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The Algebraic Theory of Torsion. II: Products

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Abstract. The algebraic K-theory product $K_0(A)\otimes K_1(B)\to K_1(A\otimes B)$ for rings A, B is given a chain complex interpretation, using the absolute torsion invariant introduced in Part I. Given a finitely dominated A-module chain complex C and a round finite B-module chain complex D, it is shown that the $A\otimes B$-module chain complex $C\otimes D$ has a round finite chain homotopy structure. Thus, if X is a finitely dominated CW complex and Y is a round finite CW complex, the product $X\times Y$ is a CW complex with a round finite homotopy structure.

Key words. Finiteness obstruction, torsion, product, chain complex.

0. Introduction

The algebraic theory of absolute torsion developed in Part I ([16]) is here applied to products of chain complexes in algebra, and products of CW complexes in topology.

Given an additive category $\mathscr{A}$ and a chain equivalence $f: C\to D$ of finite chain complexes in $\mathscr{A}$ with $[C]=[D]=0\in K_0(\mathscr{A})$ there was defined in Part I a torsion invariant $\tau(f)\in K_1^{\mathrm{iso}}(\mathscr{A})$ in the isomorphism torsion group of $\mathscr{A}$. Here, we shall only be concerned with the case of the additive category $\mathscr{A}$ of based f.g. free A-modules, for some ring A such that the rank of f.g. free A-modules is well-defined. Thus, the natural map $K_0(\mathbb{Z})=\mathbb{Z}\to K_0(A)$ is injective, and the Euler characteristic of a finite chain complex C in $\mathscr{A}$

$$\chi(C)=\Sigma_{r=0}^{\infty}(-)^r\operatorname{rank}(C_r)\in\mathbb{Z}$$

is a chain homotopy invariant which can be identified with the class $[C]\in K_0(\mathscr{A})=\mathbb{Z}$, and also the projective class $[C]\in K_0(A)$. Isomorphic objects in $\mathscr{A}$ are related by a canonical isomorphism, so there is defined a natural split surjection $K_1^{\mathrm{iso}}(\mathscr{A})\to K_1^{\mathrm{aut}}(\mathscr{A})=K_1(A)$. Given a chain equivalence $f:C\to D$ of finite chain complexes of based f.g. free A-modules such that $\chi(C)=\chi(D)=0\in\mathbb{Z}$ we thus have an invariant $\tau(f)\in K_1(A)$, the *torsion* of f. The definition of $\tau(f)$ is recalled in Section 1 below.

The absolute projective class of a finitely dominated CW complex X is defined to be the projective class of the finitely dominated cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\tilde{X})$ of the universal cover $\tilde{X}$

$$[X]=[C(\tilde{X})]\in K_0(\mathbb{Z}[\pi_1(X)]),$$

and consists of the Euler characteristic $\chi(X)=\chi(C(\tilde{X}))\in K_0(\mathbb{Z})=\mathbb{Z}$ and the finiteness obstruction $[\tilde{X}]\in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ of Wall [21]

$$[X] = (\chi(X), [\tilde{X}])\in K_0(\mathbb{Z}[\pi_1(X)]) = K_0(\mathbb{Z})\oplus \tilde{K}_0(\mathbb{Z}[\pi_1(X)]).$$

The product of finitely dominated CW complexes X, Y is a finitely dominated CW complex $X \times Y$ with universal cover $\widetilde{X \times Y} = \tilde{X} \times \tilde{Y}$, such that

$$\mathbb{Z}[\pi_1(X \times Y)] = \mathbb{Z}[\pi_1(X) \times \pi_1(Y)] = \mathbb{Z}[\pi_1(X)] \otimes \mathbb{Z}[\pi_1(Y)],$$

with a natural identification

$$C(\widetilde{X \times Y}) = C(\tilde{X}) \otimes C(\tilde{Y}).$$

The projective class product formula of Gersten [7] and Siebenmann [18]

$$[X \times Y] = [X] \otimes [Y] \in K_0(\mathbb{Z}[\pi_1(X \times Y)])$$

showed that for a finite CW complex Y with $\chi(Y) = 0\in \mathbb{Z}$ the product $X \times Y$ has Wall finiteness obstruction $[\widetilde{X \times Y}] = 0\in \tilde{K}_0(\mathbb{Z}[\pi_1(X \times Y)])$, and so $X \times Y$ has the homotopy type of a finite CW complex. This was first proved geometrically by Mather [12], in the important special case $Y = S^1$.

For any rings A, B there is defined a product in the absolute algebraic K-groups

$$\otimes : K_0(A) \otimes K_1(B) \to K_1(A \otimes B),$$
$$[P] \otimes \tau(f : Q \to Q) \mapsto \tau(1 \otimes f : P \otimes Q \to P \otimes Q),$$

in particular for group rings $A = \mathbb{Z}[\pi]$, $B = \mathbb{Z}[\rho]$, with $A \otimes B = \mathbb{Z}[\pi \times \rho]$. In general, there is no such product in the reduced K-groups, although if $\mathrm{Wh}(\rho) = 0$ there is a product $\tilde{K}_0(\mathbb{Z}[\pi]) \otimes K_1(\mathbb{Z}[\rho]) \to \mathrm{Wh}(\pi \times \rho)$. It is therefore quite reasonable that the absolute torsion should enter into the consideration of finite CW complexes in the homotopy type of CW complex products $X \times Y$.

Define a *finite structure* on a CW complex X to be an equivalence class of pairs

(finite CW complex F, homotopy equivalence $\phi : F \to X$)

under the equivalence relation

$$(F_1, \phi_1) \sim (F_2, \phi_2) \quad \text{if} \quad \tau(\phi_2^{-1}\phi_1 : F_1 \to F_2) = 0\in \mathrm{Wh}(\pi_1(X)).$$

The Whitehead torsion $\tau(f)\in \mathrm{Wh}(\pi_1(X))$ of a homotopy equivalence $f : X \to X'$ of CW complexes with given finite structures (F, ϕ), (F', ϕ') is defined by

$$\tau(f) = \tau(\phi'^{-1} f \phi : F \to X \to X' \to F')\in \mathrm{Wh}(\pi_1(X)).$$

A finite CW complex F has the canonical finite structure $(F, 1)$.

Ferry [6] proved geometrically that the mapping torus construction of Mather [12] defines a canonical finite structure on $X \times S^1$ for any finitely dominated CW complex X, which is independent of the finite domination used in the construction, and that the geometrically defined Abelian group morphism

$$\bar{B}' : \tilde{K}_0(\mathbb{Z}[\pi]) \to \mathrm{Wh}(\pi \times \mathbb{Z});\ [X] \mapsto \tau(1 \times -1 : X \times S^1 \to X \times S^1) \ \ (\pi = \pi_1(X))$$

is an injection. Now $-1: S^1 \to S^1$ is a simple homotopy equivalence (i.e., $\tau(-1) = 0 \in \mathrm{Wh}(\pi_1(S^1)) = \mathrm{Wh}(\mathbb{Z}) = 0$), so that the canonical finite structure on $X \times S^1$ depends on more than just the canonical finite structure on S^1. We shall show that it depends on the canonical 'round finite structure' on S^1.

A finite chain complex C of based f.g. free A-modules is *round* if $\chi(C) = 0 \in \mathbb{Z}$, or equivalently if $[C] = 0 \in K_0(A)$. The torsion $\tau(f) \in K_1(A)$ defined in Part I for a chain equivalence $f: C \to D$ of round finite chain complexes has the logarithmic property

$$\tau(gf: C \to D \to E) = \tau(f: C \to D) + \tau(g: D \to E) \in K_1(A).$$

In general, absolute torsion is nonadditive

$$\tau(f \oplus f': C \oplus C' \to D \oplus D') \neq \tau(f: C \to D) + \tau(f': C' \to D') \in K_1(A).$$

A *round finite structure* on an A-module chain complex C is an equivalence class of pairs

(round finite chain complex F of based f.g. free A-modules,
chain equivalence $\phi: F \to C$)

under the equivalence relation

$$(F_1, \phi_1) \sim (F_2, \phi_2) \quad \text{if } \tau(\phi_2^{-1}\phi_1: F_1 \to F_2) = 0 \in K_1(A).$$

The torsion of a chain equivalence $f: C \to C'$ of A-module chain complexes C, C' with prescribed round finite structures (F, ϕ), (F', ϕ') is defined by

$$\tau(f) = \tau(\phi'^{-1} f \phi: F \to C \to C' \to F') \in K_1(A).$$

The main result of the paper is the following chain complex interpretation of the product $K_0(A) \otimes K_1(B) \to K_1(A \otimes B)$.

ALGEBRAIC PRODUCT STRUCTURE THEOREM. *The product of a finitely dominated A-module chain complex C and a B-module chain complex D with a round finite structure (F, ϕ) is an $A \otimes B$-module chain complex $C \otimes D$ with a round finite structure $C \otimes (F, \phi)$.*

If $f: C \to C'$ is a chain equivalence of finitely dominated A-module chain complexes and $g: D \to D'$ is a chain equivalence of B-module chain complexes D, D' with round finite structures

$$\tau(f \otimes g: C \otimes D \to C' \otimes D') = [C] \otimes \tau(g) \in K_1(A \otimes B)$$

with $[C] = [C'] \in K_0(A)$ the projective class and $\tau(g) \in K_1(B)$ the torsion. □

This will be proved in Section 3, and translated into topology in Section 4.

A finite CW complex X with universal cover $\tilde{X}$ and fundamental group $\pi_1(X) = \pi$ determines a class of bases for the cellular f.g. free $\mathbb{Z}[\pi]$-module chain complex $C(\tilde{X})$, the elements of which are determined up to multiplication by $\pm g \in \mathbb{Z}[\pi]$ $(g \in \pi)$. Define a *round finite CW complex* to be a finite CW complex X such that the Euler characteristic $\chi(X) = \chi(C(\tilde{X})) \in \mathbb{Z}$ vanishes, $\chi(X) = 0 \in \mathbb{Z}$, together with a choice of base for $C(\tilde{X})$ in the canonical class. The *torsion* of a homotopy equivalence $f: X \to Y$ of round finite CW complexes is defined by

$$\tau(f) = \tau(\tilde{f}: C(\tilde{X}) \to C(\tilde{Y})) \in K_1(\mathbb{Z}[\pi]) \quad (\pi = \pi_1(X)),$$

with the image $\tau(f) \in \mathrm{Wh}(\pi)$ the usual Whitehead torsion of f.

A *round finite structure* on a CW complex X is a round finite structure on $C(\tilde{X})$, or equivalently an equivalence class of pairs

$$(\text{round finite CW complex } F, \text{ homotopy equivalence } \phi: F \to X)$$

under the equivalence relation

$$(F_1, \phi_1) \sim (F_2, \phi_2) \quad \text{if } \tau(\phi_2^{-1}\phi_1: F_1 \to F_2) = 0 \in K_1(\mathbb{Z}[\pi_1(X)]).$$

The torsion $\tau(f) \in K_1(\mathbb{Z}[\pi_1(X)])$ of a homotopy equivalence $f: X \to Y$ of CW complexes with prescribed round finite structures is defined in the obvious manner.

The main topological result of this paper is the following CW complex interpretation of the product $K_0(A) \otimes K_1(B) \to K_1(A \otimes B)$.

GEOMETRIC PRODUCT STRUCTURE THEOREM. *The product of a finitely dominated CW complex X and a CW complex Y with round finite structure (F, ϕ) is a CW complex $X \times Y$ with a round finite structure $X \times (F, \phi)$. If $f: X \to X'$ is a homotopy equivalence of finitely dominated CW complexes and $g: Y \to Y'$ is a homotopy equivalence of CW complexes with round finite structures then*

$$\tau(f \times g: X \times Y \to X' \times Y') = [X] \otimes \tau(g) \in K_1(\mathbb{Z}[\pi_1(X \times Y)]),$$

with $[X] = [X'] \in K_0(\mathbb{Z}[\pi_1(X)])$ the projective class and $\tau(g) \in K_1(\mathbb{Z}[\pi_1(Y)])$ the torsion. □

The torsion product formulae of Kwun and Szczarba [10] and Gersten [8] are special cases of the geometric product structure theorem, with X finite in [10] and $Y' = Y$ in [8].

As already noted in the introduction to Part I ([16]), the algebraic description due to Lück [11] of the transfer maps induced in the algebraic K-groups

$$p_i^!: K_i(\mathbb{Z}[\pi_1(B)]) \to K_i(\mathbb{Z}[\pi_1(E)]) \quad (i = 0, 1)$$

by a Hurewicz fibration

$$F \to E \xrightarrow{p} B$$

with finitely dominated fibre F allows the extension of the geometric product structure theorem to the twisted case: if the base B is also finitely dominated then so is the total space E, with projective class

$$[E] = p_0^!([B]) \in K_0(\mathbb{Z}[\pi_1(E)]),$$

and a round finite structure on B determines a round finite structure on E, a variation by $\tau \in K_1(\mathbb{Z}[\pi_1(B)])$ in the base leading to a variation of

$p_1^!(\tau) \in K_1(\mathbb{Z}[\pi_1(E)])$ in the total space. In the case of a trivial fibration $E = B \times F$ the transfer maps are given by product with the projective class $[F] \in K_0(\mathbb{Z}[\pi_1(F)])$

$$p_i^! = - \otimes [F]: K_i(\mathbb{Z}[\pi_1(B)]) \to K_i(\mathbb{Z}[\pi_1(B \times F)]) \quad (i = 0, 1).$$

In Section 5 we shall compare the absolute torsion invariant $\tau(f) \in K_1(\mathbb{Z}[\pi_1(X)])$ defined by Gersten [8] for a self homotopy equivalence $f: X \to X$ of a finitely dominated CW complex X with $f_* = 1: \pi_1(X) \to \pi_1(X)$ with our notion of absolute torsion, showing that they coincide when both are defined (i.e., when $[X] = 0 \in K_0(\mathbb{Z}[\pi_1(X)])$).

Finally, in Section 6 we shall show that for a particular choice of round finite structure Σ^1 on S^1 the product round finite structure $X \times \Sigma^1$ on $X \times S^1$ for a finitely dominated CW complex X reduces to the canonical finite structure obtained geometrically by Mather [12] and Ferry [6]. With respect to this choice

$$\begin{aligned} &\tau(-1: S^1 \to S^1) \\ &= \tau(-z: \mathbb{Z}[z, z^{-1}] \to \mathbb{Z}[z, z^{-1}]) \in K_1(\mathbb{Z}[\pi_1(S^1)]) \\ &= K_1(\mathbb{Z}[z, z^{-1}]), \end{aligned}$$

so that the geometric injection of [6]

$$\bar{B}': \tilde{K}_0(\mathbb{Z}[\pi]) \to \mathrm{Wh}(\pi \times \mathbb{Z}); \qquad [X] \mapsto \tau(1 \times -1: X \times S^1 \to X \times S^1)$$

may be identified with the algebraic injection of Ranicki [22]

$$\begin{aligned} \bar{B}' = - \otimes \tau(-z)&: \tilde{K}_0(\mathbb{Z}[\pi]) \to \mathrm{Wh}(\pi \times \mathbb{Z}); \\ &[P] \mapsto \tau(-z: P[z, z^{-1}] \to P[z, z^{-1}]). \end{aligned}$$

Thus, $\bar{B}'$ is a variant of the algebraic injection defined by Bass *et al.* [2]

$$\begin{aligned} \bar{B} = - \otimes \tau(z)&: \tilde{K}_0(\mathbb{Z}[\pi]) \to \mathrm{Wh}(\pi \times \mathbb{Z}); \\ &[P] \mapsto \tau(z: P[z, z^{-1}] \to P[z, z^{-1}]). \end{aligned}$$

Part III of the paper [17] deals with lower K-theory, including some further discussion of $\bar{B}$ and $\bar{B}'$.

See [9] for an application of the algebraic theory of torsion to L-theory.

1. Finite and Round Finite Structures

We shall now apply the general theory of torsion developed in Part I for any additive category to the most important special case $\mathscr{A} = \{$based f.g. free A-modules$\}$, for any ring A such that the rank of f.g. free A-modules is well-defined. In the first instance we recall from [15] the abstract chain complex version of the finiteness obstruction theory of Wall [21], and extend it to round finiteness.

A *chain complex over A* is a positive chain complex of (left) A-modules and A-module morphisms

$$C: \cdots \to C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \to \cdots \to C_1 \xrightarrow{d} C_0.$$

The chain complex C is *n-dimensional* if $C_r = 0$ for $r > n$. The chain complex C is *finite* if it is a finite-dimensional complex of based f.g. free A-modules, that is if it is a finite chain complex in the category $\mathscr{A} = \{$based f.g. free A-modules$\}$. The *Euler characteristic* of a finite chain complex C is defined by

$$\chi(C) = \Sigma_{r=0}^{\infty}(-)^r \operatorname{rank}_A(C_r) \in \mathbb{Z},$$

and C is *round* if $\chi(C) = 0 \in \mathbb{Z}$.

A *finite domination* (D, f, g, h) of a chain complex C over A consists of a finite chain complex D over A, chain maps

$$f: C \to D, \qquad g: D \to C$$

and a chain homotopy

$$h: gf \simeq 1: C \to C.$$

A chain complex is *finitely dominated* if it admits a finite domination. It was shown in [15] that a chain complex C is finitely dominated if and only if it is chain equivalent to a finite dimensional f.g. projective chain complex

$$P: \cdots \to 0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0.$$

The *projective class* of a finitely dominated chain complex C is defined using any such P to be

$$[C] = [P] = \Sigma_{r=0}^{\infty}(-)^r[P_r] \in K_0(A).$$

The projective class is a chain homotopy invariant such that for finite C

$$[C] = \chi(C) \in \operatorname{im}(K_0(\mathbb{Z}) \to K_0(A)) = \mathbb{Z} \subseteq K_0(A).$$

Thus the *reduced projective class*

$$[C] \in \tilde{K}_0(A) = \operatorname{coker}(K_0(\mathbb{Z}) \to K_0(A))$$

vanishes for finite C.

PROPOSITION 1.1. (i) *A finitely dominated chain complex C over A is chain equivalent to a finite chain complex if and only if $[C] = 0 \in \tilde{K}_0(A)$. Thus $[C] \in \tilde{K}_0(A)$ is the finiteness obstruction of C.*

(ii) *A finitely dominated chain complex C over A is chain equivalent to a round finite chain complex if and only if $[C] = 0 \in K_0(A)$. Thus $[C] \in K_0(A)$ is the round finiteness obstruction of C.*

Proof. (i) See [15]. (ii) Immediate from (i). □

The *torsion* of a contractible finite chain complex C over A is defined by

$$\tau(C) = \tau(d + \Gamma = \begin{pmatrix} d & 0 & 0 & \cdots \\ \Gamma & d & 0 & \cdots \\ 0 & \Gamma & d & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}:$$

$$C_{\mathrm{odd}} = C_1 \oplus C_3 \oplus C_5 \oplus \cdots \to C_{\mathrm{even}} = C_0 \oplus C_2 \oplus C_4 \oplus \cdots) \in K_1(A)$$

as usual, with $\Gamma: 0 \simeq 1: C \to C$ any chain contraction of C.

The *algebraic mapping cone* of a chain map $f: C \to D$ of finite chain complexes over A is the finite chain complex $C(f)$ defined as usual (up to sign conventions) by

$$d_{C(f)} = \begin{pmatrix} d_D & (-)^{r-1} f \\ 0 & d_C \end{pmatrix} : C(f)_r = D_r \oplus C_{r-1} \to C(f)_{r-1} = D_{r-1} \oplus C_{r-2}.$$

The following signs occur in the composition and sum formulae obtained in Part I [16], as recalled in Proposition 1.2 below.

Given based f.g. free A-modules M, N let

$$\varepsilon(M, N) = \mathrm{rank}_A(M)\, \mathrm{rank}_A(N) \in \mathbb{Z}_2,$$

so that

$$\tau\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : M \oplus N \to N \oplus M = \varepsilon(M, N)\tau(-1: A \to A) \in K_1(A).$$

Given finite chain complexes C, D over A let

$$\beta(C, D) = \Sigma_{i>j}(\varepsilon(C_{2i}, D_{2j}) + \varepsilon(C_{2i+1}, D_{2j+1})) \in \mathbb{Z}_2.$$

For any A-module chain complex C let SC denote the A-module chain complex with

$$d_{SC} = d_C: SC_r = C_{r-1} \to SC_{r-1} = C_{r-2}.$$

Given finite chain complexes C, D, E over A let

$$\begin{aligned} &\gamma(C, D, E) \\ &\quad = \beta(E, SC) - \beta(D, SC) - \beta(E, SD) + \\ &\qquad + (\varepsilon(D_{\mathrm{even}}, C_{\mathrm{odd}}) - \varepsilon(D_{\mathrm{odd}}, C_{\mathrm{even}})) + (\varepsilon(D_{\mathrm{even}}, E_{\mathrm{even}}) - \varepsilon(D_{\mathrm{odd}}, E_{\mathrm{odd}})) + \\ &\qquad + (\varepsilon(C_{\mathrm{odd}}, E_{\mathrm{even}}) - \varepsilon(C_{\mathrm{even}}, E_{\mathrm{odd}})) + (\varepsilon(D_{\mathrm{even}}, D_{\mathrm{odd}}) - \varepsilon(D_{\mathrm{even}}, D_{\mathrm{even}})) \in \mathbb{Z}_2. \end{aligned}$$

PROPOSITION 1.2. (i) *The torsion of the algebraic mapping cone $C(gf)$ of the composite $gf: C \to E$ of chain equivalences $f: C \to D$, $g: D \to E$ of finite chain complexes over A is given by*

$$\tau(C(gf)) = \tau(C(f)) + \tau(C(g)) + \gamma(C, D, E)\tau(-1: A \to A) \in K_1(A).$$

(ii) *The torsion of the algebraic mapping cone $C(f \oplus f')$ of the sum $f \oplus f': C \oplus C' \to D \oplus D'$ of chain equivalences $f: C \to D$, $f': C' \to D'$ of finite chain complexes over A is given by*

$$\begin{aligned} \tau(C(f \oplus f')) = \tau(C(f)) + \tau(C(f')) + \beta(D \oplus SC, D' \oplus SC')\tau(-1: A \to A) + \\ + (\Sigma_r(-)^r \varepsilon(C_{r-1}, D'_r))\tau(-1: A \to A) \in K_1(A). \end{aligned}$$

Proof. See Proposition 2.5 of Part I. □

The *reduced torsion* of a chain equivalence $f: C \to D$ of finite chain complexes over A is defined by

$$\tau(f) = \tau(C(f)) \in \tilde{K}_1(A),$$

the reduction of $\tau(C(f)) \in K_1(A)$ in $\tilde{K}_1(A) = \mathrm{coker}(K_1(\mathbb{Z}) \to K_1(A))$.

PROPOSITION 1.3. *The reduced torsion is such that*

(i) $\tau(gf: C \to D \to E) = \tau(f) + \tau(g) \in \tilde{K}_1(A)$
(ii) $\tau(f \oplus f': C \oplus C' \to D \oplus D') = \tau(f) + \tau(f') \in \tilde{K}_1(A)$
(iii) $\tau(f: C \to D) = \tau(D) - \tau(C) \in \tilde{K}_1(A)$ *if C and D are chain contractible.*

Proof. See Proposition 2.6 of Part I. □

The *torsion* of a chain equivalence $f: C \to D$ of round finite chain complexes over A is defined by

$$\tau(f) = \tau(C(f)) - \beta(D, SC)\tau(-1: A \to A) \in K_1(A).$$

PROPOSITION 1.4. *The torsion is such that*

(i) $\tau(gf: C \to D \to E) = \tau(f) + \tau(g) \in K_1(A)$,
(ii) $\tau(f \oplus f': C \oplus C' \to D \oplus D')$
$= \tau(f) + \tau(f') +$
$+ (\beta(D, D') - \beta(C, C'))\tau(-1: A \to A) \in K_1(A)$,
(iii) $\tau(f: C \to D) = \tau(D) - \tau(C) \in K_1(A)$ *if C and D are chain contractible.*

Proof. See Proposition 2.7 of Part I. □

The reduction of the torsion $\tau(f) \in K_1(A)$ is, of course, the reduced torsion $\tau(f) \in \tilde{K}_1(A)$.

A *finite structure* on a chain complex C over A is an equivalence class of pairs

(finite chain complex F over A, chain equivalence $\phi: F \to C$)

under the equivalence relation

$$(F, \phi) \sim (F', \phi') \quad \text{if } \tau(\phi'^{-1}\phi: F \to F') = 0 \in \tilde{K}_1(A).$$

The *finite structure set* $\mathscr{F}(C)$ of a chain complex C over A is the set (possibly empty) of finite structures on C.

PROPOSITION 1.5. (i) *The finite structure set $\mathscr{F}(C)$ is nonempty if and only if C is finitely dominated and $[C] = 0 \in \tilde{K}_0(A)$.*

(ii) *If $\mathscr{F}(C)$ is nonempty it is an affine $\tilde{K}_1(A)$-set, with a transitive $\tilde{K}_1(A)$-action defined by*

$$\tilde{K}_1(A) \times \mathscr{F}(C) \to \mathscr{F}(C); \qquad (\tau(D), (F, \phi)) \mapsto (F \oplus D, \phi \oplus 0)$$

with $\tau(D) \in \tilde{K}_1(A)$ the reduced torsion of a contractible finite chain complex D over A. A choice of base point $(F_0, \phi_0) \in \mathscr{F}(C)$ determines an Abelian group structure on $\mathscr{F}(C)$

with an isomorphism

$$\mathscr{F}(C) \to \tilde{K}_1(A); \qquad (F, \phi) \mapsto \tau(\phi^{-1}\phi_0 : F_0 \to F).$$

Proof. Immediate from Proposition 1.1 (i). □

Given a chain equivalence $f: C \to D$ of chain complexes over A with finite structures $(F, \phi) \in \mathscr{F}(C)$, $(G, \theta) \in \mathscr{F}(D)$ define the *reduced torsion*

$$\tau(f) = \tau(\theta^{-1} f \phi : F \xrightarrow{\phi} C \xrightarrow{f} D \xrightarrow{\theta^{-1}} G) \in \tilde{K}_1(A).$$

This evidently depends on the choices of finite structures as well as f, with the reduced torsion $\tau'(f) \in \tilde{K}_1(A)$ determined by different choices $(F', \phi') \in \mathscr{F}(C)$, $(G', \theta') \in \mathscr{F}(D)$ such that

$$\tau'(f) - \tau(f) = \tau(\theta^{-1}\theta' : G' \to G) - \tau(\phi^{-1}\phi' : F' \to F) \in \tilde{K}_1(A),$$

by the logarithmic property of reduced torsion.

A f.g. free A-module M is *even* if $\operatorname{rank}_A(M) \equiv 0 (\bmod 2)$. Thus, if either M or N is even $\varepsilon(M, N) = 0 \in \mathbb{Z}_2$.

A finite chain complex C over A is *even* if each $C_r (r \geqslant 0)$ is an even f.g. free A-module. Thus, if either C or D is even $\beta(C, D) = 0 \in \mathbb{Z}_2$.

(Let $\mathscr{C}^e(A)$ be the additive category of even finite chain complexes over A and chain homotopy classes of chain maps. The torsion function

$$\tau: \operatorname{iso}(\mathscr{C}^e(A)) \to K_1(A); \quad f \mapsto \tau(f) = \tau(C(f))$$

is both logarithmic ($\tau(gf) = \tau(f) + \tau(g)$) and additive ($\tau(f \oplus f') = \tau(f) + \tau(f')$), agreeing with the torsion $\tau: \operatorname{iso}(\mathscr{C}^r(A)) \to K_1(A)$; $f \mapsto \tau(f)$ defined above for the additive category $\mathscr{C}^r(A)$ of round finite chain complexes over A and chain homotopy classes of chain maps.)

A *round finite structure* on a chain complex C over A is an equivalence class of pairs

(round finite chain complex C over A,
chain equivalence $\phi: F \to C$)

under the equivalence relation

$$(F, \phi) \sim (F', \phi') \quad \text{if } \tau(\phi'^{-1}\phi : F \to F') = 0 \in K_1(A).$$

The *round finite structure set* $\mathscr{F}^r(C)$ of a chain complex C over A is the set (possibly empty) of round finite structures on C.

PROPOSITION 1.6. (i) *The round finite structure set $\mathscr{F}^r(C)$ is nonempty if and only if C is finitely dominated and $[C] = 0 \in K_0(A)$.*

(ii) *If $\mathscr{F}^r(C)$ is nonempty it is an affine $K_1(A)$-set, with a transitive $K_1(A)$-action* *defined by*

$$K_1(A) \times \mathscr{F}^r(C) \to \mathscr{F}^r(C); \quad (\tau(D), (F, \phi)) \mapsto (F \oplus D, \phi \oplus 0)$$

with $\tau(D) \in K_1(A)$ *the torsion of a contractible even finite chain complex* D *over* A. *A choice of base point* $(F_0, \phi_0) \in \mathscr{F}^r(C)$ *determines an Abelian group structure on* $\mathscr{F}^r(C)$ *with an isomorphism*

$$\mathscr{F}^r(C) \to K_1(A); \qquad (F, \phi) \mapsto \tau(\phi^{-1}\phi_0: F_0 \to F).$$

Proof. By analogy with Proposition 1.5. □

Given a chain equivalence $f: C \to D$ of chain complexes over A with round finite structures $(F, \phi) \in \mathscr{F}^r(C)$, $(G, \theta) \in \mathscr{F}^r(D)$ define the *torsion*

$$\tau(f) = \tau(\theta^{-1} f \phi: F \xrightarrow{\phi} C \xrightarrow{f} D \xrightarrow{\theta^{-1}} G) \in K_1(A).$$

This evidently depends on the choices of round finite structures as well as f, with the torsion $\tau'(f) \in K_1(A)$ determined by different choices $(F', \phi') \in \mathscr{F}^r(C)$, $(G', \theta') \in \mathscr{F}^r(D)$ such that

$$\tau'(f) - \tau(f) = \tau(\theta^{-1}\theta': G' \to G) - \tau(\phi^{-1}\phi': F' \to F) \in K_1(A)$$

by the logarithmic property of torsion.

The absolute K_1-group $K_1(A)$ behaves better under products than the reduced K_1-group $\tilde{K}_1(A)$, so that round finite structures behave better under products than finite structures. In Section 3 below we shall investigate this behaviour in some detail, using the following sharper version of the condition $\chi(C) = 0 \in \mathbb{Z}$ for a finite chain complex C to be round.

Given a finite chain complex C over A define the integers $e_r(C) = \text{rank}_A(C_r) - \text{rank}_A(C_{r-1}) + \cdots + (-)^r \text{rank}_A(C_0) \in \mathbb{Z}$ $(r \geqslant 0)$, uniquely characterized by

$$\text{rank}_A(C_r) = e_r(C) + e_{r-1}(C) \quad (r \geqslant 0, e_{-1}(C) = 0).$$

If C is n-dimensional, then for $r \geqslant n$

$$e_r(C) = (-)^r \chi(C) \in \mathbb{Z}.$$

A finite chain complex C over A is *rounded* if $e_r(C) \geqslant 0$ $(r \geqslant 0)$. If C is n-dimensional $e_n(C)e_{n+1}(C) = -\chi(C)^2 \geqslant 0$, so that $\chi(C) = 0$ and C is round. However, a round finite chain complex need not be rounded, as is clear from the example

$$C: \cdots \to 0 \to A \to A \to 0.$$

PROPOSITION 1.7. (i) *A finite chain complex* C *over* A *is rounded if and only if there is defined a contractible finite chain complex* C_Δ *over* A *with the same chain modules* $\{C_r | r \geqslant 0\}$

$$C_\Delta: \cdots \to C_{r+1} \xrightarrow{d_\Delta} C_r \xrightarrow{d_\Delta} C_{r-1} \to \cdots \xrightarrow{d_\Delta} C_0.$$

(ii) *For any round finite chain complex C over A there exists a contractible finite chain complex C' over A such that $C \oplus C'$ is rounded and*

$$\tau\left(\begin{pmatrix}1\\0\end{pmatrix}: C \to C \oplus C'\right) = 0 \in K_1(A).$$

Proof. (i) Given a contraction $\Gamma: 0 \simeq 1: D \to D$ of a finite chain complex D over A there are defined stably f.g. free A-modules

$$E_r = \ker(d: D_r \to D_{r-1}) = \operatorname{im}(d: D_{r+1} \to D_r) \quad (r \geqslant 0)$$

and isomorphisms

$$f: D_r \to E_r \oplus E_{r-1}; \qquad x \mapsto (d\Gamma(x), d(x)) \quad (r \geqslant 0)$$

such that

$$fdf^{-1} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}: E_r \oplus E_{r-1} \to E_{r-1} \oplus E_{r-2}.$$

Now

$$e_r(D) = \operatorname{rank}_A(E_r) \geqslant 0 \quad (r \geqslant 0),$$

so that D is rounded.

Thus, if C is such that there exists a contractible finite chain complex C_Δ with the same chain modules

$$e_r(C) = e_r(C_\Delta) \geqslant 0 \quad (r \geqslant 0),$$

and C is rounded.

Conversely, if C is a rounded finite chain complex over A define $d_\Delta \in \operatorname{Hom}_A(C_r, C_{r-1}) (r \geqslant 1)$ by

$$d_\Delta(k\text{th base element of } C_r)$$
$$= \begin{cases} 0 \in C_{r-1} & \text{if } 1 \leqslant k \leqslant e_r(C) \\ (k - e_r(C))\text{th base element} \in C_{r-1} & \text{if } e_r(C) + 1 \leqslant k \leqslant \operatorname{rank}_A(C_r). \end{cases}$$

Then C_Δ is a contractible chain complex, with a chain contraction $\Gamma: 0 \simeq 1: C_\Delta \to C_\Delta$ defined by

$$\Gamma(k\text{th base element of } C_r)$$
$$= \begin{cases} (e_{r+1}(C) + k)\text{th base element} \in C_{r+1} & \text{if } 1 \leqslant k \leqslant e_r(C) \\ 0 \in C_{r+1} & \text{if } e_r(C) \leqslant k \leqslant \operatorname{rank}_A(C_{r+1}). \end{cases}$$

(ii) Let C be n-dimensional, and let $\{C'_r \mid r \geqslant 0\}$ be a sequence of based f.g. free A-modules with the ranks

$$\operatorname{rank}_A(C') = e_r(C') + e_{r-1}(C') \quad (r \geqslant 0, e_{-1}(C') = 0)$$

determined by the nonnegative integers

$$e_r(C') = \begin{cases} \text{rank}_A(C_{r-1}) + \text{rank}_A(C_{r-3}) + \cdots & \text{if } r \leqslant n \\ 0 \quad \text{if } r > n. \end{cases}$$

Then $\{C_r \oplus C'_r \mid r \geqslant 0\}$ is a sequence of based f.g. free A-modules such that the ranks

$$\text{rank}_A(C_r \oplus C'_r) = e_r(C \oplus C') + e_{r-1}(C \oplus C') \quad (r \geqslant 0)$$

are determined by the nonnegative integers

$$\begin{aligned} e_r(C \oplus C') &= e_r(C) + e_r(C') \\ &= \begin{cases} \text{rank}_A(C_r) + \text{rank}_A(C_{r-2}) + \cdots & \text{if } r \leqslant n \\ 0 \quad \text{if } r > n. \end{cases} \end{aligned}$$

By (i) differentials $\{d_{C'} \in \text{Hom}_A(C'_r, C'_{r-1}) | r \geqslant 0\}$ may be chosen such that C' is a contractible finite chain complex over A, and in particular such that

$$\tau(C') = \beta(C, C') \in K_1(A).$$

By the sum formula of Proposition 1.2 (ii)

$$\begin{aligned} \tau\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} : C \to C \oplus C'\right) &= \tau(1 \oplus 0 : C \oplus 0 \to C \oplus C') \\ &= \tau(1 : C \to C) + \tau(0 : 0 \to C') - \beta(C, 0) + \beta(C, C') \\ &= 0 \in K_1(A). \end{aligned}$$

□

2. Change of Rings

In the applications we shall be dealing not only with the algebraic K-groups $K_0(A)$, $K_1(A)$ of a single ring A, but also with the morphisms of K-groups induced by a morphism of rings $f: A \to B$. As usual, given such a ring morphism regard B as a (B, A)-bimodule by

$$B \times B \times A \to B; \quad (b, x, a) \mapsto bxf(a),$$

so that there is defined a functor

$$f_!: (A\text{-modules}) \to (B\text{-modules}); \quad M \mapsto f_! M = B \otimes_A M$$

sending f.g. projective (resp. free) A-modules to f.g. projective (resp. free) B-modules. Given a finitely dominated (resp. contractible finite) chain complex C over A there is induced a finitely dominated (resp. contractible finite) chain complex $f_! C = B \otimes_A C$ over B, and the induced morphisms of K-groups are such that

$$\begin{aligned} &f_!: K_0(A) \to K_0(B); \quad [C] \mapsto [f_! C] \\ &f_!: K_1(A) \to K_1(B); \quad \tau(C) \mapsto \tau(f_! C). \end{aligned}$$

We shall be particularly concerned with the case in which $f: A \to B$ is an isomorphism, when it is possible to identify the B-module $f_!M$ induced by an A-module M with the B-module defined by the additive group of M with B acting by

$$B \times f_!M \to f_!M; \quad (b, x) \mapsto f^{-1}(b)x.$$

For the inner automorphism of a ring A

$$f: A \to A; \quad a \mapsto z^{-1}az$$

defined by conjugation by a unit $z \in A$ there is defined a natural equivalence of functors

$$z: 1 \rightsquigarrow f_!: (A\text{-modules}) \to (A\text{-modules}),$$

with a natural A-module isomorphism

$$z: M \to f_!M; \quad x \mapsto zx$$

for any A-module M. Thus, for any chain complex C over A there is defined an isomorphism

$$z: C \to f_!C; \quad x \mapsto zx.$$

If C is finitely dominated

$$f_![C] = [f_!C] = [C] \in K_0(A).$$

If C is finite then

$$\begin{aligned}\tau(z: C \to f_!C) &= \Sigma_{r=0}^{\infty}(-)^r\tau(z: C_r \to f_!C_r)\\ &= \chi(C)\tau(z: A \to A; a \mapsto az) \in K_1(A),\end{aligned}$$

so that if C is contractible finite

$$f_!\tau(C) = \tau(f_!C) = \tau(C) \in K_1(A).$$

Thus for an inner automorphism $f: A \to A$

$$f_! = 1: K_0(A) \to K_0(A).$$
$$f_! = 1: K_1(A) \to K_1(A).$$

A *stable isomorphism* of f.g. projective A-modules $[\phi]: P \to Q$ is an equivalence class of isomorphisms $\phi: P \oplus X \to Q \oplus X$ for f.g. projective A-modules X, defined exactly as in Section 1 for the additive category of f.g. projective A-modules, with

$$(\phi: P \oplus X \to Q \oplus X) \sim (\theta: P \oplus Y \to Q \oplus Y)$$

if

$$\tau\Bigg(P\oplus X\oplus Y\xrightarrow{\phi\oplus 1_Y}Q\oplus X\oplus Y\xrightarrow{1_Q\oplus\begin{pmatrix}0&1_Y\\1_X&0\end{pmatrix}}Q\oplus Y\oplus X$$

$$\xrightarrow{\theta^{-1}\oplus 1_X}P\oplus Y\oplus X\xrightarrow{1_P\oplus\begin{pmatrix}0&1_X\\1_Y&0\end{pmatrix}}P\oplus X\oplus Y\Bigg)$$

$$=0\in K_1(A).$$

Note that f.g. projective A-modules P, P', Q, Q' are such that

$$[P]-[Q]=[P']-[Q']\in K_0(A)$$

if and only if $P\oplus Q'$ is stably isomorphic to $Q\oplus P'$.

Define the *relative* K_1-*group* $K_1(f)$ of a morphism $f\colon A\to B$ of rings to be the Abelian group of equivalence classes of triples $(P,Q,[\phi])$ defined by f.g. projective A-modules P, Q and a stable isomorphism $[\phi]\colon f_!P\to f_!Q$ of the induced f.g. projective B-modules, under the equivalence relation

$(P,Q,[\phi])\sim(P',Q',[\phi'])$ if there exists a stable isomorphism

$[\theta]\colon P\oplus Q'\to Q\oplus P'$ such that

$$\tau(f_!P\oplus f_!Q'\xrightarrow{f_![\theta]}f_!Q\oplus f_!P'\xrightarrow{[\phi]^{-1}\oplus[\phi']}f_!P\oplus f_!Q')$$

$$=0\in K_1(B)$$

with addition by

$$(P,Q,[\phi])+(R,S,[\psi])=(P\oplus R,Q\oplus S,[\phi]\oplus[\psi])\in K_1(f).$$

$K_1(f)$ is isomorphic to the relative K_1-group defined by Bass [1]. Note the logarithmic property

$$(P,Q,[\phi])+(Q,R,[\psi])=(P,R,[\psi][\phi])\in K_1(f),$$

so that inverses are given by

$$-(P,Q,[\phi])=(Q,P,[\phi]^{-1})\in K_1(f).$$

PROPOSITION 2.1. *The relative* K_1-*group* $K_1(f)$ *fits into an exact sequence*

$$K_1(A)\xrightarrow{f_!}K_1(B)\xrightarrow{j}K_1(f)\xrightarrow{\partial}K_0(A)\xrightarrow{f_!}K_0(B)$$

with

$$j\colon K_1(B)\to K_1(f);\quad \tau(\phi\colon X\to X)\mapsto(0,0,[\phi])$$
$$\partial\colon K_1(f)\to K_0(A);\quad (P,Q,[\phi])\mapsto[Q]-[P].$$

Proof. Trivial. □

Given finitely dominated chain complexes C, D over A and a chain equivalence of the induced chain complexes over B

$$\zeta : f_! C \to f_! D$$

there is defined an element $(C, D, \zeta) \in K_1(f)$ such that

$$\partial(C, D, \zeta) = [D] - [C] \in K_0(A)$$

as follows. Choose chain equivalences $\theta : C \to P$, $\psi : D \to Q$ to bounded f.g. projective chain complexes P, Q over A and define a chain equivalence of the induced chain complexes over B

$$\phi = (f_!\psi)\zeta(f_!\theta^{-1}) : f_!P \xrightarrow{f_!\theta^{-1}} f_!C \xrightarrow{\zeta} f_!D \xrightarrow{f_!\psi} f_!Q.$$

Using any chain contraction $\Gamma : 0 \simeq 1 : C(\phi) \to C(\phi)$ and the isomorphism of f.g. projective B-modules

$$d + \Gamma : C(\phi)_{\mathrm{odd}} = f_!P_{\mathrm{even}} \oplus f_!Q_{\mathrm{odd}} \to C(\phi)_{\mathrm{even}} = f_!P_{\mathrm{odd}} \oplus f_!Q_{\mathrm{even}}$$

define an element

$$(C, D, \zeta) = (P_{\mathrm{even}} \oplus Q_{\mathrm{odd}}, P_{\mathrm{odd}} \oplus Q_{\mathrm{even}}, d + \Gamma) \in K_1(f)$$

which is independent of the choices of θ, ψ, Γ. The definition of $(C, D, \zeta) \in K_1(f)$ is a mild generalization of a construction of Smith [20]. In Section 4 below we shall use the construction to define a relative K_1-theory invariant $(X, Y, \zeta) \in K_1(f)$ for a map $\zeta : X \to Y$ of finitely dominated CW complexes which is a B-homology equivalence, for some morphism of rings $f : A = \mathbb{Z}[\pi_1(Y)] \to B$. (More generally, there is defined an invariant $(C, D, \xi) \in K_1(f)$ for any chain equivalence

$$\zeta : f_!C \oplus E \to f_!D \oplus F$$

with C, D finitely dominated chain complexes over A and E, F round finite chain complexes over B. The element is such that

$$(C, D, \zeta) = [D] - [C] \in K_0(A),$$

and

$$j : K_1(B) \to K_1(f); \quad \tau(\zeta : E \to F) \mapsto (0, 0, \zeta).$$

We need only consider $(C, D, \zeta) \in K_1(f)$ for $E = 0$, $F = 0$ here.)

Given two ring morphisms f, $g : A \to B$ define the *relative K_1-group* $K_1(f, g)$ to be the Abelian group with one generator $(P, [\phi])$ for each f.g. projective A-module P with a stable isomorphism $[\phi] : g_!P \to f_!P$ of the induced f.g. projective B-modules, subject to the relations

$$(P,[\phi])=(P',[\phi']) \text{ if there exists a stable isomorphism } [\theta]:P\to P' \text{ such that } \tau(g_![\theta]^{-1}[\phi]^{-1}f_![\theta][\phi]:g_!P\to f_!P\to f_!P'\to g_!P'\to g_!P) = 0\in K_1(B),$$

$$(P,[\phi])+(P',[\phi])=(P\oplus P',[\phi]\oplus[\phi'])\in K_1(f,g).$$

PROPOSITION 2.2. *The relative K_1-group $K_1(f,g)$ fits into an exact sequence*

$$K_1(A)\xrightarrow{f_!-g_!}K_1(B)\xrightarrow{j}K_1(f,g)\xrightarrow{\partial}K_0(A)\xrightarrow{f_!-g_!}K_0(B)$$

with

$$j:K_1(B)\to K_1(f,g);\quad \tau(\phi:B^n\to B^n)\mapsto(A^n,[\phi])-(A^n,[1])$$

$$\partial:K_1(f,g)\to K_0(A);\quad (P,[\phi])\mapsto[P].$$

Proof. Define $K_1'(f,g)$ to be the Abelian group of equivalence classes of triples $(P,Q,[\phi])$ consisting of f.g. projective A-modules P, Q and a stable isomorphism of f.g. projective B-modules

$$[\phi]:g_!P\oplus f_!Q\to f_!P\oplus g_!Q$$

under the equivalence relation

$$(P,Q,[\phi])\sim(P',Q',[\phi']) \text{ if there exists a stable isomorphism } [\theta]:P\oplus Q'\to P'\oplus Q \text{ such that } \tau((f_![\theta]^{-1}\oplus g_![\theta])([\phi]\oplus[\phi]^{-1}): g_!P\oplus f_!Q\oplus g_!P'\oplus f_!Q'\to f_!P\oplus g_!Q\oplus f_!P'\oplus g_!Q' \to g_!P\oplus f_!Q\oplus g_!P'\oplus f_!Q') = 0\in K_1(B).$$

It follows from the logarithmic property

$$(P,Q,[\phi])\oplus(Q,R,[\psi])=(P,R,[\psi][\phi])\in K_1'(f,g)$$

that inverses are given by

$$-(P,Q,[\phi])=(Q,P,[\phi]^{-1})\in K_1'(f,g).$$

Now $K_1'(f,g)$ fits into an exact sequence

$$K_1(A)\xrightarrow{f_!-g_!}K_1(B)\xrightarrow{j'}K_1'(f,g)\xrightarrow{\partial'}K_0(A)\xrightarrow{f_!-g_!}K_0(B)$$

with

$$j':K_1(B)\to K_1'(f,g);\quad \tau(\phi:B^n\to B^n)\mapsto(0,0,[\phi])$$

$$\partial':K_1'(f,g)\to K_0(A);\quad (P,Q,[\phi])\mapsto[P]-[Q],$$

and there is defined an isomorphism of Abelian groups

$$h: K_1(f,g) \to K'_1(f,g); \quad (P,[\phi]) \mapsto (P,0,[\phi])$$

with inverse

$$\begin{aligned} h^{-1}&: K'_1(f,g) \to K_1(f,g); \\ &(P,Q,[\phi]) \mapsto (P \oplus -Q,[\phi]) - (Q \oplus -Q,[1]) \\ &\quad (\text{for any } -Q \text{ such that } Q \oplus -Q = A^n) \end{aligned}$$

such that $hj = j'$, $\partial' h = \partial$. □

In the applications we shall use the isomorphism $h: K_1(f,g) \to K'_1(f,g)$ as an identification, representing elements of $K_1(f,g)$ both as pairs $(P,[\phi]: g_! P \to f_! P)$ and as triples $(P,Q,[\phi]: g_! P \oplus f_! Q \to f_! P \oplus g_! Q)$.

(Given ring morphisms $f_1: A \to B_1$, $f_2: A \to B_2$ define ring morphisms from A to the product ring $B_1 \times B_2$

$$\begin{aligned} f&: A \to B_1 \times B_2; \quad a \mapsto (f_1(a), 0), \\ g&: A \to B_1 \times B_2; \quad a \mapsto (0, f_2(a)). \end{aligned}$$

For such f, g the exact sequence of Proposition 2.2 can be written as

$$K_1(A) \xrightarrow{\begin{pmatrix} f_{1!} \\ -f_{2!} \end{pmatrix}} K_1(B_1) \oplus K_1(B_2) \xrightarrow{j} K_1(f,g)$$

$$\xrightarrow{\partial} K_0(A) \xrightarrow{\begin{pmatrix} f_{1!} \\ -f_{2!} \end{pmatrix}} K_0(B_1) \oplus K_0(B_2)$$

and $K_1(f,g)$ is isomorphic to the relative K_1-group defined by Casson [3].)

Given a finitely dominated chain complex C over A and a chain equivalence of the induced chain complexes over B for some ring morphisms $f, g: A \to B$

$$\zeta: g_! C \to f_! C$$

there is defined an element $(C,\zeta) \in K_1(f,g)$ such that

$$\partial(C,\zeta) = [C] \in K_0(A)$$

as follows. Choose a chain equivalence $\psi: C \to P$ to a bounded f.g. projective chain complex P over A and define a chain equivalence of bounded f.g. projective chain complexes over B

$$\phi = (f_! \psi)\zeta(g_! \psi^{-1}): g_! P \xrightarrow{g_! \psi^{-1}} g_! C \xrightarrow{\zeta} f_! C \xrightarrow{f_! \psi} f_! P.$$

Using any chain contraction $\Gamma: 0 \simeq 1: C(\phi) \to C(\phi)$ and the isomorphism of f.g.

projective B-modules

$$d + \Gamma: C(\phi)_{\text{odd}} = g_! P_{\text{even}} \oplus f_! P_{\text{odd}} \to C(\phi)_{\text{even}} = f_! P_{\text{odd}} \oplus g_! P_{\text{even}}$$

define an element

$$(C, \zeta) = (P_{\text{even}}, P_{\text{odd}}, d + \Gamma) \in K_1(f, g)$$

which is independent of the choices of ψ, Γ. In Section 5 below we shall use the construction to define an invariant $(X, \zeta) \in K_1(1_A, \alpha)$ for any self-homotopy equivalence $\zeta: X \to X$ of a finitely dominated CW complex X, with $A = \mathbb{Z}[\pi_1(X)]$ and $\alpha: A \to A$ the automorphism induced by $\zeta_*: \pi_1(X) \to \pi_1(X)$. (More generally, there is defined an invariant $(C, D, \zeta) \in K_1(f, g)$ for any chain equivalence

$$\zeta: g_! C \oplus f_! D \oplus E \to f_! C \oplus g_! D \oplus F$$

with C, D finitely dominated chain complexes C, D over A and E, F round finite chain complexes over B. The element is such that

$$\partial(C, D, \zeta) = [C] - [D] \in K_0(A),$$

and

$$j: K_1(B) \to K_1(f, g); \tau(\zeta: E \to F) \mapsto (0, 0, \zeta).$$

We need only consider the case $D = 0$, $E = 0$, $F = 0$ here, with $(C, 0, \zeta) = (C, \zeta) \in K_1(f, g)$).

3. Products in K-Theory

Given rings A, B let $A \otimes B$, $B \otimes A$ be the product rings, where the tensor product is taken over $\mathbb{Z}$. The transposition isomorphisms

$$T: B \otimes A \to A \otimes B; \quad b \otimes a \mapsto a \otimes b$$
$$U: A \otimes B \to B \otimes A; \quad a \otimes b \mapsto b \otimes a$$

are inverse to each other.

The product of an A-module M and a B-module N is an $A \otimes B$-module $M \otimes N$, with $A \otimes B$ acting by

$$A \otimes B \times M \otimes N \to M \otimes N; \quad (a \otimes b, x \otimes y) \mapsto ax \otimes by,$$

and the $B \otimes A$-module $N \otimes M$ is defined similarly. If M is a f.g. projective A-module and N is a f.g. projective B-module then $M \otimes N$ is a f.g. projective $A \otimes B$-module. If M and N are f.g. free then so is $M \otimes N$, and

$$\text{rank}_{A \otimes B}(M \otimes N) = \text{rank}_A(M)\text{rank}_B(N).$$

In dealing with based f.g. free modules we adopt the convention that a base $\{x_i \mid 1 \leqslant i \leqslant m\}$ for M and a base $\{y_j \mid 1 \leqslant j \leqslant n\}$ for N determine the base

$\{z_k \mid 1 \leqslant k \leqslant mn\}$ for $M \otimes N$ defined by

$$z_k = x_i \otimes y_j \quad \text{if } k = i + m(j-1),$$

so that

$$\{z_1, z_2, \ldots, z_{mn}\} = \{x_1 \otimes y_1, x_2 \otimes y_1, \ldots, x_m \otimes y_1, x_1 \otimes y_2, \ldots, x_m \otimes y_n\}.$$

The isomorphism of based f.g. free $A \otimes B$-modules

$$M \otimes N \to T_!(N \otimes M); \quad x \otimes y \mapsto y \otimes x$$

has torsion

$$\begin{aligned}&\tau(M \otimes N \to T_!(N \otimes M)) = \tfrac{1}{4}m(m-1)n(n-1)\tau(-1 : A \otimes B \to A \otimes B)\\ &\in K_1(A \otimes B),\end{aligned}$$

the sign of the permutation

$$\begin{aligned}&\{1, 2, \ldots, mn\} \to \{1, 2, \ldots, mn\};\\ &\quad k = i + m(j-1) \mapsto k' = j + n(i-1) (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n).\end{aligned}$$

Furthermore, for based f.g. free A-modules M, M_1, M_2 and based f.g. free B-modules N, N_1, N_2 the evident isomorphisms of based f.g. free $A \otimes B$-modules have torsions

$$\begin{aligned}&\tau(M \otimes (N_1 \oplus N_2) \to (M \otimes N_1) \oplus (M \otimes N_2)) = 0 \in K_1(A \otimes B)\\ &\tau((M_1 \oplus M_2) \otimes N \to (M_1 \otimes N) \oplus (M_2 \otimes N))\\ &\quad = \tfrac{1}{2}m_1 m_2 n(n-1)\tau(-1 : A \otimes B \to A \otimes B) \in K_1(A \otimes B)\end{aligned}$$

with $m_1 = \operatorname{rank}_A(M_1)$, $m_2 = \operatorname{rank}_A(M_2)$, $n = \operatorname{rank}_B(N)$. The sign is obtained by considering the commutative diagram of isomorphisms

$$\begin{array}{ccc}(M_1 \oplus M_2) \otimes N & \longrightarrow & (M_1 \otimes N) \oplus (M_2 \otimes N)\\ \big\downarrow & & \big\downarrow\\ T_!(N \otimes (M_1 \oplus M_2)) & \longrightarrow & T_!((N \otimes M_1) \oplus N \otimes M_2)),\end{array}$$

and noting that

$$\begin{aligned}&\tfrac{1}{4}(m_1 + m_2)(m_1 + m_2 - 1)n(n-1) -\\ &\quad - \tfrac{1}{4}m_1(m_1 - 1)n(n-1) - \tfrac{1}{4}m_2(m_2 - 1)n(n-1)\\ &= \tfrac{1}{2}m_1 m_2 n(n-1).\end{aligned}$$

The product operation on modules is functorial, and as usual there are defined products in the algebraic K-groups

$$\begin{aligned}&K_0(A) \otimes K_0(B) \to K_0(A \otimes B); \quad [P] \otimes [Q] \mapsto [P \otimes Q]\\ &K_1(A) \otimes K_0(B) \to K_1(A \otimes B);\\ &\quad \tau(f : P \to P) \otimes [Q] \mapsto \tau(f \otimes 1 : P \otimes Q \to P \otimes Q)\\ &K_0(A) \otimes K_1(B) \to K_1(A \otimes B);\\ &\quad [P] \otimes \tau(g : Q \to Q) \mapsto \tau(1 \otimes g : P \otimes Q \to P \otimes Q)\end{aligned}$$

with P a f.g. projective A-module, Q a f.g. projective B-module, and $f \in \mathrm{Hom}_A(P, P)$, $g \in \mathrm{Hom}_B(Q, Q)$ automorphisms.

The product of an A-module chain complex C and a B-module chain complex D is the $A \otimes B$-module chain complex $C \otimes D$ defined by

$$d_{C \otimes D}: (C \otimes D)_r = \Sigma_{s=-\infty}^{\infty} C_s \otimes D_{r-s} \to (C \otimes D)_{r-1};$$
$$x \otimes y \mapsto x \otimes d_D(y) + (-)^{r-s} d_C(x) \otimes y.$$

If C and D are finitely dominated, then so is $C \otimes D$, and if either C or D is contractible then so is $C \otimes D$. If C and D are finite, then so is $C \otimes D$, as in $D \otimes C$, and the transposition isomorphism of finite chain complexes over $A \otimes B$

$$C \otimes D \to T_!(D \otimes C); \quad x \otimes y \mapsto (-)^{st} y \otimes x \quad (x \in C_s, y \in D_t)$$

has torsion

$$\tau(C \otimes D \to T_!(D \otimes C))$$
$$= \zeta(C, D)\tau(-1: A \otimes B \to A \otimes B) \in K_1(A \otimes B),$$

where

$$\zeta(C, D) = \nu(C)\nu(D) + \chi_{\mathrm{odd}}(C)\chi_{\mathrm{odd}}(D) +$$
$$+ \Sigma_{r=0}^{\infty} \Sigma_{0 \leqslant s < t \leqslant r} c_s c_t d_{r-s} d_{r-t} \in \mathbb{Z}_2,$$

with

$$c_s = \mathrm{rank}_A(C_s), \qquad d_t = \mathrm{rank}_B(D_t),$$
$$\nu(C) = \Sigma_{s=0}^{\infty} \tfrac{1}{2} c_s(c_s - 1),$$
$$\chi_{\mathrm{odd}}(C) = \Sigma_{i=0}^{\infty} c_{2i+1} \in \mathbb{Z}_2.$$

(Further below we shall also use $\chi_{\mathrm{even}}(C) = \Sigma_{i=0}^{\infty} c_{2i} \in \mathbb{Z}_2$.) If C, C' are finite chain complexes over A and D, D' are finite chain complexes over B the rearrangement isomorphisms have torsions

$$\tau(C \otimes (D \oplus D') \to (C \otimes D) \oplus (C \otimes D'))$$
$$= \lambda(C, D, D')\tau(-1: A \otimes B \to A \otimes B) \in K_1(A \otimes B)$$
$$\tau((C \oplus C') \otimes D \to (C \otimes D) \oplus (C \otimes D'))$$
$$= \mu(C, C', D)\tau(-1: A \otimes B \to A \otimes B) \in K_1(A \otimes B)$$

with λ, μ defined by

$$\lambda(C, D, D') = (\Sigma_{s=0}^{\infty} c_s c_{s+1})(\Sigma_{t=0}^{\infty} d_t d'_{t+1}),$$
$$\mu(C, C', D) = \lambda(D, C, C') + \varepsilon(C \oplus C', D) + \varepsilon(C, D) + \varepsilon(C', D) \in \mathbb{Z}_2.$$

For any finite chain complex C over A and any chain map $g: D \to D'$ of finite chain complexes over B, the rearrangement isomorphism $C(1 \otimes g: C \otimes D \to C \otimes D') \to$

$C \otimes C(g: D \to D')$ has torsion

$$\begin{aligned}&\tau(C(1 \otimes g) \to C \otimes C(g))\\&\quad = \lambda(D', SD, C)\tau(-1: A \otimes B \to A \otimes B) \in K_1(A \otimes B).\end{aligned}$$

For any chain map $f: C \to C'$ of finite chain complexes over A and any finite chain complex D over B the rearrangement isomorphism $C(f \otimes 1: C \otimes D \to C' \otimes D) \to C(f: C) \to C') \otimes D$ has torsion

$$\begin{aligned}&\tau(C(f \otimes 1) \to C(f) \otimes D)\\&\quad = \mu(C', SC, D)\tau(-1: A \otimes B \to A \otimes B) \in K_1(A \otimes B).\end{aligned}$$

PROPOSITION 3.1. (i) *The projective class of the product $C \otimes D$ of a finitely dominated chain complex C over A and a finitely dominated chain complex D over B is given by*

$$[C \otimes D] = [C] \otimes [D] \in K_0(A \otimes B).$$

(ii) *The torsion of the product $C \otimes D$ of a contractible finite chain complex C over A and a finite chain complex D over B is given by*

$$\tau(C \otimes D) = \tau(C) \otimes [D] + \eta(C, D)\tau(-1: A \otimes B \to A \otimes B) \in K_1(A \otimes B)$$

where $[D] = \chi(D) \in K_0(B)$ and η is defined by

$$\begin{aligned}\eta(C, D) = \beta(C, C)\nu(D) + \Sigma_{i>j}\beta(C \otimes S^i D_i, C \otimes S^j D_j) +\\ + \chi_{\mathrm{odd}}(C)\chi_{\mathrm{odd}}(D) \in \mathbb{Z}_2.\end{aligned}$$

(iii) *The torsion of the product $C \otimes D$ of a finite chain complex C over A and a contractible finite chain complex D over B is given by*

$$\begin{aligned}\tau(C \otimes D) = [C] \otimes \tau(D) + (\eta(D, C) +\\ + \zeta(D, C))\tau(-1: A \otimes B \to A \otimes B) \in K_1(A \otimes B),\end{aligned}$$

where $[C] = \chi(C) \in K_0(A)$. If C is even the sign term vanishes and

$$\tau(C \otimes D) = [C] \otimes \tau(D) \in K_1(A \otimes B).$$

(iv) *The reduced torsion of the product $f \otimes g: C \otimes D \to C' \otimes D'$ of a chain equivalence $f: C \to C'$ of finite chain complexes over A and a chain equivalence $g: D \to D'$ of finite chain complexes over B is given by*

$$\tau(f \otimes g) = [C] \otimes \tau(g) + \tau(f) \otimes [D] \in \tilde{K}_1(A \otimes B),$$

where

$$[C] = \chi(C) \in \mathbb{Z} \subset K_0(A), \qquad [D] = \chi(D) \in \mathbb{Z} \subset K_0(B).$$

Proof. (i) By the chain homotopy invariance of the projective class it may be assumed that C and D are bounded positive complexes of f.g. projective modules, in which case so is $C\otimes D$ and

$$\begin{aligned}[C\otimes D] &= \Sigma_{r=0}^{\infty}(-)^r[(C\otimes D)_r]\\ &= \Sigma_{r=0}^{\infty}\Sigma_{s+t=r}(-)^{s+t}[C_s\otimes D_t]\\ &= \Sigma_{r=0}^{\infty}\Sigma_{s+t=r}(-)^{s+t}[C_s\otimes[D_t]]\\ &= (\Sigma_{s=0}^{\infty}(-)^s[C_s])\otimes(\Sigma_{t=0}^{\infty}(-)^t[D_t])\\ &= [C]\otimes[D]\in K_0(A\otimes B).\end{aligned}$$

(ii) If D is 0-dimensional, then by definition

$$\tau(C\otimes D)=\tau((d+\Gamma)\otimes 1:(C\otimes D)_{\text{odd}}\to(C\otimes D)_{\text{even}})\in K_1(A\otimes B),$$

for any chain contraction $\Gamma:0\simeq 1:C\to C$. The rearrangement isomorphisms have torsions

$$\begin{aligned}&\tau((C\otimes D)_{\text{odd}}\to C_{\text{odd}}\otimes D_0)\\ &\quad=(\Sigma_{i>j}c_{2i+1}c_{2j+1})\tfrac{1}{2}d_0(d_0-1)\tau(-1:A\otimes B\to A\otimes B),\\ &\tau((C\otimes D)_{\text{even}}\to C_{\text{even}}\otimes D_0)\\ &\quad=(\Sigma_{i>j}c_{2i}c_{2j})\tfrac{1}{2}d_0(d_0-1)\tau(-1:A\otimes B\to A\otimes B)\in K_1(A\otimes B)\end{aligned}$$

and

$$(d+\Gamma)\otimes 1:(C\otimes D)_{\text{odd}}\to C_{\text{odd}}\otimes D_0\xrightarrow{(d+\Gamma)\otimes 1}C_{\text{even}}\otimes D_0\to(C\otimes D)_{\text{even}},$$

so that

$$\begin{aligned}\tau(C\otimes D) &= \tau((C\otimes D)_{\text{odd}}\to C_{\text{odd}}\otimes D_0)+\tau((d+\Gamma)\otimes 1:C_{\text{odd}}\otimes D_0\to C_{\text{even}}\otimes D_0)-\\ &\quad-\tau((C\otimes D)_{\text{even}}\to C_{\text{even}}\otimes D_0)\\ &= \tau(C)\otimes[D]+\beta(C,C)\nu(D)\tau(-1:A\otimes B\to A\otimes B)\\ &= \tau(C)\otimes[D]+\eta(C,D)\tau(-1:A\otimes B\to A\otimes B)\in K_1(A\otimes B).\end{aligned}$$

Assume inductively that $\tau(C\otimes D)=\tau(C)\otimes[D]+\eta(C,D)\tau(-1)$ if D is of dimension $<n$. If D is n-dimensional, let D' be the $(n-1)$-skeleton, so that there is defined a short exact sequence of finite chain complexes over B

$$0\to D'\xrightarrow{i}D\xrightarrow{j}S^nD_n\to 0$$

with

$$(S^nD_n)_r=D_n\quad\text{if } r=n,\ =0 \text{ if } r\neq n.$$

Applying $C\otimes$- there is obtained a short exact sequence of finite chain complexes over $A\otimes B$

$$0\to C\otimes D'\xrightarrow{1\otimes i}C\otimes D\xrightarrow{1\otimes j}C\otimes S^nD_n\to 0.$$

By the sum formula of Proposition 2.3 of [16] and the inductive hypothesis

$$\tau(C\otimes D)=\tau(C\otimes D')+\tau(C\otimes S^nD_n)+\beta(C\otimes D',C\otimes S^nD_n)\tau(-1:A\otimes B\to A\otimes B)$$

$$= \tau(C)\otimes[D'] + (-)^n\tau(C)\otimes[D_n] +$$
$$+ (\beta(C\otimes D', C\otimes S^n D_n) + n\chi_{\text{odd}}(C) + \sum_{m<n}\beta(C\otimes S^m D_m, C\otimes S^n D_n))\tau(-1)$$
$$= \tau(C)\otimes[D] + \eta(C,D)\tau(-1\colon A\otimes B\to A\otimes B)\in K_1(A\otimes B),$$

establishing the inductive step.

(iii) Using the transposition isomorphisms

$$T\colon B\otimes A\to A\otimes B,\qquad U\colon A\otimes B\to B\otimes A$$

and the result of (ii) we have

$$\begin{aligned}\tau(C\otimes D) &= T_! U_! \tau(C\otimes D)\\ &= T_!(\tau(D\otimes C) + \eta(D,C)\tau(-1\colon B\otimes A\to B\otimes A))\\ &= T_!(\tau(D)\otimes[C] + (\zeta(D,C)+\eta(D,C))\tau(-1\colon B\otimes A\to B\otimes A))\\ &= [C]\otimes\tau(D) + (\zeta(D,C)+\eta(D,C))\tau(-1\colon A\otimes B\to A\otimes B)\in K_1(A\otimes B).\end{aligned}$$

LEMMA. *For any finite chain complex C over A*

$$\beta(C,C) = \nu(C) + \tfrac{1}{2}\chi_{\text{even}}(C)(\chi_{\text{even}}(C)-1) + \tfrac{1}{2}\chi_{\text{odd}}(C)(\chi_{\text{odd}}(C)-1)\in\mathbb{Z}_2.$$

Thus if C is round $\beta(C,C) = \nu(C)\in\mathbb{Z}_2$. *If C is even*

$$\beta(C,C) = 0 = \nu(C) + \tfrac{1}{2}\chi(C)\in\mathbb{Z}_2.$$

Proof. If C is such that $C_r = 0$ for $r\neq n$, both sides of the identity are zero.

If the identity holds for finite chain complexes C, C' then it also holds for their sum $C\oplus C'$, since

$$\begin{aligned}&\beta(C\oplus C', C\oplus C') - \beta(C,C) - \beta(C',C') = \beta(C,C') + \beta(C',C)\\ &= \sum_r c_r c'_r + \chi_{\text{even}}(C)\chi_{\text{odd}}(C') + \chi_{\text{odd}}(C)\chi_{\text{even}}(C')\\ &= (\nu(C\oplus C') + \tfrac{1}{2}\chi_{\text{even}}(C\oplus C')(\chi_{\text{even}}(C\oplus C')-1) + \tfrac{1}{2}\chi_{\text{odd}}(C\oplus C')(\chi_{\text{odd}}(C\oplus C')-1) -\\ &\quad - (\nu(C) + \tfrac{1}{2}\chi_{\text{even}}(C)(\chi_{\text{even}}(C)-1) + \tfrac{1}{2}\chi_{\text{odd}}(C)(\chi_{\text{odd}}(C)-1)) -\\ &\quad - (\nu(C') + \tfrac{1}{2}\chi_{\text{even}}(C')(\chi_{\text{even}}(C')-1) + \tfrac{1}{2}\chi_{\text{odd}}(C')(\chi_{\text{odd}}(C')-1))\in\mathbb{Z}_2.\end{aligned}$$

Ignoring boundaries $C = C_0\oplus SC_1\oplus S^2C_2\oplus\ldots\oplus S^nC_n$, for some $n\geqslant 0$, so that the identity holds for all finite complexes C. □

Applying the Lemma we have that for even C

$$\eta(D,C) = \beta(D,D)\nu(C),\qquad \varepsilon(D,C) = \nu(D)\nu(C)\in\mathbb{Z}_2$$

and as D is round $\beta(D,D) = \nu(D)$, so that

$$\eta(D,C) + \varepsilon(D,C) = 0\in\mathbb{Z}_2.$$

(iv) Expressing $f\otimes g$ as the composite

$$f\otimes g\colon C\otimes D\xrightarrow{f\otimes 1_D} C'\otimes D\xrightarrow{1_{C'}\otimes g} C'\otimes D'$$

we have by the logarithmic property of reduced torsion

$$\tau(f \otimes g) = \tau(f \otimes 1_D) + \tau(1_{C'} \otimes g) \in \tilde{K}_1(A \otimes B).$$

The sign terms may be ignored in the reduced K_1-group, so that

$$\tau(f \otimes 1_D) = \tau(C(f \otimes 1_D)) = \tau(C(f) \otimes D) \in \tilde{K}_1(A \otimes B).$$

By (ii) above

$$\tau(C(f) \otimes D) = \tau(C(f)) \otimes [D] + \text{sign term} \in K_1(A \otimes B),$$

so that

$$\tau(f \otimes 1_D) = \tau(C(f) \otimes D) = \tau(f) \otimes [D] \in \tilde{K}_1(A \otimes B).$$

Similarly, by (iii)

$$\tau(1_{C'} \otimes g) = [C'] \otimes \tau(g) = [C] \otimes \tau(g) \in \tilde{K}_1(A \otimes B). \qquad \square$$

The product formula of Proposition 3.1(i) was first obtained by Gersten [7] (although of course well known prior to that for χ), and that of Proposition 3.1 (iv) by Kwun and Szczarba [10]. The topological interpretations are recalled in Proposition 4.5 below.

Proposition 3.1(i) shows that the product $C \otimes D$ of a finitely dominated chain complex C over A and a chain complex D over B which admits a round finite structure is a chain complex over $A \otimes B$ such that

$$[C \otimes D] = [C] \otimes [D] = [C] \otimes 0 = 0 \in K_0(A \otimes B),$$

so that $C \otimes D$ also admits a round finite structure. More precisely:

PROPOSITION 3.2.(i) *The product of a finitely dominated chain complex C over A and a chain complex D over B with a round finite structure $(G, \theta) \in \mathscr{F}^r(D)$ is a chain complex $C \otimes D$ over $A \otimes B$ with a canonical product round finite structure $C \otimes (G, \theta) \in \mathscr{F}^r(C \otimes D)$.*

(ii) *The product $f \otimes g: C \otimes D \to C' \otimes D'$ of a chain equivalence $f: C \to C'$ of finitely dominated chain complexes over A and a chain equivalence $g: D \to D'$ of chain complexes over B with round finite structures $(G, \theta) \in \mathscr{F}^r(D)$, $(G', \theta') \in \mathscr{F}^r(D')$ is a chain equivalence of chain complexes over $A \otimes B$ with torsion*

$$\tau(f \otimes g) = [C] \otimes \tau(g) \in K_1(A \otimes B)$$

with respect to the product round finite structures

$$C \otimes (G, \theta) \in \mathscr{F}^r(C \otimes D), \qquad C' \otimes (G', \theta') \in \mathscr{F}^r(C' \otimes D'),$$

where $[C] = [C'] \in K_0(A)$ and $\tau(g) \in K_1(B)$.

Proof. This occupies the rest of the Section. In (i) we shall define the product round finite structure $C \otimes (D, 1) \in \mathscr{F}^r(C \otimes D)$ for a round finite chain complex D over B.

Then in (ii) we shall prove the torsion product formula

$$\tau(f\otimes g: C\otimes D\to C'\otimes D') = [C]\otimes\tau(g: D\to D')\in K_1(A\otimes B)$$

for any chain equivalence $g: D\to D'$ of round finite chain complexes, with respect to the round finite structures $C\otimes(D,1)\in\mathscr{F}^r(C\otimes D)$, $C'\otimes(D',1)\in\mathscr{F}^r(C'\otimes D')$.

For any chain complex D over B with a round finite structure $(G,\theta)\in\mathscr{F}^r(D)$ the product round finite structure $C\otimes(G,\theta)\in\mathscr{F}^r(C\otimes D)$ can then be defined using $C\otimes(G,1) = (F,\phi)\in\mathscr{F}^r(C\otimes G)$ to be

$$C\otimes(G,\theta) = ((1\otimes\theta)\phi: F\to C\otimes G\to C\otimes D)\in\mathscr{F}^r(C\otimes D).$$

(i) It suffices to consider only the case of a rounded finite chain complex D over B, since by Proposition 1.7(ii) for any round finite chain complex D over B there exists a contractible finite chain complex D' such that $D\otimes D'$ is rounded and

$$\tau((1\,0): D\oplus D'\to D) = 0\in K_1(B).$$

If $C\otimes(D\oplus D',1) = (F,\phi)\in\mathscr{F}^r(C\otimes(D\oplus D'))$ is already defined let $C\otimes(D,1) = (F,(1\otimes(1,0))\phi: F\to C\otimes(D\oplus D')\to C\otimes D)\in\mathscr{F}^r(C\otimes D)$.

Let then D be a rounded finite chain complex over B. By Proposition 1.7(i) there exists a contractible finite chain complex D_Δ over B with the same chain modules $\{D_r|r\geqslant 0\}$, and the differentials $\{d_\Delta\in\mathrm{Hom}_B(D_r,D_{r-1})|r\geqslant 1\}$ can be chosen such that

$$\tau(D_\Delta) = 0\in K_1(B).$$

In dealing with the finitely dominated chain complex C over A it is convenient to work with the *idempotent completion* $\underline{\underline{P}}(A)$ of the additive category $\mathscr{A} = \underline{\underline{F}}(A)$ of based f.g. free A-modules. An object in $\underline{\underline{P}}(A)$ is a pair (E,p) consisting of a based f.g. free A-module E and an A-module morphism $p\in\mathrm{Hom}_A(E,E)$ which is a projection

$$p^2 = p: E\to E.$$

A morphism in $\underline{\underline{P}}(A)$

$$f: (E,p)\to(E',p')$$

is an A-module morphism $f\in\mathrm{Hom}_A(E,E')$ such that

$$p'fp = f: E\to E'.$$

The additive functor

$$P(A)\to\{\text{f.g. projective } A\text{-modules}\};\ (E,p)\to\mathrm{im}(p: E\to E)$$

is an equivalence of additive categories.

A *finite idempotent chain complex over* A (E,p) is a finite chain complex in $\underline{\underline{P}}(A)$

$$(E,p):\ldots\to 0\to(E_n,p_n)\xrightarrow{d}(E_{n-1},p_{n-1})\to\ldots\to(E_0,p_0).$$

The chain homotopy theory of finite idempotent chain complexes is defined in the

obvious way, with a bijection of sets of chain equivalence classes
{finite idempotent chain complexes over A}

$\to$ {finitely dominated chain complexes over A};

$$(E, p) \to \mathrm{im}(p\colon E \to E).$$

See Ranicki [15] for a detailed exposition.

Given a finite idempotent chain complex (E,p) over A and a rounded finite chain complex D over B define a round finite chain complex over $A \otimes B$

$$F = (E,p) \otimes D$$

by

$$d_F\colon F_r = (E \otimes D)_r = \Sigma_{s=0}^{r} E_s \otimes D_{r-s} \to F_{r-1};$$

$$x \otimes y \to p(x) \otimes d_D(y) + (1-p)(x) \otimes d_\Delta(y) + (-)^{r-s} d_E(x) \otimes y,$$

with $\{d_D \in \mathrm{Hom}_B(D_r, D_{r-1})^\pi r \geqslant 1\}$ the differentials of D and $\{d_\Delta \in \mathrm{Hom}_B(D_r, D_{r-1}) | r \geqslant 1\}$ the differentials of D_Δ (as above). For example, if $p = 1\colon E \to E$ then $F = F \otimes D$. As an unbased chain complex over B

$$F = \mathrm{im}(p) \otimes D \oplus \mathrm{im}(1-p) \otimes D_\Delta,$$

and the projection

$$F \to \mathrm{im}(p) \otimes D; \quad x \otimes y \to p(x) \otimes y$$

is a chain equivalence (since it has contractible kernel $\mathrm{im}(1-p) \otimes D_\Delta$).

A finite idempotent chain complex (E,p) over A is *even* if

$$\mathrm{rank}_A(E_r) \equiv 0 (\mathrm{mod}\, 2) (r \geqslant 0).$$

For any finitely dominated chain complex C over A there exists a triple (E,p,θ) consisting of an even idempotent finite chain complex (E,p) over A and a chain equivalence $\theta\colon \mathrm{im}(p) \to C$. (Choose a bounded f.g. projective chain complex P over A chain equivalent to C, and let $\{Q_r | r \geqslant 0\}$ be a sequence of f.g. projective A-modules such that $P_r \oplus Q_r$ is a f.g. free A-module of even rank if P_r is non-zero and $Q_r = 0$ if $P_r = 0$. Then $E = P \oplus Q$ as an unbased chain complex, with

$$d_E = \begin{pmatrix} d_P & 0 \\ 0 & 0 \end{pmatrix} \colon E_r = P_r \oplus Q_r \to E_{r-1} = P_{r-1} \oplus Q_{r-1},$$

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \colon E_r = P_r \oplus Q_r \to E_r = P_r \oplus Q_r, \mathrm{im}(p\colon E \to E) = P).$$

The product round finite structure $C \otimes (D,1) = (F,\phi) \in \mathscr{F}^r(C \otimes D)$ is defined using any such triple (E,p,θ) by

$$\phi : F = (E,p)\otimes D \xrightarrow{\text{projection}} \mathrm{im}(p)\otimes D \xrightarrow{\theta\otimes 1} C\otimes D.$$

We have to show that $(F,\phi)\in \mathscr{F}^r(C\otimes D)$ is independent of the choice of (E,p,θ). If $(E,p,\theta),(E',p',\theta')$ are two such choices the chain equivalence of even idempotent finite complexes

$$f = \phi'^{-1}\phi \oplus 0 : (E,p)\to (E',p')$$

is such that

$$(F,\phi)-(F',\phi') = \tau(f\otimes 1 : F = (E,p)\otimes D \to F' = (E',p')\otimes D)$$
$$\in K_1(A\otimes B).$$

We thus have to show that $\tau(f\otimes 1) = 0 \in K_1(A\otimes B)$. We consider first the special case of contractible C:

LEMMA *If (E,p) is an even finite idempotent chain complex over A such that $P = \mathrm{im}(p : E\to E)$ is a contractible chain complex over A then $F = (E,p)\otimes D$ is a contractible finite chain complex over $A\otimes B$ with torsion $\tau(F) = 0 \in K_1(A\otimes B)$.*

Proof. Choose a chain contraction $\Gamma : 0 \simeq 1 : P\to P$ and define an isomorphism of contractible finite chain complexes over $A\otimes B$

$$h : F \to E\otimes D_\Delta$$

by the $A\otimes B$-module automorphisms

$$h_r : F_r = \Sigma_{s=0}^{r} E_s\otimes D_{r-s} \to (E\otimes D_\Delta)_r = \Sigma_{s=0}^{r} E_s\otimes D_{r-s};$$
$$x\otimes y \to x\otimes y + (-)^{r-s}\Gamma p(x)\otimes (d_\Delta - d_D)(y))(r\geqslant 0)$$

so that

$$\tau(F) = \tau(E\otimes D_\Delta) - \Sigma_{r=0}^{\infty}(-)^r\tau(h_r : (E\otimes D)_r \to (E\otimes D)_r)\in K_1(A\otimes B).$$

As E is even

$$\tau(E\otimes D_\Delta) = [E]\otimes \tau(D_\Delta) \text{ (by Proposition 3.1(ii))}$$
$$= [E]\otimes 0 = 0 \in K_1(A\otimes B).$$

The f.g. projective $A\otimes B$-modules $M_r, N_r (r\geqslant 0)$ defined by

$$M_r = \Sigma_{s=0}^{r}\ker(d_P : P_s\to P_{s-1})\otimes D_{r-s},$$
$$N_r = \Sigma_{s=0}^{r}(\ker(\Gamma : P_s\to P_{s+1})\oplus \mathrm{im}(1-p : E_s\to E_s))\otimes D_{r-s}$$

are such that

$$h_r = \begin{pmatrix} 1 & 0 \\ \Sigma_{s=0}^{r}(-)^{r-s}\Gamma\otimes(d_\Delta - d_D) & 1\end{pmatrix}$$
$$:(E\otimes D)_r = M_r\oplus N_r \to M_r\oplus N_r \quad (r\geqslant 0).$$

Thus $\tau(h_r) = 0 \in K_1(A \otimes B)$, and $\tau(F) = 0 \in K_1(A \otimes B)$. □

The algebraic mapping cone of a chain map of even idempotent finite chain complexes over A

$$f: (E, p) \to (E', p')$$

is an even idempotent finite complex $(C(f),q)$ with

$$q = \begin{pmatrix} p' & 0 \\ 0 & p \end{pmatrix}: C(f)_r = E'_r \oplus E_{r-1} \to E'_r \oplus E_{r-1} \quad (r \geqslant 0).$$

The rearrangement isomorphism

$$(C(f),q) \otimes D \to C(f \otimes 1\colon (E,p) \otimes D \to (E',p') \otimes D)$$

has torsion $\mu\ (E', SE, D)\tau(-1\colon A \otimes B \to A \otimes B) \in K_1(A \otimes B)$, which is 0 since E and E' are even. If f is a chain equivalence (i.e. if $f|\colon \mathrm{im}(p')$ is a chain equivalence) then $\mathrm{im}(q)$ is contractible and

$$f \otimes 1\colon F = (E,p) \otimes D \to F' = (E',p') \otimes D$$

is a chain equivalence of even round finite chain complexes over $A \otimes B$ with torsion

$$\begin{aligned} \tau(f \otimes 1) &= \tau(C(f \otimes 1)) \\ &= \tau((C(f),q) \otimes D) \\ &= 0 \in K_1(A \otimes B), \text{ by the Lemma,} \end{aligned}$$

It follows that $(F,\phi) = (F',\phi') \in \mathscr{F}^r(C \otimes D)$, so that the round infinite structure defined on $C \otimes D$ is indeed canonical.

(ii) As for (i) it suffices to consider the special case when D and D' are rounded finite chain complexes over B. By the logarithmic property of torsion

$$\begin{aligned} \tau(f \otimes g\colon C \otimes D \to C' \otimes D') &= \tau(f \otimes g\colon C \otimes D \xrightarrow{1 \otimes g} C \otimes D' \xrightarrow{f \otimes 1} C' \otimes D) \\ &= \tau(f \otimes 1\colon C \otimes D' \to C' \otimes D') + \\ &\quad + \tau(1 \otimes g\colon C \otimes D \to C \otimes D') \in K_1(A \otimes B). \end{aligned}$$

Let $(F,\phi) \in \mathscr{F}^r(C \otimes D)$, $(F',\phi') \in \mathscr{F}^r(C' \otimes D)$ be the product round finite structures. By definition

$$\tau(f \otimes 1) = \tau(F \xrightarrow{\phi} C \otimes D \xrightarrow{f \otimes 1} C' \otimes D \xrightarrow{\phi'^{-1}} F') \in K_1(A \otimes B).$$

The proof in (i) above that $(F = (E,p) \otimes D, \phi) \in \mathscr{F}^r(C \otimes D)$ is independent of the choice of (E,p) includes a proof that $\tau(f \otimes 1) = 0 \in K_1(A \otimes B)$.

We shall prove that $\tau(1 \otimes g) = [C] \otimes \tau(g) \in K_1(A \otimes B)$ using the following generalization of the product formula of Proposition 3.1(iii), which is the special case $D = 0$.

LEMMA. *The product $F=(E,p)\otimes D$ of an even finite idempotent chain complex (E,p) over A and a contractible finite chain complex D over B is a contractible finite chain complex over $A\otimes B$ with torsion*

$$\tau(F)=[\mathrm{im}(p)]\otimes\tau(D)\in K_1(A\otimes B).$$

Proof. Choose chain contractions

$$\Gamma_D:0\simeq 1:D\to D,\quad \Gamma_\Delta:0\simeq 1:D_\Delta\to D_\Delta,$$

and use them to define a chain contraction

$$\Gamma_F=p\otimes\Gamma_D+(1-p)\otimes\Gamma_\Delta:0\simeq 1:F\to F.$$

If E is 0-dimensional the rearrangement isomorphisms are such that

$$\tau(((E\otimes D)_{\mathrm{even}}\to E_0\otimes D_{\mathrm{even}})=0,\tau((E\otimes D)_{\mathrm{odd}}\to E_0\otimes D_{\mathrm{odd}})=0\in K_1(A\otimes B),$$

so that

$$\begin{aligned}\tau(F)&=\tau(d_F+\Gamma_F:F_{\mathrm{odd}}=(E\otimes D)_{\mathrm{odd}}\to F_{\mathrm{even}}=(E\otimes D)_{\mathrm{even}})\\&=\tau(p\otimes(d_D+\Gamma_D)+(1-p)\otimes(d_\Delta+\Gamma_\Delta):E_0\otimes D_{\mathrm{odd}}\to E_0\otimes D_{\mathrm{even}})\\&=\tau(p\otimes(d_D+\Gamma_D)(d_\Delta+\Gamma_\Delta)^{-1}+(1-p)\otimes 1:E_0\otimes D_{\mathrm{even}}\to E_0\otimes D_{\mathrm{even}})\\&\qquad(\text{since }\tau(d_\Delta+\Gamma_\Delta:D_{\mathrm{odd}}\to D_{\mathrm{even}})=\tau(D_\Delta)=0\in K_1(B))\\&=[\mathrm{im}(p)]\otimes\tau((d_D+\Gamma_D)(d_\Delta+\Gamma_\Delta)^{-1}:D_{\mathrm{even}}\to D_{\mathrm{even}})\\&=[\mathrm{im}(p)]\otimes\tau(d_D+\Gamma_D:D_{\mathrm{odd}}\to D_{\mathrm{even}})\\&=[\mathrm{im}(p)]\otimes\tau(D)\in K_1(A\otimes B).\end{aligned}$$

Assume inductively that $\tau(F)=[\mathrm{im}(p:E\to E)]\otimes\tau(D)\in K_1(A\otimes B)$ if E is of dimension $<n$, and that the dimension of E is n. Let E' be the $(n-1)$-skelton of E, so that there is defined a short exact sequence of finite idempotent chain complexes over A

$$0\to(E',p')\xrightarrow{i}(E,p)\xrightarrow{j}(S^nE_n,p_n)\to 0.$$

Applying $-\otimes D$ there is obtained a short exact sequence of finite chain complexes over $A\otimes B$

$$0\to(E',p')\otimes D\xrightarrow{i\otimes 1}(E,p)\otimes D\xrightarrow{j\otimes 1}(S^nE_n,p_n)\otimes D\to 0.$$

By the torsion sum formula of Proposition 2.3 of Part I and the inductive hypothesis

$$\begin{aligned}\tau((E,p)\otimes D)&=\tau((E',p')\otimes D)+\tau((S^nE_n,p_n)\otimes D)\\&\qquad(\text{the sign term vanishes since }E\text{ is even})\\&=[\mathrm{im}(p')]\otimes\tau(D)+(-)^n[\mathrm{im}(p_n)]\otimes\tau(D)\\&=[\mathrm{im}(p)]\otimes\tau(D)\in K_1(A\otimes B).\end{aligned}$$

□

The algebraic mapping cone of a chain equivalence $g: D \to D'$ of round finite chain complexes over B is a contractible finite chain complex $C(g)$ over B, so that

$$\tau((E,p) \otimes C(g)) = [\mathrm{im}(p)] \otimes \tau(C(g)) \in K_1(A \otimes B)$$

by the Lemma. The round finite complexes $(E,p) \otimes D$, $(E,p) \otimes D'$ over $A \otimes B$ are constructed using any contractible chain complexes D_Δ, D'_Δ over B with the chain modules of D, D' respectively, and such that

$$\tau(D_\Delta) = \tau(D'_\Delta) = 0 \in K_1(B).$$

Now $C(g)$ has the chain modules of $D' \oplus SD$, but

$$\tau(D'_\Delta \oplus SD_\Delta) = \beta(D', SD)\tau(-1: B \to B) \in K_1(B)$$

so that $C(g)_\Delta$ cannot in general be chosen to be $D'_\Delta \oplus SD_\Delta$. We shall construct $(E,p) \otimes C(g)$ using the acyclic finite complex

$$C(g)_\Delta = D'_{\Delta'} \oplus D_\Delta$$

with $D'_{\Delta'}$ defined as follows. Choose an automorphism $\alpha \in \mathrm{Hom}_B(D'_n, D'_n)$ of a chain module D'_n of D' such that

$$\tau(\alpha) = \beta(D', SD)\tau(-1: B \to B) \in K_1(B).$$

Define $D'_{\Delta'}$ by

$$d_{\Delta'} = \begin{cases} d_\Delta \\ d_\Delta \alpha^{-1}: D'_r \to D'_{r-1} \\ \alpha d_\Delta \end{cases} \text{if} \begin{cases} r \neq n, n+1 \\ r = n \\ r = n+1, \end{cases}$$

so that there is defined an isomorphism of contractible finite chain complexes over B

$$h: D'_\Delta \xrightarrow{\sim} D'_{\Delta'}$$

with

$$h = \begin{cases} 1 \\ \alpha \end{cases}: D'_r \to D'_r \quad \text{if} \begin{cases} r \neq n \\ r = n \end{cases}$$

The torsion of h is given by

$$\tau(h) = \tau(D'_{\Delta'}) = (-)^r \tau(\alpha) = \beta(D', SD)\tau(-1: B \to B) \in K_1(B)$$

and

$$\begin{aligned} \tau(C(g)_\Delta) &= \tau(D'_{\Delta'}) + \tau(SD_\Delta) + \beta(D', SD)\tau(-1: B \to B) \\ &= 0 \in K_1(B). \end{aligned}$$

The isomorphism of contractible finite chain complexes over $A \otimes B$

$$k: (E,p) \otimes C(g) \to C(1 \otimes g: (E,p) \otimes D \to (E,p) \otimes D');$$
$$x \otimes (y', y) \to p(x) \otimes (y', y) + (1-p)(x) \otimes (h(y'), y)$$

has torsion

$$\begin{aligned}
\tau(k) &= \tau(C(1 \otimes g)) - \tau((E,p) \otimes C(g)) \\
&= \Sigma_{r=0}^{\infty}(-)^r \tau(k_r: (E,p) \otimes C(g))_r \to C(1 \otimes g)_r) \\
&\qquad \text{(by Proposition 1.2(iii))} \\
&= \Sigma_{r=n}^{\infty}(-)^r \tau(k_r: E_{r-n} \otimes D'_n \to E_{r-n} \otimes D'_n; \\
&\qquad x \otimes y' \to p(x) \otimes y' + (1-p)(x) \otimes h(y')) \\
&= \Sigma_{r=n}^{\infty}(-)^r [\mathrm{im}(1-p: E_{r-n} \to E_{r-n})] \otimes \tau(\alpha: D'_n \to D'_n) \\
&= [\mathrm{im}(p)] \otimes \beta(D', SD)\tau(-1: B \to B) \in K_1(A \otimes B).
\end{aligned}$$

Thus

$$\begin{aligned}
\tau(1 \otimes g: C \otimes D \to C \otimes D') &= \tau(1 \otimes g: (E,p) \otimes D \to (E,p) \otimes D') \\
&= \tau(C(1 \otimes g)) \\
&= \tau((E,p) \otimes C(g)) + [\mathrm{im}(p)] \otimes (\beta(D', SD)\tau(-1: B \to B)) \\
&= [\mathrm{im}(p)] \otimes (\tau(C(g)) + \beta(D', SD)\tau(-1: B \to B)) \\
&= [C] \otimes \tau(g) \in K_1(A \otimes B).
\end{aligned}$$

□

In the special case when $f: C \to C' = C, g: D \to D' = D$ the product formula of Proposition 3.2(ii) agrees with the product formula obtained by Gersten [8] (cf. Proposition 5.2 below).

4. Torsion for CW Complexes

Let $\tilde{X}$ be a regular cover of a CW complex X with group of covering translations π. The *cellular chain complex* of $\tilde{X}$ is the free chain complex over $\mathbb{Z}[\pi]$

$$C(\tilde{X}): \cdots \to C_{r+1}(\tilde{X}) \xrightarrow{d} C_r(\tilde{X}) \xrightarrow{d} C_{r-1}(\tilde{X}) \to \cdots \to C_0(\tilde{X})$$

defined in the usual manner, with

$$C_r(\tilde{X}) = H_r(\tilde{X}^{(r)}, \tilde{X}^{(r-1)}) \quad (r \geqslant 0)$$

a free $\mathbb{Z}[\pi]$-module with one generator for each r-cell of X.

We shall be mainly concerned with connected CW complexes X, with $\tilde{X}$ the universal cover and $\pi = \pi_1(X)$ the fundamental group. A *geometric base* for X is a base for the free $\mathbb{Z}[\pi]$-module $\Sigma_{r=0}^{\infty} C_r(\tilde{X})$ such that each base element is the Hurewicz image $\tilde{\phi}_*[e^r] \in C_r(\tilde{X})$ of a fundamental class $[e^r] = \pm 1 \in H_r(e^r, \partial e^r) = \mathbb{Z}$ under a lift $\tilde{\phi}: (e^r, \partial e^r) \to (\tilde{X}^{(r)}, \tilde{X}^{(r-1)})$ of a characteristic map $\phi: (e^r, \partial e^r) \to (X^{(r)}, X^{(r-1)})$. Geometric base elements are unique up to mulitplication by $\pm g (g \in \pi)$. A geometric base for a finite CW complex X determines a finite chain complex $C(\tilde{X})$ over $\mathbb{Z}[\pi]$.

A map of (connected) CW complexes $f: X \to Y$ induces a morphism of fundamental groups

$$f_* = \alpha: \pi_1(X) = \pi \to \pi_1(Y) = \rho$$

which is unique up to composition with inner automorphisms if base points are ignored. The universal cover $\tilde{Y}$ of Y pulls back to a cover $f^*\tilde{Y}$ of X such that f lifts to a ρ-equivariant map $\tilde{f}: f^*\tilde{Y} \to \tilde{X}$ inducing a chain map over $\mathbb{Z}[\rho]$

$$\tilde{f}: C(f^*\tilde{Y}) = \alpha_! C(\tilde{X}) \to C(\tilde{Y}).$$

The map $f: X \to Y$ is a homotopy equivalence if and only if $\alpha: \pi \to \rho$ is an isomorphism and $\tilde{f}: \alpha_! C(\tilde{X}) \to C(\tilde{Y})$ is a chain equivalence.

A *finite domination* (Y, f, g, h) of a CW complex X consists of a finite CW complex Y, maps

$$f: X \to Y, \qquad g: Y \to X$$

and a homotopy

$$h: gf \simeq 1: X \to X.$$

A CW complex X is *finitely dominated* if it admits a finite domination.

Let X be a connected CW complex with universal cover $\tilde{X}$ and fundamental group $\pi_1(X) = \pi$. A finite domination (Y, f, g, h) of X and a choice of geometric base for Y determine a finite domination of the chain complex $C(\tilde{X})$ over $\mathbb{Z}[\pi]$

$$(C(\tilde{Y}), \tilde{f}: C(\tilde{X}) \to C(\tilde{Y}), g: C(\tilde{Y}) \to C(\tilde{X}), \tilde{h}: \tilde{g}\tilde{f} \simeq 1: C(\tilde{X}) \to C(\tilde{X})),$$

where $\tilde{Y} = g^*\tilde{X}$ is the pullback cover of Y. The *projective class* of a finitely dominated CW complex X is defined by

$$[X] = [C(\tilde{X})] \in K_0(\mathbb{Z}[\pi]).$$

This is a homotopy invariant which can be expressed as

$$[X] = (\chi(X), [X]) \in K_0(\mathbb{Z}[\pi]) = K_0(\mathbb{Z}) \oplus \tilde{K}_0(\mathbb{Z}[\pi]),$$

with $\chi(X) = \chi(C(X)) \in K_0(\mathbb{Z}) = \mathbb{Z}$ the Euler characteristic of X and $[X] \in \tilde{K}_0(\mathbb{Z}[\pi]$ the reduced projective class.

PROPOSITION 4.1 (Wall [21]). (i) *A CW complex X is finitely dominated if and only if $\pi_1(X) = \pi$ is finitely presented and $C(\tilde{X})$ is finitely dominated.*

(ii) *A finitely dominated CW complex X is homotopy equivalent to a finite CW complex if and only if $[X] = 0 \in \tilde{K}_0(\mathbb{Z}[\pi])$,* i.e., *if and only if $C(\tilde{X})$ is chain equivalent to a finite complex. The reduced projective class $[X] \in \tilde{K}_0(\mathbb{Z}[\pi])$ is the finiteness obstruction of X.* □

The *Whitehead group* of a group π is defined as usual by

$$\mathrm{Wh}(\pi) = K_1(\mathbb{Z}[\pi])/\{\pm\pi\} \quad .$$

If X is a connected finite CW complex with $\pi_1(X) = \pi$ and C, C' are the finite chain complexes over $\mathbb{Z}[\pi]$ defined by the cellular chain complex $C(\tilde{X})$ of the universal cover $\tilde{X}$ and two different choices of geometric base then

$$\tau(1: C \to C') \in \{\pm\pi\} \subseteq K_1(\mathbb{Z}[\pi]),$$

and so has image $0 \in \mathrm{Wh}(\pi)$.

The (*Whitehead*) *torsion* of a homotopy equivalence $f: X \to Y$ of finite CW complexes is defined as usual by

$$\tau(f) = \tau(\tilde{f}: C(\tilde{X}) \to C(\tilde{Y})) \in \mathrm{Wh}(\pi)$$

with $\tilde{f}: \tilde{X} \to \tilde{Y}$ any lift of f to a π-equivariant map of the universal covers, identifying $\pi = \pi_1(X)$ with $\pi_1(Y)$ via the isomorphism $f_*: \pi_1(X) \to \pi_1(Y)$, with any geometric bases for $C(\tilde{X})$ and $C(\tilde{Y})$. The element $\tau(f) \in \mathrm{Wh}(\pi)$ is independent of the choices made in its definition.

A *finite structure* on a CW complex X is an equivalence class of pairs

(finite CW complex F, homotopy equivalence $f: F \to X$)

under the equivalence relation

$$(F,f) \sim (F',f') \quad \text{if } \tau(f'^{-1}f: F \to F') = 0 \in \mathrm{Wh}(\pi) \quad (\pi = \pi_1(X)).$$

The *finite structure set* $\mathscr{F}(X)$ of a CW complex X is the set (possibly empty) of finite structures on X.

PROPOSITION 4.2. (i) *$\mathscr{F}(X)$ is nonempty if and only if X is finitely dominated and* $[X] = 0 \in \tilde{K}_0(\mathbb{Z}[\pi])$.

(ii) *If $\mathscr{F}(X)$ is nonempty there is defined a transitive* $\mathrm{Wh}(\pi)$*-action* $\mathrm{Wh}(\pi) \times \mathscr{F}(X) \to \mathscr{F}(X)$;

$$(\tau(g: G \to F),(F,f)) \to (G,fg: G \to X).$$

A choice of base point (F_0,f_0) determines an abelian group structure of $\mathscr{F}(X)$ with an isomorphism

$$\mathscr{F}(X) \to \mathrm{Wh}(\pi);\ (F,f) \to \tau(f^{-1}f_0: F_0 \to F). \qquad \square$$

A (*Whitehead*) *finite structure* on a $\mathbb{Z}[\pi]$-module chain complex C is an equivalence class of pairs

(finite $\mathbb{Z}\{\pi\}$-module chain complex F, chain equivalence $\phi: F \to C$)

under the equivalence relation

$$(F,\phi) \sim (F',\phi') \quad \text{if } \tau(\phi'^{-1}\phi: F \to F') = 0 \in \mathrm{Wh}(\pi).$$

The (*Whitehead*) *finite structure set* $\mathscr{F}^{\mathrm{Wh}}(C)$ of a $\mathbb{Z}[\pi]$-module chain complex C is the set (possibly empty) of Whitehead finite structures on C. The evident analogue of Proposition 1.6 holds with $\mathrm{Wh}(\pi)$ and $\mathscr{F}^{\mathrm{Wh}}(C)$ in place of $K_1(A)$ and $\mathscr{F}(C)$.

PROPOSITION 4.3. *The finite structure set $\mathscr{F}(X)$ of a CW complex X is in natural bijective correspondence with the finite structure set $\mathscr{F}^{\mathrm{Wh}}(C(\tilde{X}))$ of the cellular $\mathbb{Z}[\pi]$-module chain complex $C(\tilde{X})$ of the universal cover $\tilde{X}$, with $\pi = \pi_1(X)$. If the sets are nonempty there is defined a natural isomorphism of affine* $\mathrm{Wh}(\pi)$*-sets*

$$\mathscr{F}(X) \to \mathscr{F}^{\mathrm{Wh}}(C(\tilde{X}));\quad (F, f: F \to X) \to (C(\tilde{F}), \tilde{f}: C(\tilde{F}) \to C(\tilde{X})). \qquad \square$$

A finite CW complex X is *round* if

$$\chi(X) = 0 \in \mathbb{Z}$$

and there is given a choice of geometric base for $C(\tilde{X})$, so that $C(\tilde{X})$ is a round finite $\mathbb{Z}[\pi]$-module chain complex. As usual, $\tilde{X}$ is the universal cover of X and $\pi = \pi_1(X)$ is the fundamental group.

The *torsion* of a homotopy equivalence $f: X \to Y$ of round finite CW complexes (meaning a homotopy equivalence of the underlying finite CW complexes) is defined by

$$\tau(f) = \tau(\tilde{f}: C(\tilde{X}) \to C(\tilde{Y})) \in K_1(\mathbb{Z}[\pi])$$

using any lift of f to a π-equivariant map $\tilde{f}: \tilde{X} \to \tilde{Y}$ of the universal covers, so that $\tilde{f}: C(\tilde{X}) \to C(\tilde{Y})$ is a chain equivalence of round finite $\mathbb{Z}[\pi]$-module chain complexes and torsion is defined as in Section 1, using the isomorphism $f_*: \pi_1(X) = \pi \to \pi_1(Y)$ as an identification. Any other lift of f is given by

$$\tilde{f}g: \tilde{X} \xrightarrow{g} \tilde{X} \xrightarrow{\tilde{f}} \tilde{Y}$$

for some $g \in \pi$, and

$$\begin{aligned}\tau(\tilde{f}g: C(\tilde{X}) \to C(\tilde{Y})) &= \tau(g: C(\tilde{X}) \to C(\tilde{X})) + \tau(\tilde{f}: C(\tilde{X}) \to C(\tilde{Y}))\\ &= \tau(g^{\chi(X)}: \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]) + \tau(f)\\ &= \tau(f) \in K_1(\mathbb{Z}[\pi]).\end{aligned}$$

Thus the torsion $\tau(f) \in K_1(\mathbb{Z}[\pi])$ is independent of the choice of lift $\tilde{f}: \tilde{X} \to \tilde{Y}$.

By the logarithmic property of torsion (Proposition 1.4(i)) the torsion of the composite $gf: X \to Z$ of homotopy equivalences $f: X \to Y$, $g: Y \to Z$ of round finite CW complexes is given by

$$\tau(gf) = \tau(f) + \tau(g) \in K_1(\mathbb{Z}[\pi]),$$

using the isomorphisms $f_*: \pi_1(X) = \pi \to \pi_1(Y), g_*: \pi_1(Y) \to \pi_1(Z)$ as identifications.

If X, X' are round finite CW complexes with the same underlying CW complex the identity map has torsion

$$\tau(1: X \to X') \in \{\pm\pi\} \subset K_1(\mathbb{Z}[\pi]),$$

measuring the difference between the two geometric bases. Thus, the image of $\tau(f: X \to Y) \in K_1(\mathbb{Z}[\pi])$ in $\mathrm{Wh}(\pi)$ is just the usual Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi)$.

A *round finite structure* on a CW complex X is an equivalence class of pairs

(round finite CW complex F, homotopy equivalence $f: F \to X$)

under the equivalence relation

$$(F, f) \sim (F', f') \quad \text{if } \tau(f'^{-1}f: F \to F') = 0 \in K_1(\mathbb{Z}[\pi]) \quad (\pi = \pi_1(X)).$$

The *round finite structure* set $\mathscr{F}^r(X)$ of a CW complex X is the set (possibly empty) of round finite structures on X.

PROPOSITION 4.4. (i) *$\mathscr{F}^r(X)$ is nonempty if and only if X is finitely dominated and* $[X] = 0 \in K_0(\mathbb{Z}[\pi])$.

(ii) *If* $\mathscr{F}^r(X)$ *is nonempty there is defined a transitive* $K_1(\mathbb{Z}[\pi])$*-action*

$$K_1(\mathbb{Z}[\pi]) \times \mathscr{F}^r(X) \to \mathscr{F}^r(X);$$
$$(\tau(g: G \to F), (F,f)) \to (G, fg: G \to X).$$

A choice of base point $(F_0, f_0) \in \mathscr{F}^r(X)$ *determines an Abelian group structure on* $\mathscr{F}^r(X)$ *with an isomorphism*

$$\mathscr{F}^r(X) \to K_1(\mathbb{Z}[\pi]); \quad (F,f) \to \tau(f^{-1}f_0: F_0 \to F).$$

(iii) $\mathscr{F}^r(X)$ *is in natural bijective correspondence with the round finite structure set* $\mathscr{F}^r(C(\tilde{X}))$ *of the cellular* $\mathbb{Z}[\pi]$*-module chain complex* $C(\tilde{X})$*. If the sets are nonempty there is defined a natural isomorphism of affine* $K_1(\mathbb{Z}[\pi])$*-sets*

$$\mathscr{F}^r(X) \to \mathscr{F}^r(C(\tilde{X})); (F, f: F \to X) \to (C(\tilde{F}), \tilde{f}: C(\tilde{F}) \to C(\tilde{X})).$$

The product $X \times Y$ of connected CW complexes X, Y is a connected CW complex with fundamental group

$$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y),$$

so that

$$\mathbb{Z}[\pi_1(X \times Y)] = \mathbb{Z}[\pi_1(X)] \otimes \mathbb{Z}[\pi_1(Y)].$$

The universal cover of $X \times Y$ is the product $\tilde{X} \times \tilde{Y}$ of the universal covers $\tilde{X}, \tilde{Y}$ of X, Y, with cellular chain complex over $\mathbb{Z}[\pi_1(X \times Y)]$

$$C(\tilde{X} \times \tilde{Y}) = C(\tilde{X}) \otimes C(\tilde{Y}).$$

The product formulae obtained for chain complexes in Section 3 above can thus be translated directly into product formulae for CW complexes.

PROPOSITION 4.5. (i) (Gersten [7], Siebenmann [18]) *The product of finitely dominated CW complexes* X, Y *is a finitely dominated CW complex* $X \times Y$ *with projective class*

$$[X \times Y] = [X] \otimes [Y] \in K_0(\mathbb{Z}[\pi_1(X \times Y)]).$$

(ii) (Kwun and Szczarba [10]) *The Whitehead torsion of the product* $f \times g: X \times Y \to X' \times Y'$ *of homotopy equivalences of finite CW complexes* $f: X \to X'$, $g: Y \to Y'$ *is given by*

$$\tau(f \times g) = \tau(f)\chi(Y) + \chi(X)\tau(g) \in \mathrm{Wh}(\pi_1(X \times Y)).$$

Proof. (i) Immediate from Proposition 4.1(i).
(ii) Immediate from Proposition 4.1(iv). □

In particular, the product $X \times Y$ of a finitely dominated CW complex X and a round finite CW complex Y has projective class

$$[X \times Y] = [X] \otimes [Y] = [X] \otimes \chi(Y) = [X] \otimes 0 = 0 \in K_0(\mathbb{Z}[\pi_1(X \times Y)]),$$

so that $X \times Y$ is homotopy equivalent to a round finite CW complex. More precisely:

PROPOSITION 4.6. (i) *The product $X \times Y$ of a finitely dominated CW complex X and a round finite CW complex Y has a canonical product round finite structure.*

(ii) *The product $f \times g: X \times Y \to X' \times Y'$ of a homotopy equivalence $f: X \to X'$ of finitely dominated CW complexes and a homotopy equivalence $g: Y \to Y'$ of round finite CW complexes is a homotopy equivalence of CW complexes with canonical round finite structures. The torsion of $f \times g$ with respect to the canonical round finite structures is the product*

$$\tau(f \times g) = [X] \otimes \tau(g) \in K_1(\mathbb{Z}[\pi_1(X \times Y)])$$

of the projective class $[X] = [X'] \in K_0(\mathbb{Z}[\pi_1(X)])$ and the torsion $\tau(g) \in K_1(\mathbb{Z}[\pi_1(Y)])$.

Proof. Immediate from Propositions 3.2, 4.4. □

The case $Y = S^1$ of Proposition 4.6 is particularly interesting, and will be dealt with separately in Section 5 below.

In the special case when $f: X \to X'$ is a homotopy equivalence of finite CW complexes the product formula of Proposition 4.6(ii) agrees with the product formula $\tau(f \times g) = \chi(X)\tau(g) \in \mathrm{Wh}(\pi_1(X \times Y))$ given by Proposition 4.5(ii).

In the special case when $f: X \to X = X'$, $g: Y \to Y = Y'$ are self-homotopy equivalences such that $f_* = 1: \pi_1(X) \to \pi_1(X)$, $g_* = 1: \pi_1(Y) \to \pi_1(Y)$ the product formula of Proposition 4.6(ii) agrees with the product formula for the torsion of self-homotopy equivalences obtained by Gersten [8], which we shall recall in Proposition 5.2 below.

Given a map $\phi: X \to Y$ of finitely dominated CW complexes, let α denote the induced morphism of fundamental groups

$$\alpha = \phi_*: \pi_1(X) \to \pi_1(Y),$$

and let $A = \mathbb{Z}[\pi_1(Y)]$, so that there is induced a chain map of finitely dominated chain complexes over A

$$\tilde{\phi}: \alpha_! C(\tilde{X}) \to C(\tilde{Y})$$

with $\tilde{X}, \tilde{Y}$ the universal covers of X, Y. If $f: A \to B$ is a ring morphism such that $\phi: X \to Y$ is a B-coefficient homology equivalence, then by the construction of Section 2 there is defined an invariant

$$(X, Y, \phi) = (\alpha_! C(\tilde{X}), C(\tilde{Y}), f_! \tilde{\phi}) \in K_1(f)$$

with image

$$\partial(X, Y, \phi) = [Y] - \alpha_![X] \in K_0(A).$$

If $\phi: X \to X = Y$ is such that $\alpha = 1: \pi_1(X) \to \pi_1(X) = \pi_1(Y)$ there is defined an element $\tau(f_! \tilde{\phi}: f_! C(\tilde{X}) \to f_! C(\tilde{X})) \in K_1(B)$ (see Section 5 below for details) with image $j\tau(f_! \tilde{\phi}) = (X, X, \phi) \in K_1(f)$.

EXAMPLE. Let $f\colon \mathbb{Z}[z,z^{-1}] \to P^{-1}\mathbb{Z}[z,z^{-1}]$ be the localization map inverting the multiplicative subset $P = \{p(z) \in \mathbb{Z}[z,z^{-1}] \mid p(1) = \pm 1 \in \mathbb{Z}\}$ of $\mathbb{Z}[z,z^{-1}]$. This has the property that a finite chain complex C over $\mathbb{Z}[z,z^{-1}]$ is such that $f_! C = P^{-1}C$ is contractible if and only if $\mathbb{Z} \otimes_{\mathbb{Z}[z,z^{-1}]} C$ is contractible (see Proposition 7.9.2 of Ranicki [14], for a proof). For any locally flat n-knot $k\colon S^n \subset S^{n+2}$ the knot complement

$$X = \text{closure of } (S^{n+2} - \text{regular neighbourhood of } k(S^n))$$

is such that the generator $1 \in H^1(X) = [X, S^1] = \mathbb{Z}$ is represented by a $\mathbb{Z}$-coefficient homology equivalence $\phi\colon X \to S^1$. The element

$$(X, S^1, \phi) \in K_1(f) = \operatorname{coker}(f_!\colon K_1(\mathbb{Z}[z,z^{-1}]) \to K_1(P^{-1}\mathbb{Z}[z,z^{-1}]))$$

is the Reidemeister torsion of the knot k. □

5. The Torsion of a Self Equivalence

We shall now compare the notion of torsion $\tau(f) \in K_1(A)$ defined in Section 1 for a chain equivalence $f\colon C \to D$ of round finite chain complexes over A with the torsion $\tau(f) \in K_1(A)$ defined by Gersten [8] for a self-chain equivalence $f\colon C \to C$ of a finitely dominated chain complex C over A. This was applied in [8] to define the absolute torsion $\tau(f) \in K_1(\mathbb{Z}[\pi_1(X)])$ of a self-homotopy equivalence $f\colon X \to X$ of a finitely dominated CW complex X such that $f_* = 1\colon \pi_1(X) \to \pi_1(X)$. In Section 6 we shall need to deal with self-homotopy equivalences $f\colon X \to X$ (notably $-1\colon S^1 \to S^1$) such that $f_* \neq 1$, so we shall consider the general case here.

In dealing with self-chain equivalences it is convenient to modify the sign conventions for the algebraic mapping cone. The *modified algebraic mapping cone* $\hat{C}(f)$ of an A-module chain map $f\colon C \to D$ is the A-module chain complex defined by

$$d_{\hat{C}(f)} = \begin{pmatrix} -d_C & 0 \\ f & d_D \end{pmatrix}\colon$$
$$\hat{C}(f)_r = C_{r-1} \oplus D_r \to \hat{C}(f)_{r-1} = C_{r-2} \oplus D_{r-1} \quad (r \in \mathbb{Z}).$$

PROPOSITION 5.1. (i) *The modified algebraic mapping cone $\hat{C}(f)$ of a chain equivalence $f\colon C \to D$ of finite chain complexes over A is a contractible finite chain complex over A such that*

$$\tau(\hat{C}(f)) - \tau(C(f))$$
$$= (\chi_{\text{odd}}(C) + \Sigma_r \operatorname{rank}_A(C_{r-1})\operatorname{rank}_A(D_r))\tau(-1\colon A \to A) \in K_1(A).$$

(ii) *For any chain equivalences $f\colon C \to D$, $g\colon D \to E$ of finite chain complexes over A the composite chain equivalence $gf\colon C \to E$ is such that*

$$\tau(\hat{C}(gf))$$
$$= \tau(\hat{C}(f)) + \tau(\hat{C}(g)) + \beta(SC \oplus SD, D \oplus E)\tau(-1\colon A \to A) \in K_1(A).$$

(iii) *For any chain equivalences $f\colon C \to D$, $f'\colon C' \to D'$ of finite chain complexes over*

A the sum chain equivalence $f \oplus f' : C \oplus C' \to D \oplus D'$ *is such that*

$$\begin{aligned}\tau(\hat{C}(f \oplus f'))\\ &= \tau(\hat{C}(f)) + \tau(\hat{C}(f')) + (\beta(D \oplus SC, D' \oplus SC') + \\ &\quad + \Sigma_r \operatorname{rank}_A(C'_{r-1})\operatorname{rank}_A D_r)\tau(-1 : A \to A) \in K_1(A).\end{aligned}$$

Proof. (i) Apply Proposition 2.2 of Part I to the isomorphism of contractible finite chain complexes

$$g : C(f) \to \hat{C}(f)$$

defined by

$$g = \begin{pmatrix} 0 & (-)^{r-1} \\ 1 & 0 \end{pmatrix} : C(f)_r = D_r \oplus C_{r-1} \to \hat{C}(f)_r = C_{r-1} \oplus D_r \quad (r \geqslant 0).$$

(ii) and (iii) Translate the formulae of Proposition 1.2 (i) and (ii) using (i) above. □

It follows from the formulae of Proposition 5.1 that for any finite chain complex C over A

$$\tau(\hat{C}(1 : C \to C)) = 0 \in K_1(A),$$

and that for any chain equivalence $f : C \to D$ of round finite chain complexes over A

$$\tau(f) = \tau(\hat{C}(f)) + \beta(SC, C \oplus D)\tau(-1 : A \to A) \in K_1(A).$$

In particular, for a self-chain equivalence $f : C \to D = C$ of a round finite chain complex C over A the sign term vanishes and

$$\tau(f) = \tau(\hat{C}(f)) \in K_1(A).$$

Following Gersten [8] define the *torsion* of a self-chain equivalence $f : C \to C$ of a finitely dominated chain complex C over A by

$$\tau(f) = \tau(\hat{C}(e)) \in K_1(A)$$

with e the composite self-chain equivalence of a finite chain complex D over A given by

$$e : D \xrightarrow{i} C \oplus C' \xrightarrow{f \oplus 1} C \oplus C' \xrightarrow{i^{-1}} D$$

for any finite chain complex D such that there exists a chain equivalence $i : D \to C \oplus C'$ with C' a finitely dominated chain complex, and any such i. (For example, if (D', f', g', h') is a finite domination of C, then $D = D'$ is such a finite chain complex, with $C' = C(f' : C \to D)$ a finitely dominated chain complex and

$$i = \begin{pmatrix} g' \\ e' \end{pmatrix} : D \to C \oplus C'$$

a chain equivalence, where $e'\colon D \to C'$ is the inclusion.) If C is a finite chain complex it is possible to choose $C' = 0$, $i = 1\colon D = C \to C$, so that $e = f\colon C \to C$ and

$$\tau(f) = \tau(\hat{C}(f)) \in K_1(A).$$

Note that $\tau(f) \in K_1(A)$ is independent of the base in C. Also, if C is round finite this is the torsion $\tau(f) \in K_1(A)$ previously defined in Section 1, by the argument above. The torsion of an automorphism $f\colon C \to C$ of a bounded f.g. projective chain complex C over A is given by

$$\tau(f) = \Sigma_{r=0}^{\infty} (-)^r \tau(f\colon C_r \to C_r) \in K_1(A).$$

Still following [8] define the *torsion* of a self-homotopy equivalence $f\colon X \to X$ of a finitely dominated CW complex X inducing $f_* = 1\colon \pi_1(X) \to \pi_1(X)$ by

$$\tau(f) = \tau(\tilde{f}\colon C(\tilde{X}) \to C(\tilde{X})) \in K_1(\mathbb{Z}[\pi_1(X)]),$$

with $\tilde{f}\colon C(\tilde{X}) \to C(\tilde{X})$ the induced self-chain equivalence of the finitely dominated cellular chain complex $C(\tilde{X})$ over $\mathbb{Z}[\pi_1(X)]$ of the universal cover $\tilde{X}$.

PROPOSITION 5.2 (Gersten [8]). (i) *The torsion of self chain equivalences of finitely dominated chain complexes over A is logarithmic and additive, with*

$$\tau(gf\colon C \to C) = \tau(f\colon C \to C) + \tau(g\colon C \to C) \in K_1(A),$$
$$\tau(f \oplus f'\colon C \oplus C' \to C \oplus C') = \tau(f\colon C \to C) + \tau(f'\colon C' \to C') \in K_1(A).$$

(ii) *The product $f \otimes g\colon C \otimes D \to C \otimes D$ of self-chain equivalences $f\colon C \to C$, $g\colon D \to D$ of finitely dominated chain complexes C, D over A, B (respectively) is a self-chain equivalence of a finitely dominated chain complex $C \otimes D$ over $A \otimes B$ with torsion*

$$\tau(f \otimes g) = [C] \otimes \tau(g) + \tau(f) \otimes [D] \in K_1(A \otimes B).$$

(iii) *The product $f \times g\colon X \times Y \to X \times Y$ of self-homotopy equivalences $f\colon X \to X$, $g\colon Y \to Y$ of finitely dominated CW complexes X, Y such that $f_* = 1\colon \pi_1(X) \to \pi_1(X)$, $g_* = 1\colon \pi_1(Y) \to \pi_1(Y)$ is a self homotopy equivalence of a finitely dominated CW complex $X \times Y$ such that*

$$(f \times g)_* = f_* \times g_* = 1\colon \pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y) \to \pi_1(X) \times \pi_1(Y),$$

with torsion

$$\tau(f \otimes g) = [X] \otimes \tau(g) + \tau(f) \otimes [Y] \in K_1(\mathbb{Z}[\pi_1(X \times Y)]). \qquad \square$$

A self-homotopy equivalence $f\colon X \to X$ of a finitely dominated CW complex X induces an automorphism of the fundamental group

$$f_* = \alpha\colon \pi_1(X) = \pi \to \pi$$

and, hence, a chain equivalence of finitely dominated chain complexes over $\mathbb{Z}[\pi]$

$$\tilde{f}\colon \alpha_! C(\tilde{X}) \to C(\tilde{X}).$$

If $[X] = 0 \in K_0(\mathbb{Z}[\pi])$ and there is given a round finite structure $(F, \phi) \in \mathscr{F}^r(C(\tilde{X}))$

$(=\mathscr{F}^r(X)$, by definition) there is defined a torsion

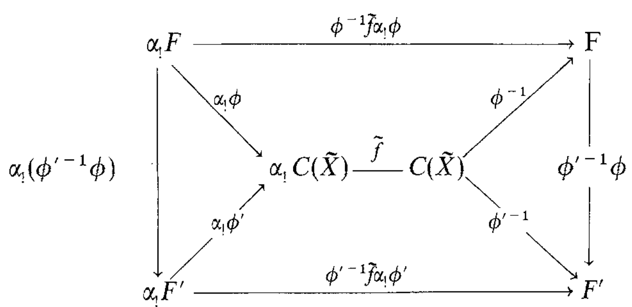
$$\tau_{(F,\phi)}(f) = \tau(\phi^{-1}\tilde{f}\alpha_! \phi : \alpha_! F \xrightarrow{\alpha_! \phi} \alpha_! C(\tilde{X}) \xrightarrow{\tilde{f}} C(\tilde{X}) \xrightarrow{\phi^{-1}} F) \in K_1(\mathbb{Z}[\pi]).$$

However, if $\alpha \neq 1$ this will in general depend on the choice of round finite structure (F, ϕ):

PROPOSITION 5.3. *The torsions associated to two different round finite structures* (F, ϕ), $(F', \phi') \in \mathscr{F}^r(X)$ *differ by*

$$\tau_{(F',\phi')}(f) - \tau_{(F,\phi)}(f) = (1 - \alpha_!)\tau(\phi'^{-1}\phi : F \to F') \in K_1(\mathbb{Z}[\pi]).$$

Proof. Consider the commutative diagram of chain complexes over $\mathbb{Z}[\pi]$ and chain equivalences



and apply the logarithmic property of torsion to the chain equivalences of round finite chain complexes on the outside. □

Given a ring A and an automorphism $\alpha: A \to A$ denote the relative K_1-group $K_1(1: A \to A, \alpha: A \to A)$ of Section 2 by $K_1(A, \alpha)$, so that there is defined an exact sequence

$$K_1(A) \xrightarrow{1-\alpha_!} K_1(A) \xrightarrow{j} K_1(A, \alpha) \xrightarrow{\partial} K_0(A) \xrightarrow{1-\alpha_!} K_0(A).$$

$K_1(A, \alpha)$ is isomorphic to the relative K_1-group defined by Siebenmann [19].

Given a finitely dominated CW complex X and a self-homotopy equivalence $f: X \to X$, let $\alpha: A \to A$ be the automorphism of the group ring $A = \mathbb{Z}[\pi_1(X)]$ induced by $f_*: \pi_1(X) \to \pi_1(X)$. Applying the construction of Section 2 to the induced chain equivalence of finitely dominated chain complexes over A

$$\tilde{f}: \alpha_! C(\tilde{X}) \to C(\tilde{X})$$

there is defined an element

$$(X, f) = (C(\tilde{X}), \tilde{f}) \in K_1(A, \alpha)$$

such that

$$\partial(X, f) = [X] \in K_0(A).$$

If $[X] = 0 \in K_0(A)$ a choice of round finite structure $(F, \phi) \in \mathscr{F}^r(X) = \mathscr{F}^r(C(\tilde{X}))$ determines an element $\tau_{(F,\phi)}(f) \in K_1(A)$ such that

$$j\tau_{(F,\phi)}(f) = (X, f) \in K_1(A, \alpha).$$

Proposition 5.3 describes the effect on $\tau_{(F,\phi)}(f) \in K_1(A)$ of a different choice of round finite structure, in precise accordance with the identity

$$\mathrm{im}(1 - \alpha_!: K_1(A) \to K_1(A)) = \ker(j: K_1(A) \to K_1(A, \alpha))$$

given by the above exact sequence.

For $\alpha = 1: A \to A$ there is defined a natural isomorphism

$$K_1(A, 1) \to K_1(A) \oplus K_0(A); \quad (P, Q, [\phi]: P \oplus Q \to P \oplus Q) \to (\tau([\phi]), [P] - [Q])$$

If $f: X \to X$ is a self homotopy equivalence of a finitely dominated CW complex X such that $f_* = 1: \pi_1(X) \to \pi_1(X)$ and $A = \mathbb{Z}[\pi_1(X)]$ the element $(X, f) \in K_1(A, 1)$ has image $(\tau(f), [X]) \in K_1(A) \oplus K_0(A)$ under this isomorphism, with $\tau(f) \in K_1(A)$ the torsion defined by Gersten [8].

The circle $S^1 = [0, 1]/(0 = 1) = e^0 \cup e^1$ is a finite CW complex such that $\chi(S^1) = 0 \in \mathbb{Z}$, with fundamental group $\pi_1(S^1) = \mathbb{Z}$ and universal cover $\tilde{S}^1 = \mathbb{R}$. Let z be the generator

$$z = (1: S^1 \to S^1) \in \pi_1(S^1),$$

so that $\pi_1(S^1) = \{z^n \mid n \in \mathbb{Z}\}$ and there is a natural identification of $\mathbb{Z}[\pi_1(S^1)]$ with the Laurent polynomial extension ring of $\mathbb{Z}$

$$\mathbb{Z}[\pi_1(S^1)] = \mathbb{Z}[z, z^{-1}].$$

Define the *canonical round finite structure* $\Sigma^1 = (D, \omega) \in \mathscr{F}^r(S^1)$ by $\omega = 1: D = C(\tilde{S}^1) \to C(\tilde{S}^1)$, with

$$D = C(\tilde{S}^1): \mathbb{Z}[z, z^{-1}] \xrightarrow{1 - z} \mathbb{Z}[z, z^{-1}].$$

The geometric base elements are oriented lifts $\tilde{e}^0, \tilde{e}^1 \subset \tilde{S}^1$ of the cells $e^0, e^1 \subset S^1$ such that $\tilde{e}^0 \subset \tilde{e}^1$.

The tensor product of a ring A and $\mathbb{Z}[z, z^{-1}]$ is the Laurent polynomial ring of A

$$A \otimes \mathbb{Z}[z, z^{-1}] = A[z, z^{-1}].$$

The tensor product of a chain complex C over A and $D = C(\tilde{S}^1)$ is the modified algebraic mapping cone chain complex over $A[z, z^{-1}]$

$$C \otimes D = \hat{C}(1 - z\colon C[z, z^{-1}] \to C[z, z^{-1}]).$$

For finite C this is an identity of round finite chain complexes. For finitely dominated C Proposition 3.2 gives the *canonical product round finite structure*

$$C \otimes \Sigma^1 = (F, \phi) \in \mathscr{F}^r(C \otimes D)$$

as defined by

$$\phi\colon F = (E, p) \otimes D \xrightarrow{\text{projection}} \operatorname{im}(p) \otimes D \xrightarrow{\theta \otimes 1} C \otimes D$$

for any projection $p = p^2\colon E \to E$ of an even finite chain complex E over A with a chain equivalence $\theta\colon \operatorname{im}(p) \simeq C$, and with

$$d_F = \begin{pmatrix} -d_E \otimes 1 & 0 \\ p \otimes d_D + (1 - p) \otimes d_\Delta & d_E \otimes 1 \end{pmatrix}\colon$$
$$F_{r+1} = E_r \otimes D_1 \otimes E_{r+1} \otimes D_0 \to F_r = E_{r-1} \otimes D_1 \otimes E_r \otimes D_0$$

for any differential $d_\Delta \in \operatorname{Hom}_{\mathbb{Z}[z, z^{-1}]}(D_1, D_0)$ such that D_Δ is a contractible finite chain complex over $\mathbb{Z}[z, z^{-1}]$ with

$$\tau(D_\Delta) = 0 \in K_1(\mathbb{Z}[z, z^{-1}]).$$

Making the obvious choice

$$d_\Delta = 1\colon D_1 = \mathbb{Z}[z, z^{-1}] \to D_0 = \mathbb{Z}[z, z^{-1}]$$

note that

$$\begin{aligned} p \otimes d_D &+ (1 - p) \otimes d_\Delta \\ &= (1 - z)p + (1 - p) \\ &= 1 - zp \\ &\colon E_r \otimes D_1 = E_r[z, z^{-1}] \to E_r \otimes D_0 = E_r[z, z^{-1}], \end{aligned}$$

and so

$$F = \hat{C}(1 - zp\colon E[z, z^{-1}] \to E[z, z^{-1}]),$$

with

$$\begin{aligned} \phi = \begin{pmatrix} \theta p & 0 \\ 0 & \theta p \end{pmatrix}\colon F_r = E_{r-1}[z, z^{-1}] \oplus E_r[z, z^{-1}] &\to (C \otimes D)_r \\ = C_{r-1}[z, z^{-1}] \oplus C_r[z, z^{-1}]. \end{aligned}$$

For any connected CW complex X the product CW complex $X \times S^1$ has fundamental group

$$\pi_1(X \times S^1) = \pi_1(X) \times \mathbb{Z}$$

and there is a natural identification of rings

$$\mathbb{Z}[\pi_1(X \times S^1)] = \mathbb{Z}[\pi_1(X)][z, z^{-1}],$$

so that the cellular chain complex of the universal cover $\widetilde{X \times S^1} = \tilde{X} \times \tilde{S}^1$ can be expressed as

$$\begin{aligned} C(\widetilde{X \times S^1}) &= C(\tilde{X} \times \tilde{S}^1) = C(\tilde{X}) \otimes C(\tilde{S}^1) \\ &= \hat{C}(1 - z\colon C(\tilde{X})[z, z^{-1}] \to C(\tilde{X})[z, z^{-1}]). \end{aligned}$$

For finitely dominated X define the *canonical round finite structure* on $X \times S^1$ by

$$X \times \Sigma^1 = C(\tilde{X}) \otimes (D, \omega) \in \mathscr{F}^r(X \times S^1) = \mathscr{F}^r(C(\widetilde{X \times S^1})).$$

In Section 6 below we shall identify the reduction of $X \times \Sigma^1$ in $\mathscr{F}(X \times S^1)$ with the canonical finite structure defined geometrically on $X \times S^1$ by Mather [12] and Ferry [6].

The self-homeomorphism of $S^1 = [0, 1]/(0 = 1)$

$$-1\colon S^1 \to S^1; \quad s \mapsto 1 - s \quad (0 \leqslant s \leqslant 1)$$

is such that

$$(-1\colon S^1 \to S^1) = z^{-1} \in \pi_1(S^1),$$

and induces the automorphism

$$(-1)_* = \alpha\colon \pi_1(S^1) \to \pi_1(S^1); \quad z^n \mapsto z^{-n}.$$

PROPOSITION 5.8. (i) *The torsion of* $-1\colon S^1 \to S^1$ *with respect to the canonical round finite structure* $\Sigma^1 \in \mathscr{F}^r(S^1)$ *is given by*

$$\tau_{\Sigma^1}(-1) = \tau(-z\colon \mathbb{Z}[z, z^{-1}] \to \mathbb{Z}[z, z^{-1}]) \in K_1(\mathbb{Z}[z, z^{-1}]).$$

(ii) *If* X *is a finitely dominated CW complex the torsion of* $1 \times -1\colon X \times S^1 \to X \times S^1$ *with respect to the canonical round finite structure* $X \times \Sigma^1 \in \mathscr{F}^r(X \times S^1)$ *is given by*

$$\tau_{X \times \Sigma^1}(1 \times -1) = [X] \otimes \tau(-z) \in K_1(\mathbb{Z}[\pi_1(X)][z, z^{-1}]).$$

Proof. (i) The induced chain equivalence $(\widetilde{-1})\colon \alpha_! D \to D$ is the isomorphism of round finite chain complexes over $\mathbb{Z}[z, z^{-1}]$

$$\begin{array}{ccc} \alpha_! D_1 = \mathbb{Z}[z, z^{-1}] & \xrightarrow{\;1\;} & D_1 = \mathbb{Z}[z, z^{-1}] \\ \Big\downarrow {\scriptstyle \alpha_! d_D = 1 - z^{-1}} & & \Big\downarrow {\scriptstyle d_D = 1 - z} \\ \alpha_! D_0 = \mathbb{Z}[z, z^{-1}] & \xrightarrow{\;-z\;} & D_0 = \mathbb{Z}[z, z^{-1}]. \end{array}$$

A direct application of Proposition 2.7 (iii) of Part I gives

$$\begin{aligned}&\tau_{\Sigma^1}(-1: S^1 \to S^1)\\&= \tau(-z: \mathbb{Z}[z, z^{-1}] \to \mathbb{Z}[z, z^{-1}]) - \tau(1: \mathbb{Z}[z, z^{-1}] \to \mathbb{Z}[z, z^{-1}])\\&= \tau(-z) \in K_1(\mathbb{Z}[z, z^{-1}]).\end{aligned}$$

(ii) Substituting the result of (i) in the product formula of Proposition 4.6 (ii)

$$\begin{aligned}&\tau_{X \times \Sigma^1}(1 \times -1: X \times S^1 \to X \times S^1)\\&= [X] \otimes \tau_{\Sigma}1(-1: S^1 \to S^1)\\&= [X] \otimes \tau(-z) \in K_1(\mathbb{Z}[\pi_1(X)][z, z^{-1}]).\end{aligned}$$ □

A noncanonical round finite structure $(D', \omega') \in \mathscr{F}^r(S^1)$ differs from the canonical structure $\Sigma^1 = (D, \omega)$ by

$$\begin{aligned}&(D, \omega) - (D', \omega')\\&= \tau(\omega'^{-1}\omega: D \to D') \in K_1(\mathbb{Z}[z, z^{-1}])\\&= \{\tau(\pm z^n: \mathbb{Z}[z, z^{-1}] \to \mathbb{Z}[z, z^{-1}]) \mid n \in \mathbb{Z}\} \quad (= \mathbb{Z} \oplus \mathbb{Z}_2),\end{aligned}$$

say $(D, \omega) - (D', \omega') = \tau(\pm z^n)$. The torsion of $-1: S^1 \to S^1$ with respect to (D', ω') is given by Propositions 5.3 and 5.4 to be

$$\begin{aligned}\tau_{(D',\omega')}(-1) &= \tau_{(D,\omega)}(-1) + (1 - \alpha_!)\tau(\omega'^{-1}\omega)\\&= \tau(-z) + (1 - \alpha_!)\tau(\pm z^n)\\&= \tau(-z^{2n+1}) \in K_1(\mathbb{Z}[z, z^{-1}]).\end{aligned}$$

It follows that for any finitely dominated CW complex X

$$\begin{aligned}&\tau_{X \otimes (D',\omega')}(1 \times -1: X \times S^1 \to X \times S^1) -\\&\quad -\tau_{X \otimes (D,\omega)}(1 \times -1: X \times S^1 \to X \times S^1)\\&= [X] \otimes (\tau_{(D',\omega')}(-1) - \tau_{(D,\omega)}(-1))\\&= [X] \otimes \tau(z^{2n}) \in K_1(\mathbb{Z}[\pi_1(X)][z, z^{-1}]).\end{aligned}$$

6. The Mapping Torus in Algebra and Topology

Actually, we shall start with the topology.

The *mapping torus* of a map $f: X \to X$ of a space X to itself is the identification space

$$T(f) = X \times [0, 1]/\{(x, 0) = (f(x), 1) \mid x \in X\}.$$

PROPOSITION 6.1. (i) *A homotopy $e: f \simeq f': X \to X$ induces a homotopy equivalence*

$$S(e): T(f) \to T(f').$$

(ii) *For any maps* $f: X \to Y$, $g: Y \to X$ *the maps*

$$S(f, g): T(gf: X \to X) \to T(fg: Y \to Y); (x, s) \mapsto (f(x), s)$$
$$S(g, f): T(fg: Y \to Y) \to T(gf: X \to X); (y, t) \to (g(y), t)$$

are inverse homotopy equivalences.

Proof. (i) Regard the mapping torus of $f: X \to X$ as the adjunction space

$$T(f) = (X \times [0, \tfrac{1}{2}]) \cup_g (X \times [\tfrac{1}{2}, 1]),$$

with the adjunction map defined by

$$g: X \times \{0, \tfrac{1}{2}\} \to X \times [\tfrac{1}{2}, 1];$$
$$(x, 0) \mapsto (f(x), 1), \quad (x, \tfrac{1}{2}) \mapsto (x, \tfrac{1}{2}).$$

A homotopy $e: f \simeq f': X \to X$ determines a homotopy of adjunction maps

$$h: g \simeq g': X \times \{0, \tfrac{1}{2}\} \to X \times [\tfrac{1}{2}, 1]$$

and, hence, a homotopy equivalence of the adjunction spaces

$$\begin{aligned} S(e) = 1 \cup_h 1: T(f) &= (X \times [0, \tfrac{1}{2}]) \cup_g (X \times [\tfrac{1}{2}, 1]) \\ \xrightarrow{\simeq} T(f') &= (X \times [0, \tfrac{1}{2}]) \cup_{g'} (X \times [\tfrac{1}{2}, 1]), \end{aligned}$$

since the pair $(X \times [0, \tfrac{1}{2}], X \times \{0, \tfrac{1}{2}\})$ has the homotopy extension property. There is no direct formula for $S(e)$, which is only determined up to homotopy.

(ii) Given a map $f: X \to X$ define a map

$$U(f): T(f) \to T(f); \quad [x, t] \mapsto [f(x), t]$$

and a homotopy

$$e: U(f) \simeq 1: T(f) \to T(f)$$

by

$$e: T(f) \times I \to T(f); \quad ([x, s], t) \mapsto \begin{cases} [f(x), s + t] & \text{if} \quad s + t \leqslant 1 \\ [x, s + t - 1] & \text{if} \quad s + t \geqslant 1 \end{cases}$$
$$(s, t \in I = [0, 1]).$$

Now for any maps $f: X \to Y$, $g: Y \to X$ the composites of

$$S(f, g): T(gf) \to T(fg), \qquad S(g, f): T(fg) \to T(gf)$$

are given by

$$S(f, g)S(g, f) = U(fg): T(fg) \to T(fg)$$
$$S(g, f)S(f, g) = U(gf): T(gf) \to T(gf),$$

so that $S(f, g)$ and $S(g, f)$ are inverse homotopy equivalences. □

We shall only be concerned with the mapping torus $T(f)$ when X is a CW complex and $f: X \to X$ is a cellular map, so that $T(f)$ is a CW complex with two r-cells $e^r \times \{0\}$, $e^r \times \{\frac{1}{2}\}$ and two $(r+1)$-cells $e^r \times [0, \frac{1}{2}]$, $e^r \times [\frac{1}{2}, 1]$ for each r-cell e^r of X. If X is a finite CW complex, then $T(f)$ is a finite CW complex such that $\chi(T(f)) = 0 \in \mathbb{Z}$, and so admits a round finite structure. We shall show that for any (cellular) map $f: X \to X$ of a finitely dominated CW complex X the mapping torus $T(f)$ has a canonical round finite structure.

PROPOSITION 6.2. (i) *For a finite CW complex X a homotopy $e: f \simeq f': X \to X$ induces a homotopy equivalence $S(e): T(f) \to T(f')$ of finite CW complexes which is simple, that is*

$$\tau(S(e)) = 0 \in \mathrm{Wh}(\pi_1(T(f))).$$

(ii) *For finite CW complexes X, Y and maps $f: X \to Y$, $g: Y \to X$ the homotopy equivalence $S(f, g): T(gf) \to T(fg)$ of finite CW complexes is simple, that is*

$$\tau(S(f, g)) = 0 \in \mathrm{Wh}(\pi_1(T(gf))).$$

Proof. This may be deduced from the material on mapping cylinders and deformations in Section 5 of Cohen [4]. □

Given a finitely dominated CW complex X and a map $\zeta: X \to X$ define a finite structure $(T(f\zeta g), \phi) \in \mathscr{F}(T(\zeta))$ for any finite domination $(Y, f: X \to Y, g: Y \to X, h: gf \simeq 1: X \to X)$ of X by

$$\begin{aligned} \phi = {} & S(\zeta h)S(\zeta g, f): \\ & T(f\zeta g: Y \to Y) \to T(\zeta g f: X \to X) \to T(\zeta: X \to X). \end{aligned}$$

PROPOSITION 6.3. *The finite structure $(T(f\zeta g), \phi) \in \mathscr{F}(T(\zeta))$ is independent of the choice of finite domination (Y, f, g, h) of X.*

Proof. The finite structures $(T(f\zeta g), \phi)$, $T(f'\zeta g'), \phi')$ on $T(\zeta)$ determined by any two finite dominations (Y, f, g, h), (Y', f', g', h') of X are such that up to homotopy

$$\begin{aligned} \phi'^{-1}\phi = {} & S(f'\zeta hg')^{-1}S(f'\zeta g, fg')S(fh'\zeta g): \\ & T(f\zeta g) \to T(fg'f'\zeta g) \to T(f'\zeta g f g') \to T(f'\zeta g'), \end{aligned}$$

a composite of simple homotopy equivalences by Proposition 6.2. It follows that $\tau(\phi'^{-1}\phi) = 0 \in \mathrm{Wh}(\pi_1(T(\zeta)))$, and so

$$(T(f\zeta g), \phi) = (T(f'\zeta g'), \phi') \in \mathscr{F}(T(\zeta)). \qquad \square$$

Call $(T(f\zeta g), \phi) \in \mathscr{F}(T(\zeta))$ the *canonical finite structure* on $T(\zeta)$. In the case $\zeta = 1: X \to X$ this is the finite structure on $T(1) = X \times S^1$ defined by Mather [12] and Ferry [6].

EXAMPLE. Let M be a compact n-manifold with a finitely dominated infinite cyclic cover $\bar{M}$, and let $\zeta\colon \bar{M} \to \bar{M}$ be a generating covering translation. Then the projection $p\colon \bar{M} \to M$ induces a homotopy equivalence of CW complexes with finite structure

$$q\colon T(\zeta) \to M; \quad (x, s) \mapsto p(x)$$

such that $\tau(q) \in \mathrm{Wh}(\pi_1(M))$ is the obstruction of Farrell [5] and Siebenmann [19] to fibering M over S^1 (assuming $n \geqslant 6$). □

In order to compare the geometrically defined canonical finite structure on $X \times S^1$ with the algebraically defined canonical round finite structure of Section 5, we shall use the following algebraic analogue of the mapping torus.

Given a ring A and a morphism $\alpha\colon A \to A$ define the *α-twisted polynomial ring of* A, $A *_\alpha[z, z^{-1}]$ to be the quotient ring of the free product $A * \mathbb{Z}[z, z^{-1}]$ given by

$$A *_\alpha[z, z^{-1}] = A * \mathbb{Z}[z, z^{-1}]/\{z^{-1}az = \alpha(a) | a \varepsilon A\}.$$

There is defined a morphism of rings

$$i\colon A \to A *_\alpha[z, z^{-1}]; \quad a \mapsto a$$

under which α becomes conjugation by z, which is injective if and only if α is injective. If $\alpha\colon A \to A$ is an automorphism $A *_\alpha[z, z^{-1}] = A_\alpha[z, z^{-1}]$ is the usual α-twisted polynomial extension ring of A, which in the untwisted case $\alpha = 1\colon A \to A$ is the Laurent polynomial extension ring $A[z, z^{-1}]$.

Let then A be a ring, $\alpha\colon A \to A$ a ring morphism, and for some chain complex C over A let $f\colon \alpha_! C \to C$ be a chain map. The *algebraic mapping torus* of f is the chain complex over $A *_\alpha[z, z^{-1}]$ defined by

$$T(f) = \hat{C}(1 - zf\colon i_! C \to i_! C),$$

using the modified algebraic mapping cone $\hat{C}$ of Section 5

$$\begin{aligned} d_{T(f)} = \begin{pmatrix} -d & 0 \\ (1 - zf) & d \end{pmatrix} &: T(f)_r = i_! C_{r-1} \oplus i_! C_r \to T(f)_{r-1} \\ &= i_! C_{r-2} \oplus i_! C_{r-1}. \end{aligned}$$

If C is finite $T(f)$ is round finite. If $\alpha = 1\colon A \to A$ there are natural identifications

$$A *_\alpha[z, z^{-1}] = A[z, z^{-1}] = A \otimes \mathbb{Z}[z, z^{-1}]$$

and for any chain complex C over A

$$T(1\colon C \to C) = C \otimes C(\tilde{S}^1),$$

which for finite C is an identity of round finite chain complexes over $A[z, z^{-1}]$, using the canonical structure $\Sigma^1 \in \mathscr{F}^r(C(\tilde{S}^1))$.

By analogy with Propositions 6.1, 6.2

PROPOSITION 6.4. (i) *A chain homotopy* $e\colon f \simeq f'\colon \alpha_! C \to C$ *induces an isomorphism of the algebraic mapping tori*

$$S(e)\colon T(f) \to T(f').$$

For finite C

$$\tau(S(e)) = 0 \in K_1(A *_\alpha [z, z^{-1}])$$

(ii) *Let* $\alpha\colon A \to B$, $\beta\colon B \to A$ *be morphisms of rings, and let* $f\colon \alpha_! C \to D$, $g\colon \beta_! D \to C$ *be chain maps for some chain complexes* C, D *over* A, B *respectively. Then there are defined an isomorphism of rings*

$$k\colon A *_{\beta\alpha}[z, z^{-1}] \to B *_{\alpha\beta}[z, z^{-1}]$$

and a chain equivalence of chain complexes over $B *_{\alpha\beta}[z, z^{-1}]$

$$S(f,g)\colon k_! T(g\beta_! f\colon (\beta\alpha)_! C \to C)$$
$$\to T(f\alpha_! g\colon (\alpha\beta)_! D \to D).$$

If C, D *are finite*

$$\tau(S(f,g)) = 0 \in K_1(B *_{\alpha\beta}[z, z^{-1}]).$$

Proof. (i) The isomorphism $S(e)\colon T(f) \to T(f')$ is defined by

$$S(e) = \begin{pmatrix} 1 & 0 \\ ze & 1 \end{pmatrix}\colon$$
$$T(f)_r = i_! C_{r-1} \oplus i_! C_r \to T(f')_r = i_! C_{r-1} \oplus i_! C_r.$$

(ii) Let

$$i\colon A \to A *_{\beta\alpha}[z, z^{-1}], \quad j\colon B \to B *_{\alpha\beta}[z, z^{-1}]$$

be the canonical ring morphisms. The isomorphism of polynomial rings

$$k\colon A *_{\beta\alpha}[z, z^{-1}] \to B *_{\alpha\beta}[z, z^{-1}]; \quad a \mapsto \alpha(a),\ z \mapsto z$$

has inverse

$$k^{-1}\colon B *_{\alpha\beta}[z, z^{-1}] \to A *_{\beta\alpha}[z, z^{-1}]; \quad b \mapsto z\beta(b)z^{-1},\ z \mapsto z,$$

and there is defined a commutative square of rings and morphisms

$$\begin{array}{ccc} A & \xrightarrow{i} & A *_{\beta\alpha}[z, z^{-1}] \\ \alpha \big\downarrow & & \big\downarrow k \\ B & \xrightarrow{i} & B *_{\alpha\beta}[z, z^{-1}]. \end{array}$$

The chain maps of chain complexes over $B *_{\alpha\beta}[z, z^{-1}]$

$$S(f, g): k_! T(g\beta_! f) \to T(f\alpha_! g)$$
$$S'(g, f): T(f\alpha_! g) \to k_! T(g\beta_! f)$$

defined by

$$\begin{aligned} S(f, g) = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} &: k_! T(g\beta_! f)_r \\ &= j_! \alpha_! C_{r-1} \oplus j_! \alpha_! C_r \to T(f\alpha_! g)_r \\ &= j_! D_{r-1} \oplus j_! D_r \end{aligned}$$

$$\begin{aligned} S'(g, f) = \begin{pmatrix} z\alpha_! g & 0 \\ 0 & z\alpha_! g \end{pmatrix} &: T(f\alpha_! g)_r \\ &= j_! D_{r-1} \oplus j_! D_r \to k_! T(g\beta_! f)_r \\ &= j_! \alpha_! C_{r-1} \oplus j_! \alpha_! C_r \end{aligned}$$

are chain homotopy inverses, with chain homotopies

$$\begin{aligned} e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &: T(g\beta_! f)_r \\ &= j_! \alpha_! C_{r-1} \oplus j_! \alpha_! C_r \to j_! \alpha_! C_r \oplus j_! \alpha_! C_{r+1}, \end{aligned}$$

$$\begin{aligned} e' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &: T(f\alpha_! g)_r \\ &= j_! D_{r-1} \oplus j_! D_r \to j_! D_r \oplus j_! D_{r+1}. \end{aligned}$$

Define a chain contraction of $C(S(f, g))$

$$\Gamma: 0 \simeq 1: C(S(f, g)) \to C(S(f, g))$$

by

$$\begin{aligned} \Gamma = \begin{pmatrix} e' & 0 \\ (-)^r S'(g, f) & e \end{pmatrix} &: C(S(f, g))_r \\ &= T(f\alpha_! g)_r \oplus k_! T(g\beta_! f)_{r-1} \to T(f\alpha_! g)_{r+1} \oplus k_! T(g\beta_! f)_r. \end{aligned}$$

Thus, if C, D are finite $S(f, g): k_! T(g\beta_! f) \to T(f\alpha_! g)$ is a chain equivalence of round finite chain complexes over $B *_{\alpha\beta}[z, z^{-1}]$ with torsion

$$\begin{aligned} \tau(S(f, g)) &= \tau(C(S(f, g))) \\ &= \tau(d + \Gamma: C(S(f, g))_{\text{odd}} \to C(S(f, g))_{\text{even}}) \end{aligned}$$

$$= \tau\left(\begin{pmatrix} d'+e' & S(f,g) \\ -S'(g,f) & d+e \end{pmatrix}: T(f\alpha_! g)_{\text{odd}} \oplus k_! T(g\beta_! f)_{\text{even}} \right.$$

$$\left. \to T(f\alpha_! g)_{\text{even}} \oplus k_! T(g\beta_! f)_{\text{odd}}\right)$$

$$= \tau\begin{pmatrix} 1 & -d & f & 0 \\ d & 1-zf\alpha_! g & 0 & f \\ -z\alpha_! g & 0 & 1-zg\beta_! f & d \\ 0 & -z\alpha_! g & -d & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ d & 1 & 0 & 0 \\ -z\alpha_! g & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & f \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -z\alpha_! g & -d & 1 \end{pmatrix}\begin{pmatrix} 1 & -d & f & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}:$$

$$: j_! D_{\text{odd}} \oplus j_! D_{\text{even}} \oplus j_! \alpha_! C_{\text{odd}} \oplus j_! \alpha_! C_{\text{even}}$$
$$\to j_! D_{\text{odd}} \oplus j_! D_{\text{even}} \oplus j_! \alpha_! C_{\text{odd}} \oplus j_! \alpha_! C_{\text{even}})$$

$$= 0 \in K_1(B^*_{\alpha\beta}[z,z^{-1}]).$$ □

Given a finitely dominated chain complex C over a ring A and a chain map $\zeta: \alpha_! C \to C$ for some morphism $\alpha: A \to A$ define a round finite structure $(T(f\zeta\alpha_! g), \phi) \in \mathscr{F}^r(T(\zeta))$ for any finite domination $(D, f: C \to D, g: D \to C, h: gf \simeq 1: C \to C)$ of C by

$$\phi = S(\zeta\alpha_! h)S(\zeta\alpha_! g, f):$$
$$T(f\zeta\alpha_! g: \alpha_! D \to D) \to T(\zeta\alpha_!(gf): \alpha_! C \to C)$$
$$\to T(\zeta: \alpha_! C \to C).$$

The round finite structures $(T(f\zeta\alpha_! g), \phi)$, $(T(f'\zeta\alpha_! g'), \phi') \in \mathscr{F}^r(T(\zeta))$ determined by two finite dominations (D, f, g, h), (D', f', g', h') of C are such that up to chain homotopy

$$\phi'^{-1}\phi = S(f'\zeta\alpha_!(hg'))^{-1}S(f'\zeta\alpha_! g, fg')S(fh'\zeta\alpha_! g):$$
$$T(f\zeta\alpha_! g) \to T(fg'f'\zeta\alpha_! g) \to T(f'\zeta\alpha_!(gfg'))$$
$$\to T(f'\zeta\alpha_! g'),$$

a composite of chain equivalences with $\tau = 0 \in K_1(A *_\alpha [z, z^{-1}])$ by Proposition 6.4, and so by analogy with Proposition 6.3

$$\begin{aligned}&(T(f\zeta\alpha_! g), \phi) - (T(f'\zeta\alpha_! g'), \phi')\\ &\quad = \tau(\phi'^{-1}\phi) = 0 \in K_1(A *_\alpha [z, z^{-1}]).\end{aligned}$$

Thus the round finite structure

$$(T(f\zeta\alpha_! g), \phi) = (T(f'\zeta\alpha_! g'), \phi') \in \mathscr{F}^r(T(\zeta))$$

is independent of the finite domination of C; we shall call this the *canonical round finite structure* on $T(\zeta)$. In particular, for $\alpha = 1\colon A \to A$, $\zeta = 1\colon \alpha_! C = C \to C$ we have:

PROPOSITION 6.5. *The canonical round finite structure $(T(fg), \phi) \in \mathscr{F}^r(T(1\colon C \to C))$ on $T(1) = C \otimes C(\tilde{S}^1)$ determined by any finite domination (D, f, g, h) of C coincides with the canonical product round finite structure*

$$(T(fg), \phi) = C \otimes \Sigma^1 \in \mathscr{F}^r(C \otimes C(\tilde{S}^1)).$$

Proof. Let $C' = \operatorname{im}(p\colon E \to E)$ be the image of a projection $p = p^2$ of an even finite chain complex E over A such that there exists a chain equivalence

$$\theta\colon C' \to C.$$

The canonical product round finite structure is defined by

$$C \otimes \Sigma^1 = (T(p), \psi) \in \mathscr{F}^r(C \otimes C(\tilde{S}^1)),$$

with

$$\begin{aligned}\psi = \begin{pmatrix}\theta q & 0\\ 0 & \theta q\end{pmatrix}\colon T(p)_r = E_{r-1}[z, z^{-1}] \oplus E_r[z, z^{-1}] &\to (C \otimes C(\tilde{S}^1))_r\\ &= C_{r-1}[z, z^{-1}] \oplus C_r[z, z^{-1}],\end{aligned}$$

where $q\colon E \to C'; x \mapsto p(x)$ is the projection.

Let $\Gamma\colon 0 \simeq 1\colon \hat{C}(\theta) \to \hat{C}(\theta)$ be a chain contraction of the modified algebraic mapping cone, so that

$$\Gamma = \begin{pmatrix}h' & \theta'\\ k & -h\end{pmatrix}\colon \hat{C}(\theta)_r = C'_{r-1} \oplus C_r \to \hat{C}(\theta)_{r+1} = C'_r \oplus C_{r+1}$$

with $\theta'\colon C \to C'$ a chain map, h, h' chain homotopies

$$h\colon \theta'\theta \simeq 1\colon C \to C, \qquad h'\colon \theta\theta' \simeq 1\colon C' \to C',$$

and k such that

$$h\theta - \theta h' = dk - kd'\colon C'_r \to C_{r+1}.$$

Use Γ to define a finite domination (D, f, g, h) of C by

$$f = q'\theta' : C \xrightarrow{\theta'} C' \xrightarrow{q' = \text{inclusion}} E = D$$

$$g = \theta q : D = E \xrightarrow{q = \text{projection}} C' \xrightarrow{\theta} C$$

$$h : gf = \theta'\theta \simeq 1 : C \to C.$$

The canonical round finite structure

$$(T(fg), \phi) \in \mathscr{F}^r(C \otimes C(\tilde{S}^1))$$

is defined by

$$\begin{aligned}\phi = S(h)S(g, f) = \begin{pmatrix} 1 & 0 \\ zh & 1 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} g & 0 \\ zhg & g \end{pmatrix} :\ & T(fg)_r = E_{r-1}[z, z^{-1}] \oplus E_r[z, z^{-1}] \\ & \to (C \otimes C(\tilde{S}^1))_r = C_{r-1}[z, z^{-1}] \oplus C_r[z, z^{-1}].\end{aligned}$$

The chain homotopy

$$e = q'h'q : fg = q'\theta'q \simeq q'q = p : E \to E$$

determines an isomorphism of round finite chain complexes

$$S(e) : T(fg) \to T(p)$$

with

$$\begin{aligned}S(e) = \begin{pmatrix} 1 & 0 \\ ze & 1 \end{pmatrix} : T(fg)_r = E_{r-1}[z, z^{-1}] \oplus E_r[z, z^{-1}] \to T(p)_r \\ = E_{r-1}[z, z^{-1}] \oplus E_r[z, z^{-1}],\end{aligned}$$

and

$$\tau(S(e)) = 0 \in K_1(A[z, z^{-1}]).$$

The diagram of chain equivalences

$$\begin{array}{ccc} T(fg) & \xrightarrow{S(e)} & T(p) \\ & \searrow^{\phi} \quad \swarrow_{\psi} & \\ & C \otimes C(\tilde{S}^1) & \end{array}$$

is chain homotopy commutative, with a chain homotopy

$$j : \psi S(e) \simeq \phi : T(fg) \to C \otimes C(S^1)$$

defined by

$$j = \begin{pmatrix} 0 & 0 \\ zk & 0 \end{pmatrix}: T(fg)_r = E_{r-1}[z, z^{-1}] \oplus E_r[z, z^{-1}]$$
$$\to (C \otimes C(\tilde{S}^1))_r = C_{r-1}[z, z^{-1}] \oplus C_r[z, z^{-1}].$$

Thus

$$(T(fg), \phi) - (T(p), \psi) = \tau(S(e)) = 0 \in K_1(A[z, z^{-1}]),$$

and

$$(T(fg), \phi) = (T(p), \psi) \in \mathscr{F}^r(C \otimes C(\tilde{S}^1)). \qquad \square$$

Given a group π and a morphism $\alpha: \pi \to \pi$, define the group

$$\pi *_\alpha \mathbb{Z} = \pi * \mathbb{Z}/\{z^{-1} g z = \alpha(g) \mid g \in \pi\},$$

denoting the generator $1 \in \mathbb{Z}$ by z. There is then a natural identification of rings

$$\mathbb{Z}[\pi *_\alpha \mathbb{Z}] = \mathbb{Z}[\pi] *_\alpha [z, z^{-1}]$$

and the canonical morphism of rings $i: \mathbb{Z}[\pi] \to \mathbb{Z}[\pi] *_\alpha [z, z^{-1}]$ is induced by a canonical morphism of groups

$$i: \pi \to \pi *_\alpha \mathbb{Z}; \quad g \mapsto g.$$

There is also defined a morphism of groups

$$j: \pi *_\alpha \mathbb{Z} \to \mathbb{Z}; \quad g \mapsto 1, z^n \mapsto z^n$$

which is onto, and induces a morphism of rings

$$j: \mathbb{Z}[\pi] *_\alpha [z, z^{-1}] \to \mathbb{Z}[z, z^{-1}]$$

which is also onto. If $\alpha: \pi \to \pi$ is an automorphism $\pi *_\alpha \mathbb{Z} = \pi \times_\alpha \mathbb{Z}$ is the α-twisted extension of π by $\mathbb{Z}$, with an

$$\{1\} \to \pi \xrightarrow{i} \pi \times_\alpha \mathbb{Z} \xrightarrow{j} \mathbb{Z} \to \{1\},$$

and

$$\mathbb{Z}[\pi]_\alpha[z, z^{-1}] = \mathbb{Z}[\pi]_\alpha[z, z^{-1}]$$

is the α-twisted polynomial extension of $\mathbb{Z}[\pi]$. On the other hand, if $\alpha(\pi) = \{1\} \subseteq \pi$ then $i(\pi) = \{1\} \subseteq \pi *_\alpha \mathbb{Z}$ and the morphisms $j: \pi *_\alpha \mathbb{Z} \to \mathbb{Z}$, $j: \mathbb{Z}[\pi] *_\alpha [z, z^{-1}] \to \mathbb{Z}[z, z^{-1}]$ are isomorphisms.

PROPOSITION 6.6 (i) *Let $f: X \to X$ be a cellular map to itself of a connected CW complex X with universal cover $\tilde{X}$, and let* $f_* = \alpha: \pi_1(X) \to \pi_1(X)$. *Then the mapping* *torus $T(f)$ is a connected CW complex with fundamental group*

$$\pi_1(\mathrm{T(f)}) = \pi_1(\mathrm{X}) *_\alpha \mathbb{Z},$$

and the cellular chain complex over $\mathbb{Z}[\pi_1(\mathrm{T(f)})]$ of the universal cover $T(f)$ is given by

$$\begin{aligned}\mathrm{C}(\widetilde{\mathrm{T(f)}}) &= \hat{\mathrm{C}}(1 - z\tilde{\mathrm{f}}: \mathrm{i}_! \mathrm{C}(\tilde{\mathrm{X}}) \to \mathrm{i}_! \mathrm{C}(\tilde{\mathrm{X}}))\\ &= \textit{the algebraic mapping torus } T(\tilde{f}) \textit{ of the}\\ &\quad \textit{induced chain map } \tilde{\mathrm{f}}: \alpha_! \mathrm{C}(\tilde{\mathrm{X}}) \to \mathrm{C}(\tilde{\mathrm{X}})\\ &\quad \textit{over } \mathbb{Z}[\pi_1(\mathrm{X})].\end{aligned}$$

(ii) *A homotopy of maps* $\mathrm{e}: \mathrm{f} \simeq \mathrm{f}': \mathrm{X} \to \mathrm{X}$ *induces a homotopy equivalence of mapping tori $S(e): T(f) \to T(f')$ and also a chain homotopy $\tilde{e}: \tilde{f} \simeq \tilde{f}': \alpha_! C(\tilde{X}) \to C(\tilde{X})$, such that*

$$\widetilde{S(e)} = S(\tilde{e}): C(\widetilde{T(f)}) = T(\tilde{f}) \to C(\widetilde{T(f')}) = T(\tilde{f}').$$

(iii) *Let $f: X \to Y$, $g: Y \to X$ be maps, and let*

$$\begin{aligned}&f_* = \alpha: \mathbb{Z}[\pi_1(X)] \to \mathbb{Z}[\pi_1(Y)],\\ &g_* = \beta: \mathbb{Z}[\pi_1(Y)] \to \mathbb{Z}[\pi_1(X)],\\ &\tilde{f}: \alpha_! C(\tilde{X}) \to C(\tilde{Y}), \quad \tilde{g}: \beta_! C(\tilde{Y}) \to C(\tilde{X}).\end{aligned}$$

The homotopy equivalence $S(f, g): T(gf) \xrightarrow{\sim} T(fg)$ induces the isomorphism of rings

$$\begin{aligned}S(f,g)_* = k: \mathbb{Z}[\pi_1(T(gf))] &= \mathbb{Z}[\pi_1(X)] *_{\beta\alpha} [z, z^{-1}]\\ \to \mathbb{Z}[\pi_1(T(fg))] &= \mathbb{Z}[\pi_1(Y)] *_{\alpha\beta} [z, z^{-1}]\\ a \mapsto \alpha(a),\ z &\mapsto z\end{aligned}$$

and the induced chain equivalence is such that

$$\begin{aligned}\widetilde{S(f,g)} = S(\tilde{f}, \tilde{g}): k_! C(\widetilde{T(gf)}) &= k_! T(\tilde{g}(\beta_! \tilde{f}))\\ \to C(\widetilde{T(fg)}) &= T(\tilde{f}(\alpha_! \tilde{g})).\end{aligned}$$

Proof. (i) The expression for $\pi_1(T(f))$ is the version of the Van Kampen theorem appropriate to the mapping torus construction, and the expression for $C(T(f))$ is the corresponding version of the Mayer–Vietoris presentation.

(ii) & (iii) follow from (i) and Propositions 6.1 and 6.4. □

Define the *canonical round finite structure* on the mapping torus $T(\zeta)$ of a self map $\zeta: X \to X$ of a finitely dominated CW complex X to be the canonical round finite structure on the chain complex $C(\widetilde{T(\zeta)}) = T(\tilde{\zeta}: \alpha_! C(\tilde{X}) \to C(\tilde{X}))$ over $\mathbb{Z}[\pi_1(T(\zeta))] = \mathbb{Z}[\pi_1(X)] *_\alpha [z, z^{-1}]$, with $\alpha = \zeta_*: \pi_1(X) \to \pi_1(X)$, using the correspondence between the algebraic and the geometric mapping torus of Proposition 6.6. A finite domination (Y, f, g, h) of X determines a (round) finite CW complex $T(f\zeta g: Y \to Y)$ and a homotopy equivalence

$$\phi = S(\zeta h) S(\zeta g, f): T(f\zeta g) \to T(\zeta),$$

such that the induced finite domination $(\mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}[\pi_1(Y)]} C(\tilde{Y}), \tilde{f}, \tilde{g}, \tilde{h})$ of $C(\tilde{X})$ determines the (round) finite chain complex $\widetilde{C(T(f\xi g))} = T(\tilde{f}\tilde{\zeta}\tilde{g})$ and the chain equivalence

$$\tilde{\phi} = S(\tilde{\zeta}\tilde{h})S(\tilde{\zeta}\tilde{g}, \tilde{f}) \colon T(\tilde{f}\tilde{\zeta}\tilde{g}) \to T(\tilde{\zeta}),$$

so that $(T(\tilde{f}\tilde{\zeta}\tilde{g}), \tilde{\phi}) \in \mathscr{F}^r(T(\tilde{\zeta})) = \mathscr{F}^r(T(\zeta))$ is the canonical round finite structure. We have proved:

PROPOSITION 6.7 *The geometric canonical finite structure* $(T(f\zeta g), \phi) \in \mathscr{F}(T(\zeta))$ *is the reduction of the algebraic canonical round finite structure* $(T(\tilde{f}\tilde{\zeta}\tilde{g}), \tilde{\phi}) \in \mathscr{F}^r(T(\tilde{\zeta}))$.

□

In particular, for $\zeta = 1 \colon X \to X$ Propositions 6.5 and 6.7 identify the geometric canonical finite structure on $T(1) = X \times S^1$ of Mather [12] and Ferry [6] with the reduction of the canonical product round finite structure $X \times \Sigma^1 \in \mathscr{F}^r(X \times S^1)$. Thus if $(F, \phi) \in \mathscr{F}(X \times S^1)$ is the canonical finite structure the Whitehead torsion of the composition homotopy equivalence of finite CW complexes

$$\phi^{-1}(1 \times -1)\phi \colon F \xrightarrow{\ \phi\ } X \times S^1 \xrightarrow{\ 1 \times -1\ } X \times S^1 \xrightarrow{\ \phi^{-1}\ } F$$

is given by Proposition 5.8 (ii) to be the reduction of

$$\begin{aligned} &\tau_{X \times \Sigma^1}(1 \times -1 \colon X \times S^1 \to X \times S^1) \\ &\quad = [X] \otimes \tau(-z) \in K_1(\mathbb{Z}[\pi_1(X)][z, z^{-1}]), \end{aligned}$$

that is

$$\tau(\phi^{-1}(1 \times -1)\phi) = [X] \otimes \tau(-z) \in \mathrm{Wh}(\pi_1(X) \times \mathbb{Z})$$

with $[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ the Wall finiteness obstruction. The geometrically defined injection of Ferry [6]

$$\bar{B}' \colon \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail \mathrm{Wh}(\pi \times \mathbb{Z});$$

$$[X] \mapsto \tau(\phi^{-1}(1 \times -1)\phi \colon F \to F) \quad (\pi_1(X) = \pi)$$

is thus given algebraically by the variant

$$\bar{B}' \colon \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail \mathrm{Wh}(\pi \times \mathbb{Z});$$

$$[P] \mapsto [P] \otimes \tau(-z) = \tau(-z \colon P[z, z^{-1}] \to P[z, z^{-1}])$$

of the original algebraic split injection of Bass, Heller and Swan [2]

$$\bar{B} \colon \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail \mathrm{Wh}(\pi \times \mathbb{Z});$$

$$[P] \mapsto [P] \otimes \tau(z) = \tau(z \colon P[z, z^{-1}] \to P[z, z^{-1}]).$$

It is $\bar{B}'$ rather than $\bar{B}$ which is geometrically significant. (See Ranicki [22]).

For example, the trivial S^1-bundle transfer maps

$$\phi_H^!: \hat{H}^n(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}[\pi])) \rightarrowtail \hat{H}^{n+1}(\mathbb{Z}_2; \mathrm{Wh}(\pi \times \mathbb{Z}))$$

on the Tate $\mathbb{Z}_2$-cohomology groups of the duality involutions which appear in the appendix of Munkholm and Ranicki [13] are induced by $\bar{B}'$ not $\bar{B}$.

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