THE ALGEBRAIC THEORY OF TORSION I. FOUNDATIONS

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Introduction

The algebraic theory of torsion developed here takes values in the absolute K_1-group $K_1(A)$ of a ring A, with a torsion invariant $\tau(f) \in K_1(A)$ for a chain equivalence $f:C \longrightarrow D$ of finite chain complexes of based f.g. free A-modules with zero Euler characteristic.

Whitehead [24] defined the torsion $\tau(C) \in K_1(A)$ of a contractible finite chain complex C of based f.g. free A-modules, assuming (as we do here) that A is such that f.g. free A-modules have well-defined rank. The algebraic mapping cone $C(f)$ of a chain equivalence $f:C \longrightarrow D$ of finite chain complexes of based f.g. free A-modules is a contractible chain complex, so that the torsion $\tau(C(f)) \in K_1(A)$ is defined. However, the expected sum formula for the composite $gf:C \longrightarrow D \longrightarrow E$ of chain equivalences $f:C \longrightarrow D$, $g:D \longrightarrow E$

$$\tau(C(gf)) = \tau(C(f)) + \tau(C(g)) \in K_1(A)$$

only holds in general on passing to the reduced K_1-group

$$\tilde{K}_1(A) = \mathrm{coker}(K_1(\mathbb{Z}) \longrightarrow K_1(A)) = K_1(A)/\{\tau(-1:A \longrightarrow A)\} \ .$$

The reduced torsion of the algebraic mapping cone

$$\tau(f) = \tau(C(f)) \in \tilde{K}_1(A)$$

is the torsion invariant usually associated to a chain equivalence f. In particular, the Whitehead torsion $\tau(f) \in Wh(\pi)$ $(\pi = \pi_1(X))$ of a homotopy equivalence $f:X \longrightarrow Y$ of finite CW complexes is the image of $\tau(\tilde{f}:C(\tilde{X}) \longrightarrow C(\tilde{Y})) \in \tilde{K}_1(\mathbb{Z}[\pi])$ in the Whitehead group $Wh(\pi) = K_1(\mathbb{Z}[\pi])/\{\pm\pi\}$. The theory of torsion developed here can be used in certain circumstances to lift the Whitehead torsion to an absolute torsion invariant $\tau(f) \in K_1(\mathbb{Z}[\pi])$, which enters into product formulae for Whitehead torsion.

The Euler characteristic of a finite chain complex C of f.g. free A-modules is defined as usual by

$$\chi(C) = \sum_{r=0}^{\infty} (-)^r \mathrm{rank}_A(C_r) \in \mathbb{Z} \ .$$

The complex C is <u>round</u> if

$$\chi(C) = 0 \in \mathbb{Z} \ .$$

The assumption on A that f.g. free A-modules have well-defined rank ensures that $K_0(\mathbb{Z}) \longrightarrow K_0(A)$ is injective, so that the Euler characteristic may be identified with the absolute projective class

$$\chi(C) = [C] \in \mathbb{Z} = K_0(\mathbb{Z}) \subseteq K_0(A) \ .$$

The <u>absolute torsion</u> of a chain equivalence $f:C \longrightarrow D$ of round finite chain complexes of based f.g. free A-modules is defined in §4 by a formula of the type

$$\tau(f) = \tau(C(f)) + \beta\tau(-1:A \longrightarrow A) \in K_1(A)$$

with the sign term $\beta = 0$ or 1 depending only on the ranks (mod 2) of the chain modules of C and D. It is quite reasonable that a K_1-valued invariant should only be defined when K_0-valued obstructions vanish! Actually, the absolute torsion is also defined if C,D are such that the Euler characteristic is 0(mod 2). For contractible C,D the torsion of f is just the difference of the torsions of C and D

$$\tau(f) = \tau(D) - \tau(C) \in K_1(A) \ .$$

The main result of Part I is the logarithmic property of absolute torsion with respect to composition

$$\tau(gf:C \longrightarrow D \longrightarrow E) = \tau(f:C \longrightarrow D) + \tau(g:D \longrightarrow E) \in K_1(A) \ .$$

As such this is not very prepossessing. The applications of absolute torsion are more interesting, but will be dealt with elsewhere. Parts II and III will deal with products and lower K-theory. Some of the applications to L-theory are contained in a forthcoming joint paper with Ian Hambleton and Larry Taylor on "Round L-theory".

The following preview of the applications of the absolute torsion to topology may help to motivate the paper.

Define a connected finite CW complex X to be <u>round</u> if $\chi(X) = 0 \in \mathbb{Z}$ and the cellular f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\widetilde{X})$ of the universal cover $\widetilde{X}$ is equipped with a choice of base in the canonical class of bases determined by the cell structure of X up to the multiplication of each base element by $\pm g$ $(g \in \pi_1(X))$. Thus $C(\widetilde{X})$ is a round finite chain complex of based f.g. free $\mathbb{Z}[\pi_1(X)]$-modules.
The <u>absolute torsion</u> of a homotopy equivalence $f:X \longrightarrow Y$ of round finite CW complexes is defined by

$$\tau(f) = \tau(\widetilde{f}:C(\widetilde{X}) \longrightarrow C(\widetilde{Y})) \in K_1(\mathbb{Z}[\pi_1(X)]) \ ,$$

and is such that the reduction $\tau(f) \in Wh(\pi_1(X))$ is the usual Whitehead torsion of f. A <u>round finite structure</u> on a topological space X is an equivalence class of pairs

(round finite CW complex K , homotopy equivalence $f : K \longrightarrow X$)

under the equivalence relation

$$(K,f) \sim (K',f') \text{ if } \tau(f'^{-1}f:K \longrightarrow X \longrightarrow K') = 0 \in K_1(\mathbb{Z}[\pi_1(X)]) .$$

For example, the mapping torus of a self map $\zeta:X \longrightarrow X$ of a finitely dominated CW complex X

$$T(\zeta) = X \times [0,1]/\{(x,0) = (\zeta(x),1) \,|\, x \in X\}$$

has a canonical round finite structure, by a generalization of the trick of Mather [9], with $T(f\zeta g:Y \longrightarrow Y)$ a round finite CW complex in the round finite homotopy type of $T(\zeta)$ for any domination of X

$$(Y , f : X \longrightarrow Y , g : Y \longrightarrow X , h : gf \simeq 1 : X \longrightarrow X)$$

by a finite CW complex Y. (Furthermore, if $X = \overline{M}$ is an infinite cyclic cover of a compact manifold M with $\zeta:X \longrightarrow X$ a generating covering translation then the projection $T(\zeta) \longrightarrow M$ is a homotopy equivalence such that the Whitehead torsion $\tau \in Wh(\pi_1(M))$ is the obstruction of Farrell [3] and Siebenmann [20] to fibering M over S^1, giving M the finite homotopy type determined by a handlebody decomposition and assuming $\dim(M) \geqslant 6$). The <u>product structure theorem</u> is that the product $F \times B$ of a finitely dominated CW complex F and a round finite CW complex B has a canonical round finite structure, such that the absolute torsion of a product homotopy equivalence is given by

$$\begin{aligned}\tau(f \times b : F \times B \longrightarrow F' \times B') &= [F]\otimes\tau(b) \\ &\in K_1(\mathbb{Z}[\pi_1(F \times B)]) = K_1(\mathbb{Z}[\pi_1(F)]\otimes\mathbb{Z}[\pi_1(B)]) \ ,\end{aligned}$$

with $[F] = [F'] \in K_0(\mathbb{Z}[\pi_1(F)])$ the absolute projective class and $\tau(b) \in K_1(\mathbb{Z}[\pi_1(B)])$ the absolute torsion. The circle

$$S^1 = T(\mathrm{id.} : \{\mathrm{pt.}\} \longrightarrow \{\mathrm{pt.}\})$$

has the canonical round finite structure in which the base elements $\tilde{e}^i \in C(\tilde{S}^1)_i = \mathbb{Z}[\pi_1(S^1)] = \mathbb{Z}[z,z^{-1}]$ $(i = 0,1)$ are such that

$$d(\tilde{e}^1) = \tilde{e}^0 - z\tilde{e}^0 .$$

For any finitely dominated CW complex F the product round finite structure on $F \times S^1 = T(1:F \longrightarrow F)$ agrees with the mapping torus round finite structure. Ferry [4] defined a geometric injection

$$\overline{B}' : \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z}) \ ; \ [F] \longmapsto \tau(1 \times -1:F \times S^1 \longrightarrow F \times S^1)$$

for any finitely presented group π, with $[F] \in \tilde{K}_0(\mathbb{Z}[\pi])$ the Wall finiteness obstruction of a finitely dominated CW complex F with

$\pi_1(F) = \pi$. The image of $\bar{B}'$ consists of the elements $\tau \in Wh(\pi \times \mathbb{Z})$ invariant under the transfer maps associated to the finite covers of S^1. The map $-1:S^1 \longrightarrow S^1$ reflecting the circle in a diameter has absolute torsion

$$\tau(-1:S^1 \longrightarrow S^1) = \tau(-z:\mathbb{Z}[z,z^{-1}] \longrightarrow \mathbb{Z}[z,z^{-1}]) \in K_1(\mathbb{Z}[z,z^{-1}]) ,$$

so that by the product structure theorem $\bar{B}'$ is given algebraically by

$$\begin{aligned} \bar{B}' = -\otimes\tau(-z) \ : \ & \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi \times \mathbb{Z}) \ ; \\ & [P] \longmapsto \tau(-z:P[z,z^{-1}] \longrightarrow P[z,z^{-1}]) \end{aligned}$$

with $[P]$ the reduced projective class of a f.g. projective $\mathbb{Z}[\pi]$-module P. Thus $\bar{B}'$ does not coincide with the traditional algebraic injection of Bass, Heller and Swan [2]

$$\begin{aligned} \bar{B} = -\otimes\tau(z) \ : \ & \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi \times \mathbb{Z}) \ ; \\ & [P] \longmapsto \tau(z:P[z,z^{-1}] \longrightarrow P[z,z^{-1}]) \ . \end{aligned}$$

The recent algebraic description due to Lück [8] of the transfer map $p_1^!:K_1(\mathbb{Z}[\pi_1(B)]) \longrightarrow K_1(\mathbb{Z}[\pi_1(E)])$ induced in the K_1-groups by a Hurewicz fibration

$$F \longrightarrow E \xrightarrow{p} B$$

with finitely dominated fibre F allows the product structure theorem to be extended to the twisted case: the total space E of a fibration with finitely dominated fibre F and round finite base B has a canonical round finite homotopy type, and if

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d] & F' \arrow[d] \\ E \arrow[r, "e"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "b"] & B' \end{tikzcd}$$

is a fibre homotopy equivalence of such fibrations the homotopy equivalence $e:E \longrightarrow E'$ has absolute torsion

$$\tau(e) = p_1^!(\tau(b)) \in K_1(\mathbb{Z}[\pi_1(E)]) \ .$$

The <u>absolute torsion</u> of a round finite n-dimensional geometric Poincaré complex B is defined by

$$\tau(B) = \tau([B]\cap -:C(\widetilde{B})^{n-*} \longrightarrow C(\widetilde{B})) \in K_1(\mathbb{Z}[\pi_1(B)]) \quad ,$$

satisfying the usual duality $\tau(B)^* = (-)^n\tau(B)$. The Poincaré complex version of the twisted product structure theorem is that the total space of a fibration $F \longrightarrow E \xrightarrow{p} B$ with a round finite n-dimensional Poincaré base B and a finitely dominated m-dimensional Poincaré fibre F is an (m+n)-dimensional Poincaré complex E with a canonical round finite structure, with respect to which the torsion of E is given by

$$\tau(E) = p_1^!(\tau(B)) \in K_1(\mathbb{Z}[\pi_1(E)]) \quad .$$

In particular, for the trivial fibration $E = F\times B$ this is a product formula

$$\tau(F\times B) = [F]\otimes\tau(B) \in K_1(\mathbb{Z}[\pi_1(F\times B)]) \quad .$$

The torsion of the circle S^1 with respect to the canonical round finite structure is

$$\tau(S^1) = \tau(-z:\mathbb{Z}[z,z^{-1}] \longrightarrow \mathbb{Z}[z,z^{-1}]) \in K_1(\mathbb{Z}[\pi_1(S^1)]) = K_1(\mathbb{Z}[z,z^{-1}]) \;,$$

so that for any finitely dominated m-dimensional Poincaré complex F

$$\begin{aligned}\tau(F\times S^1) &= [F]\otimes\tau(S^1) = [F]\otimes\tau(-z) = \overline{B}'([F]) \\ &\in K_1(\mathbb{Z}[\pi\times\mathbb{Z}]) = K_1(\mathbb{Z}[\pi][z,z^{-1}]) \qquad (\pi = \pi_1(F))\end{aligned}$$

with $\overline{B}':K_0(\mathbb{Z}[\pi]) \rightarrowtail K_1(\mathbb{Z}[\pi][z,z^{-1}])\ ;\ [P] \longmapsto \tau(-z:P[z,z^{-1}] \longrightarrow P[z,z^{-1}])$ the absolute version of the injection $\overline{B}':K_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z})$ described above. More generally, the mapping torus $T(\zeta)$ of a self homotopy equivalence $\zeta:F \longrightarrow F$ is the total space of a fibration over S^1

$$F \longrightarrow T(\zeta) \xrightarrow{p} S^1$$

such that $\pi_1(T(\zeta)) = \pi\times_\alpha\mathbb{Z}$ $(\alpha = \zeta_* : \pi \longrightarrow \pi)$, and $T(\zeta)$ is an (m+1)-dimensional geometric Poincaré complex with a canonical round finite structure with respect to which

$$\begin{aligned}\tau(T(\zeta)) &= p_1^!\tau(S^1) = \tau(-z\widetilde{\zeta}:C(\widetilde{F})_\alpha[z,z^{-1}] \longrightarrow C(\widetilde{F})_\alpha[z,z^{-1}]) \\ &\in K_1(\mathbb{Z}[\pi\times_\alpha\mathbb{Z}]) = K_1(\mathbb{Z}[\pi]_\alpha[z,z^{-1}])\end{aligned}$$

$$(gz = z\alpha(g)\ \ (g\in\pi),\ \widetilde{\zeta} : \alpha_! C(\widetilde{F}) \longrightarrow C(\widetilde{F}))\quad .$$

The algebraic theory of surgery of Ranicki [17] has a version for round finite algebraic Poincaré complexes, corresponding to the variant L-groups of Wall [22] in which only based f.g. free modules of even rank are considered (cf. the joint work with Hambleton and Taylor mentioned above). In particular, the round L-theory shows that the algebraic injections of Ranicki [16]

$$\bar{B} : L_n^j(\pi) \rightarrowtail L_{n+1}^k(\pi \times \mathbb{Z}) \quad ((j,k) = (h,s) \text{ or } (p,h))$$

do not coincide with the geometric injections

$$\bar{B}' : L_n^j(\pi) \rightarrowtail L_{n+1}^k(\pi \times \mathbb{Z}) \; ; \; \sigma_*^j((f,b):M \longrightarrow X) \longmapsto \sigma_*^k((f,b)\times 1 : M\times S^1 \longrightarrow X\times S^1)$$

of Shaneson [19] (for (h,s)) and Pedersen and Ranicki [14] (for (p,h)). The algebraic expression for $\bar{B}'$ is given by product with the round finite symmetric Poincaré complex of S^1, defined using the canonical round finite structure on S^1.

This paper is a sequel to the algebraic theory of the Wall finiteness obstruction developed in Ranicki [18]. As there we work with chain complexes in an arbitrary additive category $\mathcal{A}$, although the case $\mathcal{A}$ = {based f.g. free A-modules} for a ring A is the one of main interest

In §1 the isomorphism torsion group $K_1^{iso}(\mathcal{A})$ of an additive category $\mathcal{A}$ is defined by analogy with the automorphism torsion group $K_1^{aut}(\mathcal{A}) = K_1(\mathcal{A})$, using all the isomorphisms in $\mathcal{A}$. §2 is devoted to the isomorphism torsion properties of the permutation isomorphisms $M\oplus N \longrightarrow N\oplus M \; ; \; (x,y) \longmapsto (y,x)$. §3 deals with the torsion of contractible chain complexes. In §4 there is defined the torsion $\tau(f) \in K_1^{iso}(\mathcal{A})$ of a chain equivalence $f:C \longrightarrow D$ of finite chain complexes in $\mathcal{A}$ which are round, that is $[C] = [D] = 0 \in K_0(\mathcal{A})$. In §5 it is shown that if $\mathcal{A}$ is such that stably isomorphic objects are related by canonical stable isomorphisms then $K_1(\mathcal{A})$ is canonically a direct summand of $K_1^{iso}(\mathcal{A})$. In particular, such is the case for $\mathcal{A}$ = {based f.g. free A-modules}, allowing the definition of the absolute torsion $\tau(f) \in K_1(\mathcal{A}) = K_1(A)$ for a chain equivalence $f:C \longrightarrow D$ of round finite chain complexes of based f.g. free A-modules.

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Contents

§1. The isomorphism torsion group $K_1^{iso}(\mathcal{A})$

In order to define the torsion of a chain equivalence it is necessary to first define the torsion of an isomorphism. To this end we shall now define the isomorphism torsion group $K_1^{iso}(\mathcal{A})$ of an additive category, by analogy with the automorphism torsion group $K_1^{aut}(\mathcal{A}) = K_1(\mathcal{A})$.

Let then $\mathcal{A}$ be an additive category, with direct sum $\oplus$.

The $\left\{\begin{matrix}\underline{\text{isomorphism}}\\ \underline{\text{automorphism}}\end{matrix}\right.$ $\underline{\text{torsion group}}$ $\begin{matrix}K_1^{iso}(\mathcal{A})\\ K_1^{aut}(\mathcal{A})\end{matrix}$ is the abelian

group with one generator $\tau(f)$ for each $\left\{\begin{matrix}\text{isomorphism } f:M \longrightarrow N\\ \text{automorphism } f:M \longrightarrow M\end{matrix}\right.$ in $\mathcal{A}$,

subject to the relations

$$\text{i)}\left\{\begin{aligned}&\tau(gf:M \longrightarrow N \longrightarrow P) = \tau(f:M \longrightarrow N) + \tau(g:N \longrightarrow P)\\ &\tau(gf:M \longrightarrow M \longrightarrow M) = \tau(f) + \tau(g),\ \tau(ifi^{-1}:M' \longrightarrow M \longrightarrow M \longrightarrow M') = \tau(f)\end{aligned}\right.$$

$$\text{ii)}\left\{\begin{aligned}&\tau(f\oplus f':M\oplus M' \longrightarrow N\oplus N') = \tau(f:M \longrightarrow N) + \tau(f':M' \longrightarrow N')\\ &\tau(f\oplus f':M\oplus M' \longrightarrow M\oplus M') = \tau(f:M \longrightarrow M) + \tau(f':M' \longrightarrow M')\ .\end{aligned}\right.$$

The automorphism torsion group $K_1^{aut}(\mathcal{A})$ is just the Whitehead group of $\mathcal{A}$ in the sense of Bass [1,p.348]. There is defined a forgetful map

$$K_1^{aut}(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A})\ ;\ \tau(f) \longmapsto \tau(f)$$

which in certain circumstances (investigated in §5 below) is a split injection.

Remark: In order to avoid having to keep track of the coherence isomorphisms $(M\oplus N)\oplus P \longrightarrow M\oplus(N\oplus P)$ in $K_1^{iso}(\mathcal{A})$ we shall assume that $\mathcal{A}$ is a permutative category, so that $(M\oplus N)\oplus P = M\oplus(N\oplus P)$. There is a standard procedure for replacing any symmetric monoidal category by an equivalent permutative category (cf. Proposition 4.2 of May [10]).

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Let now $\mathcal{E}$ be an exact category. The torsion group $K_1(\mathcal{E})$ was defined by Bass [1,p.390] to be the abelian group with one generator $\tau(f)$ for each automorphism $f:M \longrightarrow M$ in $\mathcal{E}$, subject to the relations

i) $\tau(gf:M \longrightarrow M) = \tau(f:M \longrightarrow M) + \tau(g:M \longrightarrow M)$

ii) $\tau(f'':M'' \longrightarrow M'') = \tau(f:M \longrightarrow M) + \tau(f':M' \longrightarrow M')$ for any automorphism of a short exact sequence in $\mathcal{E}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & M'' & \xrightarrow{j} & M' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \\ 0 & \longrightarrow & M & \xrightarrow{i} & M'' & \xrightarrow{j} & M' & \longrightarrow & 0 \end{array}$$

An additive category $\mathcal{A}$ can be given the structure of an exact category by declaring a sequence in $\mathcal{A}$

$$0 \longrightarrow M \xrightarrow{i} M'' \xrightarrow{j} M' \longrightarrow 0$$

to be exact if $ji = 0 : M \longrightarrow M'$ and there exists a morphism $k : M' \longrightarrow M''$ such that

i) $jk = 1_{M'} : M' \longrightarrow M'$

ii) $(i\ k) : M \oplus M' \longrightarrow M''$ is an isomorphism.

We shall always use this exact structure.

Weibel [23] showed that the torsion group $K_1(\mathcal{A})$ of an additive category $\mathcal{A}$ with the above exact structure agrees with the case $i = 1$ of the general definition $K_i(\mathcal{E}) = \pi_{i+1}(B\mathcal{E}^{-1}\mathcal{E})$ $(i \geqslant 0)$ due to Quillen (Grayson [6]) of the algebraic K-groups of an exact category $\mathcal{E}$.

Proposition 1.1 (Bass [1,p.397]) There is a natural identification of torsion groups $K_1^{aut}(\mathcal{A}) = K_1(\mathcal{A})$ for an additive category $\mathcal{A}$.

Proof: In order to verify that the natural abelian group morphism

$$K_1^{aut}(\mathcal{A}) \longrightarrow K_1(\mathcal{A}) \ ; \ \tau(f) \longmapsto \tau(f)$$

is an isomorphism it suffices to show that for any morphism $e: M' \longrightarrow M$ in $\mathcal{A}$ the elementary automorphism

$$f = \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} : M \oplus M' \longrightarrow M \oplus M'$$

is such that $\tau(f) = 0 \in K_1^{aut}(\mathcal{A})$. The automorphisms

$$g = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : M \oplus M' \oplus M \longrightarrow M \oplus M' \oplus M$$

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e & 1 \end{pmatrix} : M \oplus M' \oplus M \longrightarrow M \oplus M' \oplus M$$

are such that

$$f\oplus 1_M = ghg^{-1}h^{-1} : M\oplus M'\oplus M \longrightarrow M\oplus M'\oplus M$$

(a particular example of a Steinberg relation). It follows that

$$\tau(f) = \tau(f\oplus 1_M) = \tau(ghg^{-1}h^{-1}) = 0 \in K_1^{aut}(\mathcal{A}) .$$

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Example Let A be an associative ring with 1 such that f.g. free A-modules have well defined rank (e.g. a group ring $\mathbb{Z}[\pi]$). Let $\mathcal{A}$ be the additive category of based f.g. free A-modules and A-module morphisms. The automorphism torsion group of $\mathcal{A}$ is just the usual Whitehead group of A

$$K_1^{aut}(\mathcal{A}) = K_1(\mathcal{A}) = K_1(A) = GL(A)/E(A) .$$

The isomorphism torsion group $K_1^{iso}(\mathcal{A})$ contains $K_1(A)$ as a direct summand, with the natural map $K_1(A) \longrightarrow K_1^{iso}(\mathcal{A})$ split by the surjection

$$K_1^{iso}(\mathcal{A}) \twoheadrightarrow K_1(A) \ ; \ \tau(f:M \longrightarrow N) \longmapsto \tau((f_{ij}))$$

sending the isomorphism torsion $\tau(f:M \longrightarrow N) \in K_1^{iso}(\mathcal{A})$ to the torsion $\tau((f_{ij})) \in K_1(A)$ of the invertible $n\times n$ matrix $(f_{ij}) \in GL_n(A)$ ($n = \mathrm{rank}_A M = \mathrm{rank}_A N$) representing f.

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The isomorphism torsion group $K_1^{iso}(\mathcal{A})$ of an additive category $\mathcal{A}$ is considerably larger than the automorphism torsion group $K_1(\mathcal{A})$, and is introduced here for the sole purpose of providing a home for the torsion $\tau(f) \in K_1^{iso}(\mathcal{A})$ of a chain equivalence.

§2. Signs

In dealing with the torsion of chain complexes and chain equivalences we shall be making frequent use of the following elements in $K_1^{iso}(\mathcal{A})$.

The sign of an ordered pair (M,N) of objects of $\mathcal{A}$ is the isomorphism torsion

$$\varepsilon(M,N) = \tau(\begin{pmatrix} 0 & 1_N \\ 1_M & 0 \end{pmatrix} : M\oplus N \longrightarrow N\oplus M) \in K_1^{iso}(\mathcal{A}) .$$

Example Let $\mathcal{A}$ = {based f.g. free A-modules}. The sign of objects M,N in $\mathcal{A}$ is given by

$$\varepsilon(M,N) = \mathrm{rank}_A(M)\,\mathrm{rank}_A(N)\,\tau(-1:A \longrightarrow A) \in K_1(A) \subset K_1^{iso}(\mathcal{A}) ,$$

depending only on the parities of the ranks of M and N.

Proposition 2.1 The sign function $(M,N) \longmapsto \varepsilon(M,N)$ has the following properties, for any additive category $\mathcal{A}$:

i) $\varepsilon(M\oplus M',N) = \varepsilon(M,N) + \varepsilon(M',N) \in K_1^{iso}(\mathcal{A})$,

ii) $\varepsilon(M,N) = \varepsilon(M',N) \in K_1^{iso}(\mathcal{A})$ if M is isomorphic to M',

iii) $\varepsilon(M,N) = -\varepsilon(N,M) \in K_1^{iso}(\mathcal{A})$,

iv) $\varepsilon(M,M) = \tau(-1_M : M \longrightarrow M) \in K_1^{iso}(\mathcal{A})$.

Proof: i) For any objects M,M',N of $\mathcal{A}$

$$\varepsilon(M\oplus M',N) = \tau\left(\begin{pmatrix} 0 & 0 & 1_N \\ 1_M & 0 & 0 \\ 0 & 1_{M'} & 0 \end{pmatrix}\right.$$

$$: M\oplus M'\oplus N \xrightarrow{1_M\oplus\begin{pmatrix} 0 & 1_N \\ 1_{M'} & 0\end{pmatrix}} M\oplus N\oplus M' \xrightarrow{\begin{pmatrix} 0 & 1_N \\ 1_M & 0\end{pmatrix}\oplus 1_{M'}} N\oplus M\oplus M'\Big)$$

$$= \varepsilon(M,N) + \varepsilon(M',N) \in K_1^{iso}(\mathcal{A}) \ .$$

ii) Let $f:M \longrightarrow M'$ be an isomorphism in $\mathcal{A}$, and let N be an object. It follows from the commutative diagram of isomorphisms in $\mathcal{A}$

$$\begin{tikzcd} M\oplus N \arrow[r, "\begin{pmatrix} 0 & 1_N \\ 1_M & 0\end{pmatrix}"] \arrow[d, "f\oplus 1_N"'] & N\oplus M \arrow[d, "1_N\oplus f"] \\ M'\oplus N \arrow[r, "\begin{pmatrix} 0 & 1_N \\ 1_{M'} & 0\end{pmatrix}"] & N\oplus M' \end{tikzcd}$$

that

$$\begin{aligned}\varepsilon(M',N) - \varepsilon(M,N) &= \tau(1_N\oplus f) - \tau(f\oplus 1_N) \\ &= \tau(f) - \tau(f) = 0 \in K_1^{iso}(\mathcal{A}) \ .\end{aligned}$$

iii) For any objects M,N in $\mathcal{A}$

$$\begin{aligned}\varepsilon(M,N) + \varepsilon(N,M) &= \tau\left(\begin{pmatrix} 0 & 1_N \\ 1_M & 0\end{pmatrix} : M\oplus N \longrightarrow N\oplus M\right) + \tau\left(\begin{pmatrix} 0 & 1_M \\ 1_N & 0\end{pmatrix} : N\oplus M \longrightarrow M\oplus N\right) \\ &= \tau\left(\begin{pmatrix} 0 & 1_M \\ 1_N & 0\end{pmatrix}\begin{pmatrix} 0 & 1_N \\ 1_M & 0\end{pmatrix} = 1_{M\oplus N} : M\oplus N \longrightarrow M\oplus N\right) \\ &= 0 \in K_1^{iso}(\mathcal{A}) \ .\end{aligned}$$

iv) It is immediate from Proposition 1.1 and the identity

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} : M\oplus M \longrightarrow M\oplus M$$

that

$$\varepsilon(M,M) = \tau\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} : M\oplus M \longrightarrow M\oplus M\right) = \tau(-1 : M \longrightarrow M) \in K_1^{iso}(\mathcal{A}) \ .$$

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The <u>isomorphism class group</u> $K_0(\mathcal{A})$ of an additive category $\mathcal{A}$ is defined as usual to be the abelian group with one generator $[M]$ for each isomorphism class of objects M in $\mathcal{A}$, subject to the relations

$$[M\oplus N] = [M] + [N] \in K_0(\mathcal{A}) \ .$$

<u>Example</u> The projective class group of a ring A is the isomorphism class group of the additive category $\mathcal{P} = \{$f.g. projective A-modules$\}$,

$$K_0(A) = K_0(\mathcal{P}) \ .$$

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<u>Example</u> The isomorphism class group $K_0(\mathcal{A})$ of the additive category $\mathcal{A} = \{$based f.g. free A-modules$\}$ is such that there is defined an isomorphism

$$K_0(\mathcal{A}) \longrightarrow \mathbb{Z} \ ; \ [M] \longmapsto \mathrm{rank}_A(M)$$

(assuming as always that the rank of a f.g. free A-module is well defined).

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<u>Proposition 2.2</u> Sign defines a symplectic form on the isomorphism class group $K_0(A)$ of an additive category $\mathcal{A}$ taking values in the isomorphism torsion group $K_1^{iso}(\mathcal{A})$

$$\varepsilon : K_0(\mathcal{A})\otimes K_0(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A}) \ ; \ [M]\otimes[N] \longmapsto \varepsilon(M,N) \ .$$

<u>Proof</u>: Immediate from Proposition 2.1.

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The <u>reduced isomorphism torsion group</u> of $\mathcal{A}$ is the quotient group of $K_1^{iso}(\mathcal{A})$ defined by

$$\tilde{K}_1^{iso}(\mathcal{A}) = \mathrm{coker}(\varepsilon : K_0(\mathcal{A})\otimes K_0(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A})) \ .$$

<u>Example</u> The reduced isomorphism torsion group $\tilde{K}_1^{iso}(\mathcal{A})$ of $\mathcal{A} = \{$based f.g. free A-modules$\}$ contains the reduced torsion group $\tilde{K}_1(A) = \mathrm{coker}(K_1(\mathbb{Z}) \longrightarrow K_1(A)) = K_1(A)/\{\tau(-1:A \longrightarrow A)\}$ as a direct summand, with the natural map $\tilde{K}_1(A) \longrightarrow \tilde{K}_1^{iso}(\mathcal{A}) \ ; \ \tilde{\tau}(f) \longmapsto \tilde{\tau}(f)$ split by

$$\tilde{K}_1^{iso}(\mathcal{A}) \longrightarrow\!\!\!\!\rightarrow \tilde{K}_1(A) \ ; \ \tilde{\tau}(f:M \longrightarrow N) \longmapsto \tilde{\tau}((f_{ij}))$$

$$(1 \leqslant i,j \leqslant n = \mathrm{rank}_A(M) = \mathrm{rank}_A(N)) \ .$$

[]

§3. Torsion for chain complexes

Let $\mathrm{iso}(\mathcal{A})$ denote the set of isomorphisms in an additive category $\mathcal{A}$, and let K be an abelian group. A function $\tau:\mathrm{iso}(\mathcal{A}) \longrightarrow K$ is logarithmic if for all $(f:M \longrightarrow N),(g:N \longrightarrow P) \in \mathrm{iso}(\mathcal{A})$

$$\tau(gf) = \tau(f) + \tau(g) \in K\ .$$

A function $\tau:\mathrm{iso}(\mathcal{A}) \longrightarrow K$ is additive if for all $(f:M \longrightarrow N)$, $(f':M' \longrightarrow N') \in \mathrm{iso}(\mathcal{A})$

$$\tau(f \oplus f') = \tau(f) + \tau(f') \in K\ .$$

The isomorphism torsion function

$$\tau : \mathrm{iso}(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A})\ ;\ f \longmapsto \tau(f)$$

is both logarithmic and additive, by construction, and is universal with respect to functions with these properties.

We shall now define logarithmic torsion functions $\tau:\mathrm{iso}(\mathcal{C}) \longrightarrow K$ for various additive categories $\mathcal{C}$ of chain complexes in an additive category $\mathcal{A}$ (with morphisms either chain maps or chain homotopy classes of chain maps), such that K is one of the K_1-groups of $\mathcal{A}$ considered in §§1,2. In general these torsion functions will not be additive.

We refer to Ranicki [18] for an exposition of the chain homotopy theory of chain complexes in an additive category $\mathcal{A}$, adopting the same terminology and sign conventions.

Let $\mathcal{C}(\mathcal{A})$ be the additive category of finite chain complexes in $\mathcal{A}$

$$C : \ldots \longrightarrow 0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{d} C_0$$

and chain maps.

The torsion of an isomorphism $f:C \longrightarrow D$ in $\mathcal{C}(\mathcal{A})$ is defined by

$$\tau(f) = \sum_{r=0}^{\infty} (-)^r \tau(f:C_r \longrightarrow D_r) \in K_1^{iso}(\mathcal{A})\ .$$

Proposition 3.1 The torsion function

$$\tau : \mathrm{iso}(\mathcal{C}(\mathcal{A})) \longrightarrow K_1^{iso}(\mathcal{A})\ ;\ f \longmapsto \tau(f)$$

is logarithmic and additive.

Proof: Immediate from the logarithmic and additive properties of $\tau:\mathrm{iso}(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A})$.

The torsion of a contractible finite chain complex C in $\mathcal{A}$ is defined by

$$\tau(C) = \tau(d+\Gamma = \begin{pmatrix} d & 0 & 0 & \dots \\ \Gamma & d & 0 & \dots \\ 0 & \Gamma & d & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$: C_{odd} = C_1 \oplus C_3 \oplus C_5 \oplus \dots \longrightarrow C_{even} = C_0 \oplus C_2 \oplus C_4 \oplus \dots)$$

$$\in K_1^{iso}(\mathcal{A}) \ ,$$

using any chain contraction $\Gamma : 0 \simeq 1 : C \longrightarrow C$ of C. The morphism $d+\Gamma : C_{odd} \longrightarrow C_{even}$ is an isomorphism since there is defined an inverse

$$(d+\Gamma)^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ \Gamma^2 & 1 & 0 & \dots \\ 0 & \Gamma^2 & 1 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}^{-1} \begin{pmatrix} \Gamma & d & 0 & \dots \\ 0 & \Gamma & d & \dots \\ 0 & 0 & \Gamma & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$: C_{even} = C_0 \oplus C_2 \oplus C_4 \oplus \dots \longrightarrow C_{odd} = C_1 \oplus C_3 \oplus C_5 \oplus \dots \ .$$

If $\Gamma' : 0 \simeq 1 : C \longrightarrow C$ is another chain contraction of C the morphisms defined by

$$\Delta = (\Gamma' - \Gamma)\Gamma : C_r \longrightarrow C_{r+2} \quad (r \geqslant 0)$$

are such that

$$\Delta d - d\Delta = \Gamma' - \Gamma : C_r \longrightarrow C_{r+1} \quad (r \geqslant 0)$$

(defining a homotopy of chain homotopies $\Delta : \Gamma \simeq \Gamma' : 0 \simeq 1 : C \longrightarrow C$). The simple automorphisms

$$h_{even} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ \Delta & 1 & 0 & \dots \\ 0 & \Delta & 1 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$: C_{even} = C_0 \oplus C_2 \oplus C_4 \oplus \dots \longrightarrow C_{even} = C_0 \oplus C_2 \oplus C_4 \oplus \dots \ ,$$

$$h_{odd} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ \Delta & 1 & 0 & \dots \\ 0 & \Delta & 1 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$: C_{odd} = C_1 \oplus C_3 \oplus C_5 \oplus \dots \longrightarrow C_{odd} = C_1 \oplus C_3 \oplus C_5 \oplus \dots$$

are such that the diagram of isomorphisms

$$\begin{tikzcd} C_{odd} \arrow[r, "d+\Gamma"] \arrow[d, "h_{odd}"'] & C_{even} \arrow[d, "h_{even}"] \\ C_{odd} \arrow[r, "d+\Gamma'"] & C_{even} \end{tikzcd}$$

commutes up to a simple automorphism of the type

$$(d+\Gamma)^{-1}h_{even}^{-1}(d+\Gamma')h_{odd} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ ? & 1 & 0 & \dots \\ ? & ? & 1 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$: C_{odd} = C_1 \oplus C_3 \oplus C_5 \oplus \dots \longrightarrow C_{odd} = C_1 \oplus C_3 \oplus C_5 \oplus \dots \ .$$

As usual, simple means $\tau = 0$. It follows that the torsion of C is independent of the choice of chain contraction Γ, with

$$\tau(C) = \tau(d+\Gamma : C_{odd} \longrightarrow C_{even}) = \tau(d+\Gamma' : C_{odd} \longrightarrow C_{even}) \in K_1^{iso}(\mathcal{A}) \ .$$

<u>Example</u> For $\mathcal{A}$ = {based f.g. free A-modules} the component of the isomorphism torsion $\tau(C) \in K_1^{iso}(\mathcal{A})$ in the automorphism torsion group is the torsion $\tau(C) \in K_1^{aut}(\mathcal{A}) = K_1(A)$ originally defined by Whitehead [24], with C a contractible finite based f.g. free A-module chain complex.

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<u>Proposition 3.2</u> The torsion of an isomorphism $f : C \longrightarrow D$ of contractible finite chain complexes in an additive category $\mathcal{A}$ is given by

$$\tau(f) = \tau(D) - \tau(C) \in K_1^{iso}(\mathcal{A}) \ .$$

<u>Proof</u>: Given a chain contraction $\Gamma_C : 0 \simeq 1 : C \longrightarrow C$ of C define a chain contraction of D by

$$\Gamma_D = f\Gamma_C f^{-1} : 0 \simeq 1 : D \longrightarrow D \ .$$

There is then defined a commutative diagram of isomorphisms in $\mathcal{A}$

$$\begin{tikzcd} C_{odd} = C_1 \oplus C_3 \oplus C_5 \oplus \dots \arrow[r, "d_C+\Gamma_C"] \arrow[d, "f_{odd} = f_1 \oplus f_3 \oplus f_5 \oplus \dots"'] & C_{even} = C_0 \oplus C_2 \oplus C_4 \oplus \dots \arrow[d, "f_{even} = f_0 \oplus f_2 \oplus f_4 \oplus \dots"] \\ D_{odd} = D_1 \oplus D_3 \oplus D_5 \oplus \dots \arrow[r, "d_D+\Gamma_D"] & D_{even} = D_0 \oplus D_2 \oplus D_4 \oplus \dots \end{tikzcd} ,$$

so that

$$\begin{aligned}\tau(D) - \tau(C) &= \tau(d_D+\Gamma_D : D_{odd} \longrightarrow D_{even}) - \tau(d_C+\Gamma_C : C_{odd} \longrightarrow C_{even}) \\ &= \tau(f_{even} : C_{even} \longrightarrow D_{even}) - \tau(f_{odd} : C_{odd} \longrightarrow D_{odd}) \\ &= \tau(f) \in K_1^{iso}(\mathcal{A}) .\end{aligned}$$

[]

The <u>intertwining</u> of finite chain complexes C,D in $\mathcal{A}$ is the linear combination of signs defined by

$$\beta(C,D) = \sum_{i>j} (\varepsilon(C_{2i},D_{2j}) - \varepsilon(C_{2i+1},D_{2j+1})) \in K_1^{iso}(\mathcal{A}) .$$

This invariant plays an important role in quantifying the failure of the torsion of chain complexes to be additive. Note that $\beta(C,D)$ is the difference of the torsions of the permutation isomorphisms $(C\oplus D)_{even} \longrightarrow C_{even}\oplus D_{even}$ and $(C\oplus D)_{odd} \longrightarrow C_{odd}\oplus D_{odd}$.

<u>Proposition 3.3</u> The torsions of contractible finite chain complexes in an additive category $\mathcal{A}$ appearing in a short exact sequence

$$0 \longrightarrow C \xrightarrow{\;i\;} C'' \xrightarrow{\;j\;} C' \longrightarrow 0$$

are related by the <u>sum formula</u>

$$\tau(C'') = \tau(C) + \tau(C') + \sum_{r=0}^{\infty} (-)^r \tau((i\ k) : C_r\oplus C'_r \longrightarrow C''_r) + \beta(C,C') \in K_1^{iso}(\mathcal{A}) ,$$

with $\{k : C'_r \longrightarrow C''_r \,|\, r \geqslant 0\}$ any sequence of splitting morphisms such that $jk = 1 : C'_r \longrightarrow C'_r$ $(r \geqslant 0)$ and each $(i\ k) : C_r\oplus C'_r \longrightarrow C''_r$ $(r \geqslant 0)$ is an isomorphism.

<u>Proof</u>: Consider first the special case

$$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C_r \longrightarrow C''_r = C_r\oplus C'_r ,$$

$$j = (0\ 1) : C''_r = C_r\oplus C'_r \longrightarrow C'_r ,$$

$$k = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : C'_r \longrightarrow C''_r = C_r\oplus C'_r ,$$

so that

$$d'' = \begin{pmatrix} d & e \\ 0 & d' \end{pmatrix} : C''_r = C_r\oplus C'_r \longrightarrow C''_{r-1} = C_{r-1}\oplus C'_{r-1}$$

for some morphisms $e : C'_r \longrightarrow C_{r-1}$ $(r \geqslant 1)$ such that $de + ed' = 0$.

Given chain contractions of C and C'

$$\Gamma : 0 \simeq 1 : C \longrightarrow C \quad , \quad \Gamma' : 0 \simeq 1 : C' \longrightarrow C'$$

define a chain contraction of C''

$$\Gamma'' : 0 \simeq 1 : C'' \longrightarrow C''$$

by

$$\Gamma'' = \begin{pmatrix} \Gamma & -\Gamma(e\Gamma'+\Gamma e) \\ 0 & \Gamma' \end{pmatrix} : C''_r = C_r \oplus C'_r \longrightarrow C''_{r+1} = C_{r+1} \oplus C'_{r+1} \quad .$$

There is then defined an isomorphism of short exact sequences in $\mathcal{A}$

$$\begin{tikzcd}
0 \arrow[r] & C_{odd} \arrow[r, "i_{odd}"] \arrow[d, "d+\Gamma"] & C''_{odd} \arrow[r, "j_{odd}"] \arrow[d, "d''+\Gamma''"] & C'_{odd} \arrow[r] \arrow[d, "d'+\Gamma'"] & 0 \\
0 \arrow[r] & C_{even} \arrow[r, "i_{even}"] & C''_{even} \arrow[r, "j_{even}"] & C'_{even} \arrow[r] & 0
\end{tikzcd} ,$$

so that

$$\begin{aligned}
\tau(C'') &= \tau(d''+\Gamma'' : C''_{odd} \longrightarrow C''_{even}) \\
&= \tau(d+\Gamma : C_{odd} \longrightarrow C_{even}) + \tau(d'+\Gamma' : C'_{odd} \longrightarrow C'_{even}) \\
&\quad + \tau((i_{even}\ k_{even}) : C_{even} \oplus C'_{even} \longrightarrow C''_{even}) \\
&\quad - \tau((i_{odd}\ k_{odd}) : C_{odd} \oplus C'_{odd} \longrightarrow C''_{odd}) \\
&= \tau(C) + \tau(C') + \beta(C,C') \in K_1^{iso}(\mathcal{A}) \ ,
\end{aligned}$$

verifying the sum formula in the special case.

In the general case let $\bar{C}''$ be the finite chain complex defined by

$$\bar{d}'' : \bar{C}''_r = C_r \oplus C'_r \xrightarrow{(i\ k)} C''_r \xrightarrow{d''} C''_{r-1} \xrightarrow{(i\ k)^{-1}} C_{r-1} \oplus C'_{r-1} = \bar{C}''_{r-1}$$

so that there are defined an isomorphism of chain complexes

$$(i\ k) : \bar{C}'' \longrightarrow C''$$

and a short exact sequence of contractible finite chain complexes

$$0 \longrightarrow C \xrightarrow{\bar{i}} \bar{C}'' \xrightarrow{\bar{j}} C' \longrightarrow 0$$

with

$$\bar{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C_r \longrightarrow \bar{C}''_r = C_r \oplus C'_r \ ,$$

$$\bar{j} = (0\ 1) : \bar{C}''_r = C_r \oplus C'_r \longrightarrow C'_r \ .$$

By the special case

$$\tau(\bar{C}'') = \tau(C) + \tau(C') + \beta(C,C') \in K_1^{iso}(\mathcal{A})$$

and by Proposition 3.2

$$\tau(C'') - \tau(\bar{C}'') = \sum_{r=0}^{\infty} (-)^r \tau((i\ k):C_r \oplus C'_r \longrightarrow C''_r) \in K_1^{iso}(\mathcal{A}) .$$

The sum formula in the general case follows.

[]

The reduced torsion $\tilde{\tau}(C) \in \tilde{K}_1^{iso}(\mathcal{A})$ of a contractible finite chain complex C in $\mathcal{A}$ is the reduction of the absolute torsion $\tau(C) \in K_1^{iso}(\mathcal{A})$. The intertwining term $\beta(C,C')$ in the sum formula of Proposition 3.3 vanishes in the reduced torsion group, so that

$$\tilde{\tau}(C'') = \tilde{\tau}(C) + \tilde{\tau}(C') + \sum_{r=0}^{\infty} (-)^r \tilde{\tau}((i\ k):C_r \oplus C'_r \longrightarrow C''_r) \in \tilde{K}_1^{iso}(\mathcal{A}) .$$

Remark For $\mathcal{A}$ = {based f.g. free A-modules} the sum formula for reduced torsions in $\tilde{K}_1(A)$ was first obtained by Milnor [11], and the sum formula for absolute torsions in $K_1(A)$ was first obtained by Fossum, Foxby and Iversen [5].

[]

Let $\mathcal{C}^f(\mathcal{A})$ be the additive category of finite chain complexes in $\mathcal{A}$ and chain homotopy classes of chain maps, i.e. the derived category. The isomorphism set $\mathrm{iso}(\mathcal{C}^f(\mathcal{A}))$ consists of the chain homotopy classes of chain equivalences. The appearance of the intertwining term $\beta(C,C')$ in the sum formula of Proposition 3.3 implies that it is not in general possible to extend the universal isomorphism torsion function

$$\tau : \mathrm{iso}(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A}) \ ; \ f \longmapsto \tau(f)$$

to an additive function

$$\tau : \mathrm{iso}(\mathcal{C}^f(\mathcal{A})) \longrightarrow K_1^{iso}(\mathcal{A})$$

such that for every contractible finite chain complex C in $\mathcal{A}$

$$\tau(0 \longrightarrow C) = \tau(C) \in K_1^{iso}(\mathcal{A}) .$$

If there were such an extension, and if C,C' are contractible finite chain complexes in $\mathcal{A}$ such that $\beta(C,C') \neq 0 \in K_1^{iso}(\mathcal{A})$, then

$$\begin{aligned} \tau(0 \longrightarrow C\oplus C') &= \tau(C\oplus C') \\ &= \tau(C) + \tau(C') + \beta(C,C') \\ &\neq \tau(0 \longrightarrow C) + \tau(0 \longrightarrow C') \in K_1^{iso}(\mathcal{A}) , \end{aligned}$$

a contradiction.

Example Let $\mathcal{A}$ = {based f.g. free A-modules} for some ring A (such as a group ring $\mathbb{Z}[\pi]$) for which $\mathbb{Z} \longrightarrow A \,;\, 1 \longmapsto 1$ induces an injection

$$K_1(\mathbb{Z}) = \mathbb{Z}_2 \rightarrowtail K_1(A) \;;\; \tau(-1:\mathbb{Z} \longrightarrow \mathbb{Z}) \longmapsto \tau(-1:A \longrightarrow A) \;.$$

The contractible finite chain complexes in $\mathcal{A}$ defined by

$$C : \ldots \longrightarrow 0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0$$

$$C' : \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow A \xrightarrow{1} A$$

are such that $\beta(C,C') \neq 0 \in K_1^{iso}(\mathcal{A})$, with automorphism torsion component

$$\beta(C,C') = \tau(-1:A \longrightarrow A) \neq 0 \in K_1^{aut}(\mathcal{A}) = K_1(A) \,.$$

(On the other hand $\beta(C',C) = 0 \in K_1^{iso}(\mathcal{A})$).

[]

In §4 below we shall define a logarithmic torsion function $\tau:\mathrm{iso}(\mathcal{C}^r(\mathcal{A})) \longrightarrow K_1^{iso}(\mathcal{A})$ on a certain full subcategory $\mathcal{C}^r(\mathcal{A}) \subset \mathcal{C}^f(\mathcal{A})$. We shall be making frequent use of the following properties of β.

Proposition 3.4 The intertwining function $(C,D) \longmapsto \beta(C,D) \in K_1^{iso}(\mathcal{A})$ is such that

i) $\beta(C\oplus C',D) = \beta(C,D) + \beta(C',D)$,

ii) $\beta(C,D\oplus D') = \beta(C,D) + \beta(C,D')$,

iii) $\beta(C,D) - \beta(D,C) + \sum\limits_{r=0}^{\infty} (-)^r \varepsilon(C_r,D_r)$

$$= \varepsilon(C_{even},D_{even}) - \varepsilon(C_{odd},D_{odd}) \;,$$

iv) $\beta(C,SC) + \sum\limits_{r=0}^{\infty} (-)^r \varepsilon(C_r,C_{r-1}) = \varepsilon(C_{even},C_{odd})$ where $SC_r = C_{r-1}$,

v) $\beta(SC,C) = \varepsilon(C_{odd},C_{even})$,

vi) $\beta(SC,SD) = -\beta(C,D)$,

vii) $\beta(C,D) = \beta(C',D')$ if C is isomorphic to C' and D is isomorphic to D'.

Proof: These properties of β follow from the properties of the sign function $(M,N) \longmapsto \varepsilon(M,N)$ obtained in Proposition 2.1.

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§4. Torsion for chain equivalences

The algebraic mapping cone of a chain equivalence $f:C\longrightarrow D$ is a contractible chain complex $C(f)$. The torsion $\tau(f)\in K_1^{iso}(\mathcal{A})$ will now be defined in the case when C and D are finite complexes such that $[C]=[D]=0\in K_0(\mathcal{A})$, as the sum of the torsion $\tau(C(f))$ and a sign term.

The algebraic mapping cone $C(f)$ of a chain map $f:C\longrightarrow D$ in $\mathcal{A}$ is the chain complex in $\mathcal{A}$ defined as usual by

$$d_{C(f)}=\begin{pmatrix} d_D & (-)^{r-1}f \\ 0 & d_C \end{pmatrix}$$

$$: C(f)_r = D_r\oplus C_{r-1}\longrightarrow C(f)_{r-1}=D_{r-1}\oplus C_{r-2}\ .$$

A chain map f is a chain equivalence if and only if $C(f)$ is chain contractible.

A chain homotopy in $\mathcal{A}$

$$g : f \simeq f' : C \longrightarrow D$$

determines an isomorphism of the algebraic mapping cones

$$h : C(f) \longrightarrow C(f')$$

with

$$h=\begin{pmatrix} 1 & (-)^r g \\ 0 & 1 \end{pmatrix} : C(f)_r = D_r\oplus C_{r-1}\longrightarrow C(f')_r = D_r\oplus C_{r-1}\ .$$

(The sign convention is that $d_D g + g d_C = f' - f : C_r\longrightarrow D_r$).

Proposition 4.1 The algebraic mapping cone $C(f)$ of a chain equivalence $f:C\longrightarrow D$ of finite chain complexes in $\mathcal{A}$ is a contractible finite chain complex $C(f)$ in $\mathcal{A}$ such that the torsion $\tau(C(f))\in K_1^{iso}(\mathcal{A})$ is a chain homotopy invariant of f, with $\tau(C(f))=\tau(C(f'))$ for chain homotopic $f,f':C\longrightarrow D$.

Proof: Given a chain homotopy $g:f\simeq f':C\longrightarrow D$ apply Proposition 3.2 to the isomorphism $h:C(f)\longrightarrow C(f')$ defined above, to obtain

$$\begin{aligned}\tau(C(f'))-\tau(C(f)) &= \tau(h)\\ &= \sum_{r=0}^{\infty}(-)^r\tau(h:C(f)_r\longrightarrow C(f')_r)\\ &= 0\in K_1^{iso}(\mathcal{A})\ .\end{aligned}$$

[]

The following results determine the behaviour of the torsion $\tau(C(f)) \in K_1^{iso}(\mathcal{A})$ under the composition and addition of chain equivalences.

Proposition 4.2 i) The torsion of the algebraic mapping cone $C(gf)$ of the composite $gf:C \longrightarrow D \longrightarrow E$ of chain equivalences $f;C \longrightarrow D$, $g:D \longrightarrow E$ of finite chain complexes in $\mathcal{A}$ is given by

$$\tau(C(gf)) = \tau(C(f)) + \tau(C(g)) + \gamma(C,D,E) \in K_1^{iso}(\mathcal{A}) ,$$

with the sign term γ defined by

$$\begin{aligned}\gamma(C,D,E) = \beta(E,SC) - \beta(D,SC) - \beta(E,SD) &\\ + (\varepsilon(D_{even},C_{odd}) - \varepsilon(D_{odd},C_{even}))&\\ + (\varepsilon(D_{even},E_{even}) - \varepsilon(D_{odd},E_{odd}))&\\ + (\varepsilon(C_{odd},E_{even}) - \varepsilon(C_{even},E_{odd}))&\\ + (\varepsilon(D_{even},D_{odd}) - \varepsilon(D_{even},D_{even}))&\\ \in \mathrm{im}(\varepsilon : K_0(\mathcal{A})\otimes K_0(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A})) &.\end{aligned}$$

ii) The torsion of the algebraic mapping cone $C(f\oplus f')$ of the sum $f\oplus f':C\oplus C' \longrightarrow D\oplus D'$ of chain equivalences $f:C \longrightarrow D$, $f':C' \longrightarrow D'$ of finite chain complexes in $\mathcal{A}$ is given by

$$\tau(C(f\oplus f')) = \tau(C(f)) + \tau(C(f')) + \beta(D\oplus SC,D'\oplus SC') + \sum_{r=0}^{\infty}(-)^r\varepsilon(C_{r-1},D'_r) \in K_1^{iso}(\mathcal{A}) .$$

iii) For a chain equivalence $f:C \longrightarrow D$ of contractible finite chain complexes in $\mathcal{A}$

$$\tau(C(f)) = \tau(D) - \tau(C) + \beta(D,SC) \in K_1^{iso}(\mathcal{A}) .$$

iv) The torsion of the algebraic mapping cone $C(1)$ of the identity chain map $1:C \longrightarrow C$ on a finite chain complex C in $\mathcal{A}$ is given by

$$\tau(C(1)) = \beta(C,SC) + \varepsilon(C_{odd},C_{odd}) - \varepsilon(C_{even},C_{odd}) \in K_1^{iso}(\mathcal{A}) .$$

Proof: i) Given a chain complex C let ΩC be the chain complex defined by

$$d_{\Omega C} = d_C : \Omega C_r = C_{r+1} \longrightarrow \Omega C_{r-1} = C_r .$$

Given chain equivalences $f:C \longrightarrow D$, $g:D \longrightarrow E$ of finite chain complexes in $\mathcal{A}$ define a chain map

$$h : \Omega C(g) \longrightarrow C(f)$$

by

$$h = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} : \Omega C(g)_r = E_{r+1} \oplus D_r \longrightarrow C(f)_r = D_r \oplus C_{r-1} .$$

The algebraic mapping cone C(h) is a contractible finite chain complex which fits into two short exact sequences of such complexes

$$0 \longrightarrow C(f) \xrightarrow{i} C(h) \xrightarrow{j} C(g) \longrightarrow 0$$

$$0 \longrightarrow C(gf) \xrightarrow{i'} C(h) \xrightarrow{j'} C(-1_D:D \longrightarrow D) \longrightarrow 0$$

with

$$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C(f)_r \longrightarrow C(h)_r = C(f)_r \oplus C(g)_r ,$$

$$j = (0 \quad 1) : C(h)_r = C(f)_r \oplus C(g)_r \longrightarrow C(g)_r ,$$

$$i' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & f \end{pmatrix} : C(gf)_r = E_r \oplus C_{r-1} \longrightarrow C(h)_r = D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1} ,$$

$$j' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -f & 0 & 1 \end{pmatrix}$$

$$: C(h)_r = D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1} \longrightarrow C(-1_D)_r = D_r \oplus D_{r-1} .$$

The morphisms j,j' are split by the morphisms

$$k = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : C(g)_r \longrightarrow C(h)_r = C(f)_r \oplus C(g)_r ,$$

$$k' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$: C(-1_D)_r = D_r \oplus D_{r-1} \longrightarrow C(h)_r = D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1}$$

and

$$\tau((i\ k)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : C(f)_r \oplus C(g)_r \longrightarrow C(h)_r \ = C(f)_r \oplus C(g)_r) = 0 \ ,$$

$$\tau((i'\ k')) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & f & 0 & 1 \end{pmatrix}$$

$$: C(gf)_r \oplus C(-1_D)_r \ = E_r \oplus C_{r-1} \oplus D_r \oplus D_{r-1}$$

$$\longrightarrow C(h)_r \ = D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1})$$

$$= \varepsilon(E_r \oplus C_{r-1}, D_r) + \varepsilon(E_r, C_{r-1}) \in K_1^{iso}(\mathcal{A}) \ .$$

Applying the sum formula of Proposition 3.3 twice

$$\tau(C(h)) = \tau(C(f)) + \tau(C(g)) + \sum_{r=0}^{\infty} (-)^r \tau((i\ k) : C(f)_r \oplus C(g)_r \longrightarrow C(h)_r)$$

$$+ \beta(C(f), C(g))$$

$$= \tau(C(gf)) + \tau(C(-1_D)) + \beta(C(gf), C(-1_D))$$

$$+ \sum_{r=0}^{\infty} (-)^r \tau((i'\ k') : C(gf)_r \oplus C(-1_D)_r \longrightarrow C(h)_r)$$

$$\in K_1^{iso}(\mathcal{A}) \ .$$

Eliminating $\tau(C(h))$, substituting the values obtained above for $\tau((i\ k))$, $\tau((i'\ k'))$ and also

$$\tau(C(-1_D)) = \varepsilon(D_{even}, D_{even}) - \sum_{r=0}^{\infty} (-)^r \varepsilon(D_r, D_{r-1}) \ ,$$

$$\tau(C(f), C(g)) = \beta(D \oplus SC, E \oplus SD) \ ,$$

$$\tau(C(gf), C(-1_D)) = \beta(E \oplus SC, D \oplus SD) \in K_1^{iso}(\mathcal{A})$$

leads to the required expression for $\tau(C(gf)) \in K_1^{iso}(\mathcal{A})$.

ii) The algebraic mapping cone $C(f \oplus f')$ of the sum $f \oplus f' : C \oplus C' \longrightarrow D \oplus D'$ of chain equivalences fits into a short exact sequence of contractible finite chain complexes

$$0 \longrightarrow C(f) \xrightarrow{\ i\ } C(f \oplus f') \xrightarrow{\ j\ } C(f') \longrightarrow 0$$

with

$$i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$: C(f)_r = D_r \oplus C_{r-1} \longrightarrow C(f\oplus f')_r = D_r \oplus D'_r \oplus C_{r-1} \oplus C'_{r-1} \ ,$$

$$j = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$: C(f\oplus f')_r = D_r \oplus D'_r \oplus C_{r-1} \oplus C'_{r-1} \longrightarrow C(f')_r = D'_r \oplus C'_{r-1} \ .$$

Define a splitting morphism for j by

$$k = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$: C(f')_r = D'_r \oplus C'_{r-1} \longrightarrow C(f\oplus f')_r = D_r \oplus D'_r \oplus C_{r-1} \oplus C'_{r-1} \ ,$$

with

$$\tau((i\ k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$: C(f)_r \oplus C(f')_r = D_r \oplus C_{r-1} \oplus D'_r \oplus C'_{r-1}$$

$$\longrightarrow C(f\oplus f')_r = D_r \oplus D'_r \oplus C_{r-1} \oplus C'_{r-1})$$

$$= \varepsilon(C_{r-1}, D'_r) \in K_1^{iso}(\mathcal{A}) \ .$$

It is now immediate from the sum formula of Proposition 3.3 that

$$\tau(C(f\oplus f')) = \tau(C(f)) + \tau(C(f')) + \beta(D\oplus SC, D'\oplus SC') + \sum_{r=0}^{\infty} (-)^r \varepsilon(C_{r-1}, D'_r) \in K_1^{iso}(\mathcal{A}) \ .$$

iii) Set $E = 0$ in the composition formula i).

iv) Set $f = 1 : C \longrightarrow D = C$, $g = 1 : D = C \longrightarrow E = C$ in the composition formula i).

[]

The <u>reduced torsion</u> of a chain equivalence $f:C \longrightarrow D$ of finite chain complexes in $\mathcal{A}$ is defined by

$$\tilde{\tau}(f) = \tilde{\tau}(C(f)) \in \tilde{K}_1^{iso}(\mathcal{A}) ,$$

that is the reduction of the absolute torsion $\tau(C(f)) \in K_1^{iso}(\mathcal{A})$ of the algebraic mapping cone $C(f)$.

<u>Example</u> For $\mathcal{A}$ = {based f.g. free A-modules} the automorphism component $\tilde{\tau}(f) \in \tilde{K}_1(A)$ of the reduced torsion is just the torsion of a chain equivalence $f:C \longrightarrow D$ in the sense of Whitehead [24] and Milnor [11].

[]

<u>Proposition 4.3</u> i) The reduced torsion function

$$\tilde{\tau} : \mathrm{iso}(\mathcal{C}^f(\mathcal{A})) \longrightarrow \tilde{K}_1^{iso}(\mathcal{A}) \; ; \; f \longmapsto \tilde{\tau}(f)$$

is logarithmic and additive.

ii) The reduced torsion of an isomorphism $f:C \longrightarrow D$ is the reduction of the absolute torsion $\tau(f) = \sum_{r=0}^{\infty} (-)^r \tau(f:C_r \longrightarrow D_r) \in K_1^{iso}(\mathcal{A})$, that is

$$\tilde{\tau}(f) = \sum_{r=0}^{\infty} (-)^r \tilde{\tau}(f:C_r \longrightarrow D_r) \in \tilde{K}_1^{iso}(\mathcal{A}) .$$

iii) The reduced torsion of a chain equivalence $f:C \longrightarrow D$ of contractible finite chain complexes is the difference of the reduced torsions of C and D

$$\tilde{\tau}(f) = \tilde{\tau}(D) - \tilde{\tau}(C) \in \tilde{K}_1^{iso}(\mathcal{A}) .$$

<u>Proof</u>: i) Immediate from the formulae of Proposition 4.2, since all the sign terms vanish on passing to the reduced torsion group $\tilde{K}_1^{iso}(\mathcal{A})$.

ii) Define an isomorphism of contractible finite chain complexes

$$1 \oplus f : C(f) \longrightarrow C(1:D \longrightarrow D)$$

and apply Proposition 3.2.

iii) Apply the logarithmic property of $\tilde{\tau}$ given by i) to the composite

$$f : C \longrightarrow 0 \longrightarrow D$$

(up to chain homotopy).

[]

The <u>class</u> of a finite chain complex C in $\mathcal{A}$ is the element of the isomorphism class group of $\mathcal{A}$ defined by

$$[C] = \sum_{r=0}^{\infty}(-)^r[C_r] = [C_{even}] - [C_{odd}] \in K_0(\mathcal{A}) \ ,$$

a chain homotopy invariant of C.

<u>Example</u> For $\mathcal{A}$ = {based f.g. free A-modules} the class of a finite chain complex C is just the Euler characteristic of C

$$[C] = \chi(C) = \sum_{r=0}^{\infty}(-)^r \mathrm{rank}_A(C_r) \in K_0(\mathcal{A}) = \mathbb{Z} \ .$$

[]

A finite chain complex C in $\mathcal{A}$ is <u>round</u> if

$$[C] = 0 \in K_0(\mathcal{A}) \ .$$

In particular, a contractible finite chain complex is round.

The <u>torsion</u> of a chain equivalence $f:C \longrightarrow D$ of round finite chain complexes in $\mathcal{A}$ is defined by

$$\tau(f) = \tau(C(f)) - \beta(D,SC) \in K_1^{iso}(\mathcal{A}) \ .$$

<u>Remark</u> This formula can be used to define the torsion $\tau(f) \in K_1^{iso}(\mathcal{A})$ of a chain equivalence $f:C \longrightarrow D$ of any finite chain complexes in $\mathcal{A}$, but the resulting function $\tau:\mathrm{iso}(\mathcal{C}^f(\mathcal{A})) \longrightarrow K_1^{iso}(\mathcal{A})$ is neither logarithmic nor additive (cf. Proposition 4.2, and the Example just before Proposition 3.4). There does not appear to be a reasonable way to define either a logarithmic or an additive torsion function $\tau:\mathrm{iso}(\mathcal{C}^f(\mathcal{A})) \longrightarrow K_1^{iso}(\mathcal{A})$ in general.

[]

Let $\mathcal{C}^r(\mathcal{A})$ be the additive category of round finite chain complexes in $\mathcal{A}$ and chain homotopy classes of chain maps, a full subcategory of the derived category $\mathcal{C}^f(\mathcal{A})$.

<u>Proposition 4.4</u> i) The torsion function

$$\tau \ : \ \mathrm{iso}(\mathcal{C}^r(\mathcal{A})) \longrightarrow K_1^{iso}(\mathcal{A}) \ ; \ f \longmapsto \tau(f)$$

is logarithmic, that is $\tau(gf) = \tau(f) + \tau(g)$.

ii) The torsion function $\tau:\mathrm{iso}(\mathcal{C}^r(\mathcal{A})) \longrightarrow K_1^{iso}(\mathcal{A})$ is not additive in general, with the torsion of a sum $f\oplus f':C\oplus C' \longrightarrow D\oplus D'$ given by

$$\tau(f\oplus f') = \tau(f) + \tau(f') - \beta(C,C') + \beta(D,D') \in K_1^{iso}(\mathcal{A}) \ .$$

iii) The torsion of an isomorphism $f:C \longrightarrow D$ of round finite chain complexes agrees with the previous definition

$$\tau(f) = \sum_{r=0}^{\infty}(-)^r \tau(f:C_r \longrightarrow D_r) \in K_1^{iso}(\mathcal{A}) \ .$$

iv) The torsion of a chain equivalence $f:C \longrightarrow D$ of contractible finite chain complexes is the difference of the torsions of C and D

$$\tau(f) = \tau(D) - \tau(C) \in K_1^{iso}(\mathcal{A}) \ .$$

v) The torsion of a chain equivalence $f:C \longrightarrow D$ of round finite chai complexes which fits into a short exact sequence

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0$$

is related to the torsion of the contractible finite chain complex E by the formula

$$\tau(f) = \tau(E) + \sum_{r=0}^{\infty} (-)^r \tau((f\ h):C_r \oplus E_r \longrightarrow D_r) + \beta(C,E) \in K_1^{iso}(\mathcal{A}) \quad ,$$

with $\{h:E_r \longrightarrow D_r \,|\, r \geqslant 0\}$ splitting morphisms for $\{g:D_r \longrightarrow E_r \,|\, r \geqslant 0\}$.

Proof: i) For round C,D,E the sign term $\gamma(C,D,E)$ in the composition formula of Proposition 4.2 i) is given by

$$\gamma(C,D,E) = \beta(E,SC) - \beta(D,SC) - \beta(E,SD) \in K_1^{iso}(\mathcal{A}) \ .$$

ii) By the sum formula of Proposition 4.2 ii)

$$\begin{aligned}
\tau(f\oplus f') &= \tau(C(f\oplus f')) - \beta(D\oplus D',SC\oplus SC') \\
&= \tau(C(f)) + \tau(C(f')) - \beta(D\oplus D',SC\oplus SC') \\
&\qquad + \beta(D\oplus SC,D'\oplus SC') + \sum_{r=0}^{\infty} (-)^r \varepsilon(C_{r-1},D'_r) \\
&= \tau(C(f)) + \tau(C(f')) - \beta(D,SC) - \beta(D',SC') \\
&\qquad\qquad - \beta(C,C') + \beta(D,D') \\
&\qquad \text{(by Proposition 3.4)} \\
&= \tau(f) + \tau(f') - \beta(C,C') + \beta(D,D') \in K_1^{iso}(\mathcal{A}) \ .
\end{aligned}$$

iii) Given an isomorphism $f:C \longrightarrow D$ of round finite chain complexes in $\mathcal{A}$ define an isomorphism of contractible finite chain complexes

$$f' = 1\oplus f \ : \ C' = C(f) \longrightarrow D' = C(1_D:D \longrightarrow D) \ .$$

By Proposition 3.2

$$\begin{aligned}
\tau(D') - \tau(C') &= \sum_{r=0}^{\infty} (-)^r \tau(f':C'_r \longrightarrow D'_r) \\
&= \sum_{r=0}^{\infty} (-)^r \tau(1\oplus f:D_r\oplus C_{r-1} \longrightarrow D_r \oplus D_{r-1}) \\
&= \sum_{r=0}^{\infty} (-)^r \tau(f:C_{r-1} \longrightarrow D_{r-1}) \\
&= -(\sum_{r=0}^{\infty} (-)^r \tau(f:C_r \longrightarrow D_r)) \in K_1^{iso}(\mathcal{A}) \ .
\end{aligned}$$

By the logarithmic property of torsion proved in i)

$$\begin{aligned}\tau(f) &= \tau(f) - \tau(1_D)\\ &= (\tau(C') - \beta(D,SC)) - (\tau(D') - \beta(D,SD))\\ &= \tau(C') - \tau(D')\\ &= \sum_{r=0}^{\infty}(-)^r\tau(f:C_r \longrightarrow D_r) \in K_1^{iso}(\mathcal{A}) .\end{aligned}$$

iv) Immediate from the logarithmic property of τ applied to the composite $0:C \xrightarrow{f} D \longrightarrow 0$, noting that $\tau(C \longrightarrow 0) = -\tau(C) \in K_1^{iso}(\mathcal{A})$.

v) Apply the sum formula of Proposition 3.3 to the short exact sequence of contractible finite chain complexes

$$0 \longrightarrow C(1_C) \xrightarrow{\quad i \quad} C(f) \xrightarrow{\quad j \quad} E \longrightarrow 0$$

with

$$i = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} : C(1_C)_r = C_r \oplus C_{r-1} \longrightarrow C(f)_r = D_r \oplus C_{r-1} ,$$

$$j = (g \;\; 0) : C(f)_r = D_r \oplus C_{r-1} \longrightarrow E_r$$

to obtain

$$\begin{aligned}\tau(C(f)) &= \tau(C(1_C)) + \tau(E) + \sum_{r=0}^{\infty}(-)^r\tau(\begin{pmatrix} f & 0 & h \\ 0 & 1 & 0\end{pmatrix}:C_r \oplus C_{r-1} \oplus E_r \longrightarrow D_r \oplus C_{r-1})\\ &\qquad + \beta(C(1_C),E)\\ &= \beta(C,SC) + \tau(E) + \sum_{r=0}^{\infty}(-)^r(\tau((f \;\; h):C_r \oplus E_r \longrightarrow D_r) + \varepsilon(C_{r-1},E_r))\\ &\qquad + \beta(C \oplus SC,E) \in K_1^{iso}(\mathcal{A}) .\end{aligned}$$

It follows that

$$\begin{aligned}\tau(f) &= \tau(C(f)) - \beta(D,SC)\\ &= \tau(E) + \sum_{r=0}^{\infty}(-)^r\tau((f \;\; h):C_r \oplus E_r \longrightarrow D_r) + \beta(C,E)\\ &\qquad + (\beta(SC,E) - \beta(E,SC) + \sum_{r=0}^{\infty}(-)^r\varepsilon(C_{r-1},E_r)) \in K_1^{iso}(\mathcal{A}) .\end{aligned}$$

By Proposition 3.4 iii)

$$\begin{aligned}\beta(SC,E) - \beta(E,SC) + \sum_{r=0}^{\infty}(-)^r\varepsilon(C_{r-1},E_r)\\ &= \varepsilon(C_{odd},E_{even}) - \varepsilon(C_{even},E_{odd})\\ &= 0 \in K_1^{iso}(\mathcal{A}) \text{ (since } C,E \text{ are round).}\end{aligned}$$

[]

An element $x \in K_0(\mathcal{A})$ is <u>even</u> if

$$\varepsilon(x,y) = 0 \in K_0(\mathcal{A}) \quad ,$$

for every $y \in K_0(\mathcal{A})$. The even elements of $K_0(\mathcal{A})$ define a subgroup, the kernel of the adjoint map of the sign form of Proposition 2.2

$$K_0(\mathcal{A}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{A}), K_1^{iso}(\mathcal{A})) \; ;$$

$$[M] \longmapsto ([N] \longmapsto \varepsilon(M,N)) \; .$$

<u>Example</u> For $\mathcal{A}$ = {based f.g. free A-modules} the isomorphism

$$K_0(\mathcal{A}) \longrightarrow \mathbb{Z} \; ; \; [M] - [N] \longmapsto \mathrm{rank}_A(M) - \mathrm{rank}_A(N)$$

sends the subgroup of even elements in $K_0(\mathcal{A})$ to the subgroup $2\mathbb{Z} \subset \mathbb{Z}$ of even integers.

[]

A finite chain complex C in $\mathcal{A}$ is <u>even</u> if the class $[C] \in K_0(\mathcal{A})$ is even. In particular, a round finite complex is even, since $0 \in K_0(\mathcal{A})$ is an even element.

Let $\mathcal{C}^e(\mathcal{A})$ be the additive category of even finite chain complexes in $\mathcal{A}$ and chain homotopy classes of chain maps. Thus $\mathcal{C}^e(\mathcal{A})$ is a full subcategory of $\mathcal{C}^f(\mathcal{A})$, and $\mathcal{C}^r(\mathcal{A})$ is a full subcategory of $\mathcal{C}^e(\mathcal{A})$.

The <u>torsion</u> of a chain equivalence $f: C \longrightarrow D$ of even finite chain complexes in $\mathcal{A}$ is defined in exactly the same way as for round complexes, by the formula

$$\tau(f) = \tau(C(f)) - \beta(D,SC) \in K_1^{iso}(\mathcal{A}) \; .$$

<u>Proposition 4.5</u> The torsion function $\tau: \mathrm{iso}(\mathcal{C}^e(\mathcal{A})) \longrightarrow K_1^{iso}(\mathcal{A})$ has all the properties stated for $\tau: \mathrm{iso}(\mathcal{C}^r(\mathcal{A})) \longrightarrow K_1^{iso}(\mathcal{A})$ in Proposition 4.4, in particular the logarithmic property.

<u>Proof</u>: The proof of Proposition 4.4 depended on the sign properties of round complexes which are the same for even complexes.

[]

Given an object A of $\mathcal{A}$ and an integer $n \geqslant 0$ define the <u>elementary</u> contractible finite chain complex in $\mathcal{A}$

$$A(n,n+1) : \ldots \longrightarrow 0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0 \longrightarrow \ldots$$

concentrated in degrees n,n+1. For any finite chain complex C in $\mathcal{A}$ the inclusion

$$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C \longrightarrow C \oplus A(n,n+1)$$

is a chain equivalence such that

$$\tau(C(i)) = \tau(C(1_C)) + (-)^n(\varepsilon(C_{n-1},A) - \varepsilon(C_n,A) + \varepsilon(C_{n+1},A))$$

$$\in \operatorname{im}(\varepsilon : K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A})) \quad ,$$

and such that for round finite C

$$\tau(i) = \sum_{r>n+1} (-)^r \varepsilon(C_r,A) \in K_1^{iso}(\mathcal{A}) \quad .$$

Working exactly as in Whitehead [24] (the special case $\mathcal{A}$ = {based f.g. free A-modules}) it can be shown that the reduced torsion $\tilde{\tau}(f) = \tilde{\tau}(C(f)) \in \tilde{K}_1^{iso}(\mathcal{A})$ of a chain equivalence $f : C \longrightarrow D$ of finite chain complexes in $\mathcal{A}$ is such that $\tilde{\tau}(f) = 0$ if and only if there exist elementary complexes $A_i(m_i,m_i+1)$ $(1 \leqslant i \leqslant p)$, $B_j(n_j,n_j+1)$ $(1 \leqslant j \leqslant q)$ such that the chain equivalence

$$f \oplus 0 : C' = C \oplus \sum_{i=1}^{p} A_i(m_i,m_i+1) \longrightarrow D' = D \oplus \sum_{j=1}^{q} B_j(n_j,n_j+1)$$

is chain homotopic to an isomorphism $f' : C' \longrightarrow D'$ such that

$$\tilde{\tau}(f' : C'_r \longrightarrow D'_r) = 0 \in \tilde{K}_1^{iso}(\mathcal{A}) \quad (r \geqslant 0) \quad .$$

There does not appear to be a corresponding interpretation of the vanishing $\tau(f) = 0$ of the absolute torsion $\tau(f) \in K_1^{iso}(\mathcal{A})$ of a chain equivalence $f : C \longrightarrow D$ of round finite chain complexes, except in the trivial case when the classes $[C_r], [D_r] \in K_0(\mathcal{A})$ $(r \geqslant 0)$ are all even and the sign terms vanish.

§5. Canonical structures

The isomorphism torsion group $K_1^{iso}(\mathcal{A})$ is too large (and insufficiently functorial) for practical applications, as compared to the automorphism torsion group $K_1^{aut}(\mathcal{A}) = K_1(\mathcal{A})$. We shall now investigate structures on an additive category $\mathcal{A}$ which ensure that the natural map $K_1(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A})$ is a canonically split injection, with a splitting map $K_1^{iso}(\mathcal{A}) \longrightarrow K_1(\mathcal{A})$ allowing an automorphism torsion component $\tau^{aut} \in K_1(\mathcal{A})$ to be split off from any isomorphism torsion $\tau \in K_1^{iso}(\mathcal{A})$.

A canonical structure ϕ on an additive category $\mathcal{A}$ is a collection of isomorphisms $\{\phi_{M,N}: M \longrightarrow N\}$, one for each ordered pair (M,N) of isomorphic objects in $\mathcal{A}$, such that

i) $\phi_{M,M} = 1 : M \longrightarrow M$,

ii) $\phi_{M,P} = \phi_{N,P}\phi_{M,N} : M \longrightarrow N \longrightarrow P$,

iii) $\phi_{M\oplus M',N\oplus N'} = \phi_{M,N}\oplus\phi_{M',N'} : M\oplus M' \longrightarrow N\oplus N'$.

Example Let $\mathcal{A}$ = {based f.g. free A-modules}, assuming (as always) that A is such that f.g. free A-modules have well defined rank. Based f.g. free A-modules M,N are isomorphic if and only if they have the same rank, n say, in which case there is defined a canonical isomorphism

$$\phi_{M,N} : M \longrightarrow N \ ; \ \sum_{r=1}^{n} a_r x_r \longmapsto \sum_{r=1}^{n} a_r y_r \qquad (a_r \in A)$$

with $(x_1,x_2,\ldots,x_n)$, $(y_1,y_2,\ldots,y_n)$ the given bases of M,N. The collection $\phi = \{\phi_{M,N}\}$ defines a canonical structure on $\mathcal{A}$.

[]

Proposition 5.1 A canonical structure ϕ on an additive category $\mathcal{A}$ determines a splitting of the natural map $K_1(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A})$

$$K_1^{iso}(\mathcal{A}) \longrightarrow\!\!\!\!\rightarrow K_1(\mathcal{A}) \ ; \ \tau(f:M \longrightarrow N) \longmapsto \tau(\phi_{N,M}f:M \longrightarrow N \longrightarrow M) \ ,$$

so that $K_1^{iso}(\mathcal{A}) = K_1(\mathcal{A})\oplus ?$.

Proof: Trivial.

[]

In fact, canonical stable isomorphisms are sufficient to split $K_1(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A})$, as follows.

A <u>stable isomorphism</u> between objects M,N in an additive category $\mathcal{A}$

$$[f] : M \longrightarrow N$$

is an equivalence class of isomorphisms $f: M\oplus X \longrightarrow N\oplus X$ under the equivalence relation

$(f: M\oplus X \longrightarrow N\oplus X) \sim (g: M\oplus Y \longrightarrow N\oplus Y)$ if the automorphism

$$h : M\oplus X\oplus Y \xrightarrow{f\oplus 1_Y} N\oplus X\oplus Y \xrightarrow{1_N\oplus\begin{pmatrix} 0 & 1_Y \\ 1_X & 0\end{pmatrix}} N\oplus Y\oplus X \xrightarrow{g^{-1}\oplus 1_X} M\oplus Y\oplus X \xrightarrow{1_M\oplus\begin{pmatrix} 0 & 1_X \\ 1_Y & 0\end{pmatrix}} M\oplus X\oplus Y$$

is simple, that is $\tau(h) = 0 \in K_1(\mathcal{A})$.

<u>Proposition 5.2</u> Stable isomorphisms are the morphisms of a category $\mathcal{A}^s$, with the same objects as $\mathcal{A}$.

<u>Proof</u>: The composite of the stable isomorphisms

$$[f] : M \longrightarrow N \quad , \quad [g] : N \longrightarrow P$$

is the stable isomorphism

$$[g][f] = [e] : M \longrightarrow P$$

represented by the isomorphism

$$e : M\oplus X\oplus Y \xrightarrow{f\oplus 1_Y} N\oplus X\oplus Y \xrightarrow{1_N\oplus\begin{pmatrix} 0 & 1_Y \\ 1_X & 0\end{pmatrix}} N\oplus Y\oplus X \xrightarrow{g\oplus 1_X} P\oplus Y\oplus X \xrightarrow{1_P\oplus\begin{pmatrix} 0 & 1_X \\ 1_Y & 0\end{pmatrix}} P\oplus X\oplus Y \ .$$

[]

Although the stable category $\mathcal{A}^s$ is not additive it is possible to define the <u>sum</u> of stable isomorphisms $[f]: M \longrightarrow N$, $[f']: M' \longrightarrow N'$ to be the stable isomorphism

$$[f]\oplus[f'] = [f''] : M\oplus M' \longrightarrow N\oplus N'$$

represented by the isomorphism

$$f'' : M\oplus M'\oplus X\oplus X' \xrightarrow{1_M\oplus\begin{pmatrix} 0 & 1_X \\ 1_{M'} & 0\end{pmatrix}\oplus 1_{X'}} M\oplus X\oplus M'\oplus X' \xrightarrow{f\oplus f'} N\oplus X\oplus N'\oplus X' \xrightarrow{1_N\oplus\begin{pmatrix} 0 & 1_{N'} \\ 1_X & 0\end{pmatrix}\oplus 1_{X'}} N\oplus N'\oplus X\oplus X' \ .$$

The <u>torsion</u> of a stable $\begin{cases} \text{isomorphism } [f]:M \longrightarrow N \\ \text{automorphism } [f]:M \longrightarrow M \end{cases}$

is defined by

$$\begin{cases} \tau([f]) = \tau(f:M\oplus X \longrightarrow N\oplus X) \in K_1^{iso}(\mathcal{A}) \\ \tau([f]) = \tau(f:M\oplus X \longrightarrow M\oplus X) \in K_1^{aut}(\mathcal{A}) = K_1(\mathcal{A}) \end{cases},$$

using any representative isomorphism f. In both cases

$$\tau([g][f]) = \tau([f]) + \tau([g]) \quad , \quad \tau([f]\oplus[f']) = \tau([f]) + \tau([f']) \ .$$

A <u>canonical stable structure</u> $[\phi]$ on an additive category $\mathcal{A}$ is a collection of stable isomorphisms $\{[\phi_{M,N}]:M \longrightarrow N\}$, one for each ordered pair (M,N) of stably isomorphic objects in $\mathcal{A}$, such that

i) $[\phi_{M,M}] = [1_M] : M \longrightarrow M$,

ii) $[\phi_{M,P}] = [\phi_{N,P}][\phi_{M,N}] : M \longrightarrow N \longrightarrow P$,

iii) $[\phi_{M\oplus M',N\oplus N'}] = [\phi_{M,N}]\oplus[\phi_{M',N'}] : M\oplus M' \longrightarrow N\oplus N'$.

Thus $[\phi]$ is a canonical structure on the stable category $\mathcal{A}^s$. An actual canonical structure ϕ on $\mathcal{A}$ determines a canonical stable structure $[\phi]$ on $\mathcal{A}$ with

$$[\phi_{M,N}] = [\phi_{M\oplus X,N\oplus X}] : M \longrightarrow N$$

for any objects M,N,X in $\mathcal{A}$ such that $M\oplus X$ is isomorphic to $N\oplus X$.

<u>Proposition 5.3</u> A canonical stable structure $[\phi]$ on an additive category $\mathcal{A}$ determines a splitting of the natural map $K_1(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A})$

$$K_1^{iso}(\mathcal{A}) \longrightarrow K_1(\mathcal{A}) \ ; \ \tau(f:M \longrightarrow N) \longmapsto \tau([\phi_{N,M}][f]:M \longrightarrow N \longrightarrow M) \ ,$$

so that $K_1^{iso}(\mathcal{A}) = K_1(\mathcal{A})\oplus ?$.

<u>Proof</u>: Trivial.

[]

An additive category $\mathcal{A}$ which is equipped with a sufficiently additive "Eilenberg swindle" has a canonical stable structure, as follows.

A <u>flasque structure</u> $\{\Sigma,\sigma,\rho\}$ on an additive category $\mathcal{A}$ consists of

i) an object ΣM for each object M of $\mathcal{A}$,

ii) an isomorphism $\sigma_M:M\oplus\Sigma M \longrightarrow \Sigma M$ for each object M of $\mathcal{A}$,

iii) an isomorphism $\rho_{M,N}:\Sigma(M\oplus N) \longrightarrow \Sigma M\oplus\Sigma N$ for each pair of objects M,N in $\mathcal{A}$, such that

$$\sigma_{M\oplus N} : M\oplus N\oplus\Sigma(M\oplus N) \xrightarrow{1_{M\oplus N}\oplus\rho_{M,N}} M\oplus N\oplus\Sigma M\oplus\Sigma N$$
$$\xrightarrow{1_M\oplus\begin{pmatrix} 0 & 1_{\Sigma M} \\ 1_N & 0\end{pmatrix}\oplus 1_{\Sigma N}} M\oplus\Sigma M\oplus N\oplus\Sigma N \xrightarrow{\sigma_M\oplus\sigma_N} \Sigma M\oplus\Sigma N \xrightarrow{\rho^{-1}_{M,N}} \Sigma(M\oplus N) .$$

The terminology derives from Karoubi [7,p.147].

An additive category $\mathcal{A}$ admits a structure $\{\Sigma,\sigma\}$ satisfying i) and ii) (but not necessarily iii)) if and only if $K_0(\mathcal{A}) = 0$, or equivalently if each object M is stably isomorphic to 0. The isomorphisms $\sigma_M : M\oplus\Sigma M \longrightarrow \Sigma M$ represent stable isomorphisms $[\sigma_M] : M \longrightarrow 0$.

Example If $\mathcal{A}$ is an additive category with countable direct sums then $K_0(\mathcal{A}) = 0$ by the original Eilenberg swindle (cf. Swan [21,p.66]), which is incorporated in the flasque structure $\{\Sigma,\sigma,\rho\}$ defined on $\mathcal{A}$ by

i) $\Sigma P = \sum\limits_1^\infty P = P\oplus P\oplus P\oplus\ldots$,

ii) $\sigma_P : P\oplus\Sigma P \longrightarrow \Sigma P$; $(x,(y_1,y_2,\ldots)) \longmapsto (x,y_1,y_2,\ldots)$

iii) $\rho_{P,Q} : \Sigma(P\oplus Q) \longrightarrow \Sigma P\oplus\Sigma Q$;
$((x_1,y_1),(x_2,y_2),\ldots) \longmapsto ((x_1,x_2,\ldots),(y_1,y_2,\ldots))$.

In particular, $\mathcal{A}$ = {projective A-modules} is an additive category with countable direct sums, for any ring A.

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Remark In the above example Σ can be extended to an exact endofunctor $\Sigma : \mathcal{A} \longrightarrow \mathcal{A}$ such that σ defines a natural equivalence of functors

$$\sigma : 1_{\mathcal{A}}\oplus\Sigma \simeq \Sigma : \mathcal{A} \longrightarrow \mathcal{A} ,$$

by defining $\Sigma(f:P \longrightarrow Q)$ to be

$$\Sigma f : \Sigma P \longrightarrow \Sigma Q \ ; \ (x_1,x_2,\ldots) \longmapsto (f(x_1),f(x_2),\ldots) \quad .$$

It follows that $K_*(\mathcal{A}) = 0$. A flasque category in the sense of Karoubi [7] is in particular an additive category $\mathcal{A}$ for which there exists an exact endofunctor $\Sigma : \mathcal{A} \longrightarrow \mathcal{A}$ such that $1_{\mathcal{A}}\oplus\Sigma$ is naturally equivalent to Σ. Such structures were considered in connection with formal delooping procedures abstracting the Bott periodicity theorem. In the lower algebraic K-theory examples below the flasque structures $\{\Sigma,\sigma,\rho\}$ are such that Σ does not in general extend to morphisms, and the flasque structure only guarantees that $K_0(\mathcal{A}) = 0$ for the additive categories $\mathcal{A}$ in question.

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Proposition 5.4 A flasque structure $\{\Sigma,\sigma,\rho\}$ on an additive category $\mathcal{A}$ determines a canonical stable structure $[\phi]$ on $\mathcal{A}$ by

$$[\phi_{M,N}] = [\sigma_N]^{-1}[\sigma_M] : M \longrightarrow O \longrightarrow N ,$$

so that the natural map $K_1(\mathcal{A}) \longrightarrow K_1^{iso}(\mathcal{A})$ splits and $K_1^{iso}(\mathcal{A}) = K_1(\mathcal{A})\oplus ?$.

Proof: The stable isomorphism $[\phi_{M,N}]:M \longrightarrow N$ is represented by the isomorphism

$$\phi_{M,N} : M\oplus\Sigma M\oplus\Sigma N \xrightarrow{\sigma_M\oplus 1_{\Sigma N}} \Sigma M\oplus\Sigma N \xrightarrow{1_{\Sigma M}\oplus\sigma_N^{-1}} \Sigma M\oplus N\oplus\Sigma N \xrightarrow{\begin{pmatrix} 0 & 1_N \\ 1_{\Sigma M} & 0 \end{pmatrix}\oplus 1_{\Sigma N}} N\oplus\Sigma M\oplus\Sigma N .$$

The conditions i) $[\phi_{M,M}] = [1_M]$, ii) $[\phi_{M,P}] = [\phi_{N,P}][\phi_{M,N}]$ for a canonical stable structure $[\phi]$ are clear from the definition of the stable category $\mathcal{A}^s$ (Proposition 5.1). As for the additivity condition iii) $[\phi_{M\oplus M',N\oplus N'}] = [\phi_{M,N}]\oplus[\phi_{M',N'}]$ this follows on observing that the isomorphism

$$f : \Sigma(M\oplus M')\oplus\Sigma(N\oplus N') \xrightarrow{\rho_{M,M'}\oplus\rho_{N,N'}} \Sigma M\oplus\Sigma M'\oplus\Sigma N\oplus\Sigma N' \xrightarrow{1_{\Sigma M}\oplus\begin{pmatrix} 0 & 1_{\Sigma N} \\ 1_{\Sigma M'} & 0 \end{pmatrix}\oplus 1_{\Sigma N'}} \Sigma M\oplus\Sigma N\oplus\Sigma M'\oplus\Sigma N'$$

is such that there is defined a commutative diagram of isomorphisms in $\mathcal{A}$

$$\begin{array}{ccc} M\oplus M'\oplus\Sigma(M\oplus M')\oplus\Sigma(N\oplus N') & \xrightarrow{1_{M\oplus M'}\oplus f} & M\oplus M'\oplus\Sigma M\oplus\Sigma N\oplus\Sigma M'\oplus\Sigma N' \\ \downarrow{\scriptstyle \phi_{M\oplus M',N\oplus N'}} & & \downarrow{\scriptstyle \phi_{M,N}\oplus\phi_{M',N'}} \\ N\oplus N'\oplus\Sigma(M\oplus M')\oplus\Sigma(N\oplus N') & \xrightarrow{1_{N\oplus N'}\oplus f} & N\oplus N'\oplus\Sigma M\oplus\Sigma N\oplus\Sigma M'\oplus\Sigma N' \end{array}$$

Thus $[\phi]$ is a canonical stable structure on $\mathcal{A}$, and Proposition 5.3 applies.

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Flasque structures arise naturally in lower algebraic K-theory, as follows.

Given a ring A let $\mathcal{C}_i(A)$, $\mathcal{P}_i(A)$ $(i \geq 1)$ be the additive categories defined by Pedersen [12]. The objects of $\mathcal{C}_i(A)$ are $\mathbb{Z}^i$-graded A-modules

$$M = \sum_{J \in \mathbb{Z}^i} M(J)$$

with each M(J) a f.g. free A-module. The morphisms of $\mathcal{C}_i(A)$ are the A-module morphisms

$$f = \sum_{J,K \in \mathbb{Z}^i} f(J,K) : M = \sum_{J \in \mathbb{Z}^i} M(J) \longrightarrow N = \sum_{K \in \mathbb{Z}^i} N(K)$$

which are bounded in the sense that there exists an integer $s \geq 0$ such that

$$f(J,K) = 0 : M(J) \longrightarrow N(K) \text{ if } J = (j_1, j_2, \ldots, j_i),\ K = (k_1, k_2, \ldots, k_i)$$
$$\text{are such that } \max\{|j_r - k_r| \mid 1 \leq r \leq i\} > s .$$

$\mathcal{P}_i(A)$ is the idempotent completion of $\mathcal{C}_i(A)$, with objects (M,p) the projections $p = p^2 : M \longrightarrow M$ in $\mathcal{C}_i(A)$, and morphisms

$$f : (M,p) \longrightarrow (N,q)$$

defined by morphisms $f: M \longrightarrow N$ in $\mathcal{C}_i(A)$ such that $qfp = f : M \longrightarrow N$. Also, let $\mathcal{C}_0(A)$ = {f.g. free A-modules}, and let $\mathcal{P}_0(A)$ be the idempotent completion of $\mathcal{C}_0(A)$, so that up to natural equivalence

$$\mathcal{P}_0(A) = \{\text{f.g. projective A-modules}\} .$$

The main result of [12] is that there are natural identifications

$$K_1(\mathcal{C}_{i+1}(A)) = K_0(\mathcal{P}_i(A)) = K_{-i}(A) \qquad (i \geq 0)$$

with $K_{-i}(A)$ $(i \geq 1)$ the lower algebraic K-groups of Bass [1].

Example The bounded $\mathbb{Z}^i$-graded A-module category $\mathcal{C}_i(A)$ $(i \geqslant 1)$ admits a flasque structure $\{\Sigma,\sigma,\rho\}$, with

$$\Sigma M(j_1,j_2,\dots,j_i) = \begin{cases} 0 & \text{if } j_1 = -1,0 \\ \sum_{k=0}^{j_1-1} M(k,j_2,\dots,j_i) & \text{if } j_1 \geqslant 1 \\ \sum_{k=j_1+1}^{-1} M(k,j_2,\dots,j_i) & \text{if } j_1 \leqslant -2 \end{cases}$$

$$\sigma_M : M(j_1,j_2,\dots,j_i)\oplus\Sigma M(j_1,j_2,\dots,j_i) \longrightarrow \Sigma M(j_1+1,j_2,\dots,j_i) ;$$

$$(x_{j_1},(x_0,x_1,\dots,x_{j_1-1})) \longmapsto (x_0,x_1,\dots,x_{j_1}) \quad \text{if } j_1 \geqslant 0$$

$$\sigma_M : M(j_1,j_2,\dots,j_i)\oplus\Sigma M(j_1,j_2,\dots,j_i) \longrightarrow \Sigma M(j_1+1,j_2,\dots,j_i) ;$$

$$(x_{j_1},(x_{j_1+1},x_{j_1+2},\dots,x_{-1})) \longmapsto (x_{j_1},x_{j_1+1},\dots,x_{-1}) \quad \text{if } j_1 \leqslant -1,$$

$$\rho_{M,N} : \Sigma(M\oplus N) \longrightarrow \Sigma M\oplus\Sigma N ; \sum_k (x_k,y_k) \longmapsto (\sum_k x_k,\sum_k y_k) .$$

This flasque structure (for which I am indebted to Chuck Weibel) determines by Proposition 5.4 a canonical stable structure $[\phi]$ on $\mathcal{C}_i(A)$, and hence a direct sum decomposition

$$K_1^{iso}(\mathcal{C}_i(A)) = K_1^{aut}(\mathcal{C}_i(A))\oplus ? .$$

The automorphism torsion component $\tau(C) \in K_1^{aut}(\mathcal{C}_i(A)) = K_{1-i}(A)$ of the isomorphism torsion $\tau(C) \in K_1^{iso}(\mathcal{C}_i(A))$ of a contractible finite chain complex C in $\mathcal{C}_i(A)$ is an absolute version of the reduced torsion invariant $\tilde{\tau}(C) \in \tilde{K}_{1-i}(A)$ $(= K_{1-i}(A)$ for $i > 1)$ obtained by Pedersen [13]. In particular, for $i = 1$ the splitting map is given explicitly by

$$K_1^{iso}(\mathcal{C}_1(A)) \longrightarrow\!\!\!\!\rightarrow K_1^{aut}(\mathcal{C}_1(A)) = K_0(A) ;$$

$$\tau(f:M \longrightarrow N) \longmapsto [(\sum_{j=-\infty}^{s-1} M(j)) \cap f^{-1}(\sum_{j=0}^{\infty} N(j))] - [\sum_{j=0}^{s-1} M(j)]$$

with $s \geqslant 0$ a bound for $f^{-1}:N \longrightarrow M$, such that

$$f^{-1}(N(j)) \subseteq \sum_{k=-s}^{s} M(j+k) \quad (j \in \mathbb{Z}) .$$

The flasque structure isomorphisms $\sigma_M : M\oplus\Sigma M \longrightarrow \Sigma M$ are such that $\sigma_M(\sum_{j=0}^{\infty} M(j)\oplus\Sigma M(j)) = \sum_{j=0}^{\infty}\Sigma M(j)$, and σ_M^{-1} has bound $s=1$, so that the isomorphism torsion $\tau(\sigma_M)\in K_1^{iso}(\mathcal{C}_1(A))$ has image 0 in $K_1^{aut}(\mathcal{C}_1(A)) = K_0(A)$.

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Given a filtered additive category $\mathcal{A}$ let $\mathcal{C}_i(\mathcal{A})$ $(i\geqslant 0)$ be the filtered additive category of $\mathbb{Z}^i$-graded objects in $\mathcal{A}$ defined by Pedersen and Weibel [15], with $\mathcal{C}_0(\mathcal{A}) = \mathcal{A}$, and let $\mathcal{P}_i(\mathcal{A})$ $(i\geqslant 0)$ be the idempotent completion of $\mathcal{C}_i(\mathcal{A})$. By the main result of [15] there are natural identifications of algebraic K-groups

$$\begin{aligned} K_{n+1}(\mathcal{C}_{i+1}(\mathcal{A})) &= K_n(\mathcal{P}_i(\mathcal{A})) = K_{n-i}(\mathcal{P}_0(\mathcal{A})) \quad \text{for } n,i\geqslant 0\\ &= K_n(\mathcal{C}_i(\mathcal{A})) \quad \text{for } n\geqslant 1\\ &= K_{n-i}(\mathcal{A}) \quad \text{for } n-i\geqslant 1\end{aligned}$$

with the higher K-groups defined using the split exact structure, and the lower K-groups $K_{-j}(\mathcal{P}_0(\mathcal{A}))$ $(j\geqslant 1)$ as defined by Karoubi [7].

<u>Example</u> The bounded $\mathbb{Z}^i$-graded category $\mathcal{C}_i(\mathcal{A})$ $(i\geqslant 1)$ admits a flasque structure $\{\Sigma,\sigma,\rho\}$, defined exactly as in the previous Example, which is the special case $\mathcal{A} = \{$f.g. free A-modules$\}$. The splitting map for $K_1^{aut} \rightarrowtail K_1^{iso}$ in the case $i = 1$ is given by

$$K_1^{iso}(\mathcal{C}_1(\mathcal{A})) \twoheadrightarrow K_1^{aut}(\mathcal{C}_1(\mathcal{A})) = K_0(\mathcal{P}_0(\mathcal{A}))\ ;$$

$$\tau(f:M\longrightarrow N) \longmapsto [\sum_{j=-s}^{s-1} M(j), f^{-1}p_{N^+}f] - [\sum_{j=0}^{s-1} M(j), 1]$$

with p_{N^+} the projection

$$p_{N^+} : N = \sum_{j=-\infty}^{\infty} N(j) \longrightarrow N\ ;\ \sum_{j=-\infty}^{\infty} x(j) \longmapsto \sum_{j=0}^{\infty} x(j)$$

and $s\geqslant 0$ a bound for $f^{-1}:N\longrightarrow M$,

$$f^{-1}(N(j)) \subseteq \sum_{k=-s}^{s} M(j+k) \qquad (j\in\mathbb{Z})\ .$$

Again, $\tau(\sigma_M)\in K_1^{iso}(\mathcal{C}_1(\mathcal{A}))$ has image $0\in K_1^{aut}(\mathcal{C}_1(\mathcal{A}))$. The case $i=1$ is the most significant one, since $\mathcal{C}_i(\mathcal{A}) = \mathcal{C}_1(\mathcal{C}_{i-1}(\mathcal{A}))$ for $i\geqslant 1$.

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A more detailed account of the applications of the algebraic theory of torsion to lower K-theory will appear elsewhere.

References

[1] H.Bass Algebraic K-theory Benjamin (1968)

[2] , A.Heller and R.G.Swan
The Whitehead group of a polynomial extension
Publ. Math. I.H.E.S. 22, 61 - 79 (1964)

[3] F.T.Farrell
The obstruction to fibering a manifold over a circle
Indiana Univ. J. 21, 315 - 346 (1971)

[4] S.Ferry A simple-homotopy approach to the finiteness obstruction
Springer Lecture Notes 870, 73 - 81 (1981)

[5] R.Fossum, H.-B.Foxby and B.Iversen
The Whitehead torsion of a bounded complex and higher order Mennicke symbols preprint

[6] D.Grayson (after D.Quillen)
Higher algebraic K-theory II
Springer Lecture Notes 551, 217 - 240 (1976)

[7] M.Karoubi Foncteurs dérivés et K-théorie
ibid. 136, 107 - 186 (1970)

[8] W.Lück Eine allgemeine Beschreibung des Transfers für Faserungen auf projektiven Klassengruppen und Whiteheadgruppen
Göttingen Ph.D. thesis (1984)

[9] M.Mather Counting homotopy types of manifolds
Topology 4, 93 - 94 (1965)

[10] J.P.May E_∞-spaces, group completions and permutative categories
L.M.S. Lecture Notes 11, 61 - 94 (1974)

[11] J.Milnor Whitehead torsion Bull. A.M.S. 72, 358 - 426 (1966)

[12] E.K.Pedersen
On the $K_{-i}(-)$ functors J. of Algebra 90, 461 - 475 (1984)

[13] K_{-i}-invariants of chain complexes
Springer Lecture Notes 1060, 174 - 186 (1984)

[14] and A.A.Ranicki
Projective surgery theory Topology 19, 239 - 254 (1980)

[15] and C.Weibel
A nonconnective delooping of algebraic K-theory
these proceedings

[16] A.A.Ranicki
Algebraic L-theory II: Laurent extensions
Proc. L.M.S. (3) 27, 126 - 158 (1973)

[17] The algebraic theory of surgery ibid. 40, 87 - 283 (1980)

[18] The algebraic theory of finiteness obstruction
Math. Scand. (to appear)

[19] J.Shaneson Wall's surgery obstruction groups for $G \times \mathbb{Z}$
Ann. of Maths. 90, 296 - 334 (1969)

[20] L.Siebenmann
A total Whitehead torsion obstruction to fibering over the circle Comm. Math. Helv. 45, 1 - 48 (1970)

[21] R.G.Swan Algebraic K-theory Springer Lecture Notes 76 (1968)

[22] C.T.C.Wall Foundations of algebraic L-theory
ibid. 343, 266 - 300 (1973)

[23] C.Weibel K-theory of Azumaya algebras
Proc. A.M.S. 81, 1 - 7 (1981)

[24] J.H.C.Whitehead
Simple homotopy types
Am. J. Math. 72, 1 - 57 (1950)

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