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# ON THE ALEXANDER POLYNOMIAL

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## Introduction

Two ordered collections of $\mu$ disjoint simple closed oriented curves in Euclidean three dimensional space $E^3$ are said to be *equivalent*, if there is an orientation-preserving homeomorphism of $E^3$ on itself, which transforms one collection into the other, preserving the orientation and order of the components.

A *knot* of *multiplicity* $\mu$ is the equivalence class of an ordered collection of $\mu$ disjoint simple closed polygons. Two collections in the same class are said to have the same *knot type*.

We shall be concerned here with an invariant of the knot type, the *Alexander polynomial*, which was first defined by Alexander [1][2] in the case $\mu = 1$ and then defined by R. H. Fox in the case $\mu > 1$[3].

The Alexander polynomial of a knot $K$ of multiplicity $\mu$, whose components are $X_1, \cdots, X_\mu$ is an integral polynomial $\Delta(t_1, \cdots, t_\mu)$ in the indeterminates $t_1, \cdots, t_\mu$ where each $t_i$ corresponds to one of the components $X_i$ of $K (i = 1, \cdots, \mu)$.

A characterization of the Alexander polynomial in the case $\mu = 1$ has been given by Seifert [9]. He proved that the polynomial $\Delta(t)$ of a knot of multiplicity 1 has the properties: $|\Delta(1)| = 1$ and $\Delta(t) = t^{2h}\Delta(1/t)$, and conversely, that every $\Delta(t)$ with these properties is the Alexander polynomial of some knot.

The main purpose of this paper is to prove that the Alexander polynomial, in the case $\mu > 1$, has the following properties:

1. There exist integers $n_1, \cdots, n_\mu$ such that $\Delta(t_1, \cdots, t_\mu) = (-1)^\mu t_1^{n_1} \cdots t_\mu^{n_\mu}\Delta(1/t_1, \cdots, 1/t_\mu)$.

2. If $\Delta(t_1, t_2)$ is the polynomial of a knot $K$ of multiplicity 2, whose components are $X_1$ and $X_2$, then: $\Delta(t_1, 1) = \Delta(t_1)(t_1^l - 1)/(t_1 - 1)$ where $\Delta(t_1)$ is the polynomial of $X_1$, and $l$ is the linking number of $X_1$ and $X_2$. If $\Delta(t_1, \cdots, t_\mu)$ is the polynomial of a knot of multiplicity $\mu > 2$, whose components are $X_1, \cdots, X_\mu$, then $\Delta(t_1, \cdots, t_{\mu-1}, 1) = (t_1^{l_1}t_2^{l_2} \cdots t_{\mu-1}^{l_{\mu-1}} - 1)\ \Delta(t_1, \cdots, t_{\mu-1})$, where $\Delta(t_1, \cdots, t_{\mu-1})$ is the polynomial of the knot which is obtained from $K$ by removing $X_\mu$, and $l_i (i = 1, \cdots, \mu - 1)$ is the linking number of $X_\mu$ and $X_i$.

3. $\Delta(1, \cdots, 1) = 0$ if $\mu > 2$, and $|\Delta(1, 1)| = |l|$ if $\mu = 2$, where $l$ is the linking number of the two components of $K$.[4]

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[1] The author gratefully acknowledges the guidance of Professor R. H. Fox in preparing this paper which was submitted as a Doctor's thesis to the faculty of Princeton University.

[2] Numbers between brackets refer to the bibliography at the end of the paper.

[3] An invariant polynomial associated with some knots of multiplicity 2, was considered by K. Reidemeister and H. G. Shumann [4] and by W. Burau [2].

[4] Property 3 is an immediate consequence of property 2. It is not known whether or not properties 1 and 2 suffice to characterize the polynomial. By a procedure completely anal-

We have included, in Chapter III, a proof of a theorem (Theorem 5), which is a generalization, for $\mu > 1$, of a theorem proved by Seifert [11] for $\mu = 1$. This theorem describes the effect produced on the polynomial of a knot $K$ which is contained in an unknotted tube $U$ by a further knotting of $U$.

## Chapter I

### 1. Derivation in a free group

In this paragraph we shall give a brief account of the theory of derivation in a free group. This concept, which has been introduced by R. H. Fox,[5] will be used throughout this paper.

Let $G$ be a multiplicative group, $R$ the ring of integers and $RG$ the integral group ring of $G$. The elements of $RG$ are of the form $\sum_{i=1}^{n} r_i g_i$ where $r_i \in R$ and $g_i \in G$. Let $\circ$ be the homomorphism of $RG$ onto $R$ defined by $(\sum_{i=1}^{n} r_i g_i)^\circ = \sum_{i=1}^{n} r_i$.

A *derivative* in $RG$ is a mapping $D: RG \to RG$ of $RG$ into itself satisfying:

(1) $D(u + v) = Du + Dv$

(2) $D(u \cdot v) = D(u) \cdot v^\circ + u \cdot Dv$

for all $u$ and $v$ in $RG$.

Let $X$ denote the free group generated by a finite set of symbols $x_1, \cdots, x_\lambda$. For each index there is a unique derivation $\partial/\partial x_j$ in $RX$ satisfying (1) and (2) and

(3) $\dfrac{\partial x_k}{\partial x_j} = \delta_{jk}$.

This is called the *derivative with respect to* $x_j$. If $X \ni w = a_0 x_j^{\varepsilon_1} a_1 \cdots a_j^{\varepsilon_k} a_k$, where $\rho_i = \pm 1$ and $a_0, a_1, \cdots, a_k$ are words not involving the generator $x_j$, then:

$$\frac{\partial w}{\partial x_j} = \sum_{i=1}^{k} \varepsilon_i (a_0 x_j^{\varepsilon_j} a_1 \cdots a_{i-1}) x_j^{\frac{1}{2}(\varepsilon_i - 1)}.$$

It can be proved that, for any derivation $D$ in $RX$,

$$Du = \sum_j \frac{\partial u}{\partial x_j} \cdot Dx_j \qquad u \in RX.$$

In particular $u \to u - u^\circ$ is a derivation. Hence:

(4) $\mu - \mu^\circ = \sum_j \dfrac{\partial \mu}{\partial x_j} \cdot (x_j - 1)$.

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ogous to the procedure used by Seifert in his proof [8], one can prove that properties 1′ and 2′, given below, suffice to characterize the polynomial $\bar{\Delta}(t)$, obtained by substituting $t$ for $t_i$ ($i = 1, \cdots, \mu$) in $\Delta(t_1, \cdots, t_\mu)$.

1.′ There exists $\alpha \equiv \mu$ (mod. 2) such that $\bar{\Delta}(t) = (-1)^\mu t^\alpha \bar{\Delta}(1/t)$.

2.′ $\bar{\Delta}(1) = 0$ if $\mu > 2$, and $|\Delta(1)| = |l|$ if $\mu = 2$, where $l$ is the linking number of the components of $K$.

[5] A complete account will appear in a paper "Free Differential Calculus" in these Annals.

The row vector $(\partial_\mu/\partial x_1, \partial_\mu/\partial x_2, \cdots, \partial_\mu/\partial x_\lambda)$ may be denoted by $du$. From (1) and (2) we get:

(5) $d(u + v) = du + dv$

(6) $d(uv) = du \cdot v^\circ + u \cdot dv$.

Every group $G$ is the image of some free group $X$ under some homomorphism $\phi$. $G$ is therefore determined by the *generators* $x_1, x_2, \cdots$ of $X$ and by a set of elements (relations) $u_1(x), u_2(x), \cdots$ of $X$ whose *consequence* $U$ (smallest normal subgroup containing $u_1, u_2, \cdots$) is the kernel of $\phi$. We shall call $\{x_1, x_2, \cdots / u_1, u_2, \cdots\} = \{x/u\}$ a *presentation* of $G$. All groups considered in this paper will be finitely presented; i.e., they will be given by a finite number of generators and relations.

There are two types of operations on a presentation which do not alter the group presented. They are *the Tietze operation of first kind*: adjoin to the relations $u_1, u_2, \cdots$ any element $v$ of $U$, i.e. any *consequent relation*, and the *Tietze operation of the second kind*: adjoin to the generators $x_1, x_2, \cdots$ a new generator $y$ and simultaneously adjoin to the relations $u_1(x), u_2(x), \cdots$ a new relation $y[g(x)]^{-1}$ defining $y$ in terms of the old generators. The Tietze operations are *complete* in the sense that, given two finite presentations of a group $G$, it is possible to pass from one presentation to the other by a finite sequence of the Tietze operations and their inverses [7].

In order that an invariant of finite presentations be a group invariant for all finitely presented groups, it is necessary and sufficient that it be invariant under the two Tietze operations.

To any finite presentation $\{x_1, \cdots, x_\lambda / u_1, \cdots, u_\mu\}$ of a group $G$ we associate the matrix $\| (\partial \mu_i / \partial x_j)^\phi \|$. The equivalence class of this matrix is an invariant of $G$. Matrices over a group ring are *equivalent* if one can be obtained from the other by a finite sequence of the following operations and their inverses.

1. $M \to (M)$ where the new row — is a left-linear combination of the rows of $M$.

2. $M \to \begin{pmatrix} M & 0 \\ * & 1 \end{pmatrix}$, $*$ denoting an arbitrary row vector. Operations 1 and 2 correspond to the Tietze operations of the first and second kind respectively.

Let $RX \xrightarrow{\phi} RG \xrightarrow{\psi} RH \xrightarrow{0} R$, where $H$ is the commutator quotient group of $G$, and $\psi$ is the natural homomorphism of $RG$ onto $RH$. The equivalence class of $\| (\partial u_i / \partial x_j)^{\psi\phi} \|$ are of $\| (\partial u_i / \partial x_j)^{o\psi\phi} \|$ and invariants of $G$. (The latter matrix is a relation matrix for the abelian group $H$. The former determines $G$ modulo its second commutator subgroup.)

For any matrix $M$ over a commutative ring $RH$ define: The *elementary ideal of column deficiency* $k$ = the ideal in $RH$ generated by the minor determinants of order $n - k$, where $n$ is the number of columns of $M$.

The elementary ideals of a matrix over a commutative ring are invariants of the equivalence class of the matrix. But they are not necessarily invariant under automorphisms of $RH$.

## 2. Calculation of the group $G$ of a knot $K$ [6]

Let $K$ be a knot of multiplicity $\mu$ in $E^3$. Let $G$ be the *group of* $K$, i.e. the fundamental group $\Pi_1(E^3 - k)$ of $E^3 - k$, where $k$ is a representative of $K$. We shall describe here a method for calculating $G$.

Consider a representative $k$ of $K$ whose components $X_1, \cdots, X_\mu$ are simple closed polygons. A central projection of $k$ is called *regular* if all projecting rays meet at most two segments of $k$. A regular projection has only two-fold multiple points (*double points or crossings*), and it has only a finite number $\nu$ of them. We *normalize* the projection by denoting which of two segments that determine a

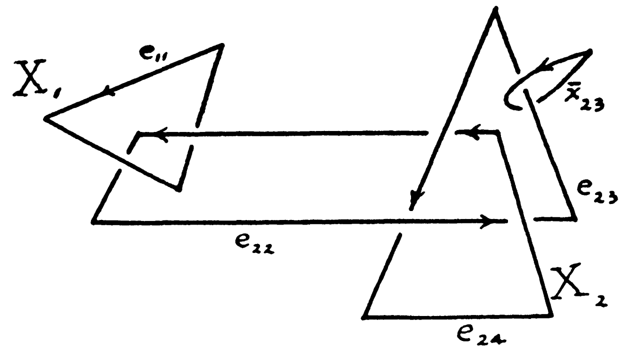


FIG. 1

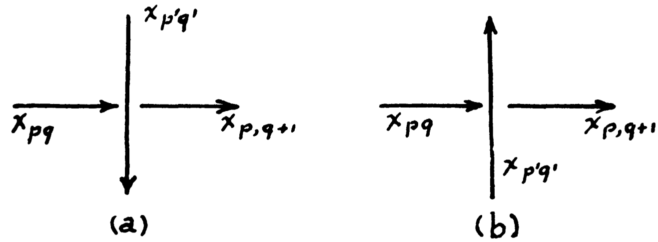


FIG. 2

double point crosses underneath the other (we shall do this as indicated in fig. 1). A normalized projection consists of a finite number $\nu$ of disjoint oriented simple polygonal arcs (fig. 1). Let us denote them by

$$e_{ij}\ (i = 1, \cdots, \mu;\ j = 1, \cdots, l_i),$$

where $e_{i1}, e_{i2}, \cdots, e_{i,l_i}$ are the arcs corresponding to $X_i$ $(i = 1, \cdots, \mu)$, as they are read off from the projection by going along $X_i$ in the positive direction (fig. 1). Then the group $G$ is generated by $\{x_{ij}\}(i = 1, \cdots, \mu; j = 1, \cdots, j_i)$, where $x_{ij}$ is represented by a loop $\bar{x}_{ij}$ which, starting from above the plane of projection, goes around $e_{ij}$ piercing the plane to the left of $e_{ij}$ and emerging to the right of $e_{ij}$ (fig. 1). There is a defining relation corresponding to each crossing, and it is of the form $r_{pq} = x_{p'q'}x_{pq}x_{p'q'}^{-1}x_{p,q+1}^{-1}$ or $r_{pq} = x_{p'q'}^{-1}x_{pq}x_{p'q'}x_{p,q+1}^{-1}$ according as the crossing is of type (a) or (b) (fig. 2). In this presentation the number of generators is equal to the number of relations. But any one of the relations is a consequence of the others, therefore $G$ is given by $\nu$ generators $\{x_{ij}\}$ and $\nu - 1$ of the relations $\{r_{ij}\}$ $(i = 1, \cdots, \mu; j = 1, \cdots, j_i)$.

## 3. The Alexander Polynomial

Now we are in a position to define the Alexander polynomial of $K$. Let $\{x_{ij}/r_{pq}\}$ be the above mentioned presentation of $G$. The commutator factor group $H$ will be the abelian group generated by the $\{x_{ij}\}$ with the abelianized relations $\{\bar{r}_{pq} = x_{pq}x_{p,q+1}^{-1}\}$ which express the equality of the generators $\{x_{ij}\}$ $(j = 1, \cdots, j_i)$ corresponding to $X_i$ $(i = 1, \cdots, \mu)$. Therefore $H$ is free abelian in $\mu$ generators $t_1, \cdots, t_\mu$, where $t_i$ is represented by each one of the loops $\{\bar{x}_{ij}\}$ $(j = 1, \cdots, j_i)$.

Consider the matrix $M(t_1, \cdots, t_\mu) = \| (\partial r_{pq}/\partial x_{ij})^{\psi\phi} \|$ over the ring $RH$ of polynomials in $t_1, \cdots, t_\mu ; t_1^{-1}, \cdots, t_\mu^{-1}$. The matrix $M(t_1, \cdots, t_\mu)$ is called the *Alexander matrix* of $K$. In order that two knots of multiplicity $\mu$ belong to the same knot type it is necessary that their matrices have the same elementary ideals in $RH$, because the basis $t_1, \cdots, t_\mu$ of $H$ is uniquely determined by the fact that $t_i (i = 1, \cdots, \mu)$ is the element of $H$ which is represented by a loop whose linking number $l_{ik} (k = 1, \cdots, \mu)$ with $X_k$ is $\delta_{ik}$.

We define the *Alexander polynomial* $\Delta(t_1, \cdots, t_\mu)$ of $K$ to be the greatest common divisor of the minor determinants of column deficiency $= 1$ of the Alexander matrix $M(t_1, \cdots, t_\mu)$. $\Delta(t_1, \cdots, t_\mu)$ is an invariant of the group $G$ of $K$ and therefore an invariant of the knot type of $K$, and is determined up to units in the ring $RH$.

The following theorem shows how to calculate $\Delta(t_1, \cdots, t_\mu)$ from a properly chosen minor of $M(t_1, \cdots, t_\mu)$.

THEOREM. *Let $M(t_1, \cdots, t_\mu)$ be an Alexander matrix of a knot $K$ of multiplicity $\mu$. Let $\nu$ be the number of columns and $\nu - 1$ be the number of rows. Denote by $\Delta_j (j = 1, \cdots, \nu)$ the determinant of the minor of $M(t_1, \cdots, t_\mu)$ obtained by deleting the* $j$th *column. There exists an element $\Delta \in RH$ such that*

$$\Delta_j = \pm \frac{x_j^{\psi\phi} - 1}{t + 1} \Delta \qquad (j = 1, \cdots, \nu) \qquad \text{if } \mu = 1$$

*and*

$$\Delta_j = \pm (x_j^{\psi\phi} - 1)\Delta \qquad (j = 1, \cdots, \nu) \text{ if } \mu \geqq 2,$$

*where $x_j$ is the generator of $G$ corresponding to the $j^{\text{th}}$ column.*

PROOF.[6] Denote by $\xi_j$ the $j^{\text{th}}$ column of $M(t_1, \cdots, t_\mu)$, $j = 1, \cdots, \nu$. By formula (4) we have, $\sum_{j=1}^{\nu} \xi_j (x_j^{\psi\phi} - 1) = 0$, hence

$$\begin{aligned} \Delta_j (x_k^{\psi\phi} - 1) &= (\xi_1, \cdots, \hat{\xi}_j, \cdots, \xi_k(x_k^{\psi\phi} - 1), \cdots, \xi_\nu) \\ &= (\xi_1, \cdots, \hat{\xi}_j, \cdots, \xi_j(x_j^{\psi\phi} - 1), \cdots, \xi_\nu) \\ &= (-1)^{j-k}(\xi_1, \cdots, \xi_j(x_j^{\psi\phi} - 1) \cdots, \hat{\xi}_k, \cdots, \xi_n) = (-1)^{j-k}\Delta_k \cdot (x_j^{\psi\phi} - 1). \end{aligned}$$

[6] In the proof we use the notation $\hat{\xi}$ meaning deletion of $\xi$.

Therefore $x_j^{\psi\phi} - 1$ divides $\Delta_j \cdot (x_k^{\psi\phi} - 1)$ for $k = 1, \cdots, \mu$, and therefore it must divide $\Delta_j \cdot \delta$ where $\delta$ is the greatest common divisor of $(x_j^{\psi\phi} - 1, \cdots, x_1^{\psi\phi} - 1)$, hence $\delta = \text{g.c.d.}_{t \epsilon H} \{t - 1\}$. But

$$(-1)^j \frac{\Delta_j \cdot \delta}{x^{\psi\phi} - 1} = (-1)^k \frac{\Delta_k \cdot \delta}{x^{\psi\phi} - 1} \qquad (k = 1, \cdots, \nu).$$

Denote the common value of $(-1)^j (\Delta_k \cdot \delta / x_k^{\psi\phi} - 1)$ $(k = 1, \cdots, \nu)$ by $\Delta$. Then $\Delta_j = (-1)^j (\psi^{\psi\phi} - 1/\delta) \cdot \Delta$.

The statement of the theorem follows from the observation that $\delta = 1$ if $\mu \geqq 2$ and $\partial = t - 1$ if $\mu = 1$.

*Note:* In the case $\mu = 1$, the matrix $M(t)$ can be interpreted as follows:

Let $G$ be the group of $K$, corresponding to each subgroup $G'$ of $G$ there is a covering space of $E^3 - k$ whose fundamental group is isomorphic to $G'$. Let $\mathfrak{M}$ be the covering corresponding to the (commutator) subgroup $G_0$ of $G$ formed by the elements of $G$ which are represented by loops whose linking numbers with $k$ are zero. The first homology group $H_1(\mathfrak{M})$ of $\mathfrak{M}$ can be given as a group with operators, with a finite number of generators $\{b_i\}$ and relations $\{r_p\}$, where the domain of operators is the ring of integral polynomials in $t$ and $t^{-1}$, where $t$ is the generator of the commutator quotient group $H$ of $G$. It can be proved that $M(t)$ is the coefficient matrix of the relations in the above mentioned presentation of $H_1(\mathfrak{M})$. This interpretation of $M(t)$ was given by Alexander [1].

## 4. Seifert's projection of a knot

We are going to describe a special type of projection of a knot. It was first described by Seifert [9] in the case of a knot of multiplicity 1, and his method can be immediately generalized to the case $\mu > 1$.

Let $K$ be a knot of multiplicity $\mu$ in $E^3$, and let $X_1, \cdots, X_\mu$ be its components. Let $F$ be an orientable surface,[7] whose boundary is $K$, and let $h$ be the genus of $F$. The surface $F$ can be deformed (the type of the knot is thereby unchanged) into a disc to which there have been attached $2h + \mu - 1$ bands $B_1, \cdots, B_{2h+\mu-1}$, which are distributed around the disc as shown in fig. 3. The corresponding projection of $K$ will be called a *Seifert projection.* The possibility of such a deformation of $F$ is illustrated in fig. 4 which represents the normal form of a surface $F$ of genus 2, whose boundary has 3 components. One of the components is the union of the arcs on the corners, the other components have been represented by circles in the interior, there have been drawn 2 canonical pairs of curves, and the dotted lines represent the boundary after the deformation.

In fig. 3 a simple closed curve $a_i$ has been drawn along each $B_i$. $a_1, \cdots, a_{2h}$ are the canonical curves which were used to direct the deformation, they are such that $a_{2k-1}$ crosses $a_{2k}(k = 1, \cdots, h)$ from left to right. The curves

$$a_{2h+1}, a_{2h+2}, \cdots, a_{2h+\mu-1}$$

[7] A procedure for spanning a knot of multiplicity 1 by an oriental surface is given by Seifert [8] and [9], and the same procedure can be applied in the case $\mu > 1$.

are disjoint and $a_{2h+\mu-1}$ separates the corresponding component $X_k$ from $X_1$, the exterior boundary of $F$. We shall refer to $B_1, \cdots, B_{2h}$ as *canonical bands* and to $B_{2h+1}, \cdots, B_{2h+\mu-1}$ as *extra bands*.

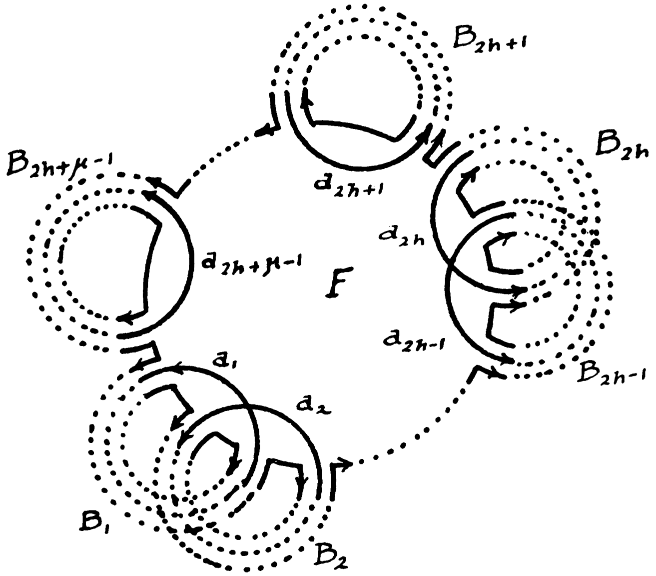


FIG. 3

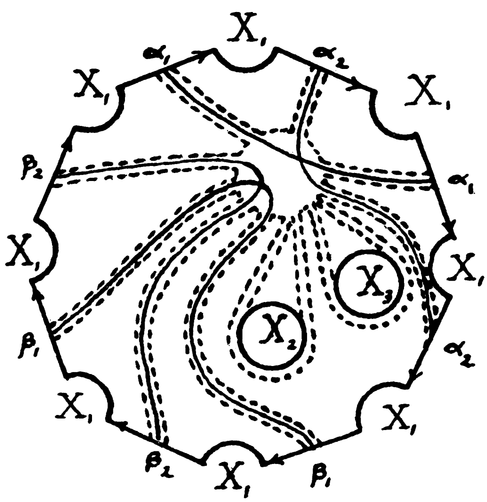


FIG. 4

We may suppose that in the projection only one face of $F$ is visible, for if a band is twisted (fig. 5(a)), the number of twistings must be even, because $F$ is orientable, and for each pair of twistings we can perform the deformation illustrated in fig. 5.

Finally let us define *crossing numbers*. Let $v_{ij}$, be equal to the number of times that $a_j$ crosses over $a_i$ from left to right minus the number of times that $a_j$ crosses over $a_i$ from right to left. The numbers $\{v_{ij}\}$ $(i, j = 1, \cdots, 2h + \mu - 1)$ are called the *crossing numbers* of the bands.

It is clear that if $a_i$ and $a_j$ are disjoint, the crossing number $v_{ij}$ is equal to the linking number of $a_i$ and $a_j$, and therefore $v_{ij} = v_{ji}$. If $a_i$ intersects $a_j$, i.e., if $i = 2k - 1$ and $j = 2k (1 \leqq k \leqq 2h)$, then we can lift $a_{2k-1}$ in a neighborhood of the intersection, obtaining a curve $a'_{2k-1}$ which does not intersect $a_{2k}$, and the linking number of $a'_{2k-1}$ and $a_{2k}$ will be equal to $v_{2k-1,2k}$ and equal to $v_{2k,2h-1} + 1$. Therefore: $v_{2k-1,2k} = v_{2k,2k-1} + 1 (0 \leqq k \leqq h)$ and $v_{ij} = v_{ji}$ otherwise.

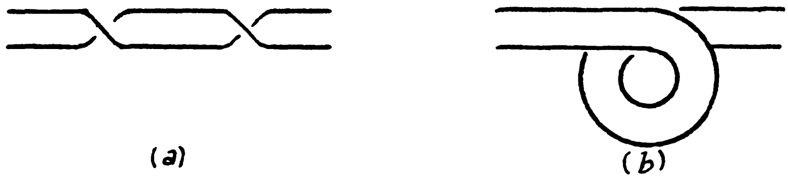


FIG. 5

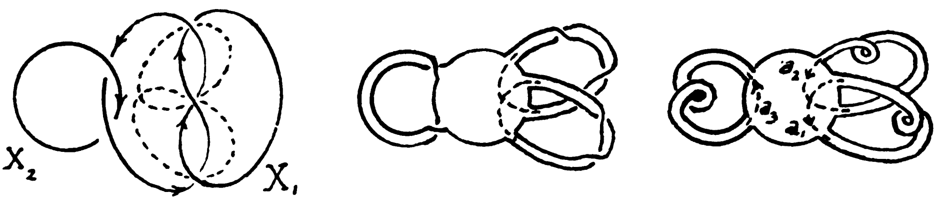


FIG. 6

In fig. 6 various stages of a deformation of a simple knot into a Seifert projection are illustrated. In the case illustrated by fig. 6, we have:

$$\begin{array}{lll} v_{11} = \phantom{-}1, & v_{12} = 0, & v_{13} = 0 \\ v_{21} = -1, & v_{22} = 1, & v_{23} = 0 \\ v_{31} = \phantom{-}0, & v_{32} = 0, & v_{33} = 0 \end{array}$$

## CHAPTER II

**1.** Let $K$ be a knot of multiplicity $\mu > 1$ in $E^3$, and let $\Delta(t_1, \cdots, t_\mu)$ be the Alexander polynomial of $K$. We shall prove:

THEOREM 1. *There exists an integer n such that*

$$\Delta(t, \cdots, t) = (-1)^\mu t^n \Delta(1/t, \cdots, 1/t),$$

*where* $\Delta(t, \cdots, t)$ *is the polynomial obtained by substituting* $t$ *for* $t_i (i = 1, \cdots, \mu)$ *in* $\Delta(t_1, \cdots, t_\mu)$.

PROOF. Let $F$ be an orientable surface of genus $h$ whose boundary is $K$, and consider a Seifert projection of $K$ obtained by using $F$ (fig. 3). We are going to use this projection to compute the group $G$ of $K$.

Let $s$ be the graph formed from the $\{a_k\}$ by retracting the central disk of $F$ to a point $P$ and retracting the bands to lines (fig. 7). Let us denote by $a_{i1}, \cdots, a_{ij_i}(i = 1, \cdots, 2h + \mu - 1)$ the edges of the projection of $s$ as they are read off from the projection by going along $a_i$ in the positive direction, the edges $a_{11}$, $a_{21}, \cdots, a_{2h+\mu-1,1}$ being the edges indicated in fig. 7.

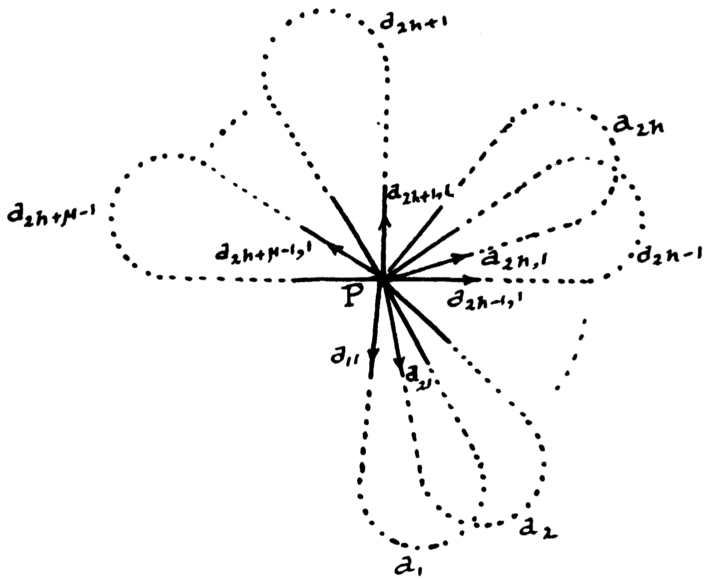


FIG. 7

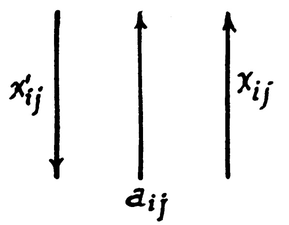


FIG. 8

Corresponding to each edge $a_{ij}$ there are two edges $x_{ij}$ and $x'_{ij}$ of the projection of $K$ (fig. 8), where $x_{ij}$ is the edge which has a similar orientation to the orientation of $a_{ij}$, and lies to the right of it. Let us denote also by $x_{ij}$ and

$$x'_{ij}(i = 1, \cdots, 2h + \mu - 1; j = 1, \cdots, j_i)$$

the generators of $G$ which correspond to the edges of $x_{ij}$ and $x'_{ij}$ respectively.

For each crossing of $a_p$ over $a_i$ we have two defining relations $R_{ij}$ and $S_{ij}$ of the form:

$$R_{ij} = (x'^{-1}_{pq} x_{pq})^{\varepsilon} x_{ij} (x'^{-1}_{pq} x_{pq})^{-\varepsilon} x^{-1}_{i,j+1}$$

$$S_{ij} = (x'^{-1}_{pq} x_{pq})^{\varepsilon} x'_{ij} (x'^{-1}_{pq} x_{pq})^{-\varepsilon} x'^{-1}_{i,j+1}$$

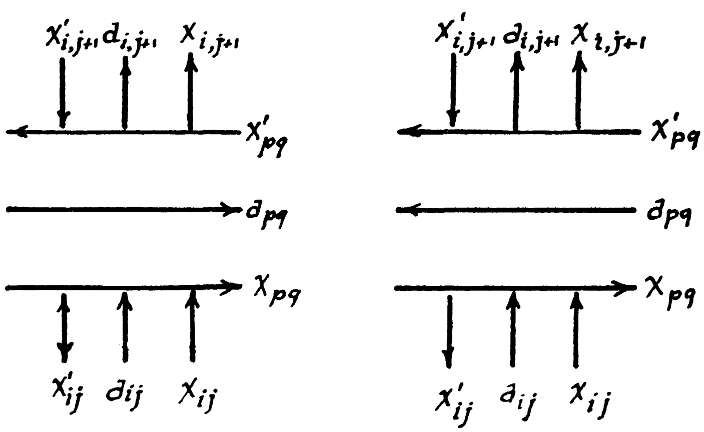


FIG. 9

where $\varepsilon = \pm 1$ according as $a_p$ crosses $a_i$ from left to right or from right to left (fig. 9). Besides these relations we have the relations:

$$\left.\begin{aligned} T'_{2l-1} &= x'_{2l-1,1}x^{-1}_{2l,1} \\ T'_{2l} &= x'_{2l,1}x'^{-1}_{2l-1,j_{2l-1}} \\ T_{2l-1} &= x_{2l-1,j_{2l-1}}x'^{-1}_{2l,j_{2l}} \\ T_{2l} &= x_{2l,j_{2l}}x^{-1}_{2l+1,1} \end{aligned}\right\} \qquad (l = 1, \cdots, k)$$

and

$$\begin{aligned} Q_t &= x_{t,j_t}x^{-1}_{t+1,1} \qquad &(t = 2h + 1, \cdots, 2h + \mu - 2) \\ Q'_{t'} &= x'_{t'1}x'^{-1}_{t',j_{t'}} \qquad &(t' = 2h + 1, \cdots, 2h + \mu - 1) \end{aligned}$$

(see figures 10 and 11). Therefore the group $G$ has the presentation:

$$\text{(I)} \qquad G\colon \{x_{ij}, x'_{ij}/R_{ij}, S_{ij}, (j < j_i)\ T_{2l-1}, T_{2l}, T'_{2l-1}, T'_{2l}, Q_t, Q'_{t'}\}$$

$$\begin{pmatrix} i &= 1, \cdots, 2h + \mu - 1 \\ j &= 1, \cdots, j_i \\ l &= 1, \cdots, k \\ t &= 2h + 1, \cdots, 2h + \mu - 2 \\ t' &= 2h + 1, \cdots, 2h + \mu - 1 \end{pmatrix}$$

This presentation of $G$ differs from the presentation we described in Chapter I, in that we have eliminated the generators corresponding to the edges which are under the bands. We have suppressed also the relation $x_{2h+\mu-1,j_{2h+\mu-1}}x^{-1}_{11}$ which, as we know, is a consequence of the other relations.

Let us introduce generators $a_{ij}(i = 1, \cdots, 2h + \mu - 1, j = 1, \cdots, j_i)$ defined by means of the relations $A'_{ij} = x'^{-1}_{ij}x_{ij}a^{-1}_{ij}$. Using $A_{ij}$ we obtain from

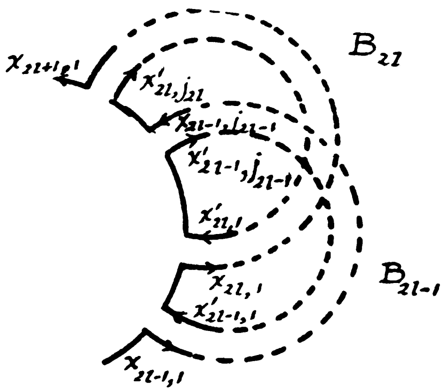

FIG. 10

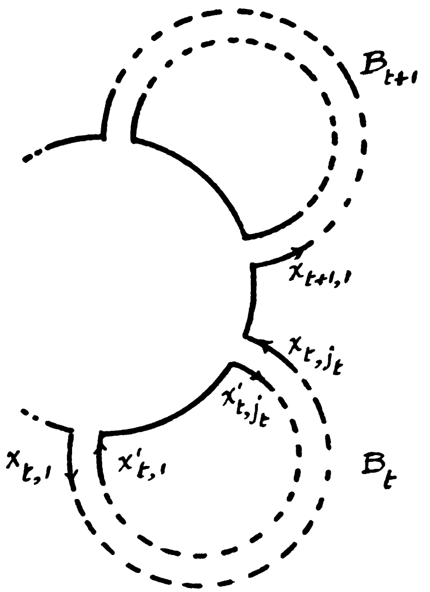


FIG. 11

$R_{ij}$ and $S_{ij}$ the relations, $R'_{ij} = a_{pq}^{\pm 1} x_{ij} a_{pq}^{\mp 1} x_{i,j+1}^{-1}$ and $S'_{ij} = a_{pq}^{\pm 1} x'_{ij} a_{pq}^{\mp 1} x'^{-1}_{i,j+1}$ respectively.

It is clear that $R_{ij}$ is a consequence of $R'_{ij}$ and $A_{ij}$, and similarly $S_{ij}$ is a consequence of $S'_{ij}$ and $A_{ij}$. Therefore we have:

(II) $\quad G: \{x_{ij}, x'_{ij}, a_{ij} / R'_{ij}, S'_{ij}, T'_{2l-1}, T'_{2l}, T_{2l-1}, T_{2l}, Q_t, Q'_{t'}, A_{ij}\}.$

From $R'_{ij}$, $S'_{ij}$, $A_{ij}$ and $A_{i,j+1}$ we obtain the relation $r_{ij} = a_{pq}^{\pm 1} a_{ij} a_{pq}^{\mp 1} a_{i,j+1}^{-1}$ $(i = 1, \cdots, 2h + \mu - 1; j = 1, \cdots, j_i)$. Introducing the relations $\{r_{ij}\}$ we can suppress the relations $\{S'_{ij}\}$, since $S'_{ij}$ is a consequence of $R'_{ij}$, $r_{ij}$, $A_{ij}$ and $A_{i,j+1}$. Each generator $x'_{ij}$ $(i = 1, \cdots, 2h + \mu - 1; 1 < j < j_i)$ appears only in the relation $A_{ij} = x'^{-1}_{ij} x_{ij} a^{-1}_{ij}$ which defines $x'_{ij}$ in terms of $x_{ij}$ and $a_{ij}$; therefore we can eliminate the generators $x'_{ij}$ and the relations $A_{ij}$ ($i = 1, \cdots, 2h + \mu - 1$; $1 < j < j_i$). The generator $x_{i1}$ appears only in the relations $R_{i1}$ and $R_{i2}$, $R_{i1} = a^{\pm 1}_{p'q'} x_{i1} a^{\mp 1}_{p'q'} x^{-1}_{i2}$ defines $x_{i2}$ as the *transformed* $a^{\pm 1}_{p'q'} x_{i1} a^{\mp 1}_{p'p'} = x_{12}$ of $x_{i1}$ by $a^{\pm 1}_{p'q'}$. If we substitute this expression of $x_{12}$ in $R_{i2} = a^{\pm 1}_{p''q''} x_{i2} a^{\mp 1}_{p''q''} x^{-1}_{i3}$ we obtain $a^{\pm 1}_{p''q''} a^{\pm 1}_{p'q'} x_{i1} a^{\mp 1}_{p'q'} a^{\mp 1}_{p''q''} x^{-1}_{i3}$, which expresses $x_{i3}$ as the transformed by $a^{\pm 1}_{p''q''} a^{\pm 1}_{p'q'}$ of $x_{i1}$. Introducing this expression, we can eliminate the generator $x_{i2}$ and the relations $R_{i1}$ and $R_{i2}$. By iteration of this process we can eliminate the generators $x_{i2}, x_{i3}, \cdots, x_{i,j_i-1}$ and the relations $R'_{i1}, R'_{i2}, \cdots, R'_{i,j_i-1}$ introducing the relation $R_i = w_i(a) x_{i1} w_i^{-1}(a) x^{-1}_{i,j_i}$ ($i = 1, \cdots, 2h + \mu - 1$), where $w_i(a)$ is the word in the $\{a_{ij}\}$ which is read off from the projection of $s$ by going along $a_i$ in the positive direction. For each crossing of an edge $a_{pq}$ over $a_i$, there is an appearance in $w(a)$ of the generator $a_i$ or its inverse $a_i^{-1}$ according as the crossing is from left to right or from right to left. Therefore, we have:

$$\text{(III)} \qquad G: \{x_{i1}, x_{ij_i}; x'_{i1}, x'_{ij_i}; a_{ij}/r_{ij}, R_i, A_{i1}, A_{i,j_i}, T'_{2l-1}, T'_{2l}, T_{2l-1}, T_{2l}, Q_t, Q'_{t'}\}.$$

Each generator $x'_{2l-1}$ ($l = 1, \cdots, k$) appears only in the relations

$$T'_{2l-1} = x'_{2l-1,1} x^{-1}_{2l,1} \qquad \text{and} \qquad A_{2l-1,1} = x'^{-1}_{2l-1,1} x_{2l-1,1} a^{-1}_{2l-1,1}.$$

From $T'_{2l-1}$ we have $x'_{2l-1} = x_{2l-1}$, and substituting in $A_{2l-1}$ we obtain the relation $U_{2l-1} = x^{-1}_{2l,1} x_{2l-1} a^{-1}_{2l-1,1}$. Analogously, from $T_{2l-1} = x_{2l-1,j_{2l-1}} x'_{2l,j_{2l}}$ and $A_{2l,j_{2l}} = x'_{2l,j_{2l}} x_{2l,j_{2l}} a^{-1}_{2l,j_{2l}}$, we obtain $U_{2l} = x^{-1}_{2l-1,j_{2l-1}} x_{2l,j_{2l}} a^{-1}_{2l,j_{2l}}$. From $T'_{2l} = x'_{2l,1} x'^{-1}_{2l-1,j_{2l-1}}$ and $A_{2l,1} = x'^{-1}_{2l,1} x_{2l,1} a^{-1}_{2l,1}$, which are the only relations involving $x'_{2l,1}$, we obtain $x'^{-1}_{2l-1,j_{2l-1}} x_{2l,1} a^{-1}_{2l,1}$, from this and

$$A_{2l-1,j_{2l-1}} = x'^{-1}_{2l-1,j_{2l-1}} x_{2l-1,j_{2l-1}} a^{-1}_{2l-1,j_{2l-1}}$$

we have $a_{2l,1} x^{-1}_{2l,1} x_{2l-1,j_{2l-1}} a^{-1}_{2l-1,j_{2l-1}} = V_{2l}$. From $Q'_{t'} = x'_{t'1} z'^{-1}_{t',j_{t'}}$ and $A_{t'1} = x'^{-1}_{t'1} x_{t'1} a^{-1}_{t'1}$ we have $x'^{-1}_{t',j_{t'}} x_{t'1} a^{-1}_{t'1}$, and from this and $A_{t'j_{t'}} = x'^{-1}_{t',j_{t'}} x_{t'j_{t'}} a^{-1}_{t'j_{t'}}$ we obtain $W_{t'} = a_{t'1} x^{-1}_{t'1} x_{t',j_{t'}} a^{-1}_{t',j_{t'}}$ ($t' = 2h + 1, \cdots, 2h + \mu - 1$). Therefore, we have the following presentation of $G$.

$$\text{(IV)} \qquad G: \{x_{i1}, x_{ij_i}, a_{ij}/r_{ij}, R_i, U_{2l-1}, Y_{2l}, V_{2l}, W_{t'}, T_{2l}, Q_t\}.$$

where
$$\begin{aligned} r_{ij} &= a^{\pm 1}_{pq}\, a_{ij}\, a^{\mp 1}_{pq}\, a^{-1}_{i,j+1} \\ R_i &= w_i(a) x_{i1}\, w_i^{-1}(a) x^{-1}_{ij_i} \\ U_{2l-1} &= x^{-1}_{2l,1}\, x_{2l-1,1}\, a^{-1}_{2l-1,1} \\ U_{2l} &= x^{-1}_{2l-1,j_{2l-1}}\, x_{2l,j_{2l}}\, a^{-1}_{2l,j_{2l}} \\ V_{2l} &= a_{2l,1}\, x^{-1}_{2l,1}\, x_{2l-1,j_{2l-1}}\, a^{-1}_{2l-1,j_{2l-1}} \\ W_{t'} &= a_{t'1}\, x^{-1}_{t'1}\, x_{t'j_{t'}}\, a^{-1}_{t'j_{t'}} \\ T_{2l} &= x_{2l,j_{2l}}\, x^{-1}_{2l+1,1} \\ Q_t &= x_{tj_t}\, x^{-1}_{t+1,1} \end{aligned} \qquad \left(\begin{aligned} i &= 1, \cdots, 2h + \mu - 1 \\ j &= 1, \cdots, j_i \\ l &= 1, \cdots, k \\ t' &= 2h + 1, \cdots, 2h + \mu - 1 \\ t &= 2h + 1, \cdots, 2h + \mu - 2 \end{aligned}\right)$$

From $W_{t'} = a_{t'1}x_{t'1}^{-1}x_{t'j_{t'}}a_{t'j_{t'}}^{-1}$ and $R_{t'} = w_{t'}(a)x_{t'1}w_{t'}^{-1}x_{t'j_{t'}}^{-1}$ we obtain the relation $R'_{t'} = w_{t'}(a)x_{t'1}w_{t'}^{-1}(a)a_{t'j_{t'}}^{-1}a_{t'1}x_{t'1}^{-1}$ $(t' = 2h + 1, \cdots, 2h + \mu - 1)$, and $R_{t'}$ is a consequence of $W_{t'}$ and $R'_{t'}$. From the relations

$$U_{2l-1} = x_{2l,1}^{-1}x_{2l-1,1}a_{2l-1,1}^{-1} \quad \text{and} \quad V_2 = a_{2l,1}x_{2l,1}^{-1}x_{2l-1,j_{2l-1}}a_{2l-1,j_{2l-1}}^{-1}$$

we obtain $a_{2l,1}a_{2l-1,1}x_{2l-1,1}^{-1}x_{2l-1,j_{2l-1}}a_{2l-1,j_{2l-1}}^{-1}$, and from this and

$$R_{2l-1} = w_{2l-1}(a)x_{2l-1,1}w_{2l-1}^{-1}(a)x_{2l-1,j_{2l-1}}^{-1},$$

we have $R'_{2l-1} = w_{2l-1}(a)x_{2l-1,1}w_{2l-1}^{-1}(a)a_{2l-1,j_{2l-1}}^{-1}a_{2l,1}a_{2l-1,1}x_{2l-1,1}^{-1}$ $(l = 1, \cdots, k)$, and $R_{2l-1}$ is a consequence of $R'_{2l-1}$, $V_{2l}$ and $U_{2l-1}$. From

$$U_{2l} = x_{2l-1,j_{2l-1}}^{-1}x_{2l,j_{2l}}a_{2l,j_{2l}}^{-1} \quad \text{and} \quad V_{2l} = a_{2l,1}x_{2l,1}^{-1}x_{2l-1,j_{2l-1}}a_{2l-1,j_{2l-1}}^{-1}$$

we have $a_{2l,1}x_{2l,1}^{-1}x_{2l,j_{2l}}a_{2l,j_{2l}}^{-1}a_{2l-1,j_{2l-1}}^{-1}$, and from this and

$$R_{2l} = w_{2l}(a)x_{2l,1}w_{2l}^{-1}(a)x_{2l,j_{2l}}^{-1}$$

we obtain the relation

$$R'_{2l} = w_{2l}(a)x_{2l,1}w_{2l}^{-1}(a)a_{2l,j_{2l}}^{-1}a_{2l-1,j_{2l-1}}^{-1}a_{2l,1}x_{2l,1}^{-1} \qquad (l = 1, \cdots, k),$$

and $R_{2l}$ is a consequence of $R'_{2l}$, $U_{2l}$ and $V_{2l}$. Therefore $G$ is given by:

$$\text{(V)} \qquad G: \{x_{i1}, x_{ij_i}, a_{ij}/r_{ij}, R'_i, U_{2l-1}, U_{2l}, V_{2l}, W_{t'}, T_{2l}, Q_t\},$$

where

$$R = \begin{cases} w_{2l-1}(a)x_{2l-1,1}w_{2l-1}^{-1}(a)a_{2l-1,j_{2l-1}}^{-1}a_{2l,1}a_{2l-1,1}x_{2l-1,1}^{-1} & (i = 2l - 1) \\ w_{2l}(a)x_{2l,1}w_{2l}^{-1}(a)a_{2l,j_{2l}}^{-1}a_{2l-1,j_{2l-1}}^{-1}a_{2l,1}x_{2l,1}^{-1} & (i = 2l) \\ w_{t'}(a)x_{t'1}w_{t'}^{-1}(a)a_{t'j_{t'}}^{-1}a_{t'1}x_{t'1}^{-1} \quad (i = t' = 2h + 1, \cdots, 2h + \mu - 1). \end{cases}$$

Let us define a homomorphism $\theta: RH \to \mathfrak{R}[t, t^{-1}]$ of the group ring $RH$ onto the ring $\mathfrak{R}[t, t^{-1}]$ of integral polynomials in $t$ and $t^{-1}$ as: $\theta(t_i^{\pm 1}) = t_i^{\pm 1}$ $(i = 1, \cdots, \mu)$. Let $\| (\partial\mu/\partial x)^{\psi\phi} \|$ be the Alexander matrix corresponding to a presentation of $G$, and let $M(t)$ be the matrix $M(t) = \| (\partial\mu/\partial\psi)^{\theta\psi\phi} \|$. We know that the Alexander polynomial $\Delta(t, \cdots, t_\mu)$ of $K$ is the g.c.d. of the minor determinants of $\| (\partial\mu/\partial x)^{\psi\phi} \|$ of column deficiency 1. Therefore the polynomial $\Delta(t, \cdots, t)$ will be the g.c.d. of the minor determinants of $M(t)$ of column deficiency equal to 1.

We are going to proceed to the calculation of the matrix $M(t)$ which corresponds to the last presentation of $G$.

Observe, first of all, that $\theta\psi\phi(x_{i1}) = \theta\psi\phi(x_{ij_i}) = t$, and $\theta\psi\phi(a_{ij}) = 1$ $(i = 1, \cdots, 2h + \mu - 1, j = 1, \cdots, j_i)$. Therefore $\phi\psi\phi w(a) = 1$ for every word $w(a)$ in the $\{a_{ij}\}$.

Let us study the contributions made to $M(t)$ by the various types of relations. (From now on, all the derivatives will be considered to be evaluated in $\mathfrak{R}[t, t^{-1}]$).

1. $r_{ij} = a_{pq}^{\pm 1}a_{ij}a_{pq}^{\mp 1}a_{i,j+1}^{-1}$

the only non-zero contributions are:

$$\frac{\partial r_{ij}}{\partial a_{ij}} = 1 \quad \text{and} \quad \frac{\partial r_{ij}}{\partial a_{i\,j+1}} = -1$$

2. (a) $R'_i = w_i(a)x_{i1}w_i^{-1}(a)a_{ij_i}^{-1}a_{i+1,1}a_{i,1}x_{i,1}^{-1}$ $(i = 2l - 1)$
the only possibly non-zero contributions are:

$$\frac{\partial R'_i}{\partial a_{pq}} = \frac{\partial w_i(a)}{\partial a_{pq}}(1 - t) \text{ if } a_{pq} \neq a_{ij_i}, a_{i+1,1}\, a_{i-1,1}$$

$$\frac{\partial R'_i}{\partial a_{ij_i}} = \frac{\partial w_i(a)}{\partial a_{ij_i}}(1 - t) - t; \qquad \frac{\partial R'_i}{\partial a_{i+1,1}} = \frac{\partial w_i(a)}{\partial a_{i+1,1}}(1 - t) + t;$$

$$\frac{\partial R_i}{\partial a_{i1}} = \frac{\partial w_i(a)}{\partial a_{i1}}(1 - t) + t$$

(b) $R'_i = w_i(a)x_{i1}w_i^{-1}(a)a_{i,j_i}^{-1}a_{i-1,j_{i-1}}^{-1}a_{i1}x_{i1}^{-1}$ $(i = 2l)$

$$\frac{\partial R'_i}{\partial a_{pq}} = \frac{\partial w_i(a)}{\partial a_{pq}}(1 - t) \text{ if } a_{pq} \neq a_{i1}, a_{ij_i}, a_{i-1,j_{i-1}}$$

$$\frac{\partial R'_i}{\partial a_{ij_i}} = \frac{\partial w_i(a)}{\partial a_{ij_i}}(1 - t) - t; \qquad \frac{\partial R'_i}{\partial a_{i-1,j_{i-1}}} = \frac{\partial w_i(a)}{\partial a_{i-1,j_{i-1}}}(1 - t) - t;$$

$$\frac{\partial R'_i}{\partial a_{i1}} = \frac{\partial w_i(a)}{\partial a_{i1}}(1 - t) + t$$

(c) $R'_i = w_i(a)x_{i1}w_i^{-1}(a)a_{ij_i}^{-1}a_{i1}x_{i1}^{-1}$ $(i = t')$

$$\frac{\partial R'_i}{\partial a_{pq}} = \frac{\partial w_i(a)}{\partial a_{pq}}(1 - t) \quad \text{if} \quad a_{pq} \neq a_{ij_i}, a_{i1}$$

$$\frac{\partial R'_i}{\partial a_{ij_i}} = \frac{\partial w_i(a)}{\partial a_{ij_i}}(1 - t) - t; \qquad \frac{\partial R_i}{\partial a_{i1}} = \frac{\partial w_i(a)}{\partial a_{i1}}(1 - t) + t.$$

3. $U_{2l-1} = x_{2l,1}^{-1}x_{2l-1,1}a_{2l-1,1}^{-1}$
the only non-zero contributions are:

$$\frac{\partial U_{2l-1}}{\partial x_{2l,1}} = -t^{-1}, \qquad \frac{\partial U_{2l-1}}{\partial x_{2l-1,1}} = t^{-1} \quad \text{and} \quad \frac{\partial U_{2l-1}}{\partial a_{2l-1,1}} = -1.$$

4. $U_{2l} = x_{2l,1}^{-1}x_{2l-1,1}a_{2l-1,1}^{-1}$
the non-zero contributions are:

$$\frac{\partial U_{2l}}{\partial x_{2l-1,j_{2l-1}}} = -t^{-1}; \qquad \frac{\partial U_{2l}}{\partial x_{2l,j_{2l}}} = t^{-1}; \qquad \frac{\partial U_{2l}}{\partial a_{2l,j_{2l}}} = -1.$$

5. $V_{2l} = a_{2l,1}x_{2l,1}^{-1}x_{2l-1,j_{2l-1}}a_{2l-1,j_{2l-1}}^{-1}$

$$\frac{\partial V_{2l}}{\partial a_{2l,1}} = 1; \qquad \frac{\partial V_{2l}}{\partial x_{2l,1}} = -t^{-1}; \qquad \frac{\partial V_{2l}}{\partial x_{2l-1,j_{2l-1}}} = t^{-1}; \qquad \frac{\partial V_{2l}}{\partial a_{2l-1,j_{2l-1}}^{-1}} = -1.$$

6. $W_{t'} = a_{t'1}x_{t'1}^{-1}x_{t'j_{t'}}a_{t'j_{t'}}^{-1}$

$$\frac{\partial W_{t'}}{\partial a_{t'1}} = 1; \qquad \frac{\partial W_{t'}}{\partial x_{t'1}} = -t^{-1}; \qquad \frac{\partial W_{t'}}{\partial x_{t'j_{t'}}} = t^{-1}; \qquad \frac{\partial W_{t'}}{\partial a_{t'j_{t'}}} = -1.$$

7. $T_{2l} = x_{2l,j_{2l}} x_{2l+1,1}^{-1}$

$$\frac{\partial T_{2l}}{2x_{2l,j_{2l}}} = 1 \quad \text{and} \quad \frac{\partial T_{2l}}{\partial x_{2l+1,1}} = -1.$$

8. $Q_t = x_{tj_t} x_{t+1,1}^{-1}$

$$\frac{\partial Q_t}{\partial x_{tj_t}} = 1 \quad \text{and} \quad \frac{\partial Q_t}{\partial x_{t+1,1}} = -1.$$

Consider the submatrix $B_i$ of $M(t)$, corresponding to the generators $a_{i1}$, $a_{i2}, \cdots, a_{ij_i}$ and the relations $r_{i1}, \cdots, r_{i,j_i-1}$. Observe that all elements in the same rows and outside $B_i$ are zero. Consider any element $g$ outside $B_i$ and in the column corresponding to $a_{ij}$. If we sum to the row to which $g$ belongs,

| | $a_{i1}$ | $a_{i2}$ | $a_{i3}$ | $\cdots$ | $a_{ij_1}$ |
|---|---|---|---|---|---|
| $r_{i1}$ | 1 | $-1$ | 0 | · | 0 |
| $r_{i2}$ | 0 | 1 | $-1$ | · | 0 |
| $r_{i3}$ | 0 | 0 | 1 | · | 0 |
| $\vdots$ | ······ | ······ | ······ | ······ | ······ |
| $r_{ij_1}$ | 0 | 0 | 0 | | 1 |

the rows corresponding to $r_{ij}, r_{i,j+1}, \cdots, r_{ij_i}$ multiplied by $-g$, we obtain a matrix in which the element appears in the column corresponding to $a_{ij_i}$. Therefore, applying this process for every element below $B_i$ $(i = 1, \cdots, 2h + \mu - 1)$, we obtain a matrix $M'(t)$ equivalent to $M(t)$ in which the only elements different from zero below $B_i$ are in the columns corresponding to $a_{ij_i}$ $(i = 1, \cdots, 2h + \mu - 1)$, and this element will be the sum of the elements of the original matrix which are in the same row and in the columns corresponding to $a_{i1}$, $a_{i2}, \cdots, a_{i,j_i}$.

Let us study the elements of $M'(t)$ in the rows corresponding to $R'_i$ $(i = 1, \cdots, 2h + \mu - 1)$.

(a) The elements different from zero in the row corresponding to $R'_{2l-1}$ $(l = 1, \cdots, h)$ are $\sum_{q=1}^{q=j_p}(\partial w_{l-1}(a)/\partial a_{pq})(1 - t)$ in the column corresponding to $a_{pj_p}$, if

$$p \neq 2l$$

and $\sum_{q=1}^{j_{2l}}(\partial w_{2l-1}(a)/\partial a_{2l,q})(1 - t) + t$ in the column corresponding to $a_{2l,j_{2l}}$ (see above).

(b) The elements different from zero in the row corresponding to $R'_{2l}$ $(l = 1, \cdots, h)$ are $\sum_{q=1}^{j_p}(\partial w_{2l}(a)/\partial a_{pq})(1 - t)$ in the column corresponding to $a_{pj_p}$, if $p \neq 2l - 1$, and $\sum_{q=1}^{j_{2l-1}}(\partial w_{2l}(a)/\partial a_{2l-1,q})(1 - t) - t$ in the column corresponding to $a_{2l-1,j_{2l-1}}$ (see above).

(c) The elements different from zero in the rows corresponding to $R'_{t'}$ $(t' = h + 1, \cdots, 2h + \mu - 1)$ are $\sum_{q=1}^{j_p}(\partial w_{t'}(a)/\partial a_{pq})(1 - t)$ $(p = 1, \cdots, 2h + \mu - 1)$. (See above 2).

In $M'(t)$, the only element different from zero in the column corresponding to $a_{ij}$ $(i = 1, \cdots, 2h + \mu - 1; j = 1, \cdots, j_i - 1)$ is in the row corresponding to

$r_{ij}$ and its value is 1, therefore it is clear that $M'(t)$ is equivalent to

$(i = 1, \cdots, 2h + \mu - 1)$

| | | $a_{ij}(j < j_i)$ | $a_{1j_1}a_{2j_2} \cdots a_{2h+\mu-1, j_{2h+\mu-1}}$ | $x_{i1}, x_{ij_i}$ |
|---|---|---|---|---|
| $M''(t) =$ | $r_{ij}$ | $E$ | 0 | 0 |
| | $R'_i$ | 0 | $A$ | 0 |
| | $u$<br>$w$<br>$v$<br>$T$<br>$Q$ | 0 | $C$ | $B$ |

where $E$ is a unit matrix.

The matrix $M''(t)$ has one column more than rows, therefore for calculating $\Delta(t)$ we may use the theorem in §3, chapter I. $(1 - t)\Delta(t)$ will be equal to the determinant of the matrix $M'''(t)$ which is obtained from $M''(t)$ by deleting the column corresponding to $x_{2h+\mu-1, j_{2h+\mu-1}}$; therefore: $(1 - t)\Delta(t) = |A||B'|$, where $B'$ is a submatrix of $B$ in $M'''(t)$. $B'$ is of the form, (see opposite page) which, clearly, is equivalent to a unit matrix. Therefore, $(1 - t)\Delta(t) = |A|$, where $A$ is the matrix corresponding to the generators $a_{ij_i}$ $(i = 1, \cdots, 2h + \mu - 1)$ and to the relations $R'_i$ $(i = 1, \cdots, 2h + \mu - 1)$. Now, since

$$\sum_{j=1}^{j=j_q} (\partial w_p(a)/\partial a_{qj}) = v_{pq} \qquad (p = 1, \cdots, 2h + \mu - 1, q = 1, \cdots, j_p)$$

(see the definition of $w_p(a)$ above, and the definition of the crossing numbers $v_{pq}$ in §4, chapter I), we have: (see last paragraph of Chapter I, page 64). Multiplying each row by $t^{-1}$, transposing, making use of the relations between the crossing numbers (§4, chapter I) and multiplying the rows by $-1$, we obtain

$$(-1)^{\mu-1}t^{-(2h+\mu-1)}(1 - t)\Delta(t) = (1 - t^{-1})\Delta(t^{-1}), \text{ or } \Delta(t) = (-1)^{\mu}t^{2h+\mu-2}\Delta(t^{-1})$$

which completes the proof of Theorem I.[8]

**2.** Theorem 2. *There exist integers* $\nu_1, \cdots, \nu_\mu$, *such that,* $\Delta(t_1, t_2, \cdots, t_\mu) = (-1)^\mu t_1^{\nu_1} \cdots t_\mu^{\nu_\mu} \Delta(t_1^{-1}, \cdots, t_\mu^{-1})$. Theorem 2 is an immediate consequence of the following two lemmas:

Lemma 1. *Let* $\Delta(t^{n_1}, \cdots, t^{n_\mu})$ *be the polynomial in* $t$, *obtained by substituting* $t^{n_i}$ *for* $t_i$ $(i = 1, \cdots, \mu)$ in $\Delta(t_1, \cdots, t_\mu)$, *where* $n_1, \cdots, n_\mu$ *are arbitrary positive integers. Then, there exists* $N$, *such that:* $\Delta(t^{n_1}, \cdots, t^{n_\mu}) = (-1)^\mu t^N (t^{-n_1}, \cdots, t^{-n_\mu})$.

Lemma 2. *If an integral polynomial* $P(t_1, \cdots, t_\mu)$ *in* $t_1, \cdots, t_\mu$ *has the property that for arbitrary positive integers* $n_1, \cdots, n_\mu$, *the polynomial*

$$Q(t) = P(t^{n_1}, \cdots, t^{n_\mu})$$

[8] The genus of $\bar{h}$ a knot $K$ is defined to be the genus of a surface of minimum genus which can span $K$. As a corollary of Theorem 1 we have: $2\bar{h} + \mu - 2 \geq \partial^\circ\Delta(t)$, where $\partial^\circ\Delta(t)$ is the degree of $\Delta(t)$. Therefore, $\partial^\circ\Delta(t) + 2 - \mu$ is a lower bound for $\bar{h}$.

$$B' = \begin{array}{c|ccccccccc|cccc|}
 & x_{11} & x_{21} & x_{1j_1} & x_{2j_1} & \cdots & x_{2h-1,1} & x_{2h,1} & x_{2h-1,j_{2h-1}} & x_{2h,j_{2h}} & x_{2h+1,1} & x_{2h+1,j_{2h+1}} & \cdots & x_{2h+\mu-1} \\ \hline
u_1 & t^{-1} & -t^{1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
v_2 & 0 & t^{-1} & t^{-1} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
u_2 & 0 & 0 & -t^{-1} & t^{-1} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
T_2 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & & \vdots & & & & & \vdots & & & \vdots & & \\
u_{2h-1} & 0 & 0 & 0 & 0 & \cdots & t & -t^{-1} & 0 & 0 & 0 & 0 & \cdots & 0 \\
v_{2h} & 0 & 0 & 0 & 0 & \cdots & 0 & -t^{-1} & t^{-1} & 0 & 0 & 0 & \cdots & 0 \\
u_{2h} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -t^{-1} & t^{-1} & 0 & 0 & \cdots & 0 \\
T_{2h} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & -1 & 0 & \cdots & 0 \\ \hline
W_{2h+1} & & & & & & & & & & -t^{-1} & t^{-1} & \cdots & 0 \\
Q_{2h+1} & & & & & & & & & & 0 & 1 & -1 & 0 \\
\vdots & & & & & 0 & & & & & & \vdots & & \\
W_{2h+\mu-1} & & & & & & & & & & 0 & 0 & -t^{-1} & t^{-1}
\end{array}$$

$$(1-t)\Delta(t) = \begin{array}{c|ccccc|cc|}
 & a_{1j_1} & a_{2j_2} & \cdots\ a_{2h-1,j_{2h-1}} & a_{2h,j_{2h}} & & a_{2h+1,j_{2h+1}} & \cdots\ a_{2h+\mu-1\ j_{2h+\mu-1}} \\ \hline
R_1^1 & v_{11}(1-t), & v_{12}(1-t)+t & \cdots\ v_{1,2h-1}(1-t), & v_{1,2h}(1-t) & & v_{1,2h+1}(1-t) & \cdots\ v_{1,2h+\mu-1}(1-t) \\
R_2^1 & v_{21}(1-t)-t, & v_{22}(1-t) & \cdots\ v_{2,2h-1}(1-t), & v_{2,2h}(1-t) & & v_{2,2h+1}(1-t) & \cdots\ v_{2,2h+\mu-1}(1-t) \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
R_{2h-1}^1 & v_{2h-1,1}(1-t), & v_{2h-1,2}(1-t) & \cdots\ v_{2h-1,2h-1}(1-t), & v_{2h-1,2h}(1-t)+t & & v_{2h-1,2h+1}(1-t) & \cdots\ v_{2h-1,2h+\mu-1}(1-t) \\
R_{2h}^1 & v_{2h,1}(1-t), & v_{2h,2}(1-t) & \cdots\ v_{2h,2h-1}(1-t)-t, & v_{2h,2h}(1-t) & & v_{2h,2h+1}(1-t) & \cdots\ v_{2h,2h+\mu-1}(1-t) \\ \hline
R_{2h+1}^1 & v_{2h+1,1}(1-t), & v_{2h+1,2}(1-t) & \cdots\ v_{2h+1,2h-1}(1-t), & v_{2h+1,2h}(1-t) & & v_{2h+1,2h+1}(1-t) & \cdots\ v_{2h+1,2h+\mu-1}(1-t) \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
R_{2h+\mu-1}^1 & v_{2h+\mu-1,2}(1-t), & v_{2h+\mu-1,2}(1-t) & \cdots\ v_{2h+\mu-1,2h-1}(1-t), & v_{2h+\mu-1,2h}(1-t) & & v_{2h+\mu-1,2h+1}(1-t) & \cdots\ v_{2h+\mu-1,2h+\mu-1}(1-t)
\end{array}$$

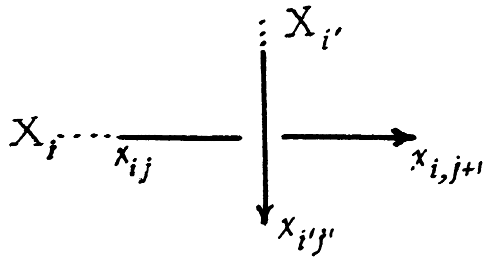


FIG. 12

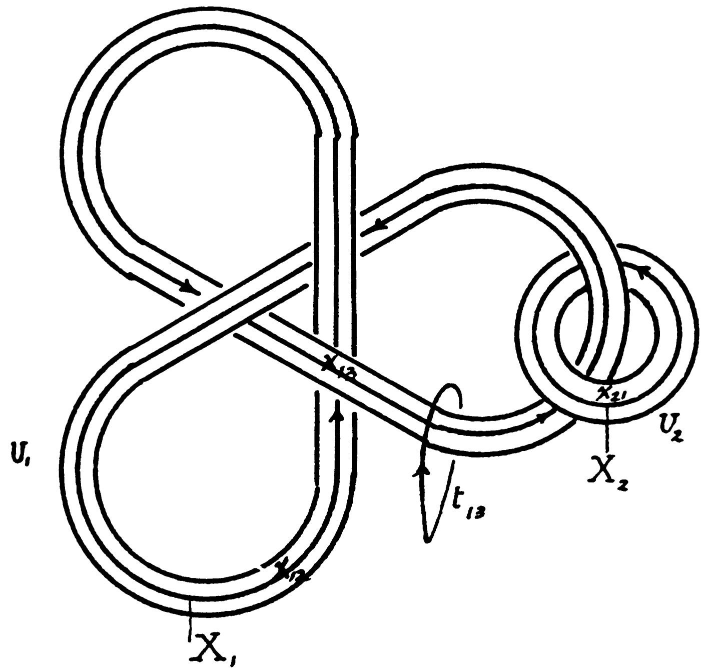


FIG. 13

*is such that there exists N for which* $Q(t) = (-1)^{\mu}t^{N}.\ Q(1/t)$. *Then, there exist integers* $\nu_1, \cdots, \nu_\mu$ *such that,* $P(t_1, \cdots, t_\mu) = (-1)^{\mu} t_1^{\nu_1} \cdots t_\mu^{\nu_\mu} P(t_1, \cdots, t_\mu)$.

PROOF OF LEMMA I. Let $\{x_{ij}/r_{ij}\}$ be a presentation of $G$ obtained from any projection of $K$, where $x_{ij}$ $(i = 1, \cdots, \mu; j = 1, \cdots, j_i)$ denotes the generator corresponding to the edge $x_{ij}$ belonging to the projection of $X_i$, and $r_{ij}$ $(i = 1, \cdots, \mu; j = 1, \cdots, j_k)$ denotes the word $x_{i'j'}^{\pm 1} x_{ij} z_{i'j'}^{\mp 1} x_{i,j+1}^{-1}$ which is the relation corresponding to a crossing of $X$ over $X_i$ (fig. 12). Consider the matrix $M(t_1, \cdots, t_u) = \| (\partial r_{pq}^{i}/\partial x_{ij})^{\psi\phi} \|$, in which we have suppressed a row corresponding to a relation which is a consequence of the other relations. We know that the minor determinant obtained by suppressing, in $M(t_1, \cdots, t_\mu)$, the column corresponding to $x_{ij}$ is

$$\pm(t_i - 1)\Delta(t_1, \cdots, t_\mu), \qquad (i = 1, \cdots, \mu; j = 1, \cdots, j_i).$$

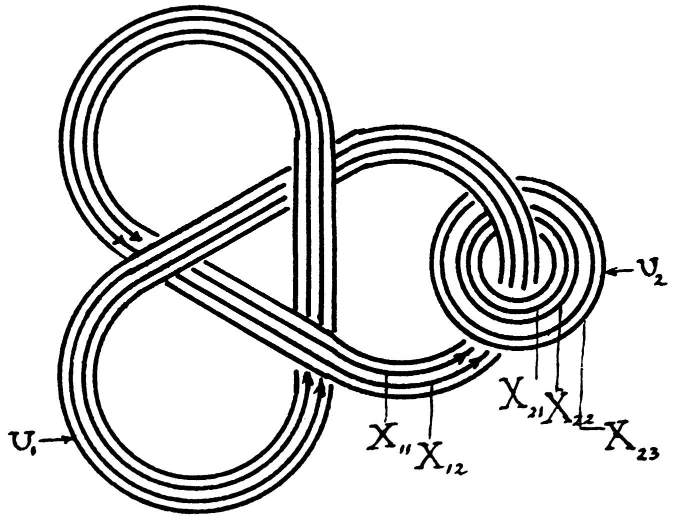


FIG. 14

We are going to construct a knot $K_{n_1,\cdots,n_\mu}$ of multiplicity $n_1 + n_2 + \cdots + n_\mu$, associated to the projection of $K$, where $n_1, \cdots, n_\mu$ are arbitrary positive integers, as follows:

Let us surround each $X_i$ by a tube $U_i$ ($i = 1, \cdots, \mu$) in such a way that the central line of $U_i$ coincides with $X_i$ (fig. 13), and $U_i \cap U_j = \emptyset$ if $i \neq j$. Consider in each $U_i$, $n_i$ copies $X_i^{(1)}, \cdots, X_i^{(n_i)}$ of $X_i$, in such a manner that the projection of $X_i^{(K+1)}$ is always on the left side of the projection of $X_i^{(K)}$ ($k = 1, \cdots, n_i - 1$) (fig. 14). Let $K_{n_1,\cdots,n_\mu}$ be the knot whose components are $X_i^{(p)}$ ($i = 1, \cdots, \mu$; $p = 1, \cdots, n_i$).

We shall proceed to the computation of the group $G'$ of $K_{n_1,\cdots,n_\mu}$.

Corresponding to each generator $x_{ij}$ of $K$, we have $n_i$ generators $x_{ij}^{(1)}, \cdots, x_{ij}^{(n_i)}$ of $G'$. Where $x_{ij}^{(p)}$ ($p = 1, \cdots, n_i$) is the generator corresponding to the edge $x_{ij}^{(p)}$ which belongs to the projection of $X_i^{(p)}$.

Corresponding to each relation $r_{ij} = x_{i'j'}^{\pm 1} x_{ij} x_{i'j'}^{\mp 1} x_{i,j+1}^{-1}$ of $G$, we have $n_i$ relations $r_{ij}^{(p)} = (x_{i'j'}^{(n_{i'})} \cdots x_{i'j'}^{(1)})^{\pm 1} x_{ij}^{(p)} (x_{i'j'}^{(n_{i'})} \cdots x_{i'j'}^{(1)})^{\mp 1} x_{i,j+1}^{(p)-1}$ ($p = 1, \cdots, n_i$) (fig. 15). Therefore $G'$ has the presentation

$$\text{(I)} \qquad G' : \{x_{ij}^{(p)} / r_{ij}^{(p)}\} \qquad \begin{pmatrix} i = 1, \cdots, \mu \\ j = 1, \cdots, j_i \\ p = 1, \cdots, n_i \end{pmatrix}.$$

Let us introduce generators $X_{ij}$ defined by means of the relations $R_{ij} = x_{ij}^{(n_i)} x_{ij}^{(n_i-1)} \cdots x_{ij}^{(1)} X_{ij}^{-1}$ ($i = 1, \cdots, \mu; j = 1, \cdots, j_i$). From $r_{ij}^{(p)}$ and $R_{i'j'}$ we obtain $s_{ij}^{(p)} = X_{i'j'}^{\pm 1} x_{ij}^{(p)} X_{i'j'}^{\mp 1} x_{i,j+1}^{(p)^{-1}}$ ($p = 1, \cdots, n_i$), and $r_{ij}^{(p)}$ is a consequence of $R_{i'j'}$ and $s_{ij}^{(p)}$. From the relations $s_{ij}^{(1)}, \cdots, s_{ij}^{(n_i)}$ we obtain $X_{i'j'}^{\pm 1} x_{ij}^{(n_i)} \cdots x_{ij}^{(1)} X_{i'j'}^{\mp 1} (x_{i,j+1}^{(n_i)} \cdots x_{i,j+1}^{(1)})^{-1}$ and using $R_{ij}$ and $R_{i,j+1}$ we have $u_{ij} = X_{i'j'}^{\pm 1} X_{ij} X_{i'j'}^{\mp 1} X_{i,j+1}^{-1}$ ($i = 1, \cdots, \mu; j = 1, \cdots, j_i$). Each relation $R_{i,j+1}$ ($j = 1, \cdots, j_i$) is a consequence of the preceding $R_{ij}$ and $s_{ij}^{(p)}$ ($p = 1, \cdots, n_i$), therefore we can delete the relations $R_{ij}$ ($j > 1$). The relation $s_{i1}^{(p)}$ expresses $x_{i2}^{(p)}$ as a transformed of

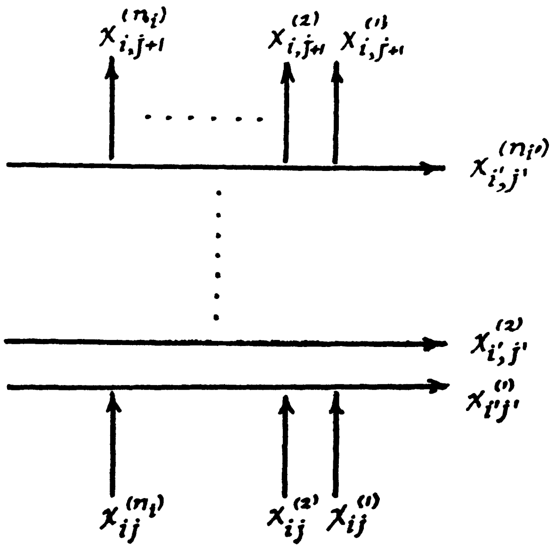


FIG. 15

$x_{i1}^{(p)}$, and $s_{i2}^{(p)}$ expresses $x_{i3}^{(p)}$ as a transformed of $x_{i2}^{(p)}$, therefore, using $s_{i,1}^{(p)}$ and $s_{i,2}^{(p)}$ we can express $x_{i,3}^{(p)}$ as a transformed of $x_{i,1}^{(p)}$, and introducing this expression we can eliminate the generator $x_{i,2}^{(p)}$ and the relations $s_{i,1}^{(p)}$ and $s_{i,2}^{(p)}$. Repeating this process, we can eliminate the generators $x_{i,2}^{(p)}, x_{i,3}^{(p)}, \cdots, x_{i,j_i}^{(p)}$ and the relations $s_{i,1}^{(p)}, \cdots, s_{ij_i}^{(p)}$ $(i = 1, \cdots, \mu)$, by introducing a relation $Q_i^{(p)} = w_i(X) x_{i1}^{(p)} w_i^{-1}(X) x_{i1}^{(p)-1}$, where $w_i(X)$ is the word in the $\{X_{ij}\}$ which is read off from the projection of $X_i$ by going all the way along $X_i$ from $x_{i1}$ to $x_{i1}$. For each crossing of $X_j$ over $X_i$, from left to right, there is an appearance of a generator $X_{jp}$ in $w_i(X)$, and for each crossing from right to left there is an appearance of $X_{jp}^{-1}$.

It is clear that from the relations $u_{i1}, \cdots, u_{i,j_i}$ we obtain $w_i(X) X_{i1} w_i^{-1}(X) X_{i1}^{-1}$, and from this and $R_{i1}$ and $Q_i^{(1)}, Q_i^{(2)}, \cdots, Q_i^{(n_i-1)}$ we obtain $Q_i^{(n_i)}$. Therefore we can delete $Q_i^{(n_i)}$ $(i = 1, \cdots, \mu)$. Finally, since the relation $u_{ij} = X_{i'j'}^{\pm 1} X_{ij} X_{i'j'}^{\mp 1} X_{i,j+1}^{-1}$ has the same form as the defining relation $r_{ij} = x_{i'j'}^{\pm 1} x_{ij} x_{i'j'}^{\mp 1} x_{i,j+1}$ of $G$, and we know that any of the $\{r_{ij}\}$ is a consequence of the others, then any of the $\{u_{ij}\}$ will be a consequence of the others, and we can delete it from the presentation of $G'$. Therefore $G'$ is given by:

$$\text{(II)} \qquad G' : \{X_{ij}\,;\, x_{i1}^{(p)} / u_{ij}\,;\, R_{i1}\,;\, Q_i^{(p)}\} \qquad \begin{pmatrix} i = 1, \cdots, \mu \\ j = 1, \cdots, j_i \\ p = 1, \cdots, n_{i-1} \end{pmatrix}.$$

and one of the $\{u_{ij}\}$ is deleted.

The commutator factor group $H'$ of $K_{n_1, \cdots, n_\mu}$ is generated by $\{t_{ip}\}$ $(i = 1, \cdots, \mu;\ p = 1, \cdots, n_i)$, where $t_{ip} = \psi\phi(x_{i1}^{(p)})$. Therefore $\psi\phi(X_{ij}) = t_{i1} t_{i2} \cdots t_{in_i}$ $(i = 1, \cdots, \mu)$.

Let us calculate the Jacobian matrix $M'$ corresponding to the presentation (II) of $G'$, (Page 77).

$$
M' = \begin{array}{l||c|cccc|cccc|c|cccc||}
 & X_{ij} & x_{11}^{(1)} & x_{11}^{(2)} & \cdots & x_{11}^{(n1)} & x_{21}^{(1)} & x_{21}^{(2)} & \cdots & x_{21}^{(n\ )} & \cdots & x_{\mu 1}^{(1)} & x_{\mu 1}^{(2)} & \cdots & x_{\mu 1}^{(n)} \\
\hline
u_{ij} & M(l) & & 0 & & & & 0 & & & \cdots & & 0 & & \\
\hline
R_{11} & & v_{11} & v_{12} & \cdots & v_{1n_1} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
R_{21} & & 0 & 0 & \cdots & 0 & v_{21} & v_{22} & \cdots & v_{2n_2} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
R_{\mu 1} & & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & v_{\mu} & v_{\mu 2} & \cdots & v_{\mu n} \\
\hline
 & & & & & & & & & & & & & & \\
\hline
Q_1^{(1)} & & \bar{w}_1 - 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
Q_1^{(2)} & & 0 & \bar{w}_1 - 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & A & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
Q_1^{(n_1-1)} & & 0 & 0 & \bar{w}_1 - 1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
Q_2^{(1)} & & 0 & 0 & \cdots & 0 & \bar{w}_2 - 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
Q_2^{(2)} & & 0 & 0 & \cdots & 0 & 0 & \bar{w}_2 - 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
Q_2^{(n_2-1)} & & 0 & 0 & \cdots & 0 & 0 & 0 & \bar{w}_2 - 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\hline
Q_{\mu}^{(1)} & & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \bar{w}_{\mu} - 1 & 0 & 0 & 0 \\
Q_{\mu}^{(2)} & & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \bar{w}_{\mu} - 1 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
Q_{\mu}^{(n-)} & & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \bar{w}_{\mu} - 1 & 0 \\
\end{array}
$$

where $M(l)$ is obtained by substituting $l_i = t_{i1}t_{i2} \cdots t_{in_i}$ for $t_i$ $(i = 1, \cdots, \mu)$ in $M(t_1, \cdots, t_\mu)$ (see §3) $v_{ip}$ $(i = 1, \cdots, \mu; p = 1, \cdots n_i)$ is a unit of $RH'$, and $\bar{w}_i = \psi\phi(w_i(X))$ $(i = 1, \cdots, \mu)$, and $A$ is a submatrix whose form we do not need to discuss.

The matrix $M'$ has one row less than columns, therefore $(1 - t_{11}t_{12} \cdots t_{1n_1})$ $\Delta' = |M''|$, where $\Delta'$ is the Alexander polynomial of $K_{n_1,\cdots,n_\mu}$ and $M''$ is the matrix which is obtained by deleting the column corresponding to $X_{11}$ in $M'$. Therefore $(1 - t_{11}t_{12} \cdots t_{1n_1})\Delta' = M^*(l) \mid B \mid$, where $M^*(l)$ is the matrix which is obtained from $M(l)$ by deleting the column corresponding to $X_{11}$, and therefore $|M^*(l)| = (1 - t_{11}t_{12} \cdots t_{1n_1})\Delta(l_1, l_2, \cdots, l_\mu)$, where $\Delta(l_1, \cdots, l_\mu)$ is the polynomial obtained from $\Delta(t_1, \cdots, t_\mu)$ by substituting $l_i$ for $t_i$ $(i = 1, \cdots, \mu)$. $B$ is the submatrix corresponding to the generators $\{x_{i1}^{(p)}\}$ and the relations $\{R_{i1}\}$ and $\{Q_i^{(p)}\}$. Therefore it is clear that $|B| = v\prod_{i=1}^{\mu} (\bar{w}_i - 1)^{n_i-1}$, where $v$ is a unit of $RH'$. Therefore: $\Delta' = \prod_{i=1}^{\mu} (\bar{w}_i - 1)^{n_i-1}\Delta(l_1, \cdots, l_\mu)$.

If we substitute $t$ for $t_{ip}$ $(i = 1, \cdots, \mu; p = 1, \cdots, n_i)$, we have: $\Delta'(t) = \prod_{i=1}^{\mu} (\bar{\bar{w}}_i - 1)^{n_i-1}\Delta(t^{n_1}, \cdots, t^{n_\mu})$, where $\bar{\bar{w}}_i$ $(i = 1, \cdots, \mu)$ is a monomial in $t$.[9]

By Theorem 1, we have

$$\prod_{i=1}^{\mu} (\bar{\bar{w}}_i - 1)^{n_i-1}\Delta(t^{n_1}, \cdots, t^{n_\mu})$$
$$= (-1)^{n_1+\cdots+n_\mu}t^N \prod_{i=1}^{\mu} (\bar{\bar{w}}_i^{-1} - 1)^{n_i-1}\Delta(t_1^{-n_1}, \cdots, t_\mu^{-n_\delta}),$$

or $\Delta(t^{n_1}, \cdots, t^{n_\mu}) = (-1)^\mu t^{N'}\Delta(t^{-n_1}, \cdots, t^{-n_\mu})$, which completes the proof of Lemma 1.

Lemma 2 is a particular case, for $P = (-1)^\mu P'$ of the following.

LEMMA 3. *If two integral polynomials $P(t_1, \cdots, t_\mu)$ and $P'(t_1, \cdots, t_\mu)$ in $t_1, \cdots, t_\mu$ have the property that for arbitrary integers $n_1, \cdots, n_\mu$ the polynomials $P_{n_1,\cdots,n_\mu}(t) = P(t^{n_1}, \cdots, t^{n_\mu})$ and $P'_{n_1,\cdots,n_\mu}(t) = P'(t^{n_1}, \cdots, t^{n_\mu})$ are such that there exists an integer $N(n_1, \cdots, n_\mu)$ for which $P_{n_1,\cdots,n_\mu}(t) = t^{N(n_1,\cdots,n_\mu)}$ $P'_{n_1,\cdots,n_\mu}(t^{-1})$. Then, there exist integers $\nu_1, \cdots, \nu_\mu$ such that*

$$P(t_1, \cdots, t_\mu) = t_1^{\nu_1} \cdots t_\mu^{\nu_\mu}P'(t_1^{-1}, \cdots, t_\mu^{-1}).$$

PROOF. We shall prove this lemma by induction on the number $\mu$ of indeterminates.

If $\mu = 1$ the lemma is obviously true. If $\mu > 1$,

$$P(t_1, \cdots, t_\mu) = \phi_r(t_1, \cdots, t_{\mu-1})\, t_\mu^r + \phi_{r+1}(t_1, \cdots, t_{\mu-1})\, t_\mu^{r+1} + \cdots$$
$$+ \phi_s(t_1, \cdots, t_{\mu-1})\, t_\mu^s,$$

and

$$P'(t_1, \cdots, t_\mu) = \psi_p(t_1, \cdots, t_{\mu-1})\, t_\mu^p + \psi_{p+1}(t_1, \cdots, t_{\mu-1})t_\mu^{p+1} + \cdots$$
$$+ \psi_q(t_1, \cdots, t_{\mu-1})\, t_\mu^q,$$

where $\phi_{r+i}$ and $\psi_{p+j}$ are polynomials in $t_1, \cdots, t_{\mu-1}$, and $\phi_r, \phi_s, \psi_p, \psi_q$ are different from zero.

---

[9] It is clear that a projection of $K$ can be taken such that $\bar{\bar{w}}_i \neq 1 (i = 1, \cdots, \mu)$.

Let us choose integers $a_1, \cdots, a_{\mu-1}$ such that

$$\phi_r(t^{a_1}, \cdots, t^{a_{\mu-1}}),\ \phi_s(t^{a_1}, \cdots, t^{a_{\mu-1}}),\ \psi_p(t^{a_1}, \cdots, t^{a_{\mu-1}}),\ \psi_q(t^{a_1}, \cdots, t^{a_{\mu-1}})$$

are different from zero. Let us denote by $\bar{\phi}_i(t)$ the polynomial

$$\phi_i(t^{a_1}, \cdots, t^{a_{\mu-1}})\ (i = r, r+1, \cdots, s)$$

and by $\bar{\psi}_j(t)$ the polynomial $\psi_j(t^{a_1}, \cdots, t^{a_{\mu-1}})$ $(j = p, p+1, \cdots, q)$.

Let us denote by $\partial_i$ and $\partial'_j$ the degrees of $\bar{\phi}_i(t)$ and $\bar{\psi}_j(t)$ respectively, and by $\delta_i$ and $\delta'_i$ the degrees of the terms of minimum degree in $\bar{\phi}_i(t)$ and $\bar{\psi}_j(t)$ respectively. Let $a_\mu$ be larger than 2 sup. $(\partial_r, \cdots, \partial_s; \partial'_p, \cdots, \partial'_q)$. We have,

$$P_{a_1,\cdots,a_\mu}(t) = \bar{\phi}_r(t)t^{ra_\mu} + \bar{\phi}_{r+1}(t)t^{(r+1)a_\mu} + \cdots + \bar{\phi}_s(t)t^{sa_\mu},$$

$$P'_{a_1,\cdots,a_\mu}(t) = \bar{\psi}_p(t)t^{pa_\mu} + \bar{\psi}_{p+1}(t)t^{(p+1)a_\mu} + \cdots + \bar{\psi}_q(t)t^{qa_\mu}.$$

Therefore:

(1) $$\bar{\phi}_r(t)t^{ra_\mu} + \cdots + \bar{\phi}_s(t)t^{sa_\mu} = t^{N(a_1,\cdots,a_\mu)}\,[\bar{\psi}_p(t^{-1})t^{-pa_\mu} + \cdots + \psi_q(t^{-1})t^{-qa_\mu}].$$

It is clear that, if $i < i'$, the degree of each term of $\bar{\phi}_i(t)t^{i'a_\mu}$ is larger than the degree of every term of $\bar{\phi}_i(t)t^{ia_\mu}$, and therefore the degree and the minimum degree of the left hand side of (1) are $\partial_s + sa_\mu$ and $\delta_r + ra_\mu$ respectively. Analogously, the degree and minimum degree of the right hand side are $N(a_1, \cdots, a_\mu) - \delta'_p - pa_\mu$ and $N(a_1, \cdots, a_\mu) - \partial'_p - qa_\mu$ respectively. We have:

$$\partial_s + sa_\mu = N(a_1, \cdots, a_\mu) - \delta'_p - pa_\mu$$

and

$$\delta_r + ra_\mu = N(a_1, \cdots, a_\mu) - \partial'_q - qa_\mu,$$

therefore

$$\partial_s - \delta_r + \delta'_p - \partial'_q = a_\mu(q - p + r - s)$$

and 2 sup $(\partial_r, \cdots, \partial_s; \partial'_p, \cdots, \partial'_q) \geqq |\partial_s - \delta_r| + |\delta'_p - \partial'_q| \geqq a_\mu|q - p + r - s|$, so we have $q - p = s - r$.

Now let $n_1, \cdots, n_{\mu-1}$ be arbitrary, and let us denote by $\bar{\phi}_i(t)$ the polynomial $\phi_i(t^{n_1}, \cdots, t^{n_{\mu-1}})$ $(i = r, \cdots, s)$ and by $\bar{\psi}_j(t)$ the polynomial $\psi_j(t^{n_1}, \cdots, t^{n_{\mu-1}})$ $(j = p, \cdots, g)$.

Let $\partial_i$ and $\partial'_j$ be the degrees of $\bar{\phi}_i(t)$ and $\bar{\psi}_j(t)$ respectively, and let $\delta_i$ and $\delta'_j$ be the minimum degree of $\bar{\phi}_i(t)$ and $\bar{\psi}_j(t)$ respectively. And let $n_\mu$ be larger than 2 sup $(\partial_r, \cdots, \partial_s, \partial'_p, \cdots, \partial'_q)$. We have,

(2) $$\bar{\phi}_r(t)t^{rn_\mu} + \cdots + \bar{\phi}_s(t)t^{sn_\mu} = t^{N(n_1,\cdots,n_\mu)}\,[\bar{\psi}_p(t^{-})t^{-pn_\mu} + \cdots + \bar{\psi}_q(t^{-1})t^{-qn_\mu}].$$

As before we have

(3) $$\delta_r + rn_\mu = N(n_1, \cdots, n_\mu) - \partial'_q - qn_\mu.$$

Let $\alpha + in_\mu$ be the degree of a term of $\bar{\phi}_i(t)t^{in_\mu}$ $(r \leqq i \leqq s)$; that term cancels against a term of degree $N(n_1, \cdots, n_\mu) - \beta - j_i n_\mu$ which belongs to $t^{N(n_1,\cdots,n_\mu)}\bar{\psi}_{j_i}(t^{-1})t^{-j_i n_\mu}$ $(p \leqq j_i \leqq q)$. Therefore

$$\alpha + in_\mu = N(n_1, \cdots, n_\mu) - \beta - j_i n_\mu . \tag{4}$$

From (3) and (4), we have

$$\delta_r - \alpha + \partial'_q - \beta = n_\mu (j_i + i - r - q)$$

and $2 \sup (\partial_r, \cdots, \partial_s ; \partial'_p, \cdots, \partial'_q) \geqq | \delta_r - \alpha | + | \partial'_q - \beta | \geqq n_\mu | j_i + i - r - q |$, therefore

$$i + j_i = r + q; \tag{5}$$

that is, every term of $\bar{\phi}_i(t)t^{in_\mu}$ cancels against a term of $t^{N(n_1,\cdots,n_\mu)}\bar{\psi}_{j_i}(t^{-1})t^{-j_i n_\mu}$, where $j_i$ is given by (5).

Thus $\bar{\phi}_i(t)t^{in_\mu} = t^{N(n_1,\cdots,n_\mu)}\bar{\psi}_{j_i}(t^{-1})t^{-j_i n_\mu}$, or

$$\bar{\phi}_i(t) = t^{N(n_1,\cdots,n_\mu)-in_\mu-j_i n_\mu}\bar{\psi}_{j_i}(t^{-1}). \tag{6}$$

By the induction hypothesis, there exist integers $\nu_{i1}, \nu_{i2}, \cdots, \nu_{i,\mu-1}$ such that

$$\phi_i(t_1, \cdots, t_{\mu-1}) = t_1^{\nu i1}t_2^{\nu i2} \cdots t_{\mu-1}^{\nu i,\mu-1}\psi_{j_i}(t_1^{-1}, \cdots, t_{\mu-1}^{-1}), \tag{7}$$

from (6) and (7) we have

$$N(n_1, \cdots, n_\mu) = \nu_{i1}n_1 + \nu_{i2}n_2 + \cdots + \nu_{i,\mu-1}n_{\mu-1} + (i + j_i)n_\mu$$

or

$$N(n_1, \cdots, n_\mu) = \nu_{i1}n_1 + \nu_{i2}n_2 + \cdots + \nu_{i,\mu-1}\, n_{\mu-1} + (r + q)n_\mu ,$$

for $i' \neq i$ we have

$$N(n_1, \cdots, n_\mu) = \nu_{i'1}n_1 + \nu_{i'2}n_2 + \cdots + \nu_{i',\mu-1}n_{\mu-1} + (r + q)n_\mu ,$$

then $n_1(\nu_{i1} - \nu_{i'1}) + \cdots + n_{\mu-1}(\nu_{i,\mu-1} - \nu_{i',\mu-1}) = 0$, since this holds for arbitrary $n_1, \cdots, n_{\mu-1}$, we must have

$$\nu_{i1} = \nu_{i'1}, \cdots, \nu_{i,\mu-1} = \nu_{i',\mu-1} ,$$

and we may write

$$N(n_1, \cdots, n_\mu) = \nu_1 n_1 + \nu_2 n_2 + \cdots + n_{\mu-1}n_{\mu-1} + \nu_\mu n_\mu$$

where the $\{\nu_i\}$ are independent of $i$, and $\nu_\mu = r + q$.

Therefore, from (7), we have

$$\begin{aligned} P(t_1, \cdots, t_\mu) & \\ &= \textstyle\sum_{i=r}^{s}\phi_i(t_1, \cdots, t_{\mu-1})t_\mu^i = t_1^{\nu_1} \cdots t_{\mu-1}^{\nu_{\mu-1}}t_\mu^{\nu_\mu} \sum_{i=r}^{s}\psi_{j_i}(t_1^{-1}, \cdots, t_{\mu-1}^{-1})t_\mu^{i-\nu_\mu} \\ &= t_1^{\nu_1} \cdots t_{\mu-1}^{\nu_{\mu-1}}t_\mu^{\nu_\mu} \textstyle\sum_{i=r}^{s}\psi_{j_i}(t_1^{-1}, \cdots, t_{\mu-1}^{-1})t_\mu^{-j_i} \\ &= t_1^{\nu_1} \cdots t_\mu^{\nu_\mu} \textstyle\sum_{j=p}^{q}\psi_j(t_1^{-1}, \cdots, t_{\mu-1}^{-1})t_\mu^{-j} = t_1^{\nu_1} \cdots t_\mu^{\nu_\mu}P'(t_1^{-1}, \cdots, t_\mu^{-1}) \end{aligned}$$

and the lemma is proved.

**3.** THEOREM 3. *Let* $\Delta(t_1, \cdots, t_\mu)$ *be the Alexander polynomial of a knot K of multiplicity* $\mu$, *let* $X_1, \cdots, X_\mu$ *be the components of K, and let* $\Delta(t_1, \cdots, t_{\mu-1})$ *be the Alexander polynomial of the knot K′ obtained by removing* $X_\mu$ *from K. Then, if* $\mu = 2$, $\Delta(t_1, 1) = (t_1^l - 1\Delta(t_1)/t_1 - 1)$, *where l is the linking number of the components of K, and if* $\mu > 2$, $\Delta(t_1, \cdots, t_{\mu-1}, 1) = (t_1^{l_1} t_2^{l_2} \cdots t_{\mu-1}^{l_{\mu-1}} - 1)$ $\Delta(t_1, \cdots, t_{\mu-1})$, *where* $l_i$ $(i = 1, \cdots, \mu - 1)$ *is the linking number of* $X_\mu$ *and* $X_i$.

PROOF. Consider any projection of $K$, and let $\{x_{ij}/r_{ij}\}$ be the presentation of the group of $K$ obtained from the projection, where $x_{i1}, \cdots, x_{i,j_i} (i = 1, \cdots, \mu)$ are the generators corresponding to $X_i$, and $r_{ij} = x_{i'j'}^{\pm 1} x_{ij} x_{i'j'}^{\mp 1} x_{i,j+1}^{-1}$ corresponds to a crossing of $X_{i'}$, over $X_i$. Consider the Jacobian matrix

$$M(t_1, \cdots, t_\mu) = \| (\partial r_{pq}/\partial x_{ij})^{\psi\phi} \| .$$

We know that the minor determinant $D_i$ obtained by deleting the column corresponding to a generator $x_{ij}$ is $(t_i - 1)\ \Delta(t_1, \cdots, t_\mu)$. Therefore if we make $t_\mu = 1$ in $M(t_1, \cdots, t_\mu)$, the value of $D_i$ will be $(t_i - 1)\ \Delta(t_1, \cdots, t_{\mu-1}, 1)$ if $i \neq \mu$. The generators $\{x_{\mu j}\}$ appear in the relations $r_{\mu j} = x_{rs}^{\pm 1} x_{\mu j} x_{rs}^{\mp 1} x_{\mu, j+1}^{-1}$ which correspond to crossings of $X_p$ over $X_\mu$, and in the relations

$$r_{pq} = x_{\mu l}^{\pm 1} x_{pq} x_{\mu l}^{\mp 1} x_{p,q+1}^{-1}$$

corresponding to crossings of $X_\mu$ over $X_p$.

Let us see which are the contributions of these relations to the matrix

$$M(t_1, \cdots, t_{\mu-1}, 1).$$

The contributions of $r_{\mu j}$ are: $(\partial r_{\mu j}/\partial x_{rs})^{\psi\phi}_{t_\mu = 1} = 0$, $(\partial r_{\mu j}/\partial x_{\mu j})^{\psi\phi}_{t_\mu = 1} = t_r^{\pm 1}$ if $\mu \neq r$, $(\partial r_{\mu j}/\partial x_{\mu j})^{\psi\phi}_{t_\mu = 1} = 1$ if $\mu = r$, and $(\partial r_{\mu j}/\partial x_{\mu, j+1})^{\psi\phi}_{t_\mu = 1} = -1$. The contributions of $r_{pq}$ are, $(\partial r_{pq}/\partial x_{\mu l})^{\psi\phi}_{t_\mu = 1} = \pm(1 - t_p)$, $(\partial r_{pq}/\partial x_{pq})^{\psi\phi}_{t_\mu = 1} = 1$ and

$$(\partial r_{pq}/\partial x_{p,q+1})_{t_\mu = 1} = -1 \text{ if } p \neq \mu;$$

the case $p = \mu$ has already been considered. Therefore, $M(t_1, \cdots, t_{\mu-1}, 1)$ is

$$M(t_1, \cdots, t_{\mu-1}, 1) = \begin{array}{c|c|ccccc|} & x_{ij}(i \neq \mu) & x_{\mu 1} & x_{\mu 2} & \cdots & & x_{\mu, j} \\ \hline \begin{array}{c} r_{ij} \\ (i \neq \mu) \end{array} & A & & & B & & \\ \hline r_{\mu 1} & & t_{i_1}^{\varepsilon_1} & -1 & 0 & \cdots & 0 \\ r_{\mu 2} & & 0 & t_{i_2}^{\varepsilon_2} & -1 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots & & \vdots \\ r_{\mu, j_\mu} & & -1 & 0 & 0 & \cdots & t_{i_\mu}^{\varepsilon_\mu} \end{array}$$

in the submatrix of the lower right corner, there is an appearance of $t_i^{\varepsilon_i}$ for each crossing of $X_i$ over $X_\mu$ $(i = 1, \cdots, \mu)$, and $\varepsilon_i = 1$ or $-1$ according as $X_i$ crosses over $X_\mu$ from left to right or from right to left. Therefore, the determinant of the matrix in the lower right corner is $t_1^{l_1} t_2^{l_2} \cdots t_{\mu-1}^{l_{\mu-1}} - 1$.

The submatrix $A$ is equivalent to the Jacobian matrix $M'(t_1, \cdots, t_{\mu-1})$ of the knot $K'$. Therefore, the determinant $D_i$ obtained from $M(t_1, \cdots, t_{\mu-1}, 1)$ by removing a column corresponding to a generator $x_{ij}(i \neq \mu)$, is

$$D_i = (t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}} - 1) \mid A' \mid,$$

where $A'$ is the matrix obtained from $A$ by removing the column corresponding to $x_{ij}$.

If $\mu = 2$, $\mid A' \mid = \Delta(t_1)$, and if $\mu > 2 \mid A' \mid = (t_i - 1)\, \Delta(t_1, \cdots, t_{\mu-1})$. Therefore, if $\mu = 2$

(1) $$(t_1 - 1)\, \Delta(t_1 - 1) = (t_1^{l_1} - 1)\, \Delta(t_1), \text{ and if } \mu > 2$$

(2) $$\Delta(t_1, \cdots, t_{\mu-1}, 1) = (t_1^{l_1}, \cdots, t_{\mu-1}^{l_{\mu-1}} - 1)\, \Delta(t_1, \cdots, t_{\mu-1}),$$

which completes the proof.

If we make $t_1 = 1$ in (1), we obtain

$$\Delta(1, 1) = l_1,$$

and if we make $t_1 = t_2 = \cdots = t_{\mu-1} = 1$ in (2) we have $\Delta(1, \cdots, 1) = 0$, therefore we have proved Theorem 4.

If $\Delta(t_1, \cdots, t_\mu)$ is the Alexander polynomial of a knot $K$ of multiplicity $\mu$, then if $\mu = 2$ $\Delta(1, 1) = l$, where $l$ is the linking number of the components of $K$, and if $\mu > 2$ $\Delta(1, \cdots, 1) = 0$.

## Chapter III

Let $C$ be a circle in euclidean 3-dimensional space, and let $\dot{T}$ be the torus whose central line is $C$ (fig. 16). The closure of the interior of $\dot{T}$ is a tube $T$. The closure of the exterior of $\dot{T}$, together with the point at $\infty$, is a tube, and it will be denoted by $E$.

Denote by $b$ the circle determined by the intersection of $\dot{T}$ and the plane of $C$. Let $a$ be a meridian circle on $\dot{T}$. Let us orient $a$ and $b$ in such a way that $a$ crosses $b$ from left to right (fig. 16).

Consider a knot $K'$ of multiplicity 1 in $E^3$. Let us surround $K'$ by a torus $\dot{T}$; whose central line is $K'$ (fig. 17). The closure of the interior of $\dot{T}'$ is a tube $T'$ which contains $K'$.

Let $K$ be a knot of multiplicity $\mu$, with components $X_1, \cdots, X_\mu$, contained in the interior of $T$ (fig. 16). Denote by $l_i$ the linking number of $a$ and $K_i (i = 1, \cdots, \mu)$.

Let $\phi$ be a homomorphism of $T$ onto $T'$, such that the linking number of $K'$ and the image $b' = \phi(b)$ of $b$ is zero.

The image $\phi(K) = K''$, of $K$ under $\phi$, is a knot of multiplicity $\mu$ contained in the interior of $T'$, (fig. 19). We shall prove:

Theorem 5. *The Alexander polynomial* $\Delta''(t_1, \cdots, t_\mu)$ *of* $K''$ *is*:

$$\Delta''(t_1, \cdots, t_\mu) = \Delta'(t_1^{l_1} t_2^{l_2} \cdots t_\mu^{l_\mu})\, \Delta(t_1, \cdots, t_\mu),$$

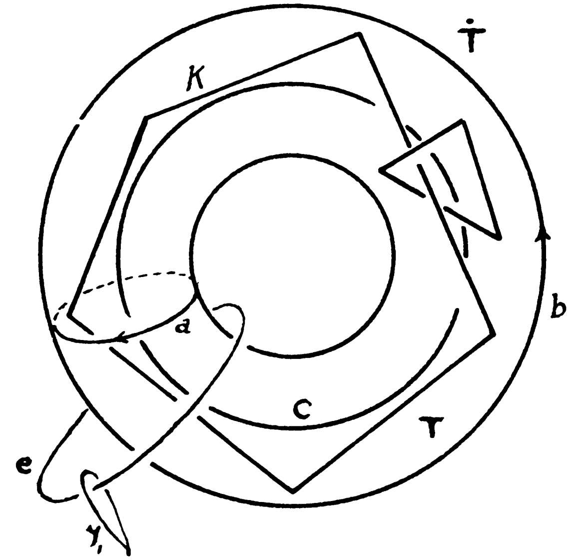


FIG. 16

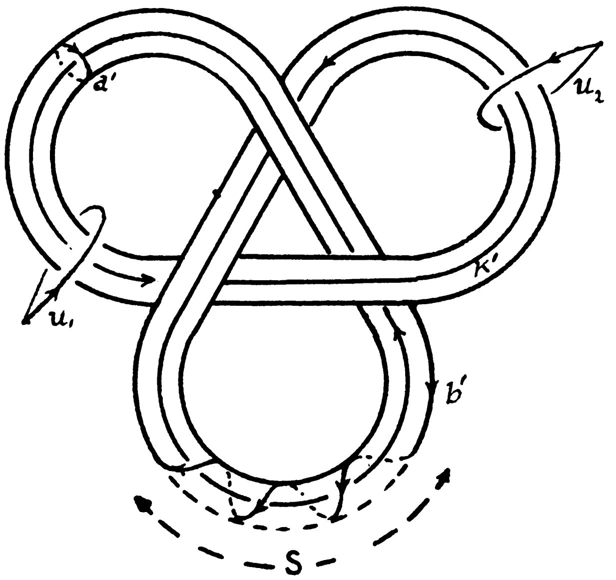


FIG. 17

*where* $\Delta(t_1, \cdots, t_\mu)$ *is the Alexander polynomial of* $K$, *and* $\Delta'(t_1^{l_1} \cdots t_\mu^{l_\mu})$ *is the polynomial obtained by substituting* $t_1^{l_1}t_2^{l_2} \cdots t_\mu^{l_\mu}$ *for* $\mu$ *in the polynomial* $\Delta'(\mu)$ *of* $K'$.

PROOF. There is no loss of generality in supposing that the projection of $b'$ is parallel to the projection of $K'$, except in a segment $S$ of $T'$ in which $b'$ and $K'$ are braided (see fig. 17).

Consider a segment $S'$ of $S$ containing two consecutive crossings of $K'$ and $b'$ (fig. 20(a)). It is clear that by applying the deformation illustrated by fig. 20 to each segment of $T'$ containing two consecutive crossings of $b'$ and $K'$, we

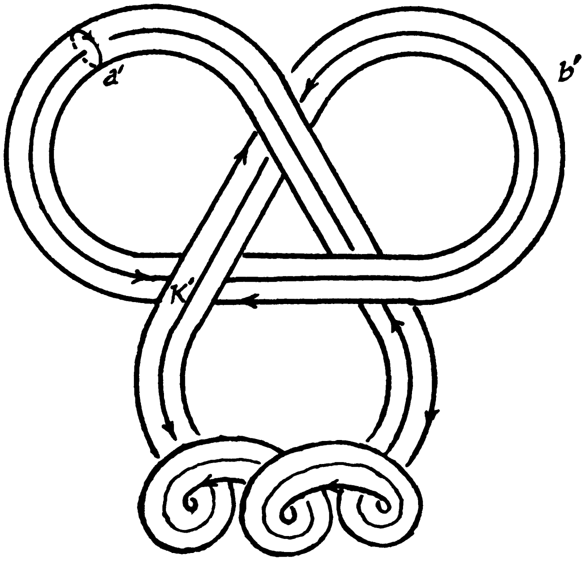


FIG. 18

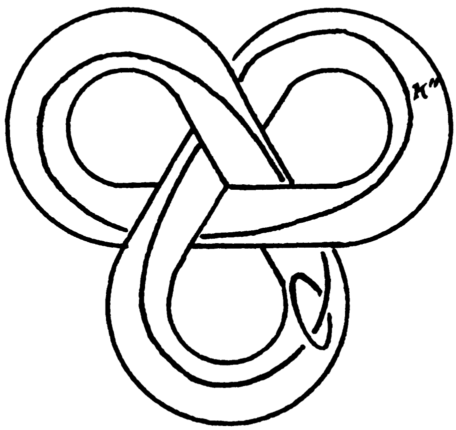


FIG. 19

obtain a projection of $T'$ for which the projections of $K'$ and $b'$ are parallel (fig. 18). Such projection will be used to compute the group $G''$ of $K''$.

We are going to compute separately the groups $\Pi_1(T' - K'')$ and $\Pi_1(E^3 - T')$. Clearly, $\Pi_1(T' - K'')$ is isomorphic to $\Pi_1(T - K)$, and since $T - K$ is homeomorphic to $S^3 - (K \cup e)$, where $e$ is a circle isotopic to $a$ in $E$, the group

$$\Pi_1(T' - K'')$$

is isomorphic to the group of the knot $K \cup e$.

Let us orient $e$ in such a way that its projection crosses over the projection of $b$ from right to left (fig. 16). From the projection of $K \cup e$ we obtain a presentation $\{x_i, y_j/r_k\}$ of $\Pi_1(T' - K'')$, where the $\{x_i\}$ are the generators corresponding to the projection of $K$, and the $\{y_j\}$ are the generators corresponding to the projection of $e$, and the number of relations is one less than the number of generators (see §2).

The group $\Pi_1(E^3 - T')$ is isomorphic to the group $G'$ of $K'$. Let us orient $K'$ in such a way that $a'$ (the image of $a$ under $\phi$) crosses over $K'$ from right to left. From the projection of $K'$ we obtain a presentation $\{u_\nu/g_\mu\}$ of $G'$, in which the

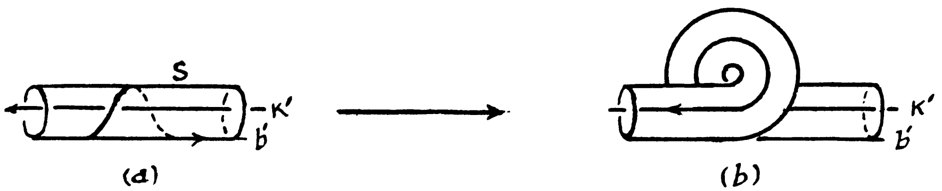


FIG. 20

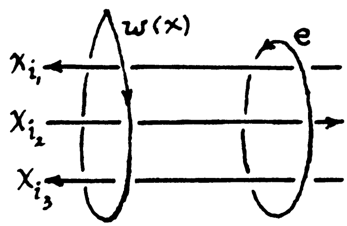


FIG. 21

number of relations is one less than the number of generators. The generators $\{u_\nu\}$, interpreted as generators of $\Pi_1(E^3 - T')$, are represented by paths which go once around $T'$ (fig. 17).

The group $G''$ is the direct product of the groups $\Pi_1(T' - K'')$ and $\Pi_1(E^3 - T')$, with identification of the elements which are represented by the same generators of $\Pi_1(\dot{T}')$ [13].

$\Pi_1(\dot{T}')$ is generated by $\alpha$ and $\beta$ which are represented by $a'$ and $b'$ respectively. $a'$ represents $w(x)$ in $\Pi_1(T' - K'')$, where $w(x)$ is the word in the $\{x_i\}$ which is read off from the projection of $K \cup e$ by going once along $e$ in the positive direction. $w(x)$ is represented by a loop which is homotopic to $e$ in $E^3 - (K \cup e)$ (see fig. 21; in the case illustrated by the figure $w(x) = x_{i_3}^{-1}x_{i_2}^{-1}x_{i_1}$). In $\Pi_1(E^3 - T')$, $a'$ represents a generator $u_\nu$, and by relabeling the $\{u_\nu\}$ we can make $a'$ represent $u_1$. $b'$ represents $y_1$ in $\Pi_1(T' - K'')$ (fig. 16), since $y_1$ is represented by a path homotopic to $b$ in $E^3 - (K \cup e)$. $b'$ represents $W(u)$ in $\Pi_1(E^3 - T')$, where $W(u)$

is the word in the $\{u_\nu\}$ which is read off from the projection of $K$ by going once along $b'$ in the positive direction.

Therefore $G''$ is given by:

$$G'' = \{x_i , y_j , u_\nu/r_k , q_\mu , w(x)u_1^{-1}, \overline{W}(u)y_1^{-1}\}.$$

Let $t_1 , \cdots , t_\mu$ be the generators of the commutator factor group of $G''$, where $t_j$ is the image $(x_i)^{\psi\phi}$ of a generator $x_i$ corresponding to the component

$$X_j (j = 1, \cdots , \mu)$$

of $K$.

The image $[W(u)]^{\psi\phi}$ of the word $W(u)$ is 1, since the linking number of $b'$ and $K'$ is zero. From the relation $W(u)y_1^{-1}$ we have $[W(u)]^{\psi\phi} = (y_1)^{\psi\phi} = 1$, and therefore $(y_j)^{\psi\phi} = 1$ for all $j$. By the definition of $w(x)$ we have

$$[w(x)]^{\psi\phi} = t_1^{l_1}t_2^{l_2} \cdots t_\mu^{l_\mu},$$

and from the relation $w(x)u_1^{-1}$ we have $(u_1)^{\psi\phi} = t_1^{l_1} \cdots t_\mu^{l_\mu}$, and therefore

$$(u_\nu)^{\psi\phi} = t_1^{l_1} \cdots t_\mu^{l_\mu}$$

for all $\nu$.

The matrix $M''(t_1 , \cdots , t_\mu)$ corresponding to the presentation of $G''$ will be:

$$M''(t_1 , \cdots , t_\mu) = \begin{array}{c} \\ q_1 \\ q_2 \\ \vdots \\ r_1 \\ r_2 \\ \vdots \\ w(x)u_1^{-1} \\ w(x)y_1^{-1} \end{array} \begin{array}{c|c|c} \mu_1 \quad \mu_2 \quad \cdots & x_1 \quad x_2 \quad \cdots & y_1 \quad y_2 \quad \cdots \\ \hline M'(t) & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{M(t_1 , \cdots , t_\mu , 1)} \\ \hline -1 \quad 0 \quad \cdots & \left(\dfrac{\partial w(x)}{\partial x_i}\right)^{\psi\phi} & 0 \\ \hline \left(\dfrac{\partial w(u)}{\partial u_\nu}\right)^{\psi\phi} & 0 & -1 \quad 0 \quad \cdots \end{array}$$

which is a square matrix. $M'(t)$ is the matrix obtained by substituting

$$t = t_1^{l_1}t_2^{l_2} \cdots t_\mu^{l_\mu}$$

for $u$ in the matrix $M'(u)$ corresponding to $G'$. $M(t_1 , \cdots , t_\mu , 1)$ is the matrix obtained by substituting 1 for $t_{\mu+1}$ in the matrix $M(t_1 , \cdots , t_\mu , t_{\mu+1})$ corresponding to the group of $K \cup e$, where $t_{\mu+1}$ is the generator of the commutator factor group of $\Pi_1(E^3 - (K \cup e))$, corresponding to $e$.

Adding to the column corresponding to $u_1$ the columns corresponding to the remaining $\{u_\nu\}$ $(\nu > 1\}$, we obtain:

$$
\text{(a)}\quad M''(t_1, \cdots, t_\mu) \sim
\begin{array}{c||c|c|c|c}
 & u_1 & u_2 \cdots & x_1 \quad x_2 \cdots & y_1 \quad y_2 \cdots \\
\hline
\begin{matrix} q_1 \\ q_2 \\ \vdots \end{matrix} & 0 & \bar{M}'(t) & 0 & 0 \\
\hline
\begin{matrix} r_1 \\ r_2 \\ \vdots \end{matrix} & 0 & 0 & \multicolumn{2}{c}{M(t_1, \cdots, t_\mu, 1)} \\
\hline
w(x)u_1^{-1} & -1 & 0 & \left(\dfrac{\partial w(x)}{\partial x_i}\right)^{\psi\phi} & 0 \\
\hline
w(u)y_1^{-1} & 0 & \left(\dfrac{\partial w(u)}{\partial u_\nu}\right)^{\psi\phi} & 0 & -1 \quad 0 \quad \cdots
\end{array}
$$

since $\sum_\nu (\partial w(u)/\partial u_\nu) = 0$, for if $(\partial w(u)/\partial u_\nu)$ is evaluated in the group ring corresponding to $K'$, we have $\sum_\nu (\partial w(u)/\partial u_\nu)\ (u - 1) = 0$ (formula (4)), and since $u - 1$ is not a divisor of zero, we have $\sum_\nu (\partial w(u)/\partial u_\nu) = 0$, and

$$\sum_\nu (\partial w(u)/\partial u_\nu)^{\psi\phi} = \sum_\nu (\partial w_{(u)}/\partial u_\nu)_{u=t} = 0.$$

$\bar{M}'(t)$ is a matrix whose determinant is $\Delta'(t)$. From (a) it is clear that:

$$
\text{(b)}\quad M'(t_1, \cdots, t_\mu) \sim
\begin{array}{c||c|c|c}
 & u_2 \quad u_3 \cdots & x_1 \quad x_2 \cdots & y_1 \quad y_2 \cdots \\
\hline
\begin{matrix} q_1 \\ q_2 \\ \vdots \end{matrix} & \bar{M}'(t) & 0 & 0 \\
\hline
\begin{matrix} r_1 \\ r_2 \\ \vdots \end{matrix} & 0 & \multicolumn{2}{c}{M(t_1, \cdots, t_\mu, 1)} \\
\hline
w(u)y_1^{-1} & \left(\dfrac{\partial w(u)}{\partial u_\nu}\right)^{\psi\phi} & 0 & 0
\end{array}
$$

Consider the matrix:

$$
T(t_1, \cdots, t_\mu) =
\begin{array}{||c|c|c}
\hline
\bar{M}'(t) & 0 & 0 \\
\hline
0 & \multicolumn{2}{c}{M(t_1, \cdots, t_\mu, 1)} \\
\hline
\left(\dfrac{\partial w(u)}{\partial u_\nu}\right)^{\psi\phi} (1 - t) & 0 & -(1 - t) \quad 0 \quad \cdots
\end{array}
$$

which is obtained by multiplying the last row of $b$) by $1 - t$.

As a consequence of the relations $\{q_\mu\}$ we have the relation

$$q = W(u)u_1W^{-1}(u)u_1^{-1},$$

whose derivatives are $\partial q/\partial u_\nu = (\partial w_{(u)}/\partial u_\nu)^{\psi\phi}\ (1 - t)$ for $\nu > 1$, and as a consequence of the $\{r_k\}$ we have $r = w(x)y_1w^{-1}(x)y_1^{-1}$, whose derivatives are $\partial r/\partial x_i = 0$ and $\partial r/\partial y_1 = t - 1$. Therefore, each row of $T(t_1, \cdots, t_\mu)$ is a linear combination of the other rows, and the ideal of column deficiency 1 will be generated by the minor determinants which are obtained by deleting one column in the matrix obtained from $T(t_1, \cdots, t_\mu)$ by deleting any row. In particular if we delete the last row, we have:

$$\left\|\begin{array}{c|c} \bar{M}''(t) & 0 \\ \hline 0 & M(t_1, \cdots, t_\mu, 1) \end{array}\right\|$$

and by Theorem 3 we have: the elementary ideal of column deficiency 1 of $T(t_1, \cdots, t_\mu)$ is generated by

$$\{(1 - t_i)(t - 1)\Delta'(t)\Delta(t_1, \cdots, t_\mu)\} \qquad (i = 1, \cdots, \mu) \text{ if } \mu > 1.$$

Therefore, the elementary ideal of column deficiency 1 in $M''(t_1, \cdots, t_\mu)$ is generated by $\{(1 - t_i)\ \Delta'(t)\ \Delta(t_1, \cdots, t_\mu)\}$ $(i = 1, \cdots, \mu)$, and:

$$\Delta''(t_1, \cdots, t_\mu) = \Delta'(t_1^{l_1}t_2^{l_2} \cdots t_\mu^{l_\mu})\Delta(t_1, \cdots, t_\mu),$$

which completes the proof of Theorem 5 in the case $\mu > 1$.

If $\mu = 1$, the matrix $M(t_1, \cdots, t_\mu, 1)$ is of the form:

$$\begin{array}{c} \quad x_1 \;\; x_2 \;\; \cdots \qquad\qquad y_1 \qquad\qquad\qquad y_1 \;\; \cdots \\ \left\|\begin{array}{c|c|cccc} & * & & & & \\ M(t_1) & * & & 0 & & \\ & \vdots & & & & \\ & * & & & & \\ \hline & t_1^{\pm 1} & -1 & 0 & \cdots & 0 \\ 0 & 0 & t_1^{\pm 1} & -1 & \cdots & 0 \\ & \vdots & \vdots & \vdots & & \vdots \\ & 0 & 0 & 0 & \cdots & -1 \end{array}\right\| \end{array}$$

where $M(t_1)$ is the matrix corresponding to $K$, and * denotes a possibly non-zero element.

Therefore, if in $M''(t_1)$ we add to each row the last row multiplied by an appropriate factor, we obtain:

$$M''(t_1) \sim \left\| \begin{array}{c|c|c|c} u_2 \quad u_3 \quad \cdots & x_1 \quad x_2 \quad \cdots & y_1 & y_2 \quad \cdots \\ \hline \bar{M}'(t) & 0 & 0 & 0 \\ \hline & M(t_1) & 0 & 0 \\ \hline \text{a} & 0 & 0 & \begin{matrix} -1 & 0 & \cdots & 0 \\ t_1^{\pm} & -1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 \end{matrix} \\ \hline \dfrac{\partial w(u)}{\partial u_\nu}(1-t) & 0 & -1 & 0 \end{array} \right\|$$

where $A$ is a certain matrix whose form we do not need to discuss. It is clear that

$$M''(t_1) \sim \left\| \begin{array}{c|c|c} \bar{M}'(t) & 0 & 0 \\ \hline & M(t_1) & 0 \\ \hline \text{a} & 0 & \begin{matrix} -1 & & 0 \\ & -1 & \ddots \\ 0 & & -1 \end{matrix} \end{array} \right\|$$

hence: $\Delta''(t_1) = \Delta'(t_1^{l_1})\ \Delta(t_1)$.[10]

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[10] Theorem 5, in the particular case $\mu = 1$, was proved by Seifert in [10].