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THE HOMOTOPY GROUPS OF A TRIAD I[1]

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One of the most urgent problems of modern topology is to devise methods for the computation of the homotopy groups of topological spaces. This paper is an attack on one phase of this general problem.

The homotopy groups resemble the homology groups in many respects. However the homology groups are computable for any triangulable space, while the homotopy group of such simple spaces as spheres have not yet been computed. This difference is undoubtedly related to the fact that the homology groups satisfy the so-called *excision axiom*, while the homotopy groups do not. The excision axiom may be stated as follows: Suppose that the topological space X is the union of two closed subspaces A and B, $X = A \cup B$. Then under fairly general circumstances the inclusion maps

$$i: (A, A \cap B) \rightarrow (X, B)$$
$$j: (B, A \cap B) \rightarrow (X, A)$$

induce isomorphisms of the relative homology groups in all dimensions. Simple examples show that this need not be true for the relative homotopy groups, even when X is a finite connected simplicial complex, and A, B, and $A \cap B$ are connected subcomplexes. The new homotopy groups defined in this paper are a measure of the amount by which the excision axiom fails to hold for relative homotopy groups. The precise meaning of this statement will be clear later.

One special case for which it is particularly important to determine the extent of the validity of the excision axiom for relative homotopy groups is the following: Let K be a cell complex[2] and let K^n, $n = 0, 1, 2, \cdots$, denote the n-skeleton. Denote the closed n-cells of K by σ_1^n, σ_2^n, $\cdots$, their boundaries by $\dot{\sigma}_1^n$, $\dot{\sigma}_2^n$, $\cdots$, and set

$$\mathcal{E}^n = \cup_i \sigma_i^n,$$
$$\dot{\mathcal{E}}^n = \cup_i \dot{\sigma}_i^n.$$

Assume that both $\mathcal{E}^n$ and $\dot{\mathcal{E}}^n$ are connected sets; actually, this is not as great a restriction as might appear at first sight. Then the inclusion map

$$i: (\mathcal{E}^n, \dot{\mathcal{E}}^n) \rightarrow (K^n, K^{n-1})$$

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[2] For the definition and properties of a cell complex, see Eilenberg and Steenrod, [5], or J. H. C. Whitehead, [20].

is an excision map. It is important to obtain information about the induced homomorphisms

$$i_* : \pi_q(\mathcal{E}^n, \dot{\mathcal{E}}^n) \to \pi_q(K^n, K^{n-1}),$$

for all values of the integers q and n.

J. H. C. Whitehead[3] [18, 19(th.8), 22] has proved a number of theorems concerning these homomorphisms. Freudenthal's "Einhängung" theorems [8] may also be interpreted as theorems of this kind, in which K is the n-sphere obtained by the non-singular adjunction of the upper hemisphere (an n-cell) to the lower hemisphere (also an n-cell). In the present paper we prove two main theorems which overlap considerably with the results of Whitehead and Freudenthal; in a subsequent paper we shall extend these results. Our methods of proof differ considerably from those used by Freudenthal and Whitehead. These latter are almost entirely geometrical and rely heavily on a good geometrical intuition. We have attempted to avoid this difficulty by extensive use of the algebraical techniques of the theory of obstructions to extensions and deformations of mappings.

This paper is subdivided into five parts. In the first we describe our notation and review many well known ideas which are used later. In the second we give several equivalent definitions for the homotopy groups of a triad. The elementary properties of these groups and their relations with existing homotopy groups are discussed in part three. The fourth part is devoted to the statement and proof of the two main theorems referred to above. The final part contains some applications of these theorems, among which are the easier cases of Freudenthal's "Einhängung" theorems. The more difficult cases of these theorems ("the critical dimensions") will be dealt with in a subsequent paper.

Part 1: Preliminaries

(1.1) Notation and Terminology

Let X and Y be topological spaces, $X_1, \cdots, X_k$ subspaces of X, and $Y_1, \cdots, Y_k$ subspaces of Y. The notation

(1.1.1) $$f: (X; X_1, \cdots, X_k) \to (Y; Y_1, \cdots, Y_k)$$

means that f is a continuous function defined on X with values in Y and satisfying the conditions $f(X_i) \subset Y_i$, $i = 1, \cdots, k$. The words "map" and "mapping" will also be used for "continuous function." Suppose that $A, A_1, \cdots, A_m$ are subsets of X, and $B, B_1, \cdots, B_m$ are subsets of Y, satisfying the conditions

$$A_i \subset A, \quad B_i \subset B, \qquad i = 1, \cdots, m,$$
$$f(A) \subset B, \quad f(A_i) \subset B_i, \qquad i = 1, \cdots, m.$$

[3] Numbers in square brackets refer to the bibliography at the end of the paper.

Then a map

$$(1.1.2) \qquad g: (A; A_1, \cdots, A_m) \to (B; B_1, \cdots, B_m)$$

is said to be *induced* by the map f if $g(x) = f(x)$ for $x \in A$. In case $X \subset Y$, $X_i \subset Y_i$, $i = 1, \cdots, k$, and $f(x) = x$ for all $x \in X$, we say that the map f in (1.1.1) is an *inclusion* map. This is sometimes indicated by the notation

$$f: (X; X_1, \cdots, X_k) \subset (Y; Y_1, \cdots, Y_k).$$

Thus an inclusion map is a map which is induced by the identity map $X \to X$.

Two maps

$$f_0, f_1 : (X; X_1, \cdots, X_k) \to (Y; Y_1, \cdots, Y_k)$$

are said to be *homotopic* (written $f_0 \simeq f_1$) if there exists a mapping

$$(1.1.3) \qquad f: (X \times I; X_1 \times I, \cdots, X_k \times I) \to (Y; Y_1, \cdots, Y_k),$$

(where $I = \{x \mid 0 \leqq x \leqq 1\}$ is the closed unit interval), such that

$$\left.\begin{aligned} f(x, 0) &= f_0(x) \\ f(x, 1) &= f_1(x) \end{aligned}\right\} \text{for all } x \in X.$$

The maps f_0, f_1 are said to be homotopic relative to a subset $A \subset X$ (Notation: $f_0 \simeq f_1$ rel A) if the map f in (1.1.3) satisfies the additional condition

$$f(x, t) = f(x, 0) \quad \text{for all} \quad x \in A, \qquad 0 \leqq t \leqq 1.$$

The existence of a homotopy (1.1.3) between the maps f_0, f_1 is often indicated by saying that there exists a continuous 1-parameter family of maps

$$f_t : (X; X_1, \cdots, X_k) \to (Y; Y_1, \cdots, Y_k), \qquad 0 \leqq t \leqq 1.$$

The relation of homotopy (1.1.3) is an equivalence relation and consequently divides the set of all maps (1.1.1) into disjoint homotopy classes. A similar statement can be made about homotopy relative to a subset.

If the sets $X_1, \cdots, X_k$ in the preceding paragraph have a non-vacuous intersection, $X_1 \cap \cdots \cap X_k \neq 0$, we call the ordered collection of spaces $(X; X_1, \cdots, X_k)$ a $(k + 1)$-ad; in particular, a 2-ad will be called a dyad, a 3-ad a triad, a 4-ad a tetrad, etc. We will also use the term *pair* for an ordered couple of topological spaces (X, A) such that $A \subset X$. A dyad is a pair in which the subspace is non-vacuous. A *triple*, (X, A, B), consists of three spaces which satisfy the condition $X \supset A \supset B$. We shall regard as distinct two $(k + 1)$-ads which are obtained by a non-trivial permutation of the sub-spaces $X_1, \cdots, X_k$.

If A is a subset of a topological space X, the notations Int A and Cl A will denote the interior and closure of A respectively.

Many of the spaces which we shall consider will be subspaces of Cartesian n-space, C^n. For convenience, we consider all such spaces C^n, $n = 1, 2, \cdots$, as subspaces of infinite dimensional Cartesian space, C^∞, whose points x are infinite sequences of real numbers $(x_1, x_2, \cdots, x_n, \cdots)$ having the property that $x_i = 0$ for all but a finite number of integers i. The real vector space C^∞ is metrized by the usual distance formula, $\rho(x, y) = \| x - y \|$, where $\| x \|$ is defined by

$$\| x \| = \left[\sum_{i=1}^{\infty} x_i^2 \right]^{\frac{1}{2}}.$$

The space $C^n \subset C^\infty$ consists of those points $(x_1, x_2, \cdots)$ for which $x_i = 0$ when $i > n$. For convenience we often shorten the notation

$$(x_1, \cdots, x_n, 0, \cdots)$$

for points of C^n to $(x_1, \cdots, x_n)$.

The *unit n-cell*, E^n, and *unit* $(n-1)$*-sphere*, S^{n-1}, of C^n are defined as follows:

$$E^n = \{x \in C^n \mid \| x \| \leqq 1\}$$
$$S^{n-1} = \{x \in C^n \mid \| x \| = 1\}.$$

Any topological space homeomorphic to E^n or S^{n-1} is called an n-cell or $(n - 1)$-sphere respectively. If $\mathcal{E}^n$ is an n-cell, and $\psi: E^n \to \mathcal{E}^n$ is a homeomorphism then the subset $\dot{\mathcal{E}}^n = \psi(S^{n-1}) \subset \mathcal{E}^n$ is called the *boundary* or *bounding sphere* of $\mathcal{E}^n$. To *orient* an n-cell $\mathcal{E}^n$ means to choose a generator of the (infinite cyclic) integral homology group, $H_n(\mathcal{E}^n, \dot{\mathcal{E}}^n)$. Similarly, an orientation of any $(n - 1)$-sphere $\mathcal{S}^{n-1}$ is given by choosing a generator of the integral homology group $H_{n-1}(\mathcal{S}^{n-1})$. If $\mathcal{E}_1^n$, $\mathcal{E}_2^n$ are n-cells with orientations $w_1^n \in H_n(\mathcal{E}_1^n, \dot{\mathcal{E}}_1^n)$, $w_2^n \in H_n(\mathcal{E}_2^n, \dot{\mathcal{E}}_2^n)$ respectively, then there exist *orientation preserving* homeomorphisms from $\mathcal{E}_1^n$ to $\mathcal{E}_2^n$; i.e., homeomorphisms h: $\mathcal{E}_1^n \to \mathcal{E}_2^n$ such that $h_*(w_1^n) = w_2^n$, where h_* is the induced isomorphism

$$h_* : H_n(\mathcal{E}_1^n, \dot{\mathcal{E}}_1^n) \approx H_n(\mathcal{E}_2^n, \dot{\mathcal{E}}_2^n).$$

It will be convenient for the later discussion of equivalent definitions for the triad homotopy groups, and for the definitions of the boundary operators, if we choose certain definite orientations for the unit n-cell E^n and unit $(n - 1)$-sphere S^{n-1}, $n \geqq 1$. The orientations which we choose are best described inductively, using the so-called "incidence isomorphisms" of homology theory (cf. Eilenberg and Steenrod, [5]). We first describe some additional notation. Let

$$E_+^{n-1} = \{x \in S^{n-1} \mid x_n \geqq 0\},$$
$$E_-^{n-1} = \{x \in S^{n-1} \mid x_n \leqq 0\}, \qquad n \geqq 2.$$

Then $S^{n-2} = \{x \in S^{n-1} \mid x_n = 0\} = E_+^{n-1} \cap E_-^{n-1}$.

Let

$$i_1 : S^{n-1} \to (S^{n-1}, E_-^{n-1}),$$
$$i_2 : S^{n-1} \to (S^{n-1}, E_+^{n-1}),$$
$$j_1 : (E_+^{n-1}, S^{n-2}) \to (S^{n-1}, E_-^{n-1}),$$
$$j_2 : (E_-^{n-1}, S^{n-2}) \to (S^{n-1}, E_+^{n-1}),$$

denote inclusion maps; let

$$\partial : H_n(E^n, S^{n-1}) \to H_{n-1}(S^{n-1})$$

denote the homology boundary operation; and let

$$p_+ : (E_+^{n-1}, S^{n-2}) \to (E^{n-1}, S^{n-2}),$$
$$p_- : (E_-^{n-1}, S^{n-2}) \to (E^{n-1}, S^{n-2}),$$

denote the projection homeomorphisms defined by

$$p_+(x_1, \cdots, x_{n-1}, x_n) = (x_1, \cdots, x_{n-1}, 0),$$
$$p_-(x_1, \cdots, x_{n-1}, x_n) = (x_1, \cdots, x_{n-1}, 0).$$

The isomorphisms induced on the integral homology groups by i_1, i_2, j_1, j_2, p_+, p_-, will be denoted by i_{1*}, i_{2*}, j_{1*}, j_{2*}, p_{+*}, p_{-*}, respectively. First, choose an orientation $w^1 \in H^1(E^1, S^0)$ as follows: Consider E^1 as an ordered 1-simplex whose first vertex is $+1$ and whose last vertex is -1. The identity map of E^1 onto itself is a singular 1-simplex, which is a relative 1-cycle modulo S^0. Then w^1 is defined to be the homology class of this relative 1-cycle. Assume that the orientation $w^{n-1} \in H_{n-1}(E^{n-1}, S^{n-2})$, of E^{n-1} has been chosen with $n \geqq 2$. Then choose the orientation $w^n \in H_n(E^n, S^{n-1})$ for E^n, such that

$$p_{+*} j_{1*}^{-1} i_{1*} \partial(w^n) = w^{n-1}.$$

It follows from the definition of the incidence isomorphisms that

$$p_{-*} j_{2*}^{-1} i_{2*} \partial(w^n) = -w^{n-1}.$$

We now choose the orientation of S^{n-1} to be that induced by the orientation of E^n, i.e., we choose the generator $w_0^{n-1} \in H_{n-1}(S^{n-1})$ where

$$w_0^{n-1} = \partial(w^n).$$

If $n = 1$ we use the reduced group $\tilde{H}_0(S^0)$ (see [5]). In Part 3 we shall need definite orientations for the $(n-1)$-cells E_+^{n-1} and E_-^{n-1}. We define these to be the generators $w_+^{n-1} \in H_{n-1}(E_+^{n-1}, S^{n-2})$, $w_-^{n-1} \in H_{n-1}(E_-^{n-1}, S^{n-2})$ given by

$$w_+^{n-1} = j_{1*}^{-1} i_{1*} \partial(w^n); \qquad w_-^{n-1} = j_{2*}^{-1} i_{2*} \partial(w^n).$$

We shall also need to choose orientations for the *unit n-cube*, I^n, of C^n, defined as follows:

$$I^n = \{x \in C^n \mid 0 \leqq x_1 \leqq 1, 0 \leqq x_2 \leqq 1, \cdots, 0 \leqq x_n \leqq 1\}.$$

The following notation will be used consistently for certain subsets of I^n:

$$\begin{aligned} I^{n-1} &= \{x \in I^n \mid x_n = 0\}, \\ \dot{I}^n &= I^n - \operatorname{Int} I^n = \text{boundary of } I^n, \\ J^{n-1} &= \dot{I}^n - \operatorname{Int} I^{n-1}, \\ I_-^{n-1} &= \{x \in I^{n-1} \mid x_1 \geqq \tfrac{1}{2}\}, \\ I_+^{n-1} &= \{x \in I^{n-1} \mid x_1 \leqq \tfrac{1}{2}\}. \end{aligned}$$

It is readily shown that I^n, J^{n-1}, I_+^{n-1}, and I_-^{n-1} are cells, and that $\dot{I}^n$ is an $(n-1)$-sphere. We shall describe an inductive method for giving a definite orientation to I^n, for all n. This method is similar to that used above in describing definite orientations for the unit cell and unit sphere. Let

$$i: (I^{n-1}, \dot{I}^{n-1}) \to (\dot{I}^n, J^{n-1})$$
$$j: \dot{I}^n \to (\dot{I}^n, J^{n-1})$$

be inclusion maps; let i_* and j_* be the corresponding induced isomorphisms of the integral homology groups, and let $\partial: H_n(I^n, \dot{I}^n) \to H_{n-1}(\dot{I}^n)$ denote the homology boundary operator. Choose an orientation v^1 for $I^1 = I$ by considering I as an ordered 1-simplex whose first vertex is 0 and whose second vertex is 1, and taking the homology class relative to $\dot{I}$ of the singular simplex which is defined by the identity map of I onto itself, as was done above for E^1. Suppose an orientation v^{n-1} already chosen from $H_{n-1}(I^{n-1}, \dot{I}^{n-1})$. Then choose $v^n \in H_n(I^n, \dot{I}^n)$ so that

$$i_*^{-1} j_* \partial(v^n) = v^{n-1}.$$

We assume henceforth that I^n has this fixed orientation.

Let $\mathcal{E}^n$ be an n-cell, $\dot{\mathcal{E}}^n$ the boundary of $\mathcal{E}^n$, and $\mathcal{E}^{n-1}$ an $(n-1)$-cell which is a subset of $\dot{\mathcal{E}}^n$. Then $\mathcal{E}^{n-1}$ is said to be a *face* of $\mathcal{E}^n$ if and only if there exists a homeomorphism $\psi: E^n \to \mathcal{E}^n$ such that

$$\psi(E_+^{n-1}) = \mathcal{E}^{n-1},$$
$$\psi(E_-^{n-1}) = \dot{\mathcal{E}}^n - \operatorname{Int} \mathcal{E}^{n-1}.$$

(1.2) Homotopy Groups and Relative Homotopy Groups

We assume that the reader is familiar with the various possible definitions, and the basic properties, of the relative and absolute homotopy groups.[4]

Let (X, A) be a dyad and choose a base point $x_0 \in A$. Then an element of the relative homotopy group $\pi_n(X, A, x_0)$, $n \geqq 2$, is determined by a homotopy class of maps

$$(\mathcal{E}^n, \dot{\mathcal{E}}^n, p_0) \to (X, A, x_0)$$

where $\mathcal{E}^n$ is an *oriented* n-cell, $\dot{\mathcal{E}}^n$ is its boundary, and $p_0 \in \dot{\mathcal{E}}^n$ is a fixed reference point. In case A consists of the single point x_0 the group is also defined for $n = 1$, and we write the group as $\pi_n(X, x_0)$ and refer to it as an *absolute* homotopy group.

The family of groups $\pi_n(X, A, x)$ for $x \in A$ forms a local system of groups in the space A, in the sense of Steenrod [15], and $\pi_1(A, x_0)$ is a group of operators on $\pi_n(X, A, x_0)$. If $\pi_1(A, x_0)$ operates trivially on $\pi_n(X, A, x_0)$ and A is arcwise connected, we say that the pair (X, A) is *simple in dimension n*, or that *X is simple relative to A in the dimension n.*

[4] For an elementary exposition of the theory of homotopy groups, see Fox [7], or Hu, [9].

As indicated, the absolute homotopy groups $\pi_n(X, x_0)$ are only defined for $n \geqq 1$, while the relative homotopy groups $\pi_n(X, A, x_0)$ are only defined for $n \geqq 2$. In what follows it will often be convenient to give a meaning to the symbols $\pi_0(X, x_0)$ and $\pi_1(X, A, x_0)$. Let S^0 be the unit 0-sphere, and p_0 the point (1, 0). Then $\pi_0(X, x_0)$ is defined to be the set of homotopy classes of mappings $(S^0, p_0) \to (X, x_0)$. We will not attempt to give this set any algebraic structure. It is clear that it has one element for each arcwise connected component of X. We shall refer to the element corresponding to the constant map $S^0 \to x_0$ as the "neutral element" or "identity element" and denote it by the symbol 0. Thus the notation $\pi_0(X, x_0) = 0$ will indicate that X is arcwise connected. In a similar manner we use the symbol $\pi_1(X, A, x_0)$ to denote the set of all homotopy classes of maps $(E^1, \dot{E}^1, p_0) \to (X, A, x_0)$. The "neutral element" of this set is the class of the constant map $E^1 \to x_0$ and is again denoted by 0. The notation $\pi_1(X, A, x_0) = 0$ indicates that the set contains only the neutral element. If X and A are arcwise connected, the condition $\pi_1(X, A) = 0$ is equivalent to the condition that the natural homomorphism $\pi_1(A) \to \pi_1(X)$ be a homomorphism onto.

In section (3.5) we will make use of the *homotopy sequence of a triple* (X, A, B), where $B \neq 0$. Choose a base point $x_0 \in B$, and consider the following sequence of groups and homomorphisms;

$$\cdots \xrightarrow{j_*} \pi_{n+1}(X, A, x_0) \xrightarrow{\beta} \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\beta} \cdots$$

The homomorphisms i_* and j_* are induced by inclusion maps i and j, while the boundary operator of the triple, β, is defined to be the composition of the boundary operator of the pair (X, A), $\beta': \pi_{n+1}(X, A, x_0) \to \pi_n(A, x_0)$, and the homomorphism $k_*: \pi_n(A, x_0) \to \pi_n(A, B, x_0)$ induced by the inclusion map k. *The homotopy sequence of a triple is exact.* This fact may be proved purely algebraically, making use only of the most elementary properties of homotopy groups of pairs, including the exactness of the homotopy sequence of a pair (cf. Eilenberg and Steenrod, [5], for the case of the homology sequence of a triple).

Part 2: The Triad Homotopy Groups

(2.1) Definition by Mappings of Cubes

Let $(X; A, B)$ be any triad, and choose a base point $x_0 \in A \cap B$. The symbol $F_n(X; A, B, x_0)$ will denote the function space of all maps

$$(I^n; I_+^{n-1}, I_-^{n-1}, J^{n-1}) \to (X; A, B, x_0)$$

with the compact open topology. (See Fox [6]). Two elements of $F_n(X; A, B, x_0)$ are homotopic if and only if they lie in the same arcwise connected component of $F_n(X; A, B, x_0)$. We introduce an operation of addition in $F_n(X; A, B, x_0)$ for $n > 2$, as follows: If $f, g \in F_n(X; A, B, x_0)$ define $h = f + g$ by

$$h(x_1, \cdots, x_n) = \begin{cases} f(x_1, 2x_2, x_3, \cdots, x_n), & 0 \leqq x_2 \leqq \frac{1}{2}, \\ g(x_1, 2x_2 - 1, x_3, \cdots, x_n), & \frac{1}{2} \leqq x_2 \leqq 1. \end{cases}$$

It is readily verified that h is a continuous map

$$h:(I^n; I_+^{n-1}, I_-^{n-1}, J^{n-1}) \to (X; A, B, x_0)$$

and hence $h \in F_n(X; A, B, x_0)$. Further, the operation of addition thus defined is continuous in the topology of $F_n(X; A, B, x_0)$; i.e., the map $F_n \times F_n \to F_n$ defined by $(f, g) \to f + g$ is continuous. Now let $\pi_n(X; A, B, x_0)$ denote the set of homotopy classes of elements of $F_n(X; A, B, x_0)$; each element is an arcwise connected component of $F_n(X; A, B, x_0)$. It follows from the continuity of the addition defined above that if $f \simeq f'$ and $g \simeq g'$ then $f + g \simeq f' + g'$. Hence the addition defined in $F_n(X; A, B, x_0)$ induces an addition in $\pi_n(X; A, B, x_0)$. We could now verify directly that with this definition of addition $\pi_n(X; A, B, x_0)$ becomes a group. The procedure would parallel that of Fox (see [7]). However it is easier to proceed as in the following section.

(2.2) The Function-space Definition

Let $k_p \in F_p(X; A, B, x_0)$ denote the constant map $I^p \to x_0$, $p \geqq 2$, and consider the (multiplicative) fundamental group $\pi_1(F_{n-1}(X; A, B, x_0), k_{n-1})$, $n \geqq 3$. An element $\alpha \in \pi_1(F_{n-1}, k_{n-1})$ is an equivalence class of closed paths.

$$a:(I, \dot{I}) \to (F_{n-1}, k_{n-1}).$$

We define a function

$$\varphi:\pi_1(F_{n-1}, k_{n-1}) \to \pi_n(X; A, B, x_0), \qquad n > 2,$$

as follows: If $a \in \alpha \in \pi_1(F_{n-1}, k_{n-1})$, $a:(I, \dot{I}) \to (F_{n-1}, k_{n-1})$, define a map

$$f:(I^n; I_+^{n-1}, I_-^{n-1}, J^{n-1}) \to (X; A, B, x_0)$$

by

$$f(x_1, \cdots, x_n) = [a(x_2)](x_1, x_3, \cdots, x_n)$$

for $x = (x_1, \cdots, x_n) \in I^n$. This mapping is well-defined since $a(x_2)$ is a mapping

$$a(x_2):(I^{n-1}; I_+^{n-2}, I_-^{n-2}, J^{n-2}) \to (X_2 A, B, x_0).$$

We now define $\varphi(\alpha)$ to be the homotopy class of f. It is readily proved that the function φ is $1 - 1$ and onto, and that, for any elements $\alpha, \beta \in \pi_1(F_{n-1}, k_{n-1})$,

$$\varphi(\alpha \cdot \beta) = \varphi(\alpha) + \varphi(\beta).$$

Since $\pi_1(F_{n-1}, k_{n-1})$ is known to be a group it follows that $\pi_n(X; A, B, x_0)$ is also a group, and that the function φ is an isomorphism of $\pi_1(F_{n-1}, k_{n-1})$ onto $\pi_n(X; A, B, x_0)$. We call the group $\pi_n(X; A, B, x_0)$ the *n^{th} or n-dimensional homotopy group of the triad $(X; A, B)$ at the base point x_0*; it is only defined for $n \geqq 3$.

In a similar way we can define isomorphisms

$$\varphi_m:\pi_m(F_n(X; A, B, x_0), k_n) \approx \pi_{m+n}(X; A, B, x_0)$$

for $m \geqq 1, n > 1$.

THEOREM 2.2.1. *The group $\pi_n(X; A, B, x_0)$ is abelian for $n > 3$.*

PROOF. This follows at once from the above, since $\pi_m(F_n, k_n)$ is abelian when $m > 1$.

We shall give examples later to show that $\pi_3(X; A, B, x_0)$ need not be abelian.

THEOREM 2.2.2. $\pi_n(X; A, B, x_0) \approx \pi_n(X; B, A, x_0)$.

PROOF. We shall exhibit a specific isomorphism in terms of mappings of cubes. Let $\alpha \in \pi_n(X; A, B, x_0)$ and $f \in \alpha$,

$$f:(I^n; I_+^{n-1}, I_-^{n-1}, J^{n-1}) \to (X; A, B, x_0).$$

Define a map

$$f':(I^n; I_+^{n-1}, I_-^{n-1}, J^{n-1}) \to (X; B, A, x_0)$$

by

$$f'(x_1, x_2, \cdots, x_n) = f(1 - x_1, x_2, \cdots, x_n).$$

Then f' determines an element $\varphi(\alpha) \in \pi_n(X; B, A, x_0)$ and the function φ thus defined,

$$\varphi:\pi_n(X; A, B, x_0) \to \pi_n(X; B, A, x_0),$$

is readily verified to be a homomorphism. Moreover, one can define, in a symmetric manner, a second homomorphism

$$\varphi':\pi_n(X; B, A, x_0) \to \pi_n(X; A, B, x_0),$$

and obviously $\varphi\varphi' = 1$, $\varphi'\varphi = 1$, so that φ is an isomorphism. If $n = 2$, φ sets up a $1 - 1$ correspondence between the elements of $\pi_2(X; A, B, x_0)$ and those of $\pi_2(X; B, A, x_0)$.

Just as in the case of the absolute and relative homotopy groups, it is frequently convenient to replace I^n by a homeomorphic copy in defining $\pi_n(X; A, B, x_0)$. This leads to a group isomorphic to that already defined, but in order to obtain a specific isomorphism, one must choose an orientation for the homeomorph of I^n.

(2.3) Definition by Mappings of Cells

Let E^n, S^{n-1}, $n \geqq 2$, be the unit n-cell and unit $(n - 1)$-sphere defined in part 1, and let

$$E_1^n = \{x \in E^n \mid x_2 \geqq 0\},$$
$$E_2^n = \{x \in E^n \mid x_2 \leqq 0\},$$
$$p_0 = (1, 0, \cdots, 0).$$

Let $F'_n(X; A, B, x_0)$ denote the set of all mappings

$$(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0),$$

and let $\pi'_n(X; A, B, x_0)$ denote the set of homotopy classes of these mappings. We have described in (1.1) an orientation $w^n \in H_n(E^n, S^{n-1})$ for E^n. Using this orientation, and the orientation $v^n \in H_n(I^n, \dot{I}^n)$ described in (1.1), we can set up a $1 - 1$ correspondence

$$\psi:\pi'_n(X; A, B, x_0) \to \pi_n(X; A, B, x_0)$$

as follows: Choose a map

$$h:(I^n; I_+^{n-1}, I_-^{n-1}, J^{n-1}) \to (E^n; E_+^{n-1}, E_-^{n-1}, p_0)$$

which maps $I^n - J^{n-1}$ homeomorphically onto $E^n - p_0$, and such that $h_*(v^n) = w^n$, where

$$h_*: H_n(I^n, \dot{I}^n) \to H_n(E^n, S^{n-1})$$

is the isomorphism induced by h. Given $f \in F'_n(X; A, B, x_0)$ define $\psi'(f)$ by

$$[\psi'(f)](x) = f[h(x)], \; x \in I^n.$$

It is readily seen that the map

$$\psi': F'_n(X; A, B, x_0) \to F_n(X; A, B, x_0)$$

is a homeomorphism onto, and hence induces a $1 - 1$ map

$$\psi: \pi'_n(X; A, B, x_0) \to \pi_n(X; A, B, x_0).$$

We can now define an addition in $\pi'_n(X; A, B, x_0)$ in such a way that this set becomes a group, by requiring that ψ shall be an isomorphism. It is useful to have a more direct, geometrical definition for the group operation in $\pi'_n(X; A, B, x_0)$. First we prove a lemma.

LEMMA 2.3.1. *Given any map $f \in F'_n(X; A, B, x_0)$, there exist homotopic maps $f', f'' \in F'_n$, such that $f'(E_1^n) = f''(E_2^n) = x_0$.*

Before proceeding with the proof we quote two well known lemmas on which the proof depends.

LEMMA 2.3.2. *Let K be a complex, L a subcomplex, $f_0: K \to Y$ a continuous map of K into an arbitrary space Y, $f_0 \mid L = g_0: L \to Y$, and $g_t: L \to Y$ a homotopy of g_0. Then there exists a homotopy $f_t: K \to Y$ of f such that $f_t \mid L = g_t$, $0 \leqq t \leqq 1$.*

This is the homotopy extension theorem (cf. Alexandroff and Hopf, [1] p. 501).

LEMMA 2.3.3. *Let $\mathcal{E}^m$ be an m-cell, $\mathcal{E}^{m-1}$ an $(m - 1)$-cell which is a face of $\mathcal{E}^m$, f_0, $f_1: \mathcal{E}^m \to Y$ two maps of $\mathcal{E}^m$ into an arbitrary space Y, and $g_0 = f_0 \mid \mathcal{E}^{m-1}$, $g_1 = f_1 \mid \mathcal{E}^{m-1}$. Then any homotopy $g_t: \mathcal{E}^{m-1} \to Y$ between g_0 and g_1 can be extended to a homotopy $f_t: \mathcal{E}^m \to Y$ between f_0 and f_1.* This lemma depends on the fact that the space $(\mathcal{E}^m \times 0) \cup (\mathcal{E}^m \times 1) \cup (\mathcal{E}^{m-1} \times I)$ is a retract of $\mathcal{E}^m \times I$.

PROOF OF (2.3.1). We shall prove only the existence of the map f'; the proof of the existence of the map f'' is entirely analogous. Let

$$E_1^{n-2} = \{x \in S^{n-2} \mid x_2 \geqq 0\},$$
$$E_2^{n-2} = \{x \in S^{n-2} \mid x_2 \leqq 0\},$$
$$E_1^{n-1} = \{x \in S^{n-1} \mid x_2 \geqq 0\},$$
$$E_{1+}^{n-1} = \{x \in E_+^{n-1} \mid x_2 \geqq 0\},$$
$$E_{1-}^{n-1} = \{x \in E_-^{n-1} \mid x_2 \geqq 0\},$$
$$E_{2+}^{n-1} = \{x \in E_+^{n-1} \mid x_2 \leqq 0\},$$
$$E_{2-}^{n-1} = \{x \in E_-^{n-1} \mid x_2 \leqq 0\}.$$

Let $a_0: S^{n-2} \to A \cap B$ be a map defined by f. It is well known (see Eilenberg [3]) that a_0 is homotopic relative to p_0 to a map $a_1: S^{n-2} \to A \cap B$ such that $a_1(E_1^{n-2}) = x_0$. Let $a_t: S^{n-2} \to A \cap B$ denote the homotopy between a_0 and a_1. Let $b_t: E_1^{n-2} \to A \cap B$ be the map induced by a_t, $0 \leqq t \leqq 1$. Next, let

$$c_0: E_{1+}^{n-1} \to A,$$

$$d_0: E_{1-}^{n-1} \to B,$$

be maps defined by f, and let

$$c_1: E_{1+}^{n-1} \to A,$$

$$d_1: E_{1-}^{n-1} \to B,$$

be the constant maps into x_0. Applying Lemma 2.3.3 we obtain homotopies

$$c_t: E_{1+}^{n-1} \to A,$$

$$d_t: E_{1-}^{n-1} \to B,$$

such that $c_t \mid E_1^{n-2} = b_t$, $d_t \mid E_1^{n-2} = b_t$. Next, let $e_0: E_+^{n-1} \to A$ be defined by f, and let $g_t: E_{1+}^{n-1} \cup S^{n-2} \to A$ be defined by: $g_t \mid E_{1+}^{n-1} = c_t$, $g_t \mid S^{n-2} = a_t$. Then $g_0 = e_0 \mid (E_{1+}^{n-1} \cup S^{n-2})$. Apply Lemma 2.3.2 to obtain a homotopy

$$e_t: E_+^{n-1} \to A,$$

which is an extension of g_t. In a similar manner, if $h_0: E_-^{n-1} \to B$ is defined by f, then there exists a homotopy $h_t: E_-^{n-1} \to B$ of h_0, such that $h_t \mid E_{1-}^{n-1} = d_t$, $h_t \mid S^{n-2} = a_t$.

Now let $i_0: E_1^n \to X$ be the map induced by f, and let $j_t: E_1^{n-1} \to X$ be defined by $j_t \mid E_{1+}^{n-1} = c_t$ and $j_t \mid E_{1-}^{n-1} = d_t$. Let $i_1: E_1^n \to X$ be the constant map. Now apply Lemma 2.3.3 to obtain a homotopy $i_t: E_1^n \to X$ which is an extension of j_t.

Finally, let $f_0: E^n \to X$ be defined by f and let $k_t: E_1^n \cup S^{n-1} \to X$ be defined by: $k_t \mid E_1^n = i_t$, $k_t \mid E_+^{n-1} = e_t$, $k_t \mid E_-^{n-1} = h_t$. Apply Lemma 2.3.2 and obtain a homotopy $f_t: E^n \to X$ of f_0, which is an extension of k_t. Let

$$f': (E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0)$$

be defined by f_1. Then f' has the required properties.

We may now define an addition in $\pi'_n(X; A, B, x_0)$ directly, as follows: Let $\alpha, \beta \in \pi'_n(X; A, B, x_0)$. Choose maps $f, g \in F'_n(X; A, B, x_0)$ representing α and β respectively, such that

$$f(E_2^n) = g(E_1^n) = x_0.$$

Define $k \in F'_n(X; A, B, x_0)$ by

$$k \mid E_1^n = f \mid E_1^n, \qquad k \mid E_2^n = g \mid E_2^n.$$

Then $\alpha + \beta$ is the homotopy class of k. It is not difficult to see that this definition is independent of the choice of the representatives f and g, and that it corresponds under the function ψ to the addition already defined in $\pi_n(X; A, B, x_0)$.

Consequently we shall usually use the symbol $\pi_n(X; A, B, x_0)$ in place of $\pi'_n(X; A, B, x_0)$. The role played by the definite orientations chosen for E^n and I^n was to ensure that ψ should be a homomorphism rather than an antihomomorphism. This has significance in dimension 3 where the triad group need not be abelian. The choice of the map h was limited by the condition $h_*(v^n) = w^n$, but the correspondence ψ was otherwise independent of the choice of h, and the condition that h map $I^n - J^{n-1}$ *homeomorphically* onto $E^n - p_0$ was superfluous.

If we are given any other n-cell, $\mathcal{E}^n$, with a definite orientation $u^n \in H_n(\mathcal{E}^n, \dot{\mathcal{E}}^n)$ chosen, with the boundary $\dot{\mathcal{E}}^n$ decomposed into the union of two faces with disjoint interiors, $\dot{\mathcal{E}}^n = \mathcal{E}_+^{n-1} \cup \mathcal{E}_-^{n-1}$, and with a base point $q_0 \in \mathcal{E}_+^{n-1} \cap \mathcal{E}_-^{n-1}$ specified, then we can define a group $\pi''_n(X; A, B, x_0)$ by properly introducing a group operation into the set of homotopy classes of maps

$$(\mathcal{E}^n; \mathcal{E}_+^{n-1}, \mathcal{E}_-^{n-1}, q_0) \to (X; A, B, x_0),$$

and the orientations of $\mathcal{E}^n$ and E^n will enable us to set up an isomorphism with the group $\pi_n(X; A, B, x_0)$. Consequently we shall regard mappings of $\mathcal{E}^n$ of the above type as determining elements of $\pi_n(X; A, B, x_0)$.

As noted above, the group $\pi_n(X; A, B, x_0)$ is only defined for $n > 2$; the symbol $\pi_2(X; A, B, x_0)$ is used for the set of homotopy classes of maps $(E^2; E_+^1, E_-^1, p_0) \to (X; A, B, x_0)$ or for the set of homotopy classes of maps $(I^2; I_+^1, I_-^1, J^1) \to (X; A, B, x_0)$, since the function ψ of (2.3) sets up a $1 - 1$ correspondence between these sets, and preserves the class of the constant map into x_0. This class will be denoted by the symbol 0 and referred to as the "neutral element" or "identity element." The notation $\pi_2(X; A, B, x_0) = 0$ means that the set $\pi_2(X; A, B, x_0)$ contains a single class, that of the identity. We know of no general procedure for introducing a group operation into the set $\pi_2(X; A, B, x_0)$, although this can be done in special cases. In spite of this the set plays a useful role in what follows.

Part 3: Elementary Properties of the Triad Homotopy Groups

(3.1) Induced Homomorphisms

Let $(X; A, B, x_0)$, $(Y; C, D, y_0)$ be triads with base points

$$x_0 \in A \cap B, \qquad y_0 \in C \cap D,$$

and let $f:(X; A, B, x_0) \to (Y; C, D, y_0)$ be a mapping. Then f *induces* homomorphisms

$$f_*: \pi_n(X; A, B, x_0) \to \pi_n(Y; C, D, y_0), \qquad n > 2.$$

These are defined as follows: Let $\alpha \in \pi_n(X; A, B, x_0)$ and let

$$g:(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0)$$

be a representative of α. Then the composite map

$$fg:(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (Y; C, D, y_0)$$

is a representative of $f_*(\alpha) \in \pi_n(Y; C, D, y_0)$. The function f_* thus defined is independent of the choice of $g \in \alpha$, and is easily verified to be a homomorphism. If $n = 2$ the function f_* is still defined and carries the neutral element of

$$\pi_2(X; A, B, x_0)$$

into the neutral element of $\pi_2(Y; C, D, y_0)$. In this case we shall continue to use the terms *kernel of f* and *image of f* for the sets $f_*^{-1}(0)$ and $f_*(\pi_2(X; A, B, x_0))$ respectively.

The following properties of f_* are obvious:

(3.1.1) *If f is the identity map, then f_* is the identity homomorphism (function when $n = 2$).*

(3.1.2) *If $f:(X; A, B, x_0) \to (Y; C, D, y_0)$ and $g:(Y; C, D, y_0) \to (Z; E, F, z_0)$, then $(gf)_* = g_*f_*$.*

(3.1.3) *If two maps*

$$f_0, f_1:(X; A, B, x_0) \to (Y; C, D, y_0)$$

are homotopic, then $f_{0*} = f_{1*}$.

(3.2) The Case $A \supset B$.

In this section we prove the following:

THEOREM 3.2.1. *If $A \supset B$ then the triad homotopy group $\pi_n(X; A, B, x_0)$ is naturally isomorphic to the relative homotopy group $\pi_n(X, A, x_0)$, $n > 2$.*

PROOF. We will consider the elements of $\pi_n(X; A, B, x_0)$ to be equivalence classes of maps $(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0)$ while the elements of $\pi_n(X, A, x_0)$ are equivalence classes of maps $(E^n, S^{n-1}, p_0) \to (X, A, x_0)$. A natural function

$$\varphi:\pi_n(X; A, B, x_0) \to \pi_n(X, A, x_0), \qquad n \geqq 2,$$

is defined by considering a representative of an element $\alpha \in \pi_n(X; A, B, x_0)$ as a representative of an element $\varphi(\alpha) \in \pi_n(X, A, x_0)$. If $n > 2$ the function φ is clearly a homomorphism. Since every homotopy class $\beta \in \pi_n(X, A, x_0)$ contains representative maps g with the property $g(E_-^{n-1}) = x_0$, we can set up a function

$$\varphi':\pi_n(X, A, x_0) \to \pi_n(X; A, B, x_0), \qquad n \geqq 2,$$

by defining $\varphi'(\beta)$ to be the element of $\pi_n(X; A, B, x_0)$ determined by considering g as a map

$$g:(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0).$$

The element $\varphi'(\beta)$ is easily seen to be independent of the choice of g. For if $g_0, g_1 \in \beta$,

$$g_0, g_1:(E^n, S^{n-1}, p_0) \to (X, A, x_0)$$

with $g_0(E_-^{n-1}) = g_1(E_-^{n-1}) = x_0$, then any homotopy g_t from g_0 to g_1 can be modified to a homotopy $g_t':(E^n, S^{n-1}, p_0) \to (X, A, x_0)$ with $g_t'(E_-^{n-1}) = x_0$, so that g'_t actually gives a homotopy between the maps

$$g_0, g_1:(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \rightarrow (X; A, B, x_0).$$

If $n > 2$, then the function φ' is a homomorphism. It follows immediately from the definitions of φ, φ' that $\varphi\varphi' = 1$. To see that $\varphi'\varphi$ is also the identity we observe that if $f:(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \rightarrow (X; A, B, x_0)$ is considered as a map

$$f_0:(E^n; S^{n-1}, p_0) \rightarrow (X, A, x_0)$$

and $f_t:(E^n, S^{n-1}, p_0) \rightarrow (X, A, x_0)$ is a homotopy to a map $f_1:(E^n, S^{n-1}, p_0) \rightarrow (X, A, x_0)$, with $f_1(E_-^{n-1}) = x_0$, then since $B \subset A$, f_t may be modified to give a homotopy f'_t from f_0 to f_1, with the property that $f'_t(E_-^{n-1}) \subset B$ Thus f'_t determines a homotopy between the mappings

$$f, f_1:(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \rightarrow (X; A, B, x_0)$$

so that $\varphi'\varphi = 1$, and hence φ is a natural isomorphism from $\pi_n(X; A, B, x_0)$ to $\pi_n(X, A, x_0)$, $n > 2$.

REMARK. If $n = 2$, φ sets up a $1 - 1$ correspondence between the set

$$\pi_2(X; A, B, x_0)$$

and the group $\pi_2(X, A, x_0)$, which preserves the identity element. Hence in this case we can define a group operation in $\pi_2(X; A, B, x_0)$ by requiring φ to be an isomorphism. This group operation could also be defined directly.

(3.3) Effect of Change of Base Point

Let x_0, x_1 be two base points in $A \cap B$, and let $\sigma:I \rightarrow A \cap B$ be a path from x_0 to x_1 in $A \cap B$. Then we can define an isomorphism

$$\varphi_\sigma:\pi_n(X; A, B, x_0) \rightarrow \pi_n(X; A, B, x_1), \qquad n > 2,$$

associated with the homotopy class of the path σ, in a manner similar to that used for the relative homotopy groups. With this definition, $\{\pi_n(X; A, B, x) \mid x \in A \cap B\}$ is a local system of groups in the space $A \cap B$, in the sense of Steenrod [15] and $\pi_1(A \cap B, x_0)$ is a group of operators on $\pi_n(X; A, B, x_0)$. If $A \cap B$ is arcwise connected, and the operators from $\pi_1(A \cap B, x_0)$ on $\pi_n(X; A, B, x_0)$ are all trivial, then we say that the triad $(X; A, B)$ is *simple in dimension n.* Since the groups $\pi_n(X; A, B, x)$, $x \in A \cap B$, are all isomorphic when $A \cap B$ is arcwise connected, we will frequently omit the base point from the discussion and write simply $\pi_n(X; A, B)$ and refer to it as the *homotopy group of the triad* $(X; A, B)$.

It should also be noted, in this connection, that the homomorphisms

$$f_*:\pi_n(X; A, B, x) \rightarrow \pi_n(Y; C, D, y), \qquad n > 2,$$

induced by mappings

$$f:(X; A, B, x) \rightarrow (Y; C, D, y)$$

are operator homomorphisms. The same remark will apply to the boundary operators which will be defined in the next section.

(3.4) The Boundary Operators

Let the elements of $\pi_n(X; A, B, x_0)$ be defined by maps

$$(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0)$$

where the cells E^n, E_+^{n-1}, E_-^{n-1} have the orientations described in (1.1). We now define functions

$$\left.\begin{array}{l} \beta_+ : \pi_n(X; A, B, x_0) \to \pi_{n-1}(A, A \cap B, x_0) \\ \beta_- : \pi_n(X; A, B, x_0) \to \pi_{n-1}(B, A \cap B, x_0) \end{array}\right\} n \geqq 2,$$

as follows: Given $\alpha \in \pi_n(X; A, B, x_0)$, choose a map $f:(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0)$ representing α. Then the induced maps

$$f \mid E_+^{n-1}:(E_+^{n-1}, S^{n-2}, p_0) \to (A, A \cap B, x_0)$$

$$f \mid E_-^{n-1}:(E_-^{n-1}, S^{n-1}, p_0) \to (B, A \cap B, x_0)$$

determine definite elements $\beta_+(\alpha)$, $\beta_-(\alpha)$ in the groups $\pi_{n-1}(A, A \cap B, x_0)$, $\pi_{n-1}(B, A \cap B, x_0)$, respectively. It is not difficult to see that these definitions for $\beta_+(\alpha)$ and $\beta_-(\alpha)$ are independent of the choice of the map f in the homotopy class α, and that with the chosen orientations for E_+^{n-1}, E_-^{n-1}, E^n, if $n > 2$, β_+ is a homomorphism and β_- is an anti-homomorphism. The functions β_+, β_- will be called the *boundary operators of the triad* $(X; A, B)$. The operator β_- will, of course, be a homomorphism if $n > 3$, but need not be when $n = 3$, since the group $\pi_2(B, A \cap B, x_0)$ need not be abelian. It is perhaps worth observing that for the corresponding boundary operators for the triad $(X; B, A)$,

$$\beta_+ : \pi_n(X; B, A, x_0) \to \pi_{n-1}(B, A \cap B, x_0)$$

$$\beta_- : \pi_n(X; B, A, x_0) \to \pi_{n-1}(A, A \cap B, x_0),$$

β_+ is again a homomorphism, while β_- is again an anti-homomorphism. Moreover the function φ of Theorem 2.2.2

$$\varphi : \pi_n(X; A, B, x_0) \to \pi_n(X; B, A, x_0)$$

induces a function

$$\varphi_1 : \pi_{n-1}(A, A \cap B, x_0) \to \pi_{n-1}(A, A \cap B, x_0)$$

which, for $n > 2$, is the anti-isomorphism which takes each element into its inverse, and we have commutativity in the following diagram:

$$\begin{array}{ccc} \pi_n(X; A, B, x_0) & \xrightarrow{\beta_+} & \pi_{n-1}(A, A \cap B, x_0) \\ \downarrow \varphi & & \downarrow \varphi_1 \\ \pi_n(X; B, A, x_0) & \xrightarrow{\beta_-} & \pi_{n-1}(A, A \cap B, x_0). \end{array}$$

If $n = 2$, β_+ and β_- are simply functions which carry the neutral element of $\pi_2(X; A, B, x_0)$ into the neutral elements of $\pi_1(A, A \cap B, x_0)$ and $\pi_1(B, A \cap B, x_0)$ respectively. The kernels of these functions are defined in the usual way.

(3.5) The Exact Sequences of a Triad

In this section we will use the following inclusion maps:

$$i_1:(A, A \cap B, x_0) \subset (X, B, x_0),$$

$$i_2:(B, A \cap B, x_0) \subset (X, A, x_0).$$

$$j_1:(X; x_0, B, x_0) \subset (X; A, B, x_0),$$

$$j_2:(X; A, x_0, x_0) \subset (X; A, B, x_0).$$

We have seen above that there exist natural isomorphisms

$$\pi_n(X; x_0, B, x_0) \approx \pi_n(X, B, x_0),$$

$$\pi_n(X; A, x_0, x_0) \approx \pi_n(X, A, x_0).$$

It will be convenient to identify such naturally isomorphic groups.

Let us consider the following infinite sequence:

$$(3.5.1) \quad \cdots \xrightarrow{\beta_+} \pi_n(A, A \cap B, x_0) \xrightarrow{i_{1*}} \pi_n(X, B, x_0) \xrightarrow{j_{1*}} \pi_n(X; A, B, x_0) \xrightarrow{\beta_+} \cdots$$

This sequence terminates with the following sets (not generally groups) and functions (not generally homomorphisms):

$$(3.5.2) \quad \cdots \xrightarrow{j_{1*}} \pi_2(X; A, B, x_0) \xrightarrow{\beta_+} \pi_1(A, A \cap B, x_0) \xrightarrow{i_{1*}} \pi_1(X, B, x_0).$$

All other terms of the sequence are groups, and all other functions are homomorphisms. This sequence is called the *upper homotopy sequence of the triad* $(X; A, B)$ *at the base point* x_0. The triad $(X; A, B)$ has a second (lower) homotopy sequence at x_0, namely

$$(3.5.3) \quad \cdots \xrightarrow{\beta_-} \pi_n(B, A \cap B, x_0) \xrightarrow{i_{2*}} \pi_n(X, A, x_0) \xrightarrow{j_{2*}} \pi_n(X; A, B, x_0) \xrightarrow{\beta_-} \cdots.$$

The most important property of the homotopy sequences of a triad is the following:

THEOREM 3.5.4. *Each homotopy sequence of a triad is exact.* (*i.e., the kernel of any homomorphism is precisely the image of the preceding homomorphism.*)

Before giving the proof we emphasize the following two facts: (a) In the proof no use is made of the group operation defined in $\pi_n(X; A, B, x_0)$ for $n > 2$ and defined in $\pi_n(A, A \cap B, x_0)$, $\pi_n(B, A \cap B, x_0)$, $\pi_n(X, B, x_0)$, $\pi_n(X, A, x_0)$ for $n > 1$; all that one needs is the natural notion of "neutral" element, so that kernels are well-defined. Hence this theorem could have been stated and proved before defining the group operation in $\pi_n(X; A, B, x_0)$. (b) Exactness of the homotopy sequence of a triad continues to hold in the lowest dimensions (3.5.2), where we do not have groups or homomorphisms. In the proof of the theorem we shall use the language of groups and homomorphisms; it will be clear that the modifications needed for the lowest dimensions are simply changes of language.

We remark that the exactness of the homotopy sequence of a pair is likewise independent of the corresponding group operations, and continues to hold in the lowest dimensions where no group operation is defined.

The following "commutativity" lemma will be of assistance in the proof of exactness:

LEMMA 3.5.5. *Consider the various groups and homomorphisms (functions) indicated by the following diagram (in which we abbreviate by omitting the base point):*

$$\begin{array}{ccccccccc}
 & & \pi_{n-1}(A, A \cap B) & & \xrightarrow{i_{1*}} & & \pi_{n-1}(X, B) & & \\
 & \beta_+ \nearrow & & \searrow i_{3*} & & \nearrow j_{4*} & & \searrow j_{1*} & \\
\pi_n(X; A, B) & & & & \pi_{n-1}(X, A \cap B) & & & & \pi_{n-1}(X; A, B) \\
 & \beta_- \searrow & & \nearrow i_{4*} & & \searrow j_{3*} & & \nearrow j_{2*} & \\
 & & \pi_{n-1}(B, A \cap B) & & \xrightarrow[i_{2*}]{} & & \pi_{n-1}(X, A) & &
\end{array}$$

(Here i_{3*}, j_{3*}, i_{4*}, j_{4*} are induced by inclusion maps and are, respectively, successive terms in the homotopy sequences of the triples $(X; A, A \cap B)$, $(X, B, A \cap B)$.) *Then the following commutativity relationships hold*:

(i) $$i_{1*} = j_{4*}i_{3*}\,; \qquad i_{2*} = j_{3*}i_{4*}\,.$$

(ii) $$j_{1*}j_{4*} = j_{2*}j_{3*}\,.$$

(iii) $$i_{3*}\beta_+ = -i_{4*}\beta_-\,.$$

PROOF OF LEMMA. (i) is obvious since all maps are inclusions; (ii) follows easily from the definition of j_{1*} and j_{2*}; (iii) follows from the choices of the orientations of E_+^{n-1}, E_-^{n-1}, and E^{n-1} in (1.1).

PROOF OF THEOREM 3.5.4. We shall prove exactness only for the upper sequence; the proof for the lower sequence is entirely analogous. The proof breaks up naturally into six parts.

(a) $$j_{1*}i_{1*} = 0.$$

We have, from the lemma above,

$$j_{1*}i_{1*} = j_{1*}j_{4*}i_{3*} = j_{2*}j_{3*}i_{3*} = 0$$

since i_{3*}, j_{3*} are successive homomorphisms in the exact sequence of the triple $(X, A, A \cap B)$.

(b) $$i_{1*}\beta_+ = 0.$$

Again, from the above lemma

$$i_{1*}\beta_+ = j_{4*}i_{3*}\beta_+ = -j_{4*}i_{4*}\beta_- = 0$$

since i_{4*}, j_{4*} are successive homomorphisms in the exact sequence of the triple $(X, B, A \cap B)$.

(c) $$\beta_+ j_{1*} = 0.$$

Let $\alpha \in \pi_n(X, B, x_0)$. Choose a map $f: (E^n, S^{n-1}, p_0) \to (X, B, x_0)$ which represents α and has the property that $f(E_+^{n-1}) = x_0$. Then f, considered as a map $(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0)$ represents $j_{1*}(\alpha)$, and $f \mid E_+^{n-1}$, considered as a map $(E_+^{n-1}, S^{n-2}, p_0) \to (A, A \cap B, x_0)$ represents $\beta_+ j_{1*}(\alpha)$ which is clearly zero.

(d) $$\textit{Kernel } \beta_+ \subset \textit{image } j_{1*}.$$

Let $\alpha \in \pi_n(X; A, B, x_0)$ be represented by a map

$$f: (E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0),$$

and let $\beta_+(\alpha) = 0$. This means that the map $f_0: (E_+^{n-1}, S^{n-2}, p_0) \to (A, A \cap B, x_0)$ defined by f is inessential. Hence there exists a homotopy

$$f_t: (E_+^{n-1}, S^{n-2}, p_0) \to (A, A \cap B, x_0)$$

of f_0, such that $f_1(E_+^{n-1}) = x_0$. Consider the induced homotopy

$$f_t \mid S^{n-2}: (S^{n-2}, p_0) \to (A \cap B, x_0).$$

By the homotopy extension theorem, this can be extended to a homotopy

$$g_t: E_-^{n-1} \to B$$

such that $g_0 = f \mid E_-^{n-1}$. Define

$$F_t: S^{n-1} \to A \cup B$$

by $F_t \mid E_+^{n-1} = f_t$, $F_t \mid E_-^{n-1} = g_t$. Apply the homotopy extension theorem again to extend F_t to a homotopy $F_t': E_n \to X$, such that $F_0' = f$. Then F_t' is a family of maps $(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0)$, so that F_1' is a representative of α. But $F_1'(E_+^{n-1}) = f_1(E_+^{n-1}) = x_0$, and hence α is the image under j_{1*} of an element of $\pi_n(X, B, x_0)$.

(e) $$\textit{Kernel } i_{1*} \subset \textit{image } \beta_+.$$

Assume $f: (E_+^{n-1}, S^{n-2}, p_0) \to (A, A \cap B, x_0)$ represents an element $\alpha \in \pi_n(A, A \cap B, x_0)$, such that $i_{1*}(\alpha) = 0$. This means that there exists a homotopy

$$F: (E_+^{n-1} \times I, S^{n-2} \times I, p_0 \times I) \to (X, B, x_0)$$

with $F(x, 0) = f(x)$, $F(x, 1) = x_0$, for $x \in E_+^{n-1}$. Now the triad $(E_+^{n-1} \times I; E_+^{n-1} \times 0, (E_+^{n-1} \times 1 \cup S^{n-2} \times I), p_0 \times 0)$ is obviously homeomorphic to the triad $(E^n; E_+^{n-1}, E_-^{n-1}, p_0)$. Let

$$h: (E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (E_+^{n-1} \times I; E_+^{n-1} \times 0, (E_+^{n-1} \times 1 \cup S^{n-2} \times I), p_0 \times 0)$$

be such a homeomorphism with $h(x) = x \times 0$ for $x \in E_+^{n-1}$. Define

$$g: (E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0)$$

by $g(x) = F(h(x))$, for $x \in E^n$. Let $\alpha' \in \pi_n(X; A, B, x_0)$ be the homotopy class of g. Then $\beta_+(\alpha') = \alpha$, as required.

(f) *Kernel* $j_{1*} \subset$ *image* i_{1*}.

This part of the exactness is the hardest to prove, but the geometrical idea involved is actually quite simple. Let $\alpha \in \pi_n(X, B, x_0)$ with $j_{1*}(\alpha) = 0$. Then α may be represented by a map

$$f:(\mathcal{E}^n, \dot{\mathcal{E}}^n, q_0) \rightarrow (X, B, x_0)$$

where $\mathcal{E}^n$ is any oriented n-cell. For the purpose of this proof we choose $\mathcal{E}^n = E^{n-1} \times I$. Then $\dot{\mathcal{E}}^n = S^{n-2} \times I \cup E^{n-1} \times \dot{I}$. For q_0 choose any point of $S^{n-2} \times 0$ and let $\mathcal{E}_+^{n-1} = E^{n-1} \times 0$, $\mathcal{E}_-^{n-1} = S^{n-2} \times I \cup E^{n-1} \times 1$. Then $j_{1*}(\alpha)$ is represented by a map

$$f':(\mathcal{E}^n; \mathcal{E}_+^{n-1}, \mathcal{E}_-^{n-1}, q_0) \rightarrow (X; A, B, x_0)$$

with $f'(\mathcal{E}_+^{n-1}) = x_0$, and $j_{1*}(\alpha) = 0$ implies the existence of a map

$$F:(\mathcal{E}^n \times J; \mathcal{E}_+^{n-1} \times J, \mathcal{E}_-^{n-1} \times J, q_0 \times J) \rightarrow (X; A, B, x_0),$$

(where $J = [0, 1]$, the closed unit interval) with

$$F(x, 0) = f'(x), x \in \mathcal{E}^n,$$

$$F(\mathcal{E}^n \times 1) = x_0 .$$

We may further assume, without loss of generality, that the homotopy is constant over the first half interval; that is, $F(x, t) = F(x, 0)$, $0 \leqq t \leqq \frac{1}{2}$. In particular, $F(\mathcal{E}_+^{n-1} \times [0, \frac{1}{2}]) = x_0$. In order to show that α is in the image of i_{1*} we must exhibit a map

$$G:(\mathcal{E}^n \times J, \dot{\mathcal{E}}^n \times J, q_0 \times J) \rightarrow (X, B, x_0)$$

such that $G(x, 0) = f(x)$, $x \in \mathcal{E}^n$, and $G(\mathcal{E}^n \times 1) \subset A$ This will be done as follows: We shall define a map $h:\mathcal{E}^n \times J \rightarrow \mathcal{E}^n \times J$ and then define G by setting $G(y) = F[h(y)]$ for $y \in \mathcal{E}^n \times J$. It will then only be necessary to verify that G has the required properties.

Now $\mathcal{E}^n \times J = E^{n-1} \times I \times J$ is an $(n + 1)$-cell. Let $K = I \times J$. Then we may write $\mathcal{E}^n \times J = E^{n-1} \times K$. We will define a homeomorphism

$$g:K \rightarrow K,$$

and then define $h:E^{n-1} \times K \rightarrow E^{n-1} \times K$ by

$$h(x, y) = (x, g(y)).$$

The map g is defined by reference to figure 1. Let $\dot{K}$ denote the boundary of the square K. We first define g on $\dot{K}$ as follows:

$$g(AC) = AB; \quad g(CD) = BC;$$

$$g(DE) = CE; \quad g(EF) = EF;$$

$$g(FA) = FA.$$

Here the notation $g(AC) = AB$ means that the oriented segment AC is mapped linearly onto the oriented segment AB. This defines a homeomorphism $\dot{K} \rightarrow \dot{K}$

which can be extended to a homeomorphism $g: K \to K$ in an arbitrary fashion. It is now a routine matter to verify that the maps h and G defined by means of g have the required properties. This we leave to the reader.

We conclude this section with two remarks:

(a) With the notation of (3.1), we have seen that a mapping $f: (X; A, B, x_0) \to (Y; C, D, y_0)$ induces a homomorphism of the corresponding triad homotopy groups. Clearly the map f induces maps $f_1: (A, A \cap B, x_0) \to (C, C \cap D, y_0)$ and $f_2: (X, B, x_0) \to (Y, D, y_0)$, which in turn induce homomorphisms of the corresponding relative homotopy groups. These homomorphisms commute with the homomorphisms of the upper sequences of the respective triads. A similar situ-

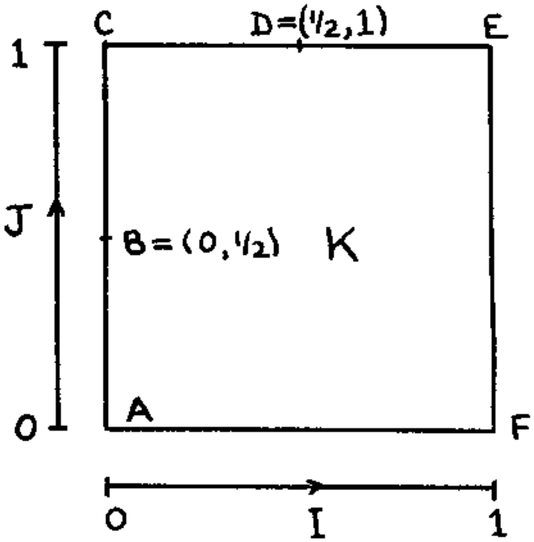

FIG. 1

ation holds for the lower sequences, and commutativity continues to hold in the lowest dimensions. We express this by saying that the mapping f induces homomorphisms of the respective homotopy sequences. In addition, the induced homomorphisms preserve the operators from $\pi_1(A \cap B, x_0)$, $\pi_1(C \cap D, y_0)$ in the following sense. If f induces the homomorphism $f'_*: \pi_1(A \cap B, x_0) \to \pi_1(C \cap D, y_0)$, and $\alpha \in \pi_1(A \cap B, x_0)$, $\beta \in \pi_n(X; A, B, x_0)$, then

$$f_*\{\alpha(\beta)\} = [f'_*(\alpha)](f_*(\beta)).$$

(b) In (3.2) we saw that if $A \supset B$ then $\pi_n(X; A, B, x_0) \approx \pi_n(X, A, x_0)$. In addition we observe that $A \cap B = B$, so that the upper sequence of such a triad becomes the well known homotopy sequence of the triple (X, A, B) at x_0, while the lower sequence has every third term trivial, and the remaining terms isomorphic in pairs.

(3.6) Relation to the Excision Axiom

In this section we will clarify the statement made in the introduction that the homotopy groups of a triad are a measure of the extent to which the excision axiom fails to hold for the relative homotopy groups. Let $(X; A, B)$ be a triad with $A \cup B = X$. Then the inclusion maps

$$i_1:(A, A \cap B) \to (X, B)$$

$$i_2:(B, A \cap B) \to (X, A)$$

are both excisions. If we consider the homotopy sequences of the triad $(X; A, B)$, we see that the homomorphisms i_{1*} and i_{2*} induced by i_1 and i_2 on the relative homotopy groups, will be isomorphisms onto if and only if all of the triad homotopy groups are trivial. Thus if some of the triad homotopy groups are non-trivial, invariance under excision cannot hold for the relative homotopy groups in all dimensions.

(3.7) The Non-abelian Character of $\pi_3(X; A, B)$

We first give an example of a triad $(X; A, B)$ for which $\pi_3(X; A, B)$ is non-abelian. Consider the following subsets of cartesian 3-space C^3:

$$\begin{aligned} y &= (1, 0, 0), \qquad z = (-1, 0, 0), \\ E_1 &= \{x \in C^3 \mid \|x - y\| \leqq 1\}, \\ E_2 &= \{x \in C^3 \mid \|x - z\| \leqq 1\}, \\ X &= E_1 \cup E_2 . \end{aligned}$$

Then X is the union of two 3-cells with a single point in common. Let $\dot{X}$ denote the boundary of X,

$$A = \{x \in \dot{X} \mid x_3 \geqq 0\}, \qquad B = \{x \in \dot{X} \mid x_3 \leqq 0\},$$

and let $x_0 = (0, 0, 0)$ be the base point. The homotopy groups of the pair (X, B) are all trivial because B is a deformation retract of X. It follows from the exactness of the upper homotopy sequence of $(X; A, B)$ that

$$\beta_+:\pi_3(X; A, B, x_0) \to \pi_2(A, A \cap B, x_0)$$

is an isomorphism onto. Since the space A is contractible to a point, $\pi_1(A, x_0)$ and $\pi_2(A, x_0)$ are trivial. Hence the boundary operator

$$\partial:\pi_2(A, A \cap B, x_0) \to \pi_1(A \cap B, x_0)$$

is also an isomorphism onto. But $A \cap B$ is a "figure 8," and its fundamental group is the free (non-abelian) group on two generators.

We return now to the case where $(X; A, B)$ is any triad, and $x_0 \in A \cap B$. We have already remarked that the group $\pi_1(A \cap B, x_0)$ acts as a group of operators on $\pi_3(X; A, B, x_0)$. Denote by $\hat{\pi}_1(A \cap B, x_0)$ the subgroup of $\pi_1(A \cap B, x_0)$ containing only those elements which operate trivially on $\pi_3(X; A, B, x_0)$. Let

$\beta_+ : \pi_3(X; A, B, x_0) \to \pi_2(A, A \cap B, x_0)$ and $\partial : \pi_2(A, A \cap B, x_0) \to \pi_1(A \cap B, x_0)$

be the boundary homomorphisms. Then we have the following result:

THEOREM 3.7.1. *The subgroup* $(\partial\beta_+)^{-1}\hat{\pi}_1(A \cap B, x_0)$ *of* $\pi_3(X; A, B, x_0)$ *is contained in the center of* $\pi_3(X; A, B, x_0)$.

PROOF. The proof is carried out in a similar manner to that of the corresponding theorem about the 2-dimensional relative homotopy group. For if γ is any element of $\pi_3(X; A, B)$, and γ is represented by a map

$$f:(E^3; E^2_+, E^2_-, p_0) \to (X; A, B, x_0)$$

then $\partial\beta_+(\gamma)$ is represented (because of our choice of orientations) by $f \mid S^1$: $(S^1, p_0) \to (A \cap B, x_0)$. Now let $\gamma_1, \gamma_2 \in \pi_3(X; A, B, x_0)$, with $\gamma = \gamma_1 + \gamma_2$, and suppose that the map f representing $\gamma_1 + \gamma_2$ is in the form obtained by adding representative mappings of γ_1 and γ_2. Let R_θ denote a rotation of E^3, $0 \leqq \theta \leqq 1$, through an angle $\pi\theta$ about the x_3 axis in the sense which takes p_0 along E^1_+. Then the function

$$f_t:(E^3; E^2_+, E^2_-) \to (X; A, B)$$

defined by $f_t(x) = fR_t(x)$, $0 \leqq t \leqq 1$, $x \in E^3$, is a homotopy of f in which the point p_0 describes a path representing $\partial\beta_+(\gamma_1)$, while the map

$$f_1:(E^3; E^2_+, E^2_-, p_0) \to (X; A, B, x_0)$$

clearly determines the element $\gamma_2 + \gamma_1 \in \pi_3(X; A, B, x_0)$. It follows from the definition of the operators that

$$[\partial\beta_+(\gamma_1)](\gamma_2 + \gamma_1) = \gamma_1 + \gamma_2 .$$

Hence, in particular, if $\gamma_1 \in (\partial\beta_+)^{-1}\hat{\pi}_1(A \cap B, x_0)$ and γ_2 is any element of $\pi_3(X; A, B, x_0)$ we get

$$\gamma_1 + \gamma_2 = \gamma_2 + \gamma_1 ,$$

i.e., γ_1 is in the center of $\pi_3(X; A, B, x_0)$.

COROLLARY 1. *If the triad is simple in dimension 3 then* $\hat{\pi}_1(A \cap B, x_0) = \pi_1(A \cap B, x_0)$, *and hence* $\pi_3(X; A, B, x_0)$ *is abelian. In particular, this is the case when* $\pi_1(A \cap B, x_0) = 0$.

COROLLARY 2. A simple extension of the above proof shows that *if* γ_1, γ_2 *are any two elements of* $\pi_3(X; A, B, x_0)$, *then* $[\partial\beta_+(\gamma_1)](\gamma_2) = -\gamma_1 + \gamma_2 + \gamma_1$.

COROLLARY 3. *The group* $\pi_3(X; A, B, x_0)$ *is a central extension of* [kernel β_+] *by* $\beta_+\pi_3(X; A, B, x_0)$.

PART 4: TWO MAIN THEOREMS

(4.1) Statement of the Theorems

In order to state these theorems simply, we need some additional definitions. A topological space X is 0-*connected* if it is arcwise connected. We recall that this may be written $\pi_0(X) = 0$. It is *n-connected* if $\pi_i(X) = 0$, $0 \leqq i \leqq n$. A pair (X, A) is said to be *n-connected*, $n > 0$, if $\pi_0(A) = \pi_0(X) = 0$ and if $\pi_i(X, A) = 0$, $1 \leqq i \leqq n$. A triad $(X; A, B)$ is said to be *n-connected*, $n > 1$, if each of the pairs $(A, A \cap B)$ and $(B, A \cap B)$ is 1-connected, and $\pi_i(X; A, B) = 0$, $1 < i \leqq n$.

Let (X^*, X) be a pair which satisfies the following conditions:
(a) The space $X^* - X$ is a union of disjoint subspaces $\mathcal{U}_i^n$, $n > 0$,

$$X^* - X = \bigcup_i \mathcal{U}_i^n ,$$

where each $\mathcal{U}_i^n$ is an open subset of X^* and is homeomorphic to $E^n - S^{n-1}$.
(b) Let $\mathcal{E}_i^n$ denote the closure of $\mathcal{U}_i^n$, and $\dot{\mathcal{E}}_i^n = \mathcal{E}_i^n \cap X$. Then it is assumed that there exists a mapping

$$\psi_i : (E^n, S^{n-1}) \to (\mathcal{E}_i^n, \dot{\mathcal{E}}_i^n)$$

which is a homeomorphism of $E^n - S^{n-1}$ onto $\mathcal{U}_i^n$. Under these conditions we say that X^* is obtained from X by simultaneous adjunction of the cells $\mathcal{E}_i^n$. The number of cells adjoined may be finite or infinite in number; in case the number adjoined is infinite, it is assumed that any compact subset of X^* intersects only a finite number of the open cells $\mathcal{U}_i^n$. Let

$$\mathcal{E}^n = \bigcup_i \mathcal{E}_i^n ; \qquad \dot{\mathcal{E}}^n = \bigcup_i \dot{\mathcal{E}}_i^n .$$

In the remainder of this paper it will be assumed that $\dot{\mathcal{E}}^n$ is arc-wise connected. An important problem is to obtain information about the homomorphisms of the relative homotopy groups induced by the inclusion map $i : (\mathcal{E}^n, \dot{\mathcal{E}}^n) \to (X^*, X)$. The following two theorems, combined with the exactness of the homotopy sequences of the triad $(X^*; \mathcal{E}^n, X)$, are often useful in this connection.

THEOREM I. *If the pair $(X, \dot{\mathcal{E}}^n)$ is m-connected $(m \geqq 1, n \geqq 2)$, then the triad $(X^*; \mathcal{E}^n, X)$ is $(m + n - 1)$-connected. If $m = 1$ we must add the hypothesis that $\pi_2(X, \dot{\mathcal{E}}^n)$ is abelian; if $n = 2$ we must add the hypothesis that X is simple relative to $\dot{\mathcal{E}}^n$ in all dimensions $< m + n$.*

THEOREM II. *If the space $\dot{\mathcal{E}}^n$ is m-connected, $(m \geqq 1, n \geqq 2)$ then the boundary homomorphism*

$$\beta_+ : \pi_{i+1}(X^*; \mathcal{E}^n, X) \to \pi_i(\mathcal{E}^n, \dot{\mathcal{E}}^n)$$

is trivial for $2 \leqq i \leqq m + n - 1$.

In a subsequent paper, we will show that Theorem I is in a certain sense a "best possible" theorem; i.e., if the pair $(X, \dot{\mathcal{E}}^n)$ is m-connected, if $\pi_{m+1}(X, \dot{\mathcal{E}}^n) \neq 0$, and if the triad $(X^*; \mathcal{E}^n, X)$ satisfies a few additional conditions of a rather general nature, then $\pi_{m+n}(X^*; \mathcal{E}^n, X) \neq 0$. However, it is not possible to make an analogous statement about Theorem II. For example, it can be shown that for the triad $(S^n; E_+^n, E_-^n)$, the boundary homomorphism

$$\beta_+ : \pi_{2n-1}(S^n; E_+^n, E_-^n) \to \pi_{2n-2}(E_+^n, S^{n-1})$$

is trivial if $n = 2, 4$, or 8. It would be interesting to extend Theorem II to higher dimensions to account for such cases.

The proofs of Theorems I and II are given in sections (4.9) and (4.10) respectively. Sections (4.2) through (4.8) contain preliminary lemmas and definitions which are used in the proofs of these theorems.

(4.2) Further Definitions

We will use the notation $| K |$ to denote the space of a simplicial complex K. A *simplicial pair* (K, L) is a finite simplicial complex K together with a subcomplex L. Given a pair (X, A), a *triangulation* $T = \{t, (K, L)\}$ of (X, A) consists of a simplicial pair (K, L) and a homeomorphism

$$t:(| K |, | L |) \to (X, A).$$

The pair (X, A) together with the triangulation T is called a *triangulated pair.* If a triangulation of (X, A) exists, then we say that (X, A) is triangulable.

If $T = \{t, K\}$ is a triangulation of a space X, a function $f:X \to C^n$ mapping X homeomorphically onto a subset of cartesian n-space C^n, is called a *linear imbedding of X in C^n*, with respect to the triangulation T, if and only if the map $ft:| K | \to C^n$ is linear; i.e., if and only if the cartesian coordinates of the point $ft(\alpha)$ are linear functions of the barycentric coordinates of the point $\alpha \in | K |$. A triangulation $T = \{t, K\}$ of an m-cell $\mathcal{E}^m$ is called *rectilinear* provided that there exists a linear imbedding $f:\mathcal{E}^m \to C^m$, with respect to T, which has the property that $f(\mathcal{E}^m)$ is a convex subset of C^m. It is known that not every triangulation of $\mathcal{E}^m$ is rectilinear if $m > 2$; however, it is obvious that rectilinear triangulations of $\mathcal{E}^m$ exist, and that successive barycentric subdivisions of a rectilinear triangulation are also rectilinear. In the remainder of this paper we shall frequently be concerned with triangulations of cells; it will always be assumed that these triangulations are rectilinear, even though this fact is not explicitly mentioned.

LEMMA 4.2.1. *Let P be a closed subset, of dimension $\leqq n - 3$, of the n-cell $\mathcal{E}^n$, and assume the existence of a rectilinear triangulation $T = \{t, (K, L)\}$ of the pair $(\mathcal{E}^n, P)$. Then $\mathcal{E}^n - P$ is simply connected.*

PROOF. Let $f:\mathcal{E}^n \to C^n$ be a linear imbedding of $\mathcal{E}^n$ in C^n, with respect to T, such that $f(\mathcal{E}^n)$ is convex. We will identify each point $x \in \mathcal{E}^n$ with its image $f(x) \in C^n$, and thus consider $\mathcal{E}^n$ as a subset of C^n. To prove $\mathcal{E}^n - P$ simply connected it suffices to prove that any closed polygonal loop in $\mathcal{E}^n - P$ can be contracted to a point in $\mathcal{E}^n - P$. Let $A_0, A_1, \cdots, A_r, A_0$ be the successive vertices of a closed polygonal loop Q in $\mathcal{E}^n - P$. Choose a point y in $\mathcal{E}^n - P$ which is in "general position" with respect to the loop Q and the triangulated space P. This condition on the point y can be stated more precisely as follows: Let A_iA_{i+1} be any segment of Q and σ^q any q-simplex of L, $0 \leqq q \leqq n - 3$. Denote the vertices of σ^q by $B_0, \cdots, B_q$. Then the points $A_i, A_{i+1}, t(B_0), t(B_1), \cdots, t(B_q)$ determine a cartesian subspace of C^n, whose dimension is at most $q + 2 \leqq n - 1$. Then the point y must not be contained in the union of such subspaces for all σ^q and for all segments of Q. We next form the "join" of the point y with the loop Q; because of convexity this join will lie in $\mathcal{E}^n$, and since y has been chosen in general position the join will not meet P. Hence we can contract the loop Q in $\mathcal{E}^n - P$ to the point y.

(4.3) Theory of Obstructions to Extensions of Mappings[5]

We shall content ourselves with a brief resume of those parts of this theory which we shall need. Let Y be an arcwise connected topological space, and let (X, A) be a triangulable pair. We assume given a fixed map

$$f: A \rightarrow Y$$

and wish to determine whether or not it is possible to extend f over all of X. We shall assume that Y is simple in all dimensions $\leqq \dim X$. Although this assumption is not essential, it will considerably simplify the discussion. Choose a fixed triangulation $T = \{t, (K, L)\}$ of (X, A) and define $\bar{K}^n = K^n \cup L$, where K^n denotes the n-dimensional skeleton of K. For convenience we will identify each point $x \in |K|$ with its image $t(x) \in X$.

DEFINITION 4.3.1. The map f is said to be *n-extensible* if it can be extended to a map $|\bar{K}^n| \rightarrow Y$.

It can be shown that the property of being n-extensible or not is independent of the choice of the triangulation T of (X, A).

We now define the *obstructions* to the extension of f. The n^{th} obstruction to the extension of f, $\mathcal{O}^n(f)$, is a subset of $H^n(X, A, \pi_{n-1}(Y))$ defined as follows:
(a) If the map f is not $(n - 1)$-extensible, then $\mathcal{O}^n(f)$ is the empty set.
(b) Assume that the map f is $(n - 1)$-extensible. Choose an extension $f': |\bar{K}^{n-1}| \rightarrow Y$ of f. The extension f' defines an n-cochain $c^n(f')$ with coefficients in $\pi_{n-1}(Y)$ as follows: Let σ^n be an oriented n-simplex of K. The orientation of σ^n determines an orientation of the boundary $\dot{\sigma}^n$ of σ^n in a natural way. The map $f' \mid \dot{\sigma}^n: \dot{\sigma}^n \rightarrow Y$ determines an element of $\pi_{n-1}(Y)$, since Y is $(n - 1)$-simple. The cochain $c^n(f')$ is now defined by assigning to σ^n the element of $\pi_{n-1}(Y)$ thus determined. This cochain is readily seen to have zero values on n-simplexes of L. Moreover, it may be shown to be an n-cocycle. Hence it determines an element of the relative cohomology group $H^n(K, L, \pi_{n-1}(Y))$, and hence an element of the cohomology group $H^n(X, A, \pi_{n-1}(Y))$. The obstruction, $\mathcal{O}^n(f)$ is defined to be the set of all such elements obtained by applying this process to all maps $|\bar{K}^{n-1}| \rightarrow Y$ which are extensions of f. It can be shown that *the obstruction $\mathcal{O}^n(f)$ does not depend on the choice of the triangulation T.*

The following important theorem can now be proved:

THEOREM 4.3.2. *The map f is n-extensible if and only if $\mathcal{O}^n(f)$ contains the zero element of $H^n(X, A, \pi_{n-1}(Y))$.*

Suppose next that the space Y is m-connected, $m \geqq 1$. Then Y is simple in all dimensions. Let $N = \dim K$. The following is a direct consequence of Theorem 4.3.2.

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[5] For a more detailed account of the material of this section, we refer the reader to references [3], [10], and [12].

THEOREM 4.3.3. *If* $H^p(X, A, \pi_{p-1}(Y)) = 0$ *for* $m + 1 < p \leqq N$, *then any map* $f: A \to Y$ *can be extended over all of* X.

(4.4) Theory of Obstructions to Deformations of Mappings[6]

The theory of obstructions to deformations parallels closely the corresponding theory for obstructions to extensions. Since no detailed account of this theory has yet appeared in print, we shall summarize the salient features.

The problem to be considered is the following. Suppose that a mapping $f:(X, A) \to (Y, B)$ of one pair into another is given. Does there exist a map $g:(X, A) \to (Y, B)$ such that $g(X) \subset B$ and $g \simeq f$? If such a map g exists, we shall say that the original map f is *deformable*. If we add the additional condition that $g \simeq f$ (rel. C), where C is a subspace of X, we say that f is *deformable* rel. C. The following lemma shows the close connection between these two concepts.

LEMMA 4.4.1. *Let* (X, A) *be a triangulable pair. Then a map* $f:(X, A) \to (Y, B)$ *is deformable if and only if it is deformable* rel. A.

The proof is a simple application of the homotopy extension theorem.

Our main concern will be to prove the following two theorems. Let (X, A) be a triangulable pair and (Y, B) a pair which is 1-connected and for which the relative homotopy group $\pi_2(Y, B)$ is abelian. Let $N = \dim\,(X - A)$.

THEOREM 4.4.2. *If the pair* (Y, B) *is p-simple for* $2 \leqq p \leqq N$, *and if* $H^q(X, A, \pi_q(Y, B)) = 0$ *for* $2 \leqq q \leqq N$, *then any map* $f:(X, A) \to (Y, B)$ *is deformable.*

THEOREM 4.4.3. *If* X *is* 1-*connected and* $H^q(X, A, \pi_q(Y, B)) = 0$ *for* $2 \leqq q \leqq N$, *then any map* $f:(X, A) \to (Y, B)$ *is deformable.*

In order to prove these theorems, we will need to develop the general theory of obstructions to deformations. With very little additional work, we could prove a great deal more than is stated in these two theorems. In a subsequent paper we shall develop and use the theory for the case in which (X, A) is an arbitrary compact pair, and (Y, B) in addition to being 1-connected, is a compact A.N.R.

Let $T = \{t, (K, L)\}$ be a definite triangulation of the pair (X, A). We shall say that the map $f:(X, A) \to (Y, B)$ is *n-deformable* if there exists a map $g:(X, A) \to (Y, B)$ such that $f \simeq g$ and $gt(\,|K^n|\,) \subset B$. The *deformation index* of f is defined to be the greatest integer n such that f is n-deformable. The concepts "f is n-deformable rel. C" and "deformation index of f rel. C," where $C \subset X$, are defined in an obvious fashion. Since (X, A) is triangulable, it follows that f is n-deformable if and only if f is n-deformable rel A. It is easily proved that all these concepts are independent of the choice of the triangulation T. Also, it is obvious that if $f_0, f_1:(X, A) \to (Y, B)$ and $f_0 \simeq f_1$, then f_0 is n-deformable if and only if f_1 is n-deformable. In other words, the property of being n-deformable is invariant under homotopies.

[6] See A. L. Blakers, Bull. Amer. Math. Soc. 54, Abstract No. 413 (1948). This theory has been described in lectures by W. Hurewicz and should appear in the forthcoming book of Hurewicz and Dugundji on homotopy theory.

First, we will develop the theory of obstructions for the case in which the hypotheses of Theorem 4.4.2 hold, i.e., it is assumed that the operations of $\pi_1(B)$ on $\pi_p(Y, B)$ are trivial for $2 \leqq p \leqq N$. Let $f:(X, A) \to (Y, B)$, and assume that $ft(|K^{n-1}|) \subset B$, where $2 \leqq n \leqq N$. Let σ^n be an oriented n-simplex of K. Define $d(f, \sigma^n) \in \pi_n(Y, B)$ to be the element determined by the map $g:(|\sigma^n|, |\dot{\sigma}^n|) \to (Y, B)$ which is defined by ft. Since (Y, B) is n-simple, it does not matter how the base points in $|\dot{\sigma}^n|$ and B are chosen in defining $d(f, \sigma^n)$. We now define an n-cochain $d^n(f) \in C^n(K, \pi_n(Y, B))$ by assigning to any oriented n-simplex σ^n of K the element $d(f, \sigma^n)$. It is obvious that $d^n(f)$ has the value zero on the n-simplexes of L and that if $d^n(f) = 0$, then f is n-deformable.

LEMMA 4.4.4. *The cochain $d^n(f)$ is a cocycle.*

PROOF. It suffices to show that the coboundary, $\delta d^n(f)$, vanishes on an arbitrary $(n + 1)$-simplex σ^{n+1} of K. Let the vertices of K be ordered. This induces an order for the vertices of any simplex of K. Let $\sigma^{n+1} = \langle p_0 \cdots p_{n+1} \rangle$, and let $\sigma_{(i)}^{n+1}$ denote the face opposite the vertex p_i. Each face of σ^{n+1} has an orientation determined by the order of the vertices, and the partial mappings

$$f \mid \sigma_{(i)}^{n+1}:(\sigma_{(i)}^{n+1}, \dot{\sigma}_{(i)}^{n+1}, p) \to (Y, B, y_0)$$

determine elements $\alpha_i \in \pi_n(Y, B)$, which do not depend on the choice of the vertex p of $\sigma_{(i)}^{n+1}$. We now have

$$\begin{aligned}(\delta d^n(f))(\sigma^{n+1}) &= d^n(f)(\partial \sigma^{n+1}) \\ &= \sum_{i=0}^{n+1} (-1)^i \alpha_i .\end{aligned}$$

It now follow from ([2], (12.1.3) and (12.1.i)), since the map f is defined over the whole of σ^{n+1}, that $\sum (-1)^i \alpha_i = 0$, which completes the proof.

We now define the n-dimensional obstruction, $\mathfrak{D}^n(f)$, to the deformation of the map $f:(X, A) \to (Y, B)$. It is a subset of the cohomology group $H^n(X, A, \pi_n(Y, B))$.

(a) In case the map f is *not* $(n - 1)$-deformable, $\mathfrak{D}^n(f)$ is defined to be the empty set.

(b) Assume f is $(n - 1)$-deformable. Then $\mathfrak{D}^n(f)$ is defined to be the set of all cohomology classes $\{d^n(f')\}$ for all maps $f':(X, A) \to (Y, B)$ such that $f't(|K^{n-1}|) \subset B$ and $f' \simeq f$.

It follows as a direct consequence of this definition that f is $(n - 1)$-deformable if and only if $\mathfrak{D}^n(f)$ is non-empty, and if f is n-deformable, then $\mathfrak{D}^n(f)$ contains the zero element. We wish to prove the converse of this latter statement, i.e., if $0 \in \mathfrak{D}^n(f)$, then f is n-deformable. This fact is a direct consequence of the following lemma:

LEMMA 4.4.5. *Assume that f is $(n - 1)$-deformable, and $u \in \mathfrak{D}^{n-1}(f)$. Then given any n-cocycle d whose cohomology class is u, there exists a map $f':(X, A) \to (Y, B)$ such that $f' \simeq f$, $f't(|K^{n-1}|) \subset B$ and $d^n(f') = d$.*

PROOF. This lemma is proved in precisely the same manner as the corresponding result in the theory of obstructions to extensions (see [3]). For, let the map f be $(n - 1)$-deformable, and let $f_0, f_1:(X, A) \to (Y, B)$ be two maps which satisfy the conditions $f_0t(|K^{n-1}|) \subset B$, $f_1t(|K^{n-1}|) \subset B$, $f_0 \simeq f_2 \simeq f$, $f_0t \mid |K^{n-2}| = f_1t \mid |K^{n-2}|$. Then it is possible to associate with any homotopy between f_0 and f_1,

$$f_\tau:(X, A) \to (Y, B), 0 \leqq \tau \leqq 1,$$

which has the property that $f_\tau(x) = f_0(x) = f_1(x)$ for $x \in t(|K^{n-2}|)$, a "separation cochain" $g^{n-1}(f_\tau) \in C^{n-1}(K, L, \pi_n(Y, B))$. This separation cochain satisfies the equation

$$\delta g^{n-1}(f_\tau) = d^n(f_0) - d^n(f_1).$$

Furthermore, if g^{n-1} is an arbitrary element of the cochain group $C^{n-1}(K, L, \pi^n(Y, B))$, and $f_0:(X, A) \to (Y, B)$ is an arbitrary map satisfying the condition $f_0t(|K^{n-1}|) \subset B$, then there exists a homotopy of f_0,

$$f_\tau:(X, A) \to (Y, B), 0 \leqq \tau \leqq 1,$$

such that $f_\tau(x) = f_0(x)$ for $x \in t(|K^{n-2}|)$, $f_1t(|K^{n-1}|) \subset B$, and $g^{n-1}(f_\tau) = g^{n-1}$.

By making use of this separation cochain, it is easy to prove the lemma.

COROLLARY 4.4.6. *The map f is n-deformable if and only if* $0 \in \mathfrak{D}^n(f)$.

The proof of (4.4.2) now follows directly from the corollary.

Next, we take up the case in which the hypotheses of Theorem 4.4.3 hold, i.e., (Y, B) is 1-connected, $\pi_2(Y, B)$ is abelian, and X is 1-connected, but it is *not* assumed that $\pi_1(B)$ operates trivially on the relative homotopy groups of (Y, B).

Let y_0 be a base point in B, and let $f:(X, A) \to (Y, B)$ be any map.

LEMMA 4.4.7. *If f is $(n - 1)$-deformable, $n \geqq 2$, then f can be deformed to a map $f':(X, A) \to (Y, B)$ such that $f't(|K^{n-1}|) \subset B$ and $f't(|K^1|) = y_0$.*

PROOF. Since K^2 is simply connected the inclusion map $j_0:|K^1| \to |K^2|$ is inessential, and hence can be deformed by a homotopy j_t, $0 \leqq t \leqq 1$, to a map j_1 with $j_1(|K^1|) = k_0$, where k_0 is any vertex of the triangulation. The homotopy j_t can now be extended to give a deformation i_t of the identity map $i_0:|K| \to |K|$ which also satisfies the condition $i_t(|K^{n-1}|) \subset |K^{n-1}|$ when $n > 2$. Suppose now that $f'':(X, A) \to (Y, B)$ is any $(n - 1)$-deformation of f. Then it follows that f'' can be deformed by composition with i_t to give the map f' required.

Now let σ^n be an oriented n-simplex of K, and $g_1:(|\sigma^n|, |\dot{\sigma}^n|, v) \to (Y, B, y_0)$ the map defined by $f't$. Here v is a vertex of σ^n. The map g_1 determines an element $d(f, \sigma^n) \in \pi_n(Y, B, y_0)$ in an obvious way. It follows from the fact that $f't(|K^1|) = y_0$, that it is immaterial which vertex $v \in \sigma^n$ we choose in order to define $d(f', \sigma^n)$. An n-cochain $d^n(f') \in C^n(K, \pi_n(Y, B, y_0))$ is now defined by assigning the element $d(f', \sigma^n)$ to the n-simplex σ^n. Again it is clear that $d^n(f')$ has the value zero on an n-simplex of L, and that if $d^n(f') = 0$, then f', (and hence f) is n-deformable.

LEMMA 4.4.8. *The cochain $d^n(f')$ is a cocycle.*

The proof proceeds in precisely the same manner as that of (4.4.4).

We are now in a position to define the n-dimensional obstruction, $\mathfrak{D}^n(f)$, to the deformation of a map $f:(X, A) \to (Y, B)$, in a manner analogous to that used in the previous case.

(a) In case the map f is *not* $(n - 1)$-deformable, $\mathfrak{D}^n(f)$ is defined to be the empty set.

(b) Assume f is $(n - 1)$-deformable. Then $\mathfrak{D}^n(f)$ is defined to be the set of all cohomology classes $\{d^n(f')\}$ for all maps $f':(X, A) \to (Y, B)$ satisfying the conditions $f' \simeq f$, $f't(\mid K^{n-1} \mid) \subset B$, and $f't(\mid K^1 \mid) = y_0$. (We have proved that maps exist satisfying all three conditions.)

We now have the analogues of (4.4.5) and (4.4.6).

Lemma 4.4.9. *Assume that f is $(n - 1)$-deformable and $u \in \mathfrak{D}^{n-1}(f)$. Then given any n-cocycle d whose cohomology class is u, there exists a map $f_1:(X, A) \to (Y, B)$ such that $f_1 \simeq f$, $f_1t(\mid K^{n-1} \mid) \subset B$, $f_1t(\mid K^1 \mid) = y_0$, and $d^n(f_1) = d$.*

Corollary 4.4.10. *The map f is $(n - 1)$-deformable if and only if $\mathfrak{D}^n(f)$ is non-empty; the map f is n-deformable if and only if $0 \in \mathfrak{D}^n(f)$.*

The proof of Theorem 4.4.3 now follows directly from this corollary.

Remarks 4.4.11. Although the definition of the obstruction $\mathfrak{D}^n(f)$ depended on the choice of a triangulation T for the pair (X, A), it may be shown that this definition is actually topologically invariant and independent of the choice of T. The definition we have given for obstructions to deformations could be generalized by introducing cohomology groups with local coefficients as was done by P. Olum (in a recent paper, [12]) for the case of obstructions to extensions.

(4.5) The Inverse Image of a Principal Simplex under a Simplicial Map

Let K and L be finite simplicial complexes, and $f:K \to L$ a simplicial map. Let σ^n be a closed n-dimensional *principal* simplex of L (that is, σ^n is not a face of a simplex of dimension $>n$) and let y be an interior point of $\mid \sigma^n \mid$. Denote the interior of $\mid \sigma^n \mid$ by V and let $P = f^{-1}(y)$, $U = f^{-1}(V)$. According to a lemma of Pontrjagin [13] P is a cell complex of dimension $\leqq r - n$, where $r = \dim K$, and hence P is triangulable. Also, there exists a homeomorphism onto,

$$h:P \times V \to U,$$

such that $fh(p, v) = v$, for any points $p \in P$, $v \in V$. Now let τ^n be a closed n-simplex contained in V and denote $f^{-1}(\tau^n)$ by A. Then A is homeomorphic to $P \times \tau^n$, and since it is the product of triangulable spaces, it too is triangulable. It does not follow that the pair $(\mid K \mid, A)$ is triangulable. However, we have the following result:

Lemma 4.5.1. *It is possible to choose $\tau^n \subset V$ so that the pair $(\mid K \mid, A)$ is triangulable.*

This lemma will be an easy consequence of the next lemma.

Definition 4.5.2. Let K and L be simplicial complexes, $f:K \to L$ a simplicial map, and ${}^mK$, ${}^mL$, the m^{th} barycentric subdivisions of K and L. Then the *first barycentric subdivision* of f

$$^1f:{}^1K \to {}^1L$$

is defined by mapping the barycenter b_s of each simplex s of K onto the barycenter of the simplex $f(s)$ of L, and extending linearly. The m^{th} barycentric subdivision of f,

$$^m f : {}^m K \to {}^m L$$

is defined inductively. Obviously $^m f$ is a simplicial map, homotopic to f.

LEMMA 4.5.3. *Let* $f:K \to L$ *be a simplicial map, and* $g: |{}^m L| \to |L|$ *the canonical homeomorphism between* $|L|$ *and* $|{}^m L|$ (see below). *Then there exists a homeomorphism* $h: |{}^m K| \to |K|$ *such that commutativity holds in the following diagram:*

$$\begin{array}{ccc} |{}^m K| & \xrightarrow{h} & |K| \\ \Big\downarrow {}^m f & & \Big\downarrow f \\ |{}^m L| & \xrightarrow[g]{} & |L| \end{array}$$

PROOF. It is clearly sufficient to prove this theorem for the case $m = 1$. We recall that the vertices of ^1K are the barycenters b_s of the simplexes $s \in K$; similarly for ^1L. The linear homeomorphism

$$g: |{}^1 L| \to |L|$$

is defined by taking the unique linear extension of the identity map of the vertices of ^1L into $|L|$.

Let s be any simplex of K; we will define a certain interior point c_s of s as follows: Under f, s is mapped onto some simplex $\sigma = f(s)$ of L. Let b_σ denote the barycenter of σ. Then $s \cap f^{-1}(b_\sigma)$ is a convex cell, linearly imbedded in s. We define c_s to be the barycenter of $s \cap f^{-1}(b_\sigma)$. We now define a complex K_1 as follows: The vertices of K_1 are the points c_s. A collection of vertices $c_{s_0}, \cdots, c_{s_q}$ spans a simplex of K_1 if for some arrangement $s_0, \cdots, s_q$ of the corresponding simplexes, it is true that s_i is a face of s_{i+1}, $i = 0, \cdots, q - 1$. Then there is a natural homeomorphism $h_1 : K_1 \to K$ which maps each vertex c_s of K_1 onto the point $c_s \in |s| \subset |K|$, and is linear.

We now define a $1 - 1$ simplicial map $h_2 : {}^1K \to K_1$ by setting $h_2(b_s) = c_s$ for any simplex s of K. Then h_2 is a homeomorphism of $|{}^1K|$ onto $|K_1|$. Finally, we define a simplicial map $f_1 : K_1 \to {}^1L$ by setting $f_1(c_s)$ equal to the barycenter of the simplex $f(s)$. Now consider the following diagram:

$$\begin{array}{ccccc} |{}^1K| & \xrightarrow{h_2} & |K_1| & \xrightarrow{h_1} & |K| \\ & {}^1f \searrow & \swarrow f_1 & & \swarrow f \\ & |{}^1L| & \xrightarrow{g} & |L| & \end{array}$$

It follows from the definitions of the various maps involved that commutativity holds around both the square and the triangle. Define $h = h_1 h_2$. Then h has the required properties.

We can now supply the proof of Lemma 4.5.1. Let ${}^2f:{}^2K \to {}^2L$ be the second barycentric subdivision of f, and let $h:{}^2K \to K$, $g:{}^2L \to L$ be the homeomorphisms of Lemma 4.5.3. Let τ^n be chosen so that it is the image under g of an n-simplex of ${}^2L$. The map h now furnishes the desired triangulation of $(\,|K|\,, A)$.

(4.6) An Extension of the Simplicial Approximation Theorem

Let $X^* = X \cup \mathcal{E}^n$ be obtained by adjoining an n-cell to X, and suppose a map $f_0 : K \to X^*$ given, where K is a finite simplicial complex.

LEMMA 4.6.1. *There exists a map $f_1 : K \to X^*$, and a closed n-cell E contained in the interior of $\mathcal{E}^n$, having the following properties:*
(1) *The pair (K, L) is triangulable, where $L = f_1^{-1}(E)$.*
(2) *$f_1 \mid L$ is a map which is simplicial with respect to some triangulation of L and E.*
(3) *There is a homotopy $f_t : K \to X^*$ between f_0 and f_1, which has the following properties:*

(a) $$f_t(f_0^{-1}\mathcal{E}^n) \subset \mathcal{E}^n, \qquad 0 \leqq t \leqq 1.$$

(b) $$f_t(x) = f_0(x) \text{ if } f_0(x) \in X, \qquad 0 \leqq t \leqq 1.$$

PROOF. By definition, there exists a map $\psi : (E^n, S^{n-1}) \to (X^*, X)$ which is a homeomorphism of $E^n - S^{n-1}$ onto $X^* - X$. Choose four real numbers $\rho_1, \cdots, \rho_4$ where $0 < \rho_1 < \rho_2 < \rho_3 < \rho_4 < 1$, and let

$$\left.\begin{aligned} D_i &= \{x \in E^n \mid \|x\| \leqq \rho_i\} \\ E_i &= \psi(D_i) \end{aligned}\right\} \qquad i = 1, \cdots, 4.$$

Then $E_1 \subset E_2 \subset E_3 \subset E_4 \subset \mathcal{E}^n$, and each E_i is a closed n-cell. Assume that E_4 is triangulated so that E_1, E_2, E_3 are subcomplexes. Let U = interior of E_4, $V = X^* - E_3$, $U' = f_0^{-1}(U)$, $V' = f_0^{-1}(V)$. Then $\{U', V'\}$ is an open covering of K. Subdivide K barycentrically so that the mesh of K is less than one half the Lebesque number of the covering $\{U', V'\}$. It then follows that the star of any vertex of K is contained entirely in one or the other of the sets U', V'; hence the image under f_0 of the star of any vertex is contained in either U or V.

Let M be the closed subcomplex of K spanned by all vertices of K whose stars map into U. It follows easily that

$$f_0^{-1}(E_4) \supset M \supset f_0^{-1}(E_3).$$

Choose a simplicial map

$$g : M \to E_4$$

which is a simplicial approximation to $f_0 \mid M$; it might be necessary to further subdivide M in order to define g. Let $E_2' = f_0^{-1}(E_2)$, $Y = f_0^{-1}(X^* - \text{Int } E_3)$. Then E_2', Y, are closed disjoint subsets of K. Let

$$\sigma : K \to [0, 1]$$

be a continuous "Urysohn function" such that $\sigma(x) = 1$ for $x \in E_2'$ and $\sigma(x) = 0$ for $x \in Y$. Let

$$F: M \times I \to E_4$$

be the homotopy between $f_0 \mid M$ and g. Define a new map

$$F': M \times I \to E_4$$

by $F'(x, t) = F(x, t\sigma(x))$. Extend F' to a map

$$F': K \times I \to X^*$$

by setting $F'(x, t) = f_0(x)$ for $x \in K - M$. Clearly F' is continuous, and is a homotopy of f_0. Define

$$f_1: K \to X^*$$

by $f_1(x) = F'(x, 1)$. Then $f_1^{-1}(E_1) = g^{-1}(E_1) = L$ is a subcomplex of K, and it is readily seen that all of the conditions of our theorem are satisfied if we take $E = E_1$.

(4.7) Two Important Lemmas

Let $(X; Y, Z)$ be an arbitrary triad, and let $\alpha \in \pi_q(X; Y, Z)$. Choose a map

$$f: (E^q; E_+^{q-1}, E_-^{q-1}) \to (X; Y, Z)$$

which represents α. Assume that E^q can be decomposed into the union of two *closed* sets A and B,

$$E^q = A \cup B$$

such that

$$E_+^{q-1} \subset A \subset f^{-1}(Y),$$
$$E_-^{q-1} \subset B \subset f^{-1}(Z).$$

Let $C = A \cap B$ and $W = Y \cap Z$. Then the map f defines maps

$$f': (B, C) \to (Z, W),$$
$$f'': C \to W.$$

In this section we prove the following two lemmas.

LEMMA 4.7.1. *If the map f' is deformable* rel C, *then* $\alpha = 0$. (If C is a subcomplex of B the condition "rel C" can be dropped.)

LEMMA 4.7.2. *If the map f'' can be extended to a map $B \to W$, then $\beta_+(\alpha) = 0$; here β_+ is the homotopy boundary operator*

$$\beta_+: \pi_q(X; Y, Z) \to \pi_{q-1}(Y, W).$$

These simple lemmas are the key to the proof of the main theorems.

PROOF OF LEMMA 4.7.1. Let

$$g_t: (B, C) \to (Z, W), \qquad 0 \leq t \leq 1,$$

be a deformation of f' rel C; this means that

$$g_0 = f', \qquad g_t \mid C = g_0 \mid C, \qquad g_1(B) \subset W.$$

We define a homotopy

$$f_t:(E^q; E_+^{q-1}, E_-^{q-1}) \to (X; Y, Z), \qquad 0 \leqq t \leqq 1,$$

by

$$f_t \mid A = f \mid A, \qquad f_t \mid B = g_t .$$

Then $f_0 = f$, hence $f_1 \simeq f$. Also $f_1(E^q) \subset Y$, and $f_1(E_-^{q-1}) \subset W$, so that f_1 is a map

$$f_1:(E^q, \dot{E}^q) \to (Y, W)$$

and therefore determines an element γ of $\pi_q(Y, W)$, whose image $j_{1*}i_{1*}(\gamma)$ in $\pi_q(X; Y, Z)$, (see (3.5)), is the element α. It now follows from the exactness of the upper homotopy sequence of $(X; Y, Z)$ that $\alpha = 0$.

Proof of Lemma 4.7.2. Let $g:B \to W$ be an extension of $f'':C \to W$; i.e., $g \mid C = f''$. We define

$$h:E^q \to Y$$

by

$$h \mid A = f \mid A, \qquad h \mid B = g.$$

Let $F:(E_+^{q-1}, S^{q-2}) \to (Y, W)$ be the map defined by f. Then F is a representative of $\beta_+(\alpha)$. Let

$$i_t:(E_+^{q-1}, S^{q-2}) \to (E^q, S^{q-2}), \qquad 0 \leqq t \leqq 1,$$

be a homotopy such that

$$\begin{cases} i_0(x) = x, & x \in E_+^{q-1}, \\ i_t(x) = x, & x \in S^{q-2}, \qquad t \in I, \\ i_1(E_+^{q-1}) \subset E_-^{q-1}. \end{cases}$$

Such a homotopy of the inclusion map $i_0:(E_+^{q-1}, S^{q-2}) \to (E^q, S^{q-2})$ clearly exists. We now define a homotopy

$$F_t:(E_+^{q-1}, S^{q-2}) \to (Y, W)$$

by setting

$$F_t(x) = h[i_t(x)], \qquad 0 \leqq t \leqq 1.$$

Then $F_0 = F$, and hence $F_1 \simeq F$. But $F_1(E_+^{q-1}) \subset W$, and therefore $\beta_+(\alpha) = 0$.

(4.8) A Normalization Process for Certain Triad Maps

We wish to apply the two lemmas of the preceding section to prove Theorems I and II. However, this is impossible in general without first making a homotopic deformation of the triad mapping which represents a given element of the group $\pi_q(X^*; \mathcal{E}^n, X)$. We will call this preliminary homotopic deformation the *normalization process.*

In this section we shall use the same notation as in (4.1), X^* is the space obtained from the space X by adjoining the cells $\mathcal{E}_1^n, \mathcal{E}_2^n, \cdots; \mathcal{E}^n = \bigcup_i \mathcal{E}_i^n$, $\dot{\mathcal{E}}^n = \bigcup_i \dot{\mathcal{E}}_i^n = \mathcal{E}^n \cap X$, and $\dot{\mathcal{E}}^n$ is assumed arcwise connected. Let α be any element of $\pi_q(X^*; \mathcal{E}^n, X)$, and

$$f: (E^q; E_+^{q-1}, E_-^{q-1}) \to (X^*; \mathcal{E}^n, X)$$

any map representing α.

By applying Lemma 4.6.1 to each of the cells $\mathcal{E}_1^n, \mathcal{E}_2^n, \cdots$, we may deform f into a map

$$f_1: (E^q; E_+^{q-1}, E_-^{q-1}) \to (X^*; \mathcal{E}^n, X)$$

having the following properties: For each subscript i there exists a closed n-simplex E_i contained in the interior of $\mathcal{E}_i^n$, such that $f_1^{-1}(E_i)$ is a subcomplex K_i of E^q, and such that $f_1 \mid K_i$ is a simplicial map. Now we may apply Lemma 4.5.1. Choose an n-simplex σ_i^n contained in the interior of E_i, such that $f_1^{-1}(\sigma_i^n)$ is a subcomplex of K_i, and hence of E^q. At this point we introduce some further notation. Let y_i be an interior point of σ_i^n, and $Y = \bigcup_i y_i$. Then Y is a closed, discrete subset of X^*. Let

$$P = f_1^{-1}(Y); \quad P' = P \cap S^{q-1}; \quad Q = f_1^{-1}(\bigcup_i \sigma_i^n); \quad Q' = Q \cap S^{q-1};$$
$$A = Q \cup E_+^{q-1}; \quad B = Cl(E^q - Q); \quad C = A \cap B = (Q \cap B) \cup Cl(E_+^{q-1} - Q').$$

Note that all of these subspaces of E^q are closed in E^q. Next, we apply the lemma of Pontrjagin, mentioned in (4.5), to the simplicial map $K_i \to E_i$ defined by f_1. It follows that P is a cell complex of dimension $\leqq q - n$, and P' is a subcomplex of dimension $\leqq q - n - 1$. Furthermore, there is a homeomorphism between the pairs (Q, Q') and $(P \times \sigma^n, P' \times \sigma^n)$, where σ^n is an n-simplex. This implies that the pair (P, P') is a deformation retract of the pair (Q, Q'), and hence (Q, Q') and (P, P') have the same homotopy type. Finally, Q, Q', A, B, and C are all subcomplexes of E^q.

Now let

$$\varphi_t: (X^*; \mathcal{E}^n, X) \to (X^*; \mathcal{E}^n, X), \qquad 1 \leqq t \leqq 2,$$

be a 1-parameter family of continuous maps having the following properties:

$$\varphi_1 = \text{identity},$$
$$\varphi_t \mid X = \varphi_1 \mid X,$$
$$\varphi_t(y_i) = y_i, \qquad i = 1, 2, \cdots,$$
$$\varphi_t(\mathcal{E}_i^n - \sigma_i^n) \subset \mathcal{E}_i^n - \sigma_i^n,$$
$$\varphi_2(\mathcal{E}_i^n - \sigma_i^n) = \dot{\mathcal{E}}_i^n,$$
$$\varphi_2(\sigma_i^n) = \mathcal{E}_i^n.$$

It is obvious that such a homotopy of the identity map exists; each simplex σ_i^n is allowed to "expand" uniformly keeping y_i fixed, until it exactly covers all of $\mathcal{E}_i^n$, and $\mathcal{E}_i^n - \sigma_i^n$ is "contracted" into $\dot{\mathcal{E}}^n$ during the process. Define $f_t(1 \leqq t \leqq 2)$ by

$$f_t = \varphi_t f_1 .$$

Then f_t is a homotopy of f_1, and we shall call f_2 a "normal form" for a representative of the homotopy class α. It is not asserted that there is a unique normal form; any normal form will suffice for the proofs of the main theorems.

(4.9) Proof of Theorem I

Let $(X^*; \mathcal{E}^n, X)$ be the triad described in (4.1), satisfying all the hypotheses of Theorem I, with the pair $(X, \dot{\mathcal{E}}^n)$ assumed to be m-connected. We will prove that if $\alpha \in \pi_q(X^*; \mathcal{E}^n, X)$ and $q < m + n$, then $\alpha = 0$. Let

$$f:(E^q; E_+^{q-1}, E_-^{q-1}) \to (X^*; \mathcal{E}^n, X)$$

be a representative of α; we assume that we have applied the normalization process of the preceding section, and f is in normal form. Let P, P', Q, Q', A, B, C, have the same meaning as in section (4.8). By using the theory of obstructions to deformations we will show that the map

$$f':(B, C) \to (X, \dot{\mathcal{E}}^n)$$

defined by f, is deformable. By Lemma 4.7.1 this will suffice to prove $\alpha = 0$.

The obstructions belong to the groups $H^j(B, C, \pi_j(X, \dot{\mathcal{E}}^n))$, $2 \leqq j \leqq q < m + n$. We will show that these groups are all trivial. It will then follow from Theorem 4.4.2 in case $n = 2$, and from Lemma 4.2.1 and Theorem 4.4.3 in case $n > 2$, that the map f' is deformable, as was to be proved.

It follows from the excision axiom (see Eilenberg-Steenrod, [5]) that

$$H^j(B, C, \pi_j(X, \dot{\mathcal{E}}^n)) \approx H^j(E^q, A, \pi_j(X, \dot{\mathcal{E}}^n)).$$

It follows from the exactness of the cohomology sequence of the triple (E^q, A, E_+^{q-1}) that

$$H^j(E^q, A) \approx H^{j-1}(A, E_+^{q-1}).$$

Next, using the excision axiom again

$$H^{j-1}(A, E_+^{q-1}) \approx H^{j-1}(Q, Q'),$$

and since (Q, Q'), (P, P') have the same homotopy type

$$H^{j-1}(Q, Q') \approx H^{j-1}(P, P').$$

Combining these isomorphisms, we have

$$H^j(B, C, \pi_j(X, \dot{\mathcal{E}}^n)) \approx H^{j-1}(P, P', \pi_j(X, \dot{\mathcal{E}}^n)).$$

Since $\dim P \leqq q - n$, it follows that these groups vanish if $j - 1 > q - n$; i.e., if $j > q - n + 1$. They vanish if $j \leqq m$, since then the coefficient group is trivial. Consequently if $q < m + n$, then $H^j(B, C, \pi_j(X, \mathcal{E}^n)) = 0$ for $2 \leqq j \leqq q$ and the proof is complete.

(4.10) Proof of Theorem II

Let $(X^*; \mathcal{E}^n, X)$ be as in section (4.1), satisfying the hypotheses of Theorem II. We shall prove that if $\alpha \in \pi_{q+1}(X^*; \mathcal{E}^n, X)$, then $\beta_+(\alpha) = 0$ for $2 \leqq q \leqq m + n - 1$. Let

$$f:(E^{q+1}; E^q_+, E^q_-) \to (X^*; \mathcal{E}^n, X)$$

be a map representing α and in normal form. By Lemma 4.7.2 it suffices to show that the map

$$f'':C \to \mathcal{E}^n$$

defined by f, can be extended to a map of B into $\mathcal{E}^n$. We shall use Theorem 4.3.3 to show the existence of such an extension.

It follows from the argument used in the preceding section, that

$$H^j(B, C, \pi_{j-1}(\mathcal{E}^n)) \approx H^{j-1}(P, P', \pi_{j-1}(\mathcal{E}^n)).$$

Because $\dim P \leqq q - n + 1$, it follows readily that the hypotheses of Theorem 4.3.3 are satisfied when $q \leqq m + n - 1$, which completes the proof.

Part 5. Applications

(5.1) Freudenthal's Einhängung Theorems

Consider the triad $(S^n; E^n_+, E^n_-)$ and assume $n \geqq 2$ (actually, the case $n = 2$ is uninteresting). Then we may apply Theorem I to this triad. Since m is clearly equal to $(n - 1)$, we conclude that the triad $(S^n; E^n_+, E^n_-)$ is $(2n - 2)$-connected. Therefore, from the exactness of the homotopy sequence, the homomorphism

$$i_{1*}:\pi_q(E^n_+, S^{n-1}) \to \pi_q(S^n, E^n_-)$$

is an isomorphism onto for $2 \leqq q \leqq 2n - 3$, and is a homomorphism onto for $q = 2n - 2$. This result may be shown to be equivalent to part of Freudenthal's results, as follows: Let

$$E:\pi_{p-1}(S^{n-1}) \to \pi_p(S^n)$$

denote the "Einhängung" homomorphism as defined by Freudenthal. Then in the diagram

$$\begin{array}{ccc} \pi_{p-1}(S^{n-1}) & \xrightarrow{E} & \pi_q(S^n) \\ \partial \uparrow & & \downarrow k_* \\ \cdots \to \pi_p(E^n_+, S^{n-1}) & \xrightarrow{i_{1*}} & \pi_p(S^n, E^n_-) \to \cdots \end{array}$$

the boundary homomorphism ∂, and the homomorphism k_* induced by the corresponding inclusion map, are isomorphisms onto, and the commutativity relation

$$i_{1*} = k_* E \, \partial$$

holds. Thus the homomorphism i_{1*} is equivalent to the Einhängung, and the statement about i_{1*} above is exactly the content of the easier part of Freudenthal's first two theorems.

(5.2) Some General Remarks on Applying Theorems I and II

In the preceding example, involving the triad $(S^n;\, E^n_+,\, E^n_-)$, it was quite obvious how our theorems should be applied. This is not always so, as the reader will see from subsequent examples. It is the purpose of this section to illustrate and discuss this point.

Let the space X^* be obtained from the space X by adjoining a *single* cell $\mathcal{E}^n$, as described in (4.1). Suppose we wish to determine the relative homotopy groups of (X^*, X). The boundary, $\dot{\mathcal{E}}^n$, of $\mathcal{E}^n$ is the continuous image of an $(n - 1)$-sphere, but of course it need not be homeomorphic to an $(n - 1)$-sphere. It may even consist of a single point. Suppose we choose a closed n-cell, σ^n, contained entirely in the interior of $\mathcal{E}^n$, and let $X' = \mathrm{Cl}(X^* - \sigma^n)$, $\dot{\sigma}^n = X' \cap \sigma^n =$ boundary of σ^n. Then it is readily seen that X is a deformation retract of X', and that the pairs (X^*, X) and (X^*, X') have the same homotopy type. Thus the relative homotopy groups of (X^*, X) and (X^*, X') are isomorphic.

One method of attacking our problem would be to consider the homomorphisms

$$i_{1*} \colon \pi_q(\mathcal{E}^n, \dot{\mathcal{E}}^n) \to \pi_q(X^*, X),$$

$$i_{2*} \colon \pi_q(\sigma^n, \dot{\sigma}^n) \to \pi_q(X^*, X'),$$

where i_1 and i_2 are inclusion maps. This leads naturally to the consideration of the exact sequences and the homotopy groups of the triads $(X^*;\, \mathcal{E}^n, X)$ and $(X^*;\, \sigma^n, X')$ respectively. Now if we wish to apply Theorem I, we have to consider the homotopy groups of the pairs $(X, \dot{\mathcal{E}}^n)$ and $(X', \dot{\sigma}^n)$ respectively. In general, these pairs will *not* have the same homotopy groups, and which pair it is most convenient or useful to consider will depend on the particular problem at hand.

Naturally, the same kind of discussion applies to the case where we adjoin several cells simultaneously to the space X.

(5.3) The Homotopy Groups of a Space Consisting of Several Spheres with a Point in Common

Let $\mathcal{S}^n_k$ be a connected cell complex consisting of a single vertex σ^0 and k n-cells, $\sigma^n_1, \cdots, \sigma^n_k$, where k may be finite or infinite. $\mathcal{S}^n_k$ may be regarded as a space consisting of a collection of n-dimensional spheres $S^n_1 \cup S^n_2 \cup \cdots \cup S^n_k$, intersecting in the unique vertex, σ^0. If the number of spheres, k, is infinite. we assume that $\mathcal{S}_k^n$ is topologized so as to become a CW-complex in the sense of J.H.C. Whitehead, [20]. It is important to determine as much information as possible about the homotopy groups of $\mathcal{S}_k^n$ for the following two reasons.

(a) If K is an arbitrary CW-complex which is $(n - 1)$-connected, then J. H. C. Whitehead [20] has shown that K has the same homotopy type as a CW-complex L with the property that the n-skeleton, L^n, of L is isomorphic to $\mathcal{S}_k^n$ for some value of k. Therefore the determination of the homotopy groups of the space $L^n = \mathcal{S}_k^n$ is a first step in the determination of the homotopy groups of K. This point is illustrated by some recent work of J. H. C. Whitehead, [21], in which he determines the third homotopy group of a simply connected complex, or, more generally, the $(n + 1)^{th}$ homotopy group of an $(n - 1)$-connected complex.

(b) Let K be a CW-complex, and K^n its n-skeleton. Then, as was remarked in the introduction, it is important to study the homomorphism

$$i_*:\pi_q(\mathcal{E}^n, \dot{\mathcal{E}}^n) \rightarrow \pi_q(K^n, K^{n-1})$$

where $\mathcal{E}^n$ is the union of all the n-cells of K. This requires in particular that we determine the homotopy groups $\pi_q(\mathcal{E}^n, \dot{\mathcal{E}}^n)$. By applying the process described in (5.2), we see that we may modify our problem slightly and assume that the space $\mathcal{E}^n$ is a union of homeomorphs of E^n, the unit n-cell in C^n, all having a single point in common. Also, $\dot{\mathcal{E}}^n$ is a union of $(n - 1)$-spheres having a single point in common, i.e., $\dot{\mathcal{E}}^n =_t \mathcal{S}_k^{n-1}$ for some value of k. Since the space $\mathcal{E}^n$ is contractible, it follows that

$$\pi_q(\mathcal{E}^n, \dot{\mathcal{E}}^n) \approx \pi_{q-1}(\dot{\mathcal{E}}^n).$$

A necessary tool in the determination of the homotopy groups of the spaces $\mathcal{S}_k^n$ is a general theorem of G. W. Whitehead about the homotopy groups of the union of two spaces with a single point in common. Let A and B be arcwise connected spaces, $a_0 \in A$, $b_0 \in B$, and let $A \vee B$ denote the subset $(A \times b_0) \cup (a_0 \times B)$ of $A \times B$. Choose the point $a_0 \times b_0$ as base point for all homotopy groups involving $A \times B$ and $A \vee B$. Define maps $\mu_1:A \rightarrow A \vee B$ and $\mu_2:B \rightarrow A \vee B$ by $\mu_1(x) = (x, b_0)$, $\mu(y) = (a_0, y)$ for $x \in A$, $y \in B$. It is an elementary matter to prove that for $n \geqq 2$, the homomorphisms

$$\mu_{1*}:\pi_n(A, a_0) \rightarrow \pi_n(A \vee B),$$

$$\mu_{2*}:\pi_n(B, b_0) \rightarrow \pi_n(A \vee B),$$

are isomorphisms into, and that the image sub-groups are direct summands of $\pi_n(A \vee B)$. Let

$$\partial:\pi_{n+1}(A \times B, A \vee B) \rightarrow \pi_n(A \vee B)$$

denote the boundary operator. Then we have the following:

THEOREM 5.3.1. *If $n \geqq 2$, then the homomorphism ∂ is an isomorphism into, and $\pi_n(A \vee B)$ splits up into the direct sum,*

$$\mu_{1*}\pi_n(A) + \mu_{2*}\pi_n(B) + \partial\pi_{n+1}(A \times B, A \vee B).$$

For the proof, see [16].

Now let $A = \mathcal{S}_{k-1}^n$, $(n > 1, k > 1, k < \infty)$, and $B = S_k^n$. Then we can identify $A \vee B$ with $\mathcal{S}_k^n$. Recall that $\mathcal{S}_{k-1}^n$ is a cell complex consisting of $(k - 1)$ n-cells, $\sigma_1^n, \cdots, \sigma_{k-1}^n$, and a single vertex, σ^0, and consider S_k^n as a cell complex consisting of a single n-cell τ^n and a single vertex, τ^0. Then $\mathcal{S}_{k-1}^n \times S_k^n$ is a cell complex consisting of $(k - 1)$ cells of dimension $2n$, $\sigma_1^n \times \tau^n, \sigma_2^n \times \tau^n, \cdots, \sigma_{k-1}^n \times \tau^n$, k cells of dimension n, $\sigma_1^n \times \tau^0, \cdots, \sigma_{k-1}^n \times \tau^0, \sigma^0 \times \tau^n$, and a single vertex, $\sigma^0 \times \tau^0$. Let $\mathcal{E}_i^{2n}$, $i = 1, \cdots, k - 1$, be a closed $2n$-cell (homeomorph of E^{2n}) such that

$$\mathcal{E}_i^{2n} \subset \sigma_i^n \times \tau^n,$$

and such that the boundary of $\mathcal{E}_i^{2n}$, denoted by $\dot{\mathcal{E}}_i^{2n}$, meets the boundary of $\sigma_i^n \times \tau^n$ in a single point, the vertex $\sigma^0 \times \tau^0$. Let

$$\mathcal{E}^{2n} = \mathcal{E}_1^{2n} \cup \mathcal{E}_2^{2n} \cup \cdots \cup \mathcal{E}_{k-1}^{2n},$$

$$\dot{\mathcal{E}}^{2n} = \dot{\mathcal{E}}_1^{2n} \cup \dot{\mathcal{E}}_2^{2n} \cup \cdots \cup \dot{\mathcal{E}}_{k-1}^{2n},$$

$$X^* = \mathcal{S}_{k-1}^n \times S_k^n,$$

$$X = \mathrm{Cl}(X^* - \mathcal{E}^{2n}).$$

Then (X^*, X) has the same homotopy type as $(\mathcal{S}_{k-1}^n \times S_k^n, \mathcal{S}_{k-1}^n \vee S_k^n)$. Furthermore, it is readily seen that $\dot{\mathcal{E}}^{2n}$ is $(2n - 2)$-connected, X is $(n - 1)$-connected, and hence $(X, \dot{\mathcal{E}}^{2n})$ is $(n - 1)$-connected. Therefore, by Theorem I, the triad $(X^*; \mathcal{E}^{2n}, X)$ is $(3n - 2)$-connected, and by Theorem II, the homomorphism

$$\beta_+ : \pi_{p+1}(X^*; \mathcal{E}^{2n}, X) \to \pi_p(\mathcal{E}^{2n}, \dot{\mathcal{E}}^{2n})$$

is trivial if $2 \leqq p \leqq 4n - 3$. Therefore

$$i_* : \pi_p(\mathcal{E}^{2n}, \dot{\mathcal{E}}^{2n}) \to \pi_p(X^*, X)$$

is an isomorphism onto for $p \leqq 3n - 2$. If now we apply (5.3.1), we obtain the following result:

LEMMA 5.3.2. *The group $\pi_p(\mathcal{S}_k^n)$ is isomorphic to the direct sum of the three groups $\pi_p(\mathcal{S}_{k-1}^n)$, $\pi_p(S_k^n)$, and $\pi_{p+1}(\mathcal{E}^{2n}, \dot{\mathcal{E}}^{2n})$, for $2 \leqq p \leqq 3n - 3$. Clearly, $\pi_{p+1}(\mathcal{E}^{2n}, \dot{\mathcal{E}}^{2n}) \approx \pi_p(\dot{\mathcal{E}}^{2n}) \approx \pi_p(\mathcal{S}_{k-1}^{2n-1})$.*

By using this lemma, we can determine the groups $\pi_p(\mathcal{S}_k^n)$ by means of an induction on k. The final result is stated most neatly in a slightly different form, which also includes the case where k is infinite. Let $\mathcal{S}_k^n = S_1^n \cup S_2^n \cup \cdots \cup S_k^n$ as before, where now k may be infinite. Let

$$\varphi_i : S_i^n \to \mathcal{S}_k^n$$

be an inclusion map, and for any two indices i, j, $i < j$, define a map

$$\varphi_{ij} : S^{2n-1} \to \mathcal{S}_k^n$$

as follows: Consider S^{2n-1} as the boundary $(E^n \times E^n)^{\cdot}$ of the $2n$-cell $E^n \times E^n$. Let

$$\psi_i : E^n \to \mathcal{S}_k^n$$

be a map such that $\psi_i(\dot{E}^n) = \sigma^0$, the single vertex of the complex $\mathcal{S}_k^n$, and $\psi_i \mid (E^n - \dot{E}^n)$ is a homeomorphism of $E^n - \dot{E}^n$ onto $S_i^n - \sigma^0$. Define

$$\varphi_{ij}(x, y) = \begin{cases} \psi_i(x), & (x, y) \in E^n \times \dot{E}^n, \\ \psi_j(y), & (x, y) \in \dot{E}^n \times E^n. \end{cases}$$

Then we know that

$$\varphi_{i*}: \pi_q(S_i^n) \to \pi_q(\mathcal{S}_k^n)$$

is an isomorphism into, and the image group is a direct summand, for all q. It follows from the above discussion that

$$\varphi_{ij*}: \pi_q(S^{2n-1}) \to \pi_q(\mathcal{S}_k^n)$$

is an isomorphism into for $q \leqq 4n - 3$, and that the image group is a direct summand for $q \leqq 3n - 3$. We now state our result as follows:

THEOREM 5.3.3. *For $q \leqq 3n - 3$, the group $\pi_q(\mathcal{S}_k^n)$ is the (weak) direct sum of the groups $\varphi_{i*}\pi_q(S_i^n)$ and $\varphi_{ij*}\pi_q(S^{2n-1})$ for all values of i and j, $i < j$.*

The proof is made first for the finite case, by an induction on k, and then the infinite case is proved by using the fact that any element α of $\pi_q(\mathcal{S}_k^n)$ has a representative map

$$f: S^q \to \mathcal{S}_k^n$$

such that $f(S^q)$ is contained in a finite sub-complex of $\mathcal{S}_k^n$.

COROLLARY 5.3.4. *For $q \leqq 2n - 2$, the group $\pi_q(\mathcal{S}_k^n)$ is the direct sum of the groups $\varphi_{i*}\pi_q(S_i^n)$.*

It is clear how these results can be generalized to the case of the union of spheres of different dimensions all having a single point in common.

In a subsequent paper, we will determine the groups $\pi_{3n-2}(\mathcal{S}_k^n)$ for the case $k < \infty$.

(5.4) Extension of Some Recent Results of G. W. Whitehead

It is the purpose of this section to discuss the results in a recent paper of G. W. Whitehead [17], and to show how the range of validity of some of his theorems can be extended by one dimension. Using the notation of this paper, X is an $(n - 1)$-connected space, $n > 1$, and A is a set of generators for the group $\pi_n(X)$. For each $\alpha \in A$, let E_α^{n+1} be an $(n + 1)$-cell with boundary S_α^n; let y_α be a fixed reference point of S_α^n, x_0 a fixed reference point of X, and let $f_\alpha: (S_\alpha^n, y_\alpha) \to (X, x_0)$ be a mapping representing the element $\alpha \in \pi_n(X)$. Suppose that $\bigcup_{\alpha \in A} E_\alpha^{n+1}$ is topologized so that the cells E_α^{n+1} are mutually separated, and let E be the space obtained from $\bigcup_{\alpha \in A} E_\alpha^{n+1}$ by identifying all the points y_α to a single point y_0. Let S be the subset of E obtained from $\bigcup_{\alpha \in A} S_\alpha^n$ by the above identification. Then the mappings f_α together define a mapping $f: (S, y_0) \to (X, x_0)$. Let X^* be the identification space obtained from $E \cup X$ by identifying each point $y \in S$ with its image $f(y) \in X$. Then X may be considered as a subspace of X^*, and the identification induces a map

$$F:(E, S) \rightarrow (X^*, X), \qquad F \mid S = f.$$

It is easily seen that X^* is n-connected. Let E'^{n+1}_α be a cell "concentric" with and contained in E^{n+1}_α, with $E'^{n+1}_\alpha \cap S^n_\alpha = y_\alpha$; let $\bigcup_{\alpha\in A} E'^{n+1}_\alpha = E'$, $\bigcup_{\alpha\in A} \dot{E}'^{n+1}_\alpha = S'$; and let $X' = \mathrm{Cl}(X^* - F(E'))$. Then X' has the same homotopy type as X and there exist homomorphisms as shown in the following diagram,

$$\begin{array}{ccc} \pi_i(E, S) & \xrightarrow{F_*} & \pi_i(X^*, X) \\ \uparrow \varphi_* & & \downarrow j_* \\ \pi_i(E', S') & \xrightarrow[F_{1*}]{} & \pi_i(X^*, X') \end{array}$$

where F_*, F_{1*} are induced by F, j_* is induced by an inclusion map, and φ_* is induced by a homeomorphism $\varphi:(E', S') \rightarrow (E, S)$ which is a "projection" from the common centers. Moreover, commutativity holds in the above diagram in the sense that $F_{1*} = j_*F_*\varphi_*$, and φ_*, j_* are isomorphisms onto. Hence the homomorphisms F_* and F_{1*} are equivalent. We shall now prove the following theorem, which is an extension of the result on p. 208 of [17] by one dimension.

THEOREM 5.4.1. (a) *F_* is an isomorphism onto for $i < 2n$.* (b) *If $i < 2n$, then $\pi_i(E, S)$ is the weak direct sum of the subgroups $p^\alpha_*(\pi_i(E^{n+1}_\alpha, S^n_\alpha))$, where $p^\alpha = F \mid E^{n+1}_\alpha$.*

PROOF. We shall prove (a) by showing the corresponding result for F_{1*}. Let $F(E') = \mathcal{E}$, $F(S') = \dot{\mathcal{E}}$. Then since X' is $(n - 1)$-connected and $\dot{\mathcal{E}}$ is $(n - 1)$-connected, the pair $(X', \dot{\mathcal{E}})$ is also $(n - 1)$-connected. In addition we have the following homomorphisms from the exact sequence of the pair $(X', \dot{\mathcal{E}})$:

$$\cdots \rightarrow \pi_n(\dot{\mathcal{E}}) \xrightarrow{i_*} \pi_n(X') \rightarrow \pi_n(X', \dot{\mathcal{E}}) \rightarrow 0.$$

Since i_* is clearly onto, it follows that $\pi_n(X', \dot{\mathcal{E}}) = 0$, so that $(X', \dot{\mathcal{E}})$ is n-connected. Now apply Theorem I to the triad $(X^*; \mathcal{E}, X')$; we conclude that this triad is $2n$-connected. It follows that the natural homomorphism

$$\pi_i(\mathcal{E}, \dot{\mathcal{E}}) \rightarrow \pi_i(X^*, X')$$

is an isomorphism onto for $i < 2n$. But F_1 is a homeomorphism, $F_1:(E', S') \rightarrow (\mathcal{E}, \dot{\mathcal{E}})$, and hence F_{1*} is also an isomorphism onto for $i < 2n$. This proves (a).

Since the space E is obviously contractible to a point, it follows that $\pi_{i+1}(E, S) \approx \pi_i(S)$. Part (b) of the theorem now follows by applying Corollary 5.3.4 to the space S.

It follows from this result that some of the theorems of [17] can be extended slightly; for example, Theorem 2 is still true for $n = 3$, and Theorem 4 can be extended to read, "If $n > 2$, then $H_{n+2}(\pi, n) \approx \pi/2\pi$."

(5.5) The Generalized Hopf Invariant

In another recent paper, [16], G. W. Whitehead has defined a generalization of the Hopf Invariant for maps of a sphere onto a sphere. By using our Theorem 5.3.3 applied to the space $\mathcal{S}_2^n = S^n \vee S^n$, the definition of the generalized Hopf Invariant can be extended by one dimension, and thus applied to new cases; in particular, it can be shown that $\pi_6(S^3) \neq 0$.

An essential part in the definition of the Generalized Hopf Invariant is played by Theorem 4.17 of [16]; according to this theorem, if $n < 3p - 3$, then

$$\pi_n(S^p \vee S^p) \approx \pi_n(S^p) + \pi_n(S^p) + \pi_n(S^{2p-1})$$

However, from Theorem 5.3.3 above, we see that this isomorphism also holds in case $n = 3p - 3$. Using this fact, the Generalized Hopf Homomorphism,

$$H:\pi_n(S^r) \rightarrow \pi_n(S^{2r-1})$$

can be defined for $n = 3r - 3$ by exactly the same method used to define it for $n < 3r - 3$ in section 5 of [16]. It can then be verified that Theorem 5.1 of [16] is still true if $n = 3r - 3$. For the case $p = 2k - 3$, Corollary 5.14 needs to be reworded slightly. A general statement of the extended Corollary 5.14, which is true for all values of $p \leqq 2k - 3$ reads as follows: "If $\alpha \in \pi_p(R_{k-1})$ and the Einhängung of $\kappa(\alpha)$ is not zero, then $J(\alpha) \neq 0$."

In particular, for the case $p = k = 3$, it is known that $\pi_3(R_2)$ is infinite cyclic, $\kappa:\pi_3(R_2) \rightarrow \pi_3(S^2)$ is an isomorphism onto, and the Einhängung of a generator of $\pi_3(S^2)$ is not zero. Hence if we choose α to be a generator of $\pi_3(R_2)$, it follows that $J(\alpha)$ is a non-zero element of $\pi_6(S^3)$. Since $\pi_6(S^3) \neq 0$, it follows that $\pi_7(S^4)$ is *not* a cyclic group, (cf. [11], corollary 6).

Appendix A. The Homology Groups of a Triad

It is reasonable to ask whether or not the concept of a homology group for triads would have interesting consequences. In terms of singular homology theory, it is natural to define the n-dimensional singular homology group for the triad $(X; A, B)$ (where $A \cap B$ need not be assumed non-vacuous) to be the group $H_n(X; A, B) = H_n(S(X), S(A) \cup S(B))$ (see [4]). The treatment for the corresponding homotopy groups can be paralleled to define upper and lower exact sequences for the triad $(X; A, B)$. The upper sequence is

$$\cdots \rightarrow H_n(X; A, B) \rightarrow H_{n-1}(A, A \cap B) \rightarrow H_{n-1}(X, B) \rightarrow H_{n-1}(X; A, B) \rightarrow \cdots .$$

In case the homology groups are invariant under the homomorphism induced by the excision map $(A, A \cap B) \rightarrow (A \cup B, B)$, it can be shown by a purely algebraic argument that $H_n(X; A, B) \approx H_n(X, A \cup B)$ and also that the excision map $(B, A \cap B) \rightarrow (A \cup B, A)$ induces an isomorphism of the relative homology groups. In terms of Čech homology groups (based on finite open coverings, with X compact Hausdorff and A and B closed in X) the role of the n-dimensional triad homology group is played by the corresponding group of the pair $(X, A \cup B)$ and is comparatively uninteresting. The singular homology groups for triads seem to find an essential use in a discussion of cup products between the singular homology groups of the pairs (X, A), (X, B) associated with an arbitrary triad $(X; A, B)$. These ideas will be taken up in detail in the forthcoming book of Eilenberg and Steenrod [5].

Appendix B. The Homotopy Groups of an N-ad

An obvious direction of generalization of the homotopy groups of a triad is to attempt to define homotopy groups for an n-ad, $(X; X_1, X_2, \cdots, X_{n-1})$. This can be done by generalizing any of the definitions used for triad homotopy groups, and homotopy groups $\pi_i(X; X_1, \cdots, X_{n-1})$ can be defined for $i \geqq n$. These groups are abelian for $i > n$ but need not be so for $i = n$. They have many properties which are analogs of the corresponding properties for the triad groups. In particular, they have operators in the group

$$\pi_1(X_1 \cap X_2 \cap \cdots \cap X_{n-1}),$$

and each n-ad has $(n - 1)$ exact sequences, whose homomorphisms are operator homomorphisms. The first exact sequence is

$$\cdots \to \pi_i(X; X_1, \cdots, X_{n-1}) \to \pi_{i-1}(X_1; X_1 \cap X_2, \cdots, X_1 \cap X_{n-1}) \\ \to \pi_{i-1}(X; X_2, \cdots, X_{n-1}) \to \pi_{i-1}(X; X_1, X_2, \cdots, X_{n-1}) \to \cdots$$

and the others are similarly formed. At the present time we know of no useful application for these groups and sequences.

Appendix C. A General Theorem on the Deformation of Triad Mappings

By exactly the same method as that used to prove Theorem I above, it is possible to prove a more general deformation theorem.

Let $(X^*; \mathcal{E}^n, X)$ be the triad of (4.1), obtained by simultaneous adjunction of the n-cells $\mathcal{E}_1^n, \mathcal{E}_2^n, \cdots$, to the space X, with $\mathcal{E}^n = \bigcup_i \mathcal{E}_i^n$, $\dot{\mathcal{E}}^n = \mathcal{E}^n \cap X$. Assume that $\dot{\mathcal{E}}^n$ is arcwise connected and that the pair $(X, \dot{\mathcal{E}}^n)$ is m-connected, $m \geqq 1$. Let $(K; L, M)$ be a triad consisting of an r-dimensional complex K and two subcomplexes L, M, and let $H^p(K, L, \pi_p(X, \dot{\mathcal{E}}^n)) = 0$ for $p > m$. (We do not need to assume $L \cap M \neq 0$, so that $(K; L, M)$ may not be a triad in the strict sense of our definition in (1.1) above.)

Thoorem. *If $r < m + n$ and the pair $(X, \dot{\mathcal{E}}^n)$ is simple in all dimensions $\leqq r$, then any map*

$$f_0: (K; L, M) \to (X^*; \mathcal{E}^n, X)$$

is homotopic to a map

$$f_1: (K; L, M) \to (X^*; \mathcal{E}^n, X)$$

which satisfies the conditions

$$f_1(K) \subset \mathcal{E}^n, \qquad f_1(M) \subset \dot{\mathcal{E}}^n.$$

We shall not give the proof since it parallels that of Theorem I at every step.

As an example of the application of this theorem we shall give an alternate proof of a lemma of Spanier ([14] lemma (15.1)). This lemma is the crucial step in the proof of exactness for the cohomotopy sequence of a pair. Let K be a finite simplicial complex of dimension $\leqq 2n - 1$, and denote by $\hat{K}$ the join of K with a point P. Then the cohomology groups $H^p(\hat{K})$ vanish in all dimensions, since $\hat{K}$ is contractible.

LEMMA. *Given any map*

$$f_0:(\hat{K}, K) \rightarrow (S^{n+1}, E_-^{n+1})$$

there exists a homotopic map f_1 *such that* $f_1(\hat{K}) \subset E_+^{n+1}$ *and* $f_1(K) \subset S^n$. This lemma is proved by considering f_0 as a triad map

$$f_0:(\hat{K}; 0, K) \rightarrow (S^{n+1}; E_+^{n+1}, E_-^{n+1}),$$

and applying the theorem above. Spanier's lemma (15.1) is now a direct consequence. For we may consider a given map $\alpha:(\hat{K}, K) \rightarrow (S^{n+1}, p)$ as a map $(\hat{K}, K) \rightarrow (S^{n+1}, E_-^{n+1})$ and apply our lemma.

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