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NON-INJECTIVITY OF THE MAP FROM THE WITT GROUP OF A VARIETY TO THE WITT GROUP OF ITS FUNCTION FIELD

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Abstract We exhibit a smooth complex affine 5-fold whose Witt group of quadratic forms does not inject into the Witt group of its function field. The dimension 5 is the smallest possible. The example depends on relating Witt groups, mod 2 Chow groups, and Steenrod operations.

Keywords: Witt group; Chow group; Steenrod operation; unramified cohomology

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0. Introduction

For a regular noetherian scheme X with 2 invertible in X, let $W(X)$ denote the Witt group of X [**9**]. By definition, the Witt group is the quotient of the Grothendieck group of vector bundles with quadratic forms over X by the subgroup generated by quadratic bundles V that admit a Lagrangian subbundle E, meaning that E is its own orthogonal complement in V. A fundamental question is whether the homomorphism from the Witt group of X to the Witt group of its function field is injective. (When X does not have this injectivity property, and X is a variety over a field so that Ojanguren's injectivity theorem for local rings applies [**15**], then X has an even-dimensional quadratic bundle that is Zariski-locally trivial but has no Lagrangian subbundle.) Ojanguren [**16**] and Pardon [**17**] proved this injectivity when the regular noetherian scheme X is affine of dimension at most three, and this was extended to all regular noetherian schemes X of dimension at most three by Balmer and Walter [**1**]. There are simple examples of four-dimensional varieties where injectivity fails; for example, the affine quadric $S^4_{\boldsymbol{R}}$ over the real numbers defined by $x_0^2 + \cdots + x_4^2 = 1$ and the smooth projective quadric of dimension four over the complex numbers. But Pardon showed that injectivity holds for all smooth complex affine varieties of dimension four [**17**]. There remained the question of whether Pardon's result could hold in a larger range of dimensions. The only known counterexamples to injectivity for smooth complex affine varieties began in dimension eight; for example $S^4_{\boldsymbol{C}} \times S^4_{\boldsymbol{C}}$, as pointed out by Knus *et al.* [**10**, § 5], as well as $S^8_{\boldsymbol{C}}$. More generally, using Karoubi's calculation of the hermitian K-group of $S^n_{\boldsymbol{C}}$ for all n [**8**, p. 141], one can check that $W(S^n_{\boldsymbol{C}}) \to W(\boldsymbol{C}(S^n_{\boldsymbol{C}}))$ fails to be injective if and only if $n \equiv 0, 1 \pmod 8$ and $n \geqslant 8$.

In this paper, we complete the story as follows.

Theorem 0.1. *There is a smooth complex affine fivefold U with $W(U) \to W(\boldsymbol{C}(U))$ not injective.*

The main technical tools are the spectral sequences of Balmer and Walter [**1**] and Pardon [**18**], which together make the relation between Witt groups and Chow groups modulo 2 fairly explicit. For example, these spectral sequences easily imply Parimala's theorem (Theorem 1.4 below, originally stated only for affine varieties) that the Witt group of a smooth complex threefold X is finite if and only if the Chow group $CH^2(X)/2$ is finite [**19**]. Schoen recently found smooth complex threefolds X with $CH^2(X)/l$ infinite for some primes l [**21**], and I expect that his method can be extended to give such examples with $l = 2$. That would give the first known examples of smooth complex varieties with infinite Witt group.

For Theorem 0.1, we need to identify the first differential in Pardon's spectral sequence with the Steenrod operation Sq^2 on Chow groups modulo 2, as defined by Voevodsky [**24**] and Brosnan [**3**]. Given that, the main problem is to exhibit a smooth complex affine variety of high dimension n whose Chow group of curves modulo 2 is non-zero. (Notice that the Chow group of zero-cycles modulo 2 is zero in this situation, as is the cohomology group $H^{2n-2}(X, \boldsymbol{F}_2)$ to which the Chow group of curves maps.) Probably one could construct such an example by the methods of Kollár and van Geemen [**12**] or Schoen [**21**]. But we use a more elementary method, based on the phenomenon that a fibration of a 'sufficiently non-singular' variety that has a multiple fibre cannot have a section.

1. The Gersten–Witt complex and Chow groups

Let X be a regular noetherian integral separated scheme of finite dimension with 2 invertible in X. Balmer and Walter [**1**] and Pardon [**18**] defined the Gersten–Witt complex W_X of X as a cochain complex

$$0 \to W(k(X)) \to \bigoplus_{x \in X^{(1)}} W(k(x)) \to \cdots \to \bigoplus_{x \in X^{(n)}} W(k(x)) \to 0,$$

where $X^{(i)}$ denotes the set of codimension-i points of the scheme X, or equivalently the set of codimension-i subvarieties of X. More canonically, as explained in Balmer and Walter's equation (6) [**1**] or Pardon's Corollary 0.11 [**18**], this sequence can be written as

$$0 \to W(k(X)) \to \bigoplus_{x \in X^{(1)}} W(k(x); \omega_{x/X}) \to \cdots \to \bigoplus_{x \in X^{(n)}} W(k(x); \omega_{x/X}) \to 0.$$

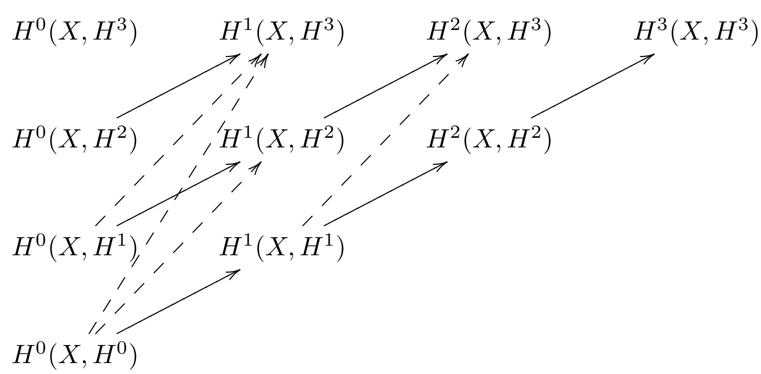
Here, $\omega_{x/X}$ denotes the one-dimensional $k(x)$-vector space $\det(m_x/m_x^2)$, where m_x is the maximal ideal in the local ring $O_{X,x}$, and $W(F; L)$ is the Witt group of quadratic forms over a field F that take values in a given one-dimensional vector space L over F. Let $H^*(W_X)$ denote the cohomology groups of this cochain complex, with $W(k(X))$ placed in degree 0. The group $H^0(W_X)$ is also known as the unramified Witt group $W_{nr}(X)$. The groups $H^*(W_X)$ are related to the 4-periodic Witt groups $W^*(X)$ by Balmer and Walter's 'Gersten–Witt spectral sequence' [**1**], which reduces to the following exact sequence for X of dimension at most seven:

$$\begin{aligned}0 &\to H^4(W) \to W^0 \to H^0(W) \to H^5(W) \to W^1 \to H^1(W)\\ &\to H^6(W) \to W^2 \to H^2(W) \to H^7(W) \to W^3 \to H^3(W) \to 0.\end{aligned}$$

In particular, this exact sequence shows that for X of dimension at most seven, the group $H^4(W_X)$ is isomorphic to the kernel of the homomorphism from the Witt group of X to the Witt group of its function field. The group $H^4(W_X)$ is obviously zero for X of dimension at most three, which gives Balmer and Walter's theorem that the homomorphism is injective for X of dimension at most three. In this paper we will prove that the homomorphism is not injective for some smooth complex affine fivefold U; equivalently, $H^4(W_U)$ can be non-zero for such a U.

To compute the groups $H^*(W_X)$, we can filter the complex W_X by the subgroups I^n in the Witt groups of fields. This gives Pardon's spectral sequence [**18**, 0.13], as follows. Write $H^i_{\text{Zar}}(X, H^j_{\boldsymbol{F}_2})$, or $H^i(X, H^j)$ for short, to mean the Zariski cohomology of X with coefficients in the presheaf $H^j_{\boldsymbol{F}_2}(V) := H^j_{\text{et}}(V, \boldsymbol{F}_2)$, for open subsets $V \subset X$. Remarkably, these are the same groups that form the E_2 term of the Bloch–Ogus spectral sequence (Theorem 1.2 below), although the following spectral sequence is different.

Theorem 1.1. *For any smooth variety X over a field k of characteristic not 2, there is a spectral sequence $E_2^{ij} = H^i_{\text{Zar}}(X, H^j_{\boldsymbol{F}_2}) \Rightarrow H^i(W_X)$. The differentials are in an unusual direction: d_r has bidegree $(1, r-1)$ for $r \geqslant 2$. The groups $H^i(X, H^j)$ are zero unless $0 \leqslant i \leqslant j$. The groups on the main diagonal are the Chow groups modulo 2: $H^i(X, H^i) = CH^i(X)/2$. Finally, the differential on this main diagonal, $d_2 : CH^i(X)/2 \to CH^{i+1}(X)/2$, is the Steenrod operation Sq^2, as defined on Chow groups by Voevodsky [**24**] and Brosnan [**3**].*



The Chow group CH^iX is defined as the group of codimension-i algebraic cycles on X modulo rational equivalence. A general reference on Chow groups is Fulton's book [**5**, 1.3].

Proof. Most of this is Pardon's Corollary 0.13 [**18**], specialized to the case where the sheaf C is the trivial line bundle O_X. Also, Pardon's grading of the spectral sequence differs from that used here. The vanishing of $H^i(X, H^j)$ for $i > j$ and the isomorphism $H^i(X, H^i_{\boldsymbol{F}_2}) = CH^i(X)/2$ were proved by Bloch and Ogus [**2**, 6.2 and proof of 7.7]). The new point is the identification of the differential $d_2 : CH^i(X)/2 \to CH^{i+1}(X)/2$ with Sq^2.

To prove this, it is convenient to reduce to the case where the field k is perfect. Indeed, for any field extension E of k, there is a natural flat pullback map of Chow groups, $CH^i(X) \to CH^i(X_E)$. Both the Pardon differential d_2 and the Steenrod operation Sq^2 are compatible with this pullback map. Furthermore, when E is a finite extension of k of odd degree, the map $CH^i(X)/2 \to CH^i(X_E)/2$ is injective, by Fulton [**5**, 15.1.5]. Since k does not have characteristic 2, any inseparable extension of k has odd degree. By taking direct limits, it follows that the natural map $CH^i(X)/2 \to CH^i(X_E)/2$ is injective when E is the perfect closure of k (that is, $E = \bigcup_r k^{1/p^r}$ for k of characteristic $p > 0$, or $E = k$ in characteristic zero). Therefore, if we prove that the Pardon differential d_2 is equal to Sq^2 for varieties over a perfect field, then the same is true for varieties over any field. Thus we can now assume that the base field k is perfect.

Let n be the dimension of the smooth variety X. Let Y be a smooth $(n-i)$-dimensional variety with a proper map $f : Y \to X$. Then the Steenrod operation

$$Sq^2 : CH^i(X)/2 \to CH^{i+1}(X)/2$$

is defined in such a way that

$$\begin{aligned} Sq^2(f_*[Y]) &= f_*(c_1 f^*(TX) - c_1 TY) \\ &= c_1(TX) f_*[Y] - f_* c_1(TY), \end{aligned}$$

by 8.10, 8.11, 9.4 in [**3**]. This property defines Sq^2 uniquely. In fact, for any codimension-i subvariety Z of X, the normalization of Z is regular in codimension one, since a normal noetherian local ring of dimension one is regular [**13**, 11.2]). Therefore, we can remove a codimension-two subset of Z from Z and from X, so as to make the normalization Y of Z regular. Since the base field k is perfect, it follows that Y is smooth over k [**13**, 26.3, Lemma 1 in § 28, 30.3]. Removing that subset does not change the group $CH^{i+1}(X)/2$, since the algebraic cycles and rational equivalences that define this group are unchanged. Having thus modified X, we can define $Sq^2[Z]$ by the above formula applied to the proper map $Y \to X$.

Furthermore, the variety Z is Cohen–Macaulay in codimension one, since any reduced noetherian local ring of dimension one is Cohen–Macaulay [**13**, Exercise 17.1 (b)]. So, after removing a suitable closed subset of codimension two in Z from Z and from X, we can assume that Z is Cohen–Macaulay, as well as having smooth normalization. As above, removing this subset does not change the group $CH^{i+1}(X)/2$ we are concerned with. With this change made, Z has a dualizing sheaf ω_Z (a short notation for $\omega_{Z/k}$). Indeed, any variety Z over k has a dualizing complex, defined as $\pi^!(k)$, where π is the map from Z to $\operatorname{Spec} k$, and Cohen–Macaulayness of Z means that this complex is a single sheaf ω_Z placed in degree $-\dim Z$. This aspect of Grothendieck–Verdier duality is described in Hartshorne's book [**7**, V, Exercise 9.7], or in more detail in Conrad's book [**4**, 3.5.1].

Having described Sq^2 of the codimension-i subvariety Z of X, we now compare it to the differential d_2. To compute $d_2[Z]$, we first lift the non-zero element of $W/I(k(z), \omega_{z/X}) \cong \boldsymbol{F}_2$ to an element of $W(k(z); \omega_{z/X})$. For example, we can do this by choosing a one-dimensional $k(z)$-vector space L together with an isomorphism $L \otimes L \cong \omega_{z/X}$. We can choose a basis element for the one-dimensional vector space L; then we get an isomorphism $k(z) \cong \omega_{z/X}$. Since Z is Cohen–Macaulay, we can view the isomorphism $k(z) \cong \omega_{z/X}$ as a non-zero rational section of the relative dualizing sheaf $\omega_{Z/X} := \omega_Z \otimes_{O_Z} \omega_X^*|_Z$.

The differential in the Gersten–Witt complex is defined by Balmer and Walter's Proposition 8.5 [**1**], as a sum over codimension-one points d in Z of elements M in the Witt group of finite-length $O_{X,d}$-modules, which can be identified with $W(k(d), \omega_{d/X})$. To compute $d_2[Z]$ in the Pardon spectral sequence, we only need to know the class of these elements in $W/I(k(d), \omega_{d/X}) \cong \boldsymbol{F}_2$, that is, the length modulo 2 of the finite-length $O_{X,d}$-module M. Thus $d_2[Z]$ is a divisor with $\boldsymbol{F}_2$ coefficients on Z. Balmer and Walter's exact sequence (35) shows that this divisor has the form

$$d_2[Z] = \omega_{Z/X} - O_Z$$

in the first graded piece of the Grothendieck group of coherent sheaves on Z, filtered by codimension of support, $(\mathrm{gr}^1 G_0 Z)/2$. Here, inspection of Quillen's spectral sequence that converges to this filtration of $G_0 Z$ shows that $\mathrm{gr}^1 G_0 Z$ is isomorphic to the divisor class group of Z [**20**, § 7, Theorem 5.4]. Also, to obtain the above formula, we use that the module $\mathrm{Ext}_O^{e-1}(A, O)$, which Balmer and Walter call ω_A in their exact sequence (35), apparently corresponding to ω_Z in the notation here, is really the relative dualizing sheaf $\omega_{Z/X}$.

Since $\omega_{Z/X}$ is the tensor product of the sheaf ω_Z with the line bundle $\omega_X^*|_Z$, it is easy to rewrite the formula for $d_2[Z]$ as

$$\begin{aligned} \omega_{Z/X} - O_Z &= -(\omega_X|_Z - O_Z) + (\omega_Z - O_Z) \\ &= c_1(TX)|_Z + (\omega_Z - O_Z) \end{aligned}$$

in the divisor class group $(\mathrm{gr}^1 G_0 Z)/2$. To show that this class equals $Sq^2[Z]$ in $CH^{i+1}(X)/2$, as defined by Brosnan's formula above, it remains to identify the class $\omega_Z - O_Z$ in $(\mathrm{gr}^1 G_0 Z)/2$ with $f_* c_1(TY)$, where $f : Y \to Z$ is the normalization of Z. Recall that we have removed a codimension-two subset of Z in such a way that the normalization Y is smooth. Here we can write $f_*(c_1 TY)$ as $f_*\omega_Y - f_* O_Y$ in $(\mathrm{gr}^1 G_0 Z)/2$.

Let A and B denote the cokernels of the natural maps of sheaves on Z,

$$0 \to O_Z \to f_* O_Y \to A \to 0$$

and

$$0 \to f_*\omega_Y \to \omega_Z \to B \to 0.$$

The equality we want, saying that $\omega_Z - O_Z$ equals $f_*\omega_Y - f_*O_Y$ in $(\mathrm{gr}^1\, G_0 Z)/2$, will clearly follow if we can show that the sheaves A and B are equal in $(\mathrm{gr}^1\, G_0 Z)/2$. Since the sheaves A and B are zero at the generic point of Z, it suffices to show that A and B have the same coefficient in $\boldsymbol{F}_2$ at each codimension-one point d in Z. Let $O_{Z,d}$ denote the local ring of Z at d. In these terms, we need to know that the $O_{Z,d}$-modules A and B have the same length modulo 2.

Using Grothendieck–Verdier duality, we find an isomorphism of O_Z-modules $B \cong \mathrm{Ext}^1_{O_Z}(A, \omega_Z)$. In more detail, consider the long exact sequence of O_Z-modules obtained from the first exact sequence above by applying $\mathrm{Hom}_{O_Z}(\cdot, \omega_Z)$,

$$\mathrm{Hom}_{O_Z}(f_*O_Y, \omega_Z) \to \mathrm{Hom}_{O_Z}(O_Z, \omega_Z) \to \mathrm{Ext}^1_{O_Z}(A, \omega_Z) \to \mathrm{Ext}^1_{O_Z}(f_*O_Y, \omega_Z).$$

We have $\mathrm{Ext}^i_{O_Z}(Rf_!M, N) \cong f_*\,\mathrm{Ext}^i_{O_Y}(M, f^!N)$ for all i, by [**23**, Theorem 1]. In the case at hand, $Rf_!O_Y$ is just the sheaf f_*O_Y because f is finite. Also, since Z is Cohen–Macaulay, we have already mentioned that the natural map $\pi_Z : Z \to \mathrm{Spec}\, k$ has the property that the complex $\pi_Z^!(k)$ is simply a sheaf ω_Z placed in degree $-\dim Z$. The same goes for the smooth variety Y. In view of the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & Z \\ & {\scriptstyle \pi_Y}\searrow & \downarrow{\scriptstyle \pi_Z} \\ & & \mathrm{Spec}\, k, \end{array}$$

it follows that $f^!\omega_Z$ is isomorphic to the sheaf ω_Y. So the above exact sequence can be rewritten as

$$f_*\omega_Y \to \omega_Z \to \mathrm{Ext}^1_{O_Z}(A, \omega_Z) \to f_*\,\mathrm{Ext}^1_{O_Y}(O_Y, \omega_Y) = 0.$$

This proves that $B \cong \mathrm{Ext}^1_{O_Z}(A, \omega_Z)$, as we claimed.

Finally, for each codimension-one point d in Z, the functor $A \mapsto \mathrm{Ext}^1_{O_{Z,d}}(A, \omega_Z)$ is a duality on the category of finite-length $O_{Z,d}$-modules, preserving the length of such modules. This is a special case of Grothendieck's local duality theorem, using that the local ring $O_{Z,d}$ is Cohen–Macaulay of dimension one [**7**, IV.6.3]. Thus A and B have the same length at each divisor in Z. As mentioned above, this completes the proof that the differential $d_2 : CH^i(X)/2 \to CH^{i+1}(X)/2$ is the Steenrod operation Sq^2. □

In order to understand the groups $H^i(X, H^j)$, and compute them in some cases, one can use the Bloch–Ogus spectral sequence [**2**, 6.3]. It has the same E_2 term as Pardon's spectral sequence, but with differentials in the usual direction.

Theorem 1.2 (Bloch–Ogus). *For any smooth variety X over a field of characteristic not 2, there is a spectral sequence $E_2^{ij} = H^i_{\mathrm{Zar}}(X, H^j_{\boldsymbol{F}_2}) \Rightarrow H^{i+j}_{\mathrm{et}}(X, \boldsymbol{F}_2)$. The groups $H^i(X, H^j)$ in the E_2 term are zero unless $0 \leqslant i \leqslant j$. The groups on the main diagonal are the Chow groups modulo 2: $H^i(X, H^i) = CH^i(X)/2$. Finally, the differentials are in* *the usual direction:* d_r *has bidegree* $(r, 1-r)$, $r \geqslant 2$.

$$\begin{array}{cccc}
H^0(X,H^3) & H^1(X,H^3) & H^2(X,H^3) & H^3(X,H^3) \\
& \searrow & & \\
H^0(X,H^2) & H^1(X,H^2) & H^2(X,H^2) & \\
& & & \\
H^0(X,H^1) & H^1(X,H^1) & & \\
& & & \\
H^0(X,H^0) & & &
\end{array}$$

For completeness, let me explain how the groups $H^i_{\mathrm{Zar}}(X, H^j_{\boldsymbol{F}_2})$ in the E_2 term of the Bloch–Ogus spectral sequence are related to motivic cohomology groups. This will not be used in the rest of the paper.

Let X be a smooth variety over a field k of characteristic not 2. There is a natural sequence of homomorphisms relating the motivic cohomology groups of X with different weights,

$$\cdots \to H^i_M(X, \boldsymbol{F}_2(j)) \to H^i_M(X, \boldsymbol{F}_2(j+1)) \to \cdots,$$

given by cup product with the non-zero element of $H^0_M(k, \boldsymbol{F}_2(1)) \cong \mu_2(k) \cong \boldsymbol{F}_2$. The motivic cohomology groups in this sequence are initially zero, namely for $i > 2j$. The first group to appear is the Chow group $CH^j(X)/2$, when $i = 2j$. Finally, the groups in the sequence are eventually isomorphic to the etale cohomology $H^i_{\mathrm{et}}(X, \boldsymbol{F}_2)$, namely when $j \geqslant i$. Now we can explain the groups $H^i_{\mathrm{Zar}}(X, H^j_{\boldsymbol{F}_2})$ in the E_2 term of the Bloch–Ogus spectral sequence: they are the differences between one motivic cohomology group and the next one, in the above sequence. More precisely, we have the following result.

Theorem 1.3. *For each* $j \geqslant 0$, *there is a long exact sequence,*

$$\to H^{i+j}_M(X, \boldsymbol{F}_2(j-1)) \to H^{i+j}_M(X, \boldsymbol{F}_2(j)) \to H^i_{\mathrm{Zar}}(X, H^j_{\boldsymbol{F}_2}) \to H^{i+j+1}_M(X, \boldsymbol{F}_2(j-1)) \to .$$

Proof. The ingredients here are deep. First, by Voevodsky, we know the Milnor conjecture, also known as the Bloch–Kato conjecture with $\boldsymbol{F}_2$ coefficients [**25**, Theorem 1.3]. Namely, for any field k of characteristic not 2, the natural map from the Milnor K-theory of k modulo 2 to the Galois cohomology of k with $\boldsymbol{F}_2$ coefficients is an isomorphism.

Next, by Geisser and Levine [**6**, Theorem 1.1], generalizing the work of Suslin and Voevodsky in characteristic zero [**22**], we know that the Bloch–Kato conjecture with $\boldsymbol{F}_p$ coefficients implies the Beilinson–Lichtenbaum conjecture with $\boldsymbol{F}_p$ coefficients. As a result, we know that the Beilinson–Lichtenbaum conjecture with $\boldsymbol{F}_2$ coefficients is true.

This means that the natural map

$$\boldsymbol{F}_2(j) \to \tau_{\leqslant j} R\alpha_*(\boldsymbol{F}_2)$$

is an isomorphism for all $j \geqslant 0$. This is a map in the derived category of sheaves in the Zariski topology on any smooth variety X over a field of characteristic not 2. The object $\boldsymbol{F}_2(j)$ is defined so that $H^i_{\text{Zar}}(X, \boldsymbol{F}_2(j))$ is the motivic cohomology of X with $\boldsymbol{F}_2$ coefficients, also written $H^i_M(X, \boldsymbol{F}_2(j))$.

On the right, $\tau_{\leqslant j}$ is a truncation, α is the projection from the etale site to the Zariski site and $\boldsymbol{F}_2$ is a sheaf in the etale topology on X. More concretely, this means that the object $\tau_{\leqslant j} R\alpha_*(\boldsymbol{F}_2)$ in the derived category of Zariski sheaves on X has cohomology sheaf in degree i given as follows. For $i > j$, this sheaf is zero; for $i \leqslant j$, it is the sheaf $H^i_{\boldsymbol{F}_2}$ of etale cohomology groups.

In particular, there is a distinguished triangle in the derived category of Zariski sheaves on X,

$$\tau_{\leqslant j-1} R\alpha_*(\boldsymbol{F}_2) \to \tau_{\leqslant j} R\alpha_*(\boldsymbol{F}_2) \to H^j_{\boldsymbol{F}_2}[-j],$$

where the shift $[-j]$ just means that the sheaf $H^j_{\boldsymbol{F}_2}$ is placed in degree j. The associated long exact sequence of Zariski cohomology groups is the result we want. □

As an application of the Balmer–Walter and Pardon spectral sequences, not needed for the rest of the paper, we can give an easy proof of the following theorem of Parimala (originally stated only for affine varieties) [**19**].

Theorem 1.4. *Let X be a smooth complex threefold. Then the Witt group $W(X)$ is finite if and only if the Chow group $CH^2(X)/2$ is finite.*

Proof of Theorem 1.4. Consider the Bloch–Ogus spectral sequence for X. Since X has dimension three over the complex numbers, the groups $H^i(X, H^j)$ in the E_2 term are zero unless $j \leqslant 3$. Combining this with the other properties listed in Theorem 1.2, we see that the only possible differential in the Bloch–Ogus spectral sequence is $d_2 : H^0(X, H^3) \to H^2(X, H^2) = CH^2(X)/2$. The spectral sequence converges to $H^*(X, \boldsymbol{F}_2)$, which is finite. So all the groups $H^i(X, H^j)$ are finite except possibly for $H^0(X, H^3)$ and $H^2(X, H^2) = CH^2(X)/2$, and $H^0(X, H^3)$ is finite if and only if $CH^2(X)/2$ is finite.

Thus, if $CH^2(X)/2$ is finite, then all the groups $H^i(X, H^j)$ are finite. By the Pardon spectral sequence, it follows that the groups $H^i(W_X)$ are finite, and by the Balmer–Walter spectral sequence, it follows that the Witt groups $W^i(X)$ are finite. In particular, the classical Witt group $W^0(X)$ is finite.

Conversely, if $CH^2(X)/2$ is infinite, then $H^0(X, H^3)$ is infinite. There are no possible differentials involving this group in the Pardon spectral sequence, and so $H^0(W_X)$ is infinite. Also, for dimension reasons, there are no differentials in the Balmer–Walter spectral sequence, and so $W^0(X)$ is infinite. □

2. The example

We come to the problem of finding a smooth complex affine fivefold U with $W(U) \to W(\boldsymbol{C}(U))$ not injective. As explained in § 1, it is equivalent to make $H^4(W_U)$ non-zero. Using Pardon's spectral sequence, Theorem 1.1, for any smooth complex affine fivefold U, we can ask when an element α of $H^4(U, H^4) = CH^4(U)/2$ corresponds to some non-zero element of $H^4(W_U)$. First, the only possible differential on α maps it to $H^5(U, H^5) = CH^5(U)/2$, but this group is zero. Indeed, more generally, the Chow group of zero-cycles on any non-compact complex variety is a divisible abelian group, by Fulton's Example 1.6.6 [**5**]. As a result, $CH_0(U)/2 = 0$. (Note that $CH^5U = CH_0U$ for U of dimension five.)

The only differential in Pardon's spectral sequence that might hit α is

$$Sq^2 : CH^3(U)/2 \to CH^4(U)/2.$$

Thus we have shown the following result.

Lemma 2.1. *Let U be a non-compact smooth complex fivefold. If $Sq^2 : CH^3(U)/2 \to CH^4(U)/2$ is not surjective, then $H^4(W_U)$ is not zero.*

We will choose U to be a suitable open subset of $Q^4 \times (A^1 - 0)$, where Q^4 denotes the smooth complex four-dimensional quadric. Any open subset U of $Q^4 \times (A^1 - 0)$ has the convenient property that the Steenrod operation $Sq^2 : CH^3(U)/2 \to CH^4(U)/2$ is identically zero, as we now show. The pullback map $CH^*(Q^4) \to CH^*(Q^4 \times A^1)$ is an isomorphism [**5**, 3.3(a)], and the restriction to an open subset $CH^*(Q^4 \times A^1) \to CH^*U$ is surjective [**5**, 1.8]. So it suffices to show that

$$Sq^2 : CH^3(Q^4)/2 \to CH^4(Q^4)/2$$

is zero. The Chow ring of a complex quadric maps isomorphically to its cohomology [**5**, 19.1.11], and so it suffices to show that

$$Sq^2 : H^6(Q^4, \boldsymbol{F}_2) \to H^8(Q^4, \boldsymbol{F}_2)$$

is zero. For this, we use Wu's formula (see [**14**, 11.14]): to show that Sq^2 is zero when mapping into the top degree of $H^*(Q^4, \boldsymbol{F}_2)$, it is equivalent to show that the Wu class $v_2(Q^4) = w_2(Q^4) + w_1^2(Q^4)$ is zero in $H^2(Q^4, \boldsymbol{F}_2)$. Since Q^4 is a complex manifold, the Wu class is the reduction modulo 2 of $c_1(Q^4)$. But $c_1(Q^4)$ is four times the hyperplane class, and so its reduction modulo 2 is zero, as we want. (In other words, we are using that an even-dimensional complex quadric is a spin manifold.)

Thus $Sq^2 : CH^3(U)/2 \to CH^4(U)/2$ is zero. Combining this with Lemma 2.1 gives the following result.

Lemma 2.2. *Let U be an open subset of $Q^4 \times (A^1 - 0)$. If $CH^4(U)/2$ is not zero, then $H^4(W_U)$ is not zero.*

It remains to find an affine open subset U of $Q^4 \times (A^1 - 0)$ such that the Chow group $CH^4(U)/2 = CH_1(U)/2$ of curves modulo 2 is not zero. Let $Q^4 \subset \boldsymbol{P}^5$ be defined by

$$x_0x_1 + x_2x_3 + x_4x_5 = 0.$$

We define U to be the complement of the hypersurface Y in $Q^4 \times (A^1 - 0)$ defined by

$$x_0^8 + tx_1^8 + t^2x_2^8 + t^3x_3^8 + t^4x_4^8 + t^5x_5^8 = 0.$$

(Equations of this form were used in a different context by Kollár [**11**, IV.6.4.3.1].) The complement of the hypersurface defined by this equation in $\boldsymbol{P}^5 \times (A^1 - 0)$ is clearly affine, and U is the closed subset of this affine variety defined by the quadric. Therefore, the smooth complex fivefold U is affine.

It remains to show that $CH_1(U)/2$ is not zero. By the exact sequence

$$CH_1(Y)/2 \to CH_1(Q^4 \times (A^1 - 0))/2 \to CH_1(U)/2 \to 0$$

from Fulton [**5**, 1.8], it is equivalent to show that $CH_1(Y)/2 \to CH_1(Q^4 \times (A^1 - 0))/2$ is not surjective. The proper pushforward map

$$CH_1(Q^4 \times (A^1 - 0))/2 \to CH_1(A^1 - 0)/2 = \boldsymbol{F}_2$$

is surjective, since the curve $p \times (A^1 - 0)$ has degree 1 over $A^1 - 0$ for any point p in Q^4. Therefore, we will be done if we can show that the pushforward map $CH_1(Y)/2 \to CH_1(A^1 - 0)/2 = \boldsymbol{F}_2$ is not surjective; in other words, that every curve in Y has even degree over $A^1 - 0$.

We can, in fact, forget about the quadric at this point, and only consider the hypersurface Z^5 in $\boldsymbol{P}^5 \times (A^1 - 0)$ defined by

$$x_0^8 + tx_1^8 + t^2x_2^8 + t^3x_3^8 + t^4x_4^8 + t^5x_5^8 = 0.$$

We will show that every curve in Z has even degree over $A^1 - 0$, which obviously implies the same statement for curves in $Y^4 \subset Z^5$. The point is that if there was a curve on Z of odd degree over $A^1 - 0$, then we could restrict this curve over a neighbourhood of 0 in A^1. It could fall into several pieces, but at least one would have odd degree over $\boldsymbol{C}((t))$. Thus we are finished if we can show that the morphism $t : Z \to A^1 - 0$ has no section over $\boldsymbol{C}((t^{1/r}))$ for any odd r.

This is immediate by a power series calculation. Let $u = t^{1/r}$, r odd, and suppose that $Z \to A^1 - 0$ has a section over $\operatorname{Spec} \boldsymbol{C}((u)) \to A^1 - 0$. This means that there are Laurent series $x_i(u)$ that satisfy the equation

$$x_0^8 + u^r x_1^8 + u^{2r}x_2^8 + u^{3r}x_3^8 + u^{4r}x_4^8 + u^{5r}x_5^8 = 0.$$

But the lowest degrees in u of these six terms are congruent to $0, r, 2r, \ldots, 5r \pmod 8$. Since r is odd, these lowest degrees are all different. So the only way the six terms can add up to zero is if all are identically zero. That would imply that $x_i(u) = 0$ for all i, but this does not correspond to a point in projective space. Thus $Z \to A^1 - 0$ has no section over $\boldsymbol{C}((t^{1/r}))$ for any odd r.

This completes the proof that the smooth complex affine fivefold U has $H^4(W_U)$ nonzero, and so the homomorphism $W(U) \to W(\boldsymbol{C}(U))$ is not injective.

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