GROTHENDIECK-WITT GROUPS OF TRIANGULATED CATEGORIES

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ABSTRACT. The basic theory of Grothendieck-Witt groups of triangulated categories with duality is developed. The main results proven include Localization, Additivity, a Fundamental Theorem, and comparison theorems with Grothendieck-Witt groups of forms and formations in exact categories.

INTRODUCTION

Balmer recently introduced triangulated-category methods into the theory of quadratic forms on vector bundles, and they have proven quite fruitful. For instance filtrations of the derived category have been used in work on the Gersten complex for Witt groups (Balmer-Walter [7]), while duality-preserving equivalences of derived categories can be used to compute the Grothendieck-Witt groups and Witt groups of projective bundles and of quadrics (Walter [34][35]). But in preparing the last two articles the author found that a number of basic results about Grothendieck-Witt groups of triangulated categories still needed to be written up, and that has led to the current paper.

The main results of the paper are as follows. We work with small triangulated categories with duality $\underline{\boldsymbol{C}} = (\boldsymbol{C}, {}^*, \delta, \varpi)$ composed of a small triangulated category $\boldsymbol{C}$ in which 2 is invertible, plus a duality functor * and biduality isomorphisms $\varpi_U : U \cong U^{**}$. The duality may be combined with powers of the translation to produce shifted dualities, and for the n-th shifted duality we define a triangulated Grothendieck-Witt group $GW^n(\underline{\boldsymbol{C}})$ similar to Balmer's triangulated Witt group $W^n(\underline{\boldsymbol{C}})$. One of our first main results is that there are forgetful and hyperbolic maps between the Grothendieck-Witt groups and Grothendieck groups which fit into exact sequences

$$GW^{n-1}(\underline{\boldsymbol{C}}) \xrightarrow{\text{forget}} K_0(\boldsymbol{C}) \xrightarrow{\mathbf{H}} GW^n(\underline{\boldsymbol{C}}) \to W^n(\underline{\boldsymbol{C}}) \to 0$$

(Fundamental Theorem 2.6). Next if $\boldsymbol{D} \subset \boldsymbol{C}$ is a thick subcategory invariant under the duality, then there are long exact sequences

$$GW^n(\underline{\boldsymbol{D}}) \to GW^n(\underline{\boldsymbol{C}}) \to GW^n(\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}}) \to W^{n+1}(\underline{\boldsymbol{D}}) \to W^{n+1}(\underline{\boldsymbol{C}}) \to \cdots$$

(Localization Theorem 2.4). Next we say (following Bondal) that $(\boldsymbol{A}_\ell, \boldsymbol{B}, \boldsymbol{A}_r)$ is an *admissible triple* of subcategories of $\boldsymbol{C}$ if they are thick subcategories which together generate $\boldsymbol{C}$ while satisfying $\mathrm{Hom}_{\boldsymbol{C}}(\boldsymbol{A}_\ell, \boldsymbol{B}) = 0$, $\mathrm{Hom}_{\boldsymbol{C}}(\boldsymbol{A}_\ell, \boldsymbol{A}_r) = 0$, and $\mathrm{Hom}_{\boldsymbol{C}}(\boldsymbol{B}, \boldsymbol{A}_r) = 0$. If the duality on $\boldsymbol{C}$ fixes $\boldsymbol{B}$ and exchanges $\boldsymbol{A}_\ell$ and $\boldsymbol{A}_r$, then we have isomorphisms

$$GW^n(\underline{\boldsymbol{C}}) \cong GW^n(\underline{\boldsymbol{B}}) \times K_0(\boldsymbol{A}_\ell), \qquad W^n(\underline{\boldsymbol{C}}) \cong W^n(\underline{\boldsymbol{B}})$$

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(Additivity Theorem 3.6). The right-to-left isomorphism of Grothendieck-Witt groups is $(\llbracket B,\phi\rrbracket,[A]) \mapsto \llbracket B,\phi\rrbracket + \llbracket A\oplus A^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi_n & 0\end{smallmatrix}\right)\rrbracket$.

In §§4–8 we compare our triangulated category version of Grothendieck-Witt groups with the versions for categories of chain complexes with weak equivalences and for forms and formations in exact categories. The methods used are not new, but we still go through all the details because the literature seems incomplete: it deals mainly with Witt groups and with categories of projective modules where all exact sequences split. We also look at admissible subcategories of categories of chain complexes with weak equivalences (§4). This is important when one wishes to apply the Additivity Theorem in explicit situations.

We return in §9 to the old theme of Witt classes as obstructions to being able to symmetrize strictly a complex which is symmetric up to quasi-isomorphism. In Theorem 9.5 we describe when one can strictly symmetrize such a complex while fixing its two ends. We apply this to strictly symmetric locally free resolutions of subcanonical subschemes (Theorem 9.6).

In §10 we define the derived Grothendieck-Witt groups of a scheme, and we describe how the deformation invariance of Kervaire semicharacteristics really amounts to having maps $GW^{4n-1}(X,L) \to \mathrm{H}^0(X_{\mathrm{Zar}},\mathbb{Z}/2)$ (Theorem 10.2). We then calculate the derived Grothendieck-Witt groups of the punctured spectrum of a regular local ring (Theorem 11.2).

The main application of this paper will be the calculation of the Grothendieck-Witt and Witt groups of projective bundles and of quadrics [34][35]. These Grothendieck-Witt groups include many terms which disappear when one passes to the Witt groups. This was the motivation to treat Grothendieck-Witt groups in this paper and not just Witt groups.

Thanks are due to P. Balmer for simplifying the proof of the Additivity Theorem.

1. Triangulated categories with duality

In this section we review some of the basic notions of triangulated duality taken from Balmer [3] [4] [5] and Balmer-Walter [7].

A triangulated category is an additive category with an automorphism (the *translation*, written T or $X \mapsto X[1]$) and a class of *exact triangles* satisfying four standard axioms. A (TR4+) *triangulated category* is a triangulated category satisfying a slightly enhanced form (TR4+) of the octahedral axiom (TR4). This axiom was first suggested by Beilinson-Bernstein-Deligne [9] Remarque 1.1.13. Certain results in this paper which rely on Balmer's sublagrangian construction (see Theorem 1.3 below) will include the enhanced axiom among their hypotheses, but we refer the reader to Balmer [4] §1 for a statement of the axiom. The standard triangulated categories of chain complexes satisfy (TR4+).

Let $\delta = \pm 1$. A covariant *δ-exact functor* $F : \boldsymbol{C} \to \boldsymbol{D}$ between triangulated categories is an covariant additive functor which commutes with the translations $FT = TF$ such that if

$$Z[-1] \xrightarrow{u} X \xrightarrow{i} Y \xrightarrow{p} Z, \tag{1}$$

is an exact triangle in $\boldsymbol{C}$, then

$$FZ[-1] \xrightarrow{\delta\cdot Fu} FX \xrightarrow{Fi} FY \xrightarrow{Fp} FZ, \tag{2}$$

is an exact triangle in $\boldsymbol{D}$. A contravariant δ-exact functor $G : \boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{D}$ is a contravariant additive functor which commutes with the translations $G(X[-1]) = (GX)[1]$ such that if (1) is an exact triangle of $\boldsymbol{C}$, then

$$GX[-1] \xrightarrow{\delta\cdot G(u[1])} GZ \xrightarrow{Gp} GY \xrightarrow{Gi} GX \tag{3}$$

is an exact triangle of $\boldsymbol{D}$. An *exact functor* is one which is either $(+1)$-exact or (-1)-exact. A *morphism of exact functors* is a morphism of functors $\alpha : F \to G$ such that $\alpha_X[1]$ is given by

(4)

	covariant	contravariant
same parity	$\alpha_{X[1]}$	$\alpha_{X[-1]}$
opposite parity	$-\alpha_{X[1]}$	$-\alpha_{X[-1]}$

with the signs determined by the variance and relative parity of F and G.

The translation functor $X \mapsto X[1]$ is (-1)-exact. The composition of a δ-exact functor with an ϵ-exact functor is $\delta\epsilon$-exact.

A *duality* on a triangulated category $\boldsymbol{C}$ is a triple $({}^*, \delta, \varpi)$ with $\delta = \pm 1$, with ${}^* : \boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{C}$ a δ-exact functor, and with $\varpi : 1_{\boldsymbol{C}} \cong {}^{**}$ an isomorphism of exact functors such that for all objects X the composition $\varpi_X^* \varpi_{X^*} : X^* \to X^{***} \to X^*$ is the identity 1_{X^*}. The * is the *duality functor* or the *functor component of the duality*, while the $\varpi_X : X \cong X^{**}$ are the *biduality maps.* The biduality maps are an intrinsic part of the structure called a *triangulated category with duality.* For instance replacing ϖ by $-\varpi$ changes the transposition operation $u \mapsto u^{\mathrm{t}}$ defined below to $u \mapsto -u^{\mathrm{t}}$, thereby changing the Witt and Grothendieck-Witt groups. In practice the δ seems generally to be more or less determined by the *, but it appears in many formulas, and it seems a good idea to insist on identifying it.

A *duality-preserving functor* $(\boldsymbol{C}, {}^*, \delta, \varpi) \to (\boldsymbol{C}', {}^\natural, \delta', \varpi')$ is a pair (F, η) with $F : \boldsymbol{C} \to \boldsymbol{C}'$ an exact functor and with $\eta : F \circ {}^* \to {}^\natural \circ F$ an isomorphism of exact functors such that the lefthand square

(5)
$$\begin{tikzcd}
FX \arrow[r, "F\varpi_X"] \arrow[d, "\varpi'_{FX}"'] & F(X^{**}) \arrow[d, "\eta_{X^*}"] \\
(FX)^{\natural\natural} \arrow[r, "\eta_X^\natural"] & F(X^*)^\natural
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F(Y^*) \arrow[r, "\eta_Y"] \arrow[d, "\alpha_{Y^*}"'] & (FY)^\natural \\
G(Y^*) \arrow[r, "\zeta_Y"] & (GY)^\natural \arrow[u, "\alpha_Y^\natural"']
\end{tikzcd}$$

commutes for all $X \in \boldsymbol{C}$. An *isomorphism of duality-preserving functors* $\alpha : (F, \eta) \cong (G, \zeta)$ is an isomorphism of exact functors $\alpha : F \cong G$ such that the righthand square commutes for all $Y \in \boldsymbol{C}$. Since η and α are morphisms of exact functors, they commute with translation up to a sign determined above (4) by the parities of * and $^\natural$, or of F and G.

Small triangulated categories with duality containing $\frac{1}{2}$, duality-preserving functors, isomorphisms of duality-preserving functors, and the natural composition laws and identities form a strict 2-category **TriCatD**. There are forgetful 2-functors **TriCatD** $\to$ **TriCat** $\to$ **Cat**. We will often abbreviate a triangulated category with duality as $\underline{\boldsymbol{C}} = (\boldsymbol{C}, {}^*, \delta, \varpi)$, with the forgetful 2-functor being $\underline{\boldsymbol{C}} \mapsto \boldsymbol{C}$.

An *equivalence of triangulated categories with duality* or *duality-preserving equivalence* is a duality-preserving functor which is invertible up to isomorphism in **TriCatD**. By Balmer-Walter [7] Lemma 4.3(d) $(F, \eta) : \underline{\boldsymbol{C}} \to \underline{\boldsymbol{D}}$ is an duality-preserving equivalence if and only if $F : \boldsymbol{C} \to \boldsymbol{D}$ is an equivalence of categories, i.e. the forgetful 2-functor **TriCatD** $\to$ **Cat** reflects equivalences.

A map of the form $u : A \to B^*$ has a *transpose* $u^{\mathrm{t}} : B \to A^*$ given by $u^{\mathrm{t}} := u^* \circ \varpi_B$. Morphisms of the form $v : A^* \to B$ also have transposes. One has $u^{\mathrm{tt}} = u$ for all u. A morphism $w : A \to A^*$ is *symmetric* if $w^{\mathrm{t}} = w$. A *symmetric object* of $\underline{\boldsymbol{C}} = (\boldsymbol{C}, {}^*, \delta, \varpi)$ is a pair (A, w) with $w : A \cong A^*$ a symmetric isomorphism. An *isomorphism* $(A, w) \cong (B, s)$ of symmetric objects is an isomorphism $r : A \cong B$ such that $w = r^* s r$. Symmetric objects and their isomorphisms form a groupoid $\mathbf{SymObj}(\underline{\boldsymbol{C}})$. A duality-preserving functor (F,η) from $\underline{\boldsymbol{C}}$ to $\underline{\boldsymbol{C}}' = (\boldsymbol{C}', \natural, \delta', \varpi')$ induces a morphism of groupoids $\mathbf{SymObj}(\underline{\boldsymbol{C}}) \to \mathbf{SymObj}(\underline{\boldsymbol{C}}')$ sending $w : A \cong A^*$ to $\eta_A \circ Fw : FA \cong F(A^*) \cong (FA)^\natural$, while an isomorphism of duality-preserving functors induces an isomorphism between morphisms of groupoids. Thus $\mathbf{SymObj} : \mathbf{TriCatD} \to \mathbf{Gpd}$ is a strict 2-functor.

A duality on a triangulated category $\underline{\boldsymbol{C}} = (\boldsymbol{C}, {}^*, \delta, \varpi)$ induces *shifted dualities* $\underline{\boldsymbol{C}}[n] = (\boldsymbol{C}, {}^*[n], \delta_n, \varpi_n)$ for all $n \in \mathbb{Z}$ defined by

$$ {}^*[n] = (X \mapsto X^*[n]), \qquad \delta_n = (-1)^n \delta, \qquad \varpi_n = (-1)^{\lceil n/2 \rceil} \delta^n \varpi. \tag{6}$$

Thus $\underline{\boldsymbol{C}}$ and $\underline{\boldsymbol{C}}[n]$ refer to the same triangulated category $\boldsymbol{C}$ equipped with different, shifted dualities. The signs in (6) are characterized by the initial values $(\delta_0, \varpi_0) = (\delta, \varpi)$ plus the recurrence relations $\delta_{n-1} = -\delta_n$ and $\varpi_{n-1} = \delta_n \cdot \varpi_n$. We therefore have $\underline{\boldsymbol{C}}[n][m] = \underline{\boldsymbol{C}}[n+m]$ for all n and m, allowing us to deduce conclusions about say the n-th and $(n-1)$-st shifted dualities on the basis of results proven for the 0-th and (-1)-st shifted dualities.

Proposition 1.1 (Periodicity). *The translation functor $A \mapsto A[1]$ plus the natural identifications of the translated duals $A^*[1] = A[1]^*[2]$ and $A^*[2] = A[2]^*[4]$ give isomorphisms of triangulated categories with duality*

$$(\boldsymbol{C}, {}^*, \delta, \varpi) \cong (\boldsymbol{C}, {}^*[2], \delta_2, -\varpi_2) \cong (\boldsymbol{C}, {}^*[4], \delta_4, \varpi_4)$$

Thus shifting a symmetric object $A \cong A^*$ in $\underline{\boldsymbol{C}}$ gives first a skew-symmetric object $A[1] \cong A[1]^*[2]$ in $\underline{\boldsymbol{C}}[2]$ then a symmetric object $A[2] \cong A[2]^*[4]$ in $\underline{\boldsymbol{C}}[4]$.

An additive category *contains* $\frac{1}{2}$ if its Hom groups are uniquely 2-divisible, or equivalently if the category is $\mathbb{Z}[\frac{1}{2}]$-linear.

Proposition 1.2 (Balmer [4] Theorem 2.6). *Let $(\boldsymbol{C}, {}^*, \delta, \varpi)$ be a triangulated category with duality containing $\frac{1}{2}$. If $u : A[-1] \to A^*$ is a map which is symmetric with respect to the (-1)-st shifted duality, then there exists an exact triangle of the form*

$$A[-1] \xrightarrow{u} A^* \xrightarrow{\phi^{-1} v^*} B \xrightarrow{v} A \tag{7}$$

with $\phi : B \cong B^*$ an isomorphism symmetric with respect to the 0-th shifted duality. Moreover, the symmetric object (B, ϕ) is uniquely determined by (A, u) up to isomorphism

In the situation of Proposition 1.2 we write $\mathrm{Cone}(A, u) = (B, \phi)$. A symmetric object is *metabolic* if it is isomorphic to a $\mathrm{Cone}(A, u)$. The object A^* is a *Lagrangian* of (B, ϕ). For any object A (and fixed duality) there is a *hyperbolic* symmetric object

$$\mathbf{H}(A) = \left(A \oplus A^*, \begin{pmatrix} 0 & 1_{A^*} \\ \varpi_A & 0 \end{pmatrix}\right) = \mathrm{Cone}(A, 0).$$

A *sublagrangian* of a symmetric object (P, ϕ) is a morphism $u : L \to B$ such that $u^* \phi u = 0$. Any sublagrangian can be completed to a commutative diagram with exact rows

$$\begin{tikzcd}
M^*[-1] \arrow[r, "s"] \arrow[d, "{r^*[-1]}"'] & L \arrow[r, "u"] \arrow[d, "r"] & P \arrow[r, "v"] \arrow[d, "\phi=\phi^{\mathrm{t}}", "\cong"'] & M^* \arrow[d, "r^*"'] \\
L^*[-1] \arrow[r, "\delta \cdot s^{\mathrm{t}}[-1]"'] & M \arrow[r, "v^{\mathrm{t}}"'] & P^* \arrow[r, "u^*"'] & L^*
\end{tikzcd} \tag{8}$$

in which all the transposes marked use the signs of the 0-th shifted duality (so $\delta \cdot s^{\mathrm{t}}[-1]$ is the transpose of s with respect to the (-1)-st shifted duality, cf. the recurrence relations of (6)).

Theorem 1.3 (Balmer [4] 4.13, 4.20)**.** *In a* (TR4+) *triangulated category containing* $\frac{1}{2}$ *with a* δ*-exact duality, given a sublagrangian and a commutative diagram with exact rows as above* (8)*, there exists a morphism* $\lambda : L \to L^*[-1]$ *such that* $s^{\mathrm{t}}[-1] \circ \lambda \circ s = 0$ *and such that, setting* $\mu = r + \delta \cdot s^{\mathrm{t}}[-1] \circ \lambda$*, the triangle*

$$M^*[-1] \xrightarrow{\left(\begin{smallmatrix} s \\ -\mu^*[-1] \end{smallmatrix}\right)} L \oplus L^*[-1] \xrightarrow{(\,\mu\ \ \delta\cdot s^{\mathrm{t}}[-1]\,)} M \xrightarrow{v\phi^{-1}v^{\mathrm{t}}} M^* \tag{9}$$

is exact. Moreover, for any such λ *and* μ*, if one replaces* r *by* μ*, one can complete* (8) *to a commutative diagram with exact rows and columns*

$$\begin{tikzcd}
M^*[-1] \arrow[r, "s"] \arrow[d, "{\mu^*[-1]}"'] & L \arrow[r, "u"] \arrow[d, "\mu"] \arrow[dl, dotted, "\lambda"'] & P \arrow[r, "v"] \arrow[d, "\phi=\phi^{\mathrm{t}}", "\cong"'] & M^* \arrow[d, "\mu^*"] \\
L^*[-1] \arrow[r, "{\delta\cdot s^{\mathrm{t}}[-1]}"'] \arrow[d, "\kappa"'] & M \arrow[r, "v^{\mathrm{t}}"'] \arrow[d, "\rho^{\mathrm{t}}"] & P^* \arrow[r, "u^*"'] & L^* \\
R \arrow[r, "\cong", "-\psi=-\psi^{\mathrm{t}}"'] \arrow[d, "\rho"'] & R^* \arrow[d, "\delta\cdot\kappa^{\mathrm{t}}"] & & \\
M^* \arrow[r, "Ts"] & L[1] & &
\end{tikzcd}$$

with ψ *a symmetric isomorphism such that* $\rho\psi^{-1}\rho^{\mathrm{t}} = v\phi^{-1}v^{\mathrm{t}}$ *and with* $(P, \phi) \perp (R, -\psi) \cong \mathrm{Cone}(M^*, \mu s)$*.*

Proof. The λ is that of [4] Lemma 4.13. The corresponding μ makes the triangle (9) exact because that is what a "very good" morphism does ([4] Definition 4.11). The ability to complete the large diagram symmetrically with $\rho\psi^{-1}\rho^{\mathrm{t}} = v\phi^{-1}v^{\mathrm{t}}$ is [4] Theorem 4.20 (except that we have changed the sign of ψ, and we have added in the sign δ relating to the parity of the exactness of the duality; Balmer assumed it was $+1$). Although the statement of Theorem 4.20 only says that (R, ψ) is Witt-equivalent to (P, ϕ), if one looks at the proof and pushes the calculations at the end one small step further one finds an exact triangle

$$M^*[-1] \xrightarrow{w} M \xrightarrow{\tau} P \oplus R \xrightarrow{\tau^*\left(\begin{smallmatrix} \phi & 0 \\ 0 & -\psi \end{smallmatrix}\right)} M^*$$

with $w = \mu s = rs$ (and with our $-\psi$ substituted for Balmer's ψ). This shows that $(P, \phi) \perp (R, -\psi) \cong \mathrm{Cone}(M^*, \mu s)$. $\square$

2. Grothendieck-Witt groups of triangulated categories

We now define triangulated Grothendieck-Witt groups along the lines of Balmer's triangulated Witt groups, and we prove some basic results including Localization and the Fundamental Theorem.

Let $\underline{\boldsymbol{C}} = (\boldsymbol{C}, {}^*, \delta, \varpi)$ be a small triangulated category with duality containing $\frac{1}{2}$. Its *Grothendieck-Witt group* $GW(\underline{\boldsymbol{C}})$ is the quotient of the free abelian group on the isomorphism classes of symmetric objects in $\underline{\boldsymbol{C}}$ by the relations of the forms $[\![(A, \alpha) \perp (B, \beta)]\!] = [\![A, \alpha]\!] + [\![B, \beta]\!]$ and $[\![\mathrm{Cone}(Y, f)]\!] = [\![\mathbf{H}(Y)]\!]$. Alternatively, the Grothendieck-Witt group is the set of equivalence classes of pairs of symmetric objects where $((A, \alpha), (B, \beta)) \sim ((A', \alpha'), (B', \beta'))$ if and only if there exist symmetric maps $c : C[-1] \to C^*$ and $d : D[-1] \to D^*$ and an object E such that the two symmetric objects

$$(A,\alpha) \perp (B',\beta') \perp \mathrm{Cone}(C,c) \perp \mathbf{H}(D) \perp \mathbf{H}(E), \tag{10a}$$

$$(A',\alpha') \perp (B,\beta) \perp \mathbf{H}(C) \perp \mathrm{Cone}(D,d) \perp \mathbf{H}(E) \tag{10b}$$

are isomorphic. The equivalence class of the pair $((A,\alpha),(B,\beta))$ is written $[\![A,\alpha]\!] - [\![B,\beta]\!]$. Addition in the group is induced by the orthogonal direct sum $\perp$.

The *Witt group* $W(\underline{\boldsymbol{C}})$ is the quotient of $GW(\underline{\boldsymbol{C}})$ by the subgroup generated by the hyperbolic classes. Equivalently, it is the group of equivalence classes of symmetric objects modulo the relation where (A,α) is equivalent to (B,β) if there exists an isomorphism of the form $(A,\alpha) \perp \mathrm{Cone}(C,c) \cong (B,\beta) \perp \mathrm{Cone}(D,d)$. The Witt equivalence class of (A,α) is written $[A,\alpha]$.

These groups depend on $\underline{\boldsymbol{C}} = (\boldsymbol{C}, {}^*, \delta, \varpi)$ in $\mathbf{TriCatD}$ and not just on $\boldsymbol{C}$. On the other hand the Grothendieck group $K_0(\boldsymbol{C})$ depends only on $\boldsymbol{C}$. The Grothendieck-Witt and Witt groups for the n-th shifted duality (6) are denoted by

$$GW^n(\underline{\boldsymbol{C}}) = GW(\underline{\boldsymbol{C}}[n]), \qquad W^n(\underline{\boldsymbol{C}}) = W(\underline{\boldsymbol{C}}[n]). \tag{11}$$

A duality-preserving functor $(F,\eta) : \underline{\boldsymbol{C}} \to \underline{\boldsymbol{C}}'$ acts on symmetric objects $\mathbf{SymObj}(\underline{\boldsymbol{C}}) \to \mathbf{SymObj}(\underline{\boldsymbol{C}}')$ and preserves cone exact triangles (7). It therefore induces well-defined morphisms of groups $GW^n(\underline{\boldsymbol{C}}) \to GW^n(\underline{\boldsymbol{C}}')$ and $W^n(\underline{\boldsymbol{C}}) \to W^n(\underline{\boldsymbol{C}}')$. We have the following properties, all of which have already been established for triangulated Witt groups (cf. Balmer [3] §1.19, Balmer-Walter [7] Lemma 4.1).

Proposition 2.1. *Triangulated Grothendieck-Witt groups form functors $GW^n : \mathbf{TriCatD} \to \mathbf{Ab}$ which send isomorphic duality-preserving functors to the same maps of Grothendieck-Witt groups. Duality-preserving equivalences induce isomorphisms of triangulated Grothendieck-Witt groups. The functors are periodic $GW^n \cong GW^{n+4}$, while the Grothendieck-Witt group of skew-symmetric objects in $\underline{\boldsymbol{C}}[n]$ is $GW^n(\boldsymbol{C}, {}^*, \delta, -\varpi) \cong GW^{n+2}(\underline{\boldsymbol{C}})$.*

We can verify the usual formulas for calculating inside a Grothendieck-Witt group.

Proposition 2.2. *Let $\underline{\boldsymbol{C}}$ be a small triangulated category with duality containing $\frac{1}{2}$.*

(a) *For any symmetric morphism $u : A[-1] \to A^*$ one has $[\![\mathrm{Cone}(A,u)]\!] = [\![\mathbf{H}(A)]\!]$.*

(b) *For any symmetric object (P,ϕ) one has $[\![P,\phi]\!] + [\![P,-\phi]\!] = [\![\mathbf{H}(P)]\!]$, and for any object M one has $[\![\mathbf{H}(M)]\!] = [\![\mathbf{H}(M^*)]\!]$.*

(c) *If $Q[-1] \to L \to A \to Q$ is an exact triangle, then one has $[\![\mathbf{H}(A)]\!] = [\![\mathbf{H}(L)]\!] + [\![\mathbf{H}(Q)]\!]$. Hence there are exact sequences*

$$K_0(\boldsymbol{C}) \xrightarrow{\mathbf{H}} GW^n(\underline{\boldsymbol{C}}) \to W^n(\underline{\boldsymbol{C}}) \to 0.$$

(d) *If $L \to (P,\phi)$ is a sublagrangian, and (R,ψ) is the associated "subquotient" of Theorem 1.3, then we have $[\![P,\phi]\!] = [\![R,\psi]\!] + [\![\mathbf{H}(L)]\!]$.*

Proof. (a) This follows easily from the definition.

(b) This is because of the isomorphisms $(P,\phi) \perp (P,-\phi) \cong \mathbf{H}(P)$ and $\mathbf{H}(M) \cong \mathbf{H}(M^*)$.

(c) We may assume $n = 0$. Name the maps in the exact triangle u, v, and w, and write

$$\mu = \begin{pmatrix} w & 0 \\ 0 & v^* \end{pmatrix}, \qquad \lambda = \begin{pmatrix} 0 & \delta \cdot T^{-1}u^* \\ \varpi_L u & 0 \end{pmatrix}.$$

If we take the direct sum of the exact triangle and its dual and then make the identification $L \oplus Q^* \cong Q^* \oplus L^{**}$ we get an exact triangle

$$Q[-1] \oplus L^*[-1] \xrightarrow{\lambda} Q^* \oplus L^{**} \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ \varpi_L & 0 \end{smallmatrix}\right)^{-1} \mu^*} A \oplus A^* \xrightarrow{\mu} Q \oplus L^*$$

Hence we have $\mathbf{H}(A) \cong \operatorname{Cone}(Q \oplus L^*, \lambda)$, and the formula $[\![\mathbf{H}(A)]\!] = [\![\mathbf{H}(L)]\!] + [\![\mathbf{H}(Q)]\!]$ then follows. The operation $\mathbf{H}$ therefore induces a well-defined morphism $K_0(\boldsymbol{C}) \to GW(\underline{\boldsymbol{C}})$. Its image is the subgroup generated by the hyperbolic classes, which is the kernel of the quotient morphism $GW(\underline{\boldsymbol{C}}) \twoheadrightarrow W(\underline{\boldsymbol{C}})$.

(d) Theorem 1.3 gives us $(P, \phi) \perp (R, -\psi) \cong \operatorname{Cone}(M^*, \mu s)$ and thus $[\![P, \phi]\!] + [\![R, -\psi]\!] = [\![\mathbf{H}(M^*)]\!]$. Parts (b) and (c) of the current proposition show that $[\![\mathbf{H}(M^*)]\!] = [\![\mathbf{H}(L)]\!] + [\![R, \psi]\!] + [\![R, -\psi]\!]$, and the assertion follows. ☐

The exact sequence of part (c) of the proposition can be extended one step to the left (Theorem 2.6).

A *thick* subcategory $\boldsymbol{D}$ of a triangulated category $\boldsymbol{C}$ is a strictly full triangulated subcategory which is closed under direct summands. The multiplicative system S of morphisms in $\boldsymbol{C}$ with mapping cone in $\boldsymbol{D}$ satisfies the Ore condition, and the quotient triangulated category is $\boldsymbol{C}/\boldsymbol{D} = \boldsymbol{C}[S^{-1}]$. A *thick invariant* subcategory $\underline{\boldsymbol{D}} = (\boldsymbol{D}, ^*, \delta, \varpi)$ of a triangulated category with duality $\underline{\boldsymbol{C}} = (\boldsymbol{C}, ^*, \delta, \varpi)$ is a thick subcategory $\boldsymbol{D} \subset \boldsymbol{C}$ which is invariant under the duality on $\boldsymbol{C}$, equipped with the restriction of the duality on $\boldsymbol{C}$. The duality on $\boldsymbol{C}$ descends to the quotient category, giving a quotient triangulated category with duality $\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}} := (\boldsymbol{C}/\boldsymbol{D}, ^*, \delta, \varpi)$. Balmer proved a localization theorem for triangulated Witt groups.

Theorem 2.3 (Balmer [4] Theorem 6.2). *Let $\underline{\boldsymbol{D}} \subset \underline{\boldsymbol{C}}$ be a thick invariant subcategory of a* (TR4+) *triangulated category with duality containing $\frac{1}{2}$. Then there is a long exact sequence of triangulated Witt groups*

$$\cdots \to W^n(\underline{\boldsymbol{D}}) \to W^n(\underline{\boldsymbol{C}}) \to W^n(\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}}) \to W^{n+1}(\underline{\boldsymbol{D}}) \to \cdots .$$

We now extend this sequence to include triangulated Grothendieck-Witt groups.

Theorem 2.4 (Localization). *The above sequence extends to an exact sequence*

$$GW^n(\underline{\boldsymbol{D}}) \to GW^n(\underline{\boldsymbol{C}}) \to GW^n(\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}}) \to W^{n+1}(\underline{\boldsymbol{D}}) \to W^{n+1}(\underline{\boldsymbol{C}}) \to \cdots .$$

For triangulated categories coming from complicial exact categories with weak equivalences and duality (see §4 below) the sequence could likely be continued to the left if one developed a higher hermitian K-theory in the style of Waldhausen [32].

To prove Theorem 2.4 we need the following notion. Let $S \subset \boldsymbol{C}$ be the multiplicative system of arrows whose mapping cones are in $\boldsymbol{D}$. An *S-symmetric* object in $\underline{\boldsymbol{C}}$ is a pair (A, α) with $\alpha : A \to A^*$ a symmetric member of S. Two S-symmetric objects (A, α) and (B, β) are *S-isomorphic* if there exists a pair of arrows in S with a common source $s : C \to A$ and $t : C \to B$ such that $s^* \alpha s = t^* \beta t$.

Lemma 2.5. *The group $GW(\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}})$ is isomorphic to the group of classes of pairs of S-symmetric objects in $\underline{\boldsymbol{C}}$ modulo the relation where $((A, \alpha), (B, \beta))$ and $((A', \alpha'), (B', \beta'))$ are equivalent if and only if there exist symmetric arrows $c : C[-1] \to C^*$ and $d : D[-1] \to D^*$ in $\boldsymbol{C}$ and an object E such that the two S-symmetric objects* (10a) *and* (10b) *are S-isomorphic.*

This is an exercise with the calculus of fractions used to define the morphisms of $\boldsymbol{C}[S^{-1}] = \boldsymbol{C}/\boldsymbol{D}$. The analogous result for Witt groups is Balmer [4] Proposition 5.5.

Proof of Theorem 2.4. Shifting the duality if necessary, we may reduce to the case where $n = 0$. We then have a commutative diagram

$$\begin{tikzcd}
K_0(\boldsymbol{C}) \arrow[r] \arrow[d] & K_0(\boldsymbol{C}/\boldsymbol{D}) \arrow[r] \arrow[d] & 0 & & \\
GW(\underline{\boldsymbol{C}}) \arrow[r] \arrow[d] & GW(\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}}) \arrow[r] \arrow[d] & W^1(\underline{\boldsymbol{D}}) \arrow[r] \arrow[d, equal] & W^1(\underline{\boldsymbol{C}}) \arrow[r] \arrow[d, equal] & \cdots \\
W(\underline{\boldsymbol{C}}) \arrow[r] \arrow[d] & W(\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}}) \arrow[r] \arrow[d] & W^1(\underline{\boldsymbol{D}}) \arrow[r] & W^1(\underline{\boldsymbol{C}}) \arrow[r] & \cdots \\
0 & 0 & & &
\end{tikzcd}$$

whose top row is exact by the localization theorem for K_0, whose bottom row is exact by Balmer's Localization Theorem 2.3, and whose columns are exact by Proposition 2.2(c). A diagram chase shows that the second line of the diagram must also be exact. Thus the sequence of Grothendieck-Witt and Witt groups is exact at $GW(\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}})$, at $W^1(\underline{\boldsymbol{D}})$, and at all places further to the right.

It remains to show that $GW(\underline{\boldsymbol{D}}) \to GW(\underline{\boldsymbol{C}}) \to GW(\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}})$ is exact in the middle. The composition clearly vanishes. Now suppose that $\xi = [\![A,\alpha]\!] - [\![B,\beta]\!] \in GW(\underline{\boldsymbol{C}})$ maps to $0 \in GW(\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}})$. By Lemma 2.5 there exist symmetric arrows $c : C[-1] \to C^*$ and $d : D[-1] \to D^*$ in $\boldsymbol{C}$ and an object E such that

$$\begin{aligned}
(X,u) &:= (A,\alpha) \perp \mathrm{Cone}(c) \perp \mathbf{H}(D) \perp \mathbf{H}(E), \\
(Y,v) &:= (B,\beta) \perp \mathbf{H}(C) \perp \mathrm{Cone}(d) \perp \mathbf{H}(E)
\end{aligned}$$

are S-isomorphic. This means that there exist morphisms $s : Z \to X$ and $t : Z \to Y$ in S such that $s^*us = t^*vt$, and therefore $Z \to (X,u) \perp (Y,-v)$ is a sublagrangian. So we can complete the diagram of Theorem 1.3, and we will then have $[\![X,u]\!] + [\![Y,-v]\!] = [\![R,\psi]\!] + [\![\mathbf{H}(Z)]\!]$ by Proposition 2.2(d), from which we deduce $\xi = [\![X,u]\!] - [\![Y,v]\!] = [\![R,\psi]\!] + [\![\mathbf{H}(Z)]\!] - [\![\mathbf{H}(Y)]\!]$. If we complete $t : Z \to Y$ to a triangle $Z \to Y \to U \to Z[1]$, then we get $\xi = [\![R,\psi]\!] - [\![\mathbf{H}(U)]\!]$. We will complete the proof of the theorem by showing that R and U are in $\boldsymbol{D}$, and therefore ξ is in the image of $GW(\underline{\boldsymbol{D}})$.

The object U is the mapping cone on $t \in S$ and hence is in $\boldsymbol{D}$.

The object R is constructed using Theorem 1.3, which means that we start by constructing diagram (8), which corresponds to the middle two rows of the following diagram.

$$\begin{tikzcd}
Z[-1] \arrow[r, "0"] \arrow[d, dotted, "{w[-1]}"'] & Z \arrow[r, "\binom{1}{1}"] \arrow[d, equal, "1"'] & Z \oplus Z \arrow[r, "(1\ -1)"] \arrow[d, "\begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}"] & Z \arrow[d, dotted, "w"] \\
M^*[-1] \arrow[r, "\sigma"] \arrow[d, "{r^*[-1]}"'] & Z \arrow[r, "\binom{s}{t}"] \arrow[d, "r"] \arrow[dl, dotted, "\lambda"'] & X \oplus Y \arrow[r] \arrow[d, "\begin{pmatrix} u & 0 \\ 0 & -v \end{pmatrix}"] & M^* \arrow[d, "r^*"] \\
Z^*[-1] \arrow[r, "{\delta\cdot\sigma^{\mathrm{t}}[-1]}"'] \arrow[d, equal] & M \arrow[r] \arrow[d, dotted, "w^{\mathrm{t}}"] & X^* \oplus Y^* \arrow[r] \arrow[d, "\begin{pmatrix} s^* & 0 \\ 0 & t^* \end{pmatrix}"] & Z^* \arrow[d, equal] \\
Z^*[-1] \arrow[r, "0"] & Z^* \arrow[r, "\binom{1}{-1}"] & Z^* \oplus Z^* \arrow[r, "(1\ 1)"] & Z^*
\end{tikzcd}$$

The first row is exact, and the top middle square commutes, so by the morphism axiom (TR3) there exists $w : Z \to M^*$ in $\boldsymbol{C}$ making all the top squares commute. Since the two solid arrows between the first and second rows become isomorphisms in $\boldsymbol{C}/\boldsymbol{D}$, so must the third. So w and w^{t} become isomorphisms in $\boldsymbol{C}/\boldsymbol{D}$. Comparing compositions along the second and third columns gives $\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right) w^{\mathrm{t}} r = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right) s^* u s$. This implies that $w^{\mathrm{t}} r = s^* u s$, and therefore r also becomes an isomorphism in $\boldsymbol{C}/\boldsymbol{D}$. Thus all the vertical arrows become isomorphisms in $\boldsymbol{C}/\boldsymbol{D}$, while σ and $\delta \cdot \sigma^{\mathrm{t}}[-1]$ become 0 in $\boldsymbol{C}/\boldsymbol{D}$. Hence the morphism $\mu = r + \delta \cdot \sigma^{\mathrm{t}}[-1] \circ \lambda$ of Theorem 1.3 also becomes an isomorphism in $\boldsymbol{C}/\boldsymbol{D}$. So μ is in S, and its mapping cone R is in $\boldsymbol{D}$. We have now proven our claim that U and R are in $\boldsymbol{D}$, and therefore $\xi = [\![R, \psi]\!] - [\![\mathbf{H}(U)]\!]$ is in the image of $GW(\underline{\boldsymbol{D}})$. □

The next theorem is a version of a standard theorem in hermitian K-theory (Karoubi [18]).

Theorem 2.6 (Fundamental Theorem)**.** *Let $\underline{\boldsymbol{C}} = (\boldsymbol{C}, {}^*, \delta, \varpi)$ be a* (TR4+) *triangulated category with duality containing $\frac{1}{2}$. Then there are exact sequences*

$$GW^{n-1}(\underline{\boldsymbol{C}}) \xrightarrow{\text{forget}} K_0(\boldsymbol{C}) \xrightarrow{\mathbf{H}} GW^n(\underline{\boldsymbol{C}}) \to W^n(\underline{\boldsymbol{C}}) \to 0.$$

We begin with a lemma extracted from the arguments of Balmer [4] §4. Suppose we have a commutative diagram with exact rows

$$\begin{tikzcd}
C^*[-1] \arrow[r, "-\delta\cdot\tau^*"] \arrow[d, dotted, "\exists\phi"', "\cong"] & A^* \arrow[r, "\nu^*"] \arrow[d, "-\sigma^{\mathrm{t}}"'] & B^* \arrow[r, "\rho^*"] \arrow[d, "\sigma"] & C^* \arrow[d, dotted, "\phi[1]"] \\
C \arrow[r, "\rho"] & B \arrow[r, "\nu"] & A \arrow[r, "\tau"] & C[1].
\end{tikzcd}$$

Lemma 2.7. *If there exists an isomorphism $\phi : C \to C^*[-1]$ making the diagram commute and such that $\tau^* \phi^{-1} \tau[-1] : A[-1] \to A^*$ is symmetric, then $\frac{1}{2}(\phi + \phi^{\mathrm{t}})$ is a symmetric isomorphism making the diagram commute.*

Proof. That ϕ makes the diagram commute is really two equations $\phi[1]\rho^* = \tau\sigma$ and $\rho\phi = \delta \cdot \sigma^{\mathrm{t}} \tau^* = \delta \cdot \varpi_B^{-1} \sigma^* \tau^*$. The second equation is equivalent to its dual

$$\phi^* \rho^* = \delta \cdot (\tau\sigma)^{**} \varpi_B^{*-1} = \delta \cdot (\tau\sigma)^{**} \varpi_{B^*} = \delta \cdot \varpi_{C[1]} \tau\sigma.$$

Since the transpose is $\phi^{\mathrm{t}} = \delta \cdot \varpi_C^{-1} \phi^*[-1]$, the commutativity of the diagram is equivalent to the two equations $\phi[1]\rho^* = \tau\sigma$ and $\phi^{\mathrm{t}}[1]\rho^* = \tau\sigma$. Therefore ϕ^{t} and $\frac{1}{2}(\phi + \phi^{\mathrm{t}})$ also make the diagram commute.

To complete the proof we show that $\frac{1}{2}(\phi + \phi^{\mathrm{t}})$ is an isomorphism by observing that ϕ and ϕ^{t} are isomorphisms making the diagram commute, and that $\tau^* \phi^{-1} \tau[-1] = \tau^* \phi^{\mathrm{t}-1} \tau[-1]$ because $\tau^* \phi^{-1} \tau[-1]$ is symmetric. So by Balmer [4] Lemma 4.6 there exists an $x : C \to C$ such that $\phi = (1 + x)\phi^{\mathrm{t}}$ and $x^3 = 0$. Therefore $\frac{1}{2}(\phi + \phi^{\mathrm{t}}) = (1 + x/2)\phi^{\mathrm{t}}$ is also an isomorphism. □

Proof of Theorem 2.6. It is enough to prove the theorem for the shift $n = 0$. Because of Proposition 2.2(c), we only need to show that the sequence is exact at $K_0(\boldsymbol{C})$.

If $\alpha : X \cong X^*[-1]$ is a symmetric isomorphism for the (-1)-st shifted duality, then $\mathrm{Cone}(X[1], \alpha) \cong 0$ because it is the mapping cone on an isomorphism. Since the composition $GW^{-1}(\underline{\boldsymbol{C}}) \to K_0(\boldsymbol{C}) \to GW(\underline{\boldsymbol{C}})$ sends

$$[\![X, \alpha]\!] \mapsto [X] \mapsto [\![\mathbf{H}(X)]\!] = -[\![\mathrm{Cone}(X[1], \alpha)]\!] = 0,$$

the composition vanishes.

Now suppose we have a class $\xi \in K_0(\boldsymbol{C})$ such that $\mathbf{H}(\xi) = 0$ in $GW(\underline{\boldsymbol{C}})$. We can write $\xi = [L] - [M]$, and we have $[\![\mathbf{H}(L)]\!] - [\![\mathbf{H}(M)]\!] = 0$, so there exist symmetric maps $c : C[-1] \to C^*$ and $d : D[-1] \to D^*$ and an object E such that

$$\mathbf{H}(L \oplus D \oplus E) \perp \operatorname{Cone}(C, c) \cong \mathbf{H}(M \oplus C \oplus E) \perp \operatorname{Cone}(D, d).$$

Call this symmetric object (Y, β), and let $\gamma = \beta^{-1}$, and $L_1 = L \oplus C \oplus D^* \oplus E$, and $M_1 = M^* \oplus C^* \oplus D \oplus E^*$. We have exact triangles

$$L_1[-1] \xrightarrow{\lambda} L_1^* \xrightarrow{\gamma i^*} Y \xrightarrow{i} L_1, \qquad M_1[-1] \xrightarrow{\mu} M_1^* \xrightarrow{\gamma j^*} Y \xrightarrow{j} M_1,$$

with λ and μ symmetric. Applying the octahedral axiom to the composable morphisms $L_1^* \oplus M_1^* \to Y \oplus Y^* \to Y^*$ gives a commutative diagram with exact rows and columns

$$\begin{tikzcd}
P \arrow[r, "\binom{p}{q}"] \arrow[d, "-\gamma i^* p = \gamma j^* q"'] & L_1^* \oplus M_1^* \arrow[r, "(i^*\ j^*)"] \arrow[d, "\begin{pmatrix} \gamma i^* & 0 \\ 0 & j^* \end{pmatrix}"] & Y^* \arrow[r, "si\gamma = tj\gamma"] \arrow[d, equal, "1"] & P[1] \arrow[d] \\
Y \arrow[r, "\binom{-1}{\beta}"] \arrow[d, "\binom{i}{-j}"'] & Y \oplus Y^* \arrow[r, "(\beta\ 1)"] \arrow[d, "\begin{pmatrix} i & 0 \\ 0 & j\gamma \end{pmatrix}"] & Y^* \arrow[r, "0"] & Y[1] \\
L_1 \oplus M_1 \arrow[r, equal, "-1"] \arrow[d, "(s\ t)"'] & L_1 \oplus M_1 \arrow[d, "\begin{pmatrix} -\lambda[1] & 0 \\ 0 & -\mu[1] \end{pmatrix}"] & & \\
P[1] \arrow[r, "\binom{p[1]}{q[1]}"] & L_1^*[1] \oplus M_1^*[1] & &
\end{tikzcd}$$

such that the two compositions $Y \oplus Y^* \to Y^* \to P[1]$ and $Y \oplus Y^* \to L_1 \oplus M_1 \to P[1]$ are equal. Here the second row and second column are exact triangles which we have been allowed to choose, while p, q, s, and t are morphisms which exist because of the axioms, and they determine the maps $P \to Y$ and $Y^* \to P[1]$ because of the commutativity conditions. The first row and first column of the diagram above become the second and third rows of the following commutative diagram with exact rows

$$\begin{tikzcd}
P^*[-1] \arrow[r] \arrow[d, dashed, "\exists\ \cong"'] & L_1^* \oplus M_1^* \arrow[r, "(i^*\ j^*)"] \arrow[d, equal, "1"'] & Y^* \arrow[r] \arrow[d, equal, "1"] & P^* \arrow[d, dashed, "\exists\ \cong"] \\
P \arrow[r, "\binom{p}{q}"] \arrow[d, equal, "1"'] & L_1^* \oplus M_1^* \arrow[r, "(i^*\ j^*)"] \arrow[d, "\frac{1}{2}(-\gamma i^*\ \gamma j^*)"'] & Y^* \arrow[r, "si\gamma = tj\gamma"] \arrow[d, "\frac{1}{2}\binom{i\gamma}{-j\gamma}"] & P[1] \arrow[d, equal, "1"] \\
P \arrow[r, "-\gamma i^* p = \gamma j^* q"'] & Y \arrow[r, "\binom{i}{j}"] & L_1 \oplus M_1 \arrow[r, "(s\ -t)"'] & P[1]
\end{tikzcd}$$

while the top row is the dual of the bottom row. The top and bottom rows of this diagram fulfill the hypotheses of Lemma 2.7 because from the commutativity of the bottom square of the earlier diagram we calculate that the composition $(L_1 \oplus M_1)[-1] \to P \cong P^*[-1] \to L_1^* \oplus M_1^*$ is $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & -\mu \end{smallmatrix}\right)$ which is symmetric. So by Lemma 2.7 there exists a symmetric isomorphism $P^*[-1] \cong P$, and therefore the class $[P] = -\xi$ is in the image of the forgetful map $GW^{-1}(\underline{\boldsymbol{C}}) \to K_0(\boldsymbol{C})$. This completes the proof of the theorem. $\square$

3. Additivity theorems

We now look at Witt and Grothendieck-Witt groups of triangulated categories with duality with n-tuples of admissible subcategories. These structures, which come from Bondal [10] and Bondal-Kapranov [11], are an analogue for triangulated categories of semi-direct products in group theory. The main results are the Additivity Theorems 3.3, 3.4 and 3.6.

Two subcategories of a triangulated category $\boldsymbol{D}$ are *orthogonal* $\boldsymbol{A} \perp \boldsymbol{B}$ if $\operatorname{Hom}_{\boldsymbol{D}}(A, B) = 0$ for all $A \in \boldsymbol{A}$ and all $B \in \boldsymbol{B}$. The right and left orthogonals of a subcategory $\boldsymbol{A}$ are the full subcategories $\boldsymbol{A}^{\perp}$ and ${}^{\perp}\boldsymbol{A}$ with objects

$$\operatorname{Ob} \boldsymbol{A}^{\perp} = \{X \in \boldsymbol{D} \mid \operatorname{Hom}_{\boldsymbol{D}}(A, X) = 0 \text{ for all } A \in \boldsymbol{A}\},$$

$$\operatorname{Ob} {}^{\perp}\boldsymbol{A} = \{Y \in \boldsymbol{D} \mid \operatorname{Hom}_{\boldsymbol{D}}(Y, A) = 0 \text{ for all } A \in \boldsymbol{A}\}.$$

An *admissible n-tuple* $(\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_n)$ of subcategories of a triangulated category $\boldsymbol{D}$ is an n-tuple of strictly full triangulated subcategories which satisfy $\boldsymbol{A}_i \perp \boldsymbol{A}_j$ for all $i < j$ and which together generate $\boldsymbol{D}$. Admissible pairs of categories can be characterized by the following proposition.

Proposition 3.1 ([10] Lemma 3.1)**.** *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be strictly full triangulated subcategories of a small triangulated category $\boldsymbol{D}$ with $\boldsymbol{A} \perp \boldsymbol{B}$. Then $(\boldsymbol{A}, \boldsymbol{B})$ is an admissible pair of subcategories of $\boldsymbol{D}$ if and only if any of the following equivalent conditions holds:*

(a) *$\boldsymbol{A}$ and $\boldsymbol{B}$ generate $\boldsymbol{D}$.*

(b) *For every $X \in \boldsymbol{D}$ there exists an exact triangle $\ell_{\boldsymbol{B}}X[-1] \to r_{\boldsymbol{A}}X \to X \to \ell_{\boldsymbol{B}}X$ with $r_{\boldsymbol{A}}X \in \boldsymbol{A}$ and $\ell_{\boldsymbol{B}}X \in \boldsymbol{B}$.*

(c) *The inclusion functor $\boldsymbol{A} \hookrightarrow \boldsymbol{D}$ has a right adjoint $r_{\boldsymbol{A}} : \boldsymbol{D} \to \boldsymbol{A}$, and $\boldsymbol{A}^{\perp} = \boldsymbol{B}$.*

(d) *The inclusion functor $\boldsymbol{B} \hookrightarrow \boldsymbol{D}$ has a left adjoint $\ell_{\boldsymbol{B}} : \boldsymbol{D} \to \boldsymbol{B}$, and $\boldsymbol{A} = {}^{\perp}\boldsymbol{B}$.*

Admissible n-tuples of subcategories behave well under quotienting (cf. [11] Proposition 1.6).

Proposition 3.2. *Let $(\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_n)$ be an admissible n-tuple of subcategories of $\boldsymbol{D}$. Then for each j:*

(a) *$\boldsymbol{A}_j$ is a thick subcategory of $\boldsymbol{D}$.*

(b) *For any $i \neq j$, the composition $\boldsymbol{A}_i \hookrightarrow \boldsymbol{D} \twoheadrightarrow \boldsymbol{D}/\boldsymbol{A}_j$ is fully faithful with essential image $\boldsymbol{B}_i := \langle \boldsymbol{A}_i, \boldsymbol{A}_j \rangle / \boldsymbol{A}_j$.*

(c) *$(\boldsymbol{B}_1, \dots, \boldsymbol{B}_{j-1}, \boldsymbol{B}_{j+1}, \dots, \boldsymbol{B}_n)$ is an admissible $(n-1)$-tuple of subcategories of $\boldsymbol{D}/\boldsymbol{A}_j$.*

Additivity theorems express the K_0, GW^n or W^n of a small triangulated category (with duality) in terms of admissible subcategories. Additivity for K_0 is well known.

Theorem 3.3 (Additivity for K_0)**.** *Let $(\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_n)$ be an admissible n-tuple of subcategories of a small triangulated category $\boldsymbol{D}$. Then the direct-sum functor $\prod_{i=1}^n \boldsymbol{A}_i \to \boldsymbol{D}$ which sends $(A_1, \dots, A_n) \mapsto \bigoplus_{i=1}^n A_i$ induces an isomorphism $\prod_{i=1}^n K_0(\boldsymbol{A}_i) \cong K_0(\boldsymbol{D})$.*

Proof. When $n = 2$ the direct-sum functor $\boldsymbol{A}_1 \times \boldsymbol{A}_2 \to \boldsymbol{D}$ and the functor $(r_{\boldsymbol{A}_1}, \ell_{\boldsymbol{A}_2}) : \boldsymbol{D} \to \boldsymbol{A}_1 \times \boldsymbol{A}_2$ induce inverse isomorphisms of Grothendieck groups. For larger n use induction. $\square$

For Grothendieck-Witt and Witt groups there are two Additivity Theorems 3.4 and 3.6.

Theorem 3.4. *Let $(\boldsymbol{A}, \boldsymbol{C})$ be an admissible pair of subcategories of a small triangulated category $\boldsymbol{D}$ containing $\frac{1}{2}$ with a duality exchanging $\boldsymbol{A}$ and $\boldsymbol{C}$. In this case:*

(a) *The duality induces a contravariant equivalence $\boldsymbol{A}^{\mathrm{op}} \simeq \boldsymbol{C}$ and hence an isomorphism $K_0(\boldsymbol{A}) \cong K_0(\boldsymbol{C})$.*

(b) *There is an isomorphism* $GW^n(\underline{\boldsymbol{D}}) \cong K_0(\boldsymbol{A})$ *given by* $[\![X,\phi]\!] \mapsto [r_{\boldsymbol{A}}X]$ *with inverse sending* $[A] \mapsto [\![\mathbf{H}(A)]\!]$.

(c) *The exact sequence of the Fundamental Theorem 2.6 reduces to a split exact sequence* $0 \to GW^{n-1}(\underline{\boldsymbol{D}}) \to K_0(\boldsymbol{D}) \to GW^n(\underline{\boldsymbol{D}}) \to 0$ *isomorphic to* $0 \to K_0(\boldsymbol{A}) \to K_0(\boldsymbol{A})^{\times 2} \to K_0(\boldsymbol{A}) \to 0$*, and we have* $W^n(\underline{\boldsymbol{D}}) = 0$.

Proof. Shifting the duality if necessary, it is enough to treat the case $n = 0$. Part (a) is left to the reader.

(b) The right-to-left map $K_0(\boldsymbol{A}) \to GW(\boldsymbol{D})$ is well-defined because it is the composition of the map $K_0(\boldsymbol{A}) \to K_0(\boldsymbol{D})$ induced by the inclusion with the map $\mathbf{H} : K_0(\boldsymbol{D}) \to GW(\underline{\boldsymbol{D}})$ of Proposition 2.2(c).

For the right-to-left map to be well-defined, we need the assignment $(X,\phi) \mapsto [r_{\boldsymbol{A}}X]$ to be compatible with isometries and orthogonal direct sums in $\underline{\boldsymbol{D}}$ (both clear), and we need that for any symmetric morphism $\alpha : Y \to Y^*[-1]$ this rule should assign the same image to the symmetric objects $\mathrm{Cone}(Y,\alpha) =: (Z,\psi)$ and $\mathbf{H}(Y) = \mathrm{Cone}(Y,0)$. But the inclusion functor $\boldsymbol{A} \hookrightarrow \boldsymbol{D}$ is exact, so its right adjoint $r_{\boldsymbol{A}}$ is also exact (Bondal-Kapranov [11] Proposition 1.4). So applying $r_{\boldsymbol{A}}$ to the exact triangle $Y[-1] \to Y^* \to Z \to Y$ of (7) gives us an exact triangle from which we deduce that $[r_{\boldsymbol{A}}Z] = [r_{\boldsymbol{A}}Y] + [r_{\boldsymbol{A}}(Y^*)]$. So the images of (Z,ψ) and $\mathbf{H}(Y)$ are indeed the same, and consequently the left-to-right map $GW(\underline{\boldsymbol{D}}) \to K_0(\boldsymbol{A})$ sending $[\![X,\phi]\!] \mapsto [r_{\boldsymbol{A}}X]$ is well-defined.

The composition $K_0(\boldsymbol{A}) \to GW(\underline{\boldsymbol{D}}) \to K_0(\boldsymbol{A})$ sends $[A] \mapsto [r_{\boldsymbol{A}}(A \oplus A^*)]$. But since A is in $\boldsymbol{A}$, and A^* is in $\boldsymbol{C}$, it follows that $r_{\boldsymbol{A}}(A \oplus A^*) \cong A$. So this composition is the identity.

The composition $GW(\underline{\boldsymbol{D}}) \to K_0(\boldsymbol{A}) \to GW(\underline{\boldsymbol{D}})$ sends $[\![X,\phi]\!] \mapsto [\![\mathbf{H}(r_{\boldsymbol{A}}X)]\!]$. But the exact triangle of Proposition 3.1(b), its dual, and $\phi : X \to X^*$ gives part of a commutative diagram with exact rows which can be completed uniquely according to Exercise 3.5 below.

$$\begin{tikzcd}
\ell_{\boldsymbol{B}}X[-1] \arrow[r, "k"] \arrow[d, dotted, "\cong"', "{\beta[-1]}"] & r_{\boldsymbol{A}}X \arrow[r, "i"] \arrow[d, dotted, "\cong"', "\alpha"] & X \arrow[r, "j"] \arrow[d, "\cong"', "\phi"] & \ell_{\boldsymbol{B}}X \arrow[d, dotted, "\cong"', "\beta"] \\
(r_{\boldsymbol{A}}X)^*[-1] \arrow[r, "{\delta\cdot k^*[-1]}"] & (\ell_{\boldsymbol{B}}X)^* \arrow[r, "j^*"] & X^* \arrow[r, "i^*"] & (r_{\boldsymbol{A}}X)^*
\end{tikzcd}$$

Applying the same exercise to similar diagrams containing ϕ^{-1}, 1_X, 1_{X^*}, and ϕ^{t} in place of ϕ, one sees that because ϕ is a symmetric isomorphism, α and β must also be isomorphisms and must satisfy $\beta = \alpha^{\mathrm{t}}$. We therefore have $(X,\phi) \cong \mathrm{Cone}(\ell_{\boldsymbol{B}}X, \alpha k)$. This gives $[\![X,\phi]\!] = [\![\mathbf{H}(\ell_{\boldsymbol{B}}X)]\!]$, and then using Proposition 2.2(b) and the isomorphism $(\ell_{\boldsymbol{B}}X)^* \cong r_{\boldsymbol{A}}X$ we get $[\![X,\phi]\!] = [\![\mathbf{H}(r_{\boldsymbol{A}}X)]\!]$. This implies that the composition $GW(\underline{\boldsymbol{D}}) \to K_0(\boldsymbol{A}) \to GW(\underline{\boldsymbol{D}})$ is also the identity and completes the proof.

(c) Theorem 3.3 and parts (a) and (b) of the present proposition allow us to identify the sequence $0 \to GW^{-1}\underline{\boldsymbol{D}} \to K_0\boldsymbol{D} \to GW\underline{\boldsymbol{D}} \to 0$ with the sequence

$$0 \to K_0(\boldsymbol{A}) \xrightarrow{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} K_0(\boldsymbol{A}) \times K_0(\boldsymbol{A}) \xrightarrow{(1\ 1)} K_0(\boldsymbol{A}) \to 0 \tag{12}$$

which is split exact. Moreover, $W(\underline{\boldsymbol{D}})$ is the cokernel of $K_0(\boldsymbol{D}) \to GW(\underline{\boldsymbol{D}})$, so it vanishes. $\square$

In the proof of the theorem we cited the following exercise.

Exercise 3.5. *Suppose one is given a morphism $X \to Y$ and two exact triangles in a triangulated category $\boldsymbol{D}$, the rows of the following diagram,*

$$\begin{tikzcd} C[-1] \arrow[r] \arrow[d, dotted] & A \arrow[r] \arrow[d, dotted, "\exists!"'] & X \arrow[r] \arrow[d] & C \arrow[d, dotted, "\exists!"'] \\ B[-1] \arrow[r] & D \arrow[r] & Y \arrow[r] & B. \end{tikzcd}$$

such that $\mathrm{Hom}_{\boldsymbol{D}}(A,B) = 0$ *and* $\mathrm{Hom}_{\boldsymbol{D}}(A,B[-1]) = 0$. *Then there is a unique morphism $A \to D$ making the middle square commute, and a unique $C \to B$ making the righthand square commute, and together they make the entire diagram into a morphism of exact triangles.*

Theorem 3.6. *Let $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$ be an admissible triple of subcategories of a small* (TR4+) *triangulated category $\boldsymbol{D}$ containing $\frac{1}{2}$ with a duality exchanging $\boldsymbol{A}$ and $\boldsymbol{C}$. Then*:

(a) *The inclusion $\underline{\boldsymbol{B}} \hookrightarrow \underline{\boldsymbol{D}}$ induces isomorphisms $W^n(\underline{\boldsymbol{B}}) \cong W^n(\underline{\boldsymbol{D}})$.*

(b) *There are isomorphisms $GW^n(\underline{\boldsymbol{B}}) \times K_0(\boldsymbol{A}) \cong GW^n(\underline{\boldsymbol{D}})$ given by $([\![B,\beta]\!],[A]) \mapsto [\![B,\beta]\!] + [\![\mathbf{H}(A)]\!]$.*

(c) *These isomorphisms identify the kernels of the forgetful maps $GW^n(\underline{\boldsymbol{D}}) \to K_0(\boldsymbol{D})$ and $GW^n(\underline{\boldsymbol{B}}) \to K_0(\boldsymbol{B})$.*

Proof. Again we may assume $n = 0$ to simplify the notation.

(a) By Proposition 3.2 the essential images of $\boldsymbol{A}$ and $\boldsymbol{C}$ in $\boldsymbol{D}/\boldsymbol{B}$ form an admissible pair of subcategories, and under our hypotheses they are exchanged by the duality. So by Theorem 3.4 we have $W^i(\underline{\boldsymbol{D}}/\underline{\boldsymbol{B}}) = 0$ for all i. So in Balmer's localization sequence (Theorem 2.3)

$$\cdots \to W^{-1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{B}}) \to W(\underline{\boldsymbol{B}}) \to W(\underline{\boldsymbol{D}}) \to W(\underline{\boldsymbol{D}}/\underline{\boldsymbol{B}}) \to \cdots$$

the maps $W(\underline{\boldsymbol{B}}) \to W(\underline{\boldsymbol{D}})$ are isomorphisms. These maps are induced by the inclusion $\underline{\boldsymbol{B}} \hookrightarrow \underline{\boldsymbol{D}}$.

(b) The localization sequences for Witt groups and for Grothendieck-Witt groups (Theorems 2.3 and 2.4) and the Fundamental Theorem 2.6 now give us a commutative diagram with exact rows and columns

$$\begin{tikzcd} GW^{-1}(\underline{\boldsymbol{B}}) \arrow[r] \arrow[d] & K_0(\boldsymbol{B}) \arrow[r, "\mathbf{H}"] \arrow[d, tail, "\text{split}"'] & GW(\underline{\boldsymbol{B}}) \arrow[r] \arrow[d] & W(\underline{\boldsymbol{B}}) \arrow[r] \arrow[d, "\cong"] & 0 \\ GW^{-1}(\underline{\boldsymbol{D}}) \arrow[r] \arrow[d] & K_0(\boldsymbol{D}) \arrow[r, "\mathbf{H}"] \arrow[d, two heads, "\text{split}"'] & GW(\underline{\boldsymbol{D}}) \arrow[r] \arrow[d] & W(\underline{\boldsymbol{D}}) \arrow[r] & 0 \\ GW^{-1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{B}}) \arrow[r, tail, "\text{split}"'] & K_0(\boldsymbol{D}/\boldsymbol{B}) \arrow[r, two heads, "\mathbf{H}", "\text{split}"'] & GW(\underline{\boldsymbol{D}}/\underline{\boldsymbol{B}}) & & \end{tikzcd}$$

We have already established that the map in the fourth column is an isomorphism. The second column is a split exact sequence by the Additivity Theorem 3.3 for K_0. By Proposition 3.2 the quotient category $\boldsymbol{D}/\boldsymbol{B}$ has an admissible pair of subcategories equivalent to $\boldsymbol{A}$ and $\boldsymbol{C}$ which are exchanged under the duality on the quotient category. So the bottom row of the diagram may be identified with the split exact sequence (12) in Theorem 3.4. Therefore the maps $K_0(\boldsymbol{D}) \twoheadrightarrow K_0(\boldsymbol{D}/\boldsymbol{B}) \twoheadrightarrow GW(\underline{\boldsymbol{D}}/\underline{\boldsymbol{B}})$ may be identified with

$$K_0(\boldsymbol{A}) \times K_0(\boldsymbol{B}) \times K_0(\boldsymbol{A}) \xrightarrow{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} K_0(\boldsymbol{A}) \times K_0(\boldsymbol{A}) \xrightarrow{(1\ 1)} K_0(\boldsymbol{A}).$$

The composition may be split by a section $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ which corresponds to the map $K_0(\boldsymbol{A}) \to K_0(\boldsymbol{D})$ induced by the inclusion $\boldsymbol{A} \hookrightarrow \boldsymbol{D}$. So the map in the third column $GW(\underline{\boldsymbol{D}}) \to GW(\underline{\boldsymbol{D}}/\underline{\boldsymbol{B}})$ can be identified with a map $GW(\underline{\boldsymbol{D}}) \to K_0(\boldsymbol{A})$ which is a surjection split by the composition $K_0(\boldsymbol{A}) \hookrightarrow K_0(\boldsymbol{D}) \xrightarrow{\mathbf{H}} GW(\underline{\boldsymbol{D}})$. Moreover, a diagram chase shows that the other map $GW(\underline{\boldsymbol{B}}) \to GW(\underline{\boldsymbol{D}})$ in the third column is an injection. Therefore the third column of the diagram is actually a short exact sequence isomorphic to $0 \to GW(\underline{\boldsymbol{B}}) \to GW(\underline{\boldsymbol{D}}) \to K_0(\boldsymbol{A}) \to 0$ with the surjection split by a section which can be identified with the map $K_0(\boldsymbol{A}) \to GW(\underline{\boldsymbol{D}})$ given by $[A] \to [\![\mathbf{H}(A)]\!]$. Part (b) follows.

Part (c) follows easily from the identifications of part (b). $\square$

4. Categories of chain complexes

Many common triangulated categories are of the form $\boldsymbol{C}[\boldsymbol{w}^{-1}]$ with $\boldsymbol{C}$ a category of chain complexes and chain maps and $\boldsymbol{w}$ a class of chain maps which one inverts formally. It is often more convenient to calculate in $\boldsymbol{C}$ keeping in mind that members of $\boldsymbol{w}$ will later become invertible rather than to work directly in $\boldsymbol{C}[\boldsymbol{w}^{-1}]$ where the morphisms are harder to control. We discuss admissible subcategories of such categories. Our basic framework is pretty much that of Thomason's complicial biWaldhausen categories ([28] Definition 1.2.11).

An *exact category with weak equivalences* $(\boldsymbol{C}, \boldsymbol{w})$ is an exact category $\boldsymbol{C}$ together with a class $\boldsymbol{w} \subset \operatorname{Mor} \boldsymbol{C}$ of 'weak equivalences' such that: (1) any isomorphism of $\boldsymbol{C}$ is in $\boldsymbol{w}$; (2) if $f : X \to Y$ and $g : Y \to Z$ are composable morphisms in $\boldsymbol{C}$, and if two of f, g, and gf are in $\boldsymbol{w}$, then so is the third; and (3) if

$$\begin{tikzcd} X' \arrow[r, rightarrowtail] \arrow[d, "h'"'] & X \arrow[r, twoheadrightarrow] \arrow[d, "h"] & X'' \arrow[d, "h''"] \\ Y' \arrow[r, rightarrowtail] & Y \arrow[r, twoheadrightarrow] & Y'' \end{tikzcd}$$

is a morphism of exact triangles of $\boldsymbol{C}$, and if two of h, h', and h'' are in $\boldsymbol{w}$, then so is the third.

A *complicial category with weak equivalences* $(\boldsymbol{C}, \boldsymbol{w})$ is an exact category with weak equivalences which is a full subcategory $\boldsymbol{C} \subset \operatorname{Ch}(\boldsymbol{E})$ of the category of chain complexes over some additive category $\boldsymbol{E}$ such that (4) $\boldsymbol{C}$ is closed under translations in both directions and under mapping cones, (5) the admissible exact sequences of $\boldsymbol{C}$ include all sequences in $\boldsymbol{C}$ which are degreewise split exact sequences in $\operatorname{Ch}(\boldsymbol{E})$, and (6) the weak equivalences $\boldsymbol{w}$ include all morphisms of $\boldsymbol{C}$ which are homotopy equivalences in $\operatorname{Ch}(\boldsymbol{E})$.

A comparison of Keller's axioms for exact categories ([19] Appendix A) with Thomason's definitions ([28] §1.2) shows that an exact category with weak equivalences is the same thing as an additive saturated extensional biWaldhausen category. A complicial biWaldhausen category is a complicial category with weak equivalences in which the category of chains $\boldsymbol{E}$ is abelian, and $\boldsymbol{w}$ contains all morphisms in $\boldsymbol{C}$ which are homology isomorphisms in $\operatorname{Ch}(\boldsymbol{E})$. These extra properties do not seem important.

Theorem 4.1. *The localization $\boldsymbol{C}[\boldsymbol{w}^{-1}]$ of a complicial category with weak equivalences $(\boldsymbol{C}, \boldsymbol{w})$ is a* (TR4+) *triangulated category.*

Theorem 4.1 is proven by imitating Verdier's two-step construction of the derived category. The homotopy category $\operatorname{Ch}(\boldsymbol{E})/\simeq$ satisfies (TR4+), and this is inherited by the subcategory $\boldsymbol{C}/\simeq$ and the localization $\boldsymbol{C}[\boldsymbol{w}^{-1}]$ created by the Ore calculus of fractions.

When a complicial category with weak equivalences $(\boldsymbol{C}, \boldsymbol{w})$ has exact subcategories which impose a natural filtration on objects of $\boldsymbol{C}$, the triangulated category $\boldsymbol{C}[\boldsymbol{w}^{-1}]$ may have admissible subcategories as in §3 above. We start with pairs of subcategories. The lemma is easy to prove and is left to the reader.

Lemma 4.2. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are full exact subcategories of an exact category $\boldsymbol{C}$ such that* (i) $\boldsymbol{A} \perp \boldsymbol{B}$, *and* (ii) *any object X of $\boldsymbol{C}$ fits into an exact sequence $r_{\boldsymbol{A}}X \rightarrowtail X \twoheadrightarrow \ell_{\boldsymbol{B}}X$ with $r_{\boldsymbol{A}}X$ in $\boldsymbol{A}$ and $\ell_{\boldsymbol{B}}X$ in $\boldsymbol{B}$, then $r_{\boldsymbol{A}}X$ and $\ell_{\boldsymbol{B}}X$ are unique up to unique isomorphism and functorial in X, and $r_{\boldsymbol{A}} : \boldsymbol{C} \to \boldsymbol{A}$ and $\ell_{\boldsymbol{B}} : \boldsymbol{C} \to \boldsymbol{B}$ are the right adjoint of the inclusion $\boldsymbol{A} \hookrightarrow \boldsymbol{C}$ and the left adjoint of the inclusion $\boldsymbol{B} \hookrightarrow \boldsymbol{C}$.*

For a subcategory $\boldsymbol{A}$ of an exact category with weak equivalences $(\boldsymbol{C}, \boldsymbol{w})$ we will write $\boldsymbol{w}_{\boldsymbol{A}} = \boldsymbol{w} \cap \boldsymbol{A}$ and similarly for other subcategories.

Theorem 4.3. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are full exact subcategories of a complicial category with weak equivalences $(\boldsymbol{C}, \boldsymbol{w})$ satisfying* (i) *and* (ii) *above and also* (iii) *$\boldsymbol{A}$ and $\boldsymbol{B}$ are translation-invariant and* (iv) *the functors $r_{\boldsymbol{A}}$ and $\ell_{\boldsymbol{B}}$ preserve weak equivalences, then $\boldsymbol{A}[\boldsymbol{w}_{\boldsymbol{A}}^{-1}] \hookrightarrow \boldsymbol{C}[\boldsymbol{w}^{-1}]$ and $\boldsymbol{B}[\boldsymbol{w}_{\boldsymbol{B}}^{-1}] \hookrightarrow \boldsymbol{C}[\boldsymbol{w}^{-1}]$ are fully faithful, and their essential images $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}})$ form an admissible pair of subcategories of the triangulated category $\boldsymbol{C}[\boldsymbol{w}^{-1}]$.*

Proof. If $f : C \xrightarrow{\sim} A$ is in $\boldsymbol{w}$ with $A \in \boldsymbol{A}$, then $r_{\boldsymbol{A}}f : r_{\boldsymbol{A}}C \xrightarrow{\sim} r_{\boldsymbol{A}}A \cong A$ is also in $\boldsymbol{w}$ by (iv) plus axiom (1) for weak equivalences. This composition also factors as $r_{\boldsymbol{A}}C \to C \xrightarrow{\sim} A$ because $r_{\boldsymbol{A}} \to 1_{\boldsymbol{C}}$ is a morphism of functors, so by axiom (2) for weak equivalences $r_{\boldsymbol{A}}C \xrightarrow{\sim} C$ is in $\boldsymbol{w}$. So for any $C \xrightarrow{\sim} A$ in $\boldsymbol{w}$ with target in $\boldsymbol{A}$ there exists an $r_{\boldsymbol{A}}C \xrightarrow{\sim} C$ in $\boldsymbol{w}$ targeting C with source in $\boldsymbol{A}$. By the calculus of fractions this implies that $\boldsymbol{A}[\boldsymbol{w}_{\boldsymbol{A}}^{-1}] \to \boldsymbol{C}[\boldsymbol{w}^{-1}]$ is fully faithful. A similar argument shows that $\boldsymbol{B}[\boldsymbol{w}_{\boldsymbol{B}}^{-1}] \to \boldsymbol{C}[\boldsymbol{w}^{-1}]$ is also fully faithful.

The essential image of $\boldsymbol{A}[\boldsymbol{w}_{\boldsymbol{A}}^{-1}] \to \boldsymbol{C}[\boldsymbol{w}^{-1}]$ is the strictly full subcategory $\overline{\boldsymbol{A}} \subset \boldsymbol{C}[\boldsymbol{w}^{-1}]$ of all objects isomorphic in $\boldsymbol{C}[\boldsymbol{w}^{-1}]$ to objects of $\boldsymbol{A}$, and $\overline{\boldsymbol{B}}$ can be characterized similarly. We will show that $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}})$ is an admissible pair of triangulated subcategories of $\boldsymbol{C}[\boldsymbol{w}^{-1}]$.

We first show that $\overline{\boldsymbol{A}} \perp \overline{\boldsymbol{B}}$ by proving that any morphism $A \to B$ in $\boldsymbol{C}[\boldsymbol{w}^{-1}]$ with $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ vanishes. Such a morphism factors as a fraction $A \xleftarrow{\sim} Y \to B$ with $Y \xrightarrow{\sim} A$ coming from $\boldsymbol{w}$ and $Y \to B$ coming from $\boldsymbol{C}$. Since the essential epimorphisms $X \twoheadrightarrow \ell_{\boldsymbol{B}}X$ are functorial by Lemma 4.2 (they form the unit of an adjunction), we get a commutative diagram

$$\begin{tikzcd} A \arrow[d, two heads] & Y \arrow[l, "\sim"'] \arrow[r] \arrow[d, two heads] & B \arrow[d, two heads, "\cong"] \\ 0 & \ell_{\boldsymbol{B}}Y \arrow[l, "\sim"'] \arrow[r] & \ell_{\boldsymbol{B}}B. \end{tikzcd}$$

Here we have $\ell_{\boldsymbol{B}}A = 0$ because $A \in \boldsymbol{A}$, while $B \cong \ell_{\boldsymbol{B}}B$ is an isomorphism because $B \in \boldsymbol{B}$. The functor $\ell_{\boldsymbol{B}}$ preserves weak equivalences, so $\ell_{\boldsymbol{B}}Y \xrightarrow{\sim} 0$ is a weak equivalence. Our morphism $A \to B$ in $\boldsymbol{C}[\boldsymbol{w}^{-1}]$ therefore factors as $A \to 0 \xleftarrow{\sim} \ell_{\boldsymbol{B}}Y \to \ell_{\boldsymbol{B}}B \cong B$, and so it vanishes.

We now claim that $\overline{\boldsymbol{A}} = {}^{\perp}\overline{\boldsymbol{B}}$. Indeed suppose X is an object such that $\mathrm{Hom}_{\boldsymbol{C}[\boldsymbol{w}^{-1}]}(X, B) = 0$ for all $B \in \boldsymbol{B}$. Then X fits into an exact triangle $(\ell_{\boldsymbol{B}}X)[-1] \to r_{\boldsymbol{A}}X \to X \to \ell_{\boldsymbol{B}}X$ in $\boldsymbol{C}[\boldsymbol{w}^{-1}]$ with $r_{\boldsymbol{A}}X$ in $\boldsymbol{A}$ and with $\ell_{\boldsymbol{B}}X$ and $(\ell_{\boldsymbol{B}}X)[-1]$ in $\boldsymbol{B}$. The third arrow of the triangle vanishes, so we have an isomorphism $r_{\boldsymbol{A}}X \cong X \oplus (\ell_{\boldsymbol{B}}X)[-1]$. But the projection $r_{\boldsymbol{A}}X \to (\ell_{\boldsymbol{B}}X)[-1]$ also vanishes, so we must have $(\ell_{\boldsymbol{B}}X)[-1] \cong 0$ and $X \cong r_{\boldsymbol{A}}X$. It follows that X is in $\overline{\boldsymbol{A}}$. So we have $\overline{\boldsymbol{A}} = {}^{\perp}\overline{\boldsymbol{B}}$. Thus $\overline{\boldsymbol{A}}$ is the left orthogonal of a translation-invariant subcategory, and it is therefore a triangulated subcategory of $\boldsymbol{C}[\boldsymbol{w}^{-1}]$. A similar argument shows that $\overline{\boldsymbol{B}}$ is also a triangulated subcategory of $\boldsymbol{C}[\boldsymbol{w}^{-1}]$. It now follows from Proposition 3.1 that $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}})$ is an admissible pair of subcategories of $\boldsymbol{C}[\boldsymbol{w}^{-1}]$. □

Now let $\boldsymbol{C}$ be an exact category with an n-tuple $(\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n)$ of strictly full exact subcategories such that (i′) $\boldsymbol{A}_i \perp \boldsymbol{A}_j$ for all $i < j$, and (ii′) any object X of $\boldsymbol{C}$ has a filtration by admissible monomorphisms

$$0 = F_0X \rightarrowtail F_1X \rightarrowtail \cdots \rightarrowtail F_{n-1}X \rightarrowtail F_nX = X \tag{13}$$

with $\mathrm{gr}_i X := F_iX/F_{i-1}X$ in $\boldsymbol{A}_i$ for each i.

Lemma 4.4. *In these circumstances the filtration $F_\bullet X$ of* (13) *is unique up to unique isomorphism and functorial.*

This lemma is proven by applying Lemma 4.2 to the categories $\boldsymbol{D}_i$ and $\boldsymbol{E}_i$ for $1 \leq i < n$ where $\boldsymbol{D}_i \subset \boldsymbol{C}$ is the full subcategory of objects Y with filtrations which stabilize after F_i

$$0 \rightarrowtail F_1Y \rightarrowtail \cdots \rightarrowtail F_iY = F_{i+1}Y = \cdots = Y,$$

and $\boldsymbol{E}_i \subset \boldsymbol{C}$ is the full subcategory of objects Z with filtrations which are trivial until F_{i+1}

$$0 = F_1Z = \cdots = F_iZ \rightarrowtail F_{i+1}Z \rightarrowtail \cdots \rightarrowtail Z.$$

The lemma gives us functors $\mathrm{gr}_i : \boldsymbol{C} \to \boldsymbol{A}_i$, which allows us to state the following theorem.

Theorem 4.5. *Let $(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n)$ be an n-tuple of subcategories of the complicial category with weak equivalences $(\boldsymbol{C}, \boldsymbol{w})$ satisfying* (i′) *and* (ii′) *above and also* (iii′) *the $\boldsymbol{A}_i$ are translation-invariant, and* (iv′) *the functors $\mathrm{gr}_i : \boldsymbol{C} \to \boldsymbol{A}_i$ preserve weak equivalences. Let $\boldsymbol{B}_i$ be the essential image of the localized functor $\boldsymbol{A}_i[\boldsymbol{w}_{\boldsymbol{A}_i}^{-1}] \to \boldsymbol{C}[\boldsymbol{w}^{-1}]$ induced by the inclusion $\boldsymbol{A}_i \hookrightarrow \boldsymbol{C}$. Then we have $\boldsymbol{A}_i[\boldsymbol{w}_{\boldsymbol{A}_i}^{-1}] \simeq \boldsymbol{B}_i$, and $(\boldsymbol{B}_1, \ldots, \boldsymbol{B}_n)$ is an admissible n-tuple of subcategories of the triangulated category $\boldsymbol{C}[\boldsymbol{w}^{-1}]$.*

Proof. Apply Theorem 4.3 first to the subcategories $\boldsymbol{D}_i \perp \boldsymbol{E}_i$ of $\boldsymbol{C}$ and then to the subcategories $\boldsymbol{D}_{i-1} \perp \boldsymbol{A}_i$ of $\boldsymbol{D}_i$. □

5. Grothendieck-Witt groups for chain complexes

Categories of chain complexes are differential graded categories, so now we consider complicial categories with weak equivalences and a differential graded duality. Our approach resembles Ranicki's stable algebraic bordism categories ([27] Definition 3.15). Our notation for the differential graded Hom in $(\boldsymbol{C}, \boldsymbol{w})$ with $\boldsymbol{C} \subset \mathrm{Ch}(\boldsymbol{E})$ is $\mathrm{Hom}^\bullet_{\boldsymbol{E}}(X, Y)$.

A covariant *δ-exact differential graded functor* $F : (\boldsymbol{C}, \boldsymbol{w}) \to (\boldsymbol{D}, \boldsymbol{v})$ between complicial categories with weak equivalences is a differential graded functor $F : \boldsymbol{C} \to \boldsymbol{D}$ compatible with the exact sequences and weak equivalences which commutes with translation and such that for any $u : X \to Y$ in $\boldsymbol{C}$ there is an isomorphism of exact sequences in $\mathrm{Ch}(\boldsymbol{E})$

$$\begin{tikzcd} FY \arrow[r, tail] \arrow[d, equal] & C(\delta \cdot Fu) \arrow[r, two heads] \arrow[d, "\cong"] & (FX)[1] \arrow[d, equal] \\ FY \arrow[r, tail] & FC(u) \arrow[r, two heads] & F(X[1]) \end{tikzcd} \tag{14}$$

(cf. Verdier [29] Chap. I, (3.4.3.1)). The letter C denotes the mapping cone of a chain map. Such a F localizes to a δ-exact functor $F : \boldsymbol{C}[\boldsymbol{w}^{-1}] \to \boldsymbol{D}[\boldsymbol{v}^{-1}]$ between the triangulated categories. Contravariant δ-exact functors between complicial categories with weak equivalences are defined analogously.

A *differential graded duality* on a complicial category with weak equivalences $(\boldsymbol{C}, \boldsymbol{w})$ is a triple $({}^*, \delta, \varpi)$ with $\delta = \pm 1$, with ${}^* : (\boldsymbol{C}, \boldsymbol{w}) \to (\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{w}^{\mathrm{op}})$ a contravariant δ-exact differential graded functor and with $\varpi_X : X \to X^{**}$ functorial weak equivalences such that the composition $\varpi_X^* \varpi_{X^*} : X^* \to X^{***} \to X^*$ is 1_{X^*} for all X in $\boldsymbol{C}$. Such a duality induces a duality on the triangulated category $\boldsymbol{C}[\boldsymbol{w}^{-1}]$. Because * is a differential graded functor, transposition gives isomorphisms $\mathrm{Hom}^\bullet_{\boldsymbol{E}}(X, Y^*) \cong \mathrm{Hom}^\bullet_{\boldsymbol{E}}(Y, X^*)$ of differential graded abelian groups. Then in $\mathrm{Hom}^\bullet_{\boldsymbol{E}}(X, X^*)$ we have not only symmetric chain maps but also symmetric homotopies and other symmetric cochains. The signs involved in the transposition $\mathrm{Hom}^n_{\boldsymbol{E}}(X, Y^*) \cong \mathrm{Hom}^n_{\boldsymbol{E}}(Y, X^*)$ of cochains of degree n are rather tricky, but they correspond in the end to the signs in the shifted duality (6). We will write $(\underline{\boldsymbol{C}}, \boldsymbol{w}) = (\boldsymbol{C}, \boldsymbol{w}, {}^*, \delta, \varpi)$ and $\underline{\boldsymbol{C}}[\boldsymbol{w}^{-1}] = (\boldsymbol{C}[\boldsymbol{w}^{-1}], {}^*, \delta, \varpi)$.

A *symmetric object* in a $(\underline{\boldsymbol{C}}, \boldsymbol{w})$ containing $\frac{1}{2}$ is a pair (X, ϕ) with $X \in \mathrm{Ob}\,\boldsymbol{C}$ and $\phi : X \xrightarrow{\sim} X^*$ a symmetric chain map which is in $\boldsymbol{w}$. Suppose we have such an (X, ϕ), a chain map $f : L \to X$, and a symmetric cochain $h \in \mathrm{Hom}^{-1}_{\boldsymbol{E}}(L, L^*)$ satisfying $d_{L^*} h + h d_L = -f^* \phi f$. Thus L is totally isotropic up to a specified symmetric homotopy. This gives a commutative diagram

$$
\begin{tikzcd}
L \arrow[r, "f"'] \arrow[d, "\varpi_L"', "\sim"] \arrow[rr, dotted, bend left, "h"] & X \arrow[r, "f^*\phi"'] \arrow[d, "\phi"', "\sim"] & L^* \arrow[d, "1"', "\cong"] \\
L^{**} \arrow[r, "\phi^* f^{**}"] \arrow[rr, dotted, bend right, "h^*"'] & X^* \arrow[r, "f^*"] & L^*
\end{tikzcd}
\tag{15}
$$

whose top row has a total complex U such that $\cdots \to U^n \to U^{n+1} \to \cdots$ is

$$
\cdots \longrightarrow L^{n+1} \oplus X^n \oplus (L^{1-n})^* \xrightarrow{\begin{pmatrix} -d_L & 0 & 0 \\ f & d_X & 0 \\ h & f^*\phi & -d_L^* \end{pmatrix}} L^{n+2} \oplus X^{n+1} \oplus (L^{-n})^* \longrightarrow \cdots .
$$

The diagram as a whole represents a symmetric weak equivalence $\psi : U \xrightarrow{\sim} U^*$. *Algebraic surgery along* $f : L \to (X, \phi)$ *using the homotopy* h is the operation which inputs those data and returns (U, ψ).

When one applies an algebraic surgery to the zero complex along $L[-1] \to (0, 0)$ using a symmetric homotopy h, the homotopy is really a symmetric chain map $h : L[-1] \to L^*$, and the surgery transforms the zero complex into a *cone symmetric object* $\mathrm{Cone}(L, h)$.

The *Grothendieck-Witt group* $GW(\underline{\boldsymbol{C}}, \boldsymbol{w})$ of a small complicial category with weak equivalences and a differential graded duality containing $\frac{1}{2}$ is the quotient of the free abelian group on the symmetric objects in $(\underline{\boldsymbol{C}}, \boldsymbol{w})$ by three kinds of relations:

(a) for the orthogonal direct sum of two symmetric objects one has $[\![(X, \phi) \perp (Y, \psi)]\!] = [\![X, \phi]\!] + [\![Y, \psi]\!]$,
(b) if (X, ϕ) is a symmetric object in $(\underline{\boldsymbol{C}}, \boldsymbol{w})$ and $\alpha : Y \xrightarrow{\sim} X$ is in $\boldsymbol{w}$, then we have $[\![X, \phi]\!] = [\![Y, \alpha^* \phi \alpha]\!]$, and

(c) if $h : L[-1] \to L^*$ is a morphism which is symmetric for the (-1)-st shifted duality, then $[\![\mathrm{Cone}(L,h)]\!] = [\![\mathbf{H}(L)]\!]$.

The *Witt group* $W(\underline{\boldsymbol{C}}, \boldsymbol{w})$ is the quotient of $GW(\underline{\boldsymbol{C}}, \boldsymbol{w})$ by the hyperbolic classes. For the shifted dualities we write $GW^n(\underline{\boldsymbol{C}}, \boldsymbol{w}) = GW(\underline{\boldsymbol{C}}[n], \boldsymbol{w})$ and $W^n(\underline{\boldsymbol{C}}, \boldsymbol{w}) = W(\underline{\boldsymbol{C}}[n], \boldsymbol{w})$.

Theorem 5.1. *If $(\underline{\boldsymbol{C}}, \boldsymbol{w})$ is a small complicial category with weak equivalences and a differential graded duality containing $\frac{1}{2}$, then the localization maps from the complicial Grothendieck-Witt groups to the triangulated Grothendieck-Witt groups are isomorphisms $GW^n(\underline{\boldsymbol{C}}, \boldsymbol{w}) \cong GW^n(\underline{\boldsymbol{C}}[\boldsymbol{w}^{-1}])$.*

Formulas 5.2. Before proving the theorem we establish a number of formulas in $GW(\underline{\boldsymbol{C}}, \boldsymbol{w})$. First of all, $[\![X,\phi]\!] + [\![X,-\phi]\!] = [\![\mathbf{H}(X)]\!]$ and $[\![\mathbf{H}(X)]\!] = [\![\mathbf{H}(X^*)]\!]$ follow from (a) and (b). Next suppose one has a degreewise split Lagrangian-up-to-weak-equivalence which is strictly totally isotropic (not just up to homotopy), i.e. a commutative diagram as on the left

$$\begin{tikzcd} K \arrow[r, tail, "i"] \arrow[d, "u"', "\sim"] & M \arrow[r, two heads] \arrow[d, "\phi=\phi^{\mathrm{t}}"', "\sim"] & L \arrow[d, "u^{\mathrm{t}}"', "\sim"] \\ L^* \arrow[r, tail] & M^* \arrow[r, two heads, "i^*"] & K^* \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, tail, "i"] \arrow[d, "u"', "\sim"] & {[M \cong C(f)]} \arrow[r, two heads] \arrow[d, "\sim"] & L \arrow[d, equal, "1"', "\cong"] \\ L^* \arrow[r, tail] & C(uf) \arrow[r, two heads] & L \end{tikzcd}$$

in which the lines are degreewise split dual exact sequences, and the vertical arrows are all in $\boldsymbol{w}$. The top line is isomorphic to the mapping cone exact sequence of some chain map $f : L[-1] \to K$, and (M,ϕ) is the pullback along a weak equivalence of $\mathrm{Cone}(L, uf)$ (on the right). So relations (b) and (c) give $[\![M,\phi]\!] = [\![\mathbf{H}(L)]\!]$. Next if $L' \rightarrowtail L \twoheadrightarrow L''$ is a degreewise split exact sequence, then we have $[\![\mathbf{H}(L')]\!] + [\![\mathbf{H}(L'')]\!] = [\![\mathbf{H}(L)]\!]$ because $L' \oplus (L'')^*$ is a degreewise split Lagrangian-up-to-weak-equivalence in $\mathbf{H}(L)$. We then get $[\![\mathbf{H}(X)]\!] + [\![\mathbf{H}(X[1])]\!] = 0$ because in the degreewise split exact sequence $X \rightarrowtail C(X) \twoheadrightarrow X[1]$ the complex $C(X)$ is contractible. The surgery formula

($*$) for an algebraic surgery along $f : L \to (X,\phi)$ using the homotopy h returning the symmetric object (U,ψ) of (15) we have $[\![X,\phi]\!] = [\![U,\psi]\!] + [\![\mathbf{H}(L)]\!]$

follows from the formulas already established because the total complex on $0 \to X \to L^*$ is a degreewise split Lagrangian-up-to-weak-equivalence of $(U,\psi) \perp (X,-\phi)$. Next

($**$) if (X,ϕ) and (X,ξ) are symmetric objects in $(\underline{\boldsymbol{C}}, \boldsymbol{w})$ with ϕ and ξ chain homotopic, then we have $[\![X,\phi]\!] = [\![X,\xi]\!]$ in $GW(\underline{\boldsymbol{C}}, \boldsymbol{w})$

follows from the surgery formula, for if $\phi - \xi = dk + kd$, then algebraic surgery along the diagonal $\Delta : X \hookrightarrow (X,\phi) \perp (X,-\xi)$ using the homotopy $-\frac{1}{2}(k + k^{\mathrm{t}})$ outputs a symmetric object weakly equivalent to 0. This gives $[\![X,\phi]\!] + [\![X,-\xi]\!] = [\![\mathbf{H}(X)]\!]$ and then ($**$). A similar application of the surgery formula shows that if $X \xrightarrow{\sim} Y$ is a weak equivalence then $[\![\mathbf{H}(X)]\!] = [\![\mathbf{H}(Y)]\!]$. Finally one shows that $[\![M,\phi]\!] = [\![\mathbf{H}(L)]\!]$ and $[\![\mathbf{H}(L)]\!] = [\![\mathbf{H}(L')]\!] + [\![\mathbf{H}(L'')]\!]$ hold even when the exact sequences are not degreewise split by pulling back to degreewise split exact sequences using mapping cones and cylinders.

Proof of Theorem 5.1. Let $\simeq$ denote homotopy equivalence. Lemma 2.5 gives a description of $GW(\underline{\boldsymbol{C}}[\boldsymbol{w}^{-1}])$ in terms of equivalence classes of pairs of $(\boldsymbol{w}/{\simeq})$-symmetric objects in the triangulated category $\underline{\boldsymbol{C}}/{\simeq}$ which makes the check that $GW(\underline{\boldsymbol{C}}, \boldsymbol{w}) \to GW(\underline{\boldsymbol{C}}[\boldsymbol{w}^{-1}])$ is an isomorphism immediate from (a), (b), (c), and ($**$). The proof of Lemma 2.5 does not require that the biduality maps $\varpi_A : A \to A^{**}$ be isomorphisms in the category called $\boldsymbol{C}$ (otherwise a general assumption in §2), merely that they be in the multiplicative system called S. $\square$

Two symmetric objects (X, ϕ) and (Y, ψ) in $(\underline{\boldsymbol{C}}, \boldsymbol{w})$ are *cobordant* if there is an algebraic surgery transforming $(X, \phi) \perp (Y, -\psi)$ into a symmetric object weakly equivalent to 0. Cobordism is an equivalence relation (Ranicki [26] Proposition 3.2, or Vogel [30] Lemme 4.3), and the equivalence classes form a group $L(\underline{\boldsymbol{C}}, \boldsymbol{w})$ with respect to the orthogonal direct sum. The groups for the shifted dualities are $L_{-n}(\underline{\boldsymbol{C}}, \boldsymbol{w}) = L(\underline{\boldsymbol{C}}[n], \boldsymbol{w})$. Because cobordant complexes are Witt equivalent by $(*)$ we have a natural map $L_{-n}(\underline{\boldsymbol{C}}, \boldsymbol{w}) \to W^n(\underline{\boldsymbol{C}}, \boldsymbol{w})$. Arguments similar to Theorem 5.1 prove the next theorem.

Theorem 5.3. *If $(\underline{\boldsymbol{C}}, \boldsymbol{w})$ is a complicial category with weak equivalences and a differential graded duality containing $\frac{1}{2}$, then the natural maps between cobordism groups and complicial and triangulated Witt groups are isomorphisms $L_{-n}(\underline{\boldsymbol{C}}, \boldsymbol{w}) \cong W^n(\underline{\boldsymbol{C}}, \boldsymbol{w}) \cong W^n(\underline{\boldsymbol{C}}[\boldsymbol{w}^{-1}])$.*

6. Grothendieck-Witt groups of exact categories

We now compare the Grothendieck-Witt groups of an exact category with duality with the even Grothendieck-Witt groups of its bounded derived category. We begin by reviewing derived categories of exact categories, for which a good reference is Neeman [23].

An *exact category* $\boldsymbol{E}$ is an additive category with a designated class of short exact sequences satisfying several axioms. The first version of these axioms was given by Quillen. If $\boldsymbol{A}$ is an additive category and $\boldsymbol{M}$ an abelian category, then examples of exact categories include: $\boldsymbol{A}$ with split exact sequences, $\mathrm{Ch}(\boldsymbol{A})$ with degreewise split exact sequences, $\boldsymbol{M}$ with all its exact sequences, and a full additive subcategory $\boldsymbol{E} \subset \boldsymbol{M}$ which is closed under extensions with the sequences which are exact in $\boldsymbol{M}$. Conversely, any small exact category $\boldsymbol{E}$ can be embedded in an abelian category $\mathbf{Lex}(\boldsymbol{E})$ with the inclusion functor $\boldsymbol{E} \hookrightarrow \mathbf{Lex}(\boldsymbol{E})$ exact and reflecting exactness and such that $\boldsymbol{E}$ is closed under extensions in $\mathbf{Lex}(\boldsymbol{E})$. If $\boldsymbol{E}$ is semisaturated (see below), then $\boldsymbol{E}$ is also closed under kernels of epimorphisms in $\mathbf{Lex}(\boldsymbol{E})$. (See Thomason [28] Appendix A for a nice exposition of the Gabriel-Quillen embedding $\boldsymbol{E} \hookrightarrow \mathbf{Lex}(\boldsymbol{E})$.)

An additive category $\boldsymbol{A}$ is *semisaturated* if every retract in $\boldsymbol{A}$ has a kernel, or equivalently if for any pair of morphisms $p : X \to Y$ and $s : Y \to X$ in $\boldsymbol{A}$ such that $ps = 1_Y$ there exists an isomorphism of the form $X \cong Y \oplus Z$ which identifies p with the projection $Y \oplus Z \to Y$ and s with the inclusion $Y \to Y \oplus Z$. Informally, any object Z of an abelian category containing $\boldsymbol{A}$ which is stably in $\boldsymbol{A}$ is already in $\boldsymbol{A}$ up to isomorphism. Any additive category has a semisaturation satisfying a universal property; one adds the missing direct summands by a formal procedure. Passing from a small exact category to its semisaturation with the inherited exact structure does not change its K-theory (Waldhausen [32] Proposition 1.5.9).

A complex $\cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots$ in an exact category $\boldsymbol{E}$ *breaks up into short exact sequences* if there exist short exact sequences $Z^n \rightarrowtail X^n \twoheadrightarrow Z^{n+1}$ in $\boldsymbol{E}$ such that each morphism in the complex decomposes into the epimorphism of one short exact sequence followed by the monomorphism of the next $X^n \twoheadrightarrow Z^{n+1} \rightarrowtail X^{n+1}$. A complex is *acyclic* if it is a direct summand of a complex which breaks up into short exact sequences. A bounded complex in $\boldsymbol{E}$ is acyclic if and only if it breaks up into short exact sequences in the semisaturation of $\boldsymbol{E}$. So for bounded complexes in semisaturated exact categories the two notions are equivalent. One may also speak of a bounded complex in $\boldsymbol{E}$ being acyclic in degrees $\leq s$ or $\geq t$. If $\boldsymbol{E}$ is semisaturated, one can truncate off acyclic ends of bounded complexes.

A *quasi-isomorphism* in $\mathrm{Ch}^{\mathrm{b}}(\boldsymbol{E})$ is a morphism whose mapping cone is acyclic. Let $\boldsymbol{w} \subset \operatorname{Mor} \mathrm{Ch}^{\mathrm{b}}(\boldsymbol{E})$ be the class of quasi-isomorphisms. Then $(\mathrm{Ch}^{\mathrm{b}}(\boldsymbol{E}), \boldsymbol{w})$ is a complicial category with weak equivalences in the sense of the previous two sections. The localization $\mathrm{D}^{\mathrm{b}}(\boldsymbol{E}) =$

$\mathrm{Ch}^{\mathrm{b}}(\boldsymbol{E})[\boldsymbol{w}^{-1}]$ is a (TR4+) triangulated category called the *bounded derived category* of $\boldsymbol{E}$. If $\boldsymbol{E}$ is a small category, then $\mathrm{D}^{\mathrm{b}}(\boldsymbol{E})$ is a small category.

A *duality* on an exact category $\boldsymbol{E}$ is a pair $(^*, \varpi)$ with $^* : \boldsymbol{E}^{\mathrm{op}} \to \boldsymbol{E}$ an exact functor and $\varpi : 1_{\boldsymbol{E}} \cong {}^{**}$ an isomorphism of functors such that for all objects X the composition $\varpi_X^* \varpi_{X^*} : X^* \to X^{***} \to X^*$ is the identity 1_{X^*}. A duality on $\boldsymbol{E}$ induces 1-exact dualities on $(\mathrm{Ch}^{\mathrm{b}}(\boldsymbol{E}), \boldsymbol{w})$ and $\mathrm{D}^{\mathrm{b}}(\boldsymbol{E})$ by letting * and ϖ act on the objects and morphisms of chain complexes without signs. (Exact functors between exact categories induce 1-exact functors between the derived categories.)

We write $\underline{\boldsymbol{E}} = (\boldsymbol{E}, ^*, \varpi)$, $(\mathrm{Ch}^{\mathrm{b}}(\underline{\boldsymbol{E}}), \boldsymbol{w}) = (\mathrm{Ch}^{\mathrm{b}}(\boldsymbol{E}), \boldsymbol{w}, ^*, 1, \varpi)$, and $\mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}}) = (\mathrm{D}^{\mathrm{b}}(\boldsymbol{E}), ^*, 1, \varpi)$ for the exact, complicial, and triangulated categories with duality.

Transposes and symmetric and skew-symmetric objects in an exact category with duality containing $\frac{1}{2}$ are defined as in a triangulated category with duality containing $\frac{1}{2}$. If (M, ϕ) is a symmetric object and $i : L \to M$ a morphism such that i and $i^*\phi$ form an exact sequence $L \rightarrowtail M \twoheadrightarrow L^*$, then L is a *Lagrangian* for the *metabolic* symmetric object (M, ϕ), and the associated *hyperbolic* object is $\mathbf{H}(L) = \left(L \oplus L^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi_L & 0 \end{smallmatrix}\right)\right)$. The Grothendieck-Witt group $GW^+(\underline{\boldsymbol{E}})$ is the quotient of the free abelian group on the isomorphism classes of symmetric objects of $\underline{\boldsymbol{E}}$ modulo relations $[\![(M, \phi) \perp (N, \psi)]\!] = [\![M, \phi]\!] + [\![N, \psi]\!]$ for orthogonal direct sums and $[\![M, \phi]\!] = [\![\mathbf{H}(L)]\!]$ for a metabolic symmetric object (M, ϕ) with a Lagrangian L. The Grothendieck-Witt group $GW^-(\underline{\boldsymbol{E}})$ of skew-symmetric objects is defined similarly.

The inclusion $\underline{\boldsymbol{E}} \hookrightarrow \mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}})$ is compatible with symmetric objects, isomorphisms, orthogonal direct sums, and Lagrangians (Balmer [3] 2.11), so it induces a natural map $GW^+(\underline{\boldsymbol{E}}) \to GW^0(\mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}}))$. The inclusion followed by the translation (cf. Proposition 1.1 above)

$$(\boldsymbol{E}, ^*, -\varpi) \hookrightarrow (\mathrm{D}^{\mathrm{b}}(\boldsymbol{E}), ^*, 1, -\varpi) \xrightarrow{X \mapsto X[1]} (\mathrm{D}^{\mathrm{b}}(\boldsymbol{E}), ^*[2], 1, \varpi_2) = \mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}})[2]$$

induces a natural map $GW^-(\underline{\boldsymbol{E}}) \to GW^2(\mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}}))$.

Theorem 6.1. *If $\underline{\boldsymbol{E}}$ is a small exact category with duality containing $\frac{1}{2}$, then the inclusion induces an isomorphism between the exact and triangulated Grothendieck-Witt groups $GW^+(\underline{\boldsymbol{E}}) \cong GW^0(\mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}}))$, and the inclusion-and-shift induces an isomorphism $GW^-(\underline{\boldsymbol{E}}) \cong GW^2(\mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}}))$.*

The isomorphisms $W^+(\underline{\boldsymbol{E}}) \cong W^0(\mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}}))$ and $W^-(\underline{\boldsymbol{E}}) \cong W^2(\mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}}))$ for Witt groups were established by Balmer [5] Theorem 4.3.

Proof. The inclusion of $\boldsymbol{E}$ in its semisaturation induces an equivalence of bounded derived categories (Neeman [23] 1.12.3) and an isomorphism of Grothendieck-Witt groups. Hence we may reduce to the case where $\boldsymbol{E}$ is semisaturated.

The inclusion of the theorem is the composition of the inclusion $\underline{\boldsymbol{E}} \hookrightarrow (\mathrm{Ch}^{\mathrm{b}}(\underline{\boldsymbol{E}}), \boldsymbol{w})$ with the localization $(\mathrm{Ch}^{\mathrm{b}}(\underline{\boldsymbol{E}}), \boldsymbol{w}) \to \mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}})$. Theorem 5.1 showed that the localization induces an isomorphism of Grothendieck-Witt groups $GW(\mathrm{Ch}^{\mathrm{b}}(\underline{\boldsymbol{E}}), \boldsymbol{w}) \cong GW(\mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}}))$. We will show that the inclusion also induces an isomorphism $GW^+(\underline{\boldsymbol{E}}) \cong GW(\mathrm{Ch}^{\mathrm{b}}(\underline{\boldsymbol{E}}), \boldsymbol{w})$ by constructing an inverse.

Suppose that $\phi : X \xrightarrow{\sim} X^*$ is a symmetric quasi-isomorphism in $\mathrm{Ch}^{\mathrm{b}}(\underline{\boldsymbol{E}})$. Let $i_X : \sigma X \hookrightarrow X$ be the inclusion map for the subcomplex which is the same as X in cochain degrees > 0 but which vanishes in degrees ≤ 0, and let $(\mathbf{s}X, \mathbf{s}\phi)$ be the symmetric object of $(\mathrm{Ch}^{\mathrm{b}}(\underline{\boldsymbol{E}}), \boldsymbol{w})$ constructed by algebraic surgery along $i_X : \sigma X \hookrightarrow (X, \phi)$ with the zero homotopy (see (15) above for algebraic surgeries). Then $\mathbf{s}X$ is the total complex of the double complex

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d, "1"] & X^2 \arrow[r] \arrow[d, "1"] & \cdots \\
\cdots \arrow[r] & X^{-2} \arrow[r] \arrow[d, "\phi"] & X^{-1} \arrow[r] \arrow[d, "\phi"] & X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & X^2 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X^{2*} \arrow[r] & X^{1*} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

so $\mathbf{s}X$ is acyclic in cochain degrees > 0. Since there exists a quasi-isomorphism $\mathbf{s}\phi : \mathbf{s}X \xrightarrow{\sim} (\mathbf{s}X)^*$, and since the duality is exact, $\mathbf{s}X$ is also acyclic in cochain degrees < 0. Since $\boldsymbol{E}$ is semisaturated, and $\mathbf{s}X$ is bounded, we may truncate off the two acyclic ends of $\mathbf{s}X$, leaving a complex concentrated in degree 0 which we call its H^0. The chain map $\mathbf{s}\phi$ has a compatible truncation, and the symmetric complex $(\mathbf{s}X, \mathbf{s}\phi)$ is quasi-isomorphic to its H^0. We now set

$$\Theta(X, \phi) = [\![H^0(\mathbf{s}X), H^0(\mathbf{s}\phi)]\!] + \sum_{i>0} (-1)^i [\![\mathbf{H}(X^i)]\!] \in GW^+(\underline{\boldsymbol{E}}).$$

To show that Θ induces a well-defined map $GW^0(\mathrm{Ch}^\mathrm{b}(\underline{\boldsymbol{E}}), \boldsymbol{w}) \to GW^+(\underline{\boldsymbol{E}})$ we have to check three conditions. Condition (a) concerning orthogonal direct sums clearly holds. For condition (c) note that if X is the mapping cone of a symmetric chain map $L[-1] \to L^*$, then the double complex above is of the form

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & (L^{-1})^* \oplus L^1 \arrow[r] \arrow[d, "1"] & \cdots \\
\cdots \arrow[r] & (L^1)^* \oplus L^{-1} \arrow[r] \arrow[d, "\cong"] & (L^0)^* \oplus L^0 \arrow[r] \arrow[d] & (L^{-1})^* \oplus L^1 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & (L^{-1})^{**} \oplus (L^1)^* \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots,
\end{tikzcd}$$

and $\Theta(X, \phi)$ reduces to $\sum_{j\in\mathbb{Z}} (-1)^j [\![\mathbf{H}(L^j)]\!]$. This class depends on L but not on the choice of the symmetric chain map $L[-1] \to L^*$, as required.

For condition (b) let (X, ϕ) be a symmetric object of $(\mathrm{Ch}^\mathrm{b}(\boldsymbol{E}), \boldsymbol{w})$, and let $\alpha : Y \xrightarrow{\sim} X$ be a quasi-isomorphism. We compare $\Theta(X, \phi)$ with $\Theta(Y, \alpha^*\phi\alpha)$. Let (T, τ) be the symmetric object constructed by surgery along $\alpha i_Y : \sigma Y \to (X, \phi)$ (lefthand diagram). Then $(\mathbf{s}Y, \mathbf{s}(\alpha^*\phi\alpha))$ is the pullback along a quasi-isomorphism $\mathbf{s}Y \xrightarrow{\sim} T$ (righthand diagram) of (T, τ).

$$\begin{tikzcd}
\sigma Y \arrow[r] \arrow[d, "\cong"] & X \arrow[r] \arrow[d, "\phi", "\sim"'] & (\sigma Y)^* \arrow[d, "="] \\
(\sigma Y)^{**} \arrow[r] & X^* \arrow[r] & (\sigma Y)^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\sigma Y \arrow[r] \arrow[d, "="] & Y \arrow[r] \arrow[d, "\alpha", "\sim"'] & (\sigma Y)^* \arrow[d, "="] \\
\sigma Y \arrow[r] & X \arrow[r] & (\sigma Y)^*
\end{tikzcd}$$

Consequently T is also quasi-isomorphic to its H^0, the symmetric objects $(H^0(T), H^0(\tau))$ and $(H^0(\mathbf{s}Y), H^0(\mathbf{s}(\alpha^*\phi\alpha)))$ are isomorphic, and we have

$$\Theta(Y, \alpha^*\phi\alpha) = [\![H^0(T), H^0(\tau)]\!] + \sum_{i>0} (-1)^i [\![\mathbf{H}(Y^i)]\!].$$

The complex T is homotopy equivalent to the total complex of

$$\begin{tikzcd}
 & (\sigma X)^* \arrow[r] \arrow[ddr, "-1"] & (\sigma Y)^* \\
 & \oplus & \oplus \\
\sigma X \arrow[r] \arrow[ddr, "-1"] & X \arrow[r] & (\sigma X)^* \\
\oplus & \oplus & \\
\sigma Y \arrow[r] & \sigma X &
\end{tikzcd}$$

This complex is filtered, with the bottom graded piece the mapping cone on $\sigma Y \to \sigma X$, the middle graded piece $\mathbf{s}X$, and top graded piece dual to the bottom graded piece. It thus has the form

$$\begin{tikzcd}
\cdots \arrow[r] & (X^2)^* \oplus (Y^3)^* \arrow[r] \arrow[ddr] & (X^1)^* \oplus (Y^2)^* \arrow[r] \arrow[ddr] & (Y^1)^* & & & \\
 & \oplus & \oplus & \oplus & & & \\
\cdots \arrow[r] & (\mathbf{s}X)^{-2} \arrow[r] & (\mathbf{s}X)^{-1} \arrow[r] & (\mathbf{s}X)^0 \arrow[r] \arrow[ddr] & (\mathbf{s}X)^1 \arrow[r] \arrow[ddr] & (\mathbf{s}X)^2 \arrow[r] & \cdots \\
 & & & \oplus & \oplus & \oplus & \\
 & & & Y^1 \arrow[r] & X^1 \oplus Y^2 \arrow[r, "d"'] & X^2 \oplus Y^3 \arrow[r] & \cdots
\end{tikzcd}$$

The mapping cone on the quasi-isomorphism $\alpha : Y \xrightarrow{\sim} X$ is acyclic, so the bottom row is acyclic in cochain degrees ≥ 2. By duality the top row is acyclic in degrees ≤ -2, and we have seen that the middle row is acylic in degrees $\neq 0$. Truncating off the acyclic parts, we see that T is quasi-isomorphic to

$$\begin{tikzcd}
(\ker d)^* \arrow[r] \arrow[ddr] & (Y^1)^* & \\
 & \oplus & \\
 & H^0(\mathbf{s}X) \arrow[ddr] & \\
 & \oplus & \\
 & Y^1 \arrow[r] & \ker d
\end{tikzcd}$$

The dual complex and the symmetric quasi-isomorphism can be treated in the same way. So $(H^0(T), H^0(\tau))$ can be obtained from $(H^0(\mathbf{s}X), H^0(\mathbf{s}\phi))$ by adding $\mathbf{H}(Y^1)$ and then removing the sublagrangian $(\ker d)^*$ and the corresponding quotient $\ker d$. This gives the equation

$$[\![H^0(T), H^0(\tau)]\!] = [\![H^0(\mathbf{s}X), H^0(\mathbf{s}\phi)]\!] + [\![\mathbf{H}(Y^1)]\!] - [\![\mathbf{H}(\ker d)]\!]$$

in $GW^+(\underline{\boldsymbol{E}})$. We also have $[\ker d] = \sum_{i\geq 2}(-1)^i[Y^i] - \sum_{j\geq 1}(-1)^j[X^j]$ in $K_0(\boldsymbol{E})$, and since $\mathbf{H}$ induces a morphism of groups $K_0(\boldsymbol{E}) \to GW^+(\underline{\boldsymbol{E}})$, this gives a formula for $[\![\mathbf{H}(\ker d)]\!]$. Putting these formulas together gives us $\Theta(X,\phi) = \Theta(Y, \alpha^*\phi\alpha)$ in $GW^+(\underline{\boldsymbol{E}})$. So condition (b) holds. Hence Θ induces a well-defined map $GW^0(\mathrm{Ch}^{\mathrm{b}}(\underline{\boldsymbol{E}}), \boldsymbol{w}) \to GW^+(\underline{\boldsymbol{E}})$, and the formulas of §5.2 may be used to verify that this morphism is the inverse of the natural map $GW^+(\underline{\boldsymbol{E}}) \to GW^0(\mathrm{Ch}^{\mathrm{b}}(\underline{\boldsymbol{E}}), \boldsymbol{w})$, so the natural map is an isomorphism. The skew-symmetric Grothendieck-Witt groups are treated similarly. □

7. Short complexes

We now look at the odd triangulated Grothendieck-Witt groups of $\mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}})$. A truncation argument as in the previous section shows that these groups are generated by the classes of short complexes with only two nonzero objects. We give the relations among these generators, basing our approach on Pardon's work on Witt groups [24] (1.10)–(1.21) of short complexes, which was based on Ranicki's work with Witt groups of projective modules. Our proof of

Theorem 7.1 is more general than Pardon's proof [24] (7.1) for Witt groups of Cohen-Macaulay modules of given dimension, which used the projective resolutions of such modules.

Let $\underline{\boldsymbol{E}} = (\boldsymbol{E}, {}^*, \varpi)$ be an exact category with duality containing $\frac{1}{2}$, and let $\varepsilon = \pm 1$. A *short complex* in $\boldsymbol{E}$ is a complex with two objects. The category $\mathbf{Short}(\boldsymbol{E})$ of short complexes has a duality functor given by

$$\big(K_\bullet : 0 \to K_1 \xrightarrow{k} K_0 \to 0\big) \mapsto \big(K^*_\bullet : 0 \to K^*_0 \xrightarrow{-k^*} K^*_1 \to 0\big)$$

and biduality maps which componentwise are $-\varpi$. We use the sign in the duality functor because it is a duality of complexes with an odd shift, and the sign in the biduality maps compensates. Then $\mathbf{Short}(\boldsymbol{E})$ is an exact category with duality and with weak equivalences (the quasi-isomorphisms). An *ε-symmetric short complex* is a quasi-isomorphism of short complexes $f_\bullet : K_\bullet \xrightarrow{\sim} K^*_\bullet$ such that $f^{\mathrm{t}}_\bullet = \varepsilon f_\bullet$, i.e. a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r, "k"] \arrow[d, "-\varepsilon f^{\mathrm{t}}"'] & K_0 \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K^*_0 \arrow[r, "-k^*"] & K^*_1 \arrow[r] & 0
\end{tikzcd}$$

such that $K_1 \rightarrowtail K_0 \oplus K^*_0 \twoheadrightarrow K^*_1$ is an exact sequence. Note that $fk : K_1 \to K^*_1$ is ε-symmetric, and if $f = 1_{K^*_1}$, then k is ε-symmetric.

An ε-symmetric homotopy $h_\bullet : K_\bullet \dashrightarrow K^*_\bullet$ is an ε-symmetric morphism $h : K_0 \to K^*_0$. An ε-symmetric homotopy transforms the ε-symmetric complex $(K_\bullet, f_\bullet)$ above into a *homotopic complex* $(K_\bullet, f_\bullet + \partial(h_\bullet))$ with both occurrences of f replaced by $f - k^*h$.

The *graph complex* of an ε-symmetric morphism $u : X \to X^*$ is the ε-symmetric complex

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, "-\varepsilon\varpi_X"', "\cong"] & X^* \arrow[r] \arrow[d, "\cong"', "1_{X^*}"] & 0 \\
0 \arrow[r] & X^{**} \arrow[r, "-u^*"] & X^* \arrow[r] & 0
\end{tikzcd}$$

which we will write as $\Gamma_{X,u}$. The associated *trivial graph complex* is $\Gamma_{X,0} = \mathbb{H}(X)$.

The *Grothendieck-Witt group of ε-symmetric short complexes* in $\underline{\boldsymbol{E}}$, denoted $GW^+_{\mathrm{short}}(\underline{\boldsymbol{E}})$ or $GW^-_{\mathrm{short}}(\underline{\boldsymbol{E}})$ depending on ε, is the quotient of the free abelian group on the isomorphism classes of ε-symmetric short complexes modulo four types of relations:

(i) the usual relation for orthogonal direct sums $[\![(K_\bullet, f_\bullet) \perp (L_\bullet, g_\bullet)]\!] = [\![K_\bullet, f_\bullet]\!] + [\![L_\bullet, g_\bullet]\!]$,
(ii) homotopic short complexes are in the same class $[\![K_\bullet, f_\bullet]\!] = [\![K_\bullet, f_\bullet + \partial(h_\bullet)]\!]$,
(iii) if $s_\bullet : H_\bullet \xrightarrow{\sim} K_\bullet$ is a quasi-isomorphism, then we have $[\![K_\bullet, f_\bullet]\!] = [\![H_\bullet, s^*_\bullet f_\bullet s_\bullet]\!]$,
(iv) a graph complex is in the same class as the associated trivial graph complex $[\![\Gamma_{X,u}]\!] = [\![\mathbb{H}(X)]\!]$.

The *Witt groups* $W^+_{\mathrm{short}}(\underline{\boldsymbol{E}})$ and $W^-_{\mathrm{short}}(\underline{\boldsymbol{E}})$ are the quotients of the Grothendieck-Witt groups by the classes of the graph complexes.

In the formulas for shifted dualities (6) one has $\varpi_1 = -\varpi$ and $\varpi_3 = \varpi$ when $\delta = 1$. So symmetric short complexes ($-\varepsilon = -1$) may be considered as symmetric complexes in $\mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}})[1]$ concentrated in cochain degrees -1 and 0, while skew-symmetric short complexes ($-\varepsilon = +1$) may be considered as symmetric complexes in $\mathrm{D}^{\mathrm{b}}(\underline{\boldsymbol{E}})[3]$ concentrated in cochain degrees -2 and -1.

Theorem 7.1. *If $\underline{\boldsymbol{E}}$ is a small exact category with duality containing $\frac{1}{2}$, then the natural maps from Grothendieck-Witt groups of short complexes to triangulated Grothendieck-Witt groups are isomorphisms $GW^{+}_{\mathrm{short}}(\underline{\boldsymbol{E}}) \cong GW^{1}(\mathrm{D^b}(\underline{\boldsymbol{E}}))$ and $GW^{-}_{\mathrm{short}}(\underline{\boldsymbol{E}}) \cong GW^{3}(\mathrm{D^b}(\underline{\boldsymbol{E}}))$. The same is true for Witt groups $W^{+}_{\mathrm{short}}(\underline{\boldsymbol{E}}) \cong W^{1}(\mathrm{D^b}(\underline{\boldsymbol{E}}))$ and $W^{-}_{\mathrm{short}}(\underline{\boldsymbol{E}}) \cong W^{3}(\mathrm{D^b}(\underline{\boldsymbol{E}}))$.*

The proof of Theorem 7.1 is nearly identical to Theorem 6.1, but it uses a number of formulas in $GW^{\pm}_{\mathrm{short}}(\underline{\boldsymbol{E}})$ which we prove for completeness's sake.

Formulas 7.2. (a) If there exists an ε-symmetric isomorphism $u : Y \cong Y^*$, then we have $[\![\mathbb{H}(Y)]\!] = [\![\Gamma_{Y,u}]\!] = 0$ because $\Gamma_{Y,u}$ is quasi-isomorphic to 0. Applying this to $Y = X \oplus X^*$ yields $[\![\mathbb{H}(X^*)]\!] = -[\![\mathbb{H}(X)]\!]$.

(b) Given an exact sequence $L \rightarrowtail M \overset{\pi}{\twoheadrightarrow} N$, let $v = \left(\begin{smallmatrix} 0 & \varepsilon\pi^* \\ \varpi\pi & 0 \end{smallmatrix}\right) : M \oplus N^* \to M^* \oplus N^{**}$. There are quasi-isomorphisms of the form $\Gamma_{M\oplus N^*,v} \xleftarrow{\sim} (K_\bullet, f_\bullet) \xrightarrow{\sim} \mathbb{H}(L)$; essentially one replaces $M \oplus N^* \to M^* \oplus N^{**}$ first by $L \oplus N^* \to M^*$ then by $L \to L^*$. This gives $[\![\mathbb{H}(M)]\!] - [\![\mathbb{H}(N)]\!] = [\![\Gamma_{M\oplus N^*,v}]\!] = [\![\mathbb{H}(L)]\!]$.

(c) Let $\mathbf{mirror}(K_\bullet, f_\bullet)$ be the ε-symmetric short complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1 & \xrightarrow{f^{\mathrm{t}}} & K_0^* & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle -k} & & \Big\downarrow{\scriptstyle \varepsilon k^*} & & \\ 0 & \longrightarrow & K_0 & \xrightarrow{-f} & K_1^* & \longrightarrow & 0 \end{array}$$

Pardon [24] (1.16) transforms $(K_\bullet, f_\bullet) \perp \mathbf{mirror}(K_\bullet, f_\bullet)$ into $\Gamma_{K_1,fk}$ by a homotopy and a quasi-isomorphism. So we get $[\![\mathbf{mirror}(K_\bullet, f_\bullet)]\!] = [\![\mathbb{H}(K_1)]\!] - [\![K_\bullet, f_\bullet]\!]$. With mirroring we can apply homotopies and quasi-isomorphisms to the columns of the diagram of $(K_\bullet, f_\bullet)$.

(d) A *sublagrangian* of an ε-symmetric complex $(K_\bullet, f_\bullet)$ is a morphism $i_\bullet : L_\bullet \rightarrowtail K_\bullet$ such that the components of $i_\bullet$ and of $f_\bullet i_\bullet$ are essential monomorphisms, and $i_\bullet^* f_\bullet i_\bullet = 0$. Then let $L_\bullet^\perp = \ker(i_\bullet^* f_\bullet : K_\bullet \to L_\bullet^*)$, and let $\overline{f}_\bullet : L_\bullet^\perp / L_\bullet \xrightarrow{\sim} (L_\bullet^\perp / L_\bullet)^*$ be the induced ε-symmetric quasi-isomorphism.

A sublagrangian of $(K_\bullet, f_\bullet)$ of the form $0 \to L_1 \to 0 \to 0$ corresponds to a sublagrangian of the mirror of the form $0 \to L_1 \to L_1 \to 0$, which is quasi-isomorphic to 0. So we have $[\![\mathbf{mirror}(K_\bullet, f_\bullet)]\!] = [\![\mathbf{mirror}(L_\bullet^\perp / L_\bullet, \overline{f}_\bullet)]\!]$. Combining this with the mirror formula (c) and with (b) gives $[\![K_\bullet, f_\bullet]\!] = [\![L_\bullet^\perp / L_\bullet, \overline{f}_\bullet]\!] + [\![\mathbb{H}(L_1)]\!]$ in this case (cf. Pardon [24] (1.18)).

If $(K_\bullet, f_\bullet)$ has a sublagrangian of the form $0 \to 0 \to L_0 \to 0$, then $(K_\bullet, f_\bullet) \perp \mathbb{H}(L_0)$ and the ε-symmetric complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1 \oplus L_0 & \xrightarrow{\left(\begin{smallmatrix} k & i \\ \varepsilon i^* f^{\mathrm{t}} & 0 \end{smallmatrix}\right)} & K_0 \oplus L_0^* & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} -\varepsilon f^{\mathrm{t}} & 0 \\ 0 & -\varepsilon \end{smallmatrix}\right)} & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)} & & \\ 0 & \longrightarrow & K_0^* \oplus L_0 & \longrightarrow & K_1^* \oplus L_0^* & \longrightarrow & 0 \end{array}$$

have homotopic mirrors and hence are in the same class. The latter ε-symmetric complex can be reduced to $(L_\bullet^\perp / L_\bullet, \overline{f}_\bullet)$ by a pair of quasi-isomorphisms. So we have $[\![K_\bullet, f_\bullet]\!] = [\![L_\bullet^\perp / L_\bullet, \overline{f}_\bullet]\!] - [\![\mathbb{H}(L_0)]\!]$ in this case.

For a general sublagrangian $0 \to L_1 \to L_0 \to 0$ one quotients out first by $0 \to 0 \to L_0 \to 0$ and then by $0 \to L_1 \to 0 \to 0$. We deduce the general sublagrangian formula $[\![K_\bullet, f_\bullet]\!] = [\![L_\bullet^\perp / L_\bullet, \overline{f}_\bullet]\!] + [\![\mathbb{H}(L_1)]\!] - [\![\mathbb{H}(L_0)]\!]$.

8. Formations

Another description of the odd Grothendieck-Witt groups of $\underline{\boldsymbol{E}}$ can be given using ε-*symmetric formations* (M, ϕ, L_1, L_2), which are ε-symmetric objects (M, ϕ) equipped with two specified Lagrangians $L_1 \rightarrowtail M$ and $L_2 \rightarrowtail M$. Then $GW^+_{\text{form}}(\underline{\boldsymbol{E}})$ and $GW^-_{\text{form}}(\underline{\boldsymbol{E}})$ are the quotients of the free abelian groups of isomorphism classes of ε-symmetric formations (for $\varepsilon = +1$ and $\varepsilon = -1$ respectively) by three kinds of relations (Karoubi [17] p. 370):

(I) $[\![(M, \phi, L_1, L_2) \perp (M', \phi', L'_1, M'_1)]\!] = [\![M, \phi, L_1, L_2]\!] + [\![M', \phi', L'_1, L'_2]\!]$,
(II) $[\![M, \phi, L_1, L_2]\!] + [\![M, \phi, L_2, L_3]\!] = [\![M, \phi, L_1, L_3]\!]$,
(III) if $L \rightarrowtail (M, \phi)$ is a common sublagrangian of L_1 and L_2, then one has $[\![M, \phi, L_1, L_2]\!] = [\![L^\perp/L, \overline{\phi}, L_1/L, L_2/L]\!]$. (A *common sublagrangian* is an essential monomorphism $L \rightarrowtail M$ which factors through both lagrangians $L \rightarrowtail L_1 \rightarrowtail M$ and $L \rightarrowtail L_2 \rightarrowtail M$.)

Note that relation (II) implies that $[\![M, \phi, L, L]\!] = 0$ and $[\![M, \phi, L_1, L_2]\!] = -[\![M, \phi, L_2, L_1]\!]$. The Witt groups $W^+_{\text{form}}(\underline{\boldsymbol{E}})$ and $W^-_{\text{form}}(\underline{\boldsymbol{E}})$ are the quotients of the Grothendieck-Witt groups by the subgroup generated by classes of the form $[\![L \oplus L^*, \left(\begin{smallmatrix} 0 & \varepsilon \\ \varpi & 0 \end{smallmatrix}\right), L, L^*]\!]$. There are maps

$$\begin{array}{rcl} \{\varepsilon\text{-symmetric short complexes}\} & \longrightarrow & \{(-\varepsilon)\text{-symmetric formations}\} \\ (K_\bullet, f_\bullet) & \longmapsto & \left(K_0 \oplus K_0^*, \left(\begin{smallmatrix} 0 & -\varepsilon \\ \varpi & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) K_0, \left(\begin{smallmatrix} k \\ f^{\mathrm{t}} \end{smallmatrix}\right) K_1\right). \end{array} \tag{16}$$

Theorem 8.1. *For any small exact category with duality $\underline{\boldsymbol{E}}$ containing $\frac{1}{2}$ the above map induces isomorphisms $GW^+_{\text{short}}(\underline{\boldsymbol{E}}) \cong GW^-_{\text{form}}(\underline{\boldsymbol{E}})$ and $GW^-_{\text{short}}(\underline{\boldsymbol{E}}) \cong GW^+_{\text{form}}(\underline{\boldsymbol{E}})$ and similarly for Witt groups.*

It is enough to establish $GW^-_{\text{short}}(\underline{\boldsymbol{E}}) \cong GW^+_{\text{form}}(\underline{\boldsymbol{E}})$ because the other isomorphism then follows if one changes the sign of the biduality maps in $\underline{\boldsymbol{E}}$. We study the map on Grothendieck-Witt groups induced by (16) by factoring it through two intermediate groups. The first group $GW^+_a(\underline{\boldsymbol{E}})$ has the same definition as $GW^+_{\text{form}}(\underline{\boldsymbol{E}})$ except that one imposes that for all formations (M, ϕ, L_1, L_2) appearing in the generators and the relations the first lagrangian L_1 should be a direct summand of M. There is a corresponding group $W^+_a(\underline{\boldsymbol{E}})$.

Lemma 8.2. *The map* (16) *induces isomorphisms $GW^-_{\text{short}}(\underline{\boldsymbol{E}}) \cong GW^+_a(\underline{\boldsymbol{E}})$ and $W^-_{\text{short}}(\underline{\boldsymbol{E}}) \cong W^+_a(\underline{\boldsymbol{E}})$.*

Proof. Since $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) K_0$ is a direct summand of $K_0 \oplus K_0^*$, (16) induces a map from the set of skew-symmetric short complexes into $GW^+_a(\underline{\boldsymbol{E}})$. We show that it factors through $GW^-_{\text{short}}(\underline{\boldsymbol{E}})$. It is certainly compatible with (i) orthogonal direct sums, while (ii) if $(K_\bullet, f_\bullet)$ and $(K_\bullet, f_\bullet + \partial(h_\bullet))$ are homotopic with $h^{\mathrm{t}} = -h$, then applying the automorphism $\left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right)$ to the symmetric object $\left(K_0 \oplus K_0^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)\right)$ and its lagrangians gives isomorphic formations and therefore equalities in $GW^+_a(\underline{\boldsymbol{E}})$

$$\begin{aligned} \left[K_0 \oplus K_0^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) K_0, \left(\begin{smallmatrix} k \\ f^{\mathrm{t}} \end{smallmatrix}\right) K_1\right] &= \left[K_0 \oplus K_0^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ h \end{smallmatrix}\right) K_0, \left(\begin{smallmatrix} k \\ f^{\mathrm{t}} + hk \end{smallmatrix}\right) K_1\right], \\ \left[K_0 \oplus K_0^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) K_0, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) K_0^*\right] &= \left[K_0 \oplus K_0^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ h \end{smallmatrix}\right) K_0, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) K_0^*\right]. \end{aligned}$$

Because of (II) the second equation gives $\left[K_0 \oplus K_0^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) K_0, \left(\begin{smallmatrix} 1 \\ h \end{smallmatrix}\right) K_0\right] = 0$, and then the first equation gives

$$\left[K_0 \oplus K_0^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) K_0, \left(\begin{smallmatrix} k \\ f^{\mathrm{t}} \end{smallmatrix}\right) K_1\right] = \left[K_0 \oplus K_0^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) K_0, \left(\begin{smallmatrix} k \\ f^{\mathrm{t}} + hk \end{smallmatrix}\right) K_1\right].$$

Since $f^{\mathrm{t}} + hk = (f - k^*h)^{\mathrm{t}}$, this means that $(K_\bullet, f_\bullet)$ and $(K_\bullet, f_\bullet + \partial(h_\bullet))$ have the same image in $GW_a^+(\underline{\boldsymbol{E}})$.

Now suppose (iii) that $s_\bullet : H_\bullet \xrightarrow{\sim} K_\bullet$ is a quasi-isomorphism. If $s_\bullet$ is an *epimorphic quasi-isomorphism*, i.e. if $s_0 : H_0 \twoheadrightarrow K_0$ and $s_1 : H_1 \twoheadrightarrow K_1$ are essential epimorphisms, then the kernel of $s_\bullet$ would be a sublagrangian of $(H_\bullet, s_\bullet^* f_\bullet s_\bullet)$ of the form $0 \to L \to L \to 0$. Then L is a common sublagrangian of the two lagrangians H_0 and H_1 of $\bigl(H_0 \oplus H_0^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)\bigr)$, and relation (III) shows that $(H_\bullet, s_\bullet^* f_\bullet s_\bullet)$ and $(K_\bullet, f_\bullet)$ have the same image in $GW_a^+(\underline{\boldsymbol{E}})$. In general $s_\bullet$ is not epimorphic, but there exist epimorphic quasi-isomorphisms $t_\bullet : G_\bullet \overset{\sim}{\twoheadrightarrow} H_\bullet$ (on the left) and $u_\bullet : G_\bullet \overset{\sim}{\twoheadrightarrow} K_\bullet$ (on the right) of short complexes such that $s_\bullet t_\bullet$ and $u_\bullet$ are homotopic.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1 \oplus K_1 & \xrightarrow{\left(\begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix}\right)} & H_0 \oplus K_1 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (1\ 0)} & & \Big\downarrow{\scriptstyle (1\ 0)} & & \\
0 & \longrightarrow & H_1 & \xrightarrow{h} & H_0 & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & H_1 \oplus K_1 & \xrightarrow{\left(\begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix}\right)} & H_0 \oplus K_1 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (s_1\ 1)} & & \Big\downarrow{\scriptstyle (s_0\ k)} & & \\
0 & \longrightarrow & K_1 & \xrightarrow{k} & K_0 & \longrightarrow & 0
\end{array}$$

Therefore $(H_\bullet, s_\bullet^* f_\bullet s_\bullet)$ has the same image in $GW_a^+(\underline{\boldsymbol{E}})$ as $(G_\bullet, t_\bullet^* s_\bullet^* f_\bullet s_\bullet t_\bullet)$, as $(G_\bullet, u_\bullet^* f_\bullet u_\bullet)$, and finally as $(K_\bullet, f_\bullet)$.

For (iv) a graph complex $\Gamma_{X,u}$ is sent to the formation $\bigl(X^* \oplus X^{**}, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) X^*, \left(\begin{smallmatrix} u \\ \varpi \end{smallmatrix}\right) X\bigr)$, which is isomorphic to $\bigl(X^* \oplus X, \left(\begin{smallmatrix} 0 & \varpi \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) X^*, \left(\begin{smallmatrix} u \\ 1 \end{smallmatrix}\right) X\bigr)$. Applying the automorphism $\left(\begin{smallmatrix} 1 & -u \\ 0 & 1 \end{smallmatrix}\right)$ of $\bigl(X^* \oplus X, \left(\begin{smallmatrix} 0 & \varpi \\ 1 & 0 \end{smallmatrix}\right)\bigr)$ gives an isomorphic formation $\bigl(X^* \oplus X, \left(\begin{smallmatrix} 0 & \varpi \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) X^*, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) X\bigr)$ which is independent of u. So $\Gamma_{X,u}$ and $\Gamma_{X,0} = \mathbb{H}(X)$ have the same image in $GW_a^+(\underline{\boldsymbol{E}})$.

We have now verified that (16) induces a well-defined map $GW^-_{\text{short}}(\underline{\boldsymbol{E}}) \to GW_a^+(\underline{\boldsymbol{E}})$. Moreover $\mathbb{H}(X)$ is sent to $\bigl(X^* \oplus X, \left(\begin{smallmatrix} 0 & \varpi \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) X^*, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) X\bigr)$, so we also have a well-defined map of Witt groups.

We now construct the inverse map $GW_a^+(\underline{\boldsymbol{E}}) \to GW^-_{\text{short}}(\underline{\boldsymbol{E}})$. Given a symmetric formation (M, ϕ, L_1, L_2) with L_1 a split lagrangian of (M, ϕ), then there exists a split lagrangian complementary to L_1 giving an identification $(M, \phi) \cong (L_1 \oplus L_1^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right))$. Then L_2 is identified with a lagrangian $\left(\begin{smallmatrix} k \\ f^{\mathrm{t}} \end{smallmatrix}\right) L_2$ of $(L_1 \oplus L_1^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right))$, and we send (M, ϕ, L_1, L_2) to the class of the skew-symmetric short complex

$$\begin{array}{ccccccc}
0 & \longrightarrow & L_2 & \xrightarrow{k} & L_1 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle -f^{\mathrm{t}}} & & \Big\downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & L_1^* & \xrightarrow{-k^*} & L_2^* & \longrightarrow & 0
\end{array}$$

If one chose a different complementary lagrangian, it would be of the form $\left(\begin{smallmatrix} u \\ 1 \end{smallmatrix}\right) L_1^*$ for some skew-symmetric $u : L_1^* \to L_1$, and the change from the new splitting to the old one would correspond to the action of the automorphism $\left(\begin{smallmatrix} 1 & -u \\ 0 & 1 \end{smallmatrix}\right)$. Thus in the new splitting L_2 would be identified with $\left(\begin{smallmatrix} k - uf^{\mathrm{t}} \\ f^{\mathrm{t}} \end{smallmatrix}\right) L_2$, and the associated short complex is in the same class in $GW^-_{\text{short}}(\underline{\boldsymbol{E}})$ as the short complex above because the mirrors are homotopic (cf. Formulas 7.2(c)). Thus the reverse map does not depend on the choice of the split lagrangian complementary to L_1.

The reverse map is (I) compatible with direct summands, and (III) if L_1 and L_2 have a common sublagrangian L, then the natural chain map from $0 \to L_2 \to L_1 \to 0$ to $0 \to L_2/L \to L_1/L \to 0$ is a quasi-isomorphism, and the formations (M, ϕ, L_1, L_2) and

$(L^\perp/L, \overline{\phi}, L_1/L, L_2/L)$ are mapped to the same class in $GW^-_{\text{short}}(\boldsymbol{E})$ because of the quasi-isomorphism relation (iii). For (II) calculations show that the formations $(M,\phi,L_1,L_2) \perp (M,\phi,L_2,L_3)$ and $(M,\phi,L_1,L_3) \perp (M,\phi,L_2,L_2)$ are sent to skew-symmetric short complexes whose mirrors are homotopic (cf. Formula (7.2)(c)), while of course (M,ϕ,L_2,L_2) is sent to the zero class.

Therefore the reverse assignment induces a well-defined map $GW^+_a(\underline{\boldsymbol{E}}) \to GW^-_{\text{short}}(\underline{\boldsymbol{E}})$. Since a formation $\left(L\oplus L^*, \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right), L, L^*\right)$ is sent to a skew-symmetric short complex isomorphic to $\mathbb{H}(L^*)$, we also have a well-defined map of Witt groups. The maps in the two directions are inverse to each other, proving the lemma. □

The second intermediate group $GW^+_b(\underline{\boldsymbol{E}})$ has the same definition as $GW^+_{\text{form}}(\underline{\boldsymbol{E}})$ except that one imposes that for all formations (M,ϕ,L_1,L_2) appearing in the generators and the relations the symmetric object (M,ϕ) should have a split lagrangian, and one imposes condition (III) only for sublagrangians common to L_1, L_2 and a split lagrangian of (M,ϕ). There is a corresponding group $W^+_b(\underline{\boldsymbol{E}})$.

Lemma 8.3. *The natural maps $GW^+_a(\underline{\boldsymbol{E}}) \to GW^+_b(\underline{\boldsymbol{E}})$ and $W^+_a(\underline{\boldsymbol{E}}) \to W^+_b(\underline{\boldsymbol{E}})$ are isomorphisms.*

Proof. If (M,ϕ) has two split lagrangians N_1 and N_2 and two other lagrangians L_1 and L_2, then in $GW^+_a(\underline{\boldsymbol{E}})$ one has

$$[M,\phi,N_1,L_i] - [M,\phi,N_2,L_i] = [M,\phi,N_1,N_2]$$

for $i=1$ and 2. It follows that the assignment

$$(M,\phi,L_1,L_2) \mapsto [M,\phi,N_j,L_2] - [M,\phi,N_j,L_1] \in GW^+_a(\underline{\boldsymbol{E}})$$

is independent of the choice of split lagrangian N_j. This assignment is compatible with the relations defining $GW^+_b(\underline{\boldsymbol{E}})$ and $W^+_b(\underline{\boldsymbol{E}})$ and defines maps $GW^+_b(\underline{\boldsymbol{E}}) \to GW^+_a(\underline{\boldsymbol{E}})$ and $W^+_b(\underline{\boldsymbol{E}}) \to W^+_a(\underline{\boldsymbol{E}})$ inverse to the natural maps in the other direction. □

The next two lemmas are adapted from Karoubi [17] Corollaire 2.5 and Théorème 2.6.

Lemma 8.4. *Let (M,ϕ,L_1,L_2) be a formation, let u_1 and u_2 be automorphisms of (M,ϕ), and let $u = u_1^{-1}u_2^{-1}u_1u_2$ be their commutator. Then we have $[M,\phi,L_1,L_2] = [M,\phi,u(L_1),L_2]$ in $GW^+_{\text{form}}(\underline{\boldsymbol{E}})$. The same relation holds in $GW^+_b(\underline{\boldsymbol{E}})$ if (M,ϕ) has a split lagrangian.*

Proof. We have

$$\begin{aligned}[M,\phi,u_1u_2(L_1),L_1] &= [M,\phi,u_1u_2(L_1),u_1(L_1)] + [M,\phi,u_1(L_1),L_1] \\ &= [M,\phi,u_2(L_1),L_1] + [M,\phi,u_1(L_1),L_1],\end{aligned}$$

i.e. the composition u_1u_2 is transformed into a sum. Consequently the commutator u is transformed into 0, i.e. $[M,\phi,u(L_1),L_1] = 0$. Since $[M,\phi,u(L_1),L_2] = [M,\phi,u(L_1),L_1] + [M,\phi,L_1,L_2]$, this proves the lemma. □

Lemma 8.5. *If L_1 and L_2 are lagrangians of (M,ϕ), then we have*

$$\begin{aligned}&[M^{\oplus 2}\oplus M^{\oplus 2}, \phi^{\oplus 2}\oplus(-\phi)^{\oplus 2}, (L_1\oplus L_2)\oplus N_1, N_2\oplus N_3] \\ &\qquad = [M^{\oplus 2}\oplus M^{\oplus 2}, \phi^{\oplus 2}\oplus(-\phi)^{\oplus 2}, (L_2\oplus L_1)\oplus N_1, N_2\oplus N_3]\end{aligned}$$

in $GW^+_b(\underline{\boldsymbol{E}})$ for any lagrangians N_1, N_2, and N_3 of $(M^{\oplus 2}, \phi^{\oplus 2})$.

Proof. It is enough to show that

$$\begin{aligned}&[M^{\oplus 3}\oplus M^{\oplus 3},\phi^{\oplus 3}\oplus(-\phi)^{\oplus 3},(L_1\oplus L_2\oplus L_1)\oplus(N_1\oplus L_1),(N_2\oplus L_1)\oplus(N_3\oplus L_1)]\\&\quad=[M^{\oplus 3}\oplus M^{\oplus 3},\phi^{\oplus 3}\oplus(-\phi)^{\oplus 3},(L_2\oplus L_1\oplus L_1)\oplus(N_1\oplus L_1),(N_2\oplus L_1)\oplus(N_3\oplus L_1)].\end{aligned}$$

But since the lefthand side may be obtained from the righthand side by letting a cyclic permutation of order 3 (thus a commutator of two transpositions) act on the first three factors of the first lagrangian, this follows from the previous lemma. □

The final stage of the proof of Theorem 8.1 is adapted from [17] Lemme 2.8.

Proof of Theorem 8.1. Because of Lemmas 8.2 and 8.3 it remains only to show that the natural maps $GW_b^+(\underline{\boldsymbol{E}})\to GW^+_{\mathrm{form}}(\underline{\boldsymbol{E}})$ and $W_b^+(\underline{\boldsymbol{E}})\to W^+_{\mathrm{form}}(\underline{\boldsymbol{E}})$ are isomorphisms. To construct the inverse maps consider the assignment

$$(M,\phi,L_1,L_2)\mapsto[M\oplus M,\phi\oplus(-\phi),L_1\oplus L_2,L_2\oplus L_2]\in GW_b^+(\underline{\boldsymbol{E}}).$$

It is compatible with direct sums (I), and it sends $(M,\phi,L_1,L_2)\perp(M,\phi,L_2,L_3)$ to

$$[M^{\oplus 2}\oplus M^{\oplus 2},\phi^{\oplus 2}\oplus(-\phi)^{\oplus 2},(L_1\oplus L_2)\oplus(L_2\oplus L_3),(L_2\oplus L_3)\oplus(L_2\oplus L_3)].$$

Because of Lemma 8.5 this class is the same as

$$\begin{aligned}&[M^{\oplus 2}\oplus M^{\oplus 2},\phi^{\oplus 2}\oplus(-\phi)^{\oplus 2},(L_2\oplus L_1)\oplus(L_2\oplus L_3),(L_2\oplus L_3)\oplus(L_2\oplus L_3)]\\&\qquad\qquad=[M\oplus M,\phi\oplus(-\phi),L_1\oplus L_3,L_3\oplus L_3]\end{aligned}$$

Thus the assignment is compatible with relation (II). If L is a common sublagrangian of L_1 and L_2, then $L\oplus L$ is the direct sum of the sublagrangians $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)L$ and $\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)L$, each of which is common to $L_1\oplus L_2$, $L_2\oplus L_2$, and to a split lagrangian $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)M$ or $\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)M$ of $(M\oplus M,\phi\oplus(-\phi))$. It follows that

$$\begin{aligned}&[M\oplus M,\phi\oplus(-\phi),L_1\oplus L_2,L_2\oplus L_2]\\&\qquad=[(L^\perp/L)\oplus(L^\perp/L),\overline{\phi}\oplus(-\overline{\phi}),(L_1/L)\oplus(L_2/L),(L_2/L)\oplus(L_2/L)].\end{aligned}$$

in $GW_b^+(\underline{\boldsymbol{E}})$. Consequently the assignment is compatible with relation (III), and it induces a map $GW^+_{\mathrm{form}}(\underline{\boldsymbol{E}})\to GW_b^+(\underline{\boldsymbol{E}})$ inverse to the natural map $GW_b^+(\underline{\boldsymbol{E}})\to GW^+_{\mathrm{form}}(\underline{\boldsymbol{E}})$. So the natural map is an isomorphism. Since the hyperbolic classes $[L\oplus L^*,\left(\begin{smallmatrix}0&1\\\varpi&0\end{smallmatrix}\right),L,L^*]$ in the two Grothendieck-Witt groups correspond under the natural map, this also induces an isomorphism of Witt groups $W_b^+(\underline{\boldsymbol{E}})\cong W^+_{\mathrm{form}}(\underline{\boldsymbol{E}})$. □

9. Strictly symmetric complexes

We return to the old theme of Witt classes as obstructions to being able to symmetrize strictly a complex which is symmetric up to quasi-isomorphism. In Theorem 9.5 we describe when one can strictly symmetrize such a complex while fixing its two ends. We apply this to strictly symmetric locally free resolutions of subcanonical subschemes (Theorem 9.6).

Let $\underline{\boldsymbol{E}}$ be an exact category with duality containing $\frac{1}{2}$, and let n be an integer. A *strictly symmetric complex* in $\mathrm{Ch}^{\mathrm{b}}(\underline{\boldsymbol{E}})[n]$ is a pair (C,u) with $C\in\mathrm{Ob}\,\mathrm{Ch}^{\mathrm{b}}(\boldsymbol{E})$ and $u:C\cong C^*[n]$ a chain isomorphism which is symmetric for the n-th shifted duality $(^*, 1, \varpi)[n]$. If n is even, then such a C is chain isomorphic to a complex

$$\begin{tikzcd}
 & & & C_{n/2} \arrow[dd, "\alpha"', "\cong"] \arrow[dr, "d_{n/2}"] & & & \\
\cdots \arrow[r, "d^*_{n/2-2}"] & C^*_{n/2-2} \arrow[r, "d^*_{n/2-1}"] & C^*_{n/2-1} \arrow[ur] \arrow[dr, "d^*_{n/2}"'] & & C_{n/2-1} \arrow[r, "d_{n/2-1}"] & C_{n/2-2} \arrow[r, "d_{n/2-2}"] & \cdots \\
 & & & C^*_{n/2} \arrow[ur] & & &
\end{tikzcd}$$

with $\alpha : C_{n/2} \cong C^*_{n/2}$ an isomorphism satisfying $\alpha^{\mathrm{t}} = (-1)^{n/2}\alpha$, and if n is odd, then such a C is chain isomorphic to a complex

$$\cdots \xrightarrow{d^*_{(n-3)/2}} C^*_{(n-3)/2} \xrightarrow{d^*_{(n-1)/2}} C^*_{(n-1)/2} \xrightarrow{\beta} C_{(n-1)/2} \xrightarrow{d_{(n-1)/2}} C_{(n-3)/2} \xrightarrow{d_{(n-3)/2}} \cdots$$

with $\beta^{\mathrm{t}} = (-1)^{\lfloor n/2 \rfloor}\beta$. The sign is the product of the sign $(-1)^n$ in the differentials in the shifted dual complex and the sign $(-1)^{\lceil n/2 \rceil}$ appearing in the shifted biduality maps.

Theorem 9.1. *If n is odd, then a symmetric object (X, ϕ) in $\mathrm{D^b}(\underline{\boldsymbol{E}})[n]$ (i.e. with the n-th shifted duality) is quasi-isomorphic to a strictly symmetric complex if and only if its Witt class $[X, \phi] \in W^n(\mathrm{D^b}(\underline{\boldsymbol{E}}))$ vanishes.*

Proof. If (X, ϕ) is quasi-isomorphic to a strictly symmetric complex (C, u), then since (C, u) has a Lagrangian $\cdots \to 0 \to C_{(n-1)/2} \to C_{(n-3)/2} \to \cdots$ in the complicial category with weak equivalences $(\mathrm{Ch^b}(\underline{\boldsymbol{E}}), \boldsymbol{w})$, we have $[X, \phi] = [C, u] = 0$. This uses the identification of Witt groups of Theorem 5.3.

Conversely, if $[X, \phi] = 0$ in the triangulated Witt group $W^n(\mathrm{D^b}(\underline{\boldsymbol{E}}))$, then by Balmer [4] Theorem 3.5 there exists morphism $h : L[-1] \to L^*[n]$ which is symmetric with respect to the $(n-1)$-st shifted duality such that $(X, \phi) \cong \mathrm{Cone}(L, h)$ in $\mathrm{D^b}(\underline{\boldsymbol{E}})[n]$. After making replacements using the calculus of fractions, we may assume that h is a chain map. The biduality maps in $\mathrm{Ch^b}(\underline{\boldsymbol{E}})$ are isomorphisms, so one sees from (15) that a surgery on a strictly symmetric complex in $\mathrm{Ch^b}(\underline{\boldsymbol{E}})$ yields a strictly symmetric complex. The cone complex $\mathrm{Cone}(L, h)$ is obtained by a surgery on the zero complex, so it is strictly symmetric. □

Before proving the corresponding theorem for even shifts we prove two preliminary results.

Lemma 9.2. *Let $(\underline{\boldsymbol{C}}, \boldsymbol{w})$ be a complicial category with weak equivalences and a differential graded duality containing $\frac{1}{2}$. If (X, ϕ) is a symmetric object in $\underline{\boldsymbol{C}}[\boldsymbol{w}^{-1}]$ and $i : L \to X$ is a morphism in $\underline{\boldsymbol{C}}[\boldsymbol{w}^{-1}]$ such that $i^*\phi i = 0$ in $\boldsymbol{C}[\boldsymbol{w}^{-1}]$, then there exists a diagram*

$$\begin{tikzcd}
 & Y \arrow[r, "\psi=\psi^{\mathrm{t}}", "\sim"'] \arrow[d, dotted, "\alpha"', "\cong"] & Y^* \arrow[dr, "j^*"] & \\
L \arrow[ur, "j"] \arrow[r, dotted, "i"'] & X \arrow[r, dotted, "\cong", "\phi=\phi^{\mathrm{t}}"'] & X^* \arrow[u, dotted, "\alpha^*", "\cong"'] \arrow[r, dotted, "i^*"'] & L^*
\end{tikzcd}$$

in which the solid arrows are morphisms in $\boldsymbol{C}$, the dotted arrows are morphisms in $\boldsymbol{C}[\boldsymbol{w}^{-1}]$, the diagram commutes in $\boldsymbol{C}[\boldsymbol{w}^{-1}]$, and $j^*\psi j = 0$ in $\boldsymbol{C}$.

Proof. With the calculus of fractions we can complete $L \to X \cong X^* \to L^*$ to a diagram

$$\begin{tikzcd}
M \arrow[r, "\beta"] \arrow[d, "f"', "\sim"] & Z \arrow[r, "\theta=\theta^{\mathrm{t}}", "\sim"'] \arrow[d, dotted, "\cong"] & Z^* \arrow[r, "\beta^*"] & M^* \\
L \arrow[r, dotted, "i"'] & X \arrow[r, dotted, "\cong", "\phi=\phi^{\mathrm{t}}"'] & X^* \arrow[r, dotted, "i^*"'] \arrow[u, dotted, "\cong"'] & L^* \arrow[u, "f^*", "\sim"']
\end{tikzcd}$$

of the same sort, with $f \in \boldsymbol{w}$ and with $-\beta^*\theta\beta = dh + hd$ for some symmetric homotopy h. Let (Y, ψ) be the symmetric object of $(\underline{\boldsymbol{C}}, \boldsymbol{w})$ constructed by algebraic surgery (see (15) above) along the composition $C(f)[-1] \twoheadrightarrow M \to (Z, \theta)$ using the homotopy induced by h. This means that Y is the total complex of the diagram

$$\begin{tikzcd}
& L^* \arrow[ddr, "f^*"] & \\
& \oplus & \\
M \arrow[r, "\beta"'] \arrow[ddr, "f"'] \arrow[rr, dotted, bend left=15, "h"] & Z \arrow[r, "\beta^*\theta"'] & M^*, \\
& \oplus & \\
& L &
\end{tikzcd}$$

so L is a subcomplex of Y, and the inclusion $j : L \hookrightarrow Y$ satisfies $j^*\psi j = 0$. Since $f : M \to L$ is in $\boldsymbol{w}$, one passes from Z to Y by adding complexes which are 0 in $\boldsymbol{C}[\boldsymbol{w}^{-1}]$, so Y and Z are isomorphic in $\boldsymbol{C}[\boldsymbol{w}^{-1}]$. $\square$

Proposition 9.3. *Let (X, ϕ) be a symmetric object in $\mathrm{D^b}(\underline{\boldsymbol{E}})[n]$, let $r \leq n/2 - 1$, and let $i : L \to X$ be a morphism in $\mathrm{D^b}(\underline{\boldsymbol{E}})$ whose mapping cone is acyclic in chain degrees $\leq r$ and such that L vanishes in chain degrees $> r$. Then (X, ϕ) is isomorphic in $\mathrm{D^b}(\underline{\boldsymbol{E}})[n]$ to a symmetric object coming from a symmetric chain map of the form*

$$\begin{tikzcd}
\cdots \arrow[r] & L^*_{r-1} \arrow[r] \arrow[d, equal, "1"'] & L^*_r \arrow[r] \arrow[d, equal, "1"'] & F_{n-r-1} \arrow[r] \arrow[d, "\pm f^{\mathrm{t}}_{r+1}"] & \cdots \arrow[r] & F_{r+1} \arrow[r] \arrow[d, "f_{r+1}"] & L_r \arrow[r] \arrow[d, "\varpi"', "\cong"] & L_{r-1} \arrow[r] \arrow[d, "\varpi"', "\cong"] & \cdots \\
\cdots \arrow[r] & L^*_{r-1} \arrow[r] & L^*_r \arrow[r] & F^*_{r+1} \arrow[r] & \cdots \arrow[r] & F^*_{n-r-1} \arrow[r] & L^{**}_r \arrow[r] & L^{**}_{r-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that i becomes identified with the inclusion of the subcomplex L if and only if $i^*\phi i = 0$ in $\mathrm{D^b}(\boldsymbol{E})$.

Proof. If (X, ϕ) is isomorphic to such a symmetric object with i identified with the inclusion, then $i^*\phi i$ becomes identified with a chain map from L to L^*. Since there is no degree in which both L and L^* are nonzero, this chain map vanishes, and $i^*\phi i = 0$.

Conversely, if $i^*\phi i = 0$ in $\mathrm{D^b}(\boldsymbol{E})$, then by Lemma 9.2 we may assume that ϕ and i are chain maps and that $i^*\phi i = 0$ as a chain map. Let (A, α) be the symmetric object of $(\mathrm{Ch^b}(\underline{\boldsymbol{E}}), \boldsymbol{w})$ constructed by surgery along $i : L \to (X, \phi)$. This surgery can be undone up to homotopy equivalence by another surgery along $L^*[-1] \to (A, \alpha)$: the first surgery transforms X into the total complex of the middle line of the following diagram, and the second surgery transforms it into the total complex of the entire diagram

$$\begin{tikzcd}
& L^* \arrow[ddr, "1"] & \\
& \oplus & \\
L \arrow[r, "i"] \arrow[ddr, "1"'] & X \arrow[r, "i^*\phi"'] & L^*. \\
& \oplus & \\
& L &
\end{tikzcd}$$

The first surgery constructs A which is the total complex of

$$\begin{tikzcd}
 & & & & & & L_r \arrow[r] \arrow[d, "i"'] & L_{r-1} \arrow[r] \arrow[d, "i"'] & \cdots \\
\cdots \arrow[r] & X_{n-r+1} \arrow[r] \arrow[d, "i^*\phi"'] & X_{n-r} \arrow[r] \arrow[d, "i^*\phi"'] & X_{n-r-1} \arrow[r] & \cdots \arrow[r] & X_{r+1} \arrow[r] & X_r \arrow[r] & X_{r-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & L^*_{r-1} \arrow[r] & L^*_r
\end{tikzcd}$$

It is acyclic in chain degrees $\leq r$ and, as in the proof of Theorem 6.1, it is therefore also also acyclic in chain degrees $\geq n-r$. The second surgery constructs the total complex of

$$\begin{tikzcd}
\cdots \arrow[r] & L^*_{r-2} \arrow[r] \arrow[d] & L^*_{r-1} \arrow[r] \arrow[d] & L^*_r \arrow[d] \\
\cdots \arrow[r] & A_{n-r+1} \arrow[r] & A_{n-r} \arrow[r, "d_{n-r}"] & A_{n-r-1} \arrow[r] & \cdots \arrow[r] & A_{r+1} \arrow[r, "d_{r+1}"] \arrow[d] & A_r \arrow[r] \arrow[d] & A_{r-1} \arrow[r] \arrow[d] & \cdots \\
 & & & & & L_r \arrow[r] & L_{r-1} \arrow[r] & L_{r-2} \arrow[r] & \cdots
\end{tikzcd}$$

Truncating off the acyclic portions of A and taking the total complex gives a complex quasi-isomorphic to X which is

$$\cdots \to L^*_{r-1} \to L^*_r \to \operatorname{coker} d_{n-r} \to A_{n-r-2} \to \cdots \to A_{r+2} \to \ker d_{r+1} \to L_r \to L_{r-1} \to \cdots$$

if $r < n/2 - 1$ and

$$\cdots \to L^*_{r-1} \to L^*_r \to H_{n/2}(A) \to L_r \to L_{r-1} \to \cdots \tag{17}$$

if $r = n/2-1$. One may treat X^* and ϕ in the same manner, and we get a symmetric complex of the required form quasi-isomorphic to (X, ϕ). □

Theorem 9.4. *For even n any symmetric object (X, ϕ) in $\mathrm{D^b}(\underline{\boldsymbol{E}})[n]$ is quasi-isomorphic to a strictly symmetric complex.*

Proof. After making replacements using the calculus of fractions we may assume that ϕ is a symmetric chain map. We then apply Proposition 9.3 to the brutal truncation

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X_{n/2-1} \arrow[r] \arrow[d, "1"] & X_{n/2-2} \arrow[r] \arrow[d, "1"] & \cdots \\
\cdots \arrow[r] & X_{n/2+2} \arrow[r] & X_{n/2+1} \arrow[r] & X_{n/2} \arrow[r] & X_{n/2-1} \arrow[r] & X_{n/2-2} \arrow[r] & \cdots ,
\end{tikzcd}$$

giving a symmetric complex as in (17). □

Theorems 9.1 and 9.4 can be combined with Proposition 9.3.

Theorem 9.5. *Let n, r, and $i : L \to (X, \phi)$ be as in Proposition 9.3. Suppose that $i^*\phi i = 0$ in $\mathrm{D^b}(\boldsymbol{E})$ and either* (i) *n is odd, and the Witt class $[X, \phi] \in W^n(\mathrm{D^b}(\underline{\boldsymbol{E}}))$ vanishes, or* (ii) *n is even. Then (X, ϕ) is isomorphic in $\mathrm{D^b}(\underline{\boldsymbol{E}})[n]$ to a strctly symmetric complex of the form*

$$\cdots \to L^*_{r-1} \to L^*_r \to C^*_{r+1} \to C^*_{r+2} \to \cdots \to C_{r+2} \to C_{r+1} \to L_r \to L_{r-1} \to \cdots .$$

Proof. First apply Proposition 9.3, and then apply Theorem 9.1 or 9.4 to the complex (F, f) consisting of $0 \to F_{n-r-1} \to \cdots \to F_{r+1} \to 0$ and the associated vertical arrows, getting a strictly symmetric complex (C, u). By the calculus of fractions, there exist quasi-isomorphisms $F \xleftarrow{\sim} G \xrightarrow{\sim} C$ such that the pullbacks of f and u to G are homotopic maps $G \xrightarrow{\sim} G^*$. Since C and G are quasi-isomorphic to F, they are acyclic in chain degrees $\leq r$ and $\geq n-r$, so truncating if necessary we may assume that C and G vanish in those degrees. If we embed $\boldsymbol{E}$ as a full exact subcategory of an abelian category, then the chain maps induce homology isomorphisms

$$\operatorname{coker}(F_{r+2} \to F_{r+1}) \cong \operatorname{coker}(G_{r+2} \to G_{r+2}) \cong \operatorname{coker}(C_{r+2} \to C_{r+1}),$$

and so there is a unique way to push forward and pull back the arrow uniting F with L so that L becomes united with G and with C and the unions are complexes.

$$\begin{tikzcd}
 & & C^*_{r+1} \arrow[r] & C^*_{r+2} \arrow[r] & \cdots \arrow[r] & C_{r+2} \arrow[r] & C_{r+1} \arrow[ddr, dotted] & & \\
 & & G_{n-r-1} \arrow[u] \arrow[d] \arrow[r] & G_{n-r-2} \arrow[u] \arrow[d] \arrow[r] & \cdots \arrow[r] & G_{r+2} \arrow[u] \arrow[d] \arrow[r] & G_{r+1} \arrow[u] \arrow[d] \arrow[dr, dotted] & & \\
\cdots \arrow[r] & L^*_r \arrow[r] \arrow[uur, dotted] \arrow[ur, dotted] & F_{n-r-1} \arrow[r] & F_{n-r-2} \arrow[r] & \cdots \arrow[r] & F_{r+2} \arrow[r] & F_{r+1} \arrow[r] & L_r \arrow[r] & \cdots
\end{tikzcd}$$

Similarly there is a unique way of pulling back to G and pushing forward to C the arrow uniting L^* with F so that the unions of L^* with G and C are complexes. Moreover, since the old attachment arrows were compatible with $f : F \xrightarrow{\sim} F^*$, the new attachment arrows are compatible with $u : C \cong C^*$. So (C, u) with L and L^* attached is a strictly symmetric complex. $\square$

We give an application of Theorem 9.5. Let X be a scheme with $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$, and let $Z \subset X$ be a closed subscheme of codimension d. We ask: *When does $\mathcal{O}_Z$ have a strictly symmetric locally free resolution of length d the form*

$$0 \to L \to \mathcal{F}_1^* \otimes L \to \mathcal{F}_2^* \otimes L \to \cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{O}_X \to \mathcal{O}_Z \to 0, \tag{18}$$

i.e. strictly symmetric of length d, with $\mathcal{O}_X$ in degree 0, and with a line bundle L in degree d?

In codimension 2 such a resolution has the form $0 \to L \xrightarrow{\psi} \mathcal{F} \xrightarrow{\psi^*\alpha} \mathcal{O}_X \to \mathcal{O}_Z \to 0$, with $\operatorname{rk}\mathcal{F} = 2$, with $L = \det F$, and $\alpha : \mathcal{F} \cong \mathcal{F}^* \otimes L$ the natural alternating isomorphism. Thus Z is the zero locus of a section of a rank 2 vector bundle. In codimension 3 the form is

$$0 \to L \xrightarrow{\operatorname{Pf}(\phi)^*} \mathcal{F}^* \otimes L \xrightarrow{\phi} \mathcal{F} \xrightarrow{\operatorname{Pf}(\phi)} \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

with ϕ alternating. The structure theorem of Buchsbaum-Eisenbud [12] then implies that $\mathcal{F}$ is a vector bundle of odd rank $2n+1$ and that the map $\mathcal{F} \to \mathcal{O}_X$ is the vector of Pfaffians of order $2n$ of ϕ. Such a Z is called a *Pfaffian subscheme*, and they have been studied by Walter [33] and Eisenbud-Popescu-Walter [13] [14].

To answer the question we use several conditions (cf. [13]):

(A) The $\mathcal{O}_X$-module $\mathcal{O}_Z$ should be of finite local projective dimension.

(B) The subscheme Z should be *subcanonical of codimension d*. This means:

  (i) Z is *relatively Cohen-Macaulay of codimension* d in X, i.e. $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X) = 0$ for all $i \neq d$, and

(ii) there exists a line bundle L on X such that the relative canonical sheaf $\omega_{Z/X} := \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X)$ is isomorphic to the sheaf of sections of $L^{-1}|_Z$.

Condition (B) implies that there is a symmetric isomorphism $\eta : \mathcal{O}_Z \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, L[d])$, so it gives us a derived Witt class $[\mathcal{O}_Z, \eta] \in W^d(X, L)$. Moreover, $1 \in \Gamma(X, \mathcal{O}_Z)$ corresponds to an $\overline{\eta} \in \mathrm{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_Z, L) = \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(\mathcal{O}_X, L[d])$.

(C) *Restriction condition.* The composition $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z \xrightarrow{\overline{\eta}} L[d]$ vanishes in $\mathrm{H}^d_{\mathrm{Zar}}(X, L) = \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(\mathcal{O}_X, L[d])$.

(D) *Witt condition.* Either (i) d is odd, and $[\mathcal{O}_Z, \eta] \in W^d(X, L)$ vanishes, or (ii) d is even.

Theorem 9.6. *Let $Z \subset X$ be a closed subscheme of codimension d in a noetherian scheme with $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. Then $\mathcal{O}_Z$ has a strictly symmetric locally free resolution of the form* (18) *if and only if conditions* (A)–(D) *above all hold.*

Proof. Condition (A) implies that $\mathcal{O}_Z$ has a finite locally free resolution, and condition (B)(i) that the resolution can be of length d. One can always put $\mathcal{O}_X$ in degree 0, and under condition (B)(ii) one can put L in degree d. According to Theorem 9.5 the restriction condition (C) allows one to put $\mathcal{O}_X$ in degree 0 and L in degree d simultaneously, and the Witt condition (D) then allows one to strictly symmetrize the rest of the resolution. $\square$

Conditions (A)–(C) are taken from Eisenbud-Popescu-Walter [13] pp. 428–429. The Restriction Condition (C) was mentioned earlier in codimension 2 in Griffiths-Harris [15] Proposition 1.33, Vogelaar [31] Theorem 2.1, and Bănică-Putinar [8] §2.1. The Witt condition (D) appears first in Walter [33] for codimension 3 subschemes of projective space. (An arbitrary ground field is discussed on the bottom of p. 674.) Pardon has also discussed the Witt condition in codimension 3 ([25] Proposition 0.18).

10. Grothendieck-Witt groups of schemes

In this section we define the derived Grothendieck-Witt groups of a scheme, and we describe how the deformation invariance of Kervaire semicharacteristics really amounts to having maps $GW^{4n-1}(X, L) \to \mathrm{H}^0(X_{\mathrm{Zar}}, \mathbb{Z}/2)$ (Theorem 10.2).

Let X be a scheme such that $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$, and let L be a line bundle over X. The exact category $\mathbf{VB}_X$ of algebraic vector bundles on X has a (twisted) duality functor $\mathcal{E} \mapsto \mathcal{E}^\natural := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}, L)$ and biduality isomorphisms $\varpi_{\mathcal{E}} : \mathcal{E} \cong \mathcal{E}^{\natural\natural}$ which are the usual evaluation maps. We write $GW^n(X, L) = GW^n(\mathrm{D^b}(\mathbf{VB}_X), {}^\natural, 1, \varpi)$ and call these the *derived Grothendieck-Witt groups* of (X, L). The derived Witt groups are defined similarly. When the duality is untwisted, i.e. when $L = \mathcal{O}_X$, we often write * instead of ${}^\natural$, and we write $GW^n(X) = GW^n(X, \mathcal{O}_X)$ and $W^n(X) = W^n(X, \mathcal{O}_X)$. If $f : Y \to X$ is a morphism of schemes, then the pullback $f^* : \mathrm{D^b}(\mathbf{VB}_X) \to \mathrm{D^b}(\mathbf{VB}_Y)$ can be made into a duality-preserving functor yielding pullback morphisms $f^* : W^n(X, L) \to W^n(Y, f^*L)$.

Stieffel-Whitney classes are defined on $GW(X)$ and so can be (and have been) extended to the isomorphic groups $GW^{4n}(X)$ by periodicity. Two other invariants are the Kervaire semicharacteristic and the Pfaffian line bundle

$$\mathrm{semi} : GW^{4n-1}(X, L) \to \mathrm{H}^0(X_{\mathrm{Zar}}, \mathbb{Z}/2), \qquad \mathrm{Pf} : GW^{4n-1}(X) \to \mathrm{Pic}(X).$$

For the Pfaffian line bundle see Laszlo-Sorger [21] §7. One may see that the line bundle they associate to a skew-symmetric short complex of vector bundles (for the untwisted duality) only depends on the class in $GW^-_{\mathrm{short}}(\mathbf{VB}_X, {}^*, 1, \varpi) = GW^{4n-1}(X)$.

The Kervaire semicharacteristic of a symmetric complex of vector bundles on a scheme was studied by Kempf [20]. Looking at Kempf's arguments in light of the current paper, one sees that they are really based on the calculation of the derived Witt and Grothendieck-Witt groups of a local ring. The former were computed by Balmer [5] Theorem 5.6, who found:

$$W^0(R) = W(R), \qquad W^1(R) = 0, \qquad W^2(R) = 0, \qquad W^3(R) = 0. \tag{19}$$

Theorem 10.1. *The derived Grothendieck-Witt groups of a commutative local ring R in which 2 is invertible are*

$$GW^0(R) = GW(R), \qquad GW^1(R) = 0, \qquad GW^2(R) = \mathbb{Z}, \qquad GW^3(R) = \mathbb{Z}/2$$

Moreover, if $R \to S$ is a morphism of commutative local rings in which 2 is invertible, then the base change maps $GW^i(R) \to GW^i(S)$ are isomorphisms for $i \not\equiv 0 \pmod 4$.

Thus given X and L and a point $x \in X$ there are restriction maps $\mathrm{semi}_x : GW^{4n-1}(X, L) \to GW^{4n-1}(k(x)) = \mathbb{Z}/2\mathbb{Z}$.

Theorem 10.2 (Invariance of semicharacteristics). *For any class $\gamma \in GW^{4n-1}(X, L)$ the function $X \to \mathbb{Z}/2\mathbb{Z}$ given by $x \mapsto \mathrm{semi}_x(\gamma)$ is locally constant in the Zariski topology. Thus the* semi_x *define a function* $\mathrm{semi} : GW^{4n-1}(X, L) \to \mathrm{H}^0(X_{\mathrm{Zar}}, \mathbb{Z}/2)$.

Proof of Theorem 10.1. Theorem 6.1 gives $GW^0(R) = GW(R)$ and $GW^2(R) = GW^-(R)$, and the latter is an infinite cyclic group generated by the class of the hyperbolic plane. The Fundamental Theorem 2.6 yields

$$GW^{2n}(R) \xrightarrow{\text{forget}} K_0(R) \xrightarrow{\mathbf{H}} GW^{2n+1}(R) \to W^{2n+1}(R) \to 0.$$

Since $W^{2n+1}(R) = 0$ by (19), and since the forgetful map into $K_0(R) = \mathbb{Z}$ measures the rank of a symmetric bilinear or symplectic module, we get $GW^1(R) = 0$ and $GW^3(R) = \mathbb{Z}/2$. All the computations except of $GW^0(R)$ depend only on ranks and so are invariant under base change. $\square$

The class in $GW^{4n-1}(R) = \mathbb{Z}/2$ of a symmetric object (B, ψ) in $(\mathrm{Ch}^{\mathrm{b}}(\mathbf{VB}_R), \boldsymbol{w}, {}^*, 1, \varpi)[4n-1]$ is computed by noting that $W^{4n-1}(R) = 0$, so by Theorem 9.1 (B, ψ) is homotopy equivalent to a strictly symmetric complex

$$\cdots \xrightarrow{d_{2n+2}} C_{2n+1} \xrightarrow{d_{2n+1}} C_{2n} \xrightarrow{\beta = -\beta^{\mathrm{t}}} C_{2n}^* \xrightarrow{d_{2n+1}^*} C_{2n+1}^* \xrightarrow{d_{2n+2}^*} \cdots.$$

Its class is then $\sum_{i \geq 2n} (-1)^i [\![\mathbf{H}(C_i)]\!]$, so we can identify $[\![B, \psi]\!] \equiv \sum_{i \geq 2n} \operatorname{rk} C_i$ in $\mathbb{Z}/2$. If $R \to F$ is a morphism to a field, then the rank of $\beta \otimes F$ is even, and we get $[\![B, \psi]\!] \equiv \sum_{i \geq 2n} \dim_F H_i(B \otimes_R F)$ in $\mathbb{Z}/2$.

Proof of Theorem 10.2. Let (A, ϕ) be a symmetric object of $(\mathrm{Ch}^{\mathrm{b}}(\mathbf{VB}_X), \boldsymbol{w}^{\natural}, 1, \varpi)[4n-1]$ such that $\gamma = [\![A, \phi]\!]$. Restricting to the local ring of $x \in X$ gives a symmetric complex $(A, \phi) \otimes \mathcal{O}_{X,x}$ which is quasi-isomorphic to a strictly symmetric bounded complex (B_x, ψ_x) of free $\mathcal{O}_{X,x}$-modules. There is some affine open neighborhood U of x where this extends to a quasi-isomorphism between $(A, \phi) \otimes \mathcal{O}_U$ and a strictly symmetric bounded complex (B_U, ψ_U) of free $\mathcal{O}_U$-modules. Since free modules over a commutative ring have a well-defined rank which is invariant under base change, the semicharacteristic of γ is constant on U. $\square$

Thus when one develops direct images for derived Grothendieck-Witt groups, one will be able to give algebraic proofs of the deformation invariance of theta-characteristics of algebraic curves (cf. Mumford [22]) and of the Atiyah-Rees invariant of rank-two vector bundles on $\mathbb{P}^3$ with $c_1 = 0$ (cf. Hartshorne [16]) which follow exactly the natural lines of the analytic/topological proofs (Atiyah [1] Theorem 1, Atiyah-Rees [2] Theorem 4.2).

11. The punctured spectrum of a regular local ring

As an application of the Localization Theorem 2.4 we calculate the derived Grothendieck-Witt groups of the punctured spectrum of a regular local ring. We follow the same general plan used for Witt groups in Balmer-Walter [7] §9. First we discuss the Gersten complex.

Theorem 11.1 (Purity and the Gersten complex). *Let R be an equicharacteristic regular local ring not of characteristic 2. Then there is an exact sequence*

$$0 \to GW(R) \xrightarrow{i} GW(K) \xrightarrow{\partial^0} \bigoplus_{\mathrm{ht}\,\mathfrak{p}=1} W(k(\mathfrak{p}), \omega_{k(\mathfrak{p})/R_\mathfrak{p}}) \xrightarrow{\partial^1} \bigoplus_{\mathrm{ht}\,\mathfrak{p}=2} W(k(\mathfrak{p}), \omega_{k(\mathfrak{p})/R_\mathfrak{p}}) \xrightarrow{\partial^2} \cdots .$$

Here K is the quotient field of R, the $\mathfrak{p}$ are the prime ideals of R, and the $k(\mathfrak{p})$ the residue fields, while i is the base-change map for the inclusion $R \hookrightarrow K$, the map ∂^0 is the direct sum of the second residue maps, and the later ∂^i are the same maps as in the Gersten-Witt complex of Balmer-Walter [7] §8. The Witt groups have coefficients in the one-dimensional vector spaces $\omega_{k(\mathfrak{p})/R_\mathfrak{p}} := \mathrm{Ext}^{\dim R_\mathfrak{p}}_{R_\mathfrak{p}}(k(\mathfrak{p}), R_\mathfrak{p})$, the duals of the $k(\mathfrak{p})$ as finite-length $R_\mathfrak{p}$-modules. Theorem 11.1 is deduced from the corresponding theorem for Witt groups, see for instance Balmer-Gille-Panin-Walter [6].

Theorem 11.2. *If $(R, \mathfrak{m}, k)$ is an equicharacteristic regular local ring of dimension $r \geq 2$ of characteristic not 2, and if $U = \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ is its punctured spectrum, then we have*

$$GW^i(U) = \begin{cases} GW^i(R) & \text{if } i \not\equiv r-1 \pmod 4, \\ GW^i(R) \oplus W(k, \omega_{k/R}) & \text{if } i \equiv r-1 \pmod 4. \end{cases}$$

Proof. Localization arguments analogous to those used for Witt groups in Balmer-Walter [7] Theorem 9.1 yield long exact sequences which are now of the form

$$GW^{i-r}(k, \omega_{k/R}) \to GW^i(R) \to GW^i(U) \to W^{i-r+1}(k, \omega_{k/R}) \to W^{i+1}(R) \to \cdots .$$

The first map $GW^{i-r}(k, \omega_{k/R}) \to GW^i(R)$ may be calculated explicitly by lifting a symmetric complex of k-modules to R and tensoring with a minimal free resolution of the R-module $k = R/\mathfrak{m}$. To show that this map vanishes, it is enough to show that the class in $GW^r(R)$ of this minimal resolution vanishes. But the minimal free resolution may be written in the form of a Koszul complex $0 \to \Lambda^r F \to \cdots \to F \to R \to 0$, where F is a free module of rank r. We will show that this Koszul complex is metabolic with a lagrangian subcomplex of rank ρ to be determined, which implies that its class is the image of $\rho \in \mathbb{Z}$ under the hyperbolic map $\mathbf{H} : \mathbb{Z} = K_0(R) \to GW^r(R)$. To show that this class vanishes, it is enough to show that $\rho = 0$ when r is even and that $\rho \equiv 0 \pmod 2$ when $r \equiv 3 \pmod 4$ (cf. Theorem 10.1). But when r is odd, one may take the righthand half of the complex $0 \to \Lambda^{(r-1)/2} F \to \cdots \to F \to R \to 0$ as a Lagrangian subcomplex, and its rank $\rho = \sum_{i=0}^{(r-1)/2} (-1)^i \binom{r}{i}$ is congruent modulo 2 to $\sum_{i=0}^{(r-1)/2} \binom{r}{i} = 2^{r-1}$. If r is even, then write $F = R \oplus G$ with G free of rank $r-1$, and one may take as a Lagrangian subcomplex the righthand part of the resolution plus a Lagrangian submodule of the middle module $0 \to \Lambda^{r/2} G \to \Lambda^{r/2-1} F \to \cdots \to F \to R \to 0$. Its rank ρ is half the rank of the entire resolution, so we have $\rho = \frac{1}{2} \sum_{i=0}^{r} (-1)^i \binom{r}{i} = 0$.

Thus the first map $GW^{i-r}(k,\omega_{k/R}) \to GW^i(R)$ in the localization sequence vanishes, and the fourth map $W^{i-r+1}(k,\omega_{k/R}) \to W^{i+1}(R)$ vanishes by a similar argument. So the localization sequences reduce to short exact sequences

$$0 \to GW^i(R) \to GW^i(U) \to W^{i-(r-1)}(k,\omega_{k/R}) \to 0.$$

From Balmer's calculation (19) we see that the base-change map $GW^i(R) \to GW^i(U)$ is an isomorphism when $i \not\equiv r-1 \pmod 4$, and the remaining case becomes an exact sequence

$$0 \to GW^{r-1}(R) \to GW^{r-1}(U) \to W(k,\omega_{k/R}) \to 0.$$

In the subcase $r \not\equiv 1 \pmod 4$ the composite base-change map $GW^{r-1}(R) \to GW^{r-1}(U) \to GW^{r-1}(K)$ is an isomorphism by Theorem 10.1, so $GW^{r-1}(R) \to GW^{r-1}(U)$ is a split injection. For $r \equiv 1 \pmod 4$ but $r > 1$, all the prime ideals $\mathfrak{p} \subset R$ of height 1 lie in U, so the image of the base-change map $GW^0(U) \to GW^0(K)$ is contained in $\ker\big(GW^0(K) \to \bigoplus_{\mathrm{ht}\,\mathfrak{p}=1} W(k(\mathfrak{p}))\big)$ by the same argument as for Witt groups. The composition

$$GW^0(R) \to GW^0(U) \to \ker\Big(GW^0(K) \to \bigoplus_{\mathrm{ht}\,\mathfrak{p}=1} W(k(\mathfrak{p}))\Big)$$

is an isomorphism by Theorem 11.1, so $GW^0(R) \to GW^0(U)$ is a split injection. In any case $GW^{r-1}(R) \to GW^{r-1}(U)$ is a split injection when $r > 1$. □

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