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THE HOMOTOPY GROUPS OF A TRIAD. II

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Introduction

The principal purpose of this paper is to state and prove a rather general theorem (Theorem I below) about triad homotopy groups. This theorem is a considerable generalization of the main theorem previously proved by the authors on this subject [1, Theorem I][1], but its proof is in many respects simpler than that of the previous theorem.

The notation and terminology of the present paper are the same as in [1]. For a proper understanding of this paper, the reader should be familiar with almost all of parts 1, 2, and 3, and the following sections of part 4 from [1]: Sections 4.1, 4.2, 4.3, 4.4, and 4.7.

The main theorem is stated in Section 1. In Section 2 several necessary lemmas are proved. The proof of Theorem I is given in Sections 3, 4, and 5. The remainder of the paper is devoted to several applications of Theorem I. Among these is Theorem III, which generalizes Theorem II of [1].

1. Statement of the main theorem

Let $(X; A, B)$ be a triad which satisfies the following conditions:

(a) A, B, and $A \cap B$ are all arc-wise connected.

(b) $X = (\text{Int } A) \cup (\text{Int } B)$.

(c) $(A, A \cap B)$ is m-connected, $(B, A \cap B)$ is n-connected, and $m \geqq n \geqq 1$. (Clearly, no generality is lost by assuming $m \geqq n$, since this condition may always be satisfied by a proper choice of notation).

(d) In case $n = 1$ and $m > n$, we assume either that $\pi_2(B, A \cap B)$ is abelian, or that $(A, A \cap B)$ is simple in dimension $m + 1$. In case $m = n = 1$, we assume that $(B, A \cap B)$ is simple in dimension 2.

It is clear that condition (d) is satisfied if $A \cap B$ is simply connected.

THEOREM I. *If the hypotheses* (a)–(d) *hold, then the triad* $(X; A, B)$ *is* $(m + n)$-*connected.*

One of the principal tools for the proof of this theorem is Lemma 4.7.1 of [1]. Before proceeding with the proof, we shall develop several auxiliary lemmas.

2. The supplement of a subcomplex

Let K be a simplicial complex and L a subcomplex. Denote by (K', L') the first barycentric subdivision of (K, L).

DEFINITION. The *supplement* of L in K, denoted by $K \div L$, is the subcomplex of K' spanned by all of the vertices of $K' - L'$; i.e., a simplex of K' belongs to $K \div L$ if and only if none of its vertices is in L'.

[1] Numbers in square brackets refer to the bibliography at the end of the paper.

It follows that every simplex of K' that is not a simplex of L' or of $K \div L$, is the join [8, p. 202] of a simplex of L' with a simplex of $K \div L$.

We will identify the spaces $|K|$ and $|K'|$, and the spaces $|L|$ and $|L'|$. Since every simplex of K' is the join of a simplex of L' and a simplex of $K \div L$, it follows that there exists a homeomorphic imbedding

$$h: |K| \to |L| * |K \div L|$$

where the symbol "$*$" denotes the join operation. By means of this homeomorphic imbedding we can introduce coordinates in $|K|$ as follows: a point $z \in |K|$ has coordinates (x, t, y) where $x \in |L|$, $y \in |K \div L|$ and $0 \leqq t \leqq 1$. If $t = 0$, then $(x, t, y) \in |L|$, while if $t = 1$, then $(x, t, y) \in |K \div L|$.

Let

$$N(L) = \{(x, t, y) \in |K| \mid 0 \leqq t < \tfrac{1}{2}\},$$

$$N(K \div L) = \{(x, t, y) \in |K| \mid \tfrac{1}{2} < t \leqq 1\}.$$

Then $N(L)$ and $N(K \div L)$ are disjoint open neighborhoods of $|L|$ and $|K \div L|$ in $|K|$. Define

$$\bar{N}(L) = \text{Cl}\, N(L) = \{(x, t, y) \in |K| \mid 0 \leqq t \leqq \tfrac{1}{2}\},$$

$$\bar{N}(K \div L) = \text{Cl}\, N(K \div L) = \{(x, t, y) \in |K| \mid \tfrac{1}{2} \leqq t \leqq 1\}.$$

Then $|K| = \bar{N}(L) \cup \bar{N}(K \div L)$. Furthermore, $|L|$ is a deformation retract[2] of $\bar{N}(L)$ and $|K \div L|$ is a deformation retract of $\bar{N}(K \div L)$.

For later use we define a deformation

$$\phi_\tau : |K| \to |K|, \qquad 1 \leqq \tau \leqq 2,$$

of $|K|$ onto itself, as follows:

$$\phi_\tau(x, t, y) = \begin{cases} (x, (2 - \tau)t, y), & 0 \leqq t \leqq \tfrac{1}{2}, \\ (x, 1 - \tau + t\tau, y), & \tfrac{1}{2} \leqq t \leqq 1. \end{cases} \tag{2.1}$$

Then $\phi_1 =$ identity, $\phi_\tau(z) = z$ if $z \in |L|$ or $z \in |K \div L|$, $\phi_2(\bar{N}(L)) \subset L$, and $\phi_2(N(K \div L)) = |K| - |L|$. We will call ϕ_τ *the deformation of* $|K|$ *on itself toward* $|L|$.

We will make use later of several additional properties of $K \div L$. These are contained in the following lemmas:

LEMMA 1. *Let A be an arbitrary subcomplex of K. Then*

$$A \div (A \cap L) = A \cap (K \div L).$$

PROOF. Let K', A', L' denote the barycentric subdivisions of K, A, L respectively. Then it is readily seen from the definition of supplement, that a simplex of K' belongs to $A \div (A \cap L)$ if and only if all its vertices are in A' but not in L'. But this is precisely the condition that the simplex belong to $A \cap (K \div L)$.

[2] Throughout this paper we use the term "deformation retract" in the strong sense, i.e., points of the retract remain fixed throughout the deformation.

Let M_1 denote $K \div L$, and $M_2 = A \div (A \cap L) = A \cap (K \div L)$. Then $\bar{N}(M_1) \subset |K|$ and $\bar{N}(M_2) \subset |A|$.

LEMMA 2. $\bar{N}(M_2) = |A| \cap \bar{N}(M_1)$.

PROOF. Let $h : |K| \to |L|^*|M_1|$ as before. Then it is readily seen that the map $h \mid |A|$ gives the homeomorphic imbedding of $|A|$ into $|A \cap L|^*|M_2|$. In other words, the coordinate system induced in $|A|$ by that in $|K|$, agrees with that defined in $|A|$ in terms of $A \cap L$ and M_2. The lemma now follows at once from the definitions of $\bar{N}(M_1)$ and $\bar{N}(M_2)$.

LEMMA 3. *The pairs* $(\bar{N}(M_1), \bar{N}(M_2))$ *and* $(|M_1|, |M_2|)$ *have the same homotopy type.*

PROOF. We define a deformation

$$\psi_\tau : (\bar{N}(M_1), \bar{N}(M_2)) \to (\bar{N}(M_1), \bar{N}(M_2),) \qquad 0 \leqq \tau \leqq 1,$$

by

$$\psi_\tau(x, t, y) = (x, \tau + t - t\tau, y)$$

for $(x, t, y) \in \bar{N}(M_1)$. Then $\psi_0 =$ identity, $\psi_\tau(z) = z$ if $z \in |M_1|$, and ψ_1 is a retraction of $(\bar{N}(M_1), \bar{N}(M_2)$ onto $(|M_1|, |M_2|)$. Hence $(|M_1|, |M_2|)$ is a deformation retract of $(\bar{N}(M_1), \bar{N}(M_2))$ and has the same homotopy type.

LEMMA 4. *If L contains the m-dimensional skeleton of K, and* $\dim K = n$, *then* $\dim (K \div L) \leqq n - m - 1$.

This follows easily from the definitions.

3. A normalization process for certain triad maps

Let $(X; A, B)$ be a triad which satisfies the conditions (a) and (b) of Section 1, and assume that $(A, A \cap B)$ is m-connected, $m \geqq 1$. Let $\alpha \in \pi_q(X; A, B)$, $q \geqq 2$, and let

$$f_0 : (E^q; E_+^{q-1}, E_-^{q-1}) \to (X; A, B)$$

represent α. We are going to define a homotopy.

$$f_t : (E^q; E_+^{q-1}, E_-^{q-1}) \to (X; A, B), \qquad 0 \leqq t \leqq 2,$$

of f_0, called the *normalization process*, and f_2 will be called a representative of α in *normal form*. It is not asserted that this normal form is unique.

Let $U = f_0^{-1}(\text{Int } A)$ and $V = f_0^{-1}(\text{Int } B)$. Then $\{U, V\}$ is an open covering of E^q. Let ε be the Lebesgue number of this covering. Choose a *rectilinear* triangulation [1, Section 4.2] of the triad $(E^q; E_+^{q-1}, E_-^{q-1})$ so fine that every simplex has diameter $< \varepsilon$. Let P_0 be the subcomplex of E^q spanned by all the simplexes contained in U, and let Q_0 be the subcomplex spanned by all the simplexes contained in V; then $E^q = P_0 \cup Q_0$. Let

$$P_1 = P_0 \cup E_+^{q-1}$$

$$Q_1 = Q_0 \cup E_-^{q-1}$$

$$R_1 = P_1 \cap Q_1$$

Consider the map $g_0 : (P_1, R_1) \to (A, A \cap B)$ defined by f_0. This map is m-deformable, since $(A, A \cap B)$ is m-connected. Let

$$g_t : (P_1, R_1) \to (A, A \cap B), \qquad 0 \leqq t \leqq 1,$$

be a homotopy of g_0 such that $g_t(x) = x$ if $x \in R_1$, and $g_1(P_1^m) \subset A \cap B$. Let L denote the closed subcomplex of P_1 consisting of all simplexes σ such that $g_1(\sigma) \subset A \cap B$. Then $L \supset (R_1 \cup P_1^m)$. Let $M = P_1 \div L$, the supplementary subcomplex of L in P_1. Let

$$\phi_t : P_1 \to P_1, \qquad 1 \leqq t \leqq 2$$

be the deformation of P_1 onto itself toward L, as described in Section 2. Define

$$g_t : (P_1, R_1) \to (A, A \cap B), \qquad 1 \leqq t \leqq 2,$$

by

$$g_t = g_1\phi_t, \qquad 1 \leqq t \leqq 2,$$

and define

$$f_t : (E^q; E_+^{q-1}, E_-^{q-1}) \to (X; A, B), \qquad 0 \leqq t \leqq 2,$$

by

$$f_t \mid P_1 = g_t, \qquad 0 \leqq t \leqq 2,$$

$$f_t \mid Q_1 = f_0 \mid Q_1, \qquad 0 \leqq t \leqq 2.$$

From Lemma 4 of Section 2 it follows, since $L \supset P_1^m$, that

$$\dim M \leqq q - m - 1. \tag{3.1}$$

We now define

$$P = \bar{N}(M) \cup E_+^{q-1},$$

$$Q = \mathrm{Cl}\,[E^q - N(M)],$$

$$R = P \cap Q.$$

Then $P \cup Q = E^q$, and

$$E_+^{q-1} \subset P \subset f_2^{-1}(A),$$

$$E_-^{q-1} \subset Q \subset f_2^{-1}(B).$$

The map

$$f = f_2 : (E^q; E_+^{q-1}, E_-^{q-1}) \to (X; A, B)$$

is the desired *normal form* for a representative of α.

4. Proof of Theorem I when $m \geqq n \geqq 2$

In this section we give the proof of the main theorem when $m \geqq n \geqq 2$. The modifications in the proof which are necessary when $n = 1$ will be described in the next section.

We assume, then, that the triad $(X; A, B)$ satisfies the hypotheses (a), (b), and (c) of Section 1, and that $m \geqq n \geqq 2$. Let $\alpha \in \pi_q(X; A, B)$, $q \leqq m + n$, and let

$$f : (E^q; E_+^{q-1}, E_-^{q-1}) \to (X; A, B)$$

be a representative of α in normal form. Let P, Q, R and M denote the subsets of E^q described at the end of Section 3. Clearly, the spaces Q and $E^q - M$ have the same homotopy type. Since $\dim M \leqq q - m - 1 \leqq q - 3$, it follows from Lemma 4.2.1 of [1] that Q is simply connected.

To prove that $\alpha = 0$, it suffices, by Lemma 4.7.1 of [1], to show that the map

$$h : (Q, R) \to (B, A \cap B)$$

defined by f, is deformable. We will do this by using Theorem 4.4.3 of [1]. We need to consider the cohomology groups $H^j(Q, R, \pi_j)$, $2 \leqq j \leqq q$, where $\pi_j = \pi_j(B, A \cap B)$. By the excision axiom,

$$H^j(Q, R) \approx H^j(E^q, P).$$

It follows from the exactness of the cohomology sequence of the triple (E^q, P, E_+^{q-1}) that

$$H^j(E^q, P) \approx H^{j-1}(P, E_+^{q-1}),$$

and by the excision axiom again,

$$H^{j-1}(P, E_+^{q-1}) \approx H^{j-1}(\bar{N}(M), \bar{N}(M) \cap E_+^{q-1}).$$

It follows from Section 2 that the pairs $(\bar{N}(M), \bar{N}(M) \cap E_+^{q-1})$ and $(M, M \cap E_+^{q-1})$ are of the same homotopy type. Therefore

$$H^{j-1}(\bar{N}(M), \bar{N}(M) \cap E_+^{q-1}) \approx H^{j-1}(M, M \cap E_+^{q-1}).$$

Combining these isomorphisms, we have,

$$H^j(Q, R, \pi_j) \approx H^{j-1}(M, M \cap E_+^{q-1}, \pi_j).$$

Now $H^j(Q, R, \pi_j) = 0$ for $j \leqq n$, because $\pi_j = 0$ for $j \leqq n$. If $j > q - m$, then $H^{j-1}(M, M \cap E_+^{q-1}, \pi_j) = 0$ because $\dim M \leqq q - m - 1$, by (3.1). Hence if $q \leqq m + n$, $H^j(Q, R, \pi_j) = 0$ for all values of j, and since Q is simply connected it follows from Theorem 4.4.3 of [1] that h is deformable and therefore that $\alpha = 0$.

5. Modifications necessary when $n = 1$

If $m > n = 1$, and $\pi_2(B, A \cap B)$ is abelian, the proof goes through exactly as before, since all hypotheses of Theorem 4.4.3 of [1] remain satisfied.

We examine next the case where $m = n = 1$ and $(B, A \cap B)$ is simple in dimension 2. We can no longer apply Lemma 4.2.1 of [1] to conclude that Q is simply connected. However it follows that $\pi_2(B, A \cap B)$ is abelian. Hence we can use Theorem 4.4.2 of [1] instead of Theorem 4.4.3, to show that the map $h : (Q, R) \to (B, A \cap B)$ is 2-deformable, and therefore $(X; A, B)$ is 2-connected.

Consider, finally, the case where $m > n = 1$, and it is assumed that $(A, A \cap B)$ is simple in dimension $(m + 1)$. By a simple change of notation we may assume the equivalent hypotheses: $n > m = 1$ and $(B, A \cap B)$ is simple in dimension $n + 1$. The proof now follows that of the case considered immediately above, using theorem (4.4.2) of [1].

6. On condition (b) of Theorem I

In many cases in which one actually wishes to apply Theorem I, there is given a triad $(X; A, B)$ such that $X = A \cup B$ and the conditions (a), (c), and (d) of Theorem I are satisfied, but condition (b) is not satisfied. Then it is not possible to apply Theorem I directly. However, it sometimes happens that there exists a subset $A' \subset X$ such that $A' \supset A$, the triads $(X; A, B)$ and $(X; A', B)$ have isomorphic homotopy sequences, and condition (b) does apply to the triad $(X; A', B)$. Then we may apply Theorem I to the triad $(X; A', B)$ to conclude that certain of its homotopy groups vanish, and hence that the homotopy groups of the triad $(X; A, B)$ vanish in the corresponding dimensions. One such case is described in the following lemma:

LEMMA 5. *Let $(X; A, B)$ be a triad with $X = A \cup B$, with A and B closed subsets of X, and such that there exists an open neighborhood N of $A \cap B$ in B with $A \cap B$ a deformation retract of N in the strong sense. Let $A' = A \cup N$. Then the triad $(X; A', B)$ satisfies condition* (b) *of Theorem I, and the triads $(X; A, B)$ and $(X; A', B)$ have isomorphic homotopy sequences.*

PROOF. It readily follows from the hypotheses that A' and $X - A$ are open subsets of X, that $(X - A) \subset \operatorname{Int} B$, and that $A' \cup (X - A) = X$. Hence the triad $(X; A', B)$ satisfies condition (b) of Theorem I.

Next, we observe that our hypotheses imply that A is a deformation retract of A', since the deformation retraction which is defined over N can be extended to all of A' in the obvious way. The continuity of this extension follows from the fact that both A and N are closed in A'. This deformation retraction is a homotopy equivalence between the pairs $(A, A \cap B)$ and $(A', A' \cap B)$, and therefore the inclusion map $(A, A \cap B) \to (A', A' \cap B)$ induces isomorphisms of the homotopy sequences of these two pairs. Next we look at the homomorphism induced by the inclusion map $(B, A \cap B) \to (B, A' \cap B)$ on the homotopy sequences of these pairs. Since the homomorphisms $\pi_p(B) \to \pi_p(B)$ and $\pi_p(A \cap B) \to \pi_p(A' \cap B)$ thus induced are isomorphisms onto in all dimensions, it follows from the purely algebraic "five lemma" [2, Lemma 3, p. 435] that the injection $\pi_p(B, A \cap B) \to \pi_p(B, A' \cap B)$ is also an isomorphism onto. This proof even goes through with minor modifications to prove that the injection $\pi_1(B, A \cap B) \to \pi_1(B, A' \cap B)$ is $1 - 1$ and onto. Another application of the "five lemma" to the homotopy sequences of the triads $(X; A, B)$ and $(X; A', B)$ enables us to prove that these triads have isomorphic homotopy sequences, as was to be proved. Here again minor modifications are necessary in the lowest dimension.

COROLLARY. *Theorem I remains true when condition* (b) *in its hypothesis is replaced by the hypothesis of Lemma* 5.

Obviously, the symmetric statement obtained by interchanging the roles of A and B in Lemma 5 is also true.

A particular case where the hypothesis of Lemma 5 is satisfied occurs when we have a triad $(X; A, B)$ such that $X = A \cup B$, A and B are closed in X, and at least one of the pairs $(A, A \cap B)$ and $(B, A \cap B)$ is triangulable. For example, if $(B, A \cap B)$ is triangulable, we can choose $N = N(A \cap B)$ as defined in Section 2.

7. Shrinking a subcomplex to a point

Let (X, A) be a pair consisting of a CW-complex,[3] X, and a closed subcomplex, A, and let $(\bar{X}, x_0)$ be the pair obtained by identifying all of A to a single point x_0; then $\bar{X}$ is a CW-complex and x_0 is a vertex of A. Assume that (X, A) is m-connected, $m \geqq 1$, and that A is n-connected, $n \geqq 1$. Let $\phi : (X, A) \to (\bar{X}, x_0)$ denote the identification map and $\phi_p : \pi_p(X, A) \to \pi_p(\bar{X}, x_0)$ the homomorphism induced by ϕ.

THEOREM II. *ϕ_p is an isomorphism onto for $p \leqq m + n$, and is a homomorphism onto for $p = m + n + 1$.*

Before proceeding with the proof, we prove a lemma. Let (X, A) be a pair consisting of a space X and closed subspace A. Let A^* be the join of A with a single point a_0, (i.e., the identification space of $A \times I$ when all of $A \times 1$ is identified to the single point a_0) and let X^* be the identification space resulting from $X \cup A^*$ by identifying each point $a \times 0 \in A^*$ with the corresponding point $a \in A$. Assume that (X, A) is m-connected, $m \geqq 1$, and that A is n-connected, $n \geqq 1$.

LEMMA 6. *With the above hypotheses, the triad $(X^*; X, A^*)$ is $(m + n + 1)$-connected.*

PROOF. Since A is n-connected and A^* is contractible, (A^*, A) is $(n + 1)$-connected. Let $N \subset A^*$ be the set $\eta(A \times [0 \leqq t < \frac{1}{2}])$ where η denotes the identification map $\eta : A \times I \to A^*$. Then N is open in A^* and A is a deformation retract of N. Moreover $X \cap A^* = A$ is simply connected and X, A^* are closed in X^*. The result now follows from the corollary to Lemma 5.

PROOF OF THEOREM II. Let us extend the map ϕ to a map $\psi : (X^*, A^*) \to (\bar{X}, x_0)$, by defining $\psi(x) = x_0$ for all points $x \in A^*$. It is readily verified that ψ is continuous. Let $\psi' : X^* \to \bar{X}$ be the map defined by ψ. Since A^* is contractible, it follows that ψ' is a homotopy equivalence [6, Theorem 12], so that $\psi'_* : \pi_p(X^*) \to \pi_p(\bar{X})$ is an isomorphism onto for all p. Also, the injection $j : \pi_p(X^*) \to \pi_p(X^*, A^*)$ is an isomorphism onto for all p. Consider the following diagram:

$$\begin{array}{ccc} \pi_p(X^*) & \xrightarrow{\ j\ } & \pi_p(X^*, A^*) \\ & \searrow^{\psi'_*} & \downarrow \psi_* \\ & & \pi_p(\bar{X}, x_0) \end{array}$$

[3] For the definition and properties of a CW-complex, see [6].

Since commutativity holds, it follows that ψ_* is an isomorphism onto in all dimensions. Consider next the following diagram:

$$\begin{array}{ccc} \pi_p(X, A) & \xrightarrow{\ i\ } & \pi_p(X^*, A^*) \\ & \searrow^{\phi_p} & \downarrow \psi_* \\ & & \pi_p(\tilde{X}, x_0) \end{array}$$

Commutativity again holds. Since ψ_* is an isomorphism onto in all dimensions, it follows that the injection i, and ϕ_p, are equivalent homomorphisms. The theorem now follows from Lemma 6 and consideration of the homotopy sequence of the triad $(X^*; X, A^*)$.

8. An application of Theorem II

Let $(X; A, B)$ be a triad such that $X = A \cup B$, A and B are *closed* subsets of X, and $(A, A \cap B)$ is a pair consisting of a CW-complex and closed subcomplex. Let

$$i_p : \pi_p(A, A \cap B) \to \pi_p(X, B), \qquad p = 1, 2, \cdots,$$

denote the injection.

THEOREM III. *If $(A, A \cap B)$ is m-connected, $m \geqq 1$, and $A \cap B$ is r-connected, $r \geqq 1$, then i_p is an isomorphism into for $p \leqq m + r$, and the image subgroup is a direct summand of $\pi_p(X, B)$.*

PROOF. Let $(\tilde{A}, x_0)$ denote the pair obtained from $(A, A \cap B)$ by identifying all of $A \cap B$ to a single point, x_0, and let $\phi : (A, A \cap B) \to (\tilde{A}, x_0)$ be the identification map. Define a map $\psi : (X, B) \to (\tilde{A}, x_0)$ by

$$\psi(x) = \begin{cases} x_0, & x \in B, \\ \phi(x), & x \in A. \end{cases}$$

The map ψ thus defined is continuous, since A and B are closed subsets of X.

Consider now the following diagram:

$$\begin{array}{ccc} \pi_p(A, A \cap B) & \xrightarrow{\ i_p\ } & \pi_p(X, B) \\ \phi_p \searrow & & \swarrow \psi_p \\ & \pi_p(\tilde{A}, x_0) & \end{array}$$

Commutativity clearly holds. It follows from Theorem II that ϕ_p is an isomorphism onto for $p \leqq m + r$, and from this the theorem follows at once.

The authors wish to acknowledge that the main idea for this proof is due to P. Hilton [5].

A closely related result is the following:

THEOREM III'. *Let $(X; A, B)$ be a triad such that $X = (\text{Int } A) \cup (\text{Int } B)$,*

$(A, A \cap B)$ is m-connected, $m \geqq 1$, and $A \cap B$ is r-connected, $r \geqq 1$. Then the injection

$$i_p : \pi_p(A, A \cap B) \to \pi_p(X, B)$$

is an isomorphism into, for $p \leqq m + r$.

The proof of this result is similar to the proof of the main theorem, except that Lemma 4.7.2 of [1] is used in place of Lemma 4.7.1; cf. the proof given in Section 4.10 of [1].

It seems probable that under the hypotheses of Theorem III′ the image subgroup is a direct summand of $\pi_p(X, B)$, but the authors are unable to prove this fact.

9. Geometric proof of an algebraic theorem of Eilenberg and MacLane

For each abelian group Π and integer $n \geqq 2$, Eilenberg and MacLane [3, p. 507] have defined an abstract complex $K(\Pi, n)$. In a recent paper [4] they have defined a "suspension" operation, which is a chain transformation (raising dimensions by 1) of $K(\Pi, n)$ into $K(\Pi, n + 1)$, and hence induces homomorphisms,

$$S_q : H_q[K(\Pi, n), G] \to H_{q+1}[K(\Pi, n + 1), G]$$

of the corresponding homology groups with G as coefficient group. Concerning these homomorphisms, they have stated a theorem which is equivalent to the following:

THEOREM IV. *The suspension homomorphism, S_q, is an isomorphism onto for $q < 2n$, and is a homomorphism onto for $q = 2n$.*

We shall give a proof this result, based on Theorem I. Let L be a CW-complex with $\pi_i(L) = 0$ for $i < n$ and $n < i < m$, $m > 2n$, and with $\pi_n(L) \approx \Pi$. That this realization is possible follows from a theorem of J. H. C. Whitehead, [7]. Let $K_n = K(\Pi, n)$, $K_{n+1} = K(\Pi, n + 1)$, $K_0 = K(0, n)$. K_0 is the complex constructed on the group consisting of the identity element only, and is homologically trivial. It may be considered to be a subcomplex of both K_n and K_{n+1}, and it is easily seen from the definition of S_q that we might equally well consider the equivalent homomorphisms induced by the suspension,

$$S : H_q(K_n, K_0) \to H_{q+1}(K_{n+1}, K_0).$$

Let a vertex $p \in L$ be chosen, and let a_1, a_2 be a pair of distinct points. Let $L_1 = L * \{a_1\}$ (the join of L with a_1), $L_2 = L * \{a_2\}$, $\bar{L} = L_1 \cup L_2$, $p_1 = \{p\} * \{a_1\}$, $p_2 = \{p\} * \{a_2\}$, $\bar{p} = p_1 \cup p_2$. Let $\tilde{L}$ be the space obtained from $\bar{L}$ by identifying the segment $\bar{p}$ to a single point, which we will also denote by p. Since $\bar{p}$ is contractible, the identification map is a homotopy equivalence [6, theorem 12], and induces isomorphisms

$$\psi : H_i(\bar{L}, \bar{p}) \approx H_i(\tilde{L}, p)$$

of the singular homology groups in all dimensions.

Consider now the homomorphisms indicated in the following diagram (singular homology groups):

$$
\begin{array}{ccccccc}
H_q(S_n(L),\, S(p)) & \xrightarrow{\kappa} & H_q(K_n\,,\, K_0) & \xrightarrow{S} & H_{q+1}(K_{n+1},\, K_0) & \xleftarrow{\kappa'} & H_{q+1}(S_{n+1}(\tilde{L}),\, S(p)) \\
\downarrow{\scriptstyle \eta} & & & & & & \downarrow{\scriptstyle \eta'} \\
H_q(L,\, p) & & & & & & H_{q+1}(L,\, p) \\
\uparrow{\scriptstyle \partial} & & & & & & \uparrow{\scriptstyle \psi}
\end{array}
$$

$$
H_{q+1}(L_1\,,\, L) \xrightarrow{i_1} H_{q+1}(L_1\,,\, L \cup p_1) \xrightarrow{j} H_{q+1}(\bar{L},\, L_2 \cup \bar{p}) \xleftarrow{i_2} H_{q+1}(\bar{L},\, \bar{p})
$$

The homomorphisms κ, κ', η, η', are defined in [3]. The homomorphisms i_1, i_2, and j are induced by inclusion maps, and ∂ is the boundary operator. By applying Theorem I to the triad $(\bar{L};\, L_1\,,\, L_2)$, we find that this triad is $2n$-connected. Hence $\pi_i(L) \approx \pi_{i+1}(\bar{L})$ for $i \leqq 2n - 1$, and $\pi_{2n}(\bar{L})$ is a homomorphic image of $\pi_{2n-1}(L)$. Therefore $\pi_i(\bar{L}) = 0$ for $i < n + 1$ and $n + 1 < i \leqq 2n$, while $\pi_{n+1}(\bar{L}) \approx \Pi$. Furthermore, $\pi_i(\bar{L}) \approx \pi_i(\tilde{L})$ for all i. Hence η and η' are isomorphisms onto for all dimensions, κ is an isomorphism onto for $q \leqq m$, and κ' is an isomorphism onto for $q + 1 \leqq 2n$, and is onto for $q + 1 = 2n + 1$. The homomorphisms ∂, i_1, i_2, and j are isomorphisms onto in all dimensions. It can be verified that the commutativity relation

$$\kappa'\eta'^{-1}\psi i_2^{-1} j i_1 \partial^{-1}\eta = S\kappa$$

holds. Therefore S is an isomorphism onto for $q < 2n$, and is onto for $q = 2n$.

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