstruction of Siebenmann [27], such that $[\overline{M}^+] = 0$ if and only if the tame end can be closed, i.e. compactified by a manifold with boundary.

Given a ring A let Ω be the set of square matrices $\omega \in M_r(A[z,z^{-1}])$ over the Laurent polynomial extension $A[z,z^{-1}]$ such that the A-module

$$P \;=\; \mathrm{coker}(\omega : A[z,z^{-1}]^r \to A[z,z^{-1}]^r)$$

is f.g. projective. The noncommutative Fredholm localization $\Omega^{-1}A[z,z^{-1}]$ has the universal property that a finite f.g. free $A[z,z^{-1}]$-module chain complex C is A-module chain equivalent to a finite f.g. projective A-module chain complex if and only if $H_*(\Omega^{-1}C) = 0$ (Ranicki [21, Proposition 13.9]), with $\Omega^{-1}C = \Omega^{-1}A[z,z^{-1}] \otimes_{A[z,z^{-1}]} C$.

Let M be a connected finite CW complex with a connected infinite cyclic cover $\overline{M}$. The fundamental group $\pi_1(M)$ fits into an extension

$$\{1\} \to \pi_1(\overline{M}) \to \pi_1(M) \to \mathbb{Z} \to \{1\}$$

and $\mathbb{Z}[\pi_1(M)]$ is a twisted Laurent polynomial extension

$$\mathbb{Z}[\pi_1(M)] \;=\; \mathbb{Z}[\pi_1(\overline{M})]_\alpha[z,z^{-1}]$$

with

$$\alpha \;:\; \pi_1(\overline{M}) \to \pi_1(\overline{M}) \;;\; g \mapsto z^{-1}gz$$

the monodromy automorphism. For the sake of simplicity only the untwisted case $\alpha = 1$ will be considered here, so that $\pi_1(M) = \pi_1(\overline{M}) \times \mathbb{Z}$. The infinite cyclic cover $\overline{M}$ is finitely dominated if and only if $H_*(M;\Omega^{-1}\mathbb{Z}[\pi_1(M)]) = 0$, with $A = \mathbb{Z}[\pi_1(\overline{M})]$ and $\mathbb{Z}[\pi_1(M)] = A[z,z^{-1}]$. The Farrell-Siebenmann obstruction $\Phi(M) \in \mathrm{Wh}(\pi_1(M))$ of an n-dimensional manifold M with finitely dominated infinite cyclic cover $\overline{M}$ is such that $\Phi(M) = 0$ if (and for $n \geqslant 6$ only if) M is a fibre bundle over S^1 – see [21, Proposition 15.16] for the expression of $\Phi(M)$ in terms of the $\Omega^{-1}\mathbb{Z}[\pi_1(M)]$-coefficient Reidemeister-Whitehead torsion

$$\tau(M;\Omega^{-1}\mathbb{Z}[\pi_1(M)]) \;=\; \tau(\Omega^{-1}C(\widetilde{M})) \in K_1(\Omega^{-1}\mathbb{Z}[\pi_1(M)])\;.$$

1.2 Codimension 1 splitting

Surgery theory asks whether a homotopy equivalence of manifolds is homotopic (or h-cobordant) to a homeomorphism – in general, the answer is no. There are obstructions in the topological K-theory of vector bundles, in the

algebraic K-theory of modules and in the algebraic L-theory of quadratic forms. The algebraic K-theory obstruction lives in the Whitehead group $Wh(\pi)$ of the fundamental group π. The L-theory obstruction lives in one of the surgery groups $L_*(\mathbb{Z}[\pi])$ of Wall [32], and is defined when the topological and algebraic K-theory obstructions vanish. The groups $L_*(\Lambda)$ are defined for any ring with involution Λ to be the generalized Witt groups of nonsingular quadratic forms over Λ. For manifolds of dimension $\geqslant 5$ the vanishing of the algebraic obstructions is both a necessary and sufficient condition for deforming a homotopy equivalence to a homeomorphism. See Ranicki [20] for the reduction of the Browder-Novikov-Sullivan-Wall surgery theory to algebra.

A homotopy equivalence of m-dimensional manifolds $f : M' \to M$ *splits* along a submanifold $N^n \subset M^m$ if f is homotopic to a map (also denoted by f) such that $N' = f^{-1}(N) \subset M'$ is also a submanifold, and the restriction $f| : N' \to N$ is also a homotopy equivalence. For codimension $m - n \geqslant 3$ the splitting obstruction is just the ordinary surgery obstruction $\sigma_*(f|) \in L_m(\mathbb{Z}[\pi_1(N)])$. For codimension $m - n = 1, 2$ the splitting obstructions involve the interplay of the knotting properties of codimension $(m - n)$ submanifolds and Mayer-Vietoris-type decompositions of the algebraic K- and L-groups of $\mathbb{Z}[\pi_1(M)]$ in terms of the groups of $\mathbb{Z}[\pi_1(N)]$, $\mathbb{Z}[\pi_1(M\backslash N)]$.

In the case $m - n = 1$ $\pi_1(M)$ is a generalized free product, i.e. either an amalgamated free product or an HNN extension, by the Seifert-van Kampen theorem. Codimension 1 splitting theorems and the algebraic K- and L-theory of generalized free products are a major ingredient of high-dimensional manifold topology, featuring in the work of Stallings, Browder, Novikov, Wall, Siebenmann, Farrell, Hsiang, Shaneson, Casson, Waldhausen, Cappell, ..., and the author. Noncommutative localization provides a systematic development of this algebra, using the intuition afforded by the topological applications – see Part 2 below for a more detailed discussion.

1.3 Homology surgery theory

For a morphism of rings with involution $\mathcal{F} : \mathbb{Z}[\pi] \to \Lambda$ Cappell and Shaneson [3] considered the problem of whether a Λ-coefficient homology equivalence of manifolds with fundamental group π is H-cobordant to a homeomorphism. Again, the answer is no in general, with obstructions in the topological K-theory of vector bundles and in the homology surgery groups $\Gamma_*(\mathcal{F})$, which are generalized Witt groups of Λ-nonsingular quadratic forms over

$\mathbb{Z}[\pi]$. Vogel [28], [29] identified the Λ-coefficient homology surgery groups with the ordinary L-groups of the localization $\Sigma^{-1}\mathbb{Z}[\pi]$ of $\mathbb{Z}[\pi]$ inverting the set Σ of Λ-invertible square matrices over $\mathbb{Z}[\pi]$

$$\Gamma_*(\mathcal{F}) \;=\; L_*(\Sigma^{-1}\mathbb{Z}[\pi])\ ,$$

and identified the relative L-groups $L_*(\mathbb{Z}[\pi] \to \Sigma^{-1}\mathbb{Z}[\pi])$ in the localization exact sequence

$$\cdots \to L_n(\mathbb{Z}[\pi]) \to L_n(\Sigma^{-1}\mathbb{Z}[\pi]) \to L_n(\mathbb{Z}[\pi] \to \Sigma^{-1}\mathbb{Z}[\pi]) \to L_{n-1}(\mathbb{Z}[\pi]) \to \dots$$

with generalized Witt groups $L_*(\mathbb{Z}[\pi], \Sigma)$ of nonsingular $\Sigma^{-1}\mathbb{Z}[\pi]/\mathbb{Z}[\pi])$-valued quadratic linking forms on Σ-torsion $\mathbb{Z}[\pi]$-modules of homological dimension 1.

1.4 Codimension 2 embeddings

Suppose given a codimension 2 embedding $N^n \subset M^{n+2}$ such as a knot or link. Let $\Sigma^{-1}A$ be the localization of $A = \mathbb{Z}[\pi_1(M\backslash N)]$ inverting the set Σ of matrices over A which become invertible over $\mathbb{Z}[\pi_1(M)]$. By Alexander duality the $\Sigma^{-1}A$-coefficient homology modules

$$H_*(M\backslash N; \Sigma^{-1}A) \;\cong\; H^{n+2-*}(M, N; \Sigma^{-1}A) \;\; (* \neq 0, n+2)$$

are determined by the homotopy class of the inclusion $N \subset M$. The A-coefficient homology groups $H_*(M\backslash N; A)$ and their Poincaré duality properties reflect more subtle invariants of $N \subset M$ such as knotting. See Ranicki [21] for a general account of high-dimensional codimension 2 embedding theory, including some of the applications of noncommutative localization.

1.5 Open books

An $(n+2)$-dimensional manifold M^{n+2} is an *open book* if there exists a codimension 2 submanifold $N^n \subset M^{n+2}$ such that the complement $M\backslash N$ is a fibre bundle over S^1. Every odd-dimensional manifold is an open book. Quinn [17] showed that for $k \geqslant 2$ a $(2k+2)$-dimensional manifold M is an open book if and only if an asymmetric form over $\mathbb{Z}[\pi_1(M)]$ associated to M represents 0 in the Witt group. This obstruction was identified in Ranicki [21] with an element in the L-group $L_{2k+2}(\Omega^{-1}\mathbb{Z}[\pi_1(M)][z, z^{-1}])$ of the Fredholm localization of $\mathbb{Z}[\pi_1(M)][z, z^{-1}]$ (cf. section 1.1 above).

1.6 Boundary link cobordism

An n-dimensional μ-component boundary link is a codimension 2 embedding

$$N^n \;=\; \bigcup_\mu S^n \subset M^{n+2} \;=\; S^{n+2}$$

with a μ-component Seifert surface, in which case the fundamental group of the complement $X = M\backslash N$ has a compatible surjection $\pi_1(X) \to F_\mu$ onto the free group on μ generators. Duval [8] used the work of Cappell and Shaneson [4] and Vogel [29] to identify the cobordism group of n-dimensional μ-component boundary links for $n \geqslant 2$ with the relative L-group $L_{n+3}(\mathbb{Z}[F_\mu], \Sigma)$ in the localization exact sequence

$$\cdots \to L_{n+3}(\mathbb{Z}[F_\mu]) \to L_{n+3}(\Sigma^{-1}\mathbb{Z}[F_\mu]) \to L_{n+3}(\mathbb{Z}[F_\mu], \Sigma) \to L_{n+2}(\mathbb{Z}[F_\mu]) \to \dots$$

with Σ the set of $\mathbb{Z}$-invertible square matrices over $\mathbb{Z}[F_\mu]$. The even-dimensional boundary link cobordism groups are $L_{2*+1}(\mathbb{Z}[F_\mu], \Sigma) = 0$. The cobordism class in $L_{2k+2}(\mathbb{Z}[F_\mu], \Sigma)$ of a $(2k-1)$-dimensional μ-component boundary link $\cup_\mu S^{2k-1} \subset S^{2k+1}$ was identified with the Witt class of a $\Sigma^{-1}\mathbb{Z}[F_\mu]/\mathbb{Z}[F_\mu]$-valued nonsingular $(-1)^{k+1}$-quadratic linking form on $H_k(X; \mathbb{Z}[F_\mu])$, generalizing the Blanchfield pairing on the homology of the infinite cyclic cover of a knot. The localization $\Sigma^{-1}\mathbb{Z}[F_\mu]$ was identified by Dicks and Sontag [7] and Farber and Vogel [11] with a ring of rational functions in μ noncommuting variables. The high odd-dimensional boundary link cobordism groups $L_{2*+2}(\mathbb{Z}[F_\mu], \Sigma)$ have been computed by Sheiham [26].

1.7 Circle-valued Morse theory

Novikov [15] proposed the study of the critical points of Morse functions $f : M \to S^1$ on compact manifolds M. The 'Novikov complex' $C(M, f)$ over $\mathbb{Z}((z)) = \mathbb{Z}[[z]][z^{-1}]$ has one generator for each critical point of f, and the 'Novikov homology'

$$H_*(C(M, f)) \;=\; H_*(M; \mathbb{Z}((z)))$$

provides lower bounds on the number of critical points of Morse functions in the homotopy class of f, generalizing the inequalities of the classical Morse theory of real-valued functions $M \to \mathbb{R}$. Suppose given a Morse function $f : M \to S^1$ with $\overline{M} = f^*\mathbb{R}$ such that $\pi_1(M) = \pi_1(\overline{M}) \times \mathbb{Z}$ (for the sake of simplicity). Let Σ be the set of square matrices over $\mathbb{Z}[\pi_1(\overline{M})][z]$ which

become invertible over $\mathbb{Z}[\pi_1(\overline{M})]$ under the augmentation $z \mapsto 0$. There is a natural morphism from the localization to the completion

$$\Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to \widehat{\mathbb{Z}[\pi_1(M)]} \;=\; \mathbb{Z}[\pi_1(\overline{M})][[z]][z^{-1}]$$

which is an injection if $\pi_1(M)$ is abelian or F_μ (Dicks and Sontag [7], Farber and Vogel [11]), but may not be an injection in general (Sheiham [25]). See Pajitnov [16], Farber and Ranicki [10], Ranicki [22], and Cornea and Ranicki [6] for the construction and properties of Novikov complexes of f over $\widehat{\mathbb{Z}[\pi_1(M)]}$ and $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$. Naturally, noncommutative localization also features in the more general Morse theory of closed 1-forms – see Novikov [15] and Farber [9].

1.8 3- and 4-dimensional manifolds

See Garoufalidis and Kricker [12], Quinn [18] for applications of noncommutative localization in the topology of 3- and 4-dimensional manifolds.

Part 2. The algebraic K- and L-theory of generalized free products via noncommutative localization

A generalized free product of groups (or rings) is either an amalgamated free product or an HNN extension. The expressions of Schofield [24] of generalized free products as noncommutative localizations of triangular matrix rings combine with the localization exact sequences of Neeman and Ranicki [14] to provide more systematic proofs of the Mayer-Vietoris decompositions of Waldhausen [30] and Cappell [2] of the algebraic K- and L-theory of generalized free products. The topological motivation for these proofs comes from a noncommutative localization interpretation of the Seifert-van Kampen and Mayer-Vietoris theorems. If $(M, N \subseteq M)$ is a two-sided pair of connected CW complexes the fundamental group $\pi_1(M)$ is a generalized free product: an amalgamated free product if N separates M, and an HNN extension otherwise. The morphisms $\pi_1(N) \to \pi_1(M\backslash N)$ determine a triangular $k \times k$ matrix ring A with universal localization the full $k \times k$ matrix ring $\Sigma^{-1}A = M_k(\mathbb{Z}[\pi_1(M)])$ ($k = 3$ in the separating case, $k = 2$ in the non-separating case), such that the corresponding presentations of the $\mathbb{Z}[\pi_1(M)]$-module chain complex $C(\widetilde{M})$ of the universal cover $\widetilde{M}$ is the

assembly of an A-module chain complex constructed from the chain complexes $C(\widetilde{N})$, $C(\widetilde{M\backslash N})$ of the universal covers $\widetilde{N}$, $\widetilde{M\backslash N}$ of N, $M\backslash N$. The two cases will be considered separately, in sections 2.3, 2.4.

2.1 The algebraic K-theory of a noncommutative localization

Given an injective noncommutative localization $A \to \Sigma^{-1}A$ let $H(A,\Sigma)$ be the exact category of homological dimension 1 A-modules T which admit a f.g. projective A-module resolution

$$0 \longrightarrow P \xrightarrow{s} Q \longrightarrow T \longrightarrow 0$$

such that $1 \otimes s : \Sigma^{-1}P \to \Sigma^{-1}Q$ is an $\Sigma^{-1}A$-module isomorphism. The algebraic K-theory localization exact sequence of Schofield [24, Theorem 4.12]

$$K_1(A) \to K_1(\Sigma^{-1}A) \to K_1(A,\Sigma) \to K_0(A) \to K_0(\Sigma^{-1}A)$$

was obtained for any injective noncommutative localization $A \to \Sigma^{-1}A$, with $K_1(A,\Sigma) = K_0(H(A,\Sigma))$. Neeman and Ranicki [14] proved that if $A \to \Sigma^{-1}A$ is injective and 'stably flat'

$$\mathrm{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A) \;=\; 0 \quad (i \geqslant 1)$$

then

(i) $\Sigma^{-1}A$ has the chain complex lifting property : every finite f.g. free $\Sigma^{-1}A$-module chain complex C is chain equivalent to $\Sigma^{-1}B$ for a finite f.g. projective A-module chain complex B,

(ii) the localization exact sequence extends to the higher K-groups

$$\cdots \to K_n(A) \to K_n(\Sigma^{-1}A) \to K_n(A,\Sigma) \to K_{n-1}(A) \to \cdots \to K_0(\Sigma^{-1}A)$$

with $K_n(A,\Sigma) = K_{n-1}(H(A,\Sigma))$.

2.2 Matrix rings

The amalgamated free product of rings and the HNN construction are special cases of the following type of noncommutative localization of triangular matrix rings.

Given rings A_1, A_2 and an (A_1, A_2)-bimodule B define the triangular 2×2 matrix ring

$$A \;=\; \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} .$$

An A-module can be written as

$$M \;=\; \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

with M_1 an A_1-module, M_2 an A_2-module, together with an A_1-module morphism $B \otimes_{A_2} M_2 \to M_1$. The injection

$$A_1 \times A_2 \to A \;;\; (a_1, a_2) \mapsto \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

induces isomorphisms of algebraic K-groups

$$K_*(A_1) \oplus K_*(A_2) \;\cong\; K_*(A) .$$

The columns of A are f.g. projective A-modules

$$P_1 \;=\; \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \;,\; P_2 \;=\; \begin{pmatrix} B \\ A_2 \end{pmatrix}$$

such that

$$\begin{aligned} &P_1 \oplus P_2 \;=\; A \;,\; \mathrm{Hom}_A(P_i, P_i) \;=\; A_i \;\; (i = 1, 2) \;, \\ &\mathrm{Hom}_A(P_1, P_2) \;=\; B \;,\; \mathrm{Hom}_A(P_2, P_1) \;=\; 0 \;. \end{aligned}$$

The noncommutative localization of A inverting a non-empty subset $\Sigma \subseteq \mathrm{Hom}_A(P_1, P_2) = B$ is the 2×2 matrix ring

$$\Sigma^{-1}A \;=\; M_2(C) \;=\; \begin{pmatrix} C & C \\ C & C \end{pmatrix}$$

with C the endomorphism ring of the induced f.g. projective $\Sigma^{-1}A$-module $\Sigma^{-1}P_1 \cong \Sigma^{-1}P_2$. The Morita equivalence

$$\{\Sigma^{-1}A\text{-modules}\} \to \{C\text{-modules}\} \;;\; L \mapsto (C \; C) \otimes_{\Sigma^{-1}A} L$$

induces isomorphisms in algebraic K-theory

$$K_*(M_2(C)) \ \cong \ K_*(C) \ .$$

The composite of the functor

$$\{A\text{-modules}\} \to \{\Sigma^{-1}A\text{-modules}\} \ ; \ M \mapsto \Sigma^{-1}M \ = \ \Sigma^{-1}A \otimes_A M$$

and the Morita equivalence is the *assembly* functor

$$\begin{aligned} &\{A\text{-modules}\} \to \{C\text{-modules}\} \ ; \\ &M \ = \ \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \mapsto (C \ C) \otimes_A M \\ &\qquad\qquad = \ \mathrm{coker}(C \otimes_{A_1} B \otimes_{A_2} M_2 \to C \otimes_{A_1} M_1 \oplus C \otimes_{A_2} M_2) \end{aligned}$$

inducing the morphisms

$$K_*(A) \ = \ K_*(A_1) \oplus K_*(A_2) \to K_*(\Sigma^{-1}A) \ = \ K_*(C)$$

in the algebraic K-theory localization exact sequence.

There are evident generalizations to $k \times k$ matrix rings for any $k \geqslant 2$.

2.3 HNN extensions

The HNN extension $R *_{\alpha,\beta} \{z\}$ is defined for any ring morphisms $\alpha, \beta : S \to R$, with

$$\alpha(s)z \ = \ z\beta(s) \in R *_{\alpha,\beta} \{z\} \ \ (s \in S) \ .$$

Define the triangular 2×2 matrix ring

$$A \ = \ \begin{pmatrix} R & R_\alpha \oplus R_\beta \\ 0 & S \end{pmatrix}$$

with R_α the (R,S)-bimodule R with S acting on R via α, and similarly for R_β. Let $\Sigma = \{\sigma_1, \sigma_2\} \subset \mathrm{Hom}_A(P_1, P_2)$, with

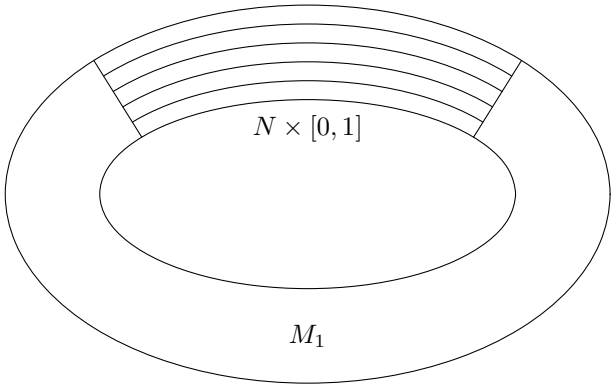
$$\sigma_1 \ = \ \begin{pmatrix} (1,0) \\ 0 \end{pmatrix} \ , \ \sigma_2 \ = \ \begin{pmatrix} (0,1) \\ 0 \end{pmatrix} \ : \ P_1 \ = \ \begin{pmatrix} R \\ 0 \end{pmatrix} \to P_2 \ = \ \begin{pmatrix} R_\alpha \oplus R_\beta \\ S \end{pmatrix} \ .$$

The A-modules P_1, P_2 are f.g. projective since $P_1 \oplus P_2 = A$. Theorem 13.1 of [24] identifies

$$\Sigma^{-1}A \ = \ M_2(R *_{\alpha,\beta} \{z\}) \ .$$

Example Let $(M, N \subseteq M)$ be a non-separating pair of connected CW complexes such that N is two-sided in M (i.e. has a neighbourhood $N \times [0,1] \subseteq M$) with $M \backslash N = M_1$ connected

$$M \ = \ M_1 \cup_{N \times \{0,1\}} N \times [0,1]$$



By the Seifert-van Kampen theorem, the fundamental group $\pi_1(M)$ is the HNN extension determined by the morphisms $\alpha, \beta : \pi_1(N) \to \pi_1(M_1)$ induced by the inclusions $N \times \{0\} \to M_1$, $N \times \{1\} \to M_1$

$$\pi_1(M) \ = \ \pi_1(M_1) *_{\alpha,\beta} \{z\} \ ,$$

so that

$$\mathbb{Z}[\pi_1(M)] \ = \ \mathbb{Z}[\pi_1(M_1)] *_{\alpha,\beta} \{z\} \ .$$

As above, define a triangular 2×2 matrix ring

$$A \ = \ \begin{pmatrix} \mathbb{Z}[\pi_1(N)] & \mathbb{Z}[\pi_1(M_1)]_\alpha \oplus \mathbb{Z}[\pi_1(M_1)]_\beta \\ 0 & \mathbb{Z}[\pi_1(M)] \end{pmatrix}$$

with noncommutative localization

$$\Sigma^{-1} A \ = \ M_2(\mathbb{Z}[\pi_1(M_1)] *_{\alpha,\beta} \{z\}) \ = \ M_2(\mathbb{Z}[\pi_1(M)]) \ .$$

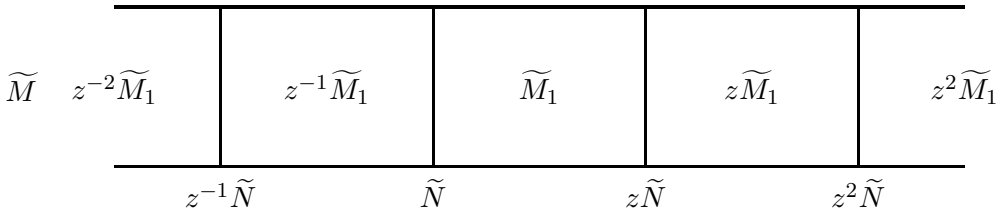
Assume that $\pi_1(N) \to \pi_1(M)$ is injective, so that the morphisms α, β are injective, and the universal cover $\widetilde{M}$ is a union

$$\widetilde{M} \ = \ \bigcup_{g \in [\pi_1(M) : \pi_1(M_1)]} g\widetilde{M}_1$$

of translates of the universal cover $\widetilde{M}_1$ of M_1, and

$$g_1\widetilde{M}_1 \cap g_2\widetilde{M}_1 \;=\; \begin{cases} h\widetilde{N} & \text{if } g_1 \cap g_2 z = h \in [\pi_1(M) : \pi_1(N)] \\ g_1\widetilde{M}_1 & \text{if } g_1 = g_2 \\ \emptyset & \text{if } g_1 \neq g_2 \text{ and } g_1 \cap g_2 z = \emptyset \end{cases}$$

with $h\widetilde{N}$ the translates of the universal cover $\widetilde{N}$ of N. In the diagram it is assumed that α, β are isomorphisms



The cellular f.g. free chain complexes $C(\widetilde{M}_1)$, $C(\widetilde{N})$ are related by $\mathbb{Z}[\pi_1(M_1)]$-module chain maps

$$\begin{aligned} i_\alpha \;&:\; \mathbb{Z}[\pi_1(M_1)]_\alpha \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \to C(\widetilde{M}_1)\ , \\ i_\beta \;&:\; \mathbb{Z}[\pi_1(M_1)]_\beta \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \to C(\widetilde{M}_1) \end{aligned}$$

defining a f.g. projective A-module chain complex $\begin{pmatrix} C(\widetilde{M}_1) \\ C(\widetilde{N}) \end{pmatrix}$ with assembly the cellular f.g. free $\mathbb{Z}[\pi_1(M)]$-module chain complex of $\widetilde{M}$

$$\begin{aligned} \mathrm{coker}\Big(i_\alpha - zi_\beta \;&: \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \to \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(M_1)]} C(\widetilde{M}_1)\Big) \\ &= \; C(\widetilde{M}) \end{aligned}$$

by the Mayer-Vietoris theorem. □

Let $R *_{\alpha,\beta} \{z\}$ be an HNN extension of rings in which the morphisms $\alpha, \beta : S \to R$ are both injections of (S,S)-bimodule direct summands, and R_α, R_β are flat S-modules. (This is the case in the above example if $\pi_1(N) \to \pi_1(M)$ is injective). Then the natural ring morphisms

$$\begin{aligned} &R \to R *_{\alpha,\beta} \{z\}\ ,\ S \to R *_{\alpha,\beta} \{z\}\ , \\ &A \;=\; \begin{pmatrix} R & R_\alpha \oplus R_\beta \\ 0 & S \end{pmatrix} \to \Sigma^{-1}A \;=\; M_2(R *_{\alpha,\beta} \{z\}) \end{aligned}$$

are injective, and $\Sigma^{-1}A$ is a stably flat universal localization, with $H(A,\Sigma) = \mathrm{Nil}(R,S,\alpha,\beta)$ the nilpotent category of Waldhausen [30]. The chain complex lifting property of $\Sigma^{-1}A$ gives a noncommutative localization proof of the existence of Mayer-Vietoris presentations for finite f.g. free $R *_{\alpha,\beta} \{z\}$-module chain complexes C

$$0 \longrightarrow R *_{\alpha,\beta} \{z\} \otimes_S E \xrightarrow{i_\alpha - z i_\beta} R *_{\alpha,\beta} \{z\} \otimes_R D \longrightarrow C \longrightarrow 0$$

with D (resp. E) a finite f.g. free R- (resp. S-) module chain complex ([30], Ranicki [23]). The algebraic K-theory localization exact sequence of [14]

$$\begin{array}{l} \cdots \to K_{n+1}(A,\Sigma) \;=\; K_n(S) \oplus K_n(S) \oplus \widetilde{\mathrm{Nil}}_n(R,S,\alpha,\beta) \\ \qquad \xrightarrow{\begin{pmatrix} \alpha & \beta & 0 \\ 1 & 1 & 0 \end{pmatrix}} K_n(A) \;=\; K_n(R) \oplus K_n(S) \\ \qquad\qquad\qquad\qquad \to K_n(\Sigma^{-1}A) \;=\; K_n(R *_{\alpha,\beta} \{z\}) \to \dots \end{array}$$

is just the stabilization by $1 : K_*(S) \to K_*(S)$ of the Mayer-Vietoris exact sequence of [30]

$$\cdots \longrightarrow K_n(S) \oplus \widetilde{\mathrm{Nil}}_n(R,\alpha,\beta) \xrightarrow{(\alpha-\beta)\oplus 0} K_n(R) \longrightarrow K_n(R *_{\alpha,\beta} \{z\}) \longrightarrow \cdots$$

In particular, for $\alpha = \beta = 1 : S = R \to R$ the HNN extension is just the Laurent polynomial extension

$$R *_{\alpha,\beta} \{z\} \;=\; R[z,z^{-1}]$$

and the Mayer-Vietoris exact sequence splits to give the original splitting of Bass, Heller and Swan [1]

$$K_1(R[z,z^{-1}]) \;=\; K_1(R) \oplus K_0(R) \oplus \widetilde{\mathrm{Nil}}_0(R) \oplus \widetilde{\mathrm{Nil}}_0(R)$$

as well as its extension to the Quillen higher K-groups K_*.

2.4 Amalgamated free products

The amalgamated free product $R_1 *_S R_2$ is defined for any ring morphisms $i_1 : S \to R_1$, $i_2 : S \to R_2$, with

$$r_1 i_1(s) * r_2 \;=\; r_1 * i_2(s) r_2 \in R_1 *_S R_2 \;\; (r_1 \in R_1, r_2 \in R_2, s \in S) \; .$$

Define the triangular 3×3 matrix ring

$$A \;=\; \begin{pmatrix} R_1 & 0 & R_1 \\ 0 & R_2 & R_2 \\ 0 & 0 & S \end{pmatrix}$$

and the A-module morphisms

$$\sigma_1 \;=\; \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;:\; P_1 \;=\; \begin{pmatrix} R_1 \\ 0 \\ 0 \end{pmatrix} \to P_3 \;=\; \begin{pmatrix} R_1 \\ R_2 \\ S \end{pmatrix} \;,$$

$$\sigma_2 \;=\; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \;:\; P_2 \;=\; \begin{pmatrix} 0 \\ R_2 \\ 0 \end{pmatrix} \to P_3 \;=\; \begin{pmatrix} R_1 \\ R_2 \\ S \end{pmatrix} \;.$$

The A-modules P_1, P_2, P_3 are f.g. projective since $P_1 \oplus P_2 \oplus P_3 = A$. The noncommutative localization of A inverting $\Sigma = \{\sigma_1, \sigma_2\}$ is the full 3×3 matrix ring

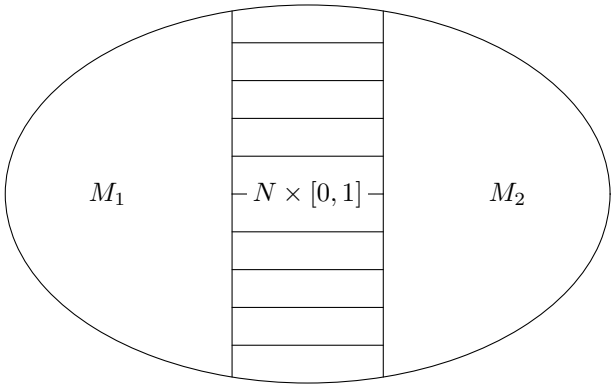
$$\Sigma^{-1}A \;=\; M_3(R_1 *_S R_2)$$

(a modification of Theorem 4.10 of [24]).

Example Let $(M, N \subseteq M)$ be a separating pair of CW complexes such that N has a neighbourhood $N \times [0,1] \subseteq M$ and

$$M \;=\; M_1 \cup_{N \times \{0\}} N \times [0,1] \cup_{N \times \{1\}} M_2$$

with M_1, M_2, N connected.



By the Seifert-van Kampen theorem, the fundamental group of M is the amalgamated free product

$$\pi_1(M) \;=\; \pi_1(M_1) *_{\pi_1(N)} \pi_1(M_2) \;,$$

so that

$$\mathbb{Z}[\pi_1(M)] \;=\; \mathbb{Z}[\pi_1(M_1)] *_{\mathbb{Z}[\pi_1(N)]} \mathbb{Z}[\pi_1(M_2)] \;.$$

As above, define a triangular matrix ring

$$A \;=\; \begin{pmatrix} \mathbb{Z}[\pi_1(M_1)] & 0 & \mathbb{Z}[\pi_1(M_1)] \\ 0 & \mathbb{Z}[\pi_1(M_2)] & \mathbb{Z}[\pi_1(M_2)] \\ 0 & 0 & \mathbb{Z}[\pi_1(N)] \end{pmatrix}$$

with noncommutative localization

$$\Sigma^{-1}A \;=\; M_3(\mathbb{Z}[\pi_1(M_1)] *_{\mathbb{Z}[\pi_1(N)]} \mathbb{Z}[\pi_1(M_2)]) \;=\; M_3(\mathbb{Z}[\pi_1(M)]) \;.$$

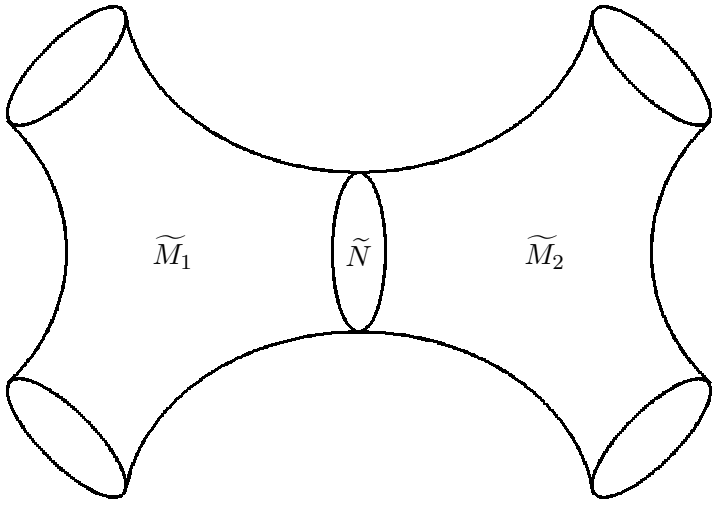
Assume that $\pi_1(N) \to \pi_1(M)$ is injective, so that the morphisms

$$\begin{aligned} &i_1 \;:\; \pi_1(N) \to \pi_1(M_1) \;,\; i_2 \;:\; \pi_1(N) \to \pi_1(M_2) \;, \\ &\pi_1(M_1) \to \pi_1(M) \;,\; \pi_1(M_2) \to \pi_1(M) \end{aligned}$$

are all injective, and the universal cover $\widetilde{M}$ of M is a union

$$\widetilde{M} \;=\; \bigcup_{g_1 \in [\pi_1(M):\pi_1(M_1)]} g_1\widetilde{M}_1 \cup \bigcup_{h \in [\pi_1(M):\pi_1(N)]} h\widetilde{N} \bigcup_{g_2 \in [\pi_1(M):\pi_1(M_2)]} g_2\widetilde{M}_2$$

of $[\pi_1(M) : \pi_1(M_1)]$ translates of the universal cover $\widetilde{M}_1$ of M_1 and $[\pi_1(M) : \pi_1(M_2)]$ translates of the universal cover $\widetilde{M}_2$ of M_2 with intersection the $[\pi_1(M) : \pi_1(N)]$ translates of the universal cover $\widetilde{N}$ of N.



The cellular f.g. free chain complexes $C(\widetilde{M}_1)$, $C(\widetilde{N})$ of the universal covers $\widetilde{M}_1$, $\widetilde{N}$ are related by $\mathbb{Z}[\pi_1(M_1)]$-module chain maps

$$\begin{aligned} i_1 &: \mathbb{Z}[\pi_1(M_1)] \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \to C(\widetilde{M}_1) \ , \\ i_2 &: \mathbb{Z}[\pi_1(M_2)] \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \to C(\widetilde{M}_2) \end{aligned}$$

defining a f.g. projective A-module chain complex $\begin{pmatrix} C(\widetilde{M}_1) \\ C(\widetilde{M}_2) \end{pmatrix}$ with assembly the cellular f.g. free $\mathbb{Z}[\pi_1(M)]$-module chain complex of $\widetilde{M}$

$$\begin{aligned} \mathrm{coker}\Big(\begin{pmatrix} 1 \otimes i_1 \\ 1 \otimes i_2 \end{pmatrix} &: \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \to \\ & \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(M_1)]} C(\widetilde{M}_1) \oplus \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(M_1)]} C(\widetilde{M}_2) \Big) \\ &= C(\widetilde{M}) \end{aligned}$$

by the Mayer-Vietoris theorem. □

Let $R_1 *_S R_2$ be an amalgamated free product of rings in which the morphisms $i_1 : S \to R_1$, $i_2 : S \to R_2$ are both injections of (S,S)-bimodule direct summands, and R_1, R_2 are flat S-modules. (This is the case in the above example if $\pi_1(N) \to \pi_1(M)$ is injective). Then the natural ring morphisms

$$R_1 \to R_1 *_S R_2 \ , \ R_2 \to R_1 *_S R_2 \ , \ S \to R_1 *_S R_2 \ ,$$

$$A \ = \ \begin{pmatrix} R_1 & 0 & R_1 \\ 0 & R_2 & R_2 \\ 0 & 0 & S \end{pmatrix} \to \Sigma^{-1}A \ = \ M_3(R_1 *_S R_2)$$

are injective, and $\Sigma^{-1}A$ is a stably flat noncommutative localization, with $H(A,\Sigma) = \mathrm{Nil}(R_1, R_2, S)$ the nilpotent category of Waldhausen [30]. The chain complex lifting property of $\Sigma^{-1}A$ gives a noncommutative localization proof of the existence of Mayer-Vietoris presentations for finite f.g. free $R_1 *_S R_2$-module chain complexes C

$$0 \longrightarrow R_1 *_S R_2 \otimes_S E \longrightarrow R_1 *_S R_2 \otimes_{R_1} D_1 \oplus R_1 *_S R_2 \otimes_{R_2} D_2 \longrightarrow C \longrightarrow 0$$

with D_i (resp. E) a finite f.g. free R_i- (resp. S-) module chain complex ([30], Ranicki [23]). The algebraic K-theory localization exact sequence of

[14]

$$\cdots \to K_{n+1}(A,\Sigma) \;=\; K_n(S) \oplus K_n(S) \oplus \widetilde{\mathrm{Nil}}_n(R_1,R_2,S)$$
$$\xrightarrow{\begin{pmatrix} i_1 & 0 & 0 \\ 0 & i_2 & 0 \\ 1 & 1 & 0 \end{pmatrix}} K_n(A) \;=\; K_n(R_1) \oplus K_n(R_2) \oplus K_n(S)$$
$$\to K_n(\Sigma^{-1}A) \;=\; K_n(R_1 *_S R_2) \to \dots$$

is just the stabilization by $1 : K_*(S) \to K_*(S)$ of the Mayer-Vietoris exact sequence of [30]

$$\cdots \longrightarrow K_n(S) \oplus \widetilde{\mathrm{Nil}}_n(R_1,R_2,S)$$
$$\xrightarrow{\begin{pmatrix} i_1 & 0 \\ i_2 & 0 \end{pmatrix}} K_n(R_1) \oplus K_n(R_2) \longrightarrow K_n(R_1 *_S R_2) \longrightarrow \cdots$$

2.5 The algebraic L-theory of a noncommutative localization

See Chapter 3 of Ranicki [19] for the algebraic L-theory of a commutative localization.

The algebraic L-theory of a ring A depends on an involution, that is a function $\bar{\ } : A \to A; a \mapsto \overline{a}$ such that

$$\overline{a+b} \;=\; \overline{a} + \overline{b}\ ,\ \overline{ab} \;=\; \overline{b}\,\overline{a}\ ,\ \overline{\overline{a}} \;=\; a\ ,\ \overline{1} \;=\; 1 \quad (a,b \in A)\ .$$

For an injective noncommutative localization $A \to \Sigma^{-1}A$ of a ring A with an involution which extends to $\Sigma^{-1}A$ Vogel [29] obtained a localization exact sequence in quadratic L-theory

$$\cdots \to L_n(A) \to L_n(\Sigma^{-1}A) \to L_n(A,\Sigma) \to L_{n-1}(A) \to \dots$$

with $L_n(A,\Sigma) = L_{n-1}(H(A,\Sigma))$. (See [14] for the symmetric L-theory localization exact sequence in the stably flat case). At first sight, it does not appear possible to apply this sequence to the triangular matrix rings of sections 2.2, 2.3, 2.4. How does one define an involution on a triangular matrix ring

$$A \;=\; \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}\ ?$$

The trick is to observe that if A_1, A_2 are rings with involution, and (B, β) is a nonsingular symmetric form over A_1 such that B is an (A_1, A_2)-bimodule then A has a *chain duality* in the sense of Definition 1.1 of Ranicki [20], sending an A-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ to the 1-dimensional A-module chain complex

$$TM \ : \ TM_1 \ = \ \begin{pmatrix} M_1^* \\ 0 \end{pmatrix} \to TM_0 \ = \ \begin{pmatrix} B \otimes_{A_2} M_2^* \\ M_2^* \end{pmatrix} \ .$$

The quadratic L-groups of A are just the relative L-groups in the exact sequence

$$\cdots \to L_n(A) \to L_n(A_2) \xrightarrow{(B,\beta)\otimes_{A_2} -} L_n(A_1) \to L_{n-1}(A) \to \ldots \ .$$

In particular, for generalized free products of rings with involution the triangular matrix rings A of section 2.3, 2.4 have such chain dualities, and in the injective case the torsion L-groups $L_*(A, \Sigma) = L_{*-1}(H(A, \Sigma))$ in the localization exact sequence

$$\cdots \to L_n(A) \to L_n(\Sigma^{-1}A) \to L_n(A, \Sigma) \to L_{n-1}(A) \to \ldots$$

are just the unitary nilpotent L-groups UNil_* of Cappell [2].

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