## THE HOMOTOPY GROUPS OF A TRIAD, III

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(Received February 5, 1953)

#### Introduction

The principal purpose of this paper is to prove a rather general theorem about the homotopy groups of a triad in what may be called the "critical dimension," i.e., the lowest dimension for which the homotopy groups of a triad are non-zero. This theorem may be stated roughly as follows. Let (X; A, B) be a triad such that  $X = A \cup B$ . If  $\pi_p(A, A \cap B) = 0$  for  $p \leq m$  and  $\pi_q(B, A \cap B) = 0$  for  $q \leq n$ , then the authors have shown previously [2] that  $\pi_r(X; A, B) = 0$  for  $r \leq m + n$  under very general conditions. We now show that  $\pi_{m+n+1}(X; A, B)$ is isomorphic to the tensor product,  $\pi_{m+1}(A, A \cap B) \otimes \pi_{n+1}(B, A \cap B)$ , under rather general conditions. Moreover, this isomorphism is defined in a very natural manner by means of a generalized Whitehead product. This theorem includes as special cases some results we have announced previously without proof.<sup>1</sup> The proof which we give below depends heavily on a recent paper<sup>2</sup> of J. C. Moore, [8]. This proof is much simpler than the authors' original, unpublished proofs for the previously announced results.

In sections 2 and 3 we give some applications of our main theorem to some problems of current interest in algebraic topology.

This paper is essentially a continuation of our earlier papers, [1], [2], and [3]. For the explanation of any terminology or notation that is not contained in the present paper, the reader is referred to these previous papers. In general, it is assumed that the reader is familiar with the basic properties of triad homotopy groups and generalized Whitehead products.

## 1. Statement and Proof of the Main Theorem

Let (X; A, B) be a triad such that X is a CW-complex,<sup>3</sup> A and B are subcomplexes, and  $X = A \cup B$ . Let C denote the intersection,  $A \cap B$ , which we will always assume to be connected and simply connected. Let

$$W: \pi_p(A, C) \otimes \pi_q(B, C) \to \pi_{p+q-1}(X; A, B)$$

be the homomorphism which is defined by the generalized Whitehead product,<sup>4</sup> i.e.,  $W(\alpha \otimes \beta) = [\alpha, \beta]$  for any elements  $\alpha \in \pi_{\alpha}(A, C)$  and  $\beta \in \pi_{\alpha}(B, C)$ .

THEOREM I. If (A, C) is (m - 1)-connected, m > 2, and (B, C) is (n - 1)connected, n > 2, then the homomorphism

<sup>&</sup>lt;sup>1</sup> Cf. [6] and various abstracts in the Bulletin of the American Mathematical Society for the years 1949-1951.

<sup>&</sup>lt;sup>2</sup> The authors are indebted to J. C. Moore for allowing them to see the manuscript of his paper before publication, and for many helpful suggestions.

<sup>&</sup>lt;sup>a</sup> For the definition and properties of a CW-complex, see [13].

<sup>\*</sup> For the definition and properties of the generalized Whitehead products, see [3].

 $W: \pi_m(A, C) \otimes \pi_n(B, C) \to \pi_{m+n-1}(X; A, B)$ 

is an isomorphism onto.5

**PROOF.** Let  $(X_0; A_0, B_0)$  be the triad obtained by identifying all of C to a single point, and let

$$f: (X; A, B) \rightarrow (X_0; A_0, B_0)$$

denote the identification map.

It follows from Theorem I of [2] that the triad (X; A, B) is m + n - 2 connected. Note that m + n - 2 > 3. Hence by considering the homotopy sequences of the triad (X; A, B), we conclude that (X, A) and (X, B) are 2-connected, at least. A similar argument shows that  $(X_0, A_0)$  and  $(X_0, B_0)$  are at least 2-connected.

Next, note that our hypotheses imply that both A and B are simply connected. Hence it follows by a well known theorem [9, §52] that X is simply connected. By using Theorem II of [2], one can prove in a similar manner that  $X_0$  is simply connected.

By the Hurewicz equivalence theorem,  $H_q(A, C) = 0$  for  $q \leq m - 1$ . By the excision property for homology,  $H_q(X, B) \approx H_q(A, C)$ . Hence  $H_q(X, B) = 0$  for  $q \leq m - 1$ . Similarly,  $H_q(X, A) = 0$  for  $q \leq n - 1$ .

Finally, note that the homomorphisms  $H_q(X, A) \to H_q(X_0, A_0)$  and

$$H_q(X, B) \to H_q(X_0, B_0)$$

induced by the identification map are isomorphisms onto in all dimensions.

We have now verified that all the hypotheses of Theorem 3.5 of [8] hold for the case R = ring of integers, r = 2. Hence we can conclude that the homomorphism

$$f'': \pi_{m+n-1}(X; A, B) \to \pi_{m+n-1}(X_0; A_0, B_0)$$

induced by f is an isomorphism onto.

The proof may now be completed by the same method used to prove Theorem I of [3].

# 2. The Relative Homotopy Groups of the Pair $(X^*, X)$

It is the purpose of this section to prove some theorems about the relative homotopy groups of a pair  $(X^*, X)$  where  $X^*$  is obtained from X by the adjunction of cells. The results obtained will be direct applications of Theorem I.

Throughout this section, X will denote a topological space which is a connected CW-complex, and  $X^*$  is obtained from X by the adjunction of *n*-cells, as explained in [1, §4.1]. As a preliminary step, we first state two lemmas.

For the statement of the first lemma, let  $X^*$  be obtained from X by the adjunction of an indexed family of open n-cells  $e_{\lambda}^n$ ,  $\lambda \in \Lambda$ , and let  $f_{\lambda} : E^n \to \tilde{e}_{\lambda}^n$  be the "characteristic map" by means of which the cell  $e_{\lambda}^n$  is attached to X. Simi-

<sup>&</sup>lt;sup>5</sup> It would be interesting to extend this theorem to the case m = 2 or n = 2. Note that since C is simply connected,  $\pi_2(A, C)$  and  $\pi_2(B, C)$  must be abelian in this case.

larly, let  $Y^*$  be obtained from X by the adjunction of *n*-cells  $\sigma_{\lambda}^n$ ,  $\lambda \in \Lambda$ , and let  $g_{\lambda} : E^n \to \tilde{\sigma}_{\lambda}^n$  be the characteristic map for the cell  $\sigma_{\lambda}^n$ .

LEMMA 1. If  $f_{\lambda} \mid \dot{E}^n$  is homotopic in X to  $g_{\lambda} \mid \dot{E}^n$  for each  $\lambda \in \Lambda$ , then there exists a homotopy equivalence  $h: (X^*, X) \to (Y^*, X)$  such that  $h \mid X$  is the identity map. This lemma is a trivial generalization of Lemma 5 of [12].

For the statement of the second lemma, we assume that  $X^*$  is obtained from X by the adjunction of the family of open *n*-cells,  $e_{\lambda}^n$ ,  $\lambda \in \Lambda$ , by means of characteristic maps,  $f_{\lambda} : E^n \to \bar{e}_{\lambda}^n$ . Let  $\dot{e}_{\lambda}^n$  denote the boundary of the cell  $e_{\lambda}^n$ .

LEMMA 2. Assume that there exists a point  $x_0$  such that  $x_0 \in \hat{e}_{\lambda}^n$  for all  $\lambda \in \Lambda$ . Then it is possible to choose subsets  $\tilde{X} \subset X^*$  and  $\tilde{e}_{\lambda}^n \subset e_{\lambda}^n$ ,  $\lambda \in \Lambda$ , such that:

(1)  $X \subset \tilde{X}$ , and X is a deformation retract of  $\tilde{X}$ .

(2)  $\tilde{X}$  and  $\tilde{e}_{\lambda}^{n}$  are disjoint.

(3)  $X^*$  is obtained from  $\hat{X}$  by the adjunction of the cells  $\tilde{e}_{\lambda}^n$ ,  $\lambda \in \Lambda$ , and the characteristic map for the adjunction of each cell is a homeomorphism. Hence the closure of each cell  $\tilde{e}_{\lambda}^n$  is homeomorphic to  $E^n$ .

(4) The inclusion map  $(X^*, X) \to (X^*, \tilde{X})$  is a homotopy equivalence of pairs.

(5) The point  $x_0$  belongs to the boundary of all the cells  $\tilde{e}_{\lambda}^n$ ; furthermore the intersection of the boundaries of any two cells consists only of the point  $x_0$ .

We will indicate the main idea of the proof, which is really quite simple. Recall that  $E^n$  is the set of all points in  $\mathbb{R}^n$ , Cartesian *n*-space, whose distance from the origin is  $\leq 1$ . Let  $a = (1, 0, \dots, 0) \epsilon E^n$ . We may assume without loss of generality that the characteristic maps,  $f_{\lambda}$ , are chosen so that  $f_{\lambda}(a) = x_0$ . Let  $D^n$  denote the subset of  $E^n$  consisting of all points whose distance from

$$(\frac{1}{2}, 0, \cdots, 0)$$

is  $<\frac{1}{2}$ . Then  $D^n$  is an open *n*-cell, and  $a \in \overline{D}^n$ Define  $\tilde{e}_{\lambda}^n = f_{\lambda}(D^n)$ , and

$$\widehat{X} = X^* - \mathsf{U}_{\lambda\epsilon\lambda} \ \widetilde{e}^n_\lambda$$
 ,

Then one readily verifies that the subsets  $\tilde{X}$  and  $\tilde{e}_{\lambda}^{n}$  have all the required properties.

For the statement of the next two theorems, we will assume that  $X^*$  is obtained from X by the adjunction of an indexed family of *n*-cells,  $e_{\lambda}^n$ ,  $\lambda \in \Lambda$ . Define homomorphisms

$$\begin{split} &\zeta\colon \pi_p(X)\,\,\otimes\,\,\pi_q(X^*,\,X)\to \pi_{p+q-1}(X^*,\,X),\\ &\psi\colon \pi_n(X^*,\,X)\,\,\otimes\,\,\pi_r(E^n,\,S^{n-1})\to \pi_r(X^*,\,X),\qquad (r\,<\,2n\,-\,2), \end{split}$$

by

$$\begin{split} \zeta(\alpha \otimes \beta) &= [\alpha, \beta], \ \alpha \in \pi_p(X), \ \beta \in \pi_q(X^*, X), \\ \psi(\alpha \otimes \beta) &= \alpha \circ \beta, \ \alpha \in \pi_n(X^*, X), \ \beta \in \pi_r(E^n, S^{n-1}), \end{split}$$

where  $\alpha \circ \beta$  denotes the composition of  $\alpha$  and  $\beta$  (cf. [7, §23]).

THEOREM II. If X is m-connected, 1 < m < n - 2, then the homomorphisms

$$\zeta: \pi_{m+1}(X) \otimes \pi_n(X^*, X) \to \pi_{m+n}(X^*, X),$$
  
$$\psi: \pi_n(X^*, X) \otimes \pi_{m+n}(E^n, \dot{E}^n) \to \pi_{m+n}(X^*, X)$$

are isomorphisms into, and  $\pi_{m+n}(X^*, X)$  is the direct sum of the two image subgroups.<sup>6</sup>

**PROOF.** First of all, observe that if  $(Y^*, Y)$  and  $(X^*, X)$  are two pairs which are of the same homotopy type and this theorem is true for the pair  $(Y^*, Y)$ , then it is also true for  $(X^*, X)$ . Therefore in proving this theorem, we may replace the pair  $(X^*, X)$  by another pair of the same homotopy type, if that is convenient. We see by Lemma 1 that we may as well assume that the boundaries of all the cells,  $e_{\lambda}^{n}$ , have a point,  $x_{0}$ , in common. Then by Lemma 2 we see that we may as well assume that all the cells are adjoined by characteristic maps which are homeomorphisms, and that their boundaries have only the point  $x_{0}$ in common.

With these assumptions, let

$$\begin{aligned} \varepsilon^{n} &= \bigcup_{\lambda \in \Lambda} \, \check{e}_{\lambda}^{n} \,, \\ \dot{\varepsilon}^{n} &= \, \varepsilon^{n} \, \cap \, X \, = \, \bigcup_{\lambda \in \Lambda} \, \check{e}_{\lambda}^{n} \,. \end{aligned}$$

Then  $\hat{\varepsilon}^n$  is a union of (n-1)-spheres having the single point  $x_0$  in common. Consider the triad  $(X^*; \varepsilon^n, X)$ . Obviously,  $(\varepsilon^n, \dot{\varepsilon}^n)$  is (n-1)-connected; since X is m-connected and  $\hat{\varepsilon}^n$  is (n-2)-connected, it follows that  $(X, \dot{\varepsilon}^n)$  is m-connected. Therefore by Theorem I, the homomorphism

$$W: \pi_{m+1}(X, \mathcal{E}^n) \otimes \pi_n(\mathcal{E}^n, \mathcal{E}^n) \to \pi_{m+n}(X^*; \mathcal{E}^n, X)$$

is an isomorphism onto.

Next, it follows from Theorem III of [2] that the injection,  $i_p: \pi_p(\mathcal{E}^n, \mathcal{E}^n) \to \pi_p(X^*, X)$  is an isomorphism into for  $p \leq 2n - 3$ ; in particular,  $i_{m+n}$  is an isomorphism into.

Let 
$$j: \pi_{m+1}(X) \to \pi_{m+1}(X, \hat{\varepsilon}^n)$$
,  $i_n: \pi_n(\hat{\varepsilon}^n, \hat{\varepsilon}^n) \to \pi_n(X^*, X)$ , and  
 $j': \pi_{m+n}(X^*, X) \to \pi_{m+n}(X^*; \hat{\varepsilon}^n, X)$ 

be injections. It is clear that j and  $i_n$  are isomorphisms onto. Furthermore, if  $\alpha \in \pi_{m+1}(X)$  and  $\beta \in \pi_n(\mathbb{S}^n, \mathbb{S}^n)$ , then

$$j'[\alpha, i_n\beta] = [j\alpha, \beta]$$

by (3.10) of [3]. Therefore if we denote by

$$\eta: \pi_{m+1}(X) \otimes \pi_n(X^*, X) \to \pi_{m+1}(X, \hat{\varepsilon}^n) \otimes \pi_n(\hat{\varepsilon}^n, \hat{\varepsilon}^n)$$

<sup>&</sup>lt;sup>6</sup> It would be interesting to extend this theorem to the case m = 1.

the isomorphism onto induced by the isomorphisms j and  $i_n^{-1}$ , it follows that commutativity holds in the following diagram:

Since  $\eta$  and W are isomorphisms, it follows that  $\zeta$  is also an isomorphism into, and that  $\pi_{m+n}(X^*, X)$  decomposes into the direct sum of the image of  $\zeta$  and the kernel of j'. But by exactness of the homotopy sequence of the triad  $(X^*; \varepsilon^n, X)$ , the kernel of j' is equal to  $i_{m+n}[\pi_{m+n}(\varepsilon^n, \varepsilon^n)]$ . We will now complete the proof by showing that  $\psi: \pi_n(X^*, X) \otimes \pi_{m+n}(\varepsilon^n, \varepsilon^n) \to \pi_{m+n}(X^*, X)$  is an isomorphism into, and that image  $\psi = \text{image } i_{m+n}$ .

To this end, consider the following diagram:

Here  $\psi'$  is defined by means of the composition operation, in a manner similar to  $\psi$ ; and  $i'_n$  is induced by  $i_n : \pi_n(\mathfrak{S}^n, \mathfrak{S}^n) \to \pi_n(X^*, X)$ . Commutativity holds around this diagram; this follows from (23.12) of [7]. Now  $\psi'$  is an isomorphism onto, because  $\mathfrak{S}^n$  is a union of *n*-cells with a single point in common,  $\mathfrak{S}^n$  is a cluster of (n-1)-spheres with a single point in common, and m + n < 2n - 2(cf. corollary 5.3.4. of [1]). Also,  $i'_n$  is obviously an isomorphism onto. These facts, together with the fact that  $i_{m+n}$  is an isomorphism into, suffice to prove the assertion made at the end of the last paragraph.

This completes the proof of Theorem II.

The next theorem takes care of the case m = n - 2.

THEOREM III. If X is (n-2)-connected, 7 n > 3, then  $\pi_{2n-2}(X^*, X)$  is generated by the image of the homomorphism

$$\zeta \colon \pi_{n-1}(X) \otimes \pi_n(X^*, X) \to \pi_{2n-2}(X^*, X)$$

and the elements  $\beta \circ \alpha$  for all  $\alpha \in \pi_{2n-2}(E^n, \dot{E}^n)$  and  $\beta \in \pi_n(X^*, X)$ .

**PROOF.** By repeating the first part of the proof of the preceding theorem, with slight modifications, one can show that  $\pi_{2n-2}(X^*, X)$  is generated by the subgroups (image  $\zeta$ ) and (image  $i_{2n-2}$ ). Of course one can not show that  $\zeta$  or  $i_{2n-2}$  is an isomorphism into, or that we have a direct sum decomposition.

To complete the proof, one needs to make use of the following facts. Let

<sup>&#</sup>x27; It would be interesting to extend this theorem to the case n = 3.

 $\zeta': \pi_{n-1}(\hat{\varepsilon}^n) \otimes \pi_n(\hat{\varepsilon}^n, \hat{\varepsilon}^n) \to \pi_{2n-2}(\hat{\varepsilon}^n, \hat{\varepsilon}^n)$  denote the homomorphism defined by the generalized Whitehead product:  $\zeta'(\alpha \otimes \beta) = [\alpha, \beta]$ . Then from the known theorems about the structure of the groups  $\pi_{2n-3}(\hat{\varepsilon}^n)$  (cf. Theorem 5.3.3. of [1]), and the fact that  $\pi_{2n-2}(\hat{\varepsilon}^n, \hat{\varepsilon}^n) \approx \pi_{2n-3}(\hat{\varepsilon}^n)$ , it follows that  $\pi_{2n-2}(\hat{\varepsilon}^n, \hat{\varepsilon}^n)$  is generated by image  $\zeta'$  and the elements  $\gamma \circ \alpha$  for all  $\alpha \in \pi_{2n-2}(E^n, \dot{E}^n)$  and

$$\gamma \ \epsilon \ \pi_n(\epsilon^n, \dot{\epsilon}^n).$$

Now consider the following diagram:

Here the vertical arrow on the left denotes the homomorphisms induced by the injections  $\pi_{n-i}(\hat{\varepsilon}^n) \to \pi_{n-1}(X)$  and  $\pi_n(\hat{\varepsilon}^n, \hat{\varepsilon}^n) \to \pi_n(X^*, X)$ . It is clear that this diagram is commutative.

The rest of the proof makes use of the same ideas as occur in the last part of the proof of Theorem II. The details are left to the reader.

ADDENDUM TO THEOREM III. According to §23 of [7], if  $\beta_1$  and  $\beta_2 \in \pi_n(X^*, X)$ and  $\alpha \in \pi_{2n-2}(E^n, \dot{E}^n)$ , then

$$(\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha + [\beta_1, \beta_2] \circ H(\alpha).$$

It is readily seen that  $[\beta_1, \beta_2] \circ H(\alpha)$  is an element of the subgroup (image  $\zeta$ ); cf. (3.11) of [3]. Hence one obtains the following stronger statement:

$$\pi_{2n-2}(X^*, X)$$

is generated by the subgroup image  $\zeta$ , and the elements  $\beta \circ \alpha$ , where  $\alpha$  ranges over  $\pi_{2n-2}(E^n, \dot{E}^n)$ , and  $\beta$  ranges over a set of generators of  $\pi_n(X^*, X)$ .

It is clear that  $\pi_n(X^*, X)$  is a free abelian group, having a set of generators in 1-1 correspondence with the cells  $e_{\lambda}^n$ ,  $\lambda \in \Lambda$ .

### 3. The Homotopy Groups of a Union of Spheres with a Single Point in Common

Let  $\mathbb{S}_{m}^{n} = S_{1}^{n} \cup \cdots \cup S_{m}^{n}$  be a space consisting of the union of *m* distinct *n*-spheres (n > 1) with a single point in common. We will consider  $\mathbb{S}_{m}^{n}$  as a cell complex consisting of the single vertex,  $\sigma^{0}$ , and a collection of *n*-cells,  $\sigma_{1}^{n}$ ,  $\cdots$ ,  $\sigma_{m}^{n}$ . Choose elements  $\iota_{1}, \cdots, \iota_{m} \in \pi_{n}(\mathbb{S}_{m}^{n})$  such that for  $1 \leq p \leq m$ ,  $\iota_{p}$  is represented by a map  $\mathbb{S}^{n} \to \mathbb{S}_{m}^{n}$  which has degree  $\pm 1$  on  $\mathbb{S}_{p}^{n}$  and degree 0 on  $\mathbb{S}_{i}^{n}$  for  $i \neq p$  (this is equivalent to choosing orientations for the spheres  $\mathbb{S}_{1}^{n}, \cdots, \mathbb{S}_{m}^{n}$ ). Then  $\iota_{1}, \cdots, \iota_{m}$  is a system of free generators for  $\pi_{n}(\mathbb{S}_{m}^{n})$ .

For any positive integer p, we define homomorphisms

$$\begin{split} \phi_i \colon \pi_p(S^n) &\to \pi_p(\mathbb{S}^n), \\ \phi_{ij} \colon \pi_p(S^{2n-1}) &\to \pi_p(\mathbb{S}^n), \\ \phi_{ijk} \colon \pi_p(S^{3n-2}) &\to \pi_p(\mathbb{S}^n), \end{split}$$

as follows:

$$\begin{split} \phi_i(\alpha) &= \iota_i \circ \alpha, \qquad \alpha \in \pi_p(S^n), \\ \phi_{ij}(\alpha) &= [\iota_i, \iota_j] \circ \alpha, \qquad \alpha \in \pi_p(S^{2n-1}), \\ \phi_{ijk}(\alpha) &= [[\iota_i, \iota_j], \iota_k] \circ \alpha, \quad \alpha \in \pi_p(S^{2n-2}). \end{split}$$

Here *i*, *j*, and *k* are any positive integers  $\leq m$ . The following facts may now be proved:

(1)  $\phi_i$  is an isomorphism, and the image subgroup is a direct summand of  $\pi_p(\mathbb{S}_m^n)$ .

(2)  $\phi_{ij} = (-1)^{ij} \phi_{ji}$ .

(3) If p < 4n - 2 and  $i \neq j$ , then  $\phi_{ij}$  is an isomorphism, and the image subgroup is a direct summand of  $\pi_p(\mathbb{S}_m^n)$ ; for the proof, see §5.3 of [1]. Whether or not this statement is true with no restrictions on p is not known.

(4) If p < 3n - 2, then  $\pi_p(S_m^n)$  is the direct sum of the subgroups  $\phi_i[\pi_p(S^n)]$  for  $1 \leq i \leq m$  and  $\phi_{ij}[\pi_p(S^{2n-1})]$  for  $1 \leq i < j \leq m$ . This is Theorem 5.3.3 of [1].

THEOREM IV. If  $i \neq j$ , then the homomorphism  $\phi_{ijk} : \pi_{3n-2}(S^{3n-2}) \to \pi_{3n-2}(S^{n}_{m})$ is an isomorphism. The homotopy group  $\pi_{3n-2}(S^{n}_{m})$  is the direct sum of the following subgroups:

(1)  $\phi_i[\pi_{3n-2}(S^n)]$  for  $1 \leq i \leq m$ . This gives m subgroups, each isomorphic to  $\pi_{3n-2}(S^n)$ .

(2)  $\phi_{ij}[\pi_{2n-2}(S^{2n-1})]$  for  $1 \leq i < j \leq m$ . This gives  $(m^2 - m)/2$  subgroups, each isomorphic to  $\pi_{3n-2}(S^{2n-1})$ .

(3)  $\phi_{ijk}[\pi_{3n-2}(S^{3n-2})]$  for  $1 \leq i < j \leq m$  and  $1 \leq k \leq j$ . This gives  $(m^3 - m)/3$  subgroups, each isomorphic to  $\pi_{3n-2}(S^{3n-2})$ .

**PROOF.** The proof is made by an induction on m. The theorem is trivial for the case m = 1. Assume that we have proved this theorem for  $\mathbb{S}_m^n$ ; we wish to prove it for  $\mathbb{S}_{m+1}^n$ . We may consider  $\mathbb{S}_{m+1}^n = \mathbb{S}_m^n \cup \mathbb{S}_{m+1}^n$ . Then by a general theorem (cf. [11], §4),  $\pi_p(\mathbb{S}_{m+1}^n)$  is the direct sum of the images of the following three isomorphisms:

$$\mu_{1}: \pi_{p}(\mathbb{S}_{m}^{n}) \to \pi_{p}(\mathbb{S}_{m+1}^{n}),$$

$$\mu_{2}: \pi_{p}(\mathbb{S}_{m+1}^{n}) \to \pi_{p}(\mathbb{S}_{m+1}^{n}),$$

$$\partial: \pi_{p+1}(\mathbb{S}_{m}^{n} \times \mathbb{S}_{m+1}^{n}, \mathbb{S}_{m}^{n} \cup \mathbb{S}_{m+1}^{n}) \to \pi_{p}(\mathbb{S}_{m+1}^{n}),$$

Here  $\mu_1$  and  $\mu_2$  are injections, and  $\partial$  is a homotopy boundary operator. Now we may consider  $S_m^n \times S_{m+1}^n$  as a cell complex of dimension 2n, as described on p. 199 of [1]. Then  $S_m^n \times S_{m+1}^n$  is obtained from  $S_{m+1}^n$  by the adjunction of *m* cells, each of dimension 2n. Obviously  $S_{m+1}^n$  is (n-1)-connected. We may now apply Theorem II to obtain the structure of the group  $\pi_{3n-1}(S_m^n \times S_{m+1}^n, S_{m+1}^n)$ . The result is that the homomorphisms

$$\zeta: \pi_{n}(\mathbb{S}_{m}^{n}+1) \otimes \pi_{2n}(\mathbb{S}_{m}^{n} \times \mathbb{S}_{m+1}^{n}, \mathbb{S}_{m+1}^{n}) \to \pi_{3n-1}(\mathbb{S}_{m}^{n} \times \mathbb{S}_{m+1}^{n}, \mathbb{S}_{m+1}^{n}), \psi: \pi_{2n}(\mathbb{S}_{m}^{n} \times \mathbb{S}_{m+1}^{n}, \mathbb{S}_{m+1}^{n}) \otimes \pi_{3n-1}(\mathbb{E}^{2n}, \mathbb{E}^{2n}) \to \pi_{3n-1}(\mathbb{S}_{m}^{n} \times \mathbb{S}_{m+1}^{n}, \mathbb{S}_{m+1}^{n}),$$

are both isomorphisms into, and  $\pi_{3n-1}(\mathbb{S}_m^n \times \mathbb{S}_{m+1}^n, \mathbb{S}_{m+1}^n)$  is the direct sum of the two images. Now  $\pi_{2n}(\mathbb{S}_m^n \times \mathbb{S}_{m+1}^n, \mathbb{S}_{m+1}^n)$  is a free abelian group on *m* generators and we may obviously choose a set of generators  $\alpha_1, \dots, \alpha_m$  such that

$$\partial(\alpha_i) = [\iota_i, \iota_{m+1}], \qquad i = 1, \cdots, m.$$

The completion of the induction is now a matter of straightforward calculation. The details are left to the reader. It is necessary to use equation (3.5) of [3] and equation (23.10) of [7] in the process.

REMARK 1. Consider the case m = 3 of the theorem. We then have that  $\pi_{3n-2}(S_3^n)$  is the direct sum of certain subgroups, as follows:

(1) Three subgroups, each isomorphic to  $\pi_{3n-2}(S^n)$ .

(2) Three subgroups, each isomorphic to  $\pi_{3n-2}(S^{2n-1})$ .

(3) A free abelian group on eight generators. As generators of this subgroup, one may take the triple Whitehead products  $[[\iota_i, \iota_j], \iota_k]$  for  $i \neq j, k = i$  or k = j, and any two of the following three:  $[[\iota_1, \iota_2], \iota_3], [[\iota_2, \iota_3], \iota_1], [[\iota_3, \iota_1], \iota_2].$ 

The fact that we may take any two of these three may be seen by permuting the subscripts before applying Theorem IV. This suggests that there must be some relation between the three last-named triple Whitehead products. As a matter of fact, the authors have been able to prove that these last three triple Whitehead products satisfy the Jacobi identity. More generally, if X is a topological space,  $\alpha \in \pi_p(X)$ ,  $\beta \in \pi_q(X)$ , and  $\gamma \in \pi_r(X)$ , where p, q, and r are all >1, then it is conjectured that the following modified form of the Jacobi identity holds:

$$(-1)^{(p+1)r}[[\alpha,\beta],\gamma] + (-1)^{(q+1)p}[[\beta,\gamma],\alpha] + (-1)^{(r+1)q}[[\gamma,\alpha],\beta] = 0.$$

This conjecture has not been proved as yet, however.

REMARK 2. Consider the case m = 2 of this formula. The group  $\pi_{3n-2}(S_2^n)$  is the direct sum of certain subgroups as follows:

(1) Two subgroups, each isomorphic to  $\pi_{3n-2}(S^n)$ 

(2) A subgroup isomorphic to  $\pi_{3n-2}(S^{2n-1})$ 

(3) A free abelian group on two generators.

Now let  $S^n$  be an *n*-sphere, and  $S^n \to S_2^n$  the map obtained by "shrinking" the equator of  $S^n$  to a point. This induces a homomorphism  $\pi_{3n-2}(S^n) \to \pi_{3n-2}(S_2^n)$ . By a theorem of Serre [10, chap. V, §§3 and 6],  $\pi_{3n-2}(S^n)$  is a group of finite order. Hence the image of this homomorphism is contained in the subgroup of  $\pi_{3n-2}(S_2^n)$  spanned by the direct summands listed under (1) and (2). By projecting onto the direct summand listed in (2), we obtain a "generalized Hopf homomorphism,"  $H: \pi_{3n-2}(S^n) \to \pi_{3n-2}(S^{2n-1})$  (cf. [11], [4] and [5]). One can use this extended definition to prove that the modified left distributive law,

 $(\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha + [\beta_1, \beta_2] \circ H(\alpha) \beta_1, \beta_2 \in \pi_q(X), \alpha \in \pi_p(S^q),$ holds for p = 3q - 2.

REMARK 3. From Theorem IV, it readily follows that any unversally defined m-ary homotopy construction (for the definition, see [3, §2]) from the dimension

n to any dimension  $\leq 3n - 2$  can be obtained by iterated use of Whitehead Products and the composition operation.

Finally, it should be mentioned that partial results on the homotopy groups  $\pi_p(\mathbb{S}_m^n)$  for p > 3n - 2 have been obtained by J. C. Moore and J. P. Serre. These results are unpublished as yet.

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