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An algebraic formulation of surgery

by
Andrew Ranicki

Introduction

A systematic attempt at the homotopy classification of compact manifolds led C.T.C. Wall to develop a surgery obstruction theory, which reduces the geometry to the K-theory of quadratic forms, that is L-theory. Here, we present a reformulation of the algebra, in terms of the hamiltonian formalism due to S.P. Novikov.

The paper is divided into three parts

Part I "Foundations of L-theory"

Functors

$$U_*, V_*, W_* : \text{rings with involution} \longrightarrow \mathbb{Z}_4\text{-graded abelian groups}$$

are defined, such that

$\begin{cases} U_n(A) \\ V_n(A) \\ W_n(A) \end{cases}$ is the L-group associated with

quadratic forms on $\begin{cases} \text{f.g. projective} \\ \text{f.g. free} \\ \text{based} \end{cases}$ A-modules.

Exact sequences

$$\ldots \longrightarrow \Sigma_{(-)^{n+1}}(A) \longrightarrow V_n(A) \longrightarrow U_n(A) \longrightarrow \Sigma_{(-)^n}(A) \longrightarrow \ldots$$
$$\ldots \longrightarrow \Omega_{(-)^{n+1}}(A) \longrightarrow W_n(A) \longrightarrow V_n(A) \longrightarrow \Omega_{(-)^n}(A) \longrightarrow \ldots \qquad (*)$$

are established, where $\begin{cases} \Sigma_\pm(A) \\ \Omega_\pm(A) \end{cases}$ are the Tate

$\mathbb{Z}_2$-cohomology groups of $\begin{cases} \tilde{K}_0(A) \\ \tilde{K}_1(A) \end{cases}$.

Part II "Algebraic L-theory"

Denoting the Laurent extension of A by A_z, natural direct sum decompositions.

$$V_n(A_z) = V_n(A) \oplus U_{n-1}(A)$$
$$W_n(A_z) = W_n(A) \oplus V_{n-1}(A) \qquad (**)$$

are established, and generalized to Laurent extension in several variables.

Part III "Geometric L-theory"

Functors

$\mathcal{L}_f$: rings with involution $\longrightarrow$ Kan Δ-sets (***)

such that

$$\pi_n(\mathcal{L}_f(A)) = U_{n+f}(A)$$
$$\Omega\mathcal{L}_f(A) \simeq \mathcal{L}_{f+1}(A),$$

are defined for $f \pmod 4$. There are then analogous versions for V, W-theories.

The exact sequences (*) and the direct sum decompositions (**) had been previously proved by geometrical methods (applicable for group rings of finitely presented groups) in

J. L. Shaneson "Wall's surgery obstruction groups for $G \times \mathbb{Z}$" Ann. of Maths. 90 (1969), 296-33[illegible]

[W]: C.T.C. Wall "Surgery on compact manifolds" (Academic Press [illegible])

[illegible] mark

The neat formulation of Theorem 3.1 of Part I is due to Andrew Casson.

Our method of proof of (* *) is a simplification - cum - generalization of

S.P. Novikov "Algebraic construction and properties of hermitian analogues of K-theories over rings with involution, from the point of view of hamiltonian formalism. Some applications to differential topology and the theory of characteristic classes".
Izv. Akad. Nauk S.S.S.R. ser mat. (1970) {I. 253-288, II. 475-500}

where a similar result is proved, assuming $\frac{1}{2} \in A$, and modulo 2-torsio

The functors (* * *) are an algebraic version of the "surgery spaces" constructed geometrically (for group rings) in

F. Quinn "A geometric formulation of surgery"

Their study was suggested by Andrew Casson.

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The work is original modulo the above acknowledgements

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The work is dedicated to the memory of J.T. Knight

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I. Foundations of L-theory

§0. Conventions

Let A be an associative ring with 1, and with an involution, that is a function

$$\bar{\ }: A \longrightarrow A;\ a \longmapsto \bar{a}$$

such that

i) $\bar{1} = 1$

ii) $\overline{(a+b)} = \bar{a} + \bar{b}$

iii) $\overline{ab} = \bar{b}.\bar{a}$

iv) $\bar{\bar{a}} = a$

for all $a, b \in A$.

It is further required that finitely generated (f.g.) free A-modules have well-defined dimension.

Example 0.1 The group ring $\mathbb{Z}[\pi]$ of a multiplicative group π, with involution

$$\bar{\ }: \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi];\ \sum_{g\in\pi} n_g\, g \longmapsto \sum_{g\in\pi} w(g)\, n_g\, g^{-1}$$

defined by a morphism

$$w: \pi \longrightarrow \mathbb{Z}_2 = \{1,-1\},$$

satisfies these conditions. □

(This is the ground ring occurring in topology, with π the fundamental group $\pi_1(M)$ of a compact manifold M, and $w: \pi_1(M) \to \mathbb{Z}_2$ the first Stiefel-Whitney class cf. [W])

We shall be dealing with left A-modules, $M, N, P, Q \ldots$

Denote by $\mathrm{Hom}_A(M,N)$ the additive group of A-module morphisms $f: M \longrightarrow N$.

Given M, define the dual A-module, M^*, to be $\mathrm{Hom}_A(M,A)$, with A acting by

$$A \times \mathrm{Hom}_A(M,A) \longrightarrow \mathrm{Hom}_A(M,A);\ (a,f) \longmapsto (x \longmapsto f(x).\bar{a})$$

Accordingly, given $f \in \mathrm{Hom}_A(M,N)$ define the dual morphism

$$f^*: N^* \longrightarrow M^*;\ g \longmapsto (x \longmapsto g(f(x)))$$

in $\mathrm{Hom}_A(N^*, M^*)$.

Morphisms in $\mathrm{Hom}_A(M\oplus N, P\oplus Q)$ can be displayed as matrices

$$f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : M \oplus N \longrightarrow P \oplus Q \,;$$
$$(x,y) \longmapsto (\alpha(x) + \beta(y), \gamma(x) + \delta(y))$$

with $\alpha \in \mathrm{Hom}_A(M,P)$, $\beta \in \mathrm{Hom}_A(N,P)$, $\gamma \in \mathrm{Hom}_A(M,Q)$, and $\delta \in \mathrm{Hom}_A(N,Q)$. Composition of such morphisms corresponds to right multiplication of the matrices. The morphism dual to f (as above) has matrix

$$f^* = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} : P^* \oplus Q^* \longrightarrow M^* \oplus N^* \,,$$

identifying $(M \oplus N)^*$ with $M^* \oplus N^*$ in the obvious way.

If Q is a f.g. free A-module, with base $f = (f_1, f_2, \ldots, f_m)$, then Q^* is a f.g. free A-module of the same dimension, with <u>dual</u> base $f^* = (f_1^*, f_2^*, \ldots, f_m^*)$ given by

$$f_i^*(f_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} . \end{cases}$$

It follows that if P is a f.g. projective A-module, so is P^*, though P and P^* are not in general isomorphic. However, the natural map

$$P \longrightarrow P^{**} \,;\; x \longmapsto (f \longmapsto \overline{f(x)})$$

is an isomorphism. It is used to identify P^{**} with P, whenever P is f.g. projective. In particular, given a morphism $f \in \mathrm{Hom}_A(Q, P^*)$ with P f.g. projective, we can write

$$f^* : P \longrightarrow Q^* \,;\; x \longmapsto (y \longmapsto \overline{f(y)(x)})$$

We shall make much use of the contravariant functors

$$\Pi, \Pi_+, \Pi_- : (\text{f.g. projective } A\text{-modules}) \longrightarrow (\text{abelian groups})$$

and natural transformations

$$\Pi_+ \rightsquigarrow \Pi \,,\; \Pi_- \rightsquigarrow \Pi$$

defined on objects by

$$\Pi(Q) = \mathrm{Hom}_A(Q,Q^*)$$

$$\Pi_\pm(Q) = \mathrm{Hom}_A(Q,Q^*) \big/ \{\chi \mp \chi^* \mid \chi \in \mathrm{Hom}_A(Q,Q^*)\}$$

$$\Pi_\pm(Q) \longrightarrow \Pi(Q)\,;\ [\phi] \longmapsto (\phi \pm \phi^*)$$

and on morphisms $f \in \mathrm{Hom}_A(P,Q)$ by

$$\Pi(f) : \Pi(Q) \longrightarrow \Pi(P)\,;\ \phi \longmapsto f^*\phi f$$

$$\Pi_\pm(f) : \Pi_\pm(Q) \longrightarrow \Pi_\pm(P)\,;\ [\phi] \longmapsto [f^*\phi f] .$$

§ 1 Forms

A <u>$\pm$ form</u> (over A), (Q,ϕ), is a f.g. projective A-module Q, together with $\phi \in \Pi(Q) = \mathrm{Hom}_A(Q,Q^*)$, and a choice of sign.

A <u>morphism</u> of $\pm$ forms

$$(f,\chi) : (P,\theta) \longrightarrow (Q,\phi)$$

is defined by $f \in \mathrm{Hom}_A(P,Q)$, $\chi \in \Pi_\mp(P)$ such that

$$f^*\phi f - \theta = \chi \mp \chi^* \in \mathrm{Hom}_A(P,P^*).$$

The composite of $\pm$ form morphisms

$$(f,\chi) : (P,\theta) \longrightarrow (Q,\phi)\ ,\quad (g,\nu) : (Q,\phi) \longrightarrow (R,\psi)$$

is the morphism

$$(g,\nu)(f,\chi) = (gf, \chi + f^*\nu f) : (P,\theta) \longrightarrow (R,\psi).$$

We thus have a category of $\pm$ forms (or rather two such, one for each choice of sign). There is a direct sum operation,

$$(P,\theta) \oplus (Q,\phi) = (P \oplus Q, \theta \oplus \phi)$$

with $(0,0) =: 0$.

A morphism of $\pm$ forms
$$(f,\chi):(P,\theta)\longrightarrow(Q,\phi)$$
is an isomorphism precisely when $f\in \mathrm{Hom}_A(P,Q)$ is an A-module isomorphism. For example,
$$(1,\chi):(Q,\phi)\longrightarrow(Q,\phi+\chi\mp\chi^*)$$
is a $\pm$ form isomorphism for all $\pm$ forms (Q,ϕ), and $\chi\in\Pi_{\mp}(Q)$.

In general, we shall be interested in $\pm$ forms up to isomorphism only.

For every $\pm$ form (Q,ϕ), $\phi\pm\phi^*\in \mathrm{Hom}_A(Q,Q^*)$ defines a $\pm$ symmetric sesquilinear function, the <u>associated pairing</u> of (Q,ϕ),
$$\langle\ \rangle_\phi : Q\times Q\longrightarrow A\ ;\ (x,y)\longmapsto \langle x,y\rangle_\phi \equiv \phi(x)(y)\pm\overline{\phi(y)(x)}$$
such that

i) $\overline{\langle x,y\rangle_\phi} = \pm\langle y,x\rangle_\phi \in A$

ii) $\langle x,ay\rangle_\phi = a\langle x,y\rangle_\phi \in A$.

for all $x,y\in Q$ $a\in A$.

A morphism of $\pm$ forms
$$(f,\chi):(P,\theta)\longrightarrow(Q,\phi)$$
preserves the associated pairings, in that
$$f^*(\phi\pm\phi^*)f=\theta\pm\theta^*\in \mathrm{Hom}_A(P,P^*),$$
which can also be expressed as
$$\langle f(x),f(y)\rangle_\phi=\langle x,y\rangle_\theta\in A\quad(\forall x,y\in P).$$

A $\pm$ form (Q,ϕ) is <u>non-singular</u> if $\phi\pm\phi^*\in \mathrm{Hom}_A(Q,Q^*)$ is an A-module isomorphism.

Define the <u>hamiltonian</u> $\pm$ form on a f.g. projective A-module P,
$$H_\pm(P)=\left(P\oplus P^*,\begin{pmatrix}0&1\\0&0\end{pmatrix}:P\oplus P^*\longrightarrow P^*\oplus P\right)$$
The associated pairing
$$\langle\ \rangle : P\oplus P^*\times P\oplus P^*\longrightarrow A\ ;$$
$$((x_1,y_1),(x_2,y_2))\longmapsto y_1(x_2)\pm\overline{y_2(x_1)}$$
is defined by the isomorphism $\begin{pmatrix}0&1\\\pm1&0\end{pmatrix}\in \mathrm{Hom}_A(P\oplus P^*,P^*\oplus P)$ so that $H_\pm(P)$ is non-singular.

A $\pm$form is <u>trivial</u> if it is isomorphic to a hamiltonian $\pm$form.

The remainder of §1 is devoted to answering the question:

when is a $\pm$ form trivial, or more generally, when does it have a trivial summand?

Given a $\pm$form (Q,ϕ) and a submodule L of Q define the <u>annihilator</u> of L in (Q,ϕ)

$$L^\perp = \{x \in Q \mid \langle L, x\rangle_\phi = 0 \in A\},$$

also a submodule of Q.

Submodules L, M of Q are <u>orthogonal</u> in the $\pm$form (Q,ϕ) if $\langle L, M\rangle_\phi = 0 \in A$, that is if $L \subseteq M^\perp$, or (equivalently) if $M \subseteq L^\perp$.

A <u>partial trivialization</u>, or <u>sublagrangian</u> $[L,\lambda]$ of a $\pm$form (Q,ϕ), is a self-orthogonal direct summand L of Q, together with $\lambda \in \Pi_{\mp}(L)$

such that

i) $j^*(\phi \pm \phi^*) : Q \longrightarrow L^*$ is onto

ii) $j^*\phi j = \lambda \mp \lambda^* : L \longrightarrow L^*$

where $j : L \longrightarrow Q$ is the inclusion. (Note that i) is always satisfied if (Q,ϕ) is non-singular).

Then

$$L^\perp = \ker(j^*(\phi \pm \phi^*) : Q \longrightarrow L^*)$$

is a direct summand of Q containing L, with

$$\langle\ \rangle_\phi : Q/L^\perp \longrightarrow L^* ;\ [x] \longmapsto (y \longmapsto \langle x, y\rangle_\phi)$$

an A-module isomorphism.

An isomorphism of $\pm$forms

$$(f,\chi) : (P,\theta) \longrightarrow (Q,\phi)$$

sends a sublagrangian $[L,\lambda]$ of (P,θ) to the sublagrangian

$$(f,\chi)[L,\lambda] = [fL, (f^{-1})^*(\lambda + j^*\chi j)(f^{-1})]$$

of (Q,ϕ), with $j : L \longrightarrow P$ the inclusion.

A sublagrangian $[L,\lambda]$ of a $\pm$form (Q,ϕ) which is maximally self-orthogonal, in the sense that

$$L^{\perp} = L\ ,$$

is a <u>trivialization</u>, or <u>lagrangian</u>, of (Q,ϕ).
Isomorphisms of $\pm$forms preserve lagrangians.
Lagrangians are maximal sublagrangians: if $[L,\lambda]$ is a lagrangian of (Q,ϕ), and $[M,\mu]$ is a sublagrangian of (Q,ϕ) with $L \subseteq M$, then

$$L \subseteq M \subseteq M^{\perp} \subseteq L^{\perp} = L\ ,$$

and $L = M$.

<u>Theorem 1.1</u> A $\pm$form is trivial iff it admits a trivialization.
<u>Proof</u>: It is clear that $[L,0]$ is a lagrangian of $H_{\pm}(L)$ for any f.g. projective L.

Conversely, let $[L,\lambda]$ be a lagrangian of a $\pm$form (Q,ϕ). Choosing a direct complement L_1 to L in Q, express ϕ as

$$\phi = \begin{pmatrix} \lambda \mp \lambda^* & \gamma \\ \delta & \theta \end{pmatrix} : L \oplus L_1 \longrightarrow L^* \oplus L_1^*\ ,$$

so that

$$\gamma \pm \delta^* = [\phi \pm \phi^*] : L_1 = Q/L \longrightarrow L^*$$

is an A-module isomorphism, and (Q,ϕ) is non-singular.
Define

$$\alpha = (\gamma \pm \delta^*)^{-1} \in \mathrm{Hom}_A(L^*, L_1)\ ,\quad \chi = \begin{pmatrix} \lambda & 0 \\ \delta & 0 \end{pmatrix} \in \Pi_{\mp}(L \oplus L_1)\ .$$

$$f = \begin{pmatrix} 1 & -\alpha^*\theta^*\alpha \\ 0 & \alpha \end{pmatrix} \in \mathrm{Hom}_A(L \oplus L^*, L \oplus L_1)\ .$$

Then

$$(f, f^*\chi f) : H_{\pm}(L) \longrightarrow (Q,\phi)$$

is an isomorphism of $\pm$forms (sending $[L,0]$ to $[L,\lambda]$). $\square$

A sublagrangian $[L,\lambda]$ of a $\pm$form (Q,ϕ) is given by a morphism of $\pm$forms

$$(j,\lambda) : (L,0) \longrightarrow (Q,\phi)$$

with $j \in \mathrm{Hom}_A(L,Q)$ split mono, $j^*(\phi \pm \phi^*) \in \mathrm{Hom}_A(Q,L^*)$ onto. Theorem 1.1 shows that the inclusion of a lagrangian

$$(j,\lambda) : (L,0) \longrightarrow (Q,\phi).$$

may be extended to an isomorphism of $\pm$ forms

$$(f,\chi): H_{\pm}(L) \longrightarrow (Q,\phi)$$

recalling Witt's theorem in the classical theory of quadratic forms. More generally:

<u>Corollary 1.2</u> A f.g. projective A-module L defines a sublagrangian $[L,0]$ of $H_{\pm}(L)\oplus(P,\theta)$, for any $\pm$ form (P,θ).

Conversely, the inclusion of a sublagrangian

$$(j,\lambda): (L,0) \longrightarrow (Q,\phi)$$

may be extended to an isomorphism of $\pm$ forms

$$(f,\chi): (L^{\perp}/L, \hat{\phi}) \oplus H_{\pm}(L) \longrightarrow (Q,\phi),$$

where $(L^{\perp}/L, \hat{\phi})$ is the $\pm$ form to which ϕ restricts on a direct complement to L in $L^{\perp}$, a different such choice leading to an isomorphic $\pm$ form.

<u>Proof</u>: The first part is obvious.

Conversely, let $[L,\lambda]$ be a sublagrangian of (Q,ϕ), so that

$$\phi = \begin{pmatrix} \hat{\phi} & \alpha & \beta \\ \mp\alpha^* & \lambda\mp\lambda^* & \gamma \\ \mp\beta^* & \delta & \theta \end{pmatrix} : L^{\perp}/L \oplus L \oplus L_1 \longrightarrow (L^{\perp}/L)^* \oplus L^* \oplus L_1^*$$

for some choice of direct complements to $L, L^{\perp}$ in $L^{\perp}, Q$ respectively.

Now

$$\left(1, \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) : \left(L^{\perp}/L \oplus L \oplus L_1, \begin{pmatrix} \hat{\phi} & 0 & 0 \\ 0 & \lambda\mp\lambda^* & \gamma \\ 0 & \delta & \theta \end{pmatrix}\right) \longrightarrow (L^{\perp}/L \oplus L \oplus L_1, \phi)$$

is an isomorphism of $\pm$ forms, with $[L,\lambda]$ a lagrangian of $\left(L\oplus L_1, \begin{pmatrix} \lambda\mp\lambda^* & \gamma \\ \delta & \theta \end{pmatrix}\right)$. Applying Theorem 1.1, we have an isomorphism of $\pm$ forms

$$(f,\chi): (L^{\perp}/L, \hat{\phi}) \oplus H_{\pm}(L) \longrightarrow (Q,\phi)$$

as required.

A different choice of direct complement to L in $L^{\perp}$ replaces $(L^{\perp}/L, \hat{\phi})$ by $(L^{\perp}/L, \hat{\phi} + \chi \mp \chi^*)$, where $\chi = \alpha h + h^*\lambda h \in \Pi_{\mp}(L^{\perp}/L)$, for some $h \in \mathrm{Hom}_A(L^{\perp}/L, L)$, clearly an isomorph. □

A <u>subhamiltonian complement</u> to a sublagrangian $[L,\lambda]$ of a $\pm$ form (Q,ϕ) is a sublagrangian $[M,\mu]$ such that

$$Q = L \oplus M^{\perp} = L^{\perp} \oplus M.$$

For example $[L,0]$, $[L^*,0]$ are subhamiltonian complements in $H_{\pm}(L)\oplus(P,\theta)$, for any f.g. projective L and $\pm$ form (P,θ).

Isomorphisms of $\pm$ forms preserve

subhamiltonian complements, so that by Corollary 1.2, every sublagrangian has a subhamiltonian complement.

Given subhamiltonian complements $[L,\lambda]$, $[M,\mu]$ in a $\pm$ form (Q,ϕ) we can identify M with L^* via the A-module isomorphism

$$M \longrightarrow L^* \ ; \ x \longmapsto (y \longmapsto \langle x,y\rangle_\phi) .$$

Then $\phi : Q \to Q^*$ can be expressed as

$$\phi = \begin{pmatrix} \hat{\phi} & \alpha & \beta \\ \mp\alpha^* & \lambda\mp\lambda^* & \gamma \\ \mp\beta^* & \delta & \mu\mp\mu^* \end{pmatrix} : L^\perp/L \oplus L \oplus L^* \longrightarrow (L^\perp/L)^* \oplus L^* \oplus L$$

with $\gamma \pm \delta^* = 1 : L^* \to L^*$.

The subhamiltonian complements of lagrangians are also lagrangians, in which case they are called <u>hamiltonian</u> complements.

Given a lagrangian $[L,\lambda]$ in a $\pm$ form (Q,ϕ) we shall in general identify $M = L^*$

for any one hamiltonian complement $[M,\mu]$ to $[L,\lambda]$ in (Q,ϕ), but having chosen one such, reserve the notation L^* for it alone.

A choice of hamiltonian complement to $[L,\lambda]$ is given by a morphism of $\pm$ forms

$$(j,\mu) : (L^*,0) \longrightarrow (Q,\phi)$$

such that

$$\langle j(g), x\rangle_\phi = g(x) \in A \quad (x \in L,\ g \in L^*) .$$

There is one such choice for every $\mu \in \Gamma_{\mp}(L^*)$, as is clear from:

<u>Lemma 1.3</u> The hamiltonian complements to $[P^*,0]$ in $H_\pm(P)$ are the <u>graphs</u>

$$\Gamma_{(P,\theta)} = \mathrm{im}\left(\left(\begin{pmatrix} 1 \\ \theta\mp\theta^* \end{pmatrix}, \theta\right) : (P,\theta) \longrightarrow H_\pm(P)\right)$$

of $\mp$ forms (P,θ).

<u>Proof</u>: The direct complements to P^* in $P\oplus P^*$ are just the graphs

$$\Gamma_h = \mathrm{im}\left(\begin{pmatrix} 1 \\ h \end{pmatrix} : P \longrightarrow P\oplus P^*\right)$$

of morphisms $h \in \mathrm{Hom}_A(P,P^*)$. Now $[\Gamma_h, \lambda]$ defines a lagrangian of $H_\pm(P)$ precisely when

$$(1\ \ h^*)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ h\end{pmatrix} \equiv h = \lambda \mp \lambda^* : P \longrightarrow P^* .$$

$\square$

The next result, corresponding to Theorem 3 in (*), is used by Wall to justify the sort of definition of quadratic form adopted above.

(*) C.T.C. Wall "On the axiomatic foundations of the theory of hermitian forms" Proc Camb Phil Soc 67 (1970) 243-250

<u>Lemma 1.4</u> The <u>diagonal</u> of a non-singular $\pm$form (Q,ϕ),

$$\Delta_{(Q,\phi)} = \mathrm{im}\left(\left(\begin{pmatrix}1\\ 1\end{pmatrix}, 0\right) : (Q,0) \longrightarrow (Q\oplus Q, \phi\oplus -\phi)\right)$$

is a lagrangian of $(Q,\phi)\oplus(Q,-\phi)$, with hamiltonian complements

$$\Delta^*_{(Q^*,\psi)} = \mathrm{im}\left(\left(\begin{pmatrix}\psi\\ \mp\psi^*\end{pmatrix}, (\psi^*\phi(\phi\pm\phi^*)^{-1}\pm\chi)\right) : (Q^*,0) \longrightarrow (Q\oplus Q, \phi\oplus -\phi)\right)$$

classified by $\pm$forms (Q^*,ψ) such that

$$\psi \pm \psi^* = (\phi\pm\phi^*)^{-1} : Q^* \longrightarrow Q ,$$

with

$$\psi - (\phi\pm\phi^*)^{-1}\phi(\phi\pm\phi^*)^{-1} = \chi \mp \chi^* : Q^* \longrightarrow Q$$

for some $\chi \in \Gamma_\mp(Q^*)$.

$\square$

In particular, the diagonal of a trivial $\pm$form (Q,ϕ) is a hamiltonian complement in $(Q\oplus Q, \phi\oplus -\phi)$ to $[F,\lambda]\oplus[G,-\mu]$ for any hamiltonian complements $[F,\lambda]$, $[G,\mu]$ in (Q,ϕ).

§2 Formations

A ***±formation*** (over A), $(Q,\phi;[F,\lambda],[G,\mu])$ is a triple consisting of

i) a trivial ±form (Q,ϕ)
ii) a lagrangian $[F,\lambda]$ of (Q,ϕ)
iii) a sublagrangian $[G,\mu]$ of (Q,ϕ).

An ***isomorphism*** of ±formations

$$(h,\nu):(Q,\phi;[F,\lambda],[G,\mu])\longrightarrow(Q',\phi';[F',\lambda'],[G',\mu'])$$

is an isomorphism of ±forms

$$(h,\nu):(Q,\phi)\longrightarrow(Q',\phi')$$

sending $[F,\lambda]$, $[G,\mu]$ to $[F',\lambda']$, $[G',\mu']$ respectively.

We thus have a category of ±formations, with every morphism an equivalence. A direct sum operation is defined by

$$(Q,\phi;[F,\lambda],[G,\mu])\oplus(Q',\phi';[F',\lambda'],[G',\mu'])$$
$$=(Q\oplus Q',\phi\oplus\phi';[F\oplus F',\lambda\oplus\lambda'],[G\oplus G',\mu\oplus\mu']),$$

with $(0,0;0,0)$ as zero.

By Theorem 1.1, every ±formation $(Q,\phi;[F,\lambda],[G,\mu])$ is isomorphic to one of the type $(H_{\pm}(F);[F,0],[G,\mu])$.

A ±formation $(Q,\phi;[F,\lambda],[G,\mu])$ is ***non-singular*** if $[G,\mu]$ is a lagrangian.

For any f.g. projective A-module P define the ***hamiltonian*** ±formation on P, $(H_{\pm}(P);[P,0],[P^*,0])$, clearly non-singular.

A ±formation is ***trivial*** if it is isomorphic to a hamiltonian ±formation.

Lemma 2.1 A ±formation $(Q,\phi;[F,\lambda],[G,\mu])$ is trivial iff it is non-singular and $[F,\lambda]$, $[G,\mu]$ are hamiltonian complements in (Q,ϕ).

Proof: Given hamiltonian complements $[F,\lambda]$, $[G,\mu]$ in a ±form (Q,ϕ) express $\phi:Q\longrightarrow Q^*$ as

$$\phi=\begin{pmatrix}\lambda\mp\lambda^* & \gamma\\ \delta & \mu\mp\mu^*\end{pmatrix}:F\oplus G\longrightarrow F^*\oplus G^*$$

Then

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & (\gamma\pm\delta^*)^{-1}\end{pmatrix}, \begin{pmatrix} \lambda^* & \pm\delta^*(\gamma\pm\delta^*)^{-1} \\ 0 & \mu \end{pmatrix}\right)$$

$$: (H_\pm(F); [F,0], [F^*,0]) \longrightarrow (Q,\phi; [F,\lambda], [G,\mu])$$

is an isomorphism of $\pm$formations.

The converse is obvious.

$\square$

Given a $\mp$ form (P,θ), define the <u>graph</u> $\pm$formation on (P,θ), $(H_\pm(P); [P,0], \Gamma_{(P,\theta)})$, where $\Gamma_{(P,\theta)}$ is as in Lemma 1.3.

A $\pm$formation isomorphic to a graph $\pm$formation is <u>elementary</u>.

<u>Lemma 2.2</u> A $\pm$formation $(Q,\phi; [F,\lambda], [G,\mu])$ is elementary iff it is non-singular and $[F,\lambda], [G,\mu]$ share a hamiltonian complement in (Q,ϕ).

<u>Proof</u>: For any $\mp$form (P,θ), $[P^*,0]$ is a hamiltonian complement in $H_\pm(P)$ to both $[P,0]$ and $\Gamma_{(P,\theta)}$.

Conversely, let $[H,\nu]$ be a hamiltonian complement to $[F,\lambda], [G,\mu]$ in a $\pm$form (Q,ϕ). By Lemma 2.1 there exists an isomorphism of $\pm$ formations

$$(f,\chi) : (H_\pm(F); [F,0], [F^*,0]) \to (Q,\phi; [F,\lambda], [H,\nu])$$

By Lemma 1.3 (f,χ) sends some $\Gamma_{(F,\theta)}$ to $[G,\mu]$, so that

$$(f,\chi) : (H_\pm(F); [F,0], \Gamma_{(F,\theta)}) \longrightarrow (Q,\phi; [F,\lambda], [G,\mu])$$

is an isomorphism of $\pm$formations.

$\square$

Lemmas 2.1, 2.2 are the special cases $P=0$, $L=0$ of

<u>Theorem 2.3</u> A $\pm$formation $(Q,\phi; [F,\lambda], [G,\mu])$ is isomorphic to the direct sum

$$(H_\pm(P); [P,0], \Gamma_{(P,\theta)}) \oplus (H_\pm(L); [L,0], [L^*,0])$$

of an elementary and a trivial $\pm$formation iff it is non-singular and $[F,\lambda]$ has a hamiltonian complement $[F^*,\lambda_1]$ in (Q,ϕ) such that the projection on F along F^*,

$$\pi : Q = F\oplus F^* \xrightarrow{(1\ 0)} F ,$$

sends G onto a direct summand $\pi(G)=P$ of F. The roles played by $[F,\lambda]$, $[G,\mu]$ may be reversed.

<u>Proof</u>: For any $\mp$form (P,θ) and f.g. projective L, $[P^*\oplus L^*, 0]$ is a hamiltonian complement to $[P\oplus L, 0]$ in $H_\pm(P\oplus L)$ such that the projection on $P\oplus L$ along $P^*\oplus L^*$ sends $\Gamma_{(P,\theta)}\oplus L$ onto P.

Conversely, let $(Q,\phi;[F,\lambda],[G,\mu])$ be a non-singular $\pm$formation, and $[F^*,\lambda_1]$ be a hamiltonian complement to $[F,\lambda]$ in (Q,ϕ) such that the projection on F along F^*,

$$\pi : Q = F\oplus F^* \xrightarrow{(1\ 0)} F ,$$

sends G onto a direct summand P of F, with $F=P\oplus L$ say.

Dualizing, we have a direct sum decompositi[on]

$$F^* = P^*\oplus L^* \subseteq Q$$

and

$$L^\perp = F\oplus P^* \ ,\ P^\perp = F\oplus L^*$$

with

$$P^* = F^*\cap L^\perp \ ,\ L^* = F^*\cap P^\perp .$$

Hence

$$\langle P\oplus P^*, L\oplus L^*\rangle_\phi = 0$$

and there is defined an isomorphism of $\pm$forms

$$(1,\chi) : (Q,\phi) \longrightarrow H_\pm(P\oplus L) ,$$

for some $\chi\in \Pi_\mp(Q)$, sending $[F,\lambda]$ to $[P\oplus L, 0]$.

Now

$$G\subseteq \pi(G)\oplus(1-\pi)(G) \subseteq P\oplus P^*\oplus L^* ,$$

so that

$$\langle G, L^*\rangle_\phi \subseteq \langle P\oplus P^*\oplus L^*, L^*\rangle_\phi = 0$$

and

$$L^* \subseteq G^\perp = G .$$

Denoting $(P\oplus P^*)\cap G$ by M, it follows that

$$G = M\oplus L^* ,$$

and so there is defined an isomorphism of $\pm$formations

$$(1,\chi_1) : (Q,\phi;[F,\lambda],[G,\mu]) \longrightarrow (H_\pm(P);[P,0],[M,\mu_1]) \oplus (H_\pm(L);[L,0],[L^*,0])$$

for some $\mu_1 \in \Pi_\mp(M)$, $\chi_1 \in \Pi_\mp(Q)$.

As π sends M onto $\pi(G)=P$, the projection on P along P^*,

$$(1\ 0) : P\oplus P^* \longrightarrow P,$$

a restriction of $\pi : Q \longrightarrow F$, does the same. Thus $[M,\mu_1]$ is a hamiltonian complement to $[P^*,0]$ in $H_\pm(P)$, necessarily the graph $\Gamma_{(P,\theta)}$ of a $\mp$form (P,θ), by Lemma 1.3.

Symmetry with respect to $[F,\lambda]$, $[G,\mu]$ follow from that of Lemmas 2.1, 2.2. $\square$

A <u>stable isomorphism</u> of $\pm$formations

$$[h,\nu] : (Q,\phi;[F,\lambda],[G,\mu]) \longrightarrow (Q',\phi';[F',\lambda'],[G',\mu'])$$

is an equivalence class of isomorphisms

$$(h,\nu) : (Q,\phi;[F,\lambda],[G,\mu]) \oplus (H_\pm(P);[P,0],[P^*,0]) \longrightarrow (Q',\phi';[F',\lambda'],[G',\mu']) \oplus (H_\pm(P');[P',0],[P'^*,0])$$

defined for f.g. projective P,P' under the equivalence relation

$$(h,\nu) \sim (h',\nu')$$
$$\Longleftrightarrow \exists \text{ A-module isomorphisms } \alpha : L \to L_1,\ \alpha' : L' \to L'_1 \text{ s.t.}$$
$$(h,\nu)\oplus\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{*-1}\end{pmatrix},0\right) = (h',\nu')\oplus\left(\begin{pmatrix}\alpha' & 0\\ 0 & \alpha'^{*-1}\end{pmatrix},0\right)$$
$$: (Q,\phi;[F,\lambda],[G,\mu]) \oplus (H_\pm(P);[P,0],[P^*,0]) \longrightarrow (Q',\phi';[F',\lambda'],[G',\mu']) \oplus (H_\pm(P');[P',0],[P'^*,0])$$

In general, we shall be interested in $\pm$formations up to stable isomorphism only.

Stable isomorphism is an equivalence relation on $\pm$formations. A $\pm$formation stably isomorphic to a trivial $\pm$formation is itself trivial, by Lemma 2.1.

It should be noted that there is only one stable isomorphism $: 0 \longrightarrow 0$, because every isomorphism of hamiltonian $\pm$formations

$$(f,\chi) : (H_\pm(L); [L,0], [L^*,0]) \longrightarrow (H_\pm(L_1); [L_1,0], [L_1^*,0])$$

is necessarily of the type

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}, 0\right) : (H_\pm(L); [L,0], [L^*,0]) \longrightarrow (H_\pm(L_1); [L_1,0], [L_1^*,0]),$$

for some isomorphism $\alpha \in \mathrm{Hom}_A(L, L_1)$.

§3 U-theory

Let I be an abelian monoid, J a submonoid. Call $i_1, i_2 \in I$ J-stably equivalent, $i_1 \sim_J i_2$, if

$$i_1 \oplus j_1 = i_2 \oplus j_2 \in I \quad \text{for some } j_1, j_2 \in J,$$

where $\oplus$ is the composition law in I.

The quotient monoid $I/\!\sim_J$ may be denoted by $I/\bar{J}$, because it depends only on the stabilization of J in I, the submonoid

$$\bar{J} = \{ i \in I \mid i \sim_J 0 \}$$

of I, containing J.

Note that $I/\bar{J}$ is an abelian group iff for every $i \in I$ there exists $i' \in I$ such that

$$i \oplus i' \in J.$$

<u>Theorem 3.1</u> For $n \pmod 4$ let $X_n(A)$ be the abelian monoid of $\left\{\begin{array}{l}\text{isomorphism}\\ \text{stable isomorphism}\end{array}\right.$ classes of $\left\{\begin{array}{l}\pm\text{forms}\\ \pm\text{formations}\end{array}\right.$ over A, under the direct sum $\oplus$, with $n = \left\{\begin{array}{l}2i\\ 2i+1\end{array}\right.$, $\pm = (-)^i$. The morphisms

$$\partial : X_n(A) \longrightarrow X_{n-1}(A)\ ;\ \left\{\begin{array}{l}(P,\theta) \longmapsto (H_{\mp}(P);[P,0],\Gamma_{(P,\theta)})\\ (Q,\phi;[F,\lambda],[G,\mu]) \longmapsto (G^{\perp}/G,\hat{\phi})\end{array}\right. \quad n = \left\{\begin{array}{l}2i\\ 2i+1\end{array}\right.$$

are well-defined and such that $\partial^2 = 0$.

The quotient monoids

$$U_n(A) = \ker(\partial : X_n(A) \longrightarrow X_{n-1}(A)) \Big/ \operatorname{im}(\partial : X_{n+1}(A) \longrightarrow X_n(A))$$

are groups.

<u>Proof</u>: i) n even

An isomorphism of $\pm$forms

$$(f,\chi) : (P,\theta) \longrightarrow (Q,\phi)$$

defines an isomorphism of $\mp$formations

$$\partial(f,\chi) = \left(\begin{pmatrix} f & 0 \\ 0 & f^{*-1}\end{pmatrix}, 0\right) : (H_{\mp}(P);[P,0],\Gamma_{(P,\theta)}) \longrightarrow (H_{\mp}(Q);[Q,0],\Gamma_{(Q,\phi)})$$

By Lemma 2.1

$$\ker(\partial : X_{2i}(A) \longrightarrow X_{2i-1}(A))$$
$$= \{(P,\theta) \in X_{2i}(A) \mid (H_{\mp}(P);[P,0],\Gamma_{(P,\theta)}) \text{ trivial}\}$$
$$= \{(P,\theta) \in X_{2i}(A) \mid P \oplus \Gamma_{(P,\theta)} = P \oplus P^*\}$$
$$= \{(P,\theta) \in X_{2i}(A) \mid (P,\theta) \text{ non-singular}\}.$$

By Corollary 1.2

$$\operatorname{im}(\partial : X_{2i+1}(A) \longrightarrow X_{2i}(A))$$
$$= \{(G^{\perp}/G,\hat{\phi}) \in X_{2i}(A) \mid (Q,\phi;[F,\lambda],[G,\mu]) \in X_{2i+1}(A)\}$$
$$= \{(P,\theta) \in X_{2i}(A) \mid \exists \text{ f.g. projective } F,G \text{ s.t. } (P,\theta) \oplus H_{\pm}(G) = H_{\pm}(F) \in X_{2i}(A)\}$$
$$= \overline{\{(P,\theta) \in X_{2i}(A) \mid (P,\theta) \text{ trivial}\}}$$
$$\subseteq \ker(\partial : X_{2i}(A) \longrightarrow X_{2i-1}(A)).$$

By Theorem 1.1 and Lemma 1.4, for every $(Q,\phi) \in \ker(\partial : X_{2i}(A) \longrightarrow X_{2i-1}(A))$,

$$(Q,\phi) \oplus (Q,-\phi) = 0 \in U_{2i}(A),$$

giving inverses for $U_{2i}(A)$.

ii) n odd

A stable isomorphism of $\pm$ formations

$$[h,\nu] : (Q,\phi;[F,\lambda],[G,\mu]) \longrightarrow (Q',\phi';[F',\lambda'],[G',\mu'])$$

sends G to G', and $G^{\perp}$ to $G'^{\perp}$, so that there is defined an isomorphism of $\pm$ forms

$$\partial[h,\nu] : (G^{\perp}/G, \hat{\phi}) \longrightarrow (G'^{\perp}/G', \hat{\phi}') .$$

Here,

$$\ker(\partial : X_{2i+1}(A) \longrightarrow X_{2i}(A))$$
$$= \{(Q,\phi;[F,\lambda],[G,\mu]) \in X_{2i+1}(A) \mid (G^{\perp}/G, \hat{\phi}) = 0 \in X_{2i}(A)\}$$
$$= \{(Q,\phi;[F,\lambda],[G,\mu]) \in X_{2i+1}(A) \mid (Q,\phi;[F,\lambda],[G,\mu]) \text{ non-singular}\}$$

and by Lemma 2.2

$$\operatorname{im}(\partial : X_{2i+2}(A) \longrightarrow X_{2i+1}(A))$$
$$= \{(H_{\pm}(P);[P,0],\Gamma_{(P,\theta)}) \in X_{2i+1}(A) \mid (P,\theta) \in X_{2i+2}(A)\}$$
$$= \{(Q,\phi;[F,\lambda],[G,\mu]) \in X_{2i+1}(A) \mid (Q,\phi;[F,\lambda],[G,\mu]) \text{ elementary}\}$$
$$\subseteq \ker(\partial : X_{2i+1}(A) \longrightarrow X_{2i}(A)) .$$

For every $(Q,\phi;[F,\lambda],[G,\mu]) \in \ker(\partial : X_{2i+1}(A) \longrightarrow X_{2i}(A))$,

$$(Q,\phi;[F,\lambda],[G,\mu]) \oplus (Q,-\phi;[F^*,-\lambda_1],[G^*,-\mu_1])$$
$$= 0 \in U_{2i+1}(A) ,$$

as the diagonal $\Delta_{(Q,\phi)}$ is a hamiltonian

complement in $(Q\oplus Q, \phi\oplus -\phi)$ to $[F\oplus F^*, \lambda\oplus -\lambda_1]$ and $[G\oplus G^*, \mu\oplus -\mu_1]$ for any pair of hamiltonian complements $([F,\lambda],[F^*,\lambda_1])$ and $([G,\mu],[G^*,\mu_1])$ in (Q,ϕ) by Lemma 1.4, giving inverses for $U_{2i+1}(A)$.

$\square$

<u>Example 3.2</u> For the ground ring $\mathbb{Z}[\pi]$ of Example 0.1

$$U_n(\mathbb{Z}[\pi]) = L^A_n(\pi) ,$$

the surgery obstruction group in the category A of §17 D in [W], of Poincaré complexes up to homotopy.

$\square$

The construction of the groups $U_*(A)$ is not unlike that of the groups $\tilde{K}_0(A)$, $\tilde{K}_1(A)$ of algebraic K-theory.

The <u>projective class group of A</u>, $\tilde{K}_0(A)$, is the group of isomorphism classes $[P]$ of f.g. projective A-modules P modulo the stably f.g.

free A-modules, under the direct sum $\oplus$.
Similarly, $U_{2i}(A)$ is the group of isomorphism classes of non-singular $\pm$forms over A, modulo the stably trivial ones.

The <u>Whitehead torsion group of A</u>

$$\tilde{K}_1(A) = GL(A)/\{E(A), -1\}$$

is a quotient of the general linear group $GL(A)$ of A by -1 and $E(A)$, the subgroup generated by the elementary matrices, those with 1's on the diagonal and at most one other non-zero entry. Whitehead's Lemma states that

$$E(A) = [GL(A), GL(A)],$$

the commutator subgroup of $GL(A)$. This allows $\tilde{K}_1(A)$ to be considered as the abelian group of isomorphism classes of triples $(Q, \underset{\sim}{f}, \underset{\sim}{g})$ consisting of a f.g. free A-module Q and bases $\underset{\sim}{f} = (f_1, \dots, f_m)$, $\underset{\sim}{g} = (g_1, \dots, g_m)$, under the direct sum

$$(Q, \underset{\sim}{f}, \underset{\sim}{g}) \oplus (Q', \underset{\sim}{f}', \underset{\sim}{g}') = (Q \oplus Q', \underset{\sim}{f} \oplus \underset{\sim}{f}', \underset{\sim}{g} \oplus \underset{\sim}{g}'),$$

modulo the elementary triples

i) $(Q, (f_1, \dots, f_m), (f_1, \dots, f_{j-1}, \varepsilon f_j + a f_k, f_{j+1}, \dots, f_m))$

$(1 \leq j, k \leq m \quad j \neq k \quad a \in A \quad \varepsilon = \pm 1)$

ii) $(Q, \underset{\sim}{f}, \underset{\sim}{g}) \oplus (Q, \underset{\sim}{g}, \underset{\sim}{h}) \oplus (Q, \underset{\sim}{h}, \underset{\sim}{f})$.

Similarly, $U_{2i+1}(A)$ is the group of stable isomorphism classes of non-singular $\pm$formations modulo the elementary ones. Although it is not possible to identify the elements of $U_{2i+1}(A)$ as the "torsions" of automorphisms of a trivial $\pm$form, they have the formal properties of such. In particular, we have the sum formula:

<u>Lemma 3.3</u> $(Q, \phi; [F, \lambda], [G, \mu]) \oplus (Q, \phi; [G, \mu], [H, \nu])$

$$= (Q, \phi; [F, \lambda], [H, \nu]) \in U_{2i+1}(A).$$

<u>Proof</u>: Consider first the special case when $[F, \lambda]$, $[G, \mu]$ have a common hamiltonian complement in (Q, ϕ), $[L, \lambda_1]$ say.
Then

$$(Q, \phi; [F, \lambda], [G, \mu]) = 0 \in U_{2i+1}(A)$$

and

$$(Q,\phi;[F,\lambda],[H,\nu])$$
$$= -(Q,-\phi;[L,-\lambda_1],[H^*,-\nu_1])$$
$$= (Q,\phi;[G,\mu],[H,\nu]) \in U_{2i+1}(A),$$

for any hamiltonian complement $[H^*,\nu_1]$ to $[H,\nu]$ in (Q,ϕ).

For general $(Q,\phi;[F,\lambda],[G,\mu]) \in U_{2i+1}(A)$,

$$(Q,\phi;[F,\lambda],[G,\mu]) \oplus (Q,\phi;[G,\mu],[H,\nu])$$
$$= (Q,\phi;[F,\lambda],[G,\mu]) \oplus (Q\oplus Q,\phi\oplus-\phi;[G\oplus G^*,\mu\oplus-\mu_1],[H\oplus G^*,\nu\oplus-\mu_1])$$

(for any hamiltonian complement $[G^*,\mu_1]$ to $[G,\mu]$ in (Q,ϕ))

$$= (Q,\phi;[F,\lambda],[G,\mu]) \oplus (Q\oplus Q,\phi\oplus-\phi;[F\oplus F^*,\lambda\oplus-\lambda_1],[H\oplus G^*,\nu\oplus-\mu_1])$$

(by special case and Lemma 1.4, for any hamiltonian complement $[F^*,\lambda_1]$ to $[F,\lambda]$ in (Q,ϕ))

$$= (Q\oplus Q,\phi\oplus-\phi;[F\oplus F^*,\lambda\oplus-\lambda_1],[G\oplus G^*,\mu\oplus-\mu_1])$$
$$\oplus (Q,\phi;[F,\lambda],[H,\nu])$$
$$= (Q,\phi;[F,\lambda],[H,\nu]) \in U_{2i+1}(A).$$

$\square$

Given lagrangians $[F,\lambda]$, $[F',\lambda']$ in a $\pm$ form (Q,ϕ) such that $F=F'$, it is clear that the hamiltonian complements to $[F,\lambda]$ in (Q,ϕ) are just those to $[F,\lambda']$, and hence, by the sum formula, that

$$(Q,\phi;[F,\lambda],[G,\mu]) = (Q,\phi;[F,\lambda'],[G,\mu]) \in U_{2i+1}(A)$$

for any lagrangian $[G,\mu]$ of (Q,ϕ).
Similarly,

$$(Q,\phi;[F,\lambda],[G,\mu]) = (Q,\phi;[F,\lambda],[G,\mu']) \in U_{2i+1}(A)$$

Therefore in dealing with elements of $U_{2i+1}(A)$ it is sufficient to consider the triples $(Q,\phi;F,G)$, also termed $\pm$formations, in which F,G are submodules of Q carrying lagrangians of the $\pm$ form (Q,ϕ), and which we shall also call lagrangians. The full force of the definitions will be exploited later on, in Part III.

§ 4. V-theory

V-theory is the analogue of U-theory obtained by considering $\pm$forms and $\pm$formations on stably f.g. free A-modules rather than f.g. projective ones. All the U-theory done above has an obvious V-theory version. In particular, we can define abelian monoids $Y_n(A)$ for $n \pmod 4$, and morphisms $\partial : Y_n(A) \longrightarrow Y_{n-1}(A)$, exactly as for $X_n(A)$, to obtain quotient groups

$$V_n(A) = \ker(\partial : Y_n(A) \longrightarrow Y_{n-1}(A)) \Big/ \operatorname{im}(\partial : Y_{n+1}(A) \longrightarrow Y_n(A))$$

<u>Example 4.1</u> For the ground ring $\mathbb{Z}[\pi]$ of Example 0.1

$$V_n(\mathbb{Z}[\pi]) = L^B_n(\pi),$$

the surgery obstruction group in the category B of §17 D in [W], of finite Poincaré complexes up to homotopy.

$\square$

The odd-dimensional groups $V_{2i+1}(A)$ will now be identified as stable unitary groups.

Define for $m \geqslant 1$ the unitary group $U_\pm(A,m)$ of automorphisms of $H_\pm(mA)$, where mA is the free A-module on m generators.

The function

$$\pi_m : U_\pm(A,m) \longrightarrow V_{2i+1}(A);$$

$$(f,\chi) \longmapsto (H_\pm(mA); mA, f(mA))$$

is a group morphism: given $(f,\chi),(g,\nu) \in U_\pm(A,m)$ we have that

$$\begin{aligned}
\pi_m((g,\nu)(f,\chi)) &= (H_\pm(mA); mA, gf(mA)) \\
&= (H_\pm(mA); mA, g(mA)) \oplus (H_\pm(mA); g(mA), gf(mA)) \\
&\qquad \text{(by the sum formula of Lemma 3.3 for V-theory)} \\
&= (H_\pm(mA); mA, g(mA)) \oplus (H_\pm(mA); mA, f(mA)) \\
&= \pi_m(g,\nu) \oplus \pi_m(f,\chi) \in V_{2i+1}(A).
\end{aligned}$$

Defining inclusions

$$U_\pm(A,m) \longrightarrow U_\pm(A,m+1);$$

$$(f,\chi) \longmapsto (f,\chi) \oplus (1,0),$$

let

$$\mathcal{U}_{\pm}(A) = \bigcup_{m=1}^{\infty} \mathcal{U}_{\pm}(A,m)$$

with the obvious multiplicative group structure.

There is induced a group morphism

$$\pi : \mathcal{U}_{\pm}(A) \longrightarrow V_{2i+1}(A)$$

which agrees with π_m on each $\mathcal{U}_{\pm}(A,m)$.

Denote the kernel of π by $\mathcal{H}_{\pm}(A)$, calling its elements the <u>hamiltonian transformations</u>.

<u>Theorem 4.2</u> The morphism

$$\pi : \mathcal{U}_{\pm}(A) \longrightarrow V_{2i+1}(A)$$

is onto, inducing an isomorphism

$$\mathcal{U}_{\pm}(A)/\mathcal{H}_{\pm}(A) \cong V_{2i+1}(A)$$

of abelian groups.

$\mathcal{H}_{\pm}(A)$ contains the commutator subgroup $[\mathcal{U}_{\pm}(A), \mathcal{U}_{\pm}(A)]$ of $\mathcal{U}_{\pm}(A)$, and it is generated by the elementary hamiltonian transformations:

i) $\left(\begin{pmatrix} 1 & 0 \\ \theta \mp \theta^* & 1 \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}\right) \in \mathcal{U}_{\pm}(A,m)$ for any $\mp$ form (mA, θ)

ii) $\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}, 0\right) \in \mathcal{U}_{\pm}(A,m)$,

for any automorphism $\alpha \in \mathrm{Hom}_A(mA, mA)$

iii) $\sigma \oplus \sigma \oplus \ldots \oplus \sigma \in \mathcal{U}_{\pm}(A,m)$,

involving m copies of

$$\sigma = \left(\begin{pmatrix} 0 & \pm\gamma^{-1} \\ \gamma & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right) \in \mathcal{U}_{\pm}(A,1)$$

where

$$\gamma : A \longrightarrow A^*; \ a \longmapsto (b \longmapsto b\bar{a})$$

<u>Proof</u>: It is sufficient to verify that π is onto.

Every $(Q,\phi;F,G) \in V_{2i+1}(A)$ may be represented by a non-singular $\pm$ formation $(H_{\pm}(mA); mA, G)$ with G free, of dimension

$$\dim_A G = \tfrac{1}{2}\dim_A(G \oplus G^*) = \tfrac{1}{2}\dim_A(mA \oplus (mA)^*) = m.$$

Choosing a hamiltonian complement G^* to G in $H_{\pm}(mA)$, and an isomorphism $h \in \mathrm{Hom}_A(mA, G)$, note that the isomorphism

$$\begin{pmatrix} h & 0 \\ 0 & h^{*-1} \end{pmatrix} : mA \oplus (mA)^* \longrightarrow G \oplus G^*$$

defines an automorphism of $H_{\pm}(mA)$ taking mA to G. Thus

$$\pi\begin{pmatrix} h & 0 \\ 0 & h^{*-1} \end{pmatrix} = (Q,\phi;F,G) \in V_{2i+1}(A).$$

V-theory differs from U-theory in 2-torsion only:

Theorem 4.3 There is an exact sequence

$$\ldots \to \underset{(\mathrm{I})}{\Sigma_+(A) \longrightarrow V_{2i+1}(A)} \longrightarrow \underset{(\mathrm{II})}{U_{2i+1}(A)} \longrightarrow \underset{(\mathrm{III})}{\Sigma_-(A)}$$

$$\longrightarrow \underset{(\mathrm{IV})}{V_{2i}(A)} \longrightarrow \underset{(\mathrm{V})}{U_{2i}(A)} \longrightarrow \underset{(\mathrm{VI})}{\Sigma_+(A)} \longrightarrow V_{2i-1}(A) \to \ldots$$

of abelian groups and morphisms, defined for $i \pmod 2$.

The groups $\Sigma_+(A)$ are defined by

$$\Sigma_\pm(A) = \{[P] \in \tilde{K}_0(A) \mid [P^*] = \pm[P]\} \big/ \{[Q] \pm [Q^*] \mid [Q] \in \tilde{K}_0(A)\}$$

and are of exponent 2.

The morphisms

$$V_n(A) \longrightarrow U_n(A)$$

are induced by the obvious inclusions of $Y_n(A)$ in $X_n(A)$.
The others are given by:

$$U_{2i}(A) \longrightarrow \Sigma_+(A) \; ; \; (P,\theta) \longmapsto [P]$$

$$U_{2i+1}(A) \longrightarrow \Sigma_-(A) \; ; \; (Q,\phi; F,G) \longmapsto [G] - [F^*]$$

$$\Sigma_-(A) \longrightarrow V_{2i}(A) \; ; \; [P] \longmapsto H_\pm(P)$$

$$\Sigma_+(A) \longrightarrow V_{2i-1}(A) \; ; \; [P] \longmapsto (H_\mp(P \oplus -P) \; ; \; P \oplus -P, P \oplus -P^*)$$

for any representative $-P$ of $-[P] \in \tilde{K}_0(A)$.

Proof: It is easy to verify that the given morphisms are well-defined, except perhaps $\Sigma_+(A) \longrightarrow V_{2i-1}(A)$. This sends $[P \oplus P^*]$ $(= 0 \in \Sigma_+(A))$ to

$$(H_\mp(P \oplus P^* \oplus -P \oplus -P^*) \; ; \; P \oplus P^* \oplus -P \oplus -P^*, P \oplus P^* \oplus (-P \oplus -P^*)^*),$$

which vanishes in $V_{2i-1}(A)$ because

$$\{(0,0,x,y,z,w,\pm y,x) \in P \oplus P^* \oplus -P \oplus -P^* \oplus (P \oplus P^* \oplus -P \oplus -P^*)^* \mid x \in -P, y \in -P^*, z \in P, w \in P^*\}$$

is a common hamiltonian complement.

Further, it is not difficult to see that the composition of successive morphisms is 0, except perhaps at (III) and (VI):

at (III) note that every $(Q,\phi; F,G) \in U_{2i+1}(A)$ has a representative $\pm$formation with G free, so that

$$H_\pm(F) - (Q,\phi) = H_\pm(G) = 0 \in V_{2i}(A)$$

at (VI) the composite $U_{2i}(A) \longrightarrow V_{2i-1}(A)$ sends $(Q,\phi) \in U_{2i}(A)$ to

$$(H_\mp(Q \oplus -Q) \; ; \; Q \oplus -Q, Q \oplus -Q^*)$$

$$= (H_\mp(Q \oplus -Q) \; ; \; Q \oplus -Q, Q^* \oplus -Q) \in V_{2i-1}(A)$$

(by Lemma 3.3 for V-theory)

and $\Gamma_{(Q,\phi)} \oplus -Q^*$ is a hamiltonian complement to both $Q \oplus -Q$ and $Q^* \oplus -Q$ in $H_\mp(Q\oplus -Q)$, as (Q,ϕ) is non-singular, so that

$$(H_\mp(Q\oplus -Q); Q\oplus -Q, Q^*\oplus -Q) = 0 \in V_{2i-1}(A).$$

We now verify exactness at each point of the sequence:

(I) $\Sigma_+(A) \longrightarrow V_{2i+1}(A) \longrightarrow U_{2i+1}(A)$

Every $(Q,\phi;F,G) \in \ker(V_{2i+1}(A) \longrightarrow U_{2i+1}(A))$ can be represented as

$$(H_\pm(P\oplus L); P\oplus L, \Gamma_{(P,\theta)}\oplus L^*)$$

for some $\mp$form (P,θ) and f.g. projective L, such that $P\oplus L$, $P\oplus L^*$ are free.

Applying the sum formula of Lemma 3.3 for V-theory

$$\begin{aligned}(Q,\phi;F,G) &= (H_\pm(P\oplus L); P\oplus L, \Gamma_{(P,\theta)}\oplus L^*) \\ &= (H_\pm(P\oplus L); P\oplus L, P^*\oplus L) \in \operatorname{im}(\Sigma_+(A) \longrightarrow V_{2i+1}(A))\end{aligned}$$

(II) $V_{2i+1}(A) \longrightarrow U_{2i+1}(A) \longrightarrow \Sigma_-(A)$

Let $(Q,\phi;F,G) \in \ker(U_{2i+1}(A) \longrightarrow \Sigma_-(A))$, so that

$$[G] - [F^*] = [P^*] - [P] \in \tilde{K}_0(A)$$

for some f.g. projective P.

Denote by M a f.g. projective A-module such that

$$[M] = -[G^*\oplus P^*] = -[F\oplus P] \in \tilde{K}_0(A).$$

Then

$$\begin{aligned}(Q,\phi;F,G) &= ((Q,\phi)\oplus H_\pm(P\oplus M); F\oplus P\oplus M, G\oplus P\oplus M^*) \\ &\in \operatorname{im}(V_{2i+1}(A) \longrightarrow U_{2i+1}(A)).\end{aligned}$$

(III) $U_{2i+1}(A) \longrightarrow \Sigma_-(A) \longrightarrow V_{2i}(A)$.

If $[P] \in \ker(\Sigma_-(A) \longrightarrow V_{2i}(A))$, it may be assumed that $H_\pm(P)$ has a free lagrangian, L say. Then $U_{2i+1}(A) \longrightarrow \Sigma_-(A)$ sends $(H_\pm(P); L, P)$ to $[P] \in \Sigma_-(A)$.

(IV) $\Sigma_-(A) \longrightarrow V_{2i}(A) \longrightarrow U_{2i}(A)$

If $(Q,\phi) \in \ker(V_{2i}(A) \longrightarrow U_{2i}(A))$, it may be assumed that (Q,ϕ) has a (projective) lagrangian, P say. Then

$$[P] + [P^*] = [Q] = 0 \in \tilde{K}_0(A)$$

and $(Q,\phi) = H_\pm(P) \in \operatorname{im}(\Sigma_-(A) \longrightarrow V_{2i}(A))$.

(V) $V_{2i}(A) \longrightarrow U_{2i}(A) \longrightarrow \Sigma_+(A)$

If $(Q,\phi) \in \ker(U_{2i}(A) \longrightarrow \Sigma_+(A))$, then

$$[Q] = [P] + [P^*] \in \tilde{K}_0(A)$$

for some f.g. projective P and

$$(Q,\phi) = (Q,\phi)\oplus H_\pm(-P) \in \operatorname{im}(V_{2i}(A) \longrightarrow U_{2i}(A))$$

(VI) $U_{2i}(A) \longrightarrow \Sigma_+(A) \longrightarrow V_{2i-1}(A)$

Given $[P] \in \ker(\Sigma_+(A) \longrightarrow V_{2i-1}(A))$ it may be assumed that up to isomorphism of $\mp$formations

$$(H_\mp(P\oplus -P); P\oplus -P, P\oplus -P^*) \oplus (H_\mp(L); L, \Gamma_{(L,\lambda)})$$
$$= (H_\mp(M); M, \Gamma_{(M,\mu)})$$

for some $\pm$forms $(L,\lambda), (M,\mu)$ defined on f.g free L, M.

Now $(Q,\phi; F, G) = (H_\mp(M); M, \Gamma_{(M,\mu)})$ is an elementary $\mp$ formation, with $H = M^*$ a hamiltonian complement to both F and G in (Q,ϕ). Moreover, $F^* = P^* \oplus -P^* \oplus L^*$ is a hamiltonian complement to $F = P\oplus -P \oplus L$ in (Q,ϕ) such that the projection on F along F^*,

$$\pi : Q = F \oplus F^* \xrightarrow{(1\ 0)} F ,$$

sends $G = P \oplus -P^* \oplus \Gamma_{(L,\lambda)}$ onto the direct summand $\pi(G) = P\oplus L$ of F. Thus $F \oplus F^*$ is a hamiltonian complement to $\Delta_{(Q,\phi)}$ in $(Q\oplus Q, \phi\oplus -\phi)$ such that the projection on $\Delta_{(Q,\phi)}$ along $F\oplus F^*$,

$$Q\oplus Q \longrightarrow \Delta_{(Q,\phi)} ;$$
$$(x,y) \longmapsto (\pi(x) + (1-\pi)(y), \pi(x) + (1-\pi)(y)) ,$$

sends $G\oplus H$ onto the submodule

$$N = \{(x,x) \in \Delta_{(Q,\phi)} \mid x \in \pi(G) \oplus (1-\pi)(H)\} \subseteq \Delta_{(Q,\phi)} .$$

As H is a hamiltonian complement to F in (Q,ϕ), $(1-\pi)(H) = F^*$, and N is a direct summand of $\Delta_{(Q,\phi)}$ isomorphic to $P\oplus L \oplus F^*$, with direct complement isomorphic to $-P$.

Applying Theorem 2.3, we have that up to isomorphism

$$(Q\oplus Q, \phi\oplus -\phi; \Delta_{(Q,\phi)}, G\oplus H) = (H_\mp(N); N, \Gamma_{(N,\nu)}) \oplus (H_\mp(-P); -P, -P^*)$$

for some $\pm$form (N,ν), which must be non-singular, as $(Q\oplus Q, \phi\oplus -\phi; \Delta_{(Q,\phi)}, G\oplus H)$ is a trivial $\pm$formation (H being a hamiltonian complement to G in (Q,ϕ)). Thus

$$[N] = [P\oplus L \oplus F^*] = [P]$$
$$\in \mathrm{im}(U_{2i}(A) \longrightarrow \Sigma_+(A)) .$$

$\square$

§5 W-theory

A *based* A-module, $\underset{\sim}{Q}$, is a f.g. free A-module Q together with a base $\underset{\sim}{b}=(b_1,b_2,\dots,b_m)$. The *dual* based A-module $\underset{\sim}{Q}^*$ is defined, with base $\underset{\sim}{b}^*=(b_1^*,b_2^*,\dots,b_m^*)$, where

$$b_j^*(b_k)=\begin{cases}1 & \text{if } j=k\\ 0 & j\neq k\end{cases},$$

and $\underset{\sim}{Q}^{**}$ is identified with $\underset{\sim}{Q}$.

The matrix representation in $GL(A)$ of an isomorphism $f\in \mathrm{Hom}_A(P,Q)$ of based A-modules $\underset{\sim}{P},\underset{\sim}{Q}$ defines the *torsion* of f, $\tau(f)\in \tilde{K}_1(A)$. The isomorphism is *simple* if $\tau(f)=0\in\tilde{K}_1(A)$.

A *based* $\pm$form (over A), $(\underset{\sim}{Q},\phi)$, is a $\pm$form (Q,ϕ) defined on a based A-module $\underset{\sim}{Q}$. An isomorphism of based $\pm$forms

$$(f,\chi):(\underset{\sim}{P},\theta)\longrightarrow(\underset{\sim}{Q},\phi)$$

is *simple* if the isomorphism

$$f:\underset{\sim}{P}\longrightarrow\underset{\sim}{Q}$$

is simple.

W-theory deals with the simple isomorphism properties of based $\pm$forms, just as U-theory considers the isomorphism of $\pm$forms, and V-theory that of $\pm$forms on stably f.g. free modules.

Define the *hamiltonian* based $\pm$form on a based A-module $\underset{\sim}{P}$, $H_\pm(\underset{\sim}{P})$, to be $H_\pm(P)$ with base $\underset{\sim}{P}\oplus\underset{\sim}{P}^*$. A based $\pm$form is *trivial* if it is simple isomorphic to a based hamiltonian one.

Let L be a free lagrangian of a trivial $\pm$form (Q,ϕ). A base $\underset{\sim}{L}$ of L, together with the dual $\underset{\sim}{L}^*$ on a hamiltonian complement L^* defines a *hamiltonian* base of (Q,ϕ), $\underset{\sim}{Q}=\underset{\sim}{L}\oplus\underset{\sim}{L}^*$. A different choice of hamiltonian complement L^* alters this hamiltonian base by a simple automorphism

$$\begin{pmatrix}1 & \theta\mp\theta^*\\ 0 & 1\end{pmatrix}:L\oplus L^*\longrightarrow L\oplus L^*$$

of Q for some $\mp$form (L^*,θ), by Lemma 1.3. Thus every base $\underset{\sim}{L}$ can be extended to a hamiltonian base of $(\underset{\sim}{Q},\phi)$ uniquely up to simple isomorphism.

A *based* lagrangian $\underset{\sim}{L}$ of a based $\pm$form $(\underset{\sim}{Q},\phi)$ is a lagrangian L of (Q,ϕ) (or, more properly, $[L,\lambda]$) together with a base $\underset{\sim}{L}$ such that $\underset{\sim}{Q}$ differs from a hamiltonian base extending $\underset{\sim}{L}$ by a simple automorphism.

By analogy with Theorem 1.1 we have:

Theorem 5.1 A based $\pm$form is trivial iff it admits a based lagrangian. $\square$

Given a non-singular based $\pm$form $(\underset{\sim}{Q},\phi)$ over A, define its *torsion* to be

$$\tau(\underset{\sim}{Q},\phi) = \tau((\phi\pm\phi^*):\underset{\sim}{Q}\longrightarrow\underset{\sim}{Q}^*) \in \tilde{K}_1(A)\,.$$

Torsion is a simple $\pm$form isomorphism invariant, and as based hamiltonian $\pm$forms have zero torsion, so do all trivial based $\pm$forms.

A *based* sublagrangian, $\underset{\sim}{L}$, of a based $\pm$form $(\underset{\sim}{Q},\phi)$ is a free sublagrangian L of (Q,ϕ) such that $L^\perp/L$ is free, together with bases $\underset{\sim}{L}$, $\underset{\sim}{L^\perp/L}$ such that the subhamiltonian base these determine on (Q,ϕ) agrees with $\underset{\sim}{Q}$ up to simple isomorphism.

By analogy with Corollary 1.2 we have:

Corollary 5.2 The inclusion of a based sublagrangian

$$(j,\lambda):(\underset{\sim}{L},0)\longrightarrow(\underset{\sim}{Q},\phi)$$

may be extended to a simple isomorphism

$$(f,\chi):(\underset{\sim}{L^\perp/L},\hat{\phi})\oplus H_\pm(\underset{\sim}{L})\longrightarrow(\underset{\sim}{Q},\phi)\,.$$

In particular, if $(\underset{\sim}{Q},\phi)$ is non-singular, then

$$\tau(\underset{\sim}{Q},\phi) = \tau(\underset{\sim}{L^\perp/L},\hat{\phi}) \in \tilde{K}_1(A)\,.$$

$\square$

A *based* hamiltonian complement to a based lagrangian $\underset{\sim}{F}$ of a $\pm$form $(\underset{\sim}{Q},\phi)$ is a based lagrangian $\underset{\sim}{G}$ such that

$$\underset{\sim}{G}\longrightarrow\underset{\sim}{F}^*\,;\ x\longmapsto(y\longmapsto\langle x,y\rangle_\phi)$$

defines a simple isomorphism (of A-modules), in which case $\underset{\sim}{G}$ may be identified with $\underset{\sim}{F}^*$.

Lemma 1.3 has based version:

<u>Lemma 5.3</u> The based hamiltonian complements to P^* in $H_\pm(P)$ are the graphs

$$\Gamma_{(P,\theta)} = \mathrm{im}\left(\left(\begin{pmatrix}1\\ \theta\mp\theta^*\end{pmatrix},\theta\right):(P,0)\longrightarrow H_\pm(P)\right)$$

of based $\mp$forms (P,θ), up to simple changes of base. $\square$

Lemma 1.4 has based version:

<u>Lemma 5.4</u> Let (Q,ϕ) be a non-singular based $\pm$form such that $\tau(Q,\phi)=0\in\tilde{K}_1(A)$. Then

$$\Delta_{(Q,\phi)} = \mathrm{im}\left(\left(\begin{pmatrix}1\\1\end{pmatrix},0\right):(Q,0)\longrightarrow(Q\oplus Q,\phi\oplus-\phi)\right)$$

is a based lagrangian of $(Q,\phi)\oplus(Q,-\phi)$, with based hamiltonian complements

$$\Delta^*_{(Q^*,\psi)} = \mathrm{im}\left(\left(\begin{pmatrix}\psi\\ \mp\psi^*\end{pmatrix},(\psi^*\phi(\phi\pm\phi^*)^{-1}\pm\chi)\right):(Q^*,0)\to(Q,\phi)\oplus(Q,-\phi)\right)$$

with

$$\psi=(\phi\pm\phi^*)^{-1}\phi(\phi\pm\phi^*)^{-1}+\chi\mp\chi^*\in\mathrm{Hom}_A(Q^*,Q)$$

for some $\chi\in\Pi_\mp(Q^*)$. $\square$

In particular, the diagonal of a trivial based $\pm$form (Q,ϕ) is a based hamiltonian complement to $F\oplus F^*$ in $(Q\oplus Q,\phi\oplus-\phi)$ for any based hamiltonian complements F,F^* in (Q,ϕ).

A <u>based</u> $\pm$formation $(Q,\phi;[F,\lambda],[G,\mu])$ is defined by

i) a trivial based $\pm$form (Q,ϕ)

ii) a based lagrangian $[F,\lambda]$ of (Q,ϕ)

iii) a based sublagrangian $[G,\mu]$ of (Q,ϕ).

An isomorphism of based $\pm$formations

$$(h,\nu):(Q,\phi;[F,\lambda],[G,\mu])\longrightarrow(Q',\phi';[F',\lambda'],[G',\mu'])$$

is <u>simple</u> if it is defined by a simple isomorphism

$$(h,\nu):(Q,\phi)\longrightarrow(Q',\phi')$$

which restricts to simple isomorphisms

$$:F\longrightarrow F',\ :G\longrightarrow G',\ :G^\perp/G\longrightarrow G'^\perp/G'.$$

The definitions and propositions of §§2,3 have obvious based analogues. In particular:

<u>Theorem 5.5</u> For $n\pmod 4$ let $Z_n(A)$ be the abelia[n] monoid of $\begin{cases}\text{simple isomorphism}\\ \text{stable simple isomorphism}\end{cases}$ classes of based $\begin{cases}\pm\text{forms}\\ \pm\text{formations}\end{cases}$ over A, under the direct sum $\oplus$, where $\pm=(-)^i$ if $n=\begin{cases}2i\\2i+1\end{cases}$.

The monoid morphisms

$$\partial : Z_n(A) \longrightarrow Z_{n-1}(A) ;$$

$$\begin{cases} (P, \theta) \longmapsto (H_{\mp}(P); [P,\theta], \Gamma_{(P,\theta)}) \\ (Q,\phi;[F,\lambda],[G,\mu]) \longmapsto (G^{\perp}/G, \hat{\phi}) \end{cases} \quad n = \begin{cases} 2i \\ 2i+1 \end{cases}$$

are well-defined and such that $\partial^2 = 0$, with the quotient monoids

$$W_n(A) = \ker(\partial : Z_n(A) \longrightarrow Z_{n-1}(A)) / \overline{\mathrm{im}(\partial : Z_{n+1}(A) \longrightarrow Z_n(A))}$$

groups.

□

As in the proof of Theorem 3.1 we can identify:

$$\ker(\partial : Z_{2i}(A) \longrightarrow Z_{2i-1}(A))$$
$$= \{ (P, \theta) \in Z_{2i}(A) \mid (P,\theta) \text{ non-singular}, \ \tau(P,\theta) = 0 \in \tilde{K}_1(A) \}$$

$$\overline{\mathrm{im}\,(\partial : Z_{2i+1}(A) \longrightarrow Z_{2i}(A))}$$
$$= \{ (Q,\phi) \in Z_{2i}(A) \mid (Q,\phi) \text{ trivial} \}$$

$$\ker(\partial : Z_{2i+1}(A) \longrightarrow Z_{2i}(A))$$
$$= \{ (Q,\phi;[F,\lambda],[G,\mu]) \in Z_{2i+1}(A) \mid (Q,\phi;[F,\lambda],[G,\mu]) \text{ non-singular} \}$$

$$\mathrm{im}(\partial : Z_{2i+2}(A) \longrightarrow Z_{2i+1}(A))$$
$$= \{ (Q,\phi;[F,\lambda],[G,\mu]) \in Z_{2i+1}(A) \mid (Q,\phi;[F,\lambda],[G,\mu]) \text{ elementary} \}$$

<u>Example 5.6</u> For the ground ring $\mathbb{Z}[\pi]$ of Example 5.1

$$W_n(\mathbb{Z}[\pi]) = L_n^E(\pi)$$

the surgery obstruction group in the category E of §17D of [W], of simple Poincaré complexes up to simple homotopy.

□

(This is slightly bowdlerized: in the geometrical case simple equivalence of bases is measured not in $\tilde{K}_1(\mathbb{Z}[\pi])$, but in the <u>Whitehead group</u> of π

$$Wh(\pi) = \tilde{K}_1(\mathbb{Z}[\pi]) / \mathrm{im}(\pi \longrightarrow U(\mathbb{Z}[\pi]) \longrightarrow \tilde{K}_1(\mathbb{Z}[\pi])) ;$$

where $U(\mathbb{Z}[\pi])$ is the multiplicative group of units of $\mathbb{Z}[\pi]$, regarded as a subgroup of $GL(\mathbb{Z}[\pi])$ in the obvious way).

The odd-dimensional groups $W_{2i+1}(A)$ will now be identified as stable special unitary groups, by analogy with Theorem 4.2 for V-theory.

Define for $m \geq 1$ the special unitary group $SU_{\pm}(A,m)$ of simple automorphisms of $H_{\pm}(A^m)$ where A^m is the based A-module on m generators

The functions

$$\pi'_m : S U_\pm(A,m) \longrightarrow W_{2i+1}(A)\ ;$$

$$(f,\chi) \longmapsto (H_\pm(\underset{\sim}{m}A);\ \underset{\sim}{m}A,\ f(\underset{\sim}{m}A))$$

are group morphisms (by Lemma 3.3 for W-theory).

Defining inclusions

$$S U_\pm(A,m) \longrightarrow S U_\pm(A,m+1)\ ;\ (f,\chi) \longmapsto (f,\chi)\oplus(1,0)$$

there is induced a group morphism

$$\pi' : S U_\pm(A) = \bigcup_{m=1}^{\infty} S U_\pm(A,m) \longrightarrow W_{2i+1}(A)$$

agreeing with π'_m on each $S U_\pm(A,m)$.

Denote the kernel of π' by $S H_\pm(A)$, calling its elements the <u>special</u> hamiltonian transformations.

<u>Theorem 5.6</u> The morphism

$$\pi' : S U_\pm(A) \longrightarrow W_{2i+1}(A)$$

is onto, inducing an isomorphism

$$S U_\pm(A)/S H_\pm(A) \cong W_{2i+1}(A)$$

of abelian groups.

Moreover, $S H_\pm(A)$ contains the commutator subgroup $[S U_\pm(A), S U_\pm(A)]$ of $S U_\pm(A)$ and it is generated by the elementary special hamiltonian transformations:

i) $\left(\begin{pmatrix} 1 & 0 \\ \theta\mp\theta^* & 1\end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & 0\end{pmatrix}\right) \in S U_\pm(A,m)$

for any based $\mp$ form $(\underset{\sim}{m}A, \theta)$

ii) $\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1}\end{pmatrix}, 0\right) \in S U_\pm(A,m)$

for any simple automorphism $\alpha : \underset{\sim}{m}A \longrightarrow \underset{\sim}{m}A$

iii) $\sigma\oplus\sigma\oplus\ldots\oplus\sigma \in S U_\pm(A,m)$

involving m copies of

$$\sigma = \left(\begin{pmatrix} 0 & \pm\gamma^{-1} \\ \gamma & 0\end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0\end{pmatrix}\right) \in S U_\pm(A,1)$$

where

$$\gamma : \underset{\sim}{A} \longrightarrow \underset{\sim}{A}^*\ ;\ a \longmapsto (b \longmapsto b\bar{a})$$

$\square$

Then Theorem 6.3 in [W] gives

<u>Corollary</u> The quotient $S H_\pm(A)/[S U_\pm(A), S U_\pm(A)]$ is generated by σ, so has order at most 2.

$\square$

W-theory is related to V-theory by:

<u>Theorem 5.7</u> There is an exact sequence

$$\ldots \longrightarrow \Omega_{(-)^{n+1}}(A) \longrightarrow W_n(A) \longrightarrow V_n(A) \longrightarrow \Omega_{(-)^n}(A) \longrightarrow \ldots$$

of abelian groups and morphisms, defined for $n \pmod 4$. The groups

$$\Omega_{\pm}(A) = \{\tau \in \tilde{K}_1(A) \mid \tau^* = \pm\tau \in \tilde{K}_1(A)\} \big/ \{\omega \pm \omega^* \mid \omega \in \tilde{K}_1(A)\}$$

are of exponent 2.

The morphisms $W_n(A) \longrightarrow V_n(A)$ are induced by the monoid morphisms $Z_n(A) \longrightarrow Y_n(A)$ which forget bases. The others are given by:

$$V_{2i}(A) \longrightarrow \Omega_+(A)\ ;\ (P,\theta) \longmapsto \tau(\underset{\sim}{P},\theta)$$

for any base $\underset{\sim}{P}$ of P, assumed free

$$V_{2i+1}(A) \longrightarrow \Omega_-(A)\ ;\ (Q,\phi;F,G) \longmapsto \tau(\pi'(Q,\phi;F,G))$$

π as in Theorem 4.2

$$\Omega_-(A) \longrightarrow W_{2i}(A)\ ;\ \tau(\alpha:\underset{\sim}{P}\to\underset{\sim}{P}) \longmapsto \left(\underset{\sim}{P}\oplus\underset{\sim}{P}^*, \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\right)$$

$$\Omega_+(A) \longrightarrow W_{2i-1}(A)\ ;\ \tau(\alpha:\underset{\sim}{P}\to\underset{\sim}{P}) \longmapsto (H_{\mp}(\underset{\sim}{P});\underset{\sim}{P},\alpha(\underset{\sim}{P})).$$

<u>Proof</u>: By analogy with that of Theorem 4.3, with torsions of automorphisms in $\Omega_{\pm}(A)$ replacing projective classes in $\Sigma_{\pm}(A)$ □

§6 <u>Functoriality</u>

All our constructions are functorial on the ground ring A. Let

$$f : A \longrightarrow B$$

be a morphism of ground rings (preserving the 1's and the involutions). Give B a (B,A)-bimodule structure by

$$B \times B \times A \longrightarrow B\ ;\ (b,x,a) \longmapsto b.x.f(a).$$

Given a f.g. projective left A-module P, let fP denote the f.g. projective left B-module $B\otimes_A P$, identifying $(fP)^*$ with $f(P^*)$. A morphism $\phi \in \mathrm{Hom}_A(P,Q)$ induces

$$f\phi = (1\otimes\phi : B\otimes_A P \longrightarrow B\otimes_A Q) \in \mathrm{Hom}_B(fP,fQ)$$

Given a $\begin{cases} \pm\text{ form} & (P,\theta) \\ \pm\text{ formation} & (Q,\phi;[F,\lambda],[G,\mu]) \end{cases}$

over A, there is defined a $\begin{cases} \pm\text{ form } f(P,\theta) = (fP, f\theta) \\ \pm\text{ formation } f(Q,\phi;[F,\lambda],[G,\mu]) = (fQ,f\phi;f[F,\lambda],f[G,\mu]) \end{cases}$

over B (with $f[F,\lambda] = [fF, f\lambda]$ etc.), and similarly for morphisms.

The induced monoid morphisms

$$f : X_n(A) \longrightarrow X_n(B)$$

are such that the squares

$$\begin{tikzcd} X_n(A) \arrow[r, "f"] \arrow[d, "\partial"'] & X_n(B) \arrow[d, "\partial"] \\ X_{n-1}(A) \arrow[r, "f"] & X_{n-1}(B) \end{tikzcd}$$

commute, inducing abelian group morphisms

$$f : U_n(A) \longrightarrow U_n(B) \qquad (n(\mathrm{mod}\ 4)).$$

Similarly for V-, W-theories.

The isomorphisms of Theorem 4.2, 5.6 and the exact sequences of Theorems 4.3, 5.7 are natural on A.

II Algebraic L-theory

§1 Laurent extensions

Let A be a ring with involution such as is considered in I., and let z be an invertible indeterminate over A, which commutes with every element of A. The Laurent extension of A by z, A_z, is the ring of polynomials $\sum_{j=-\infty}^{\infty} a_j z^j$ in z, z^{-1} with only a finite number of the coefficients $a_j \in A$ non-zero. Then A_z is an associative ring with a 1, under the usual addition and multiplication of polynomials. The function

$$\bar{\ } : A_z \longrightarrow A_z \ ; \ a = \sum_{j=-\infty}^{\infty} a_j z^j \longmapsto \bar{a} = \sum_{j=-\infty}^{\infty} \bar{a}_j z^{-j}$$

is an involution of A_z. The projection

$$\varepsilon : A_z \longrightarrow A \ ; \ \sum_{j=-\infty}^{\infty} a_j z^j \longmapsto \sum_{j=-\infty}^{\infty} a_j$$

is a ring morphism which preserves 1's (and the involutions), so every f.g. free A_z-module Q has a well-defined dimension, namely that of the f.g. free A-module εQ.

Thus A_z satisfies all the conditions imposed on A.

For example, if $A = \mathbb{Z}[\pi]$ (as in Example 0.1 of I), with $\pi = \pi_1(M)$ for some compact manifold M, then $A_z = \mathbb{Z}[\pi \times \mathbb{Z}]$, with $\pi \times \mathbb{Z} = \pi_1(M \times S^1)$.

Denote (left) A_z-modules by

$$M, N, P, Q \ldots$$

in general.

The injection

$$\bar{\varepsilon} : A \longrightarrow A_z \,;\, a \longmapsto a$$

splits ε, that is $\varepsilon \bar{\varepsilon} = 1$, and $\bar{\varepsilon} A$ is identified with A.

Every A_z-module Q can be regarded as an A-module by restricting the action of A_z to one of A.

A <u>modular A-base</u> of an A_z-module Q is an A-submodule Q of Q such that every

$x \in Q$ has a unique expression as

$$x = \sum_{j=-\infty}^{\infty} z^j x_j \in Q \quad (x_j \in Q)$$

with $\{x_j \in Q \mid x_j \neq 0\}$ finite, corresponding to an infinite direct sum

$$Q = \sum_{j=-\infty}^{\infty} z^j Q$$

of A-modules isomorphic to Q. Hence there is an A-module isomorphism

$$(\varepsilon Q =)\ Q/(z-1)Q \cong Q$$

and modular A-bases of isomorphic A_z-modules are isomorphic.

Given an A-module Q, define the A_z-module <u>freely generated</u> by Q, Q_z, to be the direct sum

$$Q_z = \sum_{j=-\infty}^{\infty} z^j Q$$

of a countably infinity of copies of Q with the action of A_z indicated – that is, $Q_z = \bar{\varepsilon} Q$. Then Q is a modular A-base of Q_z.

It is convenient to list here several properties of modular A-bases:

i) Every modular A-base Q of an A_z-module Q determines a dual modular A-base Q^* of Q^*, with

$$(z^k g)(z^j x) = g(x) \cdot z^{j-k} \in A_z$$
$$(g \in Q^*, x \in Q \quad j,k \in \mathbb{Z})$$

ii) For any A-modules P, Q give $\mathrm{Hom}_A(P,Q)$ a (left) A-module structure by

$$A \times \mathrm{Hom}_A(P,Q) \longrightarrow \mathrm{Hom}_A(P,Q)\,;\,(a,f) \longmapsto (x \mapsto a.f(x))$$

and similarly for A_z-modules.

Every $f \in \mathrm{Hom}_{A_z}(P_z, Q_z)$ defines $\sum_{j=-\infty}^{\infty} z^j f_j \in (\mathrm{Hom}_A(P,Q))_z$ by

$$f(x) = \sum_{j=-\infty}^{\infty} z^j f_j(x) \in A_z \quad (x \in P \quad f_j(x) \in Q)$$

and conversely, so that we may identify

$$\mathrm{Hom}_{A_z}(P_z, Q_z) = (\mathrm{Hom}_A(P,Q))_z \ .$$

Given $f \in \mathrm{Hom}_A(P,Q)$, let f also denote the element of $\mathrm{Hom}_{A_z}(P_z, Q_z)$ defined by

$$f : P_z \longrightarrow Q_z\,;$$
$$\sum_{j=-\infty}^{\infty} z^j x_j \longmapsto \sum_{j=-\infty}^{\infty} z^j f(x_j) \quad (x_j \in P)$$

iii) The A_z-module Q_z is $\begin{cases} \text{f.g. projective} \\ \text{f.g. free} \end{cases}$ iff Q is a $\begin{cases} \text{f.g. projective} \\ \text{f.g. free} \end{cases}$ A-module.

A based A-module Q generates a based A_z-module Q_z in the obvious way. Conversely, a based A_z-module Q determines a based modular A-base $\underset{\sim}{Q}$.

iv) Given an A-module Q define A-submodules

$$Q^+ = \sum_{j=0}^{\infty} z^j Q \qquad Q^- = \sum_{j=-\infty}^{-1} z^j Q$$

of Q_z. Then

$$\nu : Q_z = Q^+ \oplus Q^- \xrightarrow{(1\ 0)} Q^+$$

is the <u>positive projection</u> on Q

v) Let F, G be two modular A-bases of a f.g. free A_z-module Q. Then F, G are f.g. free A-modules and

$$z^N F^+ \subseteq G^+$$

for sufficiently large integers $N \geq 0$. For such N define the A-module

$$B_N^+(F,G) = z^N F^- \cap G^+ ,$$

a direct summand of Q (regarded as an A-mod[illegible]) with

$$G^+ = z^N F^+ \oplus B_N^+(F,G) .$$

If H is another modular A-base of Q, and if $M \geqslant 0$ is so large that $z^M G^+ \subseteq H^+$, then

$$B^+_{M+N}(F,H) = z^M B^+_N(F,G) \oplus B^+_M(G,H).$$

In particular, for $N_1 \geqslant 0$ so large that $z^{N_1} G^+ \subseteq F^+$,

$$z^N B^+_{N_1}(G,F) \oplus B^+_N(F,G) = B^+_{N+N_1}(G,G) = \sum_{j=0}^{N+N_1-1} z^j G\,,$$

so that, as G is f.g. free, $B^+_N(F,G)$ is a f.g. projective A-module.

Moreover, as

$$B^+_{N+1}(F,G) = B^+_N(F,G) \oplus z^N F,$$

and F is f.g. free, the projective class $[B^+_N(F,G)] \in \tilde{K}_0(A)$ does not depend on N.

The A-module isomorphism

$$B^+_N(F^*,G^*) \longrightarrow B^+_N(F,G)^*\,;$$
$$g \longmapsto (x \longmapsto [g(x)]_0)$$

is used as an identification, where $[a]_0 = a_0 \in A$ if $a = \sum_{j=-\infty}^{\infty} a_j z^j \in A_z$.

We now quote a principal result of algebraic K-theory (Chapter XII of Bass' "Algebraic K-theory").

Theorem There exists a natural direct sum decomposition

$$\tilde{K}_1(A_z) = \tilde{K}_1(A) \oplus \tilde{K}_0(A) \oplus \mathrm{Nil}^+(A) \oplus \mathrm{Nil}^-(A)$$

where $\mathrm{Nil}^\pm(A)$ is the subgroup of $\tilde{K}_1(A_z)$ generated by

$$\{\tau((1+\nu z^{\pm 1}): P_z \to P_z) \in \tilde{K}_1(A_z) \mid \nu \in \mathrm{Hom}_A(P,P) \text{ nilpotent}\}$$

The splitting is by injections

$$\bar{\varepsilon} : \tilde{K}_1(A) \longrightarrow \tilde{K}_1(A_z)\,;\ \tau(\alpha : F \to F) \longmapsto \tau(\alpha : F_z \to F_z)$$

$$\bar{B} : \tilde{K}_0(A) \longrightarrow \tilde{K}_1(A_z)\,;\ [P] \longmapsto \tau\left(\xi = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} : (P \oplus -P)_z \to (P \oplus -P)_z\right)$$

and by projections

$$\varepsilon : \tilde{K}_1(A_z) \longrightarrow \tilde{K}_1(A)\,;\ \tau\Big(\sum_{j=-\infty}^{\infty} z^j \alpha_j : F_z \to F_z\Big) \longmapsto \tau\Big(\sum_{j=-\infty}^{\infty} \alpha_j : F \to F\Big).$$

$$B : \tilde{K}_1(A_z) \longrightarrow \tilde{K}_0(A)\,;\ \tau(\alpha : F_z \to F_z) \longmapsto [B^+_N(F, \alpha(F))]$$

$\square$

Corollary The diagram of abelian groups and morphisms

$$\begin{array}{ccc}
\tilde{K}_1(A_z) & \xrightarrow{\ *\ } & \tilde{K}_1(A_z) \\
\varepsilon \big\downarrow \big\uparrow \bar{\varepsilon} & & \varepsilon \big\downarrow \big\uparrow \bar{\varepsilon} \\
\tilde{K}_1(A) & \xrightarrow{\ *\ } & \tilde{K}_1(A)
\end{array}$$

commutes (in the sense that $*\varepsilon = \varepsilon *$, $*\bar{\varepsilon} = \bar{\varepsilon} *$) and

$$\begin{tikzcd}
\tilde{K}_1(A_z) \arrow[r, "*"] \arrow[d, shift right, "B"'] & \tilde{K}_1(A_z) \arrow[d, shift right, "B"'] \\
\tilde{K}_0(A) \arrow[r, "*"'] \arrow[u, shift right, "\bar{B}"'] & \tilde{K}_0(A) \arrow[u, shift right, "\bar{B}"']
\end{tikzcd}$$

skewcommutes ($*B = -B*$, $*\bar{B} = -\bar{B}*$), where

$$*: \tilde{K}_1(A) \longrightarrow \tilde{K}_1(A)\ ;\ \tau(\alpha: E \to E) \longmapsto \tau(\alpha^*: E^* \to E^*)$$

$$*: \tilde{K}_0(A) \longrightarrow \tilde{K}_0(A)\ ;\ [P] \longmapsto [P^*]$$

are the duality involutions.

Moreover,

$$*: \tilde{K}_1(A_z) \longrightarrow \tilde{K}_1(A_z)$$

sends $Nil^{\pm}(A)$ onto $Nil^{\mp}(A)$.

□

Recalling the definitions of the groups

$$\Omega_{\pm}(A) = \{\tau \in \tilde{K}_1(A) \mid \tau^* = \pm\tau\} / \{\omega \pm \omega^* \mid \omega \in \tilde{K}_1(A)\}$$

$$\Sigma_{\pm}(A) = \{[P] \in \tilde{K}_0(A) \mid [P^*] = \pm[P]\} / \{[Q] \pm [Q^*] \mid [Q] \in \tilde{K}_0(A)\}$$

from I., it follows that there are defined morphisms

$$\Omega_{\pm}(A) \underset{\varepsilon}{\overset{\bar{\varepsilon}}{\rightleftarrows}} \Omega_{\pm}(A_z) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} \Sigma_{\mp}(A)$$

and hence a splitting

$$\Omega_{\pm}(A_z) = \Omega_{\pm}(A) \oplus \Sigma_{\mp}(A).$$

We wish to establish an analogous result in algebraic L-theory:

<u>Theorem 1.1</u> There exists a natural diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega_{(-)^{n+1}}(A) \arrow[r] \arrow[d, shift right, "\bar{\varepsilon}"'] & W_n(A) \arrow[r] \arrow[d, shift right, "\bar{\varepsilon}"'] & V_n(A) \arrow[r] \arrow[d, shift right, "\bar{\varepsilon}"'] & \Omega_{(-)^n}(A) \arrow[r] \arrow[d, shift right, "\bar{\varepsilon}"'] & \cdots \\
\cdots \arrow[r] & \Omega_{(-)^{n+1}}(A_z) \arrow[r] \arrow[u, shift right, "\varepsilon"'] \arrow[d, shift right, "B"'] & W_n(A_z) \arrow[r] \arrow[u, shift right, "\varepsilon"'] \arrow[d, shift right, "B"'] & V_n(A_z) \arrow[r] \arrow[u, shift right, "\varepsilon"'] \arrow[d, shift right, "B"'] & \Omega_{(-)^n}(A_z) \arrow[r] \arrow[u, shift right, "\varepsilon"'] \arrow[d, shift right, "B"'] & \cdots \\
\cdots \arrow[r] & \Sigma_{(-)^n}(A) \arrow[r] \arrow[u, shift right, "\bar{B}"'] & V_{n-1}(A) \arrow[r] \arrow[u, shift right, "\bar{B}"'] & U_{n-1}(A) \arrow[r] \arrow[u, shift right, "\bar{B}"'] & \Sigma_{(-)^{n-1}}(A) \arrow[r] \arrow[u, shift right, "\bar{B}"'] & \cdots
\end{tikzcd}$$

of abelian groups and morphisms, defined for $n \pmod 4$, with the squares of shape $\downarrow\overset{\longrightarrow}{\underset{\longrightarrow}{\ }}\downarrow$, $\uparrow\overset{\longrightarrow}{\underset{\longrightarrow}{\ }}\uparrow$ commuting.

The rows are the exact sequences of Theorems 4.3, 5.7 in I. The columns are split short exact, with $\varepsilon\bar{\varepsilon} = 1$, $B\bar{B} = 1$ whenever defined, corresponding to direct sums

$$W_n(A_z) = W_n(A) \oplus V_{n-1}(A)$$

$$V_n(A_z) = V_n(A) \oplus U_{n-1}(A).$$

□

§2. Proof of Theorem 1.1 (n odd)

Given A_z-modules $\mathcal{P}, \mathcal{Q}$ and $\Theta \in \mathrm{Hom}_{A_z}(\mathcal{P}, \mathcal{Q}^*)$ define $[\Theta]_0 \in \mathrm{Hom}_A(\mathcal{P}, \mathrm{Hom}_A(\mathcal{Q}, A))$ by

$$[\Theta]_0(x)(y) = [\Theta(x)(y)]_0 \in A \quad (x \in \mathcal{P}, y \in \mathcal{Q})$$

where $[a]_0 = a_0 \in A$ if $a = \sum_{j=-\infty}^{\infty} a_j z^j \in A_z$.

Given A-modules P, Q and $\Theta = \sum_{j=-\infty}^{\infty} z^j \Theta_j \in \mathrm{Hom}_{A_z}(P_z, Q_z^*)$ (with $\Theta_j \in \mathrm{Hom}_A(P, Q^*)$), $[\Theta]_0 \in \mathrm{Hom}_A(P_z, Q_z^*)$ is given by

$$[\Theta]_0(z^j x)(z^k y) = \Theta_{k-j}(x)(y) \in A \quad (x \in P, y \in Q \ \ j,k \in \mathbb{Z})$$

and

$$\Theta(x)(y) = \sum_{j=-\infty}^{\infty} z^j([\Theta]_0(x)(z^j y)) \in A_z \, (x \in P, y \in Q).$$

<u>Lemma 2.1</u> Let $(\mathcal{Q}, \Phi)$ be a non-singular $\pm$ form over A_z, and let C, D be complementary A-submodules of $\mathcal{Q}$ such that C is finitely generated and

$$[\langle C, D \rangle_\Phi]_0 = \{0\} \subseteq A.$$

Then $(C, i^*[\Phi]_0 i)$ is a non-singular $\pm$ form over A, where $i : C \longrightarrow \mathcal{Q}$ is the inclusion. $\square$

In general, $(C, i^*[\Phi]_0 i)$ will be denoted by $(C, [\Phi]_0)$.

Define

$$B : V_{2i+1}(A_z) \longrightarrow U_{2i}(A) ;$$

$$(\mathcal{Q}, \Phi ; \mathcal{F}, \mathcal{G}) \longmapsto (B_N^+(F \oplus F^*, G \oplus G^*), [\Phi]_0)$$

where $\mathcal{F}$ and $\mathcal{G}$ are free, with modular A-bases F, G respectively and $N \geqslant 0$ so large that

$$z^N(F \oplus F^*)^+ \subseteq (G \oplus G^*)^+$$

for some choice of hamiltonian complements $\mathcal{F}^*, \mathcal{G}^*$ to $\mathcal{F}, \mathcal{G}$ in $(\mathcal{Q}, \Phi)$ with dual modular A-bases F^*, G^*. Now

$$[\langle B_N^+(F \oplus F^*, G \oplus G^*), z^N(F \oplus F^*)^+ \oplus (G \oplus G^*)^- \rangle_\Phi]_0 = \{0\} \subseteq A$$

so that the hypotheses of Lemma 2.1 are satisfied, and $(B_N^+(F \oplus F^*, G \oplus G^*), [\Phi_0])$ is a non-singular $\pm$ form over A, and does represent an element of $U_{2i}(A)$. It does not depend on N because increasing N by 1 adds on $H_\pm(z^N F) = 0 \in U_{2i}(A)$. Nor does the choice of $\mathcal{F}^*$ matter: for $N \geqslant 0$ so large that

$$z^N F^+ \subseteq (G \oplus G^*)^+$$

define the A-module

$$E_N^+(F, G\oplus G^*) = \{x \in (G\oplus G^*)^+ \mid [\langle z^N F^+, x\rangle_\Phi]_0 = \{0\} \subseteq A\}$$

and observe that the $\pm$ form defined over A by

$$(E_N^+(F, G\oplus G^*)/z^N F^+ \,, \, [\Phi]_0)$$

coincides with $(B_N^+(F\oplus F^*, G\oplus G^*), [\Phi]_0)$ when N is so large that $z^N(F\oplus F^*)^+ \subseteq (G\oplus G^*)^+$, as then

$$E_N^+(F, G\oplus G^*) = (\mathcal{F}\oplus z^N F^{*-}) \cap (G\oplus G^*)^+$$
$$= z^N F^+ \oplus B_N^+(F\oplus F^*, G\oplus G^*).$$

The choice of $\mathcal{F}^*$ did not enter in this new definition. The choice of $\mathcal{G}^*$ may be dealt with similarly.

Next, suppose $(Q, \Phi; \mathcal{F}, \mathcal{G}) = 0 \in V_{2i+1}(A)$, and consider the generic cases:

i) $\mathcal{F}$ and $\mathcal{G}$ are hamiltonian complements in (Q, Φ). Set $F^* = G$, $G^* = F$, $N = 0$ to obtain $B_N^+(F\oplus F^*, G\oplus G^*) = 0$, and so $B(Q, \Phi; \mathcal{F}, \mathcal{G}) = 0 \in U_{2i}(A)$

ii) $\mathcal{F}$ and $\mathcal{G}$ share a hamiltonian complement in (Q, Φ). Set $F^* = G^*$ to obtain

$$B(Q, \Phi; \mathcal{F}, \mathcal{G}) = B(Q, \Phi; \mathcal{F}^*, \mathcal{G}^*) \quad \text{(by symmetry of definition)}$$
$$= 0 \in U_{2i}(A) \quad (\text{taking } N = 0).$$

It now only remains to verify that the choice of modular A-bases F, G of $\mathcal{F}, \mathcal{G}$ is immaterial to $B(Q, \Phi; \mathcal{F}, \mathcal{G}) \in U_{2i}(A)$.

Let $\hat{F}$ be another modular A-base of $\mathcal{F}$, with dual modular A-base $\hat{F}^*$ of $\mathcal{F}^*$, and let $\hat{N} \geq 0$ be so large that

$$z^{\hat{N}}(\hat{F}\oplus\hat{F}^*) \subseteq (F\oplus F^*)^+ .$$

Then

$$(B_{N+\hat{N}}^+(\hat{F}\oplus\hat{F}^*, G\oplus G^*), [\Phi]_0)$$
$$= (z^N B_{\hat{N}}^+(\hat{F}\oplus\hat{F}^*, F\oplus F^*), [\Phi]_0) \oplus (B_N^+(F\oplus F^*, G\oplus G^*), [\Phi]_0)$$
$$= H_\pm(z^N B_{\hat{N}}^+(\hat{F}, F)) \oplus (B_N^+(F\oplus F^*, G\oplus G^*), [\Phi]_0)$$
$$= (B_N^+(F\oplus F^*, G\oplus G^*), [\Phi]_0) \in U_{2i}(A),$$

so that $\hat{F}$ will do as well as F.

Similarly for choice of G.

Hence

$$B : V_{2i+1}(A_z) \longrightarrow U_{2i}(A)$$

is well-defined.

The composite

$$V_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} V_{2i+1}(A_z) \xrightarrow{B} U_{2i}(A)$$

is 0 because

$$B\bar{\varepsilon}(Q, \phi; F, G) = B(Q_z, \phi; F_z, G_z) = (B_0^+(F\oplus F^*, G\oplus G^*), \phi) = 0 \in U_{2i}$$

The diagram

$$\begin{array}{ccc} V_{2i+1}(A_z) & \longrightarrow & \Omega_-(A_z) \\ \Big\downarrow B & & \Big\downarrow B \\ U_{2i}(A) & \longrightarrow & \Sigma_+(A) \end{array}$$

commutes, because given $(Q,\mathfrak{X};\mathcal{F},\mathcal{G}) \in V_{2i+1}(A_z)$, with

$$\pi^{-1}(Q,\mathfrak{X};\mathcal{F},\mathcal{G}) = ((\alpha,\nu):(Q,\mathfrak{X}) \longrightarrow (Q,\mathfrak{X})) \in U_\pm(A_z)/\mathcal{K}_\pm(A_z)$$

such that $\alpha(\mathcal{F}) = \mathcal{G}$ (in the notation of Theorem 4.2 of I) then

$$\begin{aligned} B(\tau(\alpha)) &= [B_N^+(F\oplus F^*, \alpha(F\oplus F^*))] \\ &= [B_N^+(F\oplus F^*, G\oplus G^*)] \in \Sigma_+(A), \end{aligned}$$

for any modular base $F \subset \mathcal{F}$, with $G = \alpha(F)$.

Define

$$\begin{aligned} \bar{B}: U_{2i}(A) &\longrightarrow V_{2i+1}(A_z); \\ (Q,\phi) &\longmapsto ((Q_z\oplus Q_z, \phi\oplus-\phi)\oplus H_\pm(-Q_z); \\ &\qquad \Delta_{(Q_z,\phi)}\oplus -Q_z, \xi\Delta_{(Q_z,\phi)}\oplus -Q_z) \end{aligned}$$

(where $-Q$ is any f.g. projective A-module such that $Q\oplus -Q$ is free) with

$$\xi = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} : Q_z\oplus Q_z \longrightarrow Q_z\oplus Q_z .$$

This is well-defined because

$$\xi\{((x,0),(0,g),(0,y)) \in (P\oplus P^*)_z\oplus(P\oplus P^*)_z\oplus(-(P\oplus P^*)\oplus(-(P\oplus P^*)_z)^*) \mid x\in P_z,\ g\in P_z^*,\ y\in -(P\oplus P^*)_z\}$$

is a hamiltonian complement in $H_\pm(P_z)\oplus -H_\pm(P_z)\oplus H_\pm(-(P\oplus P^*)_z)$, to both $\Delta_{H_\pm(P_z)}\oplus-(P\oplus P^*)_z$ and $\xi\Delta_{H_\pm(P_z)}\oplus -(P\oplus P^*)_z$, so that

$$\bar{B}H_\pm(P) = 0 \in V_{2i+1}(A_z)$$

for any f.g. projective A-module P.

The composite

$$U_{2i}(A) \xrightarrow{\bar{B}} V_{2i+1}(A_z) \xrightarrow{\varepsilon} V_{2i+1}(A)$$

is 0 because

$$\begin{aligned} \varepsilon\bar{B}(Q,\phi) &= ((Q\oplus Q, \phi\oplus-\phi)\oplus H_\pm(-Q); \Delta_{(Q,\phi)}\oplus -Q, \Delta_{(Q,\phi)}\oplus -Q) \\ &= 0 \in V_{2i+1}(A). \end{aligned}$$

The diagram

$$\begin{array}{ccc} U_{2i}(A) & \longrightarrow & \Sigma_+(A) \\ \Big\downarrow \bar{B} & & \Big\downarrow \bar{B} \\ V_{2i+1}(A_z) & \longrightarrow & \Omega_-(A_z) \end{array}$$

commutes, because given $(Q,\phi) \in U_{2i}(A)$

$$\begin{aligned} \tau(\pi^{-1}\bar{B}(Q,\phi)) &= \tau(\xi\oplus 1 : (Q_z\oplus Q_z)\oplus(-Q_z\oplus -Q_z^*) \longrightarrow (Q_z\oplus Q_z)\oplus(-Q_z\oplus -Q_z^*)) \\ &= \bar{B}([Q]) \in \Omega_-(A_z). \end{aligned}$$

The composite

$$U_{2i}(A) \xrightarrow{\bar{B}} V_{2i+1}(A) \xrightarrow{B} U_{2i}(A)$$

is the identity, because for each $(Q,\phi) \in U_{2i}(A)$

$$\begin{aligned} B\bar{B}(Q,\phi) &= B((Q_z \oplus Q_z, \phi \oplus -\phi) \oplus H_\pm(-Q_z); \Delta_{(Q_z,\phi)} \oplus -Q_z, \xi\Delta_{(Q_z,\phi)} \oplus -Q_z) \\ &= (B_1^+(\Delta_{(Q,\phi)} \oplus \Delta^*_{(Q^*,\psi)}, \xi(\Delta_{(Q,\phi)} \oplus \Delta^*_{(Q^*,\psi)})), \phi \oplus -\phi) \oplus H_\pm(-Q) \\ &= (B_1^+(Q \oplus Q, Q \oplus zQ), \phi \oplus -\phi) \oplus H_\pm(-Q) \\ &= (Q,\phi) \in U_{2i}(A) \end{aligned}$$

where $\Delta^*_{(Q^*,\psi)}$ is any hamiltonian complement to $\Delta_{(Q,\phi)}$ in $(Q \oplus Q, \phi \oplus -\phi)$, as given by Lemma 1.4 of I.

It now only remains to verify the exactness of the sequence

$$V_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} V_{2i+1}(A_z) \xrightarrow{B} U_{2i}(A) .$$

This will be done by characterizing the $\pm$formations over A_z which are isomorphic to the ones obtained from $\pm$formations over A via $\bar{\varepsilon}: A \to A_z$ (in Lemma 2.2, below), and then using the hamiltonian transformation of Lemma 2.3 to show that every element of $\ker(B: V_{2i+1}(A_z) \to U_{2i}(A))$ has a representative satisfying that criterion.

<u>Lemma 2.2</u> A $\pm$formation $(Q,\Phi;\mathcal{F},\mathcal{G})$ over A_z is isomorphic to $\bar{\varepsilon}(Q,\phi;F,G)$ for some $\pm$formation $(Q,\phi;F,G)$ over A iff $\mathcal{F}$ has a modular A-base F such that for some hamiltonian complement $\mathcal{F}^*$ to $\mathcal{F}$ in (Q,Φ) the positive projection on $F \oplus F^*$

$$\nu : Q = \mathcal{F} \oplus \mathcal{F}^* \longrightarrow (F \oplus F^*)^+$$

preserves $\mathcal{G}$, that is $\nu(\mathcal{G}) \subseteq \mathcal{G}$.

<u>Proof</u>: It is clear that $\bar{\varepsilon}(Q,\phi;F,G)$ satisfies the condition, for any $\pm$formation $(Q,\phi;F,G)$ over A.

Conversely, assume the condition holds for $(Q,\Phi;\mathcal{F},\mathcal{G})$. The A-module morphism

$$\xi = z(1-\nu)z^{-1}\nu : Q \longrightarrow Q$$

sends Q onto $F \oplus F^*$, and has the property that

$$x = \sum_{j=-\infty}^{\infty} z^j \xi z^{-j} x \in (F \oplus F^*)_z = Q$$

for every $x \in Q$.

Now $\nu(\mathcal{G}) \subseteq \mathcal{G}$, so that

$$\xi(\mathcal{G}) = \mathcal{G} \cap (F \oplus F^*) ,$$

and is therefore a modular A-base G of $\mathcal{G}$ contained in $F \oplus F^*$. Thus, up to isomorphism of $\pm$formations over A_z,

$$(Q,\Phi;\mathcal{F},\mathcal{G}) = (H_\pm(\mathcal{F});\mathcal{F},\mathcal{G}) = \bar{\varepsilon}(H_\pm(F);F,G).$$

□

Lemma 2.3 Given a morphism of $\pm$forms over A

$$(f,\chi):(P,\theta)\longrightarrow(Q,\phi)$$

define the automorphism

$$H(f,\chi)=\left(\begin{pmatrix}1 & -f & 0\\ 0 & 1 & 0\\ f^*(\phi\pm\phi^*) & -\theta & 1\end{pmatrix},\begin{pmatrix}0 & -\phi f & 0\\ 0 & \chi & 0\\ 0 & 0 & 0\end{pmatrix}\right),$$

$$:(Q,\phi)\oplus H_\pm(P)\longrightarrow(Q,\phi)\oplus H_\pm(P).$$

If (Q,ϕ) is non-singular, the automorphism

$$h'=H(f)\oplus(1,0)$$

$$:(Q',\phi')=((Q,\phi)\oplus H_\pm(P))\oplus((Q,-\phi)\oplus H_\pm(-P\oplus -Q))\longrightarrow(Q',\phi')$$

is a hamiltonian transformation, that is

$$(Q',\phi';L',h'(L'))=0\in V_{2i+1}(A)$$

for any free lagrangian L' of (Q',ϕ').

Proof: The automorphism $h':(Q',\phi')\rightarrow(Q',\phi')$ preserves the free lagrangian

$$L=\{(x,y,x)\in Q\oplus(P\oplus P^*)\oplus Q\mid x\in Q,y\in \overset{*}{P}\}\oplus -P^*\oplus -Q$$

so that it is necessarily the composite

$$\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{*-1}\end{pmatrix}\begin{pmatrix}1 & \beta\mp\beta^*\\ 0 & 1\end{pmatrix}:L\oplus L^*\longrightarrow L\oplus L^*$$

of elementary hamiltonian transformations, for any hamiltonian complement L^* (cf. Theorem 4.2 of I.)

We now prove that

$$V_{2i+1}(A)\xrightarrow{\varepsilon}V_{2i+1}(A_z)\xrightarrow{B}U_{2i}(A)$$

is exact. Given $(Q,\Phi;F,G)\in\ker(B:V_{2i+1}(A_z)\rightarrow U_{2i}(A))$ there exists $N\geq 0$ so large that $(B_N^+(F\oplus F^*,G\oplus G^*),[\Phi]_0)$ is trivial. Denoting the A-module $B_N^+(F\oplus F^*,G\oplus G^*)$ by P, let $\mathcal{P}=P_z$, the f.g. projective A_z-module freely generated by P. Define an A_z-module morphism

$$f:\mathcal{P}\longrightarrow Q$$

by sending elements of the modular A-base P to themselves in Q, and extending A_z-linearly.

Then $f^*\Phi f\in \mathrm{Hom}_{A_z}(\mathcal{P},\mathcal{P}^*)$ can be expressed as

$$f^*\Phi f=[\Phi]_++[\Phi]_0+[\Phi]_-\in\mathrm{Hom}_{A_z}(\mathcal{P},\mathcal{P}^*)$$

with

$$[\Phi]_+(P)\subseteq\sum_{j=1}^{\infty}z^jP^*,\ [\Phi]_-(P)\subseteq\sum_{j=-\infty}^{-1}z^jP^*,\ [\Phi]_0(P)\subseteq P^*.$$

Choose hamiltonian complements L,L^* in $(P,[\Phi]_0)$.

Denote $H_\pm(L_z)$ by $(\mathcal{P},\Psi)$, so that

$$[\Phi]_0-\Psi=\chi\mp\chi^*:\mathcal{P}\longrightarrow\mathcal{P}^*$$

for some $\mp$form $(\mathcal{P},\chi)$ over A_z.

Consider now the composite automorphism

$$(h,\rho) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\begin{smallmatrix} 0 & z \\ 0 & 0 \end{smallmatrix}\right) \end{pmatrix} \right)$$

$$\cdot \left(\begin{pmatrix} 1 & -f\xi & 0 \\ 0 & 1 & 0 \\ \xi^* f^*(\Phi \pm \Phi^*) & -\xi^* \Theta \xi & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\Phi f \xi & 0 \\ 0 & \xi^*([\Phi]_- + \eta)\xi & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

$$: (Q,\Phi) \oplus H_\pm(\mathcal{P}) \longrightarrow (Q,\Phi) \oplus H_\pm(\mathcal{P})$$

where

$$\eta = \begin{pmatrix} 0 & \mp z \\ z^{-1} & 0 \end{pmatrix} : \mathcal{P}^* = L_z^* \oplus L_z \longrightarrow L_z \oplus L_z^* = \mathcal{P} ,$$

$$\xi = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} : \mathcal{P} = L_z \oplus L_z^* \longrightarrow L_z \oplus L_z^* = \mathcal{P}$$

and

$$\Theta = [\Phi]_+ \pm [\Phi]_-^* + \Psi \in \mathrm{Hom}_{A_z}(\mathcal{P}, \mathcal{P}^*) .$$

Defining the positive projection

$$\nu : Q \oplus (\mathcal{P} \oplus \mathcal{P}^*) \longrightarrow ((G \oplus G^*) \oplus (P \oplus P^*))^+$$

and the A-module projection

$$\beta : Q = P \oplus (z^N (F \oplus F^*)^+ \oplus (G \oplus G^*)^-) \xrightarrow{(1\ 0)} P ,$$

note that

$$\nu h(x,y) = \begin{cases} h(x,y) & x \in z^N F^+ ,\ y \in P^+ \\ h(0, \xi^{-1}\beta(f\xi(y) - x)) & x \in z^N F^- ,\ y \in P^- \end{cases}$$

whence

$$\nu h(\mathcal{F} \oplus \mathcal{P}) \subseteq h(\mathcal{F} \oplus \mathcal{P}) .$$

The product decomposition used to define (h,ρ) shows that the automorphism $(h',\rho') = (h,\rho) \oplus (1,0)$ of $(Q',\Phi') = ((Q,\Phi) \oplus H_\pm(\mathcal{P})) \oplus H_\pm(-\mathcal{P})$ is a hamiltonian transformation over A_z : the matrix involving η is an elementary hamiltonian transformation, while the other corresponds to the hamiltonian transformation generated by the morphism of $\pm$ forms over A_z

$$(f\xi, \xi^*([\Phi]_- + \eta)\xi) : (\mathcal{P}, \xi^* \Theta \xi) \longrightarrow (Q,\Phi)$$

in the sense of Lemma 2.3.

The lagrangians $\mathcal{F}' = \mathcal{F} \oplus \mathcal{P} \oplus -\mathcal{P}$, $\mathcal{G}' = \mathcal{G} \oplus \mathcal{P} \oplus -\mathcal{P}$ of (Q',Φ') are such that

$$(Q,\Phi;\mathcal{F},\mathcal{G}) = (Q',\Phi';\mathcal{F}',\mathcal{G}') = (Q',\Phi'; h'(\mathcal{F}'),\mathcal{G}') \in V_{2i+1}(A$$

using the V-theory sum formula of Lemma 3.3 of I. The last representative $\pm$formation satisfies the hypothesis of Lemma 2.2 with the roles played by $\mathcal{F}$ and $\mathcal{G}$ reversed – this is clearly all right for non-singular $\pm$formations.

Thus

$$(Q, \Phi; \mathcal{F}, \mathcal{G}) \in \mathrm{im}(\Xi : V_{2i+1}(A) \longrightarrow V_{2i+1}(A_Z)),$$

completing the proof of the part of Theorem 1.1 relating to $V_n(A_Z)$ with n odd.

We now give the analogous constructions for W-theory.

Define

$$B : W_{2i+1}(A_Z) \longrightarrow V_{2i}(A);$$

$$(\underset{\sim}{Q}, \Phi; \underset{\sim}{\mathcal{F}}, \underset{\sim}{\mathcal{G}}) \longmapsto (B_N^{+}(F \oplus F^*, G \oplus G^*), [\Phi]_0)$$

where F is the modular A-base of $\mathcal{F}$ generated by the given A_Z-base, and similarly for $G, \mathcal{G}$. Then

$$[B_N^{+}(F \oplus F^*, G \oplus G^*)] = 0 \in \tilde{K}_0(A),$$

as required for V-theory, because it is the image under $B : \tilde{K}_1(A_Z) \longrightarrow \tilde{K}_0(A)$ of the automorphism of $\underset{\sim}{Q}$ taking $\underset{\sim}{\mathcal{F}} \oplus \underset{\sim}{\mathcal{F}}^*$ to $\underset{\sim}{\mathcal{G}} \oplus \underset{\sim}{\mathcal{G}}^*$, which is simple by construction (cf. §5 of I.), so that $B : W_{2i+1}(A_Z) \longrightarrow V_{2i}(A)$ is well-defined.

The composite

$$W_{2i+1}(A) \xrightarrow{\Xi} W_{2i+1}(A_Z) \xrightarrow{B} V_{2i}(A)$$

is 0, as for V-theory.

The square

$$\begin{tikzcd}
\Omega_+(A_Z) \arrow[r] \arrow[d, "B"'] & W_{2i+1}(A_Z) \arrow[d, "B"] \\
\Sigma_-(A) \arrow[r] & V_{2i}(A)
\end{tikzcd}$$

commutes, sending $\tau(\alpha : \underset{\sim}{F}_Z \longrightarrow \hat{F}_Z) \in \Omega_+(A_Z)$ to $H_\pm(B_N^{+}(F, \alpha(F))) \in V_{2i}(A)$ both ways.

Define

$$\bar{B} : V_{2i}(A) \longrightarrow W_{2i+1}(A_Z);$$

$$(Q, \phi) \longmapsto ((\underset{\sim}{Q \oplus Q})_Z, \phi \oplus -\phi; \Delta_{(\underset{\sim}{Q}_Z, \phi)}, \xi \Delta_{(\underset{\sim}{Q}_Z, \phi)})$$

where $\xi = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} : Q_Z \oplus Q_Z \longrightarrow Q_Z \oplus Q_Z$,

with $\underset{\sim}{Q}$ any base of Q (assumed free), and $(\underset{\sim}{Q \oplus Q}, \phi \oplus -\phi)$ any hamiltonian base extending $\Delta_{(\underset{\sim}{Q}, \phi)}$

Then $\bar{B}(Q,\phi)$ is just

$$\pi'((\xi,0):(Q_z\oplus Q_z,\phi\oplus-\phi)\longrightarrow(Q_z\oplus Q_z,\phi\oplus-\phi)),$$

in the terminology of Theorem 5.6 of I., as

$$\tau(\xi)=\bar{B}(-[Q])=0\in\tilde{K}_1(A_z),$$

so that we are dealing with an element of the special unitary group $SU_{\pm}(A_z)$.

The composites

$$V_{2i}(A)\xrightarrow{\bar{B}}W_{2i+1}(A_z)\xrightarrow{\varepsilon}W_{2i+1}(A)$$
$$V_{2i}(A)\xrightarrow{\bar{B}}W_{2i+1}(A_z)\xrightarrow{B}V_{2i}(A)$$

are 0, 1 as before.

The square

$$\begin{tikzcd}
\Sigma_-(A) \arrow[r] \arrow[d, "\bar{B}"'] & V_{2i}(A) \arrow[d, "\bar{B}"] \\
\Omega_+(A_z) \arrow[r] & W_{2i+1}(A_z)
\end{tikzcd}$$

commutes, sending $[P]\in\Sigma_-(A)$ to

$$((\underset{\sim}{Q\oplus Q})_z,\phi\oplus-\phi;(\underset{\sim}{P\oplus P^*})_z,\xi(\underset{\sim}{P\oplus P^*})_z)\in W_{2i+1}(A_z)$$

both ways round, where $(Q,\phi)=H_{\pm}(P)$, and $(\underset{\sim}{P\oplus P^*})$ is any base.

The proof that

$$0\longrightarrow W_{2i+1}(A)\xrightarrow{\varepsilon}W_{2i+1}(A_z)\xrightarrow{B}V_{2i}(A)\longrightarrow 0$$

is a split short exact sequence is as for V-theory, using the following versions of Lemmas 2.2, 2.3:

Lemma 2.4 Given $(Q,\Phi,\mathcal{F},\mathcal{G})\in\ker(B:W_{2i+1}(A_z)\to V_{2i}(A))$ and a positive projection

$$\nu:Q=\mathcal{G}\oplus\mathcal{G}^*\longrightarrow(G\oplus G^*)^+$$

such that $\nu(\mathcal{F})\subseteq\mathcal{F}$, for some modular A-base G of $\mathcal{G}$ and hamiltonian complement $\mathcal{G}^*$, then

$$(Q,\Phi;\mathcal{F},\mathcal{G})\in\operatorname{im}(\varepsilon:W_{2i+1}(A)\longrightarrow W_{2i+1}(A_z))$$

Proof: As in the proof of Lemma 2.2

$$F'=\mathcal{F}\cap(G\oplus G^*)=z(1-\nu)z^{-1}\nu(\mathcal{F})$$

is a modular A-base of $\mathcal{F}$. It is by no means clear, however, that we can choose an A-base $\underset{\sim}{F}'$ such that

$$(\underset{\sim}{Q},\Phi;\underset{\sim}{F},\underset{\sim}{F}')=0\in W_{2i+1}(A_z),$$

but if such an A-base exists it is immediate from the W-theory analogue of the sum formula of Lemma 3.3 of I, that

$$(\underset{\sim}{Q}, \Phi; \mathcal{F}, \underset{\sim}{g}) = (\underset{\sim}{Q}, \Phi; \mathcal{F}', \underset{\sim}{g})$$
$$= \bar{\varepsilon}(\underset{\sim}{Q}, \phi; \underset{\sim}{F}', \underset{\sim}{G}) \in \operatorname{im}(\bar{\varepsilon}: W_{2i+1}(A) \to W_{2i+1}(A_z))$$

where $Q = G \oplus G^*$.

Let $\underset{\sim}{f} = (f_1, f_2, \ldots, f_m)$ be the given A_z-base of $\mathcal{F}$.

Let $\underset{\sim}{f}' = (f_1', f_2', \ldots, f_m')$ be any A-base of F'.

Then $\underset{\sim}{f}'$ is also an A_z-base of $\underset{\sim}{F}'$.

Recall from §1 that there exists a direct sum decomposition

$$\tilde{K}_1(A_z) = \bar{\varepsilon}\tilde{K}_1(A) \oplus \bar{B}\tilde{K}_0(A) \oplus \mathrm{Nil}^+(A) \oplus \mathrm{Nil}^-(A),$$

and from §3 of I. that we can regard $\tilde{K}_1(A)$ as the group of stable isomorphism classes of triples $(Q, \underset{\sim}{f}, \underset{\sim}{g})$ given by a f.g. free A-module Q and A-bases $\underset{\sim}{f}, \underset{\sim}{g}$.

Express $\tau' = (\mathcal{F}, \underset{\sim}{f}, \underset{\sim}{f}') \in \tilde{K}_1(A_z)$

as

$$\tau' = \bar{\varepsilon}(a) \oplus \bar{B}(b) \oplus c^+ \oplus c^-$$
$$\in \bar{\varepsilon}\tilde{K}_1(A) \oplus \bar{B}\tilde{K}_0(A) \oplus \mathrm{Nil}^+(A) \oplus \mathrm{Nil}^-(A).$$

A judicious choice of $\underset{\sim}{f}'$ ensures that

$$\varepsilon\tau' = a = 0 \in \tilde{K}_1(A)$$

at least (allowing stabilization, if necessary).

Let $\mathcal{F}^*$ be a hamiltonian complement to $\mathcal{F}$ in (Q, Φ), with $\underset{\sim}{f}^*, \underset{\sim}{f}'^*$ the A_z-bases of $\mathcal{F}^*$ dual to $\underset{\sim}{f}, \underset{\sim}{f}'$ respectively. Now $\underset{\sim}{f}' \subseteq G \oplus G^*$, so

$$\langle z(1-\nu)z^{-1}\nu f_j'^*, f_k' \rangle_{[\Phi]_0} = \langle f_j'^*, f_k' \rangle_{[\Phi]_0} = \left\{\begin{matrix} 1 & \text{if } j=k \\ 0 & \text{otherwise} \end{matrix}\right\} \in A$$

Thus $z(1-\nu)z^{-1}\nu\underset{\sim}{f}'^*$ is the A-base of a direct complement to F' in $G \oplus G^*$ (not a lagrangian, in general) and F' is a lagrangian of $(G \oplus G^*, [\Phi]_0)$.

Let $\underset{\sim}{f}''^*$ be the A-base dual to $\underset{\sim}{f}'$ of some hamiltonian complement to F' in $(G \oplus G^*, [\Phi]_0)$.

Now $\underset{\sim}{f}' \oplus \underset{\sim}{f}'^*$ and $\underset{\sim}{f}' \oplus \underset{\sim}{f}''^*$ are both hamiltonian A_z-bases of (Q, Φ) extending $\underset{\sim}{f}'$, so that

$$(Q, \underset{\sim}{f}' \oplus \underset{\sim}{f}'^*, \underset{\sim}{f}' \oplus \underset{\sim}{f}''^*) = 0 \in \tilde{K}_1(A_z)$$

Let $\underset{\sim}{g}$ be the given A_z-base of G, with $\underset{\sim}{g}^*$ the dual A_z-base of G^*, the given hamiltonian complement to G in (Q,Φ). Then

$$(Q,\ \underset{\sim}{f}\oplus\underset{\sim}{f}^*,\ \underset{\sim}{g}\oplus\underset{\sim}{g}^*) = 0 \in \tilde{K}_1(A_z)$$

(by construction of $Z_{2i+1}(A_z)$ – c.f. §5 of I.)

Hence

$$\begin{aligned}\tau'-\tau'^* &= (Q,\ \underset{\sim}{f}\oplus\underset{\sim}{f}^*,\ \underset{\sim}{f}'\oplus\underset{\sim}{f}'^*)\\ &= (Q,\ \underset{\sim}{g}\oplus\underset{\sim}{g}^*,\ \underset{\sim}{f}'\oplus\underset{\sim}{f}'^*)\\ &\in \operatorname{im}(\bar{\varepsilon}:\hat{K}_1(A)\longrightarrow\hat{K}_1(A_z))\cap\ker(\varepsilon:\tilde{K}_1(A_z)\longrightarrow\tilde{K}_1(A))\\ &= \{0\}\end{aligned}$$

and

$$\tau'^* = \tau' \in \hat{K}_1(A_z).$$

We can now express τ' as

$$\tau' = \bar{B}(b)\oplus c^+\oplus(c^+)^* \in \bar{B}\hat{K}_0(A)\oplus Nil^+(A)\oplus Nil^-(A)$$

where $b^* = -b\in\tilde{K}_0(A)$ (using all the information given by the Corollary quoted in §1).

Calculating directly,

$$\begin{aligned}b = B\tau' &= [B_N^+(F,F')]\\ &= [z^N\hat{F}^-\cap F'^+] = [\nu(z^NF^-)]\in\tilde{K}_0(A).\end{aligned}$$

Note that $\nu(z^NF^-)$ is a lagrangian of $(B_N^+(F\oplus F^*, G\oplus G^*), [\Phi]_0)$, which vanishes in $V_{2i}(A)$ (by hypothesis) and so may be assumed to have a free lagrangian, L say.

Finally, consider the diagram

$$\begin{array}{ccccc} U_{2i+1}(A) & \longrightarrow & \Sigma_-(A) & \longrightarrow & V_{2i}(A)\\ & & \Big\downarrow{\scriptstyle\bar{B}} & & \Big\downarrow{\scriptstyle\bar{B}}\\ & & \Omega_+(A_z) & \longrightarrow & W_{2i+1}(A_z)\end{array},$$

in which the square commutes, and the (top) row is exact (by Theorem 4.3 of I), so that the composite $U_{2i+1}(A)\longrightarrow W_{2i+1}(A_z)$ is 0.

Now $U_{2i+1}(A)\longrightarrow\Sigma_-(A)$ sends

$$(B_N^+(F\oplus F^*, G\oplus G^*), [\Phi]_0;\ L, \nu(z^NF^-))\in U_{2i+1}(A)$$

to

$$[\nu(z^NF^-)] = b\in\Sigma_-(A),$$

and $\bar{B}b = \tau'\in\Omega_+(A_z)$ is sent by $\Omega_+(A_z)\longrightarrow W_{2i+1}(A_z)$ to

$$(\underset{\sim}{Q},\Phi;\ \mathcal{F},\mathcal{F}') = 0\in W_{2i+1}(A_z)$$

as required.

Lemma 2.5 Let

$$(f,\chi):(\underset{\sim}{P},\theta)\longrightarrow(\underset{\sim}{Q},\phi)$$

be a morphism of based $\pm$forms over A, with $(\underset{\sim}{Q},\phi)$ non-singular and such that $\tau(\underset{\sim}{Q},\phi)=0\in\tilde{K}_1(A)$.

Then the automorphism

$$H(f)\oplus(1,0):((\underset{\sim}{Q},\phi)\oplus H_\pm(\underset{\sim}{P}))\oplus(\underset{\sim}{Q},-\phi)$$
$$\longrightarrow((\underset{\sim}{Q},\phi)\oplus H_\pm(\underset{\sim}{P}))\oplus(\underset{\sim}{Q},-\phi)$$

is a special hamiltonian transformation, where $H(f)$ is as in Lemma 2.3.

$\square$

This completes the proof of Theorem 1.1 for n odd.

§3. Proof of Theorem 1.1 (n even)

We define $B: V_{2i}(A_z)\longrightarrow U_{2i-1}(A)$ using:

Lemma 3.1 Given a non-singular $\pm$form (Q,Φ) over A_z, with Q free, and a modular A-base $\mathcal{Q}$ of Q, let

$$\nu: Q\oplus Q^*\longrightarrow(Q\oplus Q^*)^+$$

be the positive projection, and let $N\geqslant 0$ be so large that

$$(\Phi\pm\Phi^*)(\mathcal{Q})\subseteq\sum_{j=-N}^{N}z^j\mathcal{Q}^*,\quad(\Phi\pm\Phi^*)^{-1}(\mathcal{Q}^*)\subseteq\sum_{j=-N}^{N}z^j\mathcal{Q}.$$

Then the A-submodule

$$B_N(\mathcal{Q},\Phi)=\{(z^N(1-\nu)z^{-N}x,\ \nu(\Phi\pm\Phi^*)x)\in Q\oplus Q^*\ |$$
$$x\in B_N^+((\Phi\pm\Phi^*)^{-1}(\mathcal{Q}^*),\mathcal{Q})\}$$

of $Q\oplus Q^*$ is a lagrangian of $H_\mp(\sum_{j=0}^{N-1}z^j\mathcal{Q})$ such that

$$(H_\mp(\sum_{j=0}^{N-1}z^j\mathcal{Q});\ \sum_{j=0}^{N-1}z^j\mathcal{Q},\ B_N(\mathcal{Q},\Phi))\in U_{2i-1}(A)$$

does not depend on the choices made of N and $\mathcal{Q}$.

<u>Proof</u>: The restriction of

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}\in \mathrm{Hom}_A\Big(\big(\sum_{j=0}^{N-1} z^jQ\big)\oplus\big(\sum_{j=0}^{N-1} z^jQ\big)^*,\ \big(\sum_{j=0}^{N-1} z^jQ\big)^*\oplus\big(\sum_{j=0}^{N-1} z^jQ\big)\Big)$$

to $B_N(Q,\Phi)$ is given by

$$B_N(Q,\Phi)\longrightarrow B_N(Q,\Phi)^*;$$

$$(z^N(1-\nu)z^{-N}x,\ \nu(\Phi\pm\Phi^*)x)\longmapsto\big((z^N(1-\nu)z^{-N}y,\ \nu(\Phi\pm\Phi^*)y)\longmapsto\langle x,y\rangle_{[\Phi]_0}\big)$$

which is of the type $\lambda\pm\lambda^*\in\mathrm{Hom}_A(B_N(Q,\Phi),B_N(Q,\Phi)^*)$, so that we are dealing with a sublagrangian of $H_\mp(\sum_{j=0}^{N-1}z^jQ)$. In fact, it is a lagrangian, a hamiltonian complement being defined by

$$B_N^*(Q,\Phi)=\{(-\nu y,\ \nu(\Phi\pm\Phi^*)(1-\nu)y)\in Q\oplus Q^*\mid y\in B_N^+(Q,(\Phi\pm\Phi^*)^{-1}Q^*)\}.$$

Every $(s,t)\in\big(\sum_{j=0}^{N-1}z^jQ\big)\oplus\big(\sum_{j=0}^{N-1}z^jQ^*\big)$ can be expressed as

$$(s,t)=(z^N(1-\nu)z^{-N}x,\ \nu(\Phi\pm\Phi^*)x)+(-\nu y,\ \nu(\Phi\pm\Phi^*)(1-\nu)y)$$
$$\in B_N(Q,\Phi)+B_N^*(Q,\Phi)$$

with

$$x=\nu(\Phi\pm\Phi^*)^{-1}\big((1-\nu)(\Phi\pm\Phi^*)s+t\big)\in B_N^+((\Phi\pm\Phi^*)^{-1}Q^*,Q)$$

$$y=\big(-(\Phi\pm\Phi^*)^{-1}\nu(\Phi\pm\Phi^*)s+z^N(1-\nu)z^{-N}(\Phi\pm\Phi^*)^{-1}t\big)$$
$$\in B_N^+(Q,(\Phi\pm\Phi^*)^{-1}Q^*)$$

and the associated pairing of $H_\mp(\sum_{j=0}^{N-1}z^jQ)$,

$$\begin{pmatrix}0&1\\\mp1&0\end{pmatrix}\in\mathrm{Hom}_A\Big(\sum_{j=0}^{N-1}z^jQ\oplus\big(\sum_{j=0}^{N-1}z^jQ\big)^*,\ \big(\sum_{j=0}^{N-1}z^jQ\big)^*\oplus\sum_{j=0}^{N-1}z^jQ\Big)$$

restricts to the A-module isomorphism

$$B_N^*(Q,\Phi)\longrightarrow B_N(Q,\Phi)^*;$$

$$(-\nu y,\ \nu(\Phi\pm\Phi^*)(1-\nu)y)\longmapsto\big((z^N(1-\nu)z^{-N}x,\ \nu(\Phi\pm\Phi^*)x)\longmapsto\langle y,x\rangle_{[\Phi]_0}\big).$$

so that we are dealing with hamiltonian complements.

Increasing N by 1 we have

$$B_{N+1}(Q,\Phi)=B_N(Q,\Phi)\oplus\{(z^{N+1}(1-\nu)z^{-(N+1)}x,\ (\Phi\pm\Phi^*)x)\mid x\in(\Phi\pm\Phi^*)^{-1}(z^NQ^*)\}$$

Now $B_N^*(Q,\Phi)\oplus z^NQ$ is a hamiltonian complement in $H_\mp(\sum_{j=0}^{N-1}z^jQ)$ to both $B_{N+1}(Q,\Phi)$ and $B_N(Q,\Phi)\oplus z^NQ^*$ so that

$$\big(H_\mp(\sum_{j=0}^{N-1}z^jQ);\ \sum_{j=0}^{N-1}z^jQ,\ B_N(Q,\Phi)\big)$$
$$=\big(H_\mp(\sum_{j=0}^{N}z^jQ);\ \sum_{j=0}^{N}z^jQ,\ B_N^*(Q,\Phi)\oplus z^NQ^*\big)$$
$$=\big(H_\mp(\sum_{j=0}^{N}z^jQ);\ \sum_{j=0}^{N}z^jQ,\ B_{N+1}(Q,\Phi)\big)\in U_{2i-1}(A).$$

Hence choice of N immaterial.

Let $\hat{Q}$ be another modular A-base of Q, with

$$\hat{\nu} : Q \oplus Q^* \longrightarrow (\hat{Q} \oplus \hat{Q}^*)^+$$

the new positive projection. Let $M \geq 0$ be so large that

$$\hat{Q} \subseteq \sum_{j=-M}^{M} z^j Q \quad , \quad Q \subseteq \sum_{j=-M}^{M} z^j \hat{Q} .$$

Then $\hat{N} = N + 2M$ is large enough for $B_{\hat{N}}(\hat{Q}, \Phi)$ to be defined, and

$$B^+_{\hat{N}}((\Phi \pm \Phi^*)^{-1}\hat{Q}^*, \hat{Q})$$
$$= (\Phi \pm \Phi^*)^{-1}(z^{M+N}B^+_M(\hat{Q}^*, Q^*)) \oplus z^M B^+_N((\Phi \pm \Phi^*)^{-1}Q^*, Q)$$
$$\oplus B^+_M(Q, \hat{Q})$$

so that

$$B_{\hat{N}}(\hat{Q}, \Phi) = \{(z^{\hat{N}}(1-\hat{\nu})z^{-\hat{N}}x, (\Phi \pm \Phi^*)x) \mid x \in (\Phi \pm \Phi^*)^{-1}(z^{M+N}B^+_M(\hat{Q}^*, Q^*))\}$$
$$\oplus \{(x, (\Phi \pm \Phi^*)x) \mid x \in z^M B^+_N((\Phi \pm \Phi^*)^{-1}Q^*, Q)\}$$
$$\oplus \{(x, \hat{\nu}(\Phi \pm \Phi^*)x) \mid x \in B^+_M(\hat{Q}, Q)\}.$$

Moreover,

$$\sum_{j=0}^{\hat{N}-1} z^j \hat{Q} = z^{M+N} B^+_M(\hat{Q}, Q) \oplus z^M (\sum_{j=0}^{N-1} z^j Q) \oplus B^+_M(Q, \hat{Q})$$

and

$$z^{M+N} B^+_M(\hat{Q}, Q) \oplus z^M B^*_N(Q, \Phi) \oplus B^+_M(Q^*, \hat{Q}^*)$$

is a hamiltonian complement in $H_{\mp}(\sum_{j=0}^{\hat{N}-1} z^j \hat{Q})$ to both $B_{\hat{N}}(\hat{Q}, \Phi)$ and $z^{M+N} B^+_M(\hat{Q}^*, Q^*) \oplus z^M B_N(Q, \Phi) \oplus B^+_M(Q, \hat{Q})$.

Thus

$$(H_{\mp}(\sum_{j=0}^{\hat{N}-1} z^j \hat{Q}); \sum_{j=0}^{\hat{N}-1} z^j \hat{Q}, B_{\hat{N}}(\hat{Q}, \Phi))$$
$$= (H_{\mp}(\sum_{j=0}^{\hat{N}-1} z^j \hat{Q}); \sum_{j=0}^{\hat{N}-1} z^j \hat{Q}, z^{M+N} B^+_M(\hat{Q}^*, Q^*) \oplus z^{M+N} B_N(Q, \Phi) \oplus B^+_M(Q, \hat{Q}))$$
$$= (H_{\mp}(\sum_{j=0}^{N-1} z^j Q); \sum_{j=0}^{N-1} z^j Q; B_N(Q, \Phi)) \in U_{2i-1}(A).$$

Hence independence of choice of Q. $\square$

Define

$$B : V_{2i}(A_z) \longrightarrow U_{2i-1}(A);$$
$$(Q, \Phi) \longmapsto (H_{\mp}(\sum_{j=0}^{N-1} z^j Q); \sum_{j=0}^{N-1} z^j Q, B_N(Q, \Phi))$$

for any modular A-base Q of Q (which may be assumed free) and sufficiently large $N \geq 0$. As shown in Lemma 3.1 this does not depend on the choices of N and Q. Given a f.g. free A_z-module $\mathcal{F}$, with modular A-base F, we have

$$B(H_{\pm}(\mathcal{F})) = (H_{\mp}(0); 0, B_0(F \oplus F^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})) = 0 \in U_{2i-1}(A)$$

Hence $B(Q,\Phi) = 0 \in U_{2i-1}(A)$ whenever $(Q,\Phi) = 0 \in V_{2i}(A_z)$, and $B : V_{2i}(A_z) \longrightarrow U_{2i-1}(A)$ is well-defined.

The composite

$$V_{2i}(A) \xrightarrow{\bar{\varepsilon}} V_{2i}(A_z) \xrightarrow{B} U_{2i-1}(A)$$

is 0 because it sends $(Q,\phi) \in V_{2i}(A)$ to

$$B\bar{\varepsilon}(Q,\phi) = (H_{\mp}(0); 0, B_0(Q,\phi)) = 0 \in U_{2i-1}(A).$$

The square

$$\begin{tikzcd} V_{2i}(A_z) \arrow[r] \arrow[d, "B"'] & \Omega_+(A_z) \arrow[d, "B"] \\ U_{2i-1}(A) \arrow[r] & \Sigma_-(A) \end{tikzcd}$$

commutes, for given $(Q_z, \Phi) \in V_{2i}(A_z)$, with Q free

$$[B_N(Q,\Phi)] = [B_N^+(Q^*, (\Phi \pm \Phi^*)Q)]$$

$$= B\tau(\underset{\sim}{Q}_z, \Phi) \in \Sigma_-(A)$$

for any A-base $\underset{\sim}{Q}$.

We define $\bar{B} : U_{2i-1}(A) \longrightarrow V_{2i}(A_z)$ using.

Lemma 3.2 Let (Q,ϕ) be a trivial $\mp$ form over A, with lagrangian L, and hamiltonian complement L^*, so that

$$\phi = \begin{pmatrix} \lambda \pm \lambda^* & \gamma \\ \delta & \lambda_1 \pm \lambda_1^* \end{pmatrix} : L \oplus L^* \longrightarrow L^* \oplus L$$

with $\gamma \mp \delta^* = 1 : L^* \longrightarrow L^*$.

Then the isomorphism class of the $\pm$ form over A_z defined by

$$(Q_z, \Theta) = \left(L_z \oplus L_z^*, \begin{pmatrix} \lambda & -z\gamma \\ \delta & (1-z)(\lambda_1 \pm \lambda_1^*) \end{pmatrix}\right)$$

does not depend on the choice of L^*.

If $(Q,\phi) = H_{\pm}(P)$, then (Q_z, Θ) is a non-singular $\pm$ form over A_z such that

$$((Q_z,\Theta) \oplus H_{\pm}(-L_z)) \in V_{2i}(A_z)$$

is sent to 0 by $\varepsilon : V_{2i}(A_z) \longrightarrow V_{2i}(A)$, and has torsion

$$\bar{B}([L] - [P^*]) \in \Omega_+(A_z).$$

Moreover,

$$((Q_z,\Theta) \oplus H_{\pm}(-L_z)) = 0 \in V_{2i}(A_z)$$

if L is a hamiltonian complement in (Q,ϕ) to either P or P*

<u>Proof</u>: Change of hamiltonian complement L^* corresponds to an automorphism

$$\alpha = \begin{pmatrix} 1 & \kappa \pm \kappa^* \\ 0 & 1 \end{pmatrix} \in \mathrm{Hom}_A(L \oplus L^*, L \oplus L^*)$$

for some $\pm$ form (L^*, κ). The $\pm$ form over A_z (Q_z, Θ') determined by this new choice of hamiltonian complement to L is given by

$$\Theta' = \begin{pmatrix} \lambda & -z\gamma' \\ \delta' & (1-z)(\lambda_1' \pm \lambda_1'^*) \end{pmatrix} \in \mathrm{Hom}_{A_z}(L_z \oplus L^*_z, L^*_z \oplus L_z)$$

where $\gamma', \delta', \lambda_1'$ are such that

$$\alpha^* \phi \alpha = \begin{pmatrix} \lambda \pm \lambda^* & \cancel{-z}\gamma' \\ \delta' & \cancel{(1-z)}(\lambda_1' \pm \lambda_1'^*) \end{pmatrix} \in \mathrm{Hom}_A(L \oplus L^*, L^* \oplus L) .$$

Now

$$\left(\begin{pmatrix} 1 & (1-z)(\kappa \pm \kappa^*) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -(\lambda \pm z\lambda^*)(\kappa \pm \kappa^*) \\ 0 & (1-z)(\kappa^* \pm \kappa)\lambda(\kappa \pm \kappa^*) \end{pmatrix} \right)$$

$$: (L_z \oplus L^*_z, \Theta) \longrightarrow (L_z \oplus L^*_z, \Theta')$$

defines an isomorphism of $\pm$ forms over A_z, so that choice of L^* immaterial.

Defining $\omega \in \mathrm{Hom}_{A_z}(Q_z, Q_z)$ by

$$\omega = \begin{pmatrix} 1 & 0 \\ 0 & 1-z \end{pmatrix} : L_z \oplus L^*_z \longrightarrow L_z \oplus L^*_z ,$$

and $\tilde{\omega} \in \mathrm{Hom}_{A_z}(Q^*_z, Q^*_z)$ by

$$\tilde{\omega} = \begin{pmatrix} 1-z^{-1} & 0 \\ 0 & 1 \end{pmatrix} : L^*_z \oplus L_z \longrightarrow L^*_z \oplus L_z ,$$

note that there is an identity

$$\tilde{\omega}(\Theta \pm \Theta^*) = (\phi \mp z^{-1}\phi^*)\omega \in \mathrm{Hom}_{A_z}(Q_z, Q^*_z) .$$

Similarly, defining

$$\tilde{\Phi} = (\phi \mp \phi^*)^{-1} \phi (\phi \mp \phi^*)^{-1} = \begin{pmatrix} \mp(\lambda_1 \pm \lambda_1^*) & \delta \\ \gamma & \mp(\lambda \pm \lambda^*) \end{pmatrix} \in \mathrm{Hom}_A(L^* \oplus L, L \oplus L^*)$$

and

$$\tilde{\Theta} = \begin{pmatrix} \mp(1-z^{-1})(\lambda_1 \pm \lambda_1^*) & \delta \\ -z^{-1}\gamma & \mp\lambda \end{pmatrix} \in \mathrm{Hom}_{A_z}(L^*_z \oplus L_z, L_z \oplus L^*_z),$$

note that

$$\omega(\tilde{\Theta} \pm \tilde{\Theta}^*) = (\tilde{\Phi} \mp z\tilde{\Phi}^*)\tilde{\omega} : Q^*_z \longrightarrow Q_z .$$

If $(Q, \phi) = H_\mp(P)$, then

$$\phi \mp z^{-1}\phi^* = \begin{pmatrix} 0 & 1 \\ \mp z^{-1} & 0 \end{pmatrix} \in \mathrm{Hom}_{A_z}(P_z \oplus P^*_z, P^*_z \oplus P_z),$$

and combining the two identities above, we obtain

$$\omega(\tilde{\Theta} \pm \tilde{\Theta}^*)(\Theta \pm \Theta^*)$$
$$= (\tilde{\phi} \mp z\tilde{\phi}^*)\,\tilde{\omega}\,(\Theta \pm \Theta^*) = (\tilde{\phi} \mp z\tilde{\phi}^*)(\phi \mp z^{-1}\phi^*)\,\omega$$
$$= \omega \in \mathrm{Hom}_{A_z}(Q_z, Q_z) \ ,$$

and similarly

$$\tilde{\omega}(\Theta \pm \Theta^*)(\tilde{\Theta} \pm \tilde{\Theta}^*) = \tilde{\omega} \in \mathrm{Hom}_{A_z}(Q_z^*, Q_z^*)$$

Both $\omega \in \mathrm{Hom}_{A_z}(Q_z, Q_z)$ and $\tilde{\omega} \in \mathrm{Hom}_{A_z}(Q_z^*, Q_z^*)$ are monomorphisms, so

$$\tilde{\Theta} \pm \tilde{\Theta}^* = (\Theta \pm \Theta^*)^{-1} \in \mathrm{Hom}_A(Q_z^*, Q_z)$$

and (Q_z, Θ) is a non-singular $\pm$form over A_z.

The projection $\varepsilon : V_{2i}(A_z) \longrightarrow V_{2i}(A)$ sends $((Q_z,\Theta) \oplus H_\pm(-L_z))$ to

$$\left(\left(L \oplus L^*, \begin{pmatrix} \lambda & -\gamma \\ \delta & 0 \end{pmatrix}\right) \oplus H_\pm(-L)\right) \in V_{2i}(A)$$

which vanishes in $V_{2i}(A)$ because $L^* \oplus -L^*$ is a free lagrangian.

Thus the component of $\tau((Q_z,\Theta) \oplus H_\pm(-L_z)) \in \Omega_+(A_z)$ in $\Xi\,\Omega_+(A)$ is 0, and

$$\tau((Q_z,\Theta) \oplus H_\pm(-L_z)) = \bar{B}B\,\tau((Q_z,\Theta) \oplus H_\pm(-L_z))$$
$$= \bar{B}[B_1^+(Q^* \oplus (-L^* \oplus -L), (\Theta \pm \Theta^*)Q \oplus (-L^* \oplus -L)]$$
$$\in \Omega_+(A_z) \, .$$

Computing directly,

$$B_1^+((\Theta \pm \Theta^*)^{-1}Q^*, Q)$$
$$= \left\{ (x,y) \in L^+ \oplus L^{*+} \;\middle|\; \begin{matrix} (\lambda \pm \lambda^*)x + (-z\gamma \pm \delta^*)y \in zL^{*-} \\ (\delta \mp z^{-1}\gamma^*)x + (1-z)(1-z^{-1})(\lambda_1 \pm \lambda_1^*)y \in zL^- \end{matrix} \right\}$$

$$= L \oplus \{ ((1-z)x, y) \in L_z \oplus L_z^* \mid x \in L, y \in L^*, \phi(x,y) = 0 \in Q^*$$

Now $\ker(\phi : Q \longrightarrow Q^*) = P$, and $Q = L \oplus L^* = P \oplus P^*$, so

$$\tau((Q_z,\Theta) \oplus H_\pm(-L_z))$$
$$= \bar{B}([L] + [P] + [-L \oplus -L^*]) = \bar{B}([L] - [P^*]) \in \Omega(A$$

Finally, suppose that L is a hamiltonian complement to either P or P^*, choosing L^* accordingly. Then $\lambda_1 = 0$ and the annihilator of L_z^* in

$$(Q_z, \Theta) = \left(L_z \oplus L_z^*, \begin{pmatrix} \lambda & -z\gamma \\ \delta & 0 \end{pmatrix}\right)$$

is given by

$$L_z^{*\perp} = \{ (x,y) \in L_z \oplus L_z^* \mid (\delta \mp z^{-1}\gamma^*)(x) = 0 \}$$
$$= L_z^* \oplus \ker((\gamma^* \mp z\delta) : L_z \longrightarrow L_z) \, .$$

As $\gamma \mp \delta^* = 1 : L^* \longrightarrow L^*$, if

$$x \in \ker((\gamma^* \mp z\delta) : L_z \longrightarrow L_z) \, ,$$

then

$$x = (z-1)(\pm \delta x) \in (z-1)L_z$$

and $(\pm\delta x) \in \ker((\gamma^* \mp z\delta) : L_z \longrightarrow L_z)$ as well.

By induction on N; $x \in (z-1)^N L_z$ for every $N \geq 1$, which is impossible unless $x = 0$.

Thus $L_z^{*\perp} = L_z^*$ and $L_z^* \oplus -L_z^*$ is a free lagrangian of

$$(L_z \oplus L_z^*, \begin{pmatrix} \lambda & -z\gamma \\ \delta & 0 \end{pmatrix}) \oplus H_\pm(-L_z),$$

making it vanish in $V_{2i}(A_z)$.

$\square$

Define

$$\bar{B} : U_{2i-1}(A) \longrightarrow V_{2i}(A_z) ;$$

$$(Q, \phi; F, G) \longmapsto ((G_z \oplus G_z^*, \begin{pmatrix} \lambda & -z\gamma \\ \delta & (1-z)(\lambda_1 \pm \lambda_1^*) \end{pmatrix}) \oplus H_\pm(-G_z))$$

by choosing hamiltonian complements F^*, G^* to F, G in (Q, ϕ), and expressing

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : F \oplus F^* \longrightarrow F^* \oplus F$$

as

$$\begin{pmatrix} \lambda \pm \lambda^* & \gamma \\ \delta & \lambda_1 \pm \lambda_1^* \end{pmatrix} : G \oplus G^* \longrightarrow G^* \oplus G.$$

We have already shown, in Lemma 3.2 above, that this does not depend on the choice of G^*, and that

$$\bar{B}(Q, \phi; F, G) = 0 \in V_{2i}(A_z) \quad \text{if } (Q, \phi; F, G) = 0 \in U_{2i-1}(A).$$

Hence the choice of $\lambda \in \Pi_\pm(G)$ is also immaterial: for

$$\bar{B}(Q \oplus Q, \phi \oplus -\phi; F \oplus F^*, G \oplus G^*) = 0 \in V_{2i}(A_z),$$

so that

$$\bar{B}(Q, \phi; F, G) = -\bar{B}(Q, -\phi; F^*, G^*) \in V_{2i}(A_z)$$

and $-\bar{B}(Q, -\phi; F^*, G^*)$ can be defined without making a choice of $\lambda \in \Pi_\pm(G)$.

It remains to verify that the definition is invariant under changes of F^*. For this, it is convenient to have available a more intrinsic characterization of $\Theta \pm \Theta^* \in \mathrm{Hom}_{A_z}(Q_z, Q_z^*)$ (as defined in Lemma 3.2):

given a lagrangian L of a $\mp$ form (Q, ϕ) over A, let $\Psi \in \mathrm{Hom}_{A_z}(L_z \oplus Q_z, L_z^* \oplus Q_z^*)$ be the unique A_z-linear extension of

$$(1-z)\phi \pm (1-z^{-1})\phi^* : Q_z \longrightarrow Q_z^*$$

such that

$$\Psi(\mathcal{R}) = 0,$$

where

$$\mathcal{R} = \{((z-1)e, e) \in L_z \oplus Q_z \mid e \in L_z\}.$$

Let

$$\Psi : L_z\oplus Q_z/R \longrightarrow (L_z\oplus Q_z/R)^* ;$$

$$[e,x] \longmapsto ([f,y] \longmapsto \Psi(e,x)(f,y))$$

be the induced A_z-module morphism, writing $[e,x]$ for the residue class mod R of $(e,x) \in L_z\oplus Q_z$.

A choice of hamiltonian complement L^* to L in (Q,ϕ) determines an A_z-module isomorphism

$$\eta : L_z\oplus L^*_z \longrightarrow L_z\oplus Q_z/R \;;\; (e,v) \longmapsto [e,v]$$

such that

$$\eta^*\Psi\eta = \Theta \pm \Theta^* \in \mathrm{Hom}_{A_z}(Q_z,Q^*_z).$$

Now let $(Q,\phi) = H_\mp(P)$, and let $\hat{P}^*$ be any hamiltonian complement to P in $H_\mp(P)$, so that $\hat{P}^* = \Gamma_{(P^*,\mu)}$ for some $\pm$form (P^*,μ) (by Lemma 1.3 of I). Let $(Q_z,\hat{\Theta})$ be the $\pm$form over A_z defined as (Q_z,Θ), but with $\hat{P}^*$ in place of P. Let

$$\beta : Q = P\oplus P^* \xrightarrow{(0\ 1)} P^*$$

be the projection on P^* along P. Then

$$L_z\oplus Q_z \longrightarrow L_z\oplus Q_z \;;$$

$$(e,x) \longmapsto (e, x+\mu(\beta(-e+(z-1)x)))$$

is an A_z-module isomorphism which induces, via η, an isomorphism

$$(Q_z,\Theta) \longrightarrow (Q_z,\hat{\Theta})$$

of $\pm$ forms over A_z.

Thus $\bar{B}(Q,\phi;F,G) \in V_{2i}(A_z)$ does not depend on the representative $\pm$formation of $(Q,\phi;F,G) \in U_{2i-1}(A)$ and

$$\bar{B} : U_{2i-1}(A) \longrightarrow V_{2i}(A_z)$$

is well-defined.

It should be noted that we can also give a more symmetric definition

$$\bar{B} : U_{2i-1}(A) \longrightarrow V_{2i}(A_z) ;$$

$$(Q,\phi;F,G) \longmapsto \left(\left(G_z\oplus G^*_z, \begin{pmatrix} \lambda & -z\gamma \\ \delta & (1-z)(\lambda_1 \pm \lambda_1^*) \end{pmatrix}\right) \oplus H_\pm(-G_z)\right)$$

$$\oplus -\left(\left(F_z\oplus F^*_z, \begin{pmatrix} \mu & -z\alpha \\ \beta & (1-z)(\mu_1 \pm \mu_1^*) \end{pmatrix}\right) \oplus H_\pm(-F_z)\right)$$

where $(Q,\phi) = H_\mp(P)$ and

$$\phi = \begin{pmatrix} \lambda \pm \lambda^* & \gamma \\ \delta & \lambda_1 \pm \lambda_1^* \end{pmatrix} : G \oplus G^* \longrightarrow G^* \oplus G$$

$$\phi = \begin{pmatrix} \mu \pm \mu^* & \alpha \\ \beta & \mu_1 \pm \mu_1^* \end{pmatrix} : F \oplus F^* \longrightarrow F^* \oplus F$$

for some hamiltonian complements F^*, G^* to F, G in (Q, ϕ).

The two definitions agree because

$$\begin{aligned}(H_\mp(P); F, G) &= (H_\mp(P); P, G) \oplus (H_\mp(P); F, P) \\ &= (H_\mp(P); P, G) \oplus -(H_\mp(P); P, F) \in U_{2i-1}(A)\end{aligned}$$

by the U-theory sum formula of Lemma 3.3 of I.

It is immediate from Lemma 3.2 that the composite

$$U_{2i-1}(A) \xrightarrow{\bar{B}} V_{2i}(A_z) \xrightarrow{\varepsilon} V_{2i}(A)$$

is 0, and that the square

$$\begin{tikzcd} U_{2i-1}(A) \arrow[r] \arrow[d, "\bar{B}"'] & \Sigma_-(A) \arrow[d, "\bar{B}"] \\ V_{2i}(A_z) \arrow[r] & \Omega_+(A_z) \end{tikzcd}$$

commutes

Lemma 3.3 The composite

$$U_{2i-1}(A) \xrightarrow{\bar{B}} V_{2i}(A_z) \xrightarrow{B} U_{2i-1}(A)$$

is the identity.

Proof: Given $(Q, \phi; F, G) \in U_{2i-1}(A)$ we may assume $(Q, \phi) = H_\mp(F)$, so that

$$\bar{B}(Q, \phi; F, G) = ((Q_z, \Theta) \oplus H_\pm(-G_z)) \in V_{2i}(A_z)$$

where

$$\Theta = \begin{pmatrix} \lambda & -z\gamma \\ \delta & (1-z)(\lambda_1 \pm \lambda_1^*) \end{pmatrix} \in \mathrm{Hom}_{A_z}(G_z \oplus G_z^*, G_z^* \oplus G_z)$$

if

$$\phi = \begin{pmatrix} \lambda \pm \lambda^* & \gamma \\ \delta & \lambda_1 \pm \lambda_1^* \end{pmatrix} \in \mathrm{Hom}_A(G \oplus G^*, G^* \oplus G)$$

for some hamiltonian complement G^* to G in (Q, ϕ).

Thus

$$\begin{aligned} B\bar{B}(Q, \phi; F, G) &= B((Q_z, \Theta) \oplus H_\pm(-F_z)) \\ &= (H_\mp(Q); Q, B_1(Q, \Theta)) \oplus (H_\mp(-F \oplus -F^*); -F \oplus -F^*, \Gamma_{H_\pm(-F)}) \\ &= (H_\mp(Q); Q, B_1(Q, \Theta)) \in U_{2i-1}(A), \end{aligned}$$

where

$$B_1(Q, \Theta) = \{ (z(1-z)z' x, z(\Theta \pm \Theta^*) x) \in Q \oplus Q^* \mid x \in B_1^+((\Theta \pm \Theta^*)^{-1} Q^*, Q) \}.$$

with
$$\nu : (Q\oplus Q^*)_z \longrightarrow (Q\oplus Q^*)^+$$
the positive projection. As in the proof of Lemma 3.2
$$B_1^+((\Theta\pm\Theta^*)^{-1}Q^*, Q) = G\oplus\{(1-z)x+y\in Q_z \mid x\in G, y\in G^*, x+y\in F\}$$
so that
$$B_1(Q,\Theta) = \{(x,\phi x)\in Q\oplus Q^* \mid x\in G\}$$
$$\oplus\{(y,\pm\phi^* y)\in Q\oplus Q^* \mid y\in F\}.$$
The isomorphism of $\mp$ forms over A
$$\left(\begin{pmatrix}1 & 1\\ \phi & \pm\phi^*\end{pmatrix}, \begin{pmatrix}0 & 0\\ \phi & \phi\end{pmatrix}\right) : (Q\oplus Q, \phi\oplus-\phi) \longrightarrow H_\mp(Q)$$
sends $F\oplus F^*$ onto Q, and $G\oplus F$ onto $B_1(Q,\Theta)$.
Thus
$$\begin{aligned} B\bar{B}(Q,\phi;F,G) &= (H_\mp(Q); Q, B_1(Q,\Theta))\\ &= (Q\oplus Q, \phi\oplus-\phi; F\oplus F^*, G\oplus F)\\ &= (Q,\phi;F,G)\in U_{2i-1}(A).\end{aligned}$$

$\square$

We need just one more result to prove that
$$0\longrightarrow V_{2i}(A) \xrightarrow{\bar{\varepsilon}} V_{2i}(A_z) \xrightarrow{B} U_{2i-1}(A)\longrightarrow 0$$
is a split short exact sequence.

Let z_1, z_2 be independent commuting indeterminates over A. The <u>double Laurent extension</u> of A by (z_1,z_2), A_{z_1,z_2}, is the ring of polynomials in $z_1, z_1^{-1}, z_2, z_2^{-1}$ with involution by $z_1\longmapsto z_1^{-1}$, $z_2\longmapsto z_2^{-1}$. It is clear that A_{z_1,z_2} may be regarded as either $(A_{z_1})_{z_2}$ or $(A_{z_2})_{z_1}$, and satisfies all the conditions imposed on A.

<u>Lemma 3.4</u> The diagram
$$\begin{array}{ccc} V_{2i}(A_{z_1}) & \xrightarrow{B(z_1)} & U_{2i-1}(A)\\ {\scriptstyle\bar{B}(z_2)}\downarrow & & \downarrow{\scriptstyle\bar{B}(z_2)}\\ W_{2i+1}(A_{z_1,z_2}) & \xrightarrow[B(z_1)]{} & V_{2i}(A_{z_2})\end{array}$$
skewcommutes.

<u>Proof</u>: Given $(Q,\Phi)\in V_{2i}(A_{z_1})$ we may assume that Q is a f.g. free A_{z_1}-module, as usual. Choose a modular A-base Q of Q, so that
$$\Delta = \{(x,x)\in Q\oplus Q \mid x\in Q\}$$
is a modular A-base of $\Delta_{(Q,\Phi)}$.

Let (Q^*, Ψ) be a $\pm$form over A_{z_1} such that there is an isomorphism of $\pm$forms

$$((\Phi \pm \Phi^*), \chi) : (Q, \Phi) \longrightarrow (Q^*, \pm\Psi)$$

(cf. Lemma 1.4 of I.). Then

$$\Psi \pm \Psi^* = (\Phi \pm \Phi^*)^{-1} \in \mathrm{Hom}_{A_{z_1}}(Q^*, Q)$$

and

$$\Delta^* = \{ (\Psi t, \mp \Psi^* t) \in Q \oplus Q^* \mid t \in Q^* \}$$

is the modular A-base dual to Δ of the hamiltonian complement $\Delta^*_{(Q^*, \Psi)}$ to $\Delta_{(Q,\Phi)}$ in $(Q \oplus Q, \Phi \oplus -\Phi)$.

Let $N \geq 0$ be an integer so large that

$$(\Phi \pm \Phi^*)(Q) \subseteq \sum_{j=-N}^{N} z_1^j Q^* \quad (\Psi \pm \Psi^*)(Q^*) \subseteq \sum_{j=-N}^{N} z_1^j Q .$$

Adding on some $\chi \mp \chi^* \in \mathrm{Hom}_{A_{z_1}}(Q^*, Q)$ to Ψ if necessary, it may be assumed that

$$\Psi(Q^*) \subseteq \sum_{j=0}^{N} z_1^j Q .$$

This ensures that

$$z_1^N \Delta^{+_1}_{z_2} \subseteq \xi_2 (\Delta \oplus \Delta^*)^{+_1}_{z_2} ,$$

where

$$\xi_2 = \begin{pmatrix} 1 & 0 \\ 0 & z_2 \end{pmatrix} \in \mathrm{Hom}_{A_{z_2}}(Q_{z_2} \oplus Q_{z_2}, Q_{z_2} \oplus Q_{z_2}) ,$$

because every $(s,s) \in z_1^N \Delta^{+_1}_{z_2}$ can be expressed as

$$(s,s) = (x, z_2 x) + (\Psi y, \mp z_2 \Psi^* y) \in \xi_2 \Delta^{+_1}_{z_2} \oplus \xi_2 \Delta^{*\,+_1}_{z_2}$$

with

$$y = (1 - z_2^{-1})(\Phi \pm \Phi^*)(s) \in Q^{*\,+_1}_{z_2} , \quad x = (s - \Psi y) \in Q^{+_1}_{z_2} .$$

For any A_{z_1}-base $\underset{\sim}{Q}$,

$$\bar{B}(z_2)(Q, \Phi) = ((\underline{Q \oplus Q})_{z_2}, \Phi \oplus -\Phi ; \Delta_{(\underset{\sim}{Q}_{z_2}, \Phi)}, \xi_2 \Delta_{(\underset{\sim}{Q}_{z_2}, \Phi)})$$
$$\in W_{2i+1}(A_{z_1}, z_2)$$

where $\underset{\sim}{Q \oplus Q} = \Delta_{(\underset{\sim}{Q}, \Phi)} \oplus \Delta^*_{(\underset{\sim}{Q}^*, \Psi)}$, defining a hamiltonian base of $(Q \oplus Q, \Phi \oplus -\Phi)$.

Thus

$$B(z_1)\bar{B}(z_2)(Q, \Phi)$$
$$= (E^{+_1}_N(\Delta_{z_2}, \xi_2 \Delta_{z_2} \oplus \xi_2 \Delta^*_{z_2}) / z_1^N \Delta^{+_1}_{z_2} , [\Phi \oplus -\Phi]_{z_1 = 0}) \in V_{2i}(A_{z_2})$$

where

$$E^{+_1}_N(\Delta_{z_2}, \xi_2 \Delta_{z_2} \oplus \xi_2 \Delta^*_{z_2})$$
$$= \{ w \in \xi_2 \Delta^{+_1}_{z_2} \oplus \xi_2 \Delta^{*\,+_1}_{z_2} \mid \langle z_1^N \Delta^{+_1}_{z_2}, w \rangle_{[\Phi \oplus -\Phi]_{z_1=0}} = \{0\} \subseteq A_{z_2} \}$$

$$= \{ (a, (\Psi \pm \Psi^*)(\nu + z_2(1-\nu))(\Phi \pm \Phi^*) a) \in Q_{z_2} \oplus Q_{z_2} \mid a \in Q^{+_1}_{z_2} \}$$
$$\oplus \{ (0, (\Psi \pm \Psi^*) b) \in Q_{z_2} \oplus Q_{z_2} \mid b \in \sum_{j=0}^{N-1} z_1^j Q_{z_2} \}$$

(using the alternative definition of

$$B : W_{2i+1}(A_z) \longrightarrow V_{2i}(A)$$

given for V-theory in §2) with

$$\nu : Q \oplus Q^* \longrightarrow (Q \oplus Q^*)^{+1}$$

the positive projection.

Next, set $P = \sum_{j=0}^{N-1} z_1^j Q$ (an A-module), and define an A_{z_2}-module isomorphism

$$f : P_{z_2} \oplus P^*_{z_2} \longrightarrow E_N^{+1}(\Delta_{z_2}, \xi_2(\Delta \oplus \Delta^*)_{z_2}) / z_1^N \Delta^{+1}_{z_2} ;$$

$$(a, b) \longmapsto (a, (\Psi \pm \Psi^*)(\nu + z_2(1-\nu))(\Phi \pm \Phi^*) a + b))$$

so that

$$B(z_1)\bar{B}(z_2)(Q, \Phi) = (P_{z_2} \oplus P^*_{z_2}, f^*[\Phi \oplus -\Phi]_{z_1=0} f) \in V_{2i}(A_{z_2}).$$

Defining $\Theta \in \mathrm{Hom}_{A_{z_2}}(P_{z_2} \oplus P^*_{z_2}, (P_{z_2} \oplus P^*_{z_2})^*)$ by

$$\Theta(a, b)(a', b')$$

$$= [((\Phi \pm \Phi^*)(a)(a') - (\nu(\Phi \pm \Phi^*)a)((\Psi \pm \Psi^*)\nu(\Phi \pm \Phi^*)a')).(1 - z_2')$$

$$- b((\Psi \pm \Psi^*)\nu(\Phi \pm \Phi^*)a')$$

$$- ((1-\nu)(\Phi \pm \Phi^*)a)((\Psi \pm \Psi^*)b').z_2' - b(\Psi b')]_{z_1=0} \in A_{z_2}$$

$$(a, a' \in P_{z_2},\ b, b' \in P^*_{z_2})$$

it is not difficult to verify that

i) Θ differs from $f^*[\Phi \oplus -\Phi]_{z_1=0} f$ by some $\chi \mp \chi^* \in \mathrm{Hom}_{A_{z_2}}(P_{z_2} \oplus P^*_{z_2}, (P_{z_2} \oplus P^*_{z_2})^*)$

ii) defining $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{Hom}_A(P \oplus P^*, P \oplus P^*)$, we have that

$$\eta^* \Theta \eta = \begin{pmatrix} (1-z_2)(\lambda_1 \pm \lambda_1^*) & \delta \\ -z_2 \gamma & \lambda \end{pmatrix} \in \mathrm{Hom}_{A_{z_2}}(P_{z_2} \oplus P^*_{z_2}, P^*_{z_2} \oplus P_{z_2})$$

where

$$\begin{pmatrix} \lambda_1 \pm \lambda_1^* & \delta \\ \gamma & \lambda \pm \lambda^* \end{pmatrix} \in \mathrm{Hom}_A(P \oplus P^*, P^* \oplus P)$$

is an expression for

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathrm{Hom}_A(B_N(Q, \Phi) \oplus B^*_N(Q, \Phi), B_N(Q, \Phi)^* \oplus B^*_N(Q, \Phi)^*)$$

with $B_N(Q, \Phi)$, $B^*_N(Q, \Phi)$ the hamiltonian complements in $H_\mp(P)$ defined in Lemma 3.1.

Thus

$$B(z_1)\bar{B}(z_2)(Q, \Phi) = (P_{z_2} \oplus P^*_{z_2}, \Theta)$$

$$= \bar{B}(z_2)(H_\mp(P); B^*_N(Q, \Phi), P^*)$$

$$= \bar{B}(z_2)(-(H_\mp(P); P, B_N(Q, \Phi)))$$

$$= -\bar{B}(z_2)B(z_1)(Q, \Phi) \in V_{2i}(A_{z_2})$$

using the U-theory sum formula of Lemma 3.3 of I.

Applying $B(z_2)$ to the decomposition

$$W_{2i+1}(A_{z_1,z_2}) = \mathcal{E}(z_1)W_{2i+1}(A_{z_2}) \oplus \bar{B}(z_1)V_{2i}(A_{z_2})$$

obtained in §2, it is now immediate that

$$V_{2i}(A_{z_1}) = \mathcal{E}(z_1)V_{2i}(A) \oplus \bar{B}(z_1)U_{2i-1}(A)\,.$$

This proves the part of Theorem 1.1 relating to $V_n(A_z)$, n even.

To complete the proof, we give analogous constructions for W-theory.

Define

$$B : W_{2i}(A_z) \longrightarrow V_{2i-1}(A);$$

$$(\underset{\sim}{Q}, \Phi) \longmapsto \Big(H_{\mp}\big(\sum_{j=0}^{N-1} z^j Q\big);\ \sum_{j=0}^{N-1} z^j Q,\ B_N(Q,\Phi)\Big)$$

with Q the modular A-base of Q generated by the given A_z-base, and $B_N(Q,\Phi)$ as in Lemma 3.1.

Then

$$[B_N(Q,\Phi)] = B\tau(\underset{\sim}{Q},\Phi) = 0 \in \tilde{K}_0(A),$$

as required for V-theory, since

$$\tau(\underset{\sim}{Q},\Phi) = 0 \in \tilde{K}_1(A_z)$$

by construction of $W_{2i}(A_z)$ (cf §5 of I.)

The composite

$$W_{2i}(A) \xrightarrow{\mathcal{E}} W_{2i}(A_z) \xrightarrow{B} V_{2i-1}(A)$$

is 0, as for V-theory.

The square

$$\begin{array}{ccc} \Omega_-(A_z) & \longrightarrow & W_{2i}(A_z) \\ \downarrow B & & \downarrow B \\ \Sigma_+(A) & \longrightarrow & V_{2i-1}(A) \end{array}$$

commutes: for

$$\Omega_-(A_z) = \mathcal{E}\,\Omega_-(A) \oplus \bar{B}\,\Sigma_+(A)$$

and elements of $\mathcal{E}\Omega_-(A)$ are sent to 0 both ways round the square, while the composition

$$\Sigma_+(A) \xrightarrow{\bar{B}} \Omega_-(A_z) \longrightarrow W_{2i}(A_z) \xrightarrow{B} V_{2i-1}(A)$$

sends $[P] \in \Sigma_+(A)$ to

$$B\left((\underset{\sim}{P \oplus -P})_z \oplus (\underset{\sim}{P \oplus -P})^*_z,\ \begin{pmatrix} 0 & 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} & 0 \end{pmatrix} \right)$$

$$= \big(H_{\mp}(P^* \oplus P \oplus -P \oplus -P^*);\ P^* \oplus P \oplus -P \oplus -P^*,\ \Gamma_{H_\pm(P^*)} \oplus (-P)^* \oplus -P^*\big)$$

$$= \big(H_{\mp}(P^* \oplus P \oplus -P \oplus -P^*);\ P^* \oplus P \oplus -P \oplus -P^*,\ (P^*)^* \oplus (P)^* \oplus -P \oplus (-P)^*\big)$$

$$= \big(H_{\mp}(P \oplus -P);\ P \oplus -P,\ P^* \oplus -P\big)$$

$$= \big(H_{\mp}(P \oplus -P);\ P \oplus -P,\ P \oplus -P^*\big) \in V_{2i-1}(A)$$

agreeing with the map $\Sigma_+(A) \longrightarrow V_{2i-1}(A)$ defined [illegible] 4.2 of I.

Next, define

$$\bar{B} : V_{2i-1}(A) \longrightarrow W_{2i}(A_Z) ;$$
$$(Q,\phi;F,G) \longmapsto ((Q_Z,\Theta)\oplus(R\oplus R^*, \begin{pmatrix} 0 & 0 \\ \Psi & 0 \end{pmatrix}))$$

where $(Q,\phi) = H_{\mp}(\underline{F})$ for any base $\underline{F}$ of F (assumed free), and (Q_Z,Θ) is the $\pm$ form over A_Z defined in Lemma 3.2 (with F, G replacing P, L respectively), so that

$$\tau(Q_Z,\Theta) = \bar{B}([G]-[F^*]) = 0 \in \Omega_+(A_Z) ,$$

and

$$\Psi : R \longrightarrow R^*$$

is an automorphism of a based A_Z-module R such that

$$\tau(Q_Z,\Theta) + \tau(\Psi) + \tau(\Psi^*) = 0 \in \tilde{K}_1(A_Z) .$$

The composites

$$V_{2i-1}(A) \xrightarrow{\bar{B}} W_{2i}(A_Z) \xrightarrow{\varepsilon} W_{2i}(A)$$
$$V_{2i-1}(A) \xrightarrow{\bar{B}} W_{2i}(A_Z) \xrightarrow{B} V_{2i-1}(A)$$

are $0, 1$ as for V-theory.

There is no need to prove directly that the square

$$\begin{tikzcd}
\Sigma_+(A) \arrow[r] \arrow[d, "\bar{B}"'] & V_{2i-1}(A) \arrow[d, "\bar{B}"] \\
\Omega_-(A_Z) \arrow[r] & W_{2i}(A_Z)
\end{tikzcd}$$

commutes: we have already shown that its commutator lies in

$$\ker(\varepsilon : W_{2i}(A_Z) \longrightarrow W_{2i}(A)) \cap \ker(B : W_{2i}(A_Z) \longrightarrow V_{2i-1}(A)) .$$

This intersection will presently be shown to be null (without using the commutativity of the square!)

The (split) exactness of

$$0 \longrightarrow W_{2i}(A) \xrightarrow{\bar{\varepsilon}} W_{2i}(A_Z) \xrightarrow{B} V_{2i-1}(A) \longrightarrow 0$$

follows from a diagram chase round:

$$\begin{tikzcd}
 & \Omega_-(A) \arrow[r, "\alpha"] \arrow[d, shift right, "\bar{\varepsilon}"'] & W_{2i}(A) \arrow[r, "\beta"] \arrow[d, "\bar{\varepsilon}"] & V_{2i}(A) \arrow[r, "\gamma"] \arrow[d, "\bar{\varepsilon}"] & \Omega_+(A) \arrow[d, "\bar{\varepsilon}"] \\
V_{2i+1}(A_Z) \arrow[r, "\delta"] & \Omega_-(A_Z) \arrow[r, "\alpha"] \arrow[u, shift right, "\varepsilon"'] \arrow[d, shift left, "B"] & W_{2i}(A_Z) \arrow[r, "\beta"] \arrow[d, "B"] & V_{2i}(A_Z) \arrow[r, "\gamma"] \arrow[d, "B"] & \Omega_+(A_Z) \\
U_{2i}(A) \arrow[r, "\lambda"] \arrow[u, "\bar{B}"] & \Sigma_+(A) \arrow[r, "\mu"] \arrow[u, shift left, "\bar{B}"] & V_{2i-1}(A) \arrow[r, "\nu"] & U_{2i-1}(A) &
\end{tikzcd}$$

in which all the squares commute, and the rows are parts of the exact sequences of Theorems 4.3, 5.7 of I. The inside left and right columns are exact - we wish to verify that the centre column is exact as well:

let $x \in W_{2i}(A_z)$ be such that $B(x) = 0 \in V_{2i-1}(A)$;

then $B\beta(x) = \nu B(x) = 0 \in U_{2i-1}(A)$, and $\beta(x) \in \ker B = \operatorname{im} \bar{\varepsilon} \subseteq V_{2i}(A_z)$;

let $y \in V_{2i}(A)$ be such that $\beta(x) = \bar{\varepsilon}(y) \in V_{2i}(A_z)$;

then $\bar{\varepsilon}\gamma(y) = \gamma\bar{\varepsilon}(y) = \gamma\beta(x) = 0 \in \Omega_+(A_z)$, and $y \in \ker\gamma = \operatorname{im}\beta \subseteq V_{2i}(A)$;

let $s \in W_{2i}(A)$ be such that $\beta(s) = y \in V_{2i}(A)$;

then $\beta(x - \bar{\varepsilon}(s)) = (y - \beta(s)) = 0 \in V_{2i}(A_z)$,

and $(x - \bar{\varepsilon}(s)) \in \ker\beta = \operatorname{im}\alpha \subseteq W_{2i}(A_z)$;

let $t \in \Omega_-(A_z)$ be such that $\alpha(t) = x - \bar{\varepsilon}(s) \in W_{2i}(A_z)$;

now $t = \bar{B}B(t) + \bar{\varepsilon}\varepsilon(t)$, so $(x - \bar{\varepsilon}(s + \alpha\varepsilon(t))) = \alpha\bar{B}B(t) \in W_{2i}(A_z)$;

also $\mu B(t) = B\alpha(t) = B(x) - B\bar{\varepsilon}(s) = 0 \in V_{2i-1}(A)$,

and $B(t) \in \ker\mu = \operatorname{im}\lambda \subseteq \Sigma_+(A)$;

let $u \in U_{2i}(A)$ be such that $\lambda(u) = B(t) \in \Sigma_+(A)$;

then $\alpha\bar{B}B(t) = \alpha\bar{B}\lambda(u) = \alpha\delta\bar{B}(u) = 0 \in W_{2i}(A_z)$;

hence

$$x = \bar{\varepsilon}(s + \alpha\varepsilon(t)) \in \operatorname{im}(\bar{\varepsilon} : W_{2i}(A) \longrightarrow W_{2i}(A_z)).$$

This completes the proof of Theorem 1.1.

§4. Multiple Laurent extensions

Let $T(p)$ be the free abelian group of rank $p (\geq 0)$, written multiplicatively. The group ring $A[T(p)]$, with involution

$$\bar{\ } : A[T(p)] \longrightarrow A[T(p)] ;$$

$$\sum_{g \in T(p)} a_g g \longmapsto \sum_{g \in T(p)} \bar{a}_g g^{-1} \quad (a_g \in A)$$

is the <u>p-fold Laurent extension of A</u>.

We may identify

$$A[T(0)] = A, \ A[T(1)] = A_z, \ A[T(2)] = A_{z_1,z_2}$$

and also

$$(A[T(p)])[T(q)] = A[T(p+q)] \quad (p,q \geq 0),$$

so that each $A[T(p)]$ satisfies the conditions imposed on the ground ring A. Denoting some set of generators of $T(p)$ by $z_1, z_2, \ldots, z_p$ (for $p \geq 1$) we can also write

$$A[T(p)] = A_{z_1,z_2,\ldots,z_p}$$

extending the previous notation.

In order to give a complete description of the L-theory of $A_{z_1,\dots,z_p}$ we recall first the "lower K-theory" of Chapter $\overline{\text{XII}}$ of Bass' "Algebraic K-theory" involving K-groups $\tilde{K}_m(A)$ for $m<0$, and subgroups $N^+_m(A)$, $N^-_m(A)$ of $\tilde{K}_{m+1}(A_z)$.
There are defined morphisms

$$\tilde{K}_{m+1}(A_z) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} \tilde{K}_m(A) \quad (m<0)$$

such that

$$B\bar{B} = 1 : \tilde{K}_m(A) \longrightarrow \tilde{K}_m(A),$$

giving natural direct sum decompositions

$$\tilde{K}_{m+1}(A_z) = \tilde{K}_{m+1}(A) \oplus \tilde{K}_m(A) \oplus N^+_m(A) \oplus N^-_m(A) \quad (m<0)$$

Duality involutions

$$* : \tilde{K}_m(A) \longrightarrow \tilde{K}_m(A)$$

are defined for all $m<0$, with

$$\begin{tikzcd}
\tilde{K}_m(A_z) \arrow[r, "*"] \arrow[d, shift left, "\varepsilon"] & \tilde{K}_m(A_z) \arrow[d, shift left, "\varepsilon"] \\
\tilde{K}_m(A) \arrow[r] \arrow[u, shift left, "\bar{\varepsilon}"] & \tilde{K}_m(A) \arrow[u, shift left, "\bar{\varepsilon}"]
\end{tikzcd}$$

commuting, and with

$$\begin{tikzcd}
\tilde{K}_{m+1}(A_z) \arrow[r, "*"] \arrow[d, shift right, "B"'] & \tilde{K}_{m+1}(A_z) \arrow[d, shift right, "B"'] \\
\tilde{K}_m(A) \arrow[r, "*"] \arrow[u, shift right, "\bar{B}"'] & \tilde{K}_m(A) \arrow[u, shift right, "\bar{B}"']
\end{tikzcd}$$

skewcommuting, and

$$*(N^{\pm}_m(A)) = N^{\mp}_m(A) \subseteq \tilde{K}_{m+1}(A_z).$$

In short, $\tilde{K}_{m+1}(A_z)$ is related to $\tilde{K}_m(A)$ in exactly the same way for $m<0$ as for $m=0$.

Regarding $\tilde{K}_m(A)$ as a $\mathbb{Z}_2$-module via $*$, there are defined Tate cohomology groups (for all $m \leq 1$)

$$H^{(m)}_n(A) \equiv \hat{H}^n(\mathbb{Z}_2; \tilde{K}_m(A)) = \{x \in \tilde{K}_m(A) \mid *x = (-)^n x\} \big/ \{y + (-)^n * y \mid y \in \tilde{K}_m(A)\}$$

depending only on $n \pmod 2$, which are abelian of exponent 2. This generalizes to $m<0$ the definitions of

$$\Omega_{(-)^n}(A) = H^{(1)}_n(A) \quad , \quad \Sigma_{(-)^n}(A) = H^{(0)}_n(A).$$

The induced maps

$$H^{(m)}_n(A) \underset{\varepsilon}{\overset{\bar{\varepsilon}}{\rightleftarrows}} H^{(m)}_n(A_z) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} H^{(m-1)}_{n-1}(A)$$

give natural splittings

$$H^{(m)}_n(A_z) = H^{(m)}_n(A) \oplus H^{(m-1)}_{n-1}(A) \quad (n \pmod 2)$$

for each $m<0$ (exactly as for $m=1$)

We now define the "lower L-groups"

$$L_n^{(m)}(A) = \ker\left(\varepsilon : L_{n+1}^{(m+1)}(A_z) \longrightarrow L_{n+1}^{(m+1)}(A)\right) \qquad (n \pmod 4)$$

inductively, for $m \leqslant 1$, where $L_*^{(2)} = W_*$.

It is clear from Theorem 1.1 that we can identify

$$L_*^{(1)} = V_* \ , \quad L_*^{(0)} = U_*$$

and that there is defined a natural exact sequence

$$\dots \longrightarrow H_{n+1}^{(m)}(A) \longrightarrow L_n^{(m+1)}(A) \longrightarrow L_n^{(m)}(A) \longrightarrow H_n^{(m)}(A) \longrightarrow \dots$$

of abelian groups and morphisms for $m = 0, 1$.

Hence all the L-theories differ in 2-torsion only. More precisely:

<u>Theorem 4.1</u> There is defined a natural exact sequence

$$\dots \longrightarrow H_{n+1}^{(m)}(A) \longrightarrow L_n^{(m+1)}(A) \longrightarrow L_n^{(m)}(A) \longrightarrow H_n^{(m)}(A) \longrightarrow \dots$$

of abelian groups and morphisms, for all $m \leqslant 1$, $n \pmod 4$.

<u>Proof</u>: By induction on m, downwards. $\square$

<u>Theorem 4.2</u> There are defined isomorphisms of graded abelian groups

$$L_*^{(*)}(A[T(p)]) \cong L_*^{(*)}(A) \otimes_{\mathbb{Z}} \Lambda_*(p)$$

where $\Lambda_*(p)$ is the graded exterior $\mathbb{Z}$-algebra on p generators $z_1, z_2, \dots, z_p$ in degree 1, which are natural in A and $T(p)$, with components

$$L_n^{(m)}(A_{z_1, z_2, \dots, z_p}) \cong \sum_{r=0}^{p} \binom{p}{r} L_{(n-r)}^{(m-r)}(A) \qquad (m \leqslant 2, n \pmod 4, p \geqslant 0)$$

<u>Proof</u>: It is sufficient to consider the case $W_*(A_{z_1, z_2})$, the others following by induction on p.

We need first the odd-dimensional counterpart of Lemma 3.4, that the diagram

$$\begin{tikzcd}
V_{2i+1}(A_{z_1}) \arrow[r, "B(z_2)"] \arrow[d, "\bar{B}(z_2)"'] & U_{2i}(A) \arrow[d, "\bar{B}(z_2)"] \\
W_{2i+2}(A_{z_1, z_2}) \arrow[r, "B(z_1)"] & V_{2i+1}(A_{z_2})
\end{tikzcd}$$

skewcommutes: the proof of this is left to the reader.

(Hint: it is known that

$$V_{2i+1}(A_{z_1}) = \bar{\varepsilon}(z_1) V_{2i+1}(A) \oplus \bar{B}(z_1) U_{2i}(A) \ .$$

The elements of $\bar{\varepsilon}(z_1) V_{2i+1}(A)$ are sent to 0 in $V_{2i+1}(A_{z_2})$

both ways round the square, so it is sufficient to verify that the composite

$$U_{2i}(A) \xrightarrow{\bar{B}(z_1)} V_{2i+1}(A_{z_1}) \xrightarrow{\bar{E}(z_2)} W_{2i+2}(A_{z_1,z_2}) \xrightarrow{B(z_1)} V_{2i+1}(A_{z_2})$$

coincides with

$$-\bar{B}(z_2) : U_{2i}(A) \longrightarrow V_{2i+1}(A_{z_2}).\ )$$

Thus

$$B(z_1)\bar{B}(z_2) = -\bar{B}(z_2)B(z_1) : V_n(A_{z_1}) \longrightarrow V_n(A_{z_2})$$

for all $n \pmod 4$, and as

$$\bar{B}B + \bar{\varepsilon}\varepsilon = 1 : W_n(A_z) \longrightarrow W_n(A_z)$$

it follows that

$$\begin{aligned} \bar{B}(z_1)\bar{B}(z_2) &= (\bar{B}(z_2)B(z_2) + \bar{\varepsilon}(z_2)\varepsilon(z_2))\bar{B}(z_1)\bar{B}(z_2) \\ &= \bar{B}(z_2)(-\bar{B}(z_1)B(z_2))\bar{B}(z_2) + (\bar{\varepsilon}(z_2)\bar{B}(z_1)).(\varepsilon(z_2)\bar{B}(z_2)) \\ &= -\bar{B}(z_2)\bar{B}(z_1) : U_{n-2}(A) \longrightarrow W_n(A_{z_1,z_2}). \end{aligned}$$

Accordingly, we have an isomorphism of abelian groups

$$L_n^{(2)}(A_{z_1,z_2}) \cong \sum_{j=0}^{2} L_{n-j}^{(2-j)}(A) \otimes_{\mathbb{Z}} \Lambda(2) \qquad (n \pmod 4)$$

sending

$$\begin{array}{lll} \bar{\varepsilon}(z_1)\bar{\varepsilon}(z_2)L_n^{(2)}(A) & \text{to} & L_n^{(2)}(A) \otimes 1 \\ \bar{B}(z_1)\bar{\varepsilon}(z_2)L_{n-1}^{(1)}(A) & & L_{n-1}^{(1)}(A) \otimes z_1 \\ \bar{\varepsilon}(z_1)\bar{B}(z_2)L_{n-1}^{(1)}(A) & & L_{n-1}^{(1)}(A) \otimes z_2 \\ \bar{B}(z_1)\bar{B}(z_2)L_{n-2}^{(0)}(A) & & L_{n-2}^{(0)}(A) \otimes (z_1 \wedge z_2). \end{array}$$

Naturality with respect to A is obvious.

Naturality with respect to $T(2)$, and more generally $T(p)$, follows on noting that a morphism

$$f : T(p) \longrightarrow T(q)$$

is determined by the $p \times q$ integer matrix

$$(f_{jk})_{1 \leq j \leq p,\, 1 \leq k \leq q}$$

such that

$$f(z_j) = \prod_{k=1}^{q} z_k^{f_{jk}} \qquad (1 \leq j \leq p,\ f_{jk} \in \mathbb{Z}),$$

the composition of such morphisms corresponding to multiplication of the matrices. Every such matrix can be expressed as the product of elementary matrices, such as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, N(\in \mathbb{Z}), \ldots$$

and their enlargements

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ldots$$

It is easy to show directly that for $p, q \leq 2$ the elementary $p \times q$ matrices induce the corresponding morphisms

$$1 \otimes f : L_*^{(x)}(A) \otimes \Lambda_*(p) \longrightarrow L_*^{(x)}(A) \otimes \Lambda_*(q)$$

in the exterior algebra, where

$$f : \Lambda_*(p) \longrightarrow \Lambda_*(q) ;$$

$$z_{j_1} \wedge \cdots \wedge z_{j_r} \longmapsto \bigwedge_{m=1}^{r} \Big(\sum_{k=1}^{q} f_{j_m} z_k \Big) \qquad (1 \leq r \leq p).$$

$\square$

Geometric L-theory

§0 Conventions

An inclusion of $\pm$ forms is a morphism

$$(f, \chi) : (P, \theta) \longrightarrow (Q, \phi)$$

such that $f \in \mathrm{Hom}_A(P, Q)$ is split mono.

A subform of a $\pm$ form (Q, ϕ) is an equivalence class of inclusions $(f, \chi) : (P, \theta) \longrightarrow (Q, \phi)$, under the relation

$$(f, \chi) : (P, \theta) \longrightarrow (Q, \phi) \quad \sim \quad (f', \chi') : (P', \theta') \longrightarrow (Q', \phi')$$

iff there exists an isomorphism $g \in \mathrm{Hom}_A(P, P')$ such that

$$\begin{tikzcd}
(P, \theta) \arrow[dr, "{(f,\chi)}"] \arrow[d, "{(g,0)}"'] & \\
(P', \theta') \arrow[r, "{(f',\chi')}"'] & (Q, \phi)
\end{tikzcd}$$

commutes. We shall use the notation $(P, \theta) \underset{(f,\chi)}{\subseteq} (Q, \phi)$ both for a particular inclusion of $\pm$ forms, and the subform of (Q, ϕ) it represents.

For example, a sublagrangian of a $\pm$ form (Q, ϕ) is a subform $(L, 0) \underset{(j,\lambda)}{\subseteq} (Q, \phi)$ such that

[illegible] $: Q \longrightarrow L^*$ is onto.

Isomorphisms of $\pm$ forms preserve subforms in the obvious way: an isomorphism

$$(g,\nu):(Q,\phi)\longrightarrow(Q',\phi')$$

sends the subform $(P,\theta)\subseteq_{(f,\chi)}(Q,\phi)$ to the subform

$$(P,\theta)\subseteq_{(g,\nu)(f,\chi)}(Q,\phi)\ .$$

Subforms $(P,\theta)\subseteq_{(f,\chi)}(Q,\phi)$, $(P',\theta')\subseteq_{(f',\chi')}(Q,\phi)$ are <u>orthogonal</u> if

$$f'^*(\phi\pm\phi^*)f=0\in \mathrm{Hom}_A(P,P'^*)$$

and <u>maximally orthogonal</u> if the sequences

$$0\to P\xrightarrow{f} Q\xrightarrow{f'^*(\phi\pm\phi^*)} P'^*\longrightarrow 0$$

$$0\to P'\xrightarrow{f'} Q\xrightarrow{f^*(\phi\pm\phi^*)} P^*\longrightarrow 0$$

are exact (that is, $fP=(f'P')^{\perp}\subseteq Q$, $f'P'=(fP)^{\perp}\subseteq Q$).
(For non-singular $\pm$ forms (Q,ϕ) the exactness of one of the sequences implies that of the other.)

For example, lagrangians $(L,0)\subseteq_{(j,\lambda)}(Q,\phi)$ are the maximally self-orthogonal subforms of (Q,ϕ).

The corresponding notions for $\pm$ formations, set out below, are rather more complicated. It is convenient to consider only the $\pm$ formations of "standard type"

$$\left(H_\pm(F);[F,0],(G,\theta)\subseteq_{(\binom{\gamma}{\mu},\theta)}H_\pm(F)\right)$$

which we shall abbreviate to

$$\left(F,\left(\binom{\gamma}{\mu},\theta\right)G\right)\ .$$

An isomorphism of $\pm$ formations

$$\left(F,\left(\binom{\gamma}{\mu},\theta\right)G\right)\longrightarrow\left(F',\left(\binom{\gamma'}{\mu'},\theta'\right)G'\right)$$

is defined by a commutative square

$$\begin{tikzcd}
G \arrow[r, "\beta"] \arrow[d, "\binom{\gamma}{\mu}"'] & G' \arrow[d, "\binom{\gamma'}{\mu'}"] \\
F\oplus F^* \arrow[r, "\begin{pmatrix}\alpha & \alpha(\psi^*\mp\psi)\\ 0 & \alpha^{*-1}\end{pmatrix}"'] & F'\oplus F'^*
\end{tikzcd}$$

for some isomorphisms $\alpha\in\mathrm{Hom}_A(F,F')$, $\beta\in\mathrm{Hom}_A(G,G')$ and some $\psi\in\Pi_{\mp}(F^*)$ such that

$$\beta^*\theta'\beta=\theta+\mu^*\psi\mu\in\Pi_{\mp}(G).$$

(This would have been

$$\left(\begin{pmatrix}\alpha & \alpha(\psi^* F \psi)\\ 0 & \alpha^{*-1}\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & \psi\end{pmatrix}\right) : (H_\pm(F); [F,0], [G,\theta]) \longrightarrow (H_\pm(F'); [F',0], [G',\theta'])$$

in the previous notation, with $\beta : G \to G'$ implicit).

We shall use the notation

$$(\alpha, \beta, \psi) : (F, (\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta)G) \longrightarrow (F', (\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}, \theta')G')$$

for such isomorphisms, retaining the square brackets, as in $[\alpha, \beta, \psi]$, for stable isomorphisms.

The hamiltonian $\pm$formations $(P, (\begin{pmatrix}0\\ 1\end{pmatrix}, 0)P^*)$ will be denoted by (P, P^*), the graph $\pm$formations $(P, (\begin{pmatrix}1\\ \theta \mp \theta^*\end{pmatrix}, \theta)P)$ by $\partial(P, \theta)$ as before.

Composition is by $(\alpha, \beta, \psi)(\alpha', \beta', \psi') = (\alpha\alpha', \beta\beta', \psi' + \alpha'^{-1}\psi\alpha'^{*-1})$.

An <u>inclusion</u> of $\pm$formations

$$(F', (\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}, \theta')G') \underset{\lambda, h}{\subseteq} (F, (\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta)G)$$

is defined by $\lambda \in \mathrm{Hom}_A(F, F')$ and an isomorphism $h \in \mathrm{Hom}_A(G, G')$ such that the squares

$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[d, "h"'] & F \arrow[d, "\lambda"] \\ G' \arrow[r, "\gamma'"] & F' \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\mu"] \arrow[d, "h"'] & F^* \\ G' \arrow[r, "\mu'"] & F'^* \arrow[u, "\lambda^*"'] \end{tikzcd}$$

commute, and

$$h^* \theta' h = \theta \in Q_\mp(G).$$

A <u>subformation</u> of a $\pm$ formation $(F, (\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta)G)$ is an equivalence class of inclusions

$$(F', (\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}, \theta')G') \underset{\lambda, h}{\subseteq} (F, (\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta)G)$$

under the relation

$$(F', (\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}, \theta')G') \underset{\lambda', h'}{\subseteq} (F, (\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta)G) \sim (F'', (\begin{pmatrix}\gamma''\\ \mu''\end{pmatrix}, \theta'')G) \underset{\lambda'', h''}{\subseteq} (F, (\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta)G)$$

iff there exists an isomorphism $\alpha \in \mathrm{Hom}_A(F', F'')$ such that

$$(\alpha, h''h'^{-1}, 0) : (F', (\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}, \theta')G') \longrightarrow (F'', (\begin{pmatrix}\gamma''\\ \mu''\end{pmatrix}, \theta'')G'')$$

defines an isomorphism of $\pm$formations such that $\begin{tikzcd} F \arrow[r, "\lambda'"] \arrow[dr, "\lambda''"'] & F' \arrow[d, "\alpha"] \\ & F'' \end{tikzcd}$ commutes

An isomorphism of $\pm$formations

$$(\alpha, \beta, \psi) : (F, (\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta)G) \longrightarrow (F', (\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}, \theta')G')$$

sends a subformation

$$(F'', (\begin{pmatrix}\gamma''\\ \mu''\end{pmatrix}, \theta'')G'') \underset{\lambda, h}{\subseteq} (F, (\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta)G)$$

to the subformation

$$(F'', (\begin{pmatrix}\lambda\alpha^{-1}\gamma'\\ \mu''h\beta^{-1}\end{pmatrix}, \theta')G') \underset{\lambda\alpha^{-1}, 1}{\subseteq} (F', (\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}, \theta')G'),$$

with

$$(1, \beta h^{-1}, \lambda\psi^*\lambda^*) : (F'', (\begin{pmatrix}\gamma''\\ \mu''\end{pmatrix}, \theta'')G'') \longrightarrow (F'', (\begin{pmatrix}\lambda\alpha^{-1}\gamma'\\ \mu''h\beta^{-1}\end{pmatrix}, \theta')G')$$

defining an isomorphism of $\pm$formations.

The annihilator of a $\pm$formation $(F,(\binom{\gamma}{\mu}),\theta)G)$ is the $\pm$formation

$$(F,(\tbinom{\gamma}{\mu}),\theta)G)^{\perp} = (F^*,((\tbinom{\mu}{\mp\gamma}),\theta)G) .$$

(This is just an expression in the new notation of an isomorph of the $\pm$formation given by $(H_{\pm}(F);[F^*,0],[G,\theta])$ in the old notation).

Note that

$$(F,((\tbinom{\gamma}{\mu}),\theta)G)^{\perp\perp} = (F^*,((\tbinom{\mu}{\mp\gamma}),\theta)G)^{\perp}$$
$$= (F,((\tbinom{\mp\gamma}{\mp\mu}),\theta)G) = (F,((\tbinom{\gamma}{\mu}),\theta)G)$$

and that if

$$(F,((\tbinom{\gamma}{\mu}),\theta)G) \underset{\lambda,h}{\subseteq} (F',((\tbinom{\gamma'}{\mu'}),\theta')G')$$

then

$$(F',((\tbinom{\gamma'}{\mu'}),\theta')G')^{\perp} \underset{\lambda^*,h^{-1}}{\subseteq} (F,((\tbinom{\gamma}{\mu}),\theta)G)^{\perp}$$

Subformations

$$(F',((\tbinom{\gamma'}{\mu'}),\theta')G') \underset{\lambda',h'}{\subseteq} (F,((\tbinom{\gamma}{\mu}),\theta)G) , \quad (F'',((\tbinom{\gamma''}{\mu''}),\theta'')G'') \underset{\lambda'',h''}{\subseteq} (F,((\tbinom{\gamma}{\mu}),\theta)G)$$

are orthogonal if there exists $\nu \in \mathrm{Hom}_A(F''^*,F')$ such that

$$(F',((\tbinom{\gamma'}{\mu'}),\theta')G') \underset{\nu,h'h''^{-1}}{\subseteq} (F'',((\tbinom{\gamma''}{\mu''}),\theta'')G'')^{\perp}$$

(in which case

$$(F'',((\tbinom{\gamma''}{\mu''}),\theta'')G'') \underset{\mp\nu^*,h''h'^{-1}}{\subseteq} (F',((\tbinom{\gamma'}{\mu'}),\theta')G')^{\perp})$$

and maximally orthogonal if, further, $\nu \in \mathrm{Hom}_A(F''^*,F')$ is an isomorphism.

In general, given an inclusion

$$(F',((\tbinom{\gamma'}{\mu'}),\theta')G') \underset{\lambda,h}{\subseteq} (F,((\tbinom{\gamma}{\mu}),\theta)G)$$

we shall use the isomorphism $h \in \mathrm{Hom}_A(G,G')$ as an identification, writing

$$(F',((\tbinom{\gamma'}{\mu'}),\theta)G) \underset{\lambda}{\subseteq} (F,((\tbinom{\gamma}{\mu}),\theta)G) .$$

For example, for any $\pm$formation $(F,((\binom{\gamma}{\mu}),\theta)G)$ we have that

$$(G^*,((\tbinom{\theta^*\mp\theta}{1}),\theta)G) \underset{\mu^*}{\subseteq} (F,((\tbinom{\gamma}{\mu}),\theta)G) \underset{\gamma}{\subseteq} (G,((\tbinom{1}{\theta\mp\theta^*}),\theta)G)$$
$$= \partial(G,\theta) .$$

§1 Cobordism of ± forms

Call ± forms $(Q_0,\phi_0),(Q_1,\phi_1)$ <u>cobordant</u> if there exist a ± formation $(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)$ and inclusions of ± forms

$$((\begin{smallmatrix}j_r\\ k_r\end{smallmatrix}),\chi_r) : (Q_r,(-)^r\phi_r) \longrightarrow H_\pm(F) \quad (r=0,1)$$

such that the sequence

$$0 \longrightarrow Q_0\oplus G \xrightarrow{\begin{pmatrix} j_0 & \gamma \\ k_0 & \mu\end{pmatrix}} F\oplus F^* \xrightarrow{\begin{pmatrix} \pm k_1^* & j_1^* \\ \pm\mu^* & \gamma^*\end{pmatrix}} Q_1^*\oplus G^* \longrightarrow 0$$

is exact, a quintuple such as

$$((F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G);(Q_0,\phi_0),(Q_1,\phi_1),((\begin{smallmatrix}j_0\\ k_0\end{smallmatrix}),\chi_0),((\begin{smallmatrix}j_1\\ k_1\end{smallmatrix}),\chi_1))$$

being a <u>cobordism from (Q_0,ϕ_0) to (Q_1,ϕ_1)</u>.

In other words, ± forms $(Q_0,\phi_0),(Q_1,\phi_1)$ are cobordant iff $(Q_0,\phi_0),(Q_1,-\phi_1)$ may be included as maximally orthogonal subforms of a stably trivial ± form

$$(G^\perp/G,\phi) = \partial(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G) ,$$

by definition the ± form to which $\begin{pmatrix}\theta & 1\\ 0 & 0\end{pmatrix}\in \mathrm{Hom}_A(F\oplus F^*,F^*\oplus F)$ restricts on a direct complement to $\mathrm{im}((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}):G\longrightarrow F\oplus F^*)$ in $\ker((\pm\mu^*\ \gamma^*):F^*\oplus F\longrightarrow G^*)$.

<u>Theorem 1.1</u> Cobordism is an equivalence relation on ± forms.

<u>Proof</u>: A cobordism of ± forms

$$((F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G);(Q_0,\phi_0),(Q_1,\phi_1),((\begin{smallmatrix}j_0\\ k_0\end{smallmatrix}),\chi_0),((\begin{smallmatrix}j_1\\ k_1\end{smallmatrix}),\chi_1))$$

may be reversed, to give a cobordism

$$((F,((\begin{smallmatrix}-\gamma\\ \mu\end{smallmatrix}),-\theta)G);(Q_1,\phi_1),(Q_0,\phi_0),((\begin{smallmatrix}-j_1\\ k_1\end{smallmatrix}),-\chi_1),((\begin{smallmatrix}-j_0\\ k_0\end{smallmatrix}),-\chi_0)),$$

so that the relation is symmetric.

An isomorphism of ± forms

$$(f,\chi):(P,\theta)\longrightarrow(Q,\phi)$$

defines a cobordism

$$((Q,0);(P,\theta),(Q,\phi),((\begin{smallmatrix}f\\ \phi f\end{smallmatrix}),\chi),((\begin{smallmatrix}1\\ \mp\phi^*\end{smallmatrix}),\phi))$$

so that, in particular, the relation is reflexive.

Finally, suppose given adjoining cobordisms

$$c = ((F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G);(Q_0,\phi_0),(Q_1,\phi_1),((\begin{smallmatrix}j_0\\ k_0\end{smallmatrix}),\chi_0),((\begin{smallmatrix}j_1\\ k_1\end{smallmatrix}),\chi_1))$$

$$c' = ((F',((\begin{smallmatrix}\gamma'\\ \mu'\end{smallmatrix}),\theta')G');(Q_1,\phi_1),(Q_2,\phi_2),((\begin{smallmatrix}j_1'\\ k_1'\end{smallmatrix}),\chi_1'),((\begin{smallmatrix}j_2'\\ k_2'\end{smallmatrix}),\chi_2')) .$$

The <u>join</u> of c and c',

$$c * c' = \left(\left(F \oplus F', \left(\left(\begin{pmatrix} \gamma & j_1 & 0 \\ 0 & j_1' & \gamma' \end{pmatrix} \atop \begin{pmatrix} \mu & k_1 & 0 \\ 0 & k_1' & \mu' \end{pmatrix} \right), \begin{pmatrix} \theta & \gamma^* k_1 & 0 \\ 0 & \chi_1 + \chi_1' & j_1'^* \mu' \\ 0 & 0 & \theta' \end{pmatrix} \right) G \oplus Q_1 \oplus G' \right); \right.$$

$$\left. (Q_0, \phi_0), (Q_2, \phi_2), \left(\left(\begin{pmatrix} j_0 \\ 0 \end{pmatrix} \atop \begin{pmatrix} k_0 \\ 0 \end{pmatrix} \right), \chi_0 \right), \left(\left(\begin{pmatrix} 0 \\ j_2' \end{pmatrix} \atop \begin{pmatrix} 0 \\ k_2' \end{pmatrix} \right), \chi_2' \right) \right)$$

defines a cobordism from (Q_0, ϕ_0) to (Q_2, ϕ_2), whence transitivity.

$\square$

Given cobordisms of $\pm$ forms

$$c = \left(\left(F, \left(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta \right) G \right); (Q_0, \phi_0), (Q_1, \phi_1), \left(\begin{pmatrix} j_0 \\ k_0 \end{pmatrix}, \chi_0 \right), \left(\begin{pmatrix} j_1 \\ k_1 \end{pmatrix}, \chi_1 \right) \right)$$

$$c' = \left(\left(F', \left(\begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}, \theta' \right) G' \right); (Q_0', \phi_0'), (Q_1', \phi_1'), \left(\begin{pmatrix} j_0' \\ k_0' \end{pmatrix}, \chi_0' \right), \left(\begin{pmatrix} j_1' \\ k_1' \end{pmatrix}, \chi_1' \right) \right)$$

we can define their direct sum $c \oplus c'$ in the obvious way, so that cobordism respects the direct sum operation on $\pm$ forms.

<u>Corollary 1.2</u> The $\pm$ forms $(Q_0, \phi_0), (Q_1, \phi_1)$ are cobordant iff

$$(Q_0, \phi_0) = (Q_1, \phi_1) \in X_{2i}(A) / \mathrm{im}(\partial : X_{2i+1}(A) \longrightarrow X_{2i}(A)) .$$

<u>Proof</u>: A hamiltonian $\pm$ form is cobordant to 0 by

$$((L, 0) ; H_{\pm}(L), 0, (1, 0), 0) .$$

As isomorphic $\pm$ forms are cobordant, and cobordism respects $\oplus$, it follows from Theorem 1.1 that $\pm$ forms $(Q_0, \phi_0), (Q_1, \phi_1)$ related by an isomorphism

$$(f, \chi) : (Q_0, \phi_0) \oplus H_{\pm}(L_0) \longrightarrow (Q_1, \phi_1) \oplus H_{\pm}(L_1)$$

are cobordant.

Conversely, suppose given a cobordism

$$\left(F, \left(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta \right) G \right); (Q_0, \phi_0), (Q_1, \phi_1), \left(\begin{pmatrix} j_0 \\ k_0 \end{pmatrix}, \chi_0 \right), \left(\begin{pmatrix} j_1 \\ k_1 \end{pmatrix}, \chi_1 \right) .$$

Now,

$$\left(\begin{pmatrix} 1 & 0 \\ j_1 & \gamma \\ k_1 & \mu \end{pmatrix}, \begin{pmatrix} \chi_1 & 0 \\ \gamma^* k_1 & \theta \end{pmatrix} \right) : (Q_1 \oplus G, 0) \longrightarrow (Q_1, \phi_1) \oplus H_{\pm}(F)$$

is the inclusion of a sublagrangian with $(Q_1 \oplus G)^{\perp} / Q_1 \oplus G = Q_0$, as is clear from the exact sequence

$$0 \to Q_0 \oplus Q_1 \oplus G \xrightarrow{\begin{pmatrix} 0 & 1 & 0 \\ j_0 & j_1 & \gamma \\ k_0 & k_1 & \mu \end{pmatrix}} Q_1 \oplus F \oplus F^* \xrightarrow{\begin{pmatrix} \phi_1 \pm \phi_1^* & \pm k_1^* & j_1^* \\ 0 & \pm \mu^* & \gamma^* \end{pmatrix}} Q_1^* \oplus G^* \longrightarrow 0$$

A direct application of Corollary 1.2 of I gives an isomorphism of $\pm$ forms

$$(Q_0, \phi_0) \oplus H_{\pm}(Q_1 \oplus G) \longrightarrow (Q_1, \phi_1) \oplus H_{\pm}(F)$$

so that

$$(Q_0, \phi_0) = (Q_1, \phi_1) \in X_{2i}(A) / \mathrm{im}(\partial : X_{2i+1}(A) \longrightarrow X_{2i}(A))$$

$\square$

In particular, non-singular $\pm$ forms $(Q_0, \phi_0), (Q_1, \phi_1)$ are cobordant iff

$$(Q_0, \phi_0) = (Q_1, \phi_1) \in U_{2i}(A) .$$

An <u>isomorphism</u> of cobordisms

$$[\alpha,\beta,\psi] : ((F,(\binom{\gamma}{\mu}),\theta)G) ; (Q_0,\phi_0),(Q_1,\phi_1),((\binom{j_0}{k_0}),\chi_0),((\binom{j_1}{k_1}),\chi_1))$$

$$\longrightarrow ((F',(\binom{\gamma'}{\mu'}),\theta')G') ; (Q'_0,\phi'_0),(Q'_1,\phi'_1),((\binom{j'_0}{k'_0}),\chi'_0),((\binom{j'_1}{k'_1}),\chi'_1))$$

is defined by a stable isomorphism of $\pm$formations

$$[\alpha,\beta,\psi] : (F,(\binom{\gamma}{\mu}),\theta)G) \longrightarrow (F',(\binom{\gamma'}{\mu'}),\theta')G')$$

such that the induced isomorphism of $\pm$ forms

$$\partial[\alpha,\beta,\psi] : (G^{\perp}/G,\phi) \longrightarrow (G'^{\perp}/G',\phi')$$

sends the subforms $(Q_r,(-)^r\phi_r) \subseteq (G^{\perp}/G,\phi)$ to $(Q'_r,(-)^r\phi'_r) \subseteq (G'^{\perp}/G',\phi')$ for $r=0,1$.

The join construction

$$(C,C') \longmapsto C*C'$$

defined in the proof of Theorem 1.1 may be characterized up to isomorphism by:

<u>Theorem 1.3</u> Suppose given orthogonal subformations

$$(F,(\binom{\gamma}{\mu}),\theta)G) \underset{\lambda}{\subseteq} (F'',(\binom{\gamma''}{\mu''}),\theta)G)$$

$$(F',(\binom{\gamma'}{\mu'}),\theta)G) \underset{\lambda'}{\subseteq} (F'',(\binom{\gamma''}{\mu''}),\theta)G)$$

with

$$(F',(\binom{\gamma'}{\mu'}),\theta)G) \underset{\nu}{\subseteq} (F^*,(\binom{\mu}{\mp\gamma}),\theta)G) ,$$

and short exact sequences

$$0 \to Q_0\oplus G \xrightarrow{\begin{pmatrix} j_0 & \gamma'' \\ k_0 & \mu \end{pmatrix}} F''\oplus F^* \xrightarrow{(\lambda' \ -\nu)} F' \longrightarrow 0$$

$$0 \to Q_1\oplus G \xrightarrow{\begin{pmatrix} j_1 & \mu \\ k_1 & \mu' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix}} F^*\oplus F'^* \xrightarrow{(\lambda^* \ -\lambda'^*)} F''^* \longrightarrow 0$$

$$0 \to Q_2\oplus G \xrightarrow{\begin{pmatrix} j_2 & \gamma'' \\ k_2 & \mu' \end{pmatrix}} F''\oplus F'^* \xrightarrow{(\lambda \ \pm\nu^*)} F \longrightarrow 0 ,$$

and some $\sigma \in \mathrm{Hom}_A(Q_1,G)$ (to act as a parameter).
Then there are defined cobordisms

$$\tilde{C} = ((F,(\binom{\gamma}{\mu}),\theta)G) ; (Q_0,\phi_0),(Q_1,\phi_1),((\binom{\lambda j_0}{k_0}),\chi_0),((\binom{\mp\nu^* k_1}{j_1}),\chi_1))$$

$$\tilde{C}' = ((F',(\binom{\gamma'}{\mu'}),\theta)G) ; (Q_1,\phi_1),(Q_2,\phi_2),((\binom{\nu(j_1+\mu\sigma)}{k_1+\mu'\sigma}),\chi'_1),((\binom{\lambda' j_2}{k_2}),\chi'_2))$$

$$\tilde{C}'' = ((F'',(\binom{\gamma''}{\mu''}),\theta)G) ; (Q_0,\phi_0),(Q_2,\phi_2),((\binom{j_0}{\lambda^* k_0}),\chi_0),((\binom{j_2}{\lambda'^* k_2}),\chi'_2))$$

and an isomorphism of cobordisms

$$\tilde{C} * \tilde{C}' \longrightarrow \tilde{C}''.$$

Proof: We are dealing with a commutative hexagon

$$\begin{tikzcd}
 & F'' \arrow[dl, "\lambda"'] \arrow[dr, "\lambda'"] & \\
F & & F' \\
 & G \arrow[u, "\gamma''"] \arrow[ul, "\gamma"] \arrow[ur, "\gamma'"'] \arrow[dl, "\mu'"] \arrow[dr, "\mu"'] \arrow[dd, "\mu''"] & \\
F'^{*} \arrow[uu, "\mp\nu^*"] \arrow[dr, "\lambda'^*"'] & & F^* \arrow[uu, "\nu"'] \arrow[dl, "\lambda^*"] \\
 & F''^{*} &
\end{tikzcd}$$

Choose any $\phi_r \in \mathrm{Hom}_A(Q_r, Q_r^*)$, $\chi_r^{(\prime)} \in \Pi_{\mp}(Q_r^*)$ $(r = 0,1,2)$ to satisfy the identities

$$j_0^* \lambda_0^* k_0 - \phi_0 = \chi_0 \mp \chi_0^* \in \mathrm{Hom}_A(Q_0, Q_0^*)$$

$$\left.\begin{aligned} \mp k_1^* \nu j_1 + \phi_1 &= \chi_1 \mp \chi_1^* \in \mathrm{Hom}_A(Q_1, Q_1^*) \\ (j_1^* + \sigma^*\mu^*)\nu^*(k_1 + \mu'\sigma) - \phi_1 &= \chi_1' \mp \chi_1'^* \in \mathrm{Hom}_A(Q_1, Q_1^*) \end{aligned}\right\}$$

(such identities may be satisfied simultaneously because the sum of the two expressions on the left is $(j_1^*\nu^*k_1 + \sigma^*\gamma^*k_1 + \sigma^*\theta'\sigma) \mp (j_1^*\nu^*k_1 + \sigma^*\gamma^*k_1 + \sigma^*\theta'\sigma)^*$)

$$j_2^* \lambda'^* k_2 + \phi_2 = \chi_2' \mp \chi_2'^* \in \mathrm{Hom}_A(Q_2, Q_2^*)$$

In order to verify that

$$\widetilde{c} = \left(\left(F, \left(\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta\right) G\right); (Q_0,\phi_0), (Q_1,\phi_1), \left(\begin{pmatrix}\lambda j_0\\ k_0\end{pmatrix}, \chi_0\right), \left(\begin{pmatrix}\mp\nu^* k_1\\ j_1\end{pmatrix}, \chi_1\right)\right)$$

does define a cobordism we have to show that the sequence

$$0 \longrightarrow Q_0 \oplus G \xrightarrow{\begin{pmatrix}\lambda j_0 & \gamma\\ k_0 & \mu\end{pmatrix}} F \oplus F^* \xrightarrow{\begin{pmatrix}\pm j_1^* & \mp k_1^*\nu\\ \pm\mu^* & \gamma^*\end{pmatrix}} Q_1^* \oplus G^* \longrightarrow 0$$

is exact.

This is done by diagram chasing round the commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & \\
0 \arrow[r] & Q_0 \oplus G \arrow[r, "{\begin{pmatrix} j_0 & \gamma'' \\ k_0 & \mu \end{pmatrix}}"] \arrow[dr, "{\begin{pmatrix} \lambda j_0 & \gamma \\ k_0 & \mu \end{pmatrix}}"'] & F'' \oplus F^* \arrow[r, "{(\lambda' \ -\nu)}"] \arrow[d, "{\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}}"] & F' \arrow[r] & 0 \\
 & & F \oplus F^* \arrow[d, "{\begin{pmatrix} \pm 1 & 0 \\ 0 & \mp\nu \end{pmatrix}}"'] \arrow[dr, "{\begin{pmatrix} \pm j_1^* & \mp k_1^*\nu \\ \pm\mu^* & \gamma^* \end{pmatrix}}"] & & \\
0 \arrow[r] & F'' \arrow[r, "{\begin{pmatrix} \lambda \\ -\lambda' \end{pmatrix}}"'] & F \oplus F' \arrow[r, "{\begin{pmatrix} j_1^* & k_1^* \\ \mu^* & \mu'^* \end{pmatrix}}"'] & Q_1^* \oplus G^* \arrow[r] \arrow[dr] & 0 \\
 & & & & 0
\end{tikzcd}$$

in which the rows are exact (by hypothesis).

For if

$$(x, y) \in \ker\left(\begin{pmatrix}\pm j_1^* & \mp k_1^*\nu\\ \pm\mu^* & \gamma^*\end{pmatrix} : F \oplus F^* \longrightarrow Q_1^* \oplus G^*\right),$$

then

$$(\pm x, \mp\nu y) \in \ker\left(\begin{pmatrix} j_1^* & k_1^*\\ \mu^* & \mu'^*\end{pmatrix} : F \oplus F' \longrightarrow Q_1^* \oplus G^*\right)$$

$$= \mathrm{im}\left(\begin{pmatrix}\lambda\\ -\lambda'\end{pmatrix} : F'' \longrightarrow F \oplus F'\right),$$

with

$$(\pm x, \mp\nu y) = (\lambda z, -\lambda' z) \in F \oplus F'$$

for some $z \in F''$. Now

$$(\pm z, y) \in \ker\left((\lambda' \ -\nu) : F'' \oplus F^* \longrightarrow F'\right)$$

$$= \mathrm{im}\left(\begin{pmatrix} j_0 & \gamma''\\ k_0 & \mu''\end{pmatrix} : Q_0 \oplus G \longrightarrow F'' \oplus F^*\right),$$

so that

$$(x,y) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} (\pm z, y) \in \operatorname{im}\left(\begin{pmatrix} \lambda j_0 & \gamma \\ k_0 & \mu \end{pmatrix} : Q_0 \oplus G \longrightarrow F \oplus F^* \right),$$

as required for the exactness of

$$Q_0 \oplus G \xrightarrow{\begin{pmatrix} \lambda j_0 & \gamma \\ k_0 & \mu \end{pmatrix}} F \oplus F^* \xrightarrow{\begin{pmatrix} \pm j_1^* & \mp k_1^* \nu \\ \pm \mu^* & \gamma^* \end{pmatrix}} Q_1^* \oplus G^* .$$

To verify that

$$\begin{pmatrix} \pm j_1^* & \mp k_1^* \nu \\ \pm \mu^* & \gamma^* \end{pmatrix} : F \oplus F^* \longrightarrow Q_1^* \oplus G^*$$

is onto, start with any

$$(a,b) \in Q_1^* \oplus G^* = \operatorname{im}\left(\begin{pmatrix} j_1^* & k_1^* \\ \mu^* & \mu'^* \end{pmatrix} : F \oplus F' \longrightarrow Q_1^* \oplus G^* \right).$$

Let $(c,d) \in F \oplus F'$ be such that

$$(a,b) = (j_1^* c + k_1^* d \,,\, \mu^* c + \mu'^* d) \in \cancel{F \oplus F'}\, Q_1^* \oplus G^*$$

Now

$$d \in F' = \operatorname{im}\left((\lambda' \ -\nu) : F'' \oplus F^* \longrightarrow F' \right)$$

so that

$$d = \lambda' e - \nu f$$

for some $e \in F''$, $f \in F^*$.

Then

$$(a,b) = \begin{pmatrix} \pm j_1^* & \mp k_1^* \nu \\ \pm \mu^* & \gamma^* \end{pmatrix} (\pm c \pm \lambda e, \pm f)$$

$$\in \operatorname{im}\left(\begin{pmatrix} \pm j_1^* & \mp k_1^* \nu \\ \pm \mu^* & \gamma^* \end{pmatrix} : F \oplus F^* \longrightarrow Q_1^* \oplus G^* \right)$$

as required.

Despite the notation we are in a situation with a high degree of symmetry (– this will be exploited more fully later on), as is clear from the hexagon drawn above.

For example, up to signs,

$$\begin{pmatrix} \lambda j_0 & \gamma \\ k_0 & \mu \end{pmatrix} : Q_0 \oplus G \longrightarrow F \oplus F^*$$

is the dual of the morphism corresponding to

$$\begin{pmatrix} \pm j_1^* & \mp k_1^* \nu \\ \pm \mu^* & \gamma^* \end{pmatrix} : F \oplus F^* \longrightarrow Q_1^* \oplus G^*$$

in the hexagon obtained by rotating the one above through 180°. It is therefore a split mono, and $\tilde{C}$ is a well-defined cobordism of $\pm$forms.

Similarly, we have that $\tilde{C}'$ and $\tilde{C}''$ are well-defined.

It now remains to exhibit an isomorphism

$$\tilde{C} * \tilde{C}' \longrightarrow \tilde{C}'' ,$$

where

$$\tilde{C} * \tilde{C}' = \left(\left(F \oplus F', \left(\left(\begin{pmatrix} \gamma & \mp \nu^* k_1 & 0 \\ 0 & \nu j_1 & \gamma' \end{pmatrix} \Sigma \atop \begin{pmatrix} \mu & j_1 & 0 \\ 0 & k_1 & \mu' \end{pmatrix} \Sigma \right), \Sigma^* \begin{pmatrix} \theta & \gamma^* j_1 & 0 \\ 0 & \lambda_1 + \lambda_1' & \mp k_1 \gamma' \\ 0 & 0 & \theta^* \end{pmatrix} \Sigma \right) G \oplus Q_1 \oplus G \right);$$

$$(Q_0, \phi_0), (Q_2, \phi_2), \left(\left(\begin{pmatrix} \lambda j_0 \\ 0 \end{pmatrix} \atop \begin{pmatrix} k_0 \\ 0 \end{pmatrix} \right), \chi_0 \right), \left(\left(\begin{pmatrix} 0 \\ \lambda' j_2 \end{pmatrix} \atop \begin{pmatrix} 0 \\ k_2 \end{pmatrix} \right), \chi_2' \right) \Bigg)$$

by definition, where $\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sigma & 1 \end{pmatrix} \in \operatorname{Hom}_A(G \oplus Q_1 \oplus G, G \oplus Q_1 \oplus G)$

Choosing some left inverse $\begin{pmatrix} l_1 \\ m_1 \end{pmatrix} : F''^* \longrightarrow F^* \oplus F'^*$ to $(\lambda^* \; -\lambda'^*) : F^* \oplus F'^* \longrightarrow F''^*$, let

$$0 \longrightarrow F''^* \xrightarrow{\begin{pmatrix} l_1 \\ m_1 \end{pmatrix}} F^* \oplus F'^* \xrightarrow{\begin{pmatrix} n_1 & p_1 \\ q_1 & r_1 \end{pmatrix}} Q_1 \oplus G \longrightarrow 0$$

be the short exact sequence defined by the corresponding right inverse $\begin{pmatrix} n_1 & p_1 \\ q_1 & r_1 \end{pmatrix} : F^* \oplus F'^* \longrightarrow Q_1 \oplus G$ to $\begin{pmatrix} j_1 & \mu \\ k_1 & \mu' \end{pmatrix} : Q_1 \oplus G \longrightarrow F^* \oplus F'^*$.

Then it may be easily verified that

$$(\alpha, \beta, \psi) = \left(\begin{pmatrix} l_1^* & m_1^* \\ j_1^* & k_1^* \\ \mu^* & \mu'^* \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ n_1\mu & 1 & -n_1\mu \\ q_1\mu & 0 & 1-q_1\mu \end{pmatrix}\Sigma, \begin{pmatrix} 0 & 0 \\ \pm\nu & 0 \end{pmatrix} \right)$$

$$: \left(F \oplus F', \left(\begin{pmatrix} \begin{pmatrix} \gamma & \mp\nu^* k_1 & 0 \\ 0 & \nu j_1 & \gamma' \end{pmatrix}\Sigma \\ \begin{pmatrix} \mu & j_1 & 0 \\ 0 & k_1 & \mu' \end{pmatrix}\Sigma \end{pmatrix}, \Sigma^* \begin{pmatrix} \theta & \gamma^* j_1 & 0 \\ 0 & \chi_1 + \chi_1' & \mp k_1^* \gamma' \\ 0 & 0 & \theta' \end{pmatrix}\Sigma \right) (G \oplus Q_1 \oplus G) \right)$$

$$\longrightarrow \left(F'', \left(\begin{pmatrix} \gamma'' \\ \mu'' \end{pmatrix}, \theta\right) G\right) \oplus (Q_1^* \oplus G^*, Q_1 \oplus G)$$

does define an isomorphism of cobordisms

$$[\alpha, \beta, \psi] : \tilde{c} * \tilde{c}' \longrightarrow \tilde{c}''. \qquad \square$$

Conversely, we have:

Theorem 1.4 Adjoining cobordisms

$$c = \left(\left(F, \left(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta\right) G\right); (Q_0, \phi_0), (Q_1, \phi_1), \left(\begin{pmatrix} j_0 \\ k_0 \end{pmatrix}, \chi_0\right), \left(\begin{pmatrix} j_1 \\ k_1 \end{pmatrix}, \chi_1\right) \right)$$

$$c' = \left(\left(F', \left(\begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}, \theta'\right) G'\right); (Q_1, \phi_1), (Q_2, \phi_2), \left(\begin{pmatrix} j_1' \\ k_1' \end{pmatrix}, \chi_1'\right), \left(\begin{pmatrix} j_2' \\ k_2' \end{pmatrix}, \chi_2'\right) \right)$$

may be replaced by isomorphic ones

$$\tilde{c} = \left(\left(F \oplus Q_1^* \oplus G'^*, \left(\begin{pmatrix} \tilde{\gamma} \\ \tilde{\mu} \end{pmatrix}, \theta''\right) G''\right); (Q_0, \phi_0), (Q_1, \phi_1), \left(\begin{pmatrix} \begin{pmatrix} j_0 \\ k_1^* j_0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} k_0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}, \chi_0\right), \left(\begin{pmatrix} \begin{pmatrix} j_1 \\ \mp j_1^* k_1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} k_1 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}, \chi_1\right) \right)$$

$$\tilde{c}' = \left(\left(G^* \oplus Q_1^* \oplus F', \left(\begin{pmatrix} \tilde{\gamma}' \\ \tilde{\mu}' \end{pmatrix}, \theta''\right) G''\right); (Q_1, \phi_1), (Q_2, \phi_2), \left(\begin{pmatrix} \begin{pmatrix} 0 \\ \mp j_1'^* k_1' \\ j_1' \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ k_1' \end{pmatrix} \end{pmatrix}, \chi_1'\right), \left(\begin{pmatrix} \begin{pmatrix} 0 \\ k_1^* j_2' \\ j_2' \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ k_2' \end{pmatrix} \end{pmatrix}, \chi_2'\right) \right)$$

where

$$(G'', \theta'') = \left(G \oplus Q_1 \oplus G', \begin{pmatrix} \theta & \gamma^* k_1 & 0 \\ 0 & \chi_1 + \chi_1' & j_1'^* \mu' \\ 0 & 0 & \theta' \end{pmatrix} \right),$$

which together with

$$\tilde{c}'' = c * c' = \left(\left(F \oplus F', \left(\begin{pmatrix} \gamma'' \\ \mu'' \end{pmatrix}, \theta''\right) G''\right); (Q_0, \phi_0), (Q_2, \phi_2), \left(\begin{pmatrix} \begin{pmatrix} j_0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} k_0 \\ 0 \end{pmatrix} \end{pmatrix}, \chi_0\right), \left(\begin{pmatrix} \begin{pmatrix} 0 \\ j_2' \end{pmatrix} \\ \begin{pmatrix} 0 \\ k_2' \end{pmatrix} \end{pmatrix}, \chi_2'\right) \right)$$

fit into the scheme of Theorem 1.3, according to the commutative hexagon

$$\begin{tikzcd}[column sep=small]
& F\oplus F' \arrow[dl, "\lambda = \begin{pmatrix} 1 & 0 \\ k_1^* & k_1'^* \\ 0 & \mu'^* \end{pmatrix}"'] \arrow[dr, "\lambda' = \begin{pmatrix} \mu^* & 0 \\ k_1^* & k_1'^* \\ 0 & 1 \end{pmatrix}"] & \\
F\oplus Q_1^*\oplus G'^* & & G^*\oplus Q_1^*\oplus F' \\
& G\oplus Q_1\oplus G' \arrow[uu, "\gamma'' = \begin{pmatrix} \gamma & j_1 & 0 \\ 0 & j_1' & \gamma' \end{pmatrix}"'] \arrow[ul, "\tilde{\gamma} = \begin{pmatrix} \gamma & j_1 & 0 \\ k_1^*\gamma & k_1^*j_1+k_1'^*j_1' & k_1'^*\gamma' \\ 0 & \mu'^*j_1' & \mu'^*\gamma' \end{pmatrix}"'] \arrow[ur, "\tilde{\gamma}' = \begin{pmatrix} \mu^*\gamma & \mu^*j_1 & 0 \\ k_1^*\gamma & k_1^*j_1+k_1'^*j_1' & k_1'^*\gamma' \\ 0 & j_1' & \gamma' \end{pmatrix}"] \arrow[dl, "\tilde{\mu}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu' \end{pmatrix}"'] \arrow[dr, "\hat{\mu} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"] \arrow[dd, "\mu'' = \begin{pmatrix} \mu & k_1 & 0 \\ 0 & k_1' & \mu' \end{pmatrix}"'] & \\
G\oplus Q_1\oplus F'^* \arrow[uu, "\begin{pmatrix} \gamma & j_1 & 0 \\ k_1^*\gamma & k_1^*j_1+k_1'^*j_1' & \mp j_1'^* \\ 0 & \mp\gamma'^*k_1' & \mp\gamma'^* \end{pmatrix} = \mp\nu^*"] \arrow[dr, "\lambda'^* = \begin{pmatrix} \mu & k_1 & 0 \\ 0 & k_1' & 1 \end{pmatrix}"'] & & F^*\oplus Q_1\oplus G' \arrow[uu, "\nu = \begin{pmatrix} \mp\gamma^* & \mp\gamma^*k_1 & 0 \\ \mp j_1^* & k_1^*j_1+k_1'^*j_1' & k_1'^*\gamma' \\ 0 & j_1' & \gamma' \end{pmatrix}"'] \arrow[dl, "\lambda^* = \begin{pmatrix} 1 & k_1 & 0 \\ 0 & k_1' & \mu' \end{pmatrix}"] \\
& F^*\oplus F'^* &
\end{tikzcd}$$

and short exact sequences

$$0 \to Q_0\oplus(G\oplus Q_1\oplus G') \xrightarrow{\begin{pmatrix} \begin{pmatrix} j_0 \\ 0 \end{pmatrix} = \tilde{j}_0 & \gamma'' \\ \begin{pmatrix} k_0 \\ 0 \\ 0 \end{pmatrix} = \tilde{k}_0 & \tilde{\mu} \end{pmatrix}} (F\oplus F')\oplus(F^*\oplus Q_1\oplus G') \xrightarrow{(\lambda' \ \ -\nu)} (G^*\oplus Q_1^*\oplus F') \to 0$$

$$0 \to Q_1\oplus(G\oplus Q_1\oplus G') \xrightarrow{\begin{pmatrix} \begin{pmatrix} k_1 \\ 0 \\ 0 \end{pmatrix} = \tilde{j}_1 & \tilde{\mu} \\ \begin{pmatrix} 0 \\ 1 \\ k_1' \end{pmatrix} = \tilde{k}_1 & \tilde{\mu}' \end{pmatrix}} (F^*\oplus Q_1\oplus G')\oplus(G\oplus Q_1\oplus F'^*) \xrightarrow{(\lambda^* \ \ -\lambda'^*)} (F^*\oplus F'^*) \to 0$$

$$0 \to Q_2\oplus(G\oplus Q_1\oplus G') \xrightarrow{\begin{pmatrix} \begin{pmatrix} 0 \\ j_2' \end{pmatrix} = \tilde{j}_2 & \gamma'' \\ \begin{pmatrix} 0 \\ 0 \\ k_2' \end{pmatrix} = \tilde{k}_2 & \tilde{\mu}' \end{pmatrix}} (F\oplus F')\oplus(G\oplus Q_1\oplus F'^*) \xrightarrow{(\lambda \ \ \pm\nu^*)} (F\oplus Q_1^*\oplus G'^*) \to 0$$

and $\sigma = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \in \mathrm{Hom}_A(Q_1, G\oplus Q_1\oplus G')$.

Proof: The commutative diagram

$$\begin{tikzcd}[column sep=large]
(Q_0\oplus Q_1)\oplus(G\oplus Q_1^*\oplus G'^*) \arrow[r, "1"] \arrow[d, "\begin{pmatrix} \begin{pmatrix} j_0 & j_1^* \\ & 0 \end{pmatrix} & \begin{pmatrix} \gamma & \\ & 0 \end{pmatrix} \\ \begin{pmatrix} k_0 & k_1 \\ & 0 \end{pmatrix} & \begin{pmatrix} \mu & & \\ & 1 & \\ 0 & & 1 \end{pmatrix} \end{pmatrix}"'] & (Q_0\oplus Q_1)\oplus(G\oplus Q_1^*\oplus G'^*) \arrow[d, "\begin{pmatrix} \begin{pmatrix} j_0 & j_1 \\ k_1^*j_0 & \mp j_1^*k_1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \gamma & j_1 & 0 \\ k_1^*\gamma & k_1^*j_1+k_1'^*j_1' & k_1'^*\gamma' \\ 0 & \mu'^*j_1' & \mu'^*\gamma' \end{pmatrix} \\ \begin{pmatrix} k_0 & k_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}"] \\
(F\oplus Q_1^*\oplus G'^*)\oplus(F^*\oplus Q_1\oplus G') \arrow[r, "\begin{pmatrix} 1 & \begin{pmatrix} 0 & j_1 & 0 \\ \mp j_1^* & k_1^*j_1+k_1'^*j_1' & k_1'^*\gamma' \\ 0 & \mp\gamma'^*k_1' & \mu'^*\gamma' \end{pmatrix} \\ 0 & 1 \end{pmatrix}"'] & (F\oplus Q_1^*\oplus G'^*)\oplus(F^*\oplus Q_1\oplus G')
\end{tikzcd}$$

shows that

$$(\alpha,\beta,\psi) = \left(1,\ 1,\ \begin{pmatrix} 0 & \mp j_1 & 0 \\ 0 & \chi_1+\chi_1' & \mp k_1'^*\gamma' \\ 0 & 0 & \theta' \end{pmatrix}\right) :$$

$$(F, ((\begin{smallmatrix}\gamma \\ \mu\end{smallmatrix}), \theta)G) \oplus (Q_1^*\oplus G'^*, Q_1\oplus G')$$

$$\longrightarrow (F\oplus Q_1^*\oplus G'^*, ((\begin{smallmatrix}\tilde{\gamma} \\ \tilde{\mu}\end{smallmatrix}), \theta'')G'')$$

defines an isomorphism of cobordisms

$$[\alpha,\beta,\psi] : c \longrightarrow \tilde{c}\,.$$

Similarly for c', $\tilde{c}'$. $\square$

For future reference, we stress here that the stable isomorphism

$$[\alpha,\beta,\psi] : (F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G) \longrightarrow (F\oplus Q_1^*\oplus G'^*,((\begin{smallmatrix}\tilde{\gamma}\\ \tilde{\mu}\end{smallmatrix}),\theta'')G'')$$

defined above is such that the triangle

$$\begin{tikzcd}[column sep=huge, row sep=large]
 & \partial(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G) \arrow[dd, "{\partial[\alpha,\beta,\psi]}"] \\
(Q_1,\phi_1) \arrow[ur, "{((\begin{smallmatrix}j_1\\ k_1\end{smallmatrix}),\chi_1)}"] \arrow[dr, swap, "{\left(\left(\begin{smallmatrix}\begin{pmatrix}j_1\\ \mp j_1^*k_1\\ 0\end{pmatrix}\\ \begin{pmatrix}k_1\\ 0\\ 0\end{pmatrix}\end{smallmatrix}\right),\chi_2\right)}"] & \\
 & \partial(F\oplus Q_1^*\oplus G'^*,((\begin{smallmatrix}\tilde{\gamma}\\ \tilde{\mu}\end{smallmatrix}),\theta'')G'')
\end{tikzcd}$$

of $\pm$forms and inclusions commutes.

For the same reason, we state

<u>Lemma 1.5</u> If $(Q_1,\phi_1) = (Q_0',\phi_0') = 0$, the join $c * c'$ of cobordisms

$$c = \{(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)\ ;\ (Q_0,\phi_0),(Q_1,\phi_1),((\begin{smallmatrix}j_0\\ k_0\end{smallmatrix}),\chi_0),((\begin{smallmatrix}j_1\\ k_1\end{smallmatrix}),\chi_1)\}$$

$$c' = \{(F',((\begin{smallmatrix}\gamma'\\ \mu'\end{smallmatrix}),\theta')G')\ ;\ (Q_0',\phi_0'),(Q_1',\phi_1'),((\begin{smallmatrix}j_0'\\ k_0'\end{smallmatrix}),\chi_0'),((\begin{smallmatrix}j_1'\\ k_1'\end{smallmatrix}),\chi_1')\}$$

is their direct sum, $c\oplus c'$.

$\square$

§2 <u>Cobordism of ± formations</u>

Call $\pm$formations $(F_0,((\begin{smallmatrix}\gamma_0\\ \mu_0\end{smallmatrix}),\theta_0)G_0)$, $(F_1,((\begin{smallmatrix}\gamma_1\\ \mu_1\end{smallmatrix}),\theta_1)G_1)$ <u>cobordant</u> if there exists a $\mp$ form (G,θ), and an inclusion of $\pm$forms

$$((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta) : (G,0) \longrightarrow H_\pm(F)$$

such that there are defined stable isomorphisms

$$[\alpha_0,\beta_0,\psi_0] : (F_0,((\begin{smallmatrix}\gamma_0\\ \mu_0\end{smallmatrix}),\theta_0)G_0) \longrightarrow (F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)$$

$$[\alpha_1,\beta_1,\psi_1] : (F_1,((\begin{smallmatrix}\gamma_1\\ \mu_1\end{smallmatrix}),\theta_1)G_1) \longrightarrow (F^*,((\begin{smallmatrix}\pm\mu\\ \gamma\end{smallmatrix}),-\theta)G)\ ,$$

a quintuple such as

$$((G,\theta)\ ;\ (F_0,((\begin{smallmatrix}\gamma_0\\ \mu_0\end{smallmatrix}),\theta_0)G_0),(F_1,((\begin{smallmatrix}\gamma_1\\ \mu_1\end{smallmatrix}),\theta_1)G_1),[\alpha_0,\beta_0,\psi_0],[\alpha_1,\beta_1,\psi_1])$$

being <u>a cobordism from $(F_0,((\begin{smallmatrix}\gamma_0\\ \mu_0\end{smallmatrix}),\theta_0)G_0)$ to $(F_1,((\begin{smallmatrix}\gamma_1\\ \mu_1\end{smallmatrix}),\theta_1)G_1)$</u>

In other words, $\pm$formations $(F_0,((\begin{smallmatrix}\gamma_0\\ \mu_0\end{smallmatrix}),\theta_0)G_0)$, $(F_1,((\begin{smallmatrix}\gamma_1\\ \mu_1\end{smallmatrix}),\theta_1)G_1)$ are cobordant iff $(F_0,((\begin{smallmatrix}\gamma_0\\ \mu_0\end{smallmatrix}),\theta_0)G_0)$, $(F_1,((\begin{smallmatrix}-\gamma_1\\ \mu_1\end{smallmatrix}),-\theta_1)G_1)$ are stably isomorphic to maximally orthogonal subformations of the graph $\pm$formation

$$\partial(G,\theta) = (G,((\begin{smallmatrix}1\\ \theta\mp\theta^*\end{smallmatrix}),\theta)G)\ ,$$

defined by some $\mp$form (G,θ).

<u>Theorem 2.1</u> Cobordism is an equivalence relation on $\pm$ formations.

<u>Proof</u>: A cobordism of $\pm$ formations

$$((G,\theta)\,;\,(F_0,(\begin{pmatrix}\gamma_0\\ \mu_0\end{pmatrix},\theta_0)G_0),\,(F_1,(\begin{pmatrix}\gamma_1\\ \mu_1\end{pmatrix},\theta_1)G_1),\,[\alpha_0,\beta_0,\psi_0],\,[\alpha_1,\beta_1,\psi_1])$$

may be reversed, to define a cobordism

$$((G,-\theta)\,;\,(F_1,(\begin{pmatrix}\gamma_1\\ \mu_1\end{pmatrix},\theta_1)G_1),\,(F_0,(\begin{pmatrix}\gamma_0\\ \mu_0\end{pmatrix},\theta_0)G_0),\,[\alpha_1,\beta_1,\psi_1],\,[\pm\alpha_0,\pm\beta_0,\pm\psi_0])$$

so that the relation is symmetric.

Every $\pm$ formation $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$ defines a cobordism

$$((G,\theta)\,;\,(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)\,,(F^*,(\begin{pmatrix}\pm\mu\\ \gamma\end{pmatrix},-\theta)G),\,1\,,1)\,,$$

so that it is sufficient to show transitivity to have reflexitivity. So let

$$C=((G,\theta)\,;\,(F_0,(\begin{pmatrix}\gamma_0\\ \mu_0\end{pmatrix},\theta_0)G_0),\,(F_1,(\begin{pmatrix}\gamma_1\\ \mu_1\end{pmatrix},\theta_1)G_1),\,[\alpha_0,\beta_0,\psi_0],\,[\alpha_1,\beta_1,\psi_1])$$

$$C'=((G',\theta')\,;\,(F_1,(\begin{pmatrix}\gamma_1\\ \mu_1\end{pmatrix},\theta_1)G_1),\,(F_2,(\begin{pmatrix}\gamma_2\\ \mu_2\end{pmatrix},\theta_2)G_2),\,[\alpha_1',\beta_1',\psi_1'],\,[\alpha_2,\beta_2,\psi_2])$$

be adjoining cobordisms. We then have a stable isomorphism

$$[\alpha,\beta,\psi]=[\alpha_1',\beta_1',\psi_1'][\alpha_1,\beta_1,\psi_1]^{-1}$$

$$:\ (F^*,(\begin{pmatrix}\pm\mu\\ \gamma\end{pmatrix},-\theta)G)\longrightarrow(F',(\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix},\theta')G')\,.$$

Let

$$(\alpha,\beta,\psi):(F^*,((\begin{pmatrix}\pm\mu\\ \gamma\end{pmatrix},-\theta)G)\oplus(P,P^*)\longrightarrow(F',((\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix},\theta')G')\oplus(P',P'^*)$$

be any representative, with

$$\alpha=\begin{pmatrix}a&a_1\\ ?&?\end{pmatrix}:F^*\oplus P\longrightarrow F'\oplus P'\qquad \alpha^{-1}=\begin{pmatrix}a'&a_1'\\ ?&?\end{pmatrix}:F'\oplus P'\longrightarrow F^*\oplus P$$

$$\beta=\begin{pmatrix}b&b_1\\ ?&?\end{pmatrix}:G\oplus P^*\longrightarrow G'\oplus P'^*\qquad \beta^{-1}=\begin{pmatrix}b'&b_1'\\ ?&?\end{pmatrix}:G'\oplus P'^*\longrightarrow G\oplus P^*$$

$$\psi=\begin{pmatrix}s&?\\ ?&?\end{pmatrix}:F\oplus P^*\longrightarrow F^*\oplus P\qquad \alpha(\psi^*\mp\psi)=\begin{pmatrix}n&?\\ ?&?\end{pmatrix}:F\oplus P^*\longrightarrow F'\oplus P'\,,$$

so that we have a commutative diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
G\oplus P^* \arrow[r, "\begin{pmatrix}b&b_1\\ ?&?\end{pmatrix}"] \arrow[d, "\begin{pmatrix}\pm\mu&0\\ 0&0\\ \gamma&0\\ 0&1\end{pmatrix}"'] & G'\oplus P'^* \arrow[d, "\begin{pmatrix}\gamma'&0\\ 0&0\\ \mu'&0\\ 0&1\end{pmatrix}"] \\
(F^*\oplus P)\oplus(F\oplus P^*) \arrow[r, "\begin{pmatrix}\begin{pmatrix}a&a_1\\ ?&?\end{pmatrix}&\begin{pmatrix}n&?\\ ?&?\end{pmatrix}\\ 0&\begin{pmatrix}a'^*&?\\ a_1'^*&?\end{pmatrix}\end{pmatrix}"'] & (F'\oplus P')\oplus(F'^*\oplus P'^*)
\end{tikzcd}$$

with

$$\begin{pmatrix}b^*&?\\ b_1^*&?\end{pmatrix}\begin{pmatrix}\theta'&0\\ 0&0\end{pmatrix}\begin{pmatrix}b&b_1\\ ?&?\end{pmatrix}=-\begin{pmatrix}\theta&0\\ 0&0\end{pmatrix}+\begin{pmatrix}\gamma^*&0\\ 0&1\end{pmatrix}\begin{pmatrix}s&?\\ ?&?\end{pmatrix}\begin{pmatrix}\gamma&0\\ 0&1\end{pmatrix}\in\Pi_{\mp}(G\oplus P^*)$$

Then

$$(G\oplus F'^*,\begin{pmatrix}\theta&0\\ a\mu&s'\end{pmatrix})\ \underset{\left(\delta'=\begin{pmatrix}b_1'a_1'^*&-b'\\ a'^*&\mu'\end{pmatrix},\begin{pmatrix}0&0\\ \mu'^*s'a'^*&b'^*\theta b'\end{pmatrix}\right)}{\overset{\left(\delta=\begin{pmatrix}\gamma&a^*\\ -b&b_1a_1^*\end{pmatrix},\begin{pmatrix}b^*\theta'b&\gamma^*sa^*\\ 0&0\end{pmatrix}\right)}{\rightleftarrows}}\ (F\oplus G',\begin{pmatrix}s&\mp a'\gamma'\\ 0&\theta'\end{pmatrix})$$

are inverse isomorphisms of $\mp$ forms over A.

Define

$$\xi=\begin{pmatrix}1&0\\ n&\gamma'\end{pmatrix}:F\oplus G'\longrightarrow F\oplus F'$$

$$\eta=\begin{pmatrix}\mu&0\\ 0&1\end{pmatrix}:G\oplus F'^*\longrightarrow F^*\oplus F'^*\,,$$

and let

$$\alpha\psi\alpha^*=\begin{pmatrix}s'&?\\ ?&?\end{pmatrix}:F'^*\oplus P'^*\longrightarrow F'\oplus P'\,.$$

Then

$$\left(\begin{pmatrix}\xi & \delta \\ \eta & \end{pmatrix}, \begin{pmatrix}\theta & 0 \\ a\mu & s'\end{pmatrix}\right) : (G \oplus F'^{*}, 0) \longrightarrow H_{\pm}(F \oplus F')$$

$$\left(\begin{pmatrix}\xi & \\ \eta & \delta'\end{pmatrix}, \begin{pmatrix}s & \mp a'\gamma' \\ 0 & \theta'\end{pmatrix}\right) : (F \oplus G', 0) \longrightarrow H_{\pm}(F \oplus F')$$

are the inclusions of a sublagrangian $(G'', 0) \underset{\left(\binom{\gamma''}{\mu''}, \theta''\right)}{\subseteq} H_{\pm}(F'')$, where $F'' = F \oplus F'$, because the diagram of $\pm$forms and morphisms

$$\begin{tikzcd}
(G \oplus F'^{*}, 0) \arrow[d, shift right, "{(\delta, 0)}"'] \arrow[drr, "{\left(\begin{pmatrix}\xi & \delta \\ \eta & \end{pmatrix}, \begin{pmatrix}\theta & 0 \\ a\mu & s'\end{pmatrix}\right)}"] & & \\
 & & H_{\pm}(F'') \\
(F \oplus G', 0) \arrow[u, shift right, "{(\delta', 0)}"'] \arrow[urr, "{\left(\begin{pmatrix}\xi & \\ \eta & \delta'\end{pmatrix}, \begin{pmatrix}s & \mp a'\gamma' \\ 0 & \theta'\end{pmatrix}\right)}"'] & &
\end{tikzcd}$$

commutes. The isomorphisms of $\pm$formations

$$\left(1, 1, \begin{pmatrix}0 & 0 \\ a & s'\end{pmatrix}\right) : \left(F, \left(\binom{\gamma}{\mu}, \theta\right) G\right) \oplus (F', F'^{*}) \longrightarrow \left(F \oplus F', \left(\begin{pmatrix}\xi & \delta \\ \eta & \end{pmatrix}, \begin{pmatrix}\theta & 0 \\ a\mu & s'\end{pmatrix}\right) G \oplus F'^{*}\right)$$

$$\left(\begin{pmatrix}1 & -n^{*} \\ 0 & 1\end{pmatrix}, 1, \begin{pmatrix}-s & \pm a' \\ 0 & 0\end{pmatrix}\right) : (F^{*}, F) \oplus \left(F'^{*}, \left(\binom{\pm\mu'}{\gamma'}, -\theta'\right) G'\right) \to \left(F^{*} \oplus F'^{*}, \left(\begin{pmatrix} & \pm\eta\delta' \\ \xi & \end{pmatrix}, -\begin{pmatrix}s & \mp a'\gamma' \\ 0 & \theta'\end{pmatrix}\right) F \oplus G'\right)$$

define stable isomorphisms

$$[\alpha_0', \beta_0', \psi_0'] : \left(F, \left(\binom{\gamma}{\mu}, \theta\right) G\right) \longrightarrow \left(F'', \left(\binom{\gamma''}{\mu''}, \theta''\right) G''\right)$$

$$[\alpha_2, \beta_2, \psi_2] : \left(F'^{*}, \left(\binom{\pm\mu'}{\gamma'}, -\theta'\right) G'\right) \longrightarrow \left(F''^{*}, \left(\binom{\pm\mu''}{\gamma''}, -\theta''\right) G''\right)$$

A different choice of representative of the stable isomorphism $[\alpha, \beta, \psi]$, as given by $(\alpha, \beta, \psi) \oplus 1_{(L, L^{*})}$ say, leads to precisely the same $[\alpha_0', \beta_0', \psi_0'], [\alpha_2, \beta_2, \psi_2]$. Thus, defining stable isomorphisms

$$[\alpha_0'', \beta_0'', \psi_0''] = [\alpha_0', \beta_0', \psi_0'][\alpha_0, \beta_0, \psi_0] : \left(F_0, \left(\binom{\gamma_0}{\mu_0}, \theta_0\right) G_0\right) \to \left(F'', \left(\binom{\gamma''}{\mu''}, \theta''\right) G''\right)$$

$$[\alpha_2'', \beta_2'', \psi_2''] = [\alpha_2, \beta_2, \psi_2][\alpha_2', \beta_2', \psi_2'] : \left(F_2, \left(\binom{\gamma_2}{\mu_2}, \theta_2\right) G_2\right) \longrightarrow \left(F''^{*}, \left(\binom{\pm\mu''}{\gamma''}, -\theta''\right) G''\right)$$

we are justified in calling the cobordism

$$c * c' = \left((G'', \theta''); \left(F_0, \left(\binom{\gamma_0}{\mu_0}, \theta_0\right) G_0\right), \left(F_2, \left(\binom{\gamma_2}{\mu_2}, \theta_2\right) G_2\right), [\alpha_0'', \beta_0'', \psi_0''], [\alpha_2'', \beta_2'', \psi_2'']\right)$$

the <u>join</u> of c and c', the only arbitrariness in the definition being of the choice of a representative G'' and $\theta'' \in \mathrm{Hom}_A(G'', G''^{*})$ to represent $\theta'' \in \Pi_{\mp}(G'')$. In particular, we have the transitivity of the relation. $\square$

<u>Corollary 2.2</u> The $\pm$formations $\left(F_0, \left(\binom{\gamma_0}{\mu_0}, \theta_0\right) G_0\right)$, $\left(F_1, \left(\binom{\gamma_1}{\mu_1}, \theta_1\right) G_1\right)$ are cobordant if

$$\left(F_0, \left(\binom{\gamma_0}{\mu_0}, \theta_0\right) G_0\right) = \left(F_1, \left(\binom{\gamma_1}{\mu_1}, \theta_1\right) G_1\right) \in X_{2i+1}(A) \big/ \mathrm{im}(\partial : X_{2i+2}(A) \longrightarrow X_{2i+1}(A)).$$

The converse holds for non-singular $\pm$ formations, so that non-singular $\left(F_0, \left(\binom{\gamma_0}{\mu_0}, \theta_0\right) G_0\right)$, $\left(F_1, \left(\binom{\gamma_1}{\mu_1}, \theta_1\right) G_1\right)$ are cobordant iff

$$\left(F_0, \left(\binom{\gamma_0}{\mu_0}, \theta_0\right) G_0\right) = \left(F_1, \left(\binom{\gamma_1}{\mu_1}, \theta_1\right) G_1\right) \in U_{2i+1}(A).$$

<u>Proof</u>: A graph $\pm$formation $\partial(P, \theta)$ is cobordant to 0 by

$$\left((P, \theta); \partial(P, \theta), 0, 1, \left[(1, 1, -\theta) : (P^{*}, P) \to \left(P^{*}, \left(\binom{\pm(\theta \mp \theta^{*})}{1}, -\theta\right) P\right)\right]\right).$$

As stably isomorphic $\pm$ formations are cobordant, and cobordism respects the direct sum operation $\oplus$, it follow from Theorem 2.1 that $\pm$ formations which represent the same element of $X_{2i+1}(A)/\mathrm{im}(\partial : X_{2i+2}(A) \longrightarrow X_{2i+1}(A))$ are cobordant.

Although it doesnot seem possible to establish a stable isomorphism of the type

$$[\alpha,\beta,\psi] : (F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G) \oplus \partial(R,\psi) \longrightarrow (F^*,(\begin{pmatrix}\pm\mu\\ \gamma\end{pmatrix},-\theta)G) \oplus \partial(R',\psi')$$

for general $\pm$formations $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$, such may be deduced for non-singular $\pm$formations from the sum formula of Lemma 3.3 of I. $\square$

(Alternatively, note that $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G) \oplus (F^*,(\begin{pmatrix}\pm\mu\\ \gamma\end{pmatrix},-\theta)G) \cong \partial(G,\theta) \oplus (G^*,G)$ by Theorem 2.3 of I.)

An <u>isomorphism</u> of cobordisms

$$(f,(g,\nu)) : ((G,\theta);(F_0,(\begin{pmatrix}\gamma_0\\ \mu_0\end{pmatrix},\theta_0)G_0),(F_1,(\begin{pmatrix}\gamma_1\\ \mu_1\end{pmatrix},\theta_1)G_1),[\alpha_0,\beta_0,\psi_0],[\alpha_1,\beta_1,\psi_1])$$
$$\longrightarrow ((G',\theta');(F'_0,(\begin{pmatrix}\gamma'_0\\ \mu'_0\end{pmatrix},\theta'_0)G'_0),(F'_1,(\begin{pmatrix}\gamma'_1\\ \mu'_1\end{pmatrix},\theta'_1)G'_1),[\alpha'_0,\beta'_0,\psi'_0],[\alpha'_1,\beta'_1,\psi'_1])$$

is an isomorphism of $\mp$ forms

$$(g,\nu) : (G,\theta) \longrightarrow (G',\theta')$$

together with an isomorphism $f \in \mathrm{Hom}_A(F,F')$ such that the squares

$$\begin{array}{ccc} G & \xrightarrow{g} & G' \\ \gamma\downarrow & & \downarrow\gamma' \\ F & \xrightarrow{f} & F' \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{g} & G' \\ \mu\downarrow & & \downarrow\mu' \\ F^* & \xleftarrow{f^*} & F'^* \end{array}$$

commute.

The join operation defined in the proof of Theorem 2.1 may be characterized up to isomorphism by the following:

<u>Theorem 2.3</u> Let (G'',θ'') be a $\mp$ form, and let

$$(G,\theta) \underset{(f,\chi)}{\subseteq} (G'',\theta'') \ , \ (G',\theta') \underset{(f',\chi')}{\subseteq} (G'',\theta'')$$

be orthogonal subforms such that

$$\begin{pmatrix} (\theta''^* \mp \theta'')f & 0 \\ f & f' \end{pmatrix} : G \oplus G' \longrightarrow G''^* \oplus G''$$

is a split mono.

Then $F = G''/f'G'$, together with

$$\gamma : G \longrightarrow G''/f'G' \ ; \ x \longmapsto [fx]$$
$$\mu : G \longrightarrow (G''/f'G')^* \ , \ x \longmapsto ([y] \longmapsto \langle fx,y\rangle_{\theta''}) \ ,$$

define a $\pm$formation $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$, as does $(F',(\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix},\theta')G')$, with

$$\gamma' : G' \longrightarrow (G''/fG)^* \ ; \ x' \longmapsto ([y'] \longmapsto \langle f'x',y'\rangle_{\theta''^*})$$
$$\mu' : G' \longrightarrow G''/fG \ ; \ x' \longmapsto [f'x'] \ ,$$

where $F' = (G''/fG)^*$.

A particular choice of direct complements to $fG, f'G'$ in G'' defines a $\pm$ formation $(F \oplus F',(\begin{pmatrix}\gamma''\\ \mu''\end{pmatrix},\theta'')G''))$, and a stable isomorphism of $\pm$ formations

$$[\tilde\alpha,\tilde\beta,\tilde\psi] : (F^*,(\begin{pmatrix}\pm\mu\\ \gamma\end{pmatrix},-\theta)G) \longrightarrow (F',(\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix},\theta')G')$$

Then there are defined cobordisms of $\pm$formations

$$\tilde{c} = ((G,\theta);(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G),(F^*,((\begin{smallmatrix}\pm\mu\\ \gamma\end{smallmatrix}),-\theta)G),[1],[1])$$

$$\tilde{c}' = ((G',\theta');(F^*,((\begin{smallmatrix}\pm\mu\\ \gamma\end{smallmatrix}),-\theta)G),(F'^*,((\begin{smallmatrix}\pm\mu'\\ \gamma'\end{smallmatrix}),-\theta')G'),[\alpha,\beta,\psi],[1])$$

$$\tilde{c}'' = ((G'',\theta'');(F\oplus F',((\begin{smallmatrix}\tilde{\gamma}''\\ \tilde{\mu}''\end{smallmatrix}),\theta'')G''),(F^*\oplus F'^*,((\begin{smallmatrix}\pm\tilde{\mu}''\\ \tilde{\gamma}''\end{smallmatrix}),-\theta'')G''),[1],[1])$$

and an isomorphism of cobordisms

$$\tilde{c} * \tilde{c}' \longrightarrow \tilde{c}''.$$

A different choice of direct complements leads to isomorphic $\tilde{c},\tilde{c}',\tilde{c}''$.

<u>Proof</u>: A particular choice of direct complements in G'' to $fG, f'G'$ corresponds to being given pairs

$$F\oplus G' \underset{\left(\begin{smallmatrix}e\\ g'\end{smallmatrix}\right)}{\overset{(h\ f')}{\rightleftarrows}} G'', \quad F'^*\oplus G \underset{\left(\begin{smallmatrix}e'\\ g\end{smallmatrix}\right)}{\overset{(h'\ f)}{\rightleftarrows}} G''$$

of inverse isomorphisms, with

$$e: G'' \to F = G''/f'G', \quad e': G'' \to F'^* = G''/fG$$

the natural projections. Thus

$$\gamma = ef : G \longrightarrow F \qquad \gamma' = h^*(\theta''^* \mp \theta'')f' : G' \to F'$$

$$\mu = h^*(\theta'' \mp \theta''^*)f : G \longrightarrow F^* \qquad \mu' = e'f' : G' \longrightarrow F'^*.$$

The isomorphism of $\pm$formations

$$(1,1,\theta''): (G''^*, ((\begin{pmatrix} -(\theta''^*\mp\theta'')f & 0 \\ f & f'\end{pmatrix}), \begin{pmatrix}-\theta & 0\\ 0 & 0\end{pmatrix})G\oplus G')$$

$$\longrightarrow (G''^*, ((\begin{pmatrix} 0 & (\theta''^*\mp\theta'')f' \\ f & f'\end{pmatrix}), \begin{pmatrix}0 & 0\\ 0 & \theta'\end{pmatrix})G\oplus G')$$

can now be expressed as

$$\left(\begin{pmatrix} h'^*e^* & h^*g'^* \\ f^*e^* & f^*g'^*\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix} h^*\theta''h & h^*\theta''f' \\ f'^*\theta''h & f'^*\theta''f'\end{pmatrix}\right)$$

$$: (F^*\oplus G'^*, ((\begin{pmatrix} -h^*(\theta''^*\mp\theta'')f & 0 \\ 0 & 0 \\ ef & 0 \\ g'f & 1\end{pmatrix}), \begin{pmatrix}-\theta & 0\\ 0 & 0\end{pmatrix})G\oplus G')$$

$$\longrightarrow (F'\oplus G^*, ((\begin{pmatrix} h'^*(\theta''^*\mp\theta'')f & 0 \\ 0 & 0 \\ e'f' & 0 \\ gf' & 1\end{pmatrix}), \begin{pmatrix}\theta' & 0\\ 0 & 0\end{pmatrix})G\oplus G')$$

We therefore have an isomorphism

$$(\tilde{\alpha},\tilde{\beta},\tilde{\psi}) = \left(\begin{pmatrix} h'^*e^* & h'^*g'^* \\ f^*e^* & f^*g'^* \end{pmatrix}, \begin{pmatrix} -g'f & 1 \\ 1-gf'g'f & gf' \end{pmatrix}, \begin{pmatrix} h'^*\theta''h & h'^*\theta''f' \\ f'^*\theta''h & f'^*\theta''f' \end{pmatrix}\right)$$

$$: \left(F^*, \left(\left(\begin{pmatrix} -h'^*(\theta''^*\mp\theta'')f \\ ef \end{pmatrix}, -\theta\right)G\right) \oplus (G'^*, G')\right)$$

$$\longrightarrow \left(F', \left(\left(\begin{pmatrix} h^*(\theta''^*\mp\theta'')f' \\ e'f' \end{pmatrix}, \theta'\right)G'\right) \oplus (G^*, G)\right),$$

and so a stable isomorphism

$$[\tilde{\alpha},\tilde{\beta},\tilde{\psi}] : \left(F^*, \left(\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta\right)G\right) \longrightarrow \left(F', \left(\begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}, \theta'\right)G'\right),$$

incidentally verifying that

$$\left(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta\right) : (G,0) \longrightarrow H_\pm(F), \quad \left(\begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}, \theta'\right) : (G',0) \longrightarrow H_\pm(F')$$

are indeed inclusions of sublagrangians.

Applying the join construction to the cobordisms

$$\tilde{c} = \left((G,\theta);\ \left(F, \left(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta\right)G\right), \left(F^*, \left(\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta\right)G\right), [1], [1]\right)$$

$$\tilde{c}' = \left((G',\theta');\ \left(F^*, \left(\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta\right)G\right), \left(F'^*, \left(\begin{pmatrix} \pm\mu' \\ \gamma' \end{pmatrix}, -\theta'\right)G'\right), [\tilde{\alpha},\tilde{\beta},\tilde{\psi}], [1]\right)$$

we obtain (choosing the obvious representatives)

$$\tilde{c} * \tilde{c}' = \left(\left(G\oplus F'^*, \begin{pmatrix} \theta & 0 \\ h'^*e^*\mu & h'^*\theta''h' \end{pmatrix}\right);\ \left(F, \left(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta\right)G\right), \left(F'^*, \left(\begin{pmatrix} \pm\mu' \\ \gamma' \end{pmatrix}, -\theta'\right)G'\right), [\alpha''_0, \beta''_0, \psi''_0], [\alpha''_2, \beta''_2, \psi''_2]\right)$$

where

$$(\alpha''_0, \beta''_0, \psi''_0) = \left(1, 1, \begin{pmatrix} 0 & 0 \\ h'^*e^* & h'^*\theta''h' \end{pmatrix}\right)$$

$$: \left(F, \left(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta\right)G\right) \oplus (F', F'^*) \longrightarrow \left(F\oplus F', \left(\left(\begin{matrix} \begin{pmatrix} \gamma & eh' \\ \mp h'^*e^*\mu & h'^*(\theta''^*\mp\theta'')h' \end{pmatrix} \\ \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \end{matrix}\right), \begin{pmatrix} \theta & 0 \\ h'^*e^*\mu & h'^*\theta''h' \end{pmatrix}\right)G\oplus F'^*\right)$$

$$(\alpha''_2, \beta''_2, \psi''_2) = \left(\begin{pmatrix} 1 & -h^*(\theta''^*\mp\theta''^*)h' \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} gh & gf' \\ e'h & e'f' \end{pmatrix}, \begin{pmatrix} -h^*\theta''h & \pm h^*e' \\ 0 & 0 \end{pmatrix}\right)$$

$$: (F^*, F) \oplus \left(F'^*, \left(\begin{pmatrix} \pm\mu' \\ \gamma' \end{pmatrix}, -\theta'\right)G'\right) \longrightarrow \left(F^*\oplus F'^*, \left(\left(\begin{matrix} \begin{pmatrix} \pm\mu & 0 \\ 0 & \pm 1 \end{pmatrix} \\ \begin{pmatrix} \gamma & eh' \\ \mp h'^*e^*\mu & h'^*(\theta''^*\mp\theta'')h' \end{pmatrix} \end{matrix}\right), -\begin{pmatrix} \theta & 0 \\ h'^*e^*\mu & h'^*\theta''h' \end{pmatrix}\right)G\oplus F'^*\right).$$

Now

$$\left((f\ h'), \begin{pmatrix} \chi & f^*\theta''h' \\ 0 & 0 \end{pmatrix}\right) : \left(G\oplus F'^*, \begin{pmatrix} \theta & 0 \\ h'^*e^*\mu & h'^*\theta''h' \end{pmatrix}\right) \longrightarrow (G'', \theta'')$$

defines an isomorphism of $\mp$ forms, so that setting

$$\tilde{\gamma}'' = \begin{pmatrix} \gamma & eh' \\ \mp h'^*e^*\mu & h'^*(\theta''^*\mp\theta'')h' \end{pmatrix}\begin{pmatrix} g \\ e' \end{pmatrix} = \begin{pmatrix} e \\ h'^*(\theta''^*\mp\theta'') \end{pmatrix} : G'' \longrightarrow F\oplus F' = F''$$

$$\tilde{\mu}'' = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g \\ e' \end{pmatrix} = \begin{pmatrix} h^*(\theta''\mp\theta''^*)g \\ e' \end{pmatrix} : G'' \longrightarrow F^*\oplus F'^* = F''^*$$

we have a cobordism of $\pm$ formations

$$\tilde{c}'' = \left((G'',\theta'')\,;\,\left(F'',\left(\begin{pmatrix}\tilde{\gamma}''\\ \tilde{\mu}''\end{pmatrix},\theta''\right)G''\right),\left(F''^*,\left(\begin{pmatrix}\pm\tilde{\mu}''\\ \tilde{\gamma}''\end{pmatrix},-\theta''\right)G''\right),[1],[1]\right)$$

isomorphic to $\tilde{c} * \tilde{c}'$.

Change of direct complements to fG, $f'G'$ in G'' affects neither

$$\tilde{c} = \left((G,\theta)\,;\,\left(F,\left(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta\right)G\right),\left(F^*,\left(\begin{pmatrix}\pm\mu\\ \gamma\end{pmatrix},-\theta\right)G\right),[1],[1]\right),$$

nor

$$\left((G',\theta')\,;\,\left(F',\left(\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix},\theta'\right)G'\right),\left(F'^*,\left(\begin{pmatrix}\pm\mu'\\ \gamma'\end{pmatrix},-\theta'\right)G'\right),[1],[1]\right),$$

an isomorph of $\tilde{c}'$.

Change of the direct complement to $f'G'$ in G'' does not affect

$$\left(\begin{pmatrix}\tilde{\gamma}''\\ \tilde{\mu}''\end{pmatrix},\theta''\right) : (G'',0) \longrightarrow H_{\pm}(F''),$$

giving the same $\tilde{c}''$.

A different choice of direct complement to fG in G'' replaces $(h'\ f) : F'^* \oplus G \longrightarrow G''$ by

$$(\tilde{h}'\ f) = (h'\ f)\begin{pmatrix}1 & 0\\ \sigma & 1\end{pmatrix} = (h'+f\sigma\ \ f) : F'^*\oplus G \longrightarrow G'',$$

for some $\sigma \in \mathrm{Hom}_A(F'^*,G)$, and $\begin{pmatrix}e'\\ g\end{pmatrix} : G'' \longrightarrow F'^*\oplus G$ by

$$\begin{pmatrix}e'\\ \tilde{g}\end{pmatrix} = \begin{pmatrix}1 & 0\\ -\sigma & 1\end{pmatrix}\begin{pmatrix}e'\\ g\end{pmatrix} = \begin{pmatrix}e'\\ -\sigma e'+g\end{pmatrix} : G'' \longrightarrow F'^*\oplus G.$$

This gives

$$\begin{pmatrix}\tilde{\tilde{\gamma}}''\\ \tilde{\tilde{\mu}}''\end{pmatrix} = \begin{pmatrix}\begin{pmatrix}e\\ \tilde{h}'^*(\theta''^*\mp\theta'')\end{pmatrix}\\ \begin{pmatrix}h^*(\theta''\mp\theta''^*)f\tilde{g}\\ e'\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\begin{pmatrix}e\\ (h'^*+\sigma^*f^*)(\theta''^*\mp\theta'')\end{pmatrix}\\ \begin{pmatrix}h^*(\theta''\mp\theta''^*)f(g-\sigma e')\\ e'\end{pmatrix}\end{pmatrix}$$

$$: G'' \longrightarrow F''\oplus F''^*$$

in place of $\begin{pmatrix}\tilde{\gamma}''\\ \tilde{\mu}''\end{pmatrix} : G'' \longrightarrow F''\oplus F''^*$.

An isomorphism of cobordisms

$$\tilde{c}'' = \left((G'',\theta'')\,;\,\left(F'',\left(\begin{pmatrix}\tilde{\gamma}''\\ \tilde{\mu}''\end{pmatrix},\theta''\right)G''\right),\left(F''^*,\left(\begin{pmatrix}\pm\tilde{\mu}''\\ \tilde{\gamma}''\end{pmatrix},-\theta''\right)G''\right),[1],[1]\right)$$

$$\longrightarrow \tilde{\tilde{c}}'' = \left((G'',\theta'')\,;\,\left(F'',\left(\begin{pmatrix}\tilde{\tilde{\gamma}}''\\ \tilde{\tilde{\mu}}''\end{pmatrix},\theta''\right)G''\right),\left(F''^*,\left(\begin{pmatrix}\pm\tilde{\tilde{\mu}}''\\ \tilde{\tilde{\gamma}}''\end{pmatrix},-\theta''\right)G''\right),[1],[1]\right)$$

is defined by the identity on (G'',θ'') together with the automorphism

$$\begin{pmatrix}1 & 0\\ -\sigma^*\mu^* & 1\end{pmatrix} : F\oplus F' \longrightarrow F\oplus F'$$

on F''. $\square$

Conversely, we have

Theorem 2.4 Adjoining cobordisms

$$c = ((G,\theta);(F_0,((\begin{smallmatrix}\gamma_0\\ \mu_0\end{smallmatrix}),\theta_0)G_0),(F_1,((\begin{smallmatrix}\gamma_1\\ \mu_1\end{smallmatrix}),\theta_1)G_1),[\alpha_0,\beta_0,\psi_0],[\alpha_1,\beta_1,\psi_1])$$

$$c' = ((G',\theta');(F_1,((\begin{smallmatrix}\gamma_1'\\ \mu_1'\end{smallmatrix}),\theta_1)G_1),(F_2,((\begin{smallmatrix}\gamma_2'\\ \mu_2'\end{smallmatrix}),\theta_2)G_2),[\alpha_1',\beta_1',\psi_1'],[\alpha_2',\beta_2',\psi_2'])$$

may be replaced by isomorphic ones

$$\tilde{c} = ((G,\theta);(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G),(F^*,((\begin{smallmatrix}\pm\mu\\ \gamma\end{smallmatrix}),-\theta)G),[1],[1])$$

$$\tilde{c}' = ((G',\theta');(F^*,((\begin{smallmatrix}\pm\mu\\ \gamma\end{smallmatrix}),-\theta)G),(F'^*,((\begin{smallmatrix}\pm\mu'\\ \gamma'\end{smallmatrix}),-\theta')G'),[\alpha,\beta,\psi],[1])$$

where

$$[\alpha,\beta,\psi] = [\alpha_1',\beta_1',\psi_1']^{-1}[\alpha_1,\beta_1,\psi_1] : (F^*,((\begin{smallmatrix}\pm\mu\\ \gamma\end{smallmatrix}),-\theta)G) \longrightarrow (F',((\begin{smallmatrix}\gamma'\\ \mu'\end{smallmatrix}),\theta')G'),$$

which together with an isomorph of $c*c'$

$$\tilde{c}'' = ((G'',\theta'');(F'',((\begin{smallmatrix}\gamma''\\ \mu''\end{smallmatrix}),\theta'')G''),(F''^*,((\begin{smallmatrix}\pm\mu''\\ \gamma''\end{smallmatrix}),-\theta'')G''),[1],[1]),$$

fit into the scheme of Theorem 2.3, according to the subform inclusions

$$((\begin{smallmatrix}1\\ 0\end{smallmatrix}),0) : (G,\theta) \longrightarrow (G\oplus F'^*, \begin{pmatrix}\theta & 0\\ a\mu & s'\end{pmatrix})$$

$$((\begin{smallmatrix}0\\ 1\end{smallmatrix}),0) : (G',\theta') \longrightarrow (F\oplus G', \begin{pmatrix}s & \mp a'\gamma'\\ 0 & \theta'\end{pmatrix})$$

in the isomorphs of (G'',θ'') occurring in the definition of $\tilde{c}*\tilde{c}'$.

Proof: For definiteness, set

$$(G'',\theta'') = (G\oplus F'^*, \begin{pmatrix}\theta & 0\\ a\mu & s'\end{pmatrix}).$$

Then

$$(f,\chi) = ((\begin{smallmatrix}1\\ 0\end{smallmatrix}),0) : (G,\theta) \longrightarrow (G\oplus F'^*, \begin{pmatrix}\theta & 0\\ a\mu & s'\end{pmatrix})$$

$$(f',\chi') = ((\begin{smallmatrix}-b'\\ \mu'\end{smallmatrix}),\chi') : (G',\theta') \longrightarrow (G\oplus F'^*, \begin{pmatrix}\theta & 0\\ a\mu & s'\end{pmatrix})$$

are the inclusions of orthogonal subforms, for some $\chi' \in \Pi_\pm(G')$, because

$$\begin{aligned} f'^*(\theta''\mp\theta''^*)f &= -b'^*(\theta\mp\theta^*) + \mu'^*a\mu \\ &= (-b'^*\gamma^* + \mu'^*a)\mu = 0 \in \mathrm{Hom}_A(G,G'^*), \end{aligned}$$

employing the standard notation of the definition of $c*c'$.

The isomorphisms

$$G\oplus F'^* \underset{\delta' = \begin{pmatrix}b_1'a_1'^* & -b'\\ a'^* & \mu'\end{pmatrix}}{\overset{\delta = \begin{pmatrix}\gamma & a^*\\ -b & b_1a_1^*\end{pmatrix}}{\rightleftarrows}} F\oplus G'$$

can now be used to define inverse pairs of isomorphisms

$$F\oplus G' \underset{(\begin{smallmatrix}e\\ g'\end{smallmatrix}) = \left(\begin{smallmatrix}(\gamma & a^*)\\ (-b & b_1a_1^*)\end{smallmatrix}\right)}{\overset{(h\ f') = \left((\begin{smallmatrix}b_1'a_1'^*\\ a'^*\end{smallmatrix})(\begin{smallmatrix}-b'\\ \mu'\end{smallmatrix})\right)}{\rightleftarrows}} G'' = G\oplus F'^* \qquad F'^*\oplus G \underset{(\begin{smallmatrix}e'\\ g\end{smallmatrix}) = \left(\begin{smallmatrix}(0 & 1)\\ (1 & 0)\end{smallmatrix}\right)}{\overset{(h'\ f) = \left((\begin{smallmatrix}0\\ 1\end{smallmatrix})(\begin{smallmatrix}1\\ 0\end{smallmatrix})\right)}{\rightleftarrows}} G'' = G\oplus F'^*$$

such that

$$\gamma = ef : G \longrightarrow F \qquad \gamma' = h'^*(\theta''^*\mp\theta'')f' : G' \longrightarrow F'$$

$$\mu = h^*(\theta''\mp\theta''^*)f : G \longrightarrow F^* \qquad \mu' = e'f' : G' \longrightarrow F'^*$$

exactly as in the proof of Theorem 2.3.

Now

$$((e^* \; g^*), \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, 0) : (F^*, (\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta)G) \oplus (G'^*, G')$$

$$\longrightarrow (G''^*, ((\begin{pmatrix} -(\theta''^* \mp \theta'')f & 0 \\ f & f' \end{pmatrix}), \begin{pmatrix} -\theta & 0 \\ 0 & 0 \end{pmatrix}) G \oplus G')$$

defines an isomorphism of $\pm$ formations, so that

$$\begin{pmatrix} (\theta''^* \mp \theta'')f & 0 \\ f & f' \end{pmatrix} : G \oplus G' \longrightarrow G''^* \oplus G''$$

is a split mono.

The join of the cobordisms

$$\tilde{c} = ((G,\theta); (F, (\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta)G), (F^*, (\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta)G), [1], [1])$$

$$\tilde{c}' = ((G',\theta'); (F^*, (\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta)G), (F'^*, (\begin{pmatrix} \pm\mu' \\ \gamma' \end{pmatrix}, -\theta')G'), [\alpha,\beta,\psi], [1])$$

is given by Theorem 2.3 to be isomorphic to

$$\tilde{c}'' = ((G'',\theta''); (F'', (\begin{pmatrix} \tilde{\gamma}'' \\ \tilde{\mu}'' \end{pmatrix}, \theta'')G''), (F''^*, (\begin{pmatrix} \pm\tilde{\mu}'' \\ \tilde{\gamma}'' \end{pmatrix}, -\theta'')G), [1], [1])$$

where $F'' = F \oplus F'$ and

$$\tilde{\gamma}'' = \begin{pmatrix} e \\ h'^*(\theta''^* \mp \theta'') \end{pmatrix} = \begin{pmatrix} \gamma & a^* \\ \mp a\mu & s'^* \mp s' \end{pmatrix} : G \oplus F'^* \longrightarrow F \oplus F'$$

$$\tilde{\mu}'' = \begin{pmatrix} h^*(\theta'' \mp \theta''^*)fg \\ e' \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} : G \oplus F'^* \longrightarrow F^* \oplus F'^*$$

Now $\tilde{\gamma}'' = \xi\delta$, $\tilde{\mu}'' = \eta$ (in the notation of Theorem 2.1), so that we do recover an isomorph of the join $c * c'$. □

Lemma 2.5 If $(F_1, (\begin{pmatrix} \gamma_1 \\ \mu_1 \end{pmatrix}, \theta_1)G_1)$ is a trivial $\pm$ formation, the join of cobordisms

$$c = ((G,\theta); (F_0, (\begin{pmatrix} \gamma_0 \\ \mu_0 \end{pmatrix}, \theta_0)G_0), (F_1, (\begin{pmatrix} \gamma_1 \\ \mu_1 \end{pmatrix}, \theta_1)G_1), [\alpha_0,\beta_0,\psi_0], [\alpha_1,\beta_1,\psi_1])$$

$$c' = ((G',\theta'); (F_1, (\begin{pmatrix} \gamma_1 \\ \mu_1 \end{pmatrix}, \theta_1)G_1), (F_2, (\begin{pmatrix} \gamma_2 \\ \mu_2 \end{pmatrix}, \theta_2)G_2), [\alpha_1',\beta_1',\psi_1'], [\alpha_2',\beta_2',\psi_2'])$$

is isomorphic to their direct sum, $c \oplus c'$.

Proof: By Theorem 2.4 we have a situation as in Theorem 2.3, with orthogonal subform inclusions

$$(f,\chi) : (G,\theta) \longrightarrow (G'',\theta''), \quad (f',\chi') : (G',\theta') \longrightarrow (G'',\theta'')$$

such that $(F_1, (\begin{pmatrix} \gamma_1 \\ \mu_1 \end{pmatrix}, \theta_1)G_1)$ is stably isomorphic to $((G''/f'G')^*, (\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta)G)$ (as defined in Theorem 2.3). Now $(F_1, (\begin{pmatrix} \gamma_1 \\ \mu_1 \end{pmatrix}, \theta_1)G)$ is trivial, so that μ_1, and hence

$$\gamma : G \longrightarrow G''/f'G' \; ; \; x \longmapsto [fx],$$

is an A-module isomorphism, allowing the identification $(G'',\theta'') = (G \oplus G', ($

We can therefore apply Theorem 2.3 with the configuration

$$G \oplus G' \underset{\begin{pmatrix} e \\ g' \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \end{pmatrix}}{\overset{(h \; f') = (\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix})}{\rightleftarrows}} G'' = G \oplus G' \qquad G' \oplus G \underset{\begin{pmatrix} e' \\ g \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{pmatrix}}{\overset{(h' \; f) = (\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix})}{\rightleftarrows}} G'' = G \oplus G'$$

This gives an isomorph of $c * c'$

$$\Big((G'',\theta'') \,;\, \Big(G\oplus G'^*, \Big(\begin{pmatrix} e \\ h'^*(\theta''^*\mp\theta'') \\ h^*(\theta'\mp\theta''^*)fg \\ e' \end{pmatrix}, \theta''\Big)G''\Big) \,,\, \Big(G^*\oplus G', \Big(\begin{pmatrix} \pm\begin{pmatrix} h^*(\theta'\mp\theta''^*)fg \\ e' \end{pmatrix} \\ \begin{pmatrix} e \\ h'^*(\theta''^*\mp\theta'') \end{pmatrix} \end{pmatrix}, \theta''\Big)G''\Big), 1, 1\Big)$$

$$= \Big((G,\theta) \,;\, \Big(G, \Big(\begin{pmatrix} 1 \\ \theta\mp\theta^* \end{pmatrix}, \theta\Big)G\Big) \,,\, \Big(G^*, \Big(\begin{pmatrix} \pm(\theta\mp\theta^*) \\ 1 \end{pmatrix}, -\theta\Big)G\Big) \,;\, [1],[1]\Big)$$

$$\oplus \Big((G',\theta') \,;\, \Big(G'^*, \Big(\begin{pmatrix} \theta'^*\mp\theta' \\ 1 \end{pmatrix}, \theta'\Big)G'\Big) \,,\, \Big(G', \Big(\begin{pmatrix} \pm 1 \\ \theta'^*\mp\theta' \end{pmatrix}, -\theta'\Big)G'\Big), [1],[1]\Big)$$

which is in turn isomorphic to $c \oplus c'$.

$\square$

In connection with the later applications of Theorem 2.4, it should be pointed out that the isomorphism of $\pm$ formations

$$(\alpha,\beta,\psi) = \left(\begin{pmatrix} a & a_1 \\ ? & ? \end{pmatrix}, \begin{pmatrix} b & b_1 \\ ? & ? \end{pmatrix}, \begin{pmatrix} s & ? \\ ? & ? \end{pmatrix}\right)$$

$$: \Big(F^*, \Big(\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta\Big)G\Big) \oplus (P, P^*)$$

$$\longrightarrow \Big(F', \Big(\begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}, \theta'\Big)G'\Big) \oplus (P', P'^*)$$

used in the definition of $c * c'$ is replaced, on passing to the isomorphs $\tilde{c}, \tilde{c}'$ of c, c' by the isomorphism

$$(\tilde{\alpha},\tilde{\beta},\tilde{\psi}) = \left(\begin{pmatrix} a & a_1 b_1^* \\ \gamma^* & -b^* \end{pmatrix}, \begin{pmatrix} b & 1 \\ 1-b'b & -b' \end{pmatrix}, \begin{pmatrix} s & \mp a'\gamma' \\ 0 & \theta' \end{pmatrix}\right)$$

$$: \Big(F^*, \Big(\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta\Big)G\Big) \oplus (G'^*, G')$$

$$\longrightarrow \Big(F', \Big(\begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}, \theta'\Big)G'\Big) \oplus (G^*, G)$$

as given by Theorem 2.3. These isomorphisms give precisely the same join but, rather inconveniently, may not in general be the representatives of the same stable isomorphism

$$: \Big(F^*, \Big(\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta\Big)G\Big) \longrightarrow \Big(F', \Big(\begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}, \theta'\Big)G'\Big).$$

Given a stable isomorphism of $\pm$ formations

$$[\alpha,\beta,\psi] : \Big(F^*, \Big(\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta\Big)G\Big) \longrightarrow \Big(F', \Big(\begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}, \theta'\Big)G'\Big)$$

call the isomorphism of $\pm$ formations $(\tilde{\alpha},\tilde{\beta},\tilde{\psi})$ (defined above) the <u>normalization</u> of $[\alpha,\beta,\psi]$. Stable isomorphisms

$$[\alpha,\beta,\psi], [\hat{\alpha},\hat{\beta},\hat{\psi}] : \Big(F^*, \Big(\begin{pmatrix} \pm\mu \\ \gamma \end{pmatrix}, -\theta\Big)G\Big) \longrightarrow \Big(F', \Big(\begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}, \theta'\Big)G'\Big)$$

are <u>coherent</u> if their normalizations coincide.

Suppose given stable isomorphisms

$$[\alpha,\beta,\psi] : \Big(F, \Big(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta\Big)G\Big) \longrightarrow \Big(F', \Big(\begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}, \theta'\Big)G'\Big)$$

$$[\hat{\alpha},\hat{\beta},\hat{\psi}] : \Big(F', \Big(\begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}, \theta'\Big)G'\Big) \longrightarrow \Big(F'', \Big(\begin{pmatrix} \gamma'' \\ \mu'' \end{pmatrix}, \theta''\Big)G''\Big)$$

with representatives

$$(\alpha,\beta,\psi)=\left(\begin{pmatrix} a & a_1 \\ a_2 & a_3\end{pmatrix},\begin{pmatrix} b & b_1 \\ b_2 & b_3\end{pmatrix},\begin{pmatrix} s & s_1 \\ s_2 & s_3\end{pmatrix}\right)$$

$$:(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)\oplus(P,P^*)\longrightarrow(F',((\begin{smallmatrix}\gamma'\\ \mu'\end{smallmatrix}),\theta')G')\oplus(P',P'^*)$$

$$(\hat\alpha,\hat\beta,\hat\psi)=\left(\begin{pmatrix} \hat a & \hat a_1 \\ \hat a_2 & \hat a_3\end{pmatrix},\begin{pmatrix} \hat b & \hat b_1 \\ \hat b_2 & \hat b_3\end{pmatrix},\begin{pmatrix} \hat s & \hat s_1 \\ \hat s_2 & \hat s_3\end{pmatrix}\right)$$

$$:(F',((\begin{smallmatrix}\gamma'\\ \mu'\end{smallmatrix}),\theta')G')\oplus(\hat P',\hat P'^*)\longrightarrow(F'',((\begin{smallmatrix}\gamma''\\ \mu''\end{smallmatrix}),\theta'')G'')\oplus(\hat P'',\hat P''^*)$$

The composite stable isomorphism

$$[\hat\alpha,\hat\beta,\hat\psi][\alpha,\beta,\psi]:(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)\longrightarrow(F'',((\begin{smallmatrix}\gamma''\\ \mu''\end{smallmatrix}),\theta'')G'')$$

has representative isomorphism

$$\left(\begin{pmatrix} \hat a & 0 & \hat a_1 \\ 0 & 1 & 0 \\ \hat a_2 & 0 & \hat a_3\end{pmatrix},\begin{pmatrix} \hat b & 0 & \hat b_1 \\ 0 & 1 & 0 \\ \hat b_2 & 0 & \hat b_3\end{pmatrix},\begin{pmatrix} \hat s & 0 & \hat s_1 \\ 0 & 0 & 0 \\ \hat s_2 & 0 & \hat s_3\end{pmatrix}\right)$$

$$\cdot\left(\begin{pmatrix} a & a_1 & 0 \\ a_2 & a_3 & 0 \\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix} b & b_1 & 0 \\ b_2 & b_3 & 0 \\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix} s & s_1 & 0 \\ s_2 & s_3 & 0 \\ 0 & 0 & 0\end{pmatrix}\right)$$

$$=\left(\begin{pmatrix} \hat a a & \hat a a_1 & \hat a_1 \\ a_2 & a_3 & 0 \\ \hat a_2 a & \hat a_2 a_1 & \hat a_3\end{pmatrix},\begin{pmatrix} \hat b b & \hat b b_1 & \hat b_1 \\ b_2 & b_3 & 0 \\ \hat b_2 b & \hat b_2 b_1 & \hat b_3\end{pmatrix},\begin{pmatrix} s+\alpha'\hat s\alpha'^* & s_1+\alpha'\hat s\alpha_2'^* & \alpha'\hat s_1 \\ s_2+\alpha_2'\hat s\alpha'^* & s_3+\alpha_2'\hat s\alpha_2'^* & \alpha_2'\hat s_1 \\ \hat s_2\alpha'^* & \hat s_2\alpha_2'^* & \hat s_3\end{pmatrix}\right)$$

$$:(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)\oplus(P\oplus\hat P',P^*\oplus\hat P'^*)\longrightarrow(F'',((\begin{smallmatrix}\gamma''\\ \mu''\end{smallmatrix}),\theta'')G'')\oplus(P'\oplus\hat P'',P'^*\oplus\hat P''^*)$$

with normalization

$$\left(\begin{pmatrix} \hat a a & \hat a_1\hat b_1^*+\hat a a_1 b_1^*\hat b^* \\ \mu^* & -b^*\hat b^*\end{pmatrix},\begin{pmatrix} \hat b b & 1 \\ 1-b'\hat b'\hat b b & -b'\hat b'\end{pmatrix},\begin{pmatrix} s+\alpha'\hat s\alpha'^* & \mp\alpha'\hat\alpha'\gamma'' \\ 0 & \theta''\end{pmatrix}\right)$$

$$:(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)\oplus(G''^*,G'')\longrightarrow(F'',((\begin{smallmatrix}\gamma''\\ \mu''\end{smallmatrix}),\theta'')G'')\oplus(G^*,G)$$

The composite of the normalizations of $[\alpha,\beta,\psi]$, $[\hat\alpha,\hat\beta,\hat\psi]$ (stabilized in the obvious way),

$$\left(\begin{pmatrix} \hat a & 0 & \hat a_1\hat b_1^* \\ 0 & 1 & 0 \\ \mu''^* & 0 & -\hat b^*\end{pmatrix},\begin{pmatrix} \hat b & 0 & 1 \\ 0 & 1 & 0 \\ 1-\hat b'\hat b & 0 & -\hat b'\end{pmatrix},\begin{pmatrix} \hat s & 0 & \mp\hat\alpha'\gamma'' \\ 0 & 0 & 0 \\ 0 & 0 & \theta''\end{pmatrix}\right)$$

$$\cdot\left(\begin{pmatrix} a & a_1 b_1^* & 0 \\ \mu^* & -b^* & 0 \\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix} b & 1 & 0 \\ 1-b'b & -b' & 0 \\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix} s & \mp\alpha'\gamma' & 0 \\ 0 & \theta' & 0 \\ 0 & 0 & 0\end{pmatrix}\right)$$

$$=\left(\begin{pmatrix} \hat a a & \hat a a_1 b_1^* & \hat a_1\hat b_1^* \\ \mu^* & -b^* & 0 \\ \mu''^*\hat a & \mu''^*\hat a_1 b_1^* & -\hat b^*\end{pmatrix},\begin{pmatrix} \hat b b & \hat b & 1 \\ 1-b'b & -b' & 0 \\ (1-\hat b'\hat b)b & 1-\hat b'\hat b & -\hat b'\end{pmatrix},\right.$$

$$\left.\begin{pmatrix} s+\alpha'\hat s\alpha'^* & \alpha'(\mp\gamma'+\hat s\mu') & \mp\alpha'\hat\alpha'\gamma'' \\ \mu'^*\hat s\alpha'^* & \theta'+\mu'^*\hat s\mu' & \mp\mu'^*\hat\alpha'\gamma'' \\ 0 & 0 & \theta''\end{pmatrix}\right)$$

$$:(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)\oplus(G'^*\oplus G''^*,G'\oplus G'')\longrightarrow(F'',((\begin{smallmatrix}\gamma''\\ \mu''\end{smallmatrix}),\theta'')G'')\oplus(G^*\oplus G'^*,G\oplus G')$$

has precisely the same normalization.

Therefore the coherence classes of stable isomorphisms are the morphisms of a category with $\pm$ formations as objects.

§3 Lattices and n-ads

A lattice of $\pm$forms, $\{(Q_J,\phi_J)\}_{J\subseteq I}$, is a finite collection of $\pm$forms (Q_J,ϕ_J), together with inclusions

$$(f^J_{J'},\chi^J_{J'}) : (Q_J,\phi_J) \longrightarrow (Q_{J'},\phi_{J'}) \quad (J\subseteq J'\subseteq I)$$

such that the triangles

$$\begin{tikzcd}
(Q_J,\phi_J) \arrow[d, "{(f^J_{J'},\chi^J_{J'})}"'] \arrow[dr, "{(f^J_{J''},\chi^J_{J''})}"] & \\
(Q_{J'},\phi_{J'}) \arrow[r, "{(f^{J'}_{J''},\chi^{J'}_{J''})}"'] & (Q_{J''},\phi_{J''})
\end{tikzcd} \qquad (J\subseteq J'\subseteq J''\subseteq I)$$

commute, with

$$(f^J_J,\chi^J_J) = (1,0) : (Q_J,\phi_J) \longrightarrow (Q_J,\phi_J),$$

and such that for disjoint $J,K\subseteq I$ the subforms

$$(f^J_{J\cup K},\chi^J_{J\cup K}) : (Q_J,\phi_J) \longrightarrow (Q_{J\cup K},\phi_{J\cup K})$$

$$(f^K_{J\cup K},\chi^K_{J\cup K}) : (Q_K,\phi_K) \longrightarrow (Q_{J\cup K},\phi_{J\cup K})$$

are orthogonal. By convention, $(Q_\emptyset,\phi_\emptyset)=0$.

An <u>n-ad of $\pm$forms</u>, $\{(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)\,;\,\{(Q_J,\phi_J)\}_{J\subseteq I}\}$, is a $\pm$formation $(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)$ together with a lattice of $\pm$forms $\{(Q_J,\phi_J)\}_{J\subseteq I}$, where $|I| = n+1$, and with an isomorphism of $\pm$forms

$$(f,\chi) : (Q_I,\phi_I) \longrightarrow \partial(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G),$$

and such that the subforms

$$(Q_J,\phi_J) \underset{(f_I^J,\chi_I^J)}{\subseteq} (Q_I,\phi_I)\ ,\ (Q_{I-J},\phi_{I-J}) \underset{(f_I^{I-J},\chi_I^{I-J})}{\subseteq} (Q_I,\phi_I)$$

are maximally orthogonal for all $J\subseteq I$. For example, a 1-ad is a cobordism of $\pm$forms.

An <u>isomorphism</u> of n-ads of $\pm$forms

$$\{[\alpha,\beta,\psi],\{(h_J,\nu_J)\}\} : \{(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)\,;\,\{(Q_J,\phi_J)\}_{J\subseteq I}\} \longrightarrow \{(F',((\begin{smallmatrix}\gamma'\\ \mu'\end{smallmatrix}),\theta')G')\,;\,\{(Q'_J,\phi'_J)\}_{J\subseteq I}\}$$

is defined by a stable isomorphism of $\pm$formations

$$[\alpha,\beta,\psi] : (F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G) \longrightarrow (F',((\begin{smallmatrix}\gamma'\\ \mu'\end{smallmatrix}),\theta')G')$$

together with isomorphisms of $\pm$forms

$$(h_J,\nu_J) : (Q_J,\phi_J) \longrightarrow (Q'_J,\phi'_J) \qquad (J\subseteq I)$$

such that the squares

$$\begin{array}{ccc} (Q_J,\phi_J) & \xrightarrow{(h_J,\nu_J)} & (Q'_J,\phi'_J) \\ {\scriptstyle (f_{J'}^J,\chi_{J'}^J)}\big\downarrow & & \big\downarrow{\scriptstyle (f'^J_{J'},\chi'^J_{J'})} \\ (Q_{J'},\phi_{J'}) & \xrightarrow[(h_{J'},\nu_{J'})]{} & (Q'_{J'},\phi'_{J'}) \end{array} \qquad \begin{array}{ccc} (Q_I,\phi_I) & \xrightarrow{(f,\chi)} & \partial(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G) \\ {\scriptstyle (h_I,\nu_I)}\big\downarrow & & \big\downarrow{\scriptstyle \partial[\alpha,\beta,\psi]} \\ (Q'_I,\phi'_I) & \xrightarrow[(f',\chi')]{} & \partial(F',((\begin{smallmatrix}\gamma'\\ \mu'\end{smallmatrix}),\theta')G') \end{array}$$

$(J\subseteq J'\subseteq I)$

commute.

A <u>lattice of $\pm$formations</u>, $\{(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G)\}_{J\subseteq I}$, is a finite collection of $\pm$formations $(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G)$, together with inclusions

$$(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G) \underset{\lambda_J^{J'}}{\subseteq} (F_{J'},((\begin{smallmatrix}\gamma_{J'}\\ \mu_{J'}\end{smallmatrix}),\theta)G) \qquad (J\subseteq J'\subseteq I)$$

such that

$$\lambda_J^{J'}\lambda_{J'}^{J''} = \lambda_J^{J''} \in \mathrm{Hom}_A(F_{J''},F_J)$$
$$\lambda_J^J = 1 \in \mathrm{Hom}_A(F_J,F_J) \qquad (J\subseteq J'\subseteq J''\subseteq I)$$

and such that for disjoint $J,K\subseteq I$ there are given orthogonality relations

$$(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G) \underset{\nu_J^K}{\subseteq} (F_K^*,((\begin{smallmatrix}\mu_K\\ \mp\gamma_K\end{smallmatrix}),\theta)G)$$

with

$$\nu_J^{K*} = \mp\nu_K^J \in \mathrm{Hom}_A(F_J^*,F_K)$$
$$\lambda_J^{J'}\nu_{J'}^K = \nu_J^K \in \mathrm{Hom}_A(F_K^*,F_J) \qquad (J\subseteq J'\subseteq I-K).$$

By convention,

$$(F_\emptyset,((\begin{smallmatrix}\gamma_\emptyset\\ \mu_\emptyset\end{smallmatrix}),\theta)G) = (G^*,((\begin{smallmatrix}\theta^*\mp\theta\\ 1\end{smallmatrix}),\theta)G)$$

with

$$\nu_J^\emptyset = \gamma_J : G \longrightarrow F_J\ ,\quad \lambda_\emptyset^J = \mu_J^* : F_J \longrightarrow G^*.$$

An <u>n-ad of ± formations</u>, $\{(G,\theta);\{(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G)\}_{J\subseteq I}\}$ is a ∓ form (G,θ) together with a lattice of ± formations $\{(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G)\}_{J\subseteq I}$ such that $(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G)$, $(F_{I-J},((\begin{smallmatrix}\gamma_{I-J}\\ \mu_{I-J}\end{smallmatrix}),\theta)G)$ are maximally orthogonal, that is

$$\nu_J^{I-J}\in \mathrm{Hom}_A(F^*_{I-J},F_J)$$

is an isomorphism, for all $J\subseteq I$. (In particular, it follows that

$$(\nu_I^{\emptyset},1,0):(F_I,((\begin{smallmatrix}\gamma_I\\ \mu_I\end{smallmatrix}),\theta)G)\longrightarrow\partial(G,\theta)$$

defines an isomorphism of ± formations).

For example, a 0-ad is (essentially) a ± formation, while a 1-ad is a cobordism of such.

An <u>isomorphism</u> of n-ads of ± formations

$\{(f,\chi),\{h_J\}\}$

$$:\{(G,\theta);\{(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G)\}_{J\subseteq I}\}\longrightarrow\{(G',\theta');\{(F'_J,((\begin{smallmatrix}\gamma'_J\\ \mu'_J\end{smallmatrix}),\theta')G'\}_{J\subseteq I}\}$$

is an isomorphism of ∓ forms

$$(f,\chi):(G,\theta)\longrightarrow(G',\theta')$$

together with isomorphisms $h_J\in\mathrm{Hom}_A(F_J,F'_J)$ $(J\subseteq I)$ such that

$$(h_J,f,0):(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G)\longrightarrow(F'_J,((\begin{smallmatrix}\gamma'_J\\ \mu'_J\end{smallmatrix}),\theta')G')$$

defines an isomorphism of ± formations for all $J\subseteq I$ and the squares

$$\begin{tikzcd} F_{J'} \arrow[r, "h_{J'}"] \arrow[d, "\lambda_J^{J'}"'] & F'_{J'} \arrow[d, "\lambda'^{J'}_J"] \\ F_J \arrow[r, "h_J"] & F'_J \end{tikzcd}
\qquad
\begin{tikzcd} F^*_K \arrow[r, "h_K^{*-1}"] \arrow[d, "\nu_J^K"'] & F'^*_K \arrow[d, "\nu'^K_J"] \\ F_J \arrow[r, "h_J"'] & F'_J \end{tikzcd}$$

commute, with

$$h_\emptyset = f^{*-1}: G^*\longrightarrow G'^*.$$

We shall now define <u>facette operations</u>, assigning to an n-ad $y=\{x,\{x_J\}_{J\subseteq I}\}$ of $\left\{\begin{matrix}\pm\text{forms}\\ \pm\text{formations}\end{matrix}\right.$ satisfying some extra conditions one m-ad of $\left\{\begin{matrix}\mp\text{formations}\\ \pm\text{forms}\end{matrix}\right.$

$$\partial^J y=\{x_J,\{x_{J,K}\}_{K\subseteq I-J}\}$$

for each subset J of I, where $m=|I-J|-1$.

Lemma 3.1 Given an n-ad of $\mp$forms

$$y = \{(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)\,;\ \{(Q_J,\phi_J)\}_{J\subseteq I}\}$$

there is defined a morphism of $\pm$forms

$$\left(\begin{pmatrix}\gamma_{J,K}\\ \mu_{J,K}\end{pmatrix},\phi_J\right) : (Q_J,0) \longrightarrow H_{\pm}(F_{J,K})$$

for disjoint $J,K\subseteq I$ by

$$\gamma_{J,K} : Q_J \longrightarrow F_{J,K} = (Q_{J\cup K}/f^{K}_{J\cup K}Q_K)^* \,;\ x \longmapsto ([y]\longmapsto \langle f^{J}_{J\cup K}x,y\rangle_{\phi^{*}_{J\cup K}})$$

$$\mu_{J,K} : Q_J \longrightarrow F^{*}_{J,K}\,;\ x \longmapsto [f^{J}_{J\cup K}x]\ ,$$

which is the inclusion of a sublagrangian iff

$$\begin{pmatrix} f^J_{J\cup K} & -f^K_{J\cup K} & 0\\ 0 & f^K_{K\cup L} & -f^L_{K\cup L}\\ -f^J_{L\cup J} & 0 & f^L_{L\cup J}\end{pmatrix} : Q_J\oplus Q_K\oplus Q_L \longrightarrow Q_{J\cup K}\oplus Q_{K\cup L}\oplus Q_{L\cup J}$$

is a split mono, where $L = I-(J\cup K)$.

If that is the case for some $J\subseteq I$, and all $K\subseteq I-J$, then there is defined an m-ad of $\pm$formations

$$\partial^J y = \{(Q_J,\phi_J)\,;\ \{(F_{J,K},((\begin{smallmatrix}\gamma_{J,K}\\ \mu_{J,K}\end{smallmatrix}),\phi_J)Q_J)\}_{K\subseteq I-J}\}$$

etc with

$$\lambda^{K'}_{J,K} : F_{J,K'} \longrightarrow F_{J,K}\,;\ x \longmapsto [f^{J\cup K}_{J\cup K'}]^* x \quad (K\subseteq K'\subseteq I-J)$$

$$\nu^{L}_{J,K} : F^{*}_{J,L} \longrightarrow F_{J,K}\,;\ [x]\longmapsto ([w]\longmapsto \langle f^{J\cup L}_{J\cup K\cup L}x, f^{J\cup K}_{J\cup K\cup L}w\rangle_{\phi^{*}_{J\cup K\cup L}})$$

$(J,K,L\subseteq I$ disjoint$)$

An isomorphism of $\mp$form n-ads

$$h : y \longrightarrow y'$$

induces an isomorphism of $\pm$formation m-ads

$$\partial^J h : \partial^J y \longrightarrow \partial^J y'$$

whenever $\partial^J y$ (and hence $\partial^J y'$) is defined.

Proof: The construction of

$$\left(\begin{pmatrix}\gamma_{J,K}\\ \mu_{J,K}\end{pmatrix},\phi_J\right) : (Q_J,0) \longrightarrow H_{\pm}(F_{J,K})$$

is exactly as in Theorem 2.3, and it was shown there that this defines an inclusion iff

$$\begin{pmatrix} (\phi^{*}_{J\cup K}\mp\phi_{J\cup K})f^{J}_{J\cup K} & 0\\ f^{J}_{J\cup K} & f^{K}_{J\cup K}\end{pmatrix} : Q_J\oplus Q_K \longrightarrow Q^{*}_{J\cup K}\oplus Q_{J\cup K}$$

is a split mono.
Now, setting $L = I-(J\cup K)$, we have that

$$\nu^{L}_{J\cup K,\emptyset} : Q_I/Q_L \longrightarrow Q^{*}_{J\cup K}\,;\ [x]\longmapsto (y\longmapsto \langle x, f^{I}_{J\cup K}y\rangle_{\phi^{*}_{I}})$$

is an A-module isomorphism (because, by definition,

$$(Q_{J\cup K},\phi_{J\cup K}) \subseteq (Q_I,\phi_I)\ ,\quad (Q_L,\phi_L)\subseteq (Q_I,\phi_I)$$

are maximally orthogonal), and we have a commutative triangle

$$\begin{tikzcd}[column sep=huge]
 & Q_I/Q_L \oplus Q_{J\cup K} \arrow[dd, "\begin{pmatrix} \nu^L_{J\cup K,\emptyset} & 0 \\ 0 & 1 \end{pmatrix}"] \\
Q_J \oplus Q_K \arrow[ur, "\begin{pmatrix} [f^J_I] & 0 \\ f^J_{J\cup K} & f^K_{J\cup K} \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} (\phi^*_{J\cup K} \pm \phi_{J\cup K}) f^J_{J\cup K} & 0 \\ f^J_{J\cup K} & f^K_{J\cup K} \end{pmatrix}"'] & \\
 & Q^*_{J\cup K} \oplus Q_{J\cup K}
\end{tikzcd}$$

with the vertical map an isomorphism. But

$$\begin{pmatrix} [f^J_I] & 0 \\ f^J_{J\cup K} & f^K_{J\cup K} \end{pmatrix} : Q_J \oplus Q_K \longrightarrow Q_I/Q_L \oplus Q_{J\cup K}$$

is a split mono iff

$$\begin{pmatrix} f^J_{J\cup K} & -f^K_{J\cup K} & 0 \\ -f^J_{L\cup J} & 0 & f^L_{L\cup J} \end{pmatrix} : Q_J \oplus Q_K \oplus Q_L \longrightarrow Q_{J\cup K} \oplus Q_{L\cup J}$$

is a split mono, and this is so iff

$$\begin{pmatrix} f^J_{J\cup K} & -f^K_{J\cup K} & 0 \\ 0 & f^K_{K\cup L} & -f^L_{K\cup L} \\ -f^J_{L\cup J} & 0 & f^L_{L\cup J} \end{pmatrix} : Q_J \oplus Q_K \oplus Q_L \longrightarrow Q_{J\cup K} \oplus Q_{K\cup L} \oplus Q_{L\cup J}$$

is a split mono.

If the condition is satisfied for some $J \subseteq I$ and all $K \subseteq I - J$, then

$$\partial^J y = \{ (Q_J, \phi_J) \,;\, \{ (F_{J,K}, ((\begin{smallmatrix} \gamma_{J,K} \\ \mu_{J,K} \end{smallmatrix}), \phi_J) Q_J) \}_{K \subseteq I-J} \}$$

defines a lattice of $\pm$formations, provided only that

$$\nu^L_{J,K} : F^*_{J,L} \longrightarrow F_{J,K}$$

is an A-module isomorphism for complementary $J, K, L \subseteq I$. But this is clear from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Q_L \arrow[r, "f^L_{J\cup L}"] \arrow[d, "\nu^{\emptyset}_{L,J\cup K}"'] & Q_{J\cup L} \arrow[r] \arrow[d, "\nu^{\emptyset}_{J\cup L,K}"] & Q_{J\cup L}/f^L_{J\cup L} Q_L \arrow[r] \arrow[d, "\nu^L_{J,K}"] & 0 \\
0 \arrow[r] & (Q_I/f^{J\cup K}_I Q_{J\cup K})^* \arrow[r] & (Q_I/f^K_I Q_K)^* \arrow[r, "{[f^{J\cup K}_I]^*}"'] & (Q_{J\cup K}/f^K_I Q_K)^* \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact and the other two vertical maps are isomorphisms.

An isomorphism of $\mp$form n-ads

$$h = \{ [\alpha, \beta, \psi], \{ (h_J, \nu_J) \} \}$$

$$: y = \{ (F, ((\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}), \theta) G) \,;\, \{ (Q_J, \phi_J) \}_{J \subseteq I} \}$$

$$\longrightarrow y' = \{ (F', ((\begin{smallmatrix} \gamma' \\ \mu' \end{smallmatrix}), \theta') G') \,;\, \{ (Q'_J, \phi'_J) \}_{J \subseteq I} \}$$

that $\partial^J y$, $\partial^J y'$ are defined, for some $J \subseteq I$, induces isomorphisms of $\pm$formation n-ads

$$\partial^J h = \{(h_J,\nu_J),\{f_{J,K}\}\}$$

$$: \partial^J y = \{(Q_J,\phi_J);\{(F_{J,K},((\begin{smallmatrix}\gamma_{J,K}\\ \mu_{J,K}\end{smallmatrix}),\phi_J)Q_J)\}_{K\subseteq I-J}\}$$

$$\longrightarrow \partial^J y' = \{(Q'_J,\phi'_J);\{(F'_{J,K},((\begin{smallmatrix}\gamma'_{J,K}\\ \mu'_{J,K}\end{smallmatrix}),\phi'_J)Q'_J)\}_{K\subseteq I-J}\}$$

with $f_{J,K}\in \mathrm{Hom}_A(F_{J,K},F'_{J,K})$ the A-module isomorphism dual to

$$Q'_{J\cup K}/f'^{K}_{J\cup K}Q'_K \longrightarrow Q_{J\cup K}/f^{K}_{J\cup K}Q_K \ ;\ [x]\longmapsto [h^{-1}_{J\cup K}x]$$

□

<u>Lemma 3.2</u> Given an n-ad of $\pm$ formations

$$y = \{(G,\theta);\{(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G)\}_{J\subseteq I}\}$$

such that the morphisms

$$(\lambda_J^{J\cup K*}\ \ -\lambda_K^{J\cup K*}) : F_J^*\oplus F_K^* \longrightarrow F^*_{J\cup K}$$

are onto for all $K\subseteq I-J$, for some $J\subseteq I$, there is defined an m-ad of $\pm$forms

$$\partial^J y = \{(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G);\{(Q_{J,K},\phi_{J,K})\}_{K\subseteq I-J}\}$$

by

$$Q_{J,K} = \ker((\lambda_J^{J\cup K*}\ -\lambda_K^{J\cup K*}) : F_J^*\oplus F_K^* \longrightarrow F^*_{J\cup K}) \Big/ \mathrm{im}((\begin{smallmatrix}\mu_J\\ \mu_K\end{smallmatrix}) : G\longrightarrow F_J^*\oplus F_K^*)$$

with $\phi_{J,K}\in \mathrm{Hom}_A(Q_{J,K},Q^*_{J,K})$ induced by

$$\begin{pmatrix}0 & 0\\ \nu^J_K & 0\end{pmatrix} \in \mathrm{Hom}_A(F_J^*\oplus F_K^*, F_J\oplus F_K).$$

An isomorphism of $\pm$formation n-ads

$$h : y \longrightarrow y'$$

induces an isomorphism of $\pm$form m-ads

$$\partial^J h : \partial^J y \longrightarrow \partial^J y'$$

whenever $\partial^J y$, $\partial^J y'$ are defined.

<u>Proof</u>: Choosing a direct complement to $\mathrm{im}((\begin{smallmatrix}\mu_J\\ \mu_K\end{smallmatrix}) : G\longrightarrow F_J^*\oplus F_K^*)$ in $\ker((\lambda_J^{J\cup K*}\ -\lambda_K^{J\cup K*}) : F_J^*\oplus F_K^*\to F^*_{J\cup K})$ we have an exact sequence

$$0\longrightarrow Q_{J,K}\oplus G \xrightarrow{\begin{pmatrix}g_J^K & \mu_J\\ g_K^J & \mu_K\end{pmatrix}} F_J^*\oplus F_K^* \xrightarrow{(\lambda_J^{J\cup K*}\ -\lambda_K^{J\cup K*})} F^*_{J\cup K}\longrightarrow 0$$

for each $K\subseteq I-J$, and we set

$$\phi_{J,K} = g_K^{J*}\,\nu^J_K\, g_J^K \in \mathrm{Hom}_A(Q_{J,K},Q^*_{J,K}).$$

For $K\subseteq K'\subseteq I-J$ let $f^K_{K'}\in \mathrm{Hom}_A(Q_{J,K},Q_{J,K'})$

be the unique morphism making the diagram

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & Q_{J,K}\oplus G \arrow[r, "\begin{pmatrix} g_J^K & \mu_J \\ g_K^J & \mu_K \end{pmatrix}"] \arrow[d, "\begin{pmatrix} f_{K'}^K & 0 \\ 0 & 1 \end{pmatrix}"'] & F_J^*\oplus F_K^* \arrow[r, "\begin{pmatrix} \lambda_J^{J\cup K*} & -\lambda_K^{J\cup K*} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & \lambda_K^{K'*} \end{pmatrix}"'] & F_{J\cup K}^* \arrow[r] \arrow[d, "\lambda_{J\cup K}^{J\cup K'*}"] & 0 \\
0 \arrow[r] & Q_{J,K'}\oplus G \arrow[r, "\begin{pmatrix} g_J^{K'} & \mu_J \\ g_{K'}^J & \mu_{K'} \end{pmatrix}"'] & F_J^*\oplus F_{K'}^* \arrow[r, "\begin{pmatrix} \lambda_J^{J\cup K'*} & -\lambda_{K'}^{J\cup K'*} \end{pmatrix}"'] & F_{J\cup K'}^* \arrow[r] & 0
\end{tikzcd}$$

commute, so that

$$f_{K''}^{K'} f_{K'}^{K} = f_{K''}^{K} \in \mathrm{Hom}_A(Q_{J,K}, Q_{J,K''})$$

for $K \subseteq K' \subseteq K'' \subseteq I-J$.

The isomorphism of short exact sequences

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & Q_{J,I-J}\oplus G \arrow[r, "\begin{pmatrix} g_J^{I-J} & \mu_J \\ g_{I-J}^J & \mu_{I-J} \end{pmatrix}"] \arrow[d, "1"'] & F_J^*\oplus F_{I-J}^* \arrow[r, "\begin{pmatrix} \lambda_J^{I*} & -\lambda_{I-J}^{I*} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & \nu_J^{I-J} \\ 1 & 0 \end{pmatrix}"'] & F_I^* \arrow[r] \arrow[d, "-\gamma_I^*"] & 0 \\
0 \arrow[r] & Q_{J,I-J}\oplus G \arrow[r, "\begin{pmatrix} \nu_J^{I-J} g_{I-J}^{J} & \gamma_J \\ g_J^{I-J} & \mu_J \end{pmatrix}"'] & F_J\oplus F_J^* \arrow[r, "\begin{pmatrix} \pm\mu_J^* & \gamma_J \end{pmatrix}"'] & G^* \arrow[r] & 0
\end{tikzcd}$$

defines an isomorphism of $\pm$ forms

$$(f, 0) : (Q_{J,I-J}, \phi_{J,I-J}) \longrightarrow \partial\left(F_J, \left(\begin{pmatrix}\gamma_J \\ \mu_J\end{pmatrix}, \theta\right) G\right).$$

For each $K \subseteq I-J$ we have a situation almost like that in Theorem 1.3, with a commutative hexagon

$$\begin{tikzcd}
 & F_{J\cup K} \arrow[dl, "\lambda_J^{J\cup K}"'] \arrow[dr, "\lambda_K^{J\cup K}"] & \\
F_J & & F_K \\
 & G \arrow[uu, "\gamma_{J\cup K}"] \arrow[ul, "\gamma_J"'] \arrow[ur, "\gamma_K"] \arrow[dl, "\mu_K"] \arrow[dr, "\mu_J"'] \arrow[dd, "\mu_{J\cup K}"] & \\
F_K^* \arrow[uu, "\nu_J^K"] \arrow[dr, "\lambda_K^{J\cup K*}"'] & & F_J^* \arrow[uu, "\nu_K^J"'] \arrow[dl, "\lambda_J^{J\cup K*}"] \\
 & F_{J\cup K}^* &
\end{tikzcd}$$

and exact sequences

$$0 \longrightarrow Q_{J,L}\oplus G \xrightarrow{\begin{pmatrix} \nu_{J\cup K}^L g_L^J & \gamma_{J\cup K} \\ g_J^L & \mu_J \end{pmatrix}} F_{J\cup K}\oplus F_J^* \xrightarrow{\begin{pmatrix} \lambda_K^{J\cup K} & -\nu_K^J \end{pmatrix}} F_K \longrightarrow 0$$

$$0 \longrightarrow Q_{J,K}\oplus G \xrightarrow{\begin{pmatrix} g_J^K & \mu_J \\ g_K^J & \mu_K \end{pmatrix}} F_J^*\oplus F_K^* \xrightarrow{\begin{pmatrix} \lambda_J^{J\cup K*} & -\lambda_K^{J\cup K*} \end{pmatrix}} F_{J\cup K}^* \longrightarrow 0,$$

where $L = I-(J\cup K)$. The relevant part of the proof of Theorem 1.3 now shows that

$$(f_K^{I-J}, 0) : (Q_{J,K}, \phi_{J,K}) \longrightarrow (Q_{J,I-J}, \phi_{J,I-J}),\quad (f_L^{I-J}, 0) : (Q_{J,L}, \phi_{J,L}) \longrightarrow (Q_{J,I-J}, \phi_{J,I-J})$$

are the inclusions of maximally orthogonal subforms. In particular, it follows that each $f_{K'}^K \in \mathrm{Hom}_A(Q_{J,K}, Q_{J,K'})$ is a split mono.

Thus

$$\partial^J y = \{ (F_J, ((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}), \theta)G) \,;\, \{(Q_{J,K}, \phi_{J,K})\}_{K \subseteq I-J} \}$$

is indeed an m-ad of $\pm$ forms.

An isomorphism of $\pm$ formation n-ads

$$h = \{ (f, \chi), \{h_J\} \}$$

$$: y = \{ (G,\theta) \,;\, \{ (F_J, ((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G) \}_{J \subseteq I} \}$$

$$\longrightarrow y' = \{ (G',\theta') \,;\, \{ (F'_J, ((\begin{smallmatrix}\gamma'_J\\ \mu'_J\end{smallmatrix}),\theta')G') \}_{J \subseteq I} \}$$

induces an isomorphism of $\pm$ form n-ads

$$\partial^J h = \{ [h_J, f, 0] \,;\, \{(f_{J,K}, 0)\} \}$$

$$: \partial^J y = \{ (F_J, ((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G) \,;\, \{(Q_{J,K}, \phi_{J,K})\}_{K \subseteq I-J} \}$$

$$\longrightarrow \partial^J y' = \{ (F'_J, ((\begin{smallmatrix}\gamma'_J\\ \mu'_J\end{smallmatrix}),\theta')G') \,;\, \{(Q'_{J,K}, \phi'_{J,K})\}_{K \subseteq I-J} \}$$

whenever $\partial^J y$, $\partial^J y'$ are defined, with $f_{J,K} \in \mathrm{Hom}_A(Q_{J,K}, Q'_{J,K})$ the A-module isomorphism induced by

$$\partial [h_J, f, 0] : \partial(F_J, ((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G) \longrightarrow \partial(F'_J, ((\begin{smallmatrix}\gamma'_J\\ \mu'_J\end{smallmatrix}),\theta')G').$$

$\square$

Given a $\begin{cases} \pm \text{ form} \\ \pm \text{ formation} \end{cases}$ $x = \begin{cases} (Q,\phi) \\ (F, ((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G) \end{cases}$

let $-x = \begin{cases} (Q,-\phi) \\ (F, ((\begin{smallmatrix}-\gamma\\ \mu\end{smallmatrix}),-\theta)G) \end{cases}$, and given $\begin{cases} \text{an isomorphism} \\ \text{a stable isomorphism} \end{cases}$

$f = \begin{cases} (g,\chi) \\ [\alpha,\beta,\psi] \end{cases} : x \longrightarrow x'$ let $-f = \begin{cases} (g,-\chi) \\ [\alpha,\beta,-\psi] \end{cases} : -x \longrightarrow -x'$.

More generally, given an n-ad of $\begin{cases} \pm \text{ forms} \\ \pm \text{ formations} \end{cases}$

$$y = \{ x \,;\, \{x_J\} \}$$

let

$$-y = \{ -x \,;\, \{-x_J\} \}$$

be the n-ad defined by the same $\begin{cases} f^J_{J'}, \\ \lambda^J_{J'} \end{cases}$,

but with $\begin{cases} -\chi^J_{J'}, \\ -\nu^J_K \end{cases}$ replacing $\begin{cases} \chi^J_{J'}, \\ \nu^J_K \end{cases}$.

Further, given an isomorphism of n-ads of $\begin{cases} \pm \text{ forms} \\ \pm \text{ formations} \end{cases}$

$$f = \begin{cases} \{ [\alpha,\beta,\psi] \,;\, \{(g_J,\chi_J)\}_{J \subseteq I} \} \\ \{ (g,\chi) \,;\, \{g_J\}_{J \subseteq I} \} \end{cases} : y \longrightarrow y'$$

there is defined an isomorphism

$$-f = \begin{cases} \{ [\alpha,\beta,-\psi] \,;\, \{(g_J,-\chi_J)\}_{J \subseteq I} \} \\ \{ (g,-\chi) \,;\, \{g_J\} \} \end{cases} : -y \longrightarrow -y'.$$

The join construction defined on cobordisms in §§1,2 will be generalized to n-ads: given

an n-ad $y = \{x; \{x_J\}_{J \subseteq I}\}$

an n′-ad $y' = \{x'; \{x'_{J'}\}_{J' \subseteq I'}\}$

an isomorphism of n″-ads

$$f : -\partial^J y \longrightarrow \partial^{J'} y'$$

for some $J \subseteq I$, $J' \subseteq I'$ such that $I - J = I' - J' = I''$, there will be defined an n″-ad

$$y *_f y' = \{x''; \{x''_{J''}\}_{J'' \subseteq I''}\}.$$

uniquely up to isomorphism.

Call two n-ad isomorphisms

$$f_1, f_2 : \{z; \{z_J\}_{J \subseteq I}\} \longrightarrow \{z'; \{z'_J\}_{J \subseteq I}\}$$

<u>coherent</u> if their definitions coincide in all respects but one: in dealing with n-ads of ±forms, the stable isomorphisms of ±formations $: z \longrightarrow z'$ need only be coherent (in the sense defined in §2). Coherence classes of isomorphisms are the morphisms of a category with n-ads as objects.

<u>Theorem 3.3</u> Given n (resp n′)-ad isomorphisms

$$g : y \longrightarrow \hat{y} \quad g' : y' \longrightarrow \hat{y}'$$

and n″-ad isomorphisms

$$f : -\partial^J y \longrightarrow \partial^{J'} y' \quad \hat{f} : -\partial^J \hat{y} \longrightarrow \partial^{J'} \hat{y}'$$

such that

$$\begin{tikzcd}
-\partial^J y \arrow[r, "f"] \arrow[d, "-\partial^J g"'] & \partial^{J'} y' \arrow[d, "\partial^{J'} g'"] \\
-\partial^J \hat{y} \arrow[r, "\hat{f}"] & \partial^{J'} \hat{y}'
\end{tikzcd}$$

commutes up to coherence, there is defined an isomorphism of n″-ads

$$g * g' : y *_f y' \longrightarrow \hat{y} *_{\hat{f}} \hat{y}'.$$

□

This will be proved together with

Theorem 3.4 Let $y = \{x; \{x_J\}_{J\subseteq I}\}$ be an n-ad such that $\partial^J y, \partial^K y, \partial^J\partial^K y$ are defined for some disjoint $J, K \subseteq I$. Then $\partial^K\partial^J y, \partial^{J\cup K} y$ are defined, and there are defined isomorphisms

$$\tau^{J,K} : -\partial^K\partial^J y \longrightarrow \partial^J\partial^K y$$

$$\sigma^{J,K} : \partial^J y *_{\tau^{J,K}} \partial^K y \longrightarrow \partial^{J\cup K} y$$

of m-ads, with $(\tau^{J,K})^{-1} = -\tau^{K,J} : \partial^J\partial^K y \longrightarrow -\partial^K\partial^J y$.

$\square$

We define first the join of n-ads of $\pm$ forms. Let then

$$y = \{(F, ((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}), \theta)G) ; \{(Q_J, \phi_J)\}_{J\subseteq I}\}$$

$$y' = \{(F', ((\begin{smallmatrix}\gamma'\\ \mu'\end{smallmatrix}), \theta')G') ; \{(Q'_{J'}, \phi'_{J'})\}_{J'\subseteq I'}\}$$

be n (resp n')-ads of $\pm$ forms, and let

$$f = \{(f, \chi) ; \{g_K\}\}$$

$$: -\partial^J y = \{(Q_J, -\phi_J) ; \{(F_{J,K}, ((\begin{smallmatrix}-\gamma_{J,K}\\ \mu_{J,K}\end{smallmatrix}), -\phi_J)Q_J)\}_{K\subseteq I-J=I''}\}$$

$$\longrightarrow \partial^{J'} y' = \{(Q'_{J'}, \phi'_{J'}) ; \{(F'_{J',K}, ((\begin{smallmatrix}\gamma'_{J',K}\\ \mu'_{J',K}\end{smallmatrix}), \phi'_{J'})Q'_{J'})\}_{K\subseteq I'-J'=I''}\}$$

be an isomorphism of n''-ads of $\mp$ formations, for so[me]

Let

$$y *_f y' = \{(F'', ((\begin{smallmatrix}\gamma''\\ \mu''\end{smallmatrix}), \theta'')G'') ; \{(Q''_K, \phi''_K)\}_{K\subseteq I''}\}$$

be the n''-ad of $\pm$ forms defined by

$$(F'', ((\begin{smallmatrix}\gamma''\\ \mu''\end{smallmatrix}), \theta'')G'')$$

$$= \left(F\oplus F', \left(\left(\begin{smallmatrix}\begin{pmatrix}\gamma & j & 0\\ 0 & j'f & \gamma'\end{pmatrix}\\ \begin{pmatrix}\mu & k & 0\\ 0 & k'f & \mu'\end{pmatrix}\end{smallmatrix}\right), \begin{pmatrix}\theta & \gamma^* k & 0\\ 0 & \lambda + \chi_1 + f^*\chi'_1 f & f^* j'^* \mu'\\ 0 & 0 & \theta'\end{pmatrix}\right) G\oplus Q_J \oplus G'\right)$$

where

$$((\begin{smallmatrix}j\\ k\end{smallmatrix}), \chi_1) : (Q_J, \phi_J) \longrightarrow H_\pm(F)$$

$$((\begin{smallmatrix}j'\\ k'\end{smallmatrix}), \chi'_1) : (Q'_{J'}, \phi'_{J'}) \longrightarrow H_\pm(F')$$

are the inclusions defined by (f_I^J, χ_I^J) etc.,

$$(Q''_K, \phi''_K)$$

$$= \left(\ker((e_K \ -g_K^* e'_K) : Q_{J\cup K}\oplus Q'_{J'\cup K} \longrightarrow F^*_{J,K}) \Big/ \mathrm{im}\left(\begin{pmatrix} f^J_{J\cup K}\\ f'^{J'}_{J'\cup K} f\end{pmatrix} : Q_J \longrightarrow Q_{J\cup K}\oplus Q'_{J'\cup K}\right), [\phi_{J\cup K}\oplus\phi'_{J'\cup K}]\right)$$

where

$$e_K : Q_{J\cup K} \longrightarrow F^*_{J,K} = Q_{J\cup K}/Q_K \ ; x \longmapsto [x]$$

$$e'_K : Q'_{J'\cup K} \longrightarrow F'^*_{J',K} \quad (K\subseteq I'')$$

are the natural projections,

$$(f''^K_{K'}, \chi''^K_{K'}) = \left(\left[\begin{pmatrix} f^{J\cup K}_{J\cup K'} & 0\\ 0 & f'^{J'\cup K}_{J'\cup K'}\end{pmatrix}\right], \left[\begin{pmatrix}\chi^{J\cup K}_{J\cup K'} & 0\\ 0 & \chi'^{J'\cup K}_{J'\cup K'}\end{pmatrix}\right]\right)$$

Diagram chasing as in the proof of Theorem 1.3 verifies that $y *_f y'$ is indeed an n''-ad of $\pm$ forms.

In proving Theorem 3.3 for n-ads of $\pm$ forms, it is sufficient to consider only those isomorphisms

$$g : y \longrightarrow \hat{y}\ ,\ g' : y' \longrightarrow \hat{y}'$$

defined using actual (rather than stable) isomorphisms of $\pm$ formations: for if $(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G)$, $(F',((\begin{smallmatrix}\gamma'\\ \mu'\end{smallmatrix}),\theta')G')$ are stabilized by trivial $\pm$ formations, so is $(F'',((\begin{smallmatrix}\gamma''\\ \mu''\end{smallmatrix}),\theta'')G'')$, in the same way. Now for such g, g' the definition of an isomorphism

$$g * g' : y *_f y' \longrightarrow \hat{y} *_{\hat{f}} \hat{y}'$$

is obvious, proving Theorem 3.3 in this case.

We next prove Theorem 3.4 for n-ads of $\pm$ formations.

Applying Lemmas 3.2, 3.1 in turn, we have that $\partial^K \partial^J y$ is defined for some n-ad of $\pm$ formations

$$y = \{(G,\theta) ;\ \{(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G)\}_{J\subset I}\}$$

and disjoint $J,K \subseteq I$ if $\partial^J y$ is defined and if

$$\begin{pmatrix} f^K_{J,K\cup L} & -f^L_{J,K\cup L} & 0 \\ 0 & f^L_{J,L\cup M} & -f^M_{J,L\cup M} \\ -f^K_{J,M\cup K} & 0 & f^M_{J,M\cup K}\end{pmatrix} : Q_{J,K} \oplus Q_{J,L} \oplus Q_{J,M}$$

$$\longrightarrow Q_{J,K\cup L} \oplus Q_{J,L\cup M} \oplus Q_{J,M\cup K}$$

is a split mono for all disjoint $L,M \subseteq I-(J\cup K)$, which corresponds to having

$$\left[\begin{pmatrix} \lambda^{J\cup K*}_J & -\lambda^{J\cup K*}_K & 0 & 0 \\ \lambda^{J\cup L*}_J & 0 & -\lambda^{J\cup L*}_L & 0 \\ \lambda^{J\cup M*}_J & 0 & 0 & -\lambda^{J\cup M*}_M \\ 0 & -\lambda^{K\cup L*}_K & \lambda^{K\cup L*}_L & 0 \\ 0 & 0 & -\lambda^{L\cup M*}_L & \lambda^{L\cup M*}_M \\ 0 & \lambda^{M\cup K*}_K & 0 & -\lambda^{M\cup K*}_M \end{pmatrix}\right]$$

$$: F^*_J \oplus F^*_K \oplus F^*_L \oplus F^*_M \Big/ \mathrm{im}\left(\begin{pmatrix}\mu_J\\ \mu_K\\ \mu_L\\ \mu_M\end{pmatrix} : G \longrightarrow F^*_J \oplus F^*_K \oplus F^*_L \oplus F^*_M\right)$$

$$\longrightarrow F^*_{J\cup K} \oplus F^*_{J\cup L} \oplus F^*_{J\cup M} \oplus F^*_{K\cup L} \oplus F^*_{L\cup M} \oplus F^*_{M\cup K}$$

split mono.

The symmetry of this condition makes apparent that $\partial^K\partial^J y$ is defined whenever $\partial^J y$, $\partial^J\partial^K y$ are defined. If that is the case then

$$\partial^K\partial^J y = \{(Q_{J,K},\phi_{J,K})\,;\,\{(F_{J,K,L},(\begin{pmatrix}\gamma_{J,K,L}\\ \mu_{J,K,L}\end{pmatrix},\phi_{J,K})Q_{J,K}\}_{L\subseteq I-(J\cup K)}\}$$

is an m-ad of $\mp$ formations which we can express more directly in terms of y, as follows:

we have a commutative diagram (for each $L\subseteq I-(J\cup K)$)

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & Q_{J,L}\oplus G \arrow[r, "\begin{pmatrix} g_J^L & \mu_J\\ g_L^J & \mu_L\end{pmatrix}"] \arrow[d, "\begin{pmatrix} f^L_{J,K\cup L} & 0\\ 0 & 1\end{pmatrix}"'] & F_J^*\oplus F_L^* \arrow[r, "(\lambda_J^{J\cup L*}\ -\lambda_L^{J\cup L*})"] \arrow[d, "\begin{pmatrix} 1 & 0\\ 0 & \lambda_L^{K\cup L*}\end{pmatrix}"'] & F^*_{J\cup L} \arrow[r] \arrow[d, "\lambda^{J\cup K\cup L*}_{J\cup L}"] & 0\\
0 \arrow[r] & Q_{J,K\cup L}\oplus G \arrow[r, "\begin{pmatrix} g_J^{K\cup L} & \mu_J\\ g^J_{K\cup L} & \mu_{K\cup L}\end{pmatrix}"'] \arrow[d, "\begin{pmatrix} e_L & 0\\ 0 & \mu_L\end{pmatrix}"'] & F_J^*\oplus F^*_{K\cup L} \arrow[r, "(\lambda_J^{J\cup K\cup L*}\ -\lambda_{K\cup L}^{J\cup K\cup L*})"] \arrow[d, "\begin{pmatrix} \lambda_J^{J\cup L*} & 0\\ 0 & 1\end{pmatrix}"'] & F^*_{J\cup K\cup L} \arrow[r] \arrow[d, "1"] & 0\\
0 \arrow[r] & F^*_{J,K,L}\oplus F_L^* \arrow[r, "\begin{pmatrix} ? & \lambda_L^{J\cup L*}\\ ? & \lambda_L^{K\cup L*}\end{pmatrix}"'] & F^*_{J\cup L}\oplus F^*_{K\cup L} \arrow[r, "(\lambda_{J\cup L}^{J\cup K\cup L*}\ -\lambda_{K\cup L}^{J\cup K\cup L*})"'] & F^*_{J\cup K\cup L} \arrow[r] & 0
\end{tikzcd}$$

($e_L : Q_{J,K\cup L}\to F^*_{J,K,L}$ is the natural projection)

with the top two rows exact, whence the exactness of the bottom row (by diagram chasing).

We can therefore identify

$$F^*_{J,K,L} = \frac{\ker\big((\lambda_{J\cup L}^{J\cup K\cup L*}\ -\lambda_{K\cup L}^{J\cup K\cup L*}) : F^*_{J\cup L}\oplus F^*_{K\cup L}\longrightarrow F^*_{J\cup K\cup L}\big)}{\mathrm{im}\Big(\begin{pmatrix}\lambda_L^{J\cup L*}\\ \lambda_L^{K\cup L*}\end{pmatrix} : F_L^*\longrightarrow F^*_{J\cup L}\oplus F^*_{K\cup L}\Big)}$$

$$\gamma_{J,K,L} : Q_{J,K}\xrightarrow{\begin{pmatrix} g_J^K\\ g_K^J\end{pmatrix}} F_J^*\oplus F_K^* \xrightarrow{\begin{pmatrix} 0 & \nu^K_{J\cup L}\\ -\nu^J_{K\cup L} & 0\end{pmatrix}} F_{J\cup L}\oplus F_{K\cup L}\xrightarrow{(?^*\ ?^*)} F_{J,K,L}$$

$$\mu_{J,K,L} : Q_{J,K}\longrightarrow F^*_{J,K,L}\ ;\ x\longmapsto [\lambda_J^{J\cup L*}g_J^K x, \lambda_K^{K\cup L*}g_K^J x].$$

It is clear that we can identify

$$Q_{J,K} = Q_{K,J}\quad F_{J,K,L} = F_{K,J,L}\quad \gamma_{J,K,L} = -\gamma_{K,J,L}\quad \mu_{J,K,L} = \mu_{K,J,L},$$

and as

$$\begin{aligned}\phi_{J,K} &= g_K^{J*}\nu^J_K g_J^K\\ &= \mp(g_J^{K*}\nu_J^K g_K^J)^* = \mp\phi^*_{K,J}\in \mathrm{Hom}_A(Q_{J,K},Q^*_{J,K})\end{aligned}$$

there is defined an isomorphism of $\pm$ forms

$$(1,\phi_{K,J}) : (Q_{K,J},-\phi_{K,J})\longrightarrow (Q_{J,K},\phi_{J,K})$$

and hence an isomorphism

$$\tau^{J,K} : -\partial^K\partial^J y\longrightarrow \partial^J\partial^K y$$

of n-ads of $\mp$ formations, with

$$(\tau^{J,K})^{-1} = -\tau^{K,J} : \partial^J\partial^K y\longrightarrow -\partial^K\partial^J y.$$

The stable isomorphism of $\pm$ forms

$[\alpha,\beta,\psi]$:

$$\left(F_J\oplus F_K, \left(\begin{pmatrix}\begin{pmatrix}\gamma_J & \nu_J^K g_J^K & 0\\ 0 & \nu_K^J g_K^J & \gamma_K\end{pmatrix}\\ \begin{pmatrix}\mu_J & g_J^K & 0\\ 0 & g_K^J & \mu_K\end{pmatrix}\end{pmatrix}, \begin{pmatrix}\theta & \gamma_J^* g_J^K & 0\\ 0 & \phi_{J,K}^* & \mp g_J^{K*}\gamma_J\\ 0 & 0 & \theta\end{pmatrix}\right) G\oplus Q_{J,K}\oplus G\right)$$

$$\longrightarrow \left(F_{J\cup K}, \left(\begin{pmatrix}\gamma_{J\cup K}\\ \mu_{J\cup K}\end{pmatrix}, \theta\right) G\right)$$

defined in the proof of Theorem 1.3 now gives an isomorphism

$$\sigma^{J,K} : \partial^J y *_{\tau^{J,K}} \partial^K y \longrightarrow \partial^{J\cup K} y$$

of m-ads of $\pm$ forms, an exact sequence

$$0 \longrightarrow Q_{J\cup K,L}\oplus Q_{J,K} \xrightarrow{\begin{pmatrix} ? & f^K_{J,K\cup L}\\ ? & f^J_{K,J\cup L}\end{pmatrix}} Q_{J,K\cup L}\oplus Q_{J\cup L,K} \longrightarrow F^*_{J,K,L} \longrightarrow 0$$

being readily established.

This proves Theorem 3.4 for n-ads of $\pm$ formations.

We next define the join of n-ads of $\pm$ formations

Let then

$$y = \{(G,\theta) ; \{(F_J, ((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}), \theta)G)\}_{J\subseteq I}\}$$

$$y' = \{(G',\theta') ; \{(F'_{J'}, ((\begin{smallmatrix}\gamma'_{J'}\\ \mu'_{J'}\end{smallmatrix}), \theta')G')\}_{J'\subseteq I'}\}$$

be n (resp. n') - ads of $\pm$ formations, and let

$$f = \{[\alpha,\beta,\psi] ; \{(f_K,\chi_K)\}\}$$

$$: -\partial^J y = \{(F_J, ((\begin{smallmatrix}-\gamma_J\\ \mu_J\end{smallmatrix}), -\theta)G) ; \{(Q_{J,K}, -\phi_{J,K})\}_{K\subseteq I-J=I''}\}$$

$$\longrightarrow \partial^{J'} y' = \{(F'_{J'}, ((\begin{smallmatrix}\gamma'_{J'}\\ \mu'_{J'}\end{smallmatrix}), \theta')G') ; \{(Q'_{J',K}, \phi'_{J',K})\}_{K\subseteq I'-J'=I''}\}$$

be an isomorphism.

Using the standard notation of the proof of Theorem 2.1 to describe $[\alpha,\beta,\psi]$, we have a commutative diagram

$$\begin{array}{ccc}
Q_{J,K}\oplus G\oplus P^* & \xrightarrow{\begin{pmatrix} f_K & 0\\ 0 & \begin{pmatrix} b & b_1\\ ? & ?\end{pmatrix}\end{pmatrix}} & Q'_{J',K}\oplus G'\oplus P'^*\\
\begin{pmatrix} -\nu_J^K g_K^J & -\gamma_J & 0\\ 0 & 0 & 0\\ g_J^K & \mu_J & 0\\ 0 & 0 & 1\end{pmatrix}\Big\downarrow & & \Big\downarrow\begin{pmatrix} \nu'^K_{J'} g'^{J'}_K & \gamma'_{J'} & 0\\ 0 & 0 & 0\\ g'^K_{J'} & \mu'_{J'} & 0\\ 0 & 0 & 1\end{pmatrix}\\
(F_J\oplus P)\oplus(F_J^*\oplus P^*) & \xrightarrow[\begin{pmatrix}\begin{pmatrix} a & a_1\\ ? & ?\end{pmatrix} & \begin{pmatrix} n & ?\\ ? & ?\end{pmatrix}\\ 0 & \begin{pmatrix} a'^* & ?\\ a_1'^* & ?\end{pmatrix}\end{pmatrix}]{} & (F'_{J'}\oplus P')\oplus(F'^*_{J'}\oplus P'^*)
\end{array}$$

$(K\subseteq I')$

(As in the proof of Theorem 1.3) choose a left inverse

$$\begin{pmatrix} l_K \\ m_K \end{pmatrix} : F^*_{J\cup K} \longrightarrow F^*_J \oplus F^*_K$$

to

$$\begin{pmatrix} \lambda_J^{J\cup K *} & -\lambda_K^{J\cup K *} \end{pmatrix} : F^*_J \oplus F^*_K \longrightarrow F^*_{J\cup K}$$

for each $K \subseteq I''$, and let

$$\begin{pmatrix} n_K & p_K \\ q_K & r_K \end{pmatrix} : F^*_J \oplus F^*_K \longrightarrow Q_{J,K} \oplus G$$

be the corresponding right inverse to

$$\begin{pmatrix} g_J^K & \mu_J \\ g_K^J & \mu_K \end{pmatrix} : Q_{J,K} \oplus G \longrightarrow F^*_J \oplus F^*_K,$$

so that

$$(Q_{J,K} \oplus G) \oplus F^*_{J\cup K} \underset{\begin{pmatrix} \begin{pmatrix} n_K & p_K \\ q_K & r_K \end{pmatrix} \\ \begin{pmatrix} \lambda_J^{J\cup K*} & -\lambda_K^{J\cup K*} \end{pmatrix} \end{pmatrix}}{\overset{\left(\begin{pmatrix} g_J^K & \mu_J \\ g_K^J & \mu_K \end{pmatrix} \begin{pmatrix} l_K \\ m_K \end{pmatrix}\right)}{\rightleftarrows}} F^*_J \oplus F^*_K$$

are inverse isomorphisms. (By convention, $l_\emptyset = 1$, $m_\emptyset = 0$, $q_\emptyset = 0$, $r_\emptyset = 1$.) It will not be true, in general, that the diagram

$$\begin{array}{ccc} F^*_{J\cup K} & \xrightarrow{\begin{pmatrix} l_K \\ m_K \end{pmatrix}} & F^*_J \oplus F^*_K \\ \downarrow \lambda_{J\cup K}^{J\cup K' *} & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & \lambda_K^{K'*} \end{pmatrix} \\ F^*_{J\cup K'} & \xrightarrow[\begin{pmatrix} l_{K'} \\ m_{K'} \end{pmatrix}]{} & F^*_J \oplus F^*_{K'} \end{array} \qquad (K \subseteq K' \subseteq I'')$$

commutes. However, the commutator lies in

$$\ker\left(\begin{pmatrix} \lambda_J^{J\cup K' *} & -\lambda_{K'}^{J\cup K' *} \end{pmatrix} : F^*_J \oplus F^*_{K'} \longrightarrow F^*_{J\cup K'}\right)$$

$$= \operatorname{im}\left(\begin{pmatrix} g_J^{K'} & \mu_J \\ g_{K'}^J & \mu_{K'} \end{pmatrix} : Q_{J,K'} \oplus G \longrightarrow F^*_J \oplus F^*_{K'}\right)$$

so that

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda_K^{K'*} \end{pmatrix}\begin{pmatrix} l_K \\ m_K \end{pmatrix} - \begin{pmatrix} l_{K'} \\ m_{K'} \end{pmatrix} \lambda_{J\cup K}^{J\cup K' *}$$

$$= \begin{pmatrix} g_J^{K'} & \mu_J \\ g_{K'}^J & \mu_{K'} \end{pmatrix}\begin{pmatrix} t_{K'}^K \\ u_{K'}^K \end{pmatrix} \in \operatorname{Hom}_A(F^*_{J\cup K}, F^*_J \oplus F^*_{K'})$$

for some uniquely determined

$$\begin{pmatrix} t_{K'}^K \\ u_{K'}^K \end{pmatrix} \in \operatorname{Hom}_A(F^*_{J\cup K}, Q_{J,K'} \oplus G).$$

Similarly, choose left inverses

$$\begin{pmatrix} l'_K \\ m'_K \end{pmatrix} : F'^{*}_{J'\cup K} \longrightarrow F'^{*}_{J'} \oplus F'^{*}_{K}$$

to

$$\begin{pmatrix} \lambda'^{J'\cup K\,*}_{J'} & -\lambda'^{J'\cup K\,*}_{K} \end{pmatrix} : F'^{*}_{J'} \oplus F'^{*}_{K} \longrightarrow F'^{*}_{J'\cup K} ,$$

let

$$\begin{pmatrix} n'_K & p'_K \\ q'_K & r'_K \end{pmatrix} : F'^{*}_{J'} \oplus F'^{*}_{K} \longrightarrow Q'_{J';K} \oplus G'$$

be the corresponding right inverses to

$$\begin{pmatrix} g'^{K}_{J'} & \mu'_{J'} \\ g'^{J'}_{K} & \mu'_{K} \end{pmatrix} : Q'_{J';K} \oplus G' \longrightarrow F'^{*}_{J'} \oplus F'^{*}_{K} ,$$

and let

$$\begin{pmatrix} t'^{K}_{K'} \\ u'^{K}_{K'} \end{pmatrix} : F'^{*}_{J'\cup K} \longrightarrow Q'_{J';K} \oplus G'$$

be such that

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{K'*}_{K} \end{pmatrix}\begin{pmatrix} l'_K \\ m'_K \end{pmatrix} - \begin{pmatrix} l'_{K'} \\ m'_{K'} \end{pmatrix} \lambda'^{J'\cup K'\,*}_{J'\cup K}$$

$$= \begin{pmatrix} g'^{K}_{J'} & \mu'_{J'} \\ g'^{J'}_{K'} & \mu'_{K'} \end{pmatrix}\begin{pmatrix} t'^{K}_{K'} \\ u'^{K}_{K'} \end{pmatrix} \in \mathrm{Hom}_A(F'^{*}_{J'\cup K}, F'^{*}_{J'} \oplus F'^{*}_{K'}) .$$

Setting

$$(G'';\theta'') = \left(G \oplus F'^{*}_{J'}, \begin{pmatrix} \theta & 0 \\ \mp a\gamma_J & s' \end{pmatrix}\right) \qquad F''_K = F_K \oplus F'_{J'\cup K} \quad (K \subseteq I'')$$

$$\gamma''_K = \begin{pmatrix} \gamma_K & \nu^J_K a^* \\ l'^{*}_K a \gamma_J & l'^{*}_K(s'^{*} \mp s') - m'^{*}_K \nu'^{J'}_K \end{pmatrix} : G \oplus F'^{*}_{J'} \longrightarrow F_K \oplus F'_{J'\cup K}$$

$$\mu''_K = \begin{pmatrix} \mu_K & \mu_K b' q'_K + g^J_K f'_K n'_K \\ 0 & \lambda'^{J'\cup K\,*}_{J'} \end{pmatrix} : G \oplus F'^{*}_{J'} \longrightarrow F^{*}_K \oplus F'^{*}_{J'\cup K}$$

where $f'_K = f_K^{-1} : Q'_{J';K} \longrightarrow Q_{J,K}$

$$\nu''^{K}_{L} = \begin{pmatrix} \nu^K_L & \nu^J_L a^* l'_K \\ l'^{*}_L a \nu^K_J & l'^{*}_L(s'^{*} \mp s') l'_K - (l'^{*}_L \nu'^{K}_{J'} m'_K + m'^{*}_L \nu'^{J'}_L l'_K) - m'^{*}_L \nu'^{K}_L m'_K \end{pmatrix}$$

$$: F^{*}_K \oplus F'^{*}_{J'\cup K} \longrightarrow F_L \oplus F'_{J'\cup L} \quad \left(\begin{matrix} K, L \subseteq I'' \\ \text{disjoint} \end{matrix}\right)$$

$$\lambda''^{K'}_{K} = \begin{pmatrix} \lambda^{K'}_K & 0 \\ u'^{K*}_{K'} b'^{*} \mu^{*}_{K'} - t'^{K*}_{K'} f'^{*}_{K'} g^{J*}_{K'} & \lambda'^{J'\cup K'}_{J'\cup K} \end{pmatrix}$$

$$: F_{K'} \oplus F'_{J'\cup K'} \longrightarrow F_K \oplus F'_{J'\cup K} \quad (K \subseteq K' \subseteq I'')$$

it is claimed that

$$y *_f y' = \left\{ (G'',\theta'') ; \left\{ \left(F''_K, \left(\begin{pmatrix} \gamma''_K \\ \mu''_K \end{pmatrix}, \theta''\right) G''\right) \right\}_{K \subseteq I''} \right\}$$

is an n''-ad of $\pm$ formations, determined uniquely up to isomorphism by u, u', f.

To see that each

$$\left(\begin{pmatrix}\gamma''_K\\ \mu''_K\end{pmatrix},\theta''\right) : (G'',0) \longrightarrow H_{\pm}(F''_K) \qquad (K \subseteq I'')$$

is the inclusion of a sublagrangian, note that

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ -p'^*_K f'^*_K g^{J*}_K & \lambda'^{J\cup K}_K & 1 \\ l'^*_K n'^*_K f'^*_K g^{J*}_K & l'^*_K \lambda'^{J\cup K}_{J'} & m'^*_K \end{pmatrix}, \begin{pmatrix} 1 & b' q'_K & 0 \\ 0 & f'_K n'_K & f'_K p'_K \\ 0 & q'_K & r'_K \\ 0 & \lambda'^{J\cup K*}_{J'} & -\lambda'^{J\cup K*}_K \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0 & \pm\nu^J_K a^* l'_K & \mp \nu^J_K g^K_J f'_K p'_K \\ 0 & l'^*_K(-s' l'_K \mp \nu'^K_{J'} m'_K) & \pm m'^*_K \nu'^{J'}_K l'_K \lambda'^{J\cup K*}_K \\ 0 & 0 & r'^*_K \theta' r'_K + (r'^*_K \gamma'^*_{J'} + p'^*_K g'^{K*}_{J'} s') g'^K_{J'} p'_K \end{pmatrix}\right)$$

$$: \left(F''_K, \left(\begin{pmatrix}\gamma''_K\\ \mu''_K\end{pmatrix},\theta''\right) G''\right) \oplus (F'_K, F'^*_K)$$

$$\longrightarrow \left(F_K \oplus F'_K, \left(\begin{pmatrix} \begin{pmatrix} \gamma_K & \nu^J_K g^K_J & 0 \\ 0 & \nu'^{J'}_K g'^K_{J'} f_K & \gamma'_K \end{pmatrix} \\ \begin{pmatrix} \mu_K & g^J_K & 0 \\ 0 & g'^{J'}_K f_K & \mu'_K \end{pmatrix} \end{pmatrix}, \begin{pmatrix} \theta & \gamma^*_K g^J_K & 0 \\ 0 & -\chi_K & \mp f^*_K g'^{K*}_{J'} \gamma'_{J'} \\ 0 & 0 & \theta' \end{pmatrix}\right) G \oplus Q_{J,K} \oplus G'\right)$$

defines a stable isomorphism of $\pm$formations (with that on the right being of the kind used to define the join of cobordisms of $\pm$forms in §1)

The inverse isomorphisms

$$\delta = \begin{pmatrix} \mu_J & a^* \\ -b & b_1 a_1^* \end{pmatrix}$$

$$G \oplus F'^*_{J'} \rightleftarrows F^*_J \oplus G'$$

$$\delta' = \begin{pmatrix} b'_1 a'^*_1 & -b' \\ a'^* & \mu'_{J'} \end{pmatrix}$$

considered in the proof of Theorem 2.1 may be generalized, to define inverse isomorphisms

$$\delta_K = \begin{pmatrix} \lambda^{J\cup K*}_K & \lambda^{J\cup K*}_J a^* l'_K \\ -(\mu'_K b r_K + g'^{J'}_K f_K p_K) & \mu'_K b_1 a_1^* l'_K + (\mu'_K b q_K + g'^{J'}_K f_K n_K) a^* l'_K - m'_K \end{pmatrix}$$

$$F^*_K \oplus F'^*_{J'\cup K} \rightleftarrows F^*_{J\cup K} \oplus F'^*_K$$

$$\delta'_K = \begin{pmatrix} \mu_K b'_1 a'^*_1 l_K + (\mu_K b' q'_K + g^J_K f'_K n'_K) a'^* l_K - m_K & -(\mu_K b' r'_K + g^J_K f'_K p'_K) \\ \lambda'^{J'\cup K*}_{J'} a'^* l_K & \lambda'^{J'\cup K*}_K \end{pmatrix}$$

for all $K \subseteq I''$, with $\delta_\emptyset = \delta$, $\delta'_\emptyset = \delta'$.

The factorization

$$\begin{tikzcd}
F_K^* \oplus F'^*_{J\cup K} \arrow[rr, "\nu'^K_L"] \arrow[dr, "\delta_K"'] & & F_L \oplus F'_{J\cup L} \\
& F^*_{J\cup K} \oplus F'^*_K \arrow[ur, "\begin{pmatrix} \nu^{J\cup K}_L & 0 \\ 0 & \nu'^K_{J\cup L} \end{pmatrix}"'] &
\end{tikzcd}$$

now shows that $\nu''^K_L \in \mathrm{Hom}_A(F''_K, F''^*_L)$ is an isomorphism whenever K, L are complementary in I''.

Therefore

$$y'' = \{(G'', \theta''); \{(F''_K, (\begin{pmatrix} \gamma''_K \\ \mu''_K \end{pmatrix}, \theta'')G'')\}_{K\subseteq I''}\}$$

does define an n''-ad of $\pm$ formations.

So far, we have made use of $\begin{pmatrix} l_K \\ m_K \end{pmatrix}, \begin{pmatrix} n_K & p_K \\ q_K & r_K \end{pmatrix}$ only in the definition of δ'_K, and not at all in that of y''. It is clear that we could equally well have defined an n''-ad

$$y''' = \{(G''', \theta'''); \{(F'''_K, (\begin{pmatrix} \gamma'''_K \\ \mu'''_K \end{pmatrix}, \theta''')G''')\}_{K\subseteq I''}\}$$

with

$$(G''', \theta''') = (F_J^* \oplus G', \begin{pmatrix} s & \mp a'\gamma'_{J'} \\ 0 & \theta' \end{pmatrix}) \quad F'''_K = F_{J\cup K} \oplus F'_K \text{ etc.}$$

using $\begin{pmatrix} l_K \\ m_K \end{pmatrix}, \begin{pmatrix} n_K & p_K \\ q_K & r_K \end{pmatrix}$ instead of $\begin{pmatrix} l'_K \\ m'_K \end{pmatrix}, \begin{pmatrix} n'_K & p'_K \\ q'_K & r'_K \end{pmatrix}$.

Now

$$y'' \underset{\{(\delta', \xi'); \{\delta^*_K\}\}}{\overset{\{(\delta, \xi); \{\delta'^*_K\}\}}{\rightleftarrows}} y'''$$

are inverse isomorphisms of n''-ads of $\pm$ formations, where

$$\xi = \begin{pmatrix} \mu_J^* s \mu_J - \theta & \mu_J^* s a^* \\ 0 & 0 \end{pmatrix} \in \Pi_{\pm}(G \oplus F'^*_{J'})$$

$$\xi' = \begin{pmatrix} 0 & 0 \\ \mu'^*_{J'} s' a^* & \mu'^*_{J'} s' \mu'_{J'} - \theta' \end{pmatrix} \in \Pi_{\pm}(F_J^* \oplus G').$$

Hence different choices of $\begin{pmatrix} l_K \\ m_K \end{pmatrix}$ etc. lead to isomorphic expressions for $y \ast_{\mp} y'$, and we are justified in calling it *the* join.

It is readily verified that this join operation is natural in the sense of Theorem 3.3.

We are now in a position to complete the proof of Theorem 3.4.

Applying Lemmas 3.1, 3.2 in turn we have that $\partial^K \partial^J y$ is defined for some n-ad of $\pm$forms

$$y = \{ (F, ((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}), \theta) G) \, ; \, \{(Q_J, \phi_J)\}_{J \subseteq I} \}$$

and disjoint $J, K \subseteq I$ if $\partial^J y$ is defined and if

$$(\lambda^{K \cup L *}_{J,K} \;\; -\lambda^{K \cup L *}_{J,L}) : F^*_{J,K} \oplus F^*_{J,L} \longrightarrow F^*_{J,K \cup L}$$

is onto for all $L \subseteq I - (J \cup K)$, which corresponds to having

$$(f^{J \cup K}_{J \cup K \cup L} \;\; f^{K \cup L}_{J \cup K \cup L} \;\; f^{L \cup J}_{J \cup K \cup L}) : Q_{J \cup K} \oplus Q_{K \cup L} \oplus Q_{L \cup J} \longrightarrow Q_{J \cup K \cup L}$$

onto. The symmetry of this condition shows that $\partial^K \partial^J y$ is defined whenever $\partial^J y$, $\partial^J \partial^K y$ are defined. If that is the case,

$$\partial^K \partial^J y = \{ (F_{J,K}, ((\begin{smallmatrix}\gamma_{J,K}\\ \mu_{J,K}\end{smallmatrix}), \phi_J) Q_J) \, ; \, \{(Q_{J,K,L}, \phi_{J,K,L})\}_{L \subseteq I - (J \cup K)} \}$$

is an m-ad of $\mp$ forms, and we can identify

$$(Q_{J,K,L}, \phi_{J,K,L})$$

$$= \left(\frac{\ker\left((f^{J \cup K}_{J \cup K \cup L} \;\; f^{K \cup L}_{J \cup K \cup L} \;\; f^{L \cup J}_{J \cup K \cup L}) : Q_{J \cup K} \oplus Q_{K \cup L} \oplus Q_{L \cup J} \longrightarrow Q_{J \cup K \cup L} \right)}{\operatorname{im}\left(\begin{pmatrix} f^J_{J \cup K} & -f^K_{J \cup K} & 0 \\ 0 & f^K_{K \cup L} & -f^L_{K \cup L} \\ -f^J_{L \cup J} & 0 & f^L_{L \cup J} \end{pmatrix} : Q_J \oplus Q_K \oplus Q_L \to Q_{J \cup K} \oplus Q_{K \cup L} \oplus Q_{L \cup J} \right)}, \right.$$

$$\left. \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ f^{L \cup J *}_{J \cup K \cup L} (\phi^*_{J \cup K \cup L} \pm \phi_{J \cup K \cup L}) f^{J \cup K}_{J \cup K \cup L} & 0 & 0 \end{pmatrix} \right] \right)$$

$$(f^L_{J,K,L'}, \chi^L_{J,K,L'})$$

$$= \left(\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & f^{K \cup L}_{K \cup L'} & 0 \\ 0 & 0 & f^{L \cup J}_{L' \cup J} \end{pmatrix} \right], 0 \right) : (Q_{J,K,L}, \phi_{J,K,L}) \longrightarrow (Q_{J,K,L'}, \phi_{J,K,L'})$$

$$(L \subseteq L' \subseteq I - (J \cup K)).$$

Letting

$$F^*_{J,K} \oplus Q_K \underset{\left(\begin{smallmatrix} e_K \\ g_K \end{smallmatrix}\right)}{\overset{(h_K \;\; f_K)}{\rightleftarrows}} Q_{J \cup K}, \quad F^*_{K,J} \oplus Q_J \underset{\left(\begin{smallmatrix} e_J \\ g_J \end{smallmatrix}\right)}{\overset{(h_J \;\; f_J)}{\rightleftarrows}} Q_{J \cup K}$$

be pairs of inverse isomorphisms, with e_J, e_K the natural projections (as before) and $f_J = f^J_{J \cup K}$, $f_K = f^J_{J \cup K}$ we have that

$$\gamma_{J,K} = h^*_K (\phi^*_{J \cup K} \pm \phi_{J \cup K}) f_J : Q_J \longrightarrow F_{J,K}$$

$$\gamma_{K,J} = h^*_J (\phi^*_{J \cup K} \pm \phi_{J \cup K}) f_K : Q_K \longrightarrow F_{K,J}$$

$$\mu_{J,K} = e_K f_J : Q_J \longrightarrow F^*_{J,K}$$

$$\mu_{K,J} = e_J f_K : Q_K \longrightarrow F^*_{K,J}.$$

We are now in the situation of Theorem 2.3 and the stable isomorphism of $\mp$formations

$$[\alpha, \beta, \psi] : (F_{J,K}, ((\begin{smallmatrix} -\gamma_{J,K} \\ \mu_{J,K} \end{smallmatrix}), -\phi_J) Q_J) \longrightarrow ($$ [illegible]

defined there, by

$$(\alpha,\beta,\psi) =$$

$$\left(\begin{pmatrix} h_J^* e_K^* & h_J^* g_K^* \\ f_J^* e_K^* & f_J^* g_K^* \end{pmatrix}, \begin{pmatrix} -g_K f_J & 1 \\ 1-g_J f_K g_K f_J & g_J f_K \end{pmatrix}, \begin{pmatrix} h_K^* \phi_{J\cup K} h_K & h_K^* \phi_{J\cup K} f_K \\ f_K^* \phi_{J\cup K} h_K & f_K^* \phi_{J\cup K} f_K \end{pmatrix}\right)$$

$$: \left(F_{J,K}, \left(\begin{pmatrix} -\gamma_{J,K} \\ \mu_{J,K} \end{pmatrix}, -\phi_J\right) Q_J\right) \oplus (Q_K^*, Q_K)$$

$$\longrightarrow \left(F_{K,J}, \left(\begin{pmatrix} \gamma_{K,J} \\ \mu_{K,J} \end{pmatrix}, \phi_K\right) Q_K\right) \oplus (Q_J^*, Q_J)$$

is such that

$$\partial[\alpha,\beta,\psi] : \partial\left(F_{J,K}, \left(\begin{pmatrix} -\gamma_{J,K} \\ \mu_{J,K} \end{pmatrix}, -\phi_J\right) Q_J\right)$$

$$\longrightarrow \partial\left(F_{K,J}, \left(\begin{pmatrix} \gamma_{K,J} \\ \mu_{K,J} \end{pmatrix}, \phi_K\right) Q_K\right)$$

restricts to isomorphisms

$$(t_L, \chi_L) : (Q_{J,K,L}, -\phi_{J,K,L}) \longrightarrow (Q_{K,J,L}, \phi_{K,J,L})$$

for each $L \subseteq I-(J\cup K)$.

We have therefore defined an isomorphism

$$\tau^{J,K} : -\partial^K \partial^J y \longrightarrow \partial^J \partial^K y$$

Note: in the applications we shall use $\tau^{J,K}$ to denote the isomorphism given by one particular choice of direct complements.

of m-ads of $\mp$ forms. The identity

$$(\tau^{J,K})^{-1} = -\tau^{K,J} : \partial^J \partial^K y \longrightarrow -\partial^K \partial^J y \quad \text{is ch}$$

Substituting in the formulae above, we have that

$$\partial^J y *_{\tau^{J,K}} \partial^K y = \left\{(Q,\Phi) ; \left\{\left(F_L, \left(\begin{pmatrix} \gamma_L \\ \mu_L \end{pmatrix}, \Phi\right) Q\right)\right\}_{L \subseteq I-(J\cup K)}\right\}$$

is the m-ad of $\mp$ formations defined by

$$(Q,\Phi) = \left(Q_J \oplus F_{K,J}^*, \begin{pmatrix} \phi_J & 0 \\ h_J^* e_K^* \gamma_{J,K} & h_J^* \phi_{J\cup K} h_J \end{pmatrix}\right)$$

$$F_L = F_{J,L} \oplus F_{K,J\cup L} \quad (L \subseteq I-(J\cup K))$$

$$\gamma_L = \begin{pmatrix} \gamma_{J,L} & \nu_{J,L}^K e_K h_J \\ l_L^* h_J^* e_K^* & l_L^*(\phi_J^* \pm \phi_J) - m_L^* \nu_{K,L}^J \end{pmatrix} : Q_J \oplus F_{K,J}^* \longrightarrow F_{J,L} \oplus F_{K,J\cup L}$$

$$\mu_L = \begin{pmatrix} \mu_{J,L} & -\mu_{J,L} g_J f_K q_L + g_{J,L}^K f_L n_L \\ 0 & \lambda_{K,J}^{J\cup L *} \end{pmatrix} : Q_J \oplus F_{K,J}^* \longrightarrow F_{J,L}^* \oplus F_{K,J\cup L}^*$$

$$\lambda_L^{L'} = \begin{pmatrix} \lambda_{J,L}^{L'} & 0 \\ -(g_{J,L'}^K f_{L'} t_{L'}^L + \mu_{J,L'} g_J f_K u_{L'}^L)^* & \lambda_{K,J\cup L}^{J\cup L'} \end{pmatrix} : F_{J,L'} \oplus F_{K,J\cup L'} \longrightarrow F_{J,L} \oplus F_{K,J\cup L}$$

$$\nu_M^L = \begin{pmatrix} \nu_{J,M}^L & \nu_{J,M}^K e_K h_J l_L \\ l_M^* h_J^* e_K^* \nu_{J,K}^L & l_M^*(\phi_J^* \pm \phi_J) l_L - (l_M^* \nu_{K,J}^L m_L + m_M^* \nu_{K,L}^J l_L) - m_M^* \nu_{K,M}^L m_L \end{pmatrix}$$

$$: F_{J,L}^* \oplus F_{K,J\cup L}^* \longrightarrow F_{J,M} \oplus F_{K,J\cup M}$$

for some pair of inverse isomorphisms

$$(Q_{J,K,L}\oplus Q_J)\oplus F^*_{J,K\cup L} \underset{\begin{pmatrix}\begin{pmatrix} n_L & p_L \\ q_L & r_L\end{pmatrix} \\ \begin{pmatrix}\lambda^{K\cup L *}_{J,K} & -\lambda^{K\cup L *}_{J,L}\end{pmatrix}\end{pmatrix}}{\overset{\left(\begin{pmatrix} g^L_{J,K} & \mu_{J,K} \\ g^K_{J,L} & \mu_{J,L}\end{pmatrix}\begin{pmatrix} \iota_L \\ m_L\end{pmatrix}\right)}{\rightleftarrows}} F^*_{J,K}\oplus F^*_{J,L}$$

with $\begin{pmatrix} t^L_{L'} \\ u^L_{L'}\end{pmatrix} \in \mathrm{Hom}_A(F^*_{K,J\cup L}, Q_{J,K,L}\oplus Q_J)$

such that

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{L'*}_{J,L}\end{pmatrix}\begin{pmatrix} \iota_L \\ m_L\end{pmatrix} - \begin{pmatrix} \iota_{L'} \\ m_{L'}\end{pmatrix}\lambda^{K\cup L' *}_{J,K\cup L}$$

$$= \begin{pmatrix} g^L_{J,K} & \mu_{J,K} \\ g^K_{J,L} & \mu_{J,L}\end{pmatrix}\begin{pmatrix} t^L_{L'} \\ u^L_{L'}\end{pmatrix} \in \mathrm{Hom}_A(F^*_{J,K\cup L}, F^*_{J,K}\oplus F^*_{J,L})$$

for all $L\subseteq L'\subseteq I-(J\cup K)$.

Now

$$\left((f_J \;\; h_J), \begin{pmatrix} \chi^J_{J\cup K} & f^*_J\phi_{J\cup K}h_J \\ 0 & 0\end{pmatrix}\right)$$

$$: \left(Q_J\oplus F^*_{K,J}, \begin{pmatrix} \phi_J & 0 \\ \pm h^*_J e^*_K \gamma_{J,K} & h^*_J\phi_{J\cup K}h_J\end{pmatrix}\right)$$

$$\longrightarrow (Q_{J\cup K}, \phi_{J\cup K})$$

defines an isomorphism of $\pm$ forms

$$: (Q,\phi) \longrightarrow (Q_{J\cup K}, \phi_{J\cup K}).$$

Further, dualizing

$$0 \longrightarrow Q_{J\cup L}/Q_L \xrightarrow{[f^{J\cup L}_{J\cup K\cup L}]} Q_{J\cup K\cup L}/Q_L \xrightarrow{[1]} Q_{J\cup K\cup L}/Q_{L\cup J} \longrightarrow 0$$

we have a short exact sequence

$$0 \longrightarrow F_{K,J\cup L} \longrightarrow F_{J\cup K,L} \longrightarrow F_{J,L} \longrightarrow 0$$

and hence isomorphisms

$$F_{J\cup K,L} \longrightarrow F_{J,L}\oplus F_{K,J\cup L} = F_L \quad (L\subseteq I-(J\cup K)).$$

An appropriate choice of such isomorphisms, together with the isomorphism of $\pm$ forms defined above, gives an isomorphism

$$\sigma^{J,K} : \partial^J y *_{\tau^{J,K}} \partial^K y = \{(Q,\phi); \{(F_L, ((\begin{smallmatrix}\gamma_L \\ \mu_L\end{smallmatrix}),\phi)Q\}_{L\subseteq I-(J\cup K)}\}$$

$$\longrightarrow \partial^{J\cup K} y = \{(Q_{J\cup K},\phi_{J\cup K}); \{(F_{J\cup K,L}, ((\begin{smallmatrix}\gamma_{J\cup K,L} \\ \mu_{J\cup K,L}\end{smallmatrix}),\phi_{J\cup K})Q_{J\cup K}\}_{L\subseteq I-(J\cup K)}\}$$

of m-ads of $\mp$ formations.

This completes the proofs of Theorems 3.3, 3.4.

The join operation commutes with the facette operations, in the following sense:

<u>Theorem 3.5</u> Let

$$y = \{x; \{x_J\}_{J\subseteq I}\}\ ,\ y' = \{x'; \{x'_{J'}\}_{J'\subseteq I'}\}$$

be n (resp n') - ads, related by an isomorphism

$$f : -\partial^J y \longrightarrow \partial^{J'} y'$$

of n''-ads, for some $J \subseteq I$, $J' \subseteq I'$ such that $I - J = I' - J' = I''$.

If $K \subseteq I''$ is such that $\partial^K y$, $\partial^K\partial^J y$ and $\partial^K y'$, $\partial^K\partial^{J'} y'$ are defined then $\partial^K(y *_f y')$ is defined, and there is defined an isomorphism of m-ads

$$\rho^K : \partial^K(y *_f y') \longrightarrow \partial^K y *_{f_1} \partial^K y'$$

for any isomorphism $f_1 : -\partial^J\partial^K y \longrightarrow \partial^{J'}\partial^K y'$ in the coherence class of the composite

$$: -\partial^J\partial^K y \xrightarrow{\tau^{K,J}} \partial^K\partial^J y \xrightarrow{-\partial^K f} -\partial^K\partial^{J'} y' \xrightarrow{\tau^{J',K}} \partial^{J'}\partial^K y'.$$

<u>Proof</u>: i) Let

$$y = \{(F, ((\gamma_\mu), \theta) G)\ ;\ \{(Q_J, \phi_J)\}_{J\subseteq I}\}$$

$$, y' = \{(F', ((\gamma'_{\mu'}), \theta') G')\ ;\ \{(Q'_{J'}, \phi'_{J'})\}_{J'\subseteq I'}\}$$

be n (resp n') - ads of $\pm$forms, and let

$$f = \{(f, \chi)\ , \{d_K\}\}$$

$$: -\partial^J y = \{(Q_J, -\phi_J)\ ;\ \{(F_{J,K}, ((\begin{smallmatrix}-\gamma_{J,K}\\ \mu_{J,K}\end{smallmatrix}), -\phi_J) Q_J)\}_{K\subseteq I''}\}$$

$$\longrightarrow \partial^{J'} y' = \{(Q'_{J'}, \phi'_{J'})\ ;\ \{(F'_{J',K}, ((\begin{smallmatrix}\gamma'_{J',K}\\ \mu'_{J',K}\end{smallmatrix}), \phi'_{J'}) Q'_{J'})\}_{K\subseteq I''}\}$$

be an isomorphism of n''-ads of $\mp$formations.

Let

$$F^*_{J,K} \oplus Q_K \underset{\binom{e_K}{g_K}}{\overset{(h_K\ f_K)}{\rightleftarrows}} Q_{J\cup K} \qquad F^*_{K,J} \oplus Q_J \underset{\binom{e_J}{g_J}}{\overset{(h_J\ f_J)}{\rightleftarrows}} Q_{J\cup K}$$

$$F'^*_{J',K} \oplus Q'_K \underset{\binom{e'_K}{g'_K}}{\overset{(h'_K\ f'_K)}{\rightleftarrows}} Q'_{J'\cup K} \qquad F'^*_{K,J'} \oplus Q'_{J'} \underset{\binom{e'_{J'}}{g'_{J'}}}{\overset{(h'_{J'}\ f'_{J'})}{\rightleftarrows}} Q'_{J'\cup K}$$

be pairs of inverse isomorphisms, with $f_K = f^K_{J\cup K}$ etc., as before.

The composite

$$f_1 : -\partial^J\partial^K y \xrightarrow{\tau^{K,J}} \partial^K\partial^J y \xrightarrow{-\partial^K f} -\partial^K\partial^{J'} y' \xrightarrow{\tau^{J',K}} \partial^{J'}\partial^K y'$$

is defined by

$$\{[\alpha,\beta,\psi]\,;\,\{(c_L,\eta_L)\}\}$$

$$: \{(F_{K,J},(\begin{pmatrix}-\gamma_{K,J}\\ \mu_{K,J}\end{pmatrix},-\phi_K)Q_K)\,;\,\{(Q_{K,J,L},-\phi_{K,J,L})\}_{L\subseteq I^n-K}\}$$

$$\longrightarrow \{(F'_{K,J'},(\begin{pmatrix}\gamma'_{K,J'}\\ \mu'_{K,J'}\end{pmatrix},\phi'_K)Q'_K)\,;\,\{(Q'_{K,J';L},\phi'_{K,J';L})\}_{L\subseteq I^n-K}\}$$

where (α,β,ψ) is the composite

$$: (F_{K,J},(\begin{pmatrix}-\gamma_{K,J}\\ \mu_{K,J}\end{pmatrix},-\phi_K)Q_K)\oplus(Q_J^*\oplus Q_K'^*,Q_J\oplus Q'_K)$$

$$\left(\begin{pmatrix} h_K^*e_J^* & h_K^*g_J^* & 0\\ f_K^*e_J^* & f_K^*g_J^* & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix} -g_Jf_K & 1 & 0\\ 1-g_Kf_Jg_Jf_K & g_Kf_J & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix} h_J^*\phi_{J\cup K}h_J & h_J^*\phi_{J\cup K}f_J & 0\\ f_J^*\phi_{J\cup K}h_J & f_J^*\phi_{J\cup K}f_J & 0\\ 0 & 0 & 0\end{pmatrix}\right) \Big\downarrow$$

$$: (F_{J,K},(\begin{pmatrix}\gamma_{J,K}\\ \mu_{J,K}\end{pmatrix},\phi_J)Q_J)\oplus(Q_K^*\oplus Q_K'^*,Q_K\oplus Q'_K)$$

$$\left(\begin{pmatrix} d_K & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix} f & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},0\right) \Big\downarrow$$

$$: (F'_{J',K},(\begin{pmatrix}-\gamma'_{J',K}\\ \mu'_{J',K}\end{pmatrix},-\phi'_{J'})Q'_{J'})\oplus(Q_K^*\oplus Q_K'^*,Q_K\oplus Q'_K)$$

$$\left(\begin{pmatrix} h_{J'}'^*e_K'^* & 0 & h_{J'}'^*g_K'^*\\ 0 & 1 & 0\\ f_{J'}'^*e_K'^* & 0 & f_{J'}'^*g_K'^*\end{pmatrix},\begin{pmatrix} -g'_Kf'_{J'} & 0 & 1\\ 0 & 1 & 0\\ 1-g'_{J'}f'_Kg'_Kf'_{J'} & 0 & g'_{J'}f'_K\end{pmatrix},\begin{pmatrix} h_K'^*\phi'_{J'\cup K}h'_K & 0 & h_K'^*\phi'_{J'\cup K}f'_K\\ 0 & 0 & -0\\ f_K'^*\phi'_{J'\cup K}h'_K & 0 & f_K'^*\phi'_{J'\cup K}f'_K\end{pmatrix}\right) \Big\downarrow$$

$$: (F'_{K,J'},(\begin{pmatrix}\gamma'_{K,J'}\\ \mu'_{K,J'}\end{pmatrix},\phi'_K)Q'_K)\oplus(Q_K^*\oplus Q_{J'}'^*,Q_K\oplus Q'_{J'})$$

and (c_L,η_L) the composite

$$(Q_{K,J,L},-\phi_{K,J,L}) \xrightarrow{(t_L,\chi_L)} (Q_{J,K,L},\phi_{J,K,L}) \xrightarrow{(u_L,0)} (Q'_{J';K,L},-\phi'_{K,J';L}) \xrightarrow{(t'_L,\chi'_L)} (Q'_{K,J';L},\phi'_{K,J';L})$$

where (t_L,χ_L), (t'_L,χ'_L) are as defined in the proof of Theorem 3.4, and $u_L \in \mathrm{Hom}_A(Q_{J,K,L},Q'_{J';K,L})$ is the unique isomorphism making the diagram

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & Q_{J,K,L}\oplus Q_J \arrow[r, "\begin{pmatrix} g^L_{J,K} & \mu_{J,K}\\ g^K_{J,L} & \mu_{J,L}\end{pmatrix}"] \arrow[d, "\begin{pmatrix} u_L & 0\\ 0 & f\end{pmatrix}"'] & F^*_{J,K}\oplus F^*_{J,L} \arrow[r, "\begin{pmatrix}\lambda^{K\cup L*}_{J,K} & -\lambda^{K\cup L*}_{J,L}\end{pmatrix}"] \arrow[d, "\begin{pmatrix} d_K^{-1*} & 0\\ 0 & d_L^{-1*}\end{pmatrix}"'] & F^*_{J,K\cup L} \arrow[r] \arrow[d, "d^{-1*}_{K\cup L}"'] & {} \\
0 \arrow[r] & Q'_{J';K,L}\oplus Q'_{J'} \arrow[r, "\begin{pmatrix} g'^L_{J';K} & \mu'_{J';K}\\ g'^K_{J';L} & \mu'_{J';L}\end{pmatrix}"'] & F'^*_{J';K}\oplus F'^*_{J';L} \arrow[r, "\begin{pmatrix}\lambda'^{K\cup L*}_{J';K} & -\lambda'^{K\cup L*}_{J';L}\end{pmatrix}"'] & F'^*_{J';K\cup L} \arrow[r] & {}
\end{tikzcd}$$

commute.

Constructing the n''-ad of $\mp$ formations

$$\partial^K y *_{f_1} \partial^K y' = \{(Q,\phi) ; \{(F_L, ((\begin{smallmatrix}\gamma_L \\ \mu_L\end{smallmatrix}), \phi)Q)\}_{L \subseteq I''-K}\}$$

we have that

$$(Q,\phi) = \left(Q_K \oplus F'^{*}_{K,J'}, \begin{pmatrix} \phi_K & 0 \\ h'^{*}_{J'} e'^{*}_K d_K h^*_K (\phi_{J\cup K} \pm \phi^*_{J\cup K}) f_K & s \end{pmatrix}\right)$$

where

$$s = h'^{*}_{J'} e'^{*}_K d_K h^*_K \phi_{J\cup K} h_K d^*_K e'_K h'_{J'} - h'^{*}_{J'} \phi'_{J\cup K} h'_{J'}$$

$$\in \mathrm{Hom}_A(F'^{*}_{K,J'}, F'^{*}_{K,J'})$$

$$F_L = F_{K,L} \oplus F'_{K,J'\cup L}$$

$$\gamma_L = \begin{pmatrix} \gamma_{K,L} & \nu^J_{K,L} e_J h_K d^*_K e'_K h'_{J'} \\ l^*_L h'^{*}_{J'} e'^{*}_K d_K h^*_K (\phi^*_{J\cup K} \pm \phi_{J\cup K}) f_K & l^*_L (s^* \pm s) - m^*_L \nu^{J'}_{K,L} \end{pmatrix}$$

$$: Q_K \oplus F'^{*}_{K,J'} \longrightarrow F_{K,L} \oplus F'_{K,J'\cup L}$$

$$\mu_L = \begin{pmatrix} \mu_{K,L} & \mu_{K,L} g_K f_J f^{-1} g_J f_K q_L + g^J_{K,L} c_L n_L \\ 0 & \chi^{J'\cup L *}_{K,J'} \end{pmatrix}$$

$$: Q_K \oplus F'^{*}_{K,J'} \longrightarrow F^*_{K,L} \oplus F'^{*}_{K,J'\cup L}$$

$$\lambda^{L'}_L = \begin{pmatrix} \lambda^{L'}_{K,L} & 0 \\ u^{L*}_{L'} f^*_K g^*_J f^{-1*} f^*_J g^*_K \mu'^{*}_{K,L'} - t^{L*}_{L'} c^{-1*}_L g^{J*}_{K,L} & \chi^{J'\cup L'}_{K,J'\cup L} \end{pmatrix}$$

$$: F_{K,L'} \oplus F'_{K,J'\cup L'} \longrightarrow F_{K,L} \oplus F'_{K,J'\cup L}$$

$$\nu^L_M = \begin{pmatrix} \nu^L_{J,M} & \nu^J_{K,M} e_J h_K d^*_K e'_K h'_{J'} l^*_L \\ l^*_M h'^{*}_{J'} e'^{*}_K d_K h^*_K e^*_J \nu^L_{K,J} & l^*_M (s^* \pm s) l_L - (l^*_M \nu'^L_{K,J'} m_L + m^*_M \nu^{J'}_{K,L} l_L) - m^*_M \nu'^L_{J,M} m_L \end{pmatrix}$$

$$: F^*_{K,L} \oplus F'^{*}_{K,J'\cup L} \longrightarrow F_{K,M} \oplus F'_{K,J'\cup M},$$

for some choice of inverse isomorphisms

$$(Q'_{K,J';L} \oplus Q'_K) \oplus F'^{*}_{K,J'\cup L} \underset{\left(\begin{pmatrix} n_L & p_L \\ q_L & r_L \end{pmatrix} \; \begin{pmatrix} \lambda^{J'\cup L *}_{K,J'} & -\lambda^{J'\cup L *}_{K,L} \end{pmatrix}\right)}{\overset{\left(\begin{pmatrix} g'^{L}_{K,J'} & \mu'_{K,J'} \\ g'^{J'}_{K,L} & \mu'_{K,L'} \end{pmatrix} \begin{pmatrix} l_L \\ m_L \end{pmatrix}\right)}{\rightleftarrows}} F'^{*}_{K,J'} \oplus F'^{*}_{K,L'}$$

and the usual definition of

$$\begin{pmatrix} t^L_{L'} \\ u^L_{L'} \end{pmatrix} \in \mathrm{Hom}_A(F'^{*}_{K,J'\cup L}, Q'_{K,J';L} \oplus Q'_K), \text{ with}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{L'*}_{K,L} \end{pmatrix} \begin{pmatrix} l_L \\ m_L \end{pmatrix} - \begin{pmatrix} l_{L'} \\ m_{L'} \end{pmatrix} \lambda^{J'\cup L'*}_{K,J'\cup L}$$

$$= \begin{pmatrix} g'^{L}_{K,J'} & \mu_{K,J'} \\ g'^{J'}_{K,L} & \mu_{K,L} \end{pmatrix} \begin{pmatrix} t^L_{L'} \\ u^L_{L'} \end{pmatrix} \in \mathrm{Hom}_A(F'^{*}_{K,J'\cup L}, F'^{*}_{K,J'} \oplus F'^{*}_{K,L'})$$

for $L \subseteq L' \subseteq I''-K$.

Now

$$y *_f y = \{(F''; ((\gamma'' \atop \mu''), \theta'')G''); \{(Q''_K, \phi''_K)\}_{K \subseteq I''}\}$$

with

$$(Q''_K, \phi''_K)$$

$$= \left(\frac{\ker((e_K \; -d^*_K e'_K) : Q_{J\cup K} \oplus Q'_{J'\cup K} \longrightarrow F^*_{J,K})}{\operatorname{im}\left(\begin{pmatrix} f_J \\ f'_{J'} \rho \end{pmatrix} : Q_J \longrightarrow Q_{J\cup K} \oplus Q'_{J'\cup K}\right)}, \left[\begin{pmatrix} \phi_{J\cup K} & 0 \\ 0 & \phi'_{J'\cup K} \end{pmatrix}\right] \right)$$

The isomorphism of $\pm$ forms considered in the proof of Theorem 3.4 (on page III 3.32) then generalizes to an isomorphism

$$\left(\left[\begin{pmatrix} f_K & h_K d^*_K e'_K h'_{J'} \\ 0 & h'_{J'} \end{pmatrix} \right], \begin{pmatrix} \chi^J_{J\cup K} & f^*_K \phi_{J\cup K} h_K d^*_K e'_K h'_{J'} \\ 0 & 0 \end{pmatrix} \right)$$

$$: (Q, \phi) \longrightarrow (Q''_K, \phi''_K).$$

The inverse of this, together with the isomorphisms

$$F''_{K,L} = (Q''_{K\cup L}/Q''_L)^* = F_{K,L} \oplus (F'_{K\cup L,J'}/F'_{L,J'}) \longrightarrow F_{K,L} \oplus F'_{K,J\cup L} = F_L$$

given by the isomorphisms $F'_{L,J'} \oplus F'_{K,J\cup L} \longrightarrow F'_{K\cup L,J'}$ also considered there, now defines an isomorphism

$$\rho^K : \partial^K(y *_f y') \longrightarrow \partial^K y *_{f_1} \partial^K y'$$

of n''-ads of $\mp$ formations, verifying that $\partial^K(y *_f y')$ is defined in the ...

ii) Let

$$y = \{(G, \theta); \{(F_J, ((\gamma_J \atop \mu_J), \theta)G)\}_{J \subseteq I}\}$$

$$y' = \{(G', \theta'); \{(F'_{J'}, ((\gamma'_{J'} \atop \mu'_{J'}), \theta')G')\}_{J' \subseteq I'}\}$$

be n (resp. n')-ads of $\pm$ formations, and let

$$f = \{[\alpha, \beta, \psi], \{(f_K, \chi_K)\}\}$$

$$: -\partial^J y = \{(F_J, ((-\gamma_J \atop \mu_J), -\theta)G); \{(Q_{J,K}, -\phi_{J,K})\}_{K \subseteq I-J = I''}\}$$

$$\longrightarrow \partial^{J'} y' = \{(F'_{J'}, ((\gamma'_{J'} \atop \mu'_{J'}), \theta')G'); \{(Q'_{J',K}, \phi'_{J',K})\}_{K \subseteq I'-J' = I''}\}$$

be an isomorphism of n''-ads of $\pm$ forms.

Let

$$y *_f y' = \{(G'', \theta''); \{(F''_K, ((\gamma''_K \atop \mu''_K), \theta'')G'')\}_{K \subseteq I''}\}$$

and note that there is a factorization

$$\begin{tikzcd}[column sep=large]
F''^*_K = F^*_K \oplus F'^*_{J'\cup K} \arrow[rr, "{\lambda''^{K'*}_K = \begin{pmatrix} \lambda^{K'*}_K & \mu_K b' u'^K_{K'} - g^J_{K'} f'_{K'} t'^K_{K'} \\ 0 & \lambda'^{J'\cup K'*}_{J'\cup K} \end{pmatrix}}"] \arrow[dr, "{\begin{pmatrix} 1 & \mu_K b' u'^K_{K'} - g^J_K f'_{K'} t'^K_{K'} \\ 0 & 1 \end{pmatrix}}"'] & & F^*_{K'} \oplus F'^*_{J'\cup K'} = F''^*_{K'} \\
& F^*_K \oplus F'^*_{J'\cup K} \arrow[ur, "{\begin{pmatrix} \lambda^{K'*}_K & 0 \\ 0 & \lambda'^{J'\cup K'*}_{J'\cup K} \end{pmatrix}}"'] &
\end{tikzcd}$$

$(\ldots K' \subseteq I'')$

It follows that

$$\left(\chi''^{K\cup L *}_{K} \quad -\chi''^{K\cup L *}_{L}\right) : F''^{*}_{K} \oplus F''^{*}_{L} \longrightarrow F''^{*}_{K\cup L}$$

is onto for all $L \subseteq I'' - K$ whenever $K \subseteq I''$ is such that $\partial^K y, \partial^K y'$ are defined, implying that $\partial^K(y *_f y')$ is defined as well.

If that is the case, we can identify

$$Q''_{K,L} = \frac{\ker\left(\left(\chi''^{K\cup L *}_{K} \quad -\chi''^{K\cup L *}_{L}\right) : F''^{*}_{K} \oplus F''^{*}_{L} \longrightarrow F''^{*}_{K\cup L}\right)}{\operatorname{im}\left(\begin{pmatrix}\mu''_K \\ \mu''_L\end{pmatrix} : G'' \longrightarrow F''^{*}_{K} \oplus F''^{*}_{L}\right)}$$

$$= \left(\frac{\ker\left(\left(\chi^{K\cup L *}_{K} \quad -\chi^{K\cup L *}_{L}\right) : F^{*}_{K} \oplus F^{*}_{L} \longrightarrow F^{*}_{K\cup L}\right)}{\operatorname{im}\left(\begin{pmatrix}\mu_K \\ \mu_L\end{pmatrix} : G \longrightarrow F^{*}_{K} \oplus F^{*}_{L}\right)}\right)$$

$$\oplus \left(\frac{\ker\left(\left(\chi^{J\cup K\cup L *}_{J\cup K} \quad -\chi^{J\cup K\cup L *}_{J\cup L}\right) : F'^{*}_{J\cup K} \oplus F'^{*}_{J\cup L} \longrightarrow F'^{*}_{J\cup K\cup L}\right)}{\operatorname{im}\left(\begin{pmatrix}\chi^{J\cup K *}_{J'} \\ \chi^{J\cup L *}_{J'}\end{pmatrix} : F'^{*}_{J'} \longrightarrow F'^{*}_{J\cup K} \oplus F'^{*}_{J\cup L}\right)}\right)$$

$$= Q_{K,L} \oplus F'^{*}_{K,L,J'} \, ,$$

using the description of $F'^{*}_{K,L,J'}$ given in the proof of Theorem 3.4.

The composite

$$f_1 : -\partial^J \partial^K y \xrightarrow{\tau_{K,J}} \partial^K \partial^J y \xrightarrow{-\partial^K f} -\partial^K \partial^{J'} y' \xrightarrow{\tau_{J',K}} \partial^{J'} \partial^K y'$$

is defined by

$$\{(f_K, -\chi_K) \; ; \; \{[f_{K\cup L}]\}\}$$

$$: \{(Q_{K,J}, -\phi_{K,J}) \; ; \; \{(F_{K,J,L}, ((\begin{pmatrix}-\gamma_{K,J,L} \\ \mu_{K,J,L}\end{pmatrix}, -\phi_{K,J})Q_{K,J})\}_{L \subseteq I''-K}\}$$

$$\longrightarrow \{(Q'_{K,J'}, \phi'_{K,J'}) \; ; \; \{(F'_{K,J',L}, ((\begin{pmatrix}\gamma'_{K,J',L} \\ \mu'_{K,J',L}\end{pmatrix}, \phi'_{K,J'})Q'_{K,J'})\}_{L \subseteq J''-K}\}$$

identifying $(Q_{K,J}, \phi_{K,J}) = (Q_{J,K}, \mp\phi^*_{J,K})$, $F_{J,K,L} = F_{K,J,L}$ etc.

Now

$$\partial^K y *_{f_1} \partial^K y'$$

$$= \{(F_K \oplus F'_K, \left(\left(\begin{pmatrix}\gamma_K & \nu^J_K g^K_J & 0 \\ 0 & \nu'^{J'}_K g'^K_{J'} f_K & \gamma'_K\end{pmatrix}\right), \begin{pmatrix}\mu_K & g^J_K & 0 \\ 0 & g'^{J'}_K f_K & \mu'_K\end{pmatrix}\right), \begin{pmatrix}\theta & \gamma^*_K g^J_K & 0 \\ 0 & -\chi_K & \mp f^*_K g'^{J'*}_{K} \gamma'_{J'} \\ 0 & 0 & \theta'\end{pmatrix}\right) G \oplus Q_{J,K} \oplus G')$$

$$\{\left(\frac{\ker\left((e_{K,L} \; -[f_{K\cup L}]^* e'_{K,L}) : Q_{K,J\cup L} \oplus Q'_{K,J'\cup L} \longrightarrow F^*_{K,J,L}\right)}{\operatorname{im}\left(\begin{pmatrix}f_{K,J} \\ f'_{K,J'} f_K\end{pmatrix} : Q_{K,J} \longrightarrow Q_{K,J\cup L} \oplus Q'_{K,J'\cup L}\right)}\right), \left[\begin{pmatrix}\phi_{K,J\cup L} & 0 \\ 0 & \phi'_{K,J'\cup L}\end{pmatrix}\right]\}_{L \subseteq I''-K}$$

is an n''-ad of $\pm$ forms.

The stable isomorphism of $\pm$ formations

$$:(F''_K,(\begin{pmatrix}\gamma''_K\\ \mu''_K\end{pmatrix}),\theta'')G'')$$

$$\longrightarrow (F_K\oplus F'_K,\left(\left(\begin{pmatrix}\gamma_K & \nu^J_K g^K_J & 0\\ 0 & \nu'^{J'}_K g'^K_{J'} f_K & \gamma'_K\end{pmatrix}\right),\left(\begin{pmatrix}\mu_K & g^J_K & 0\\ 0 & g'^{J'}_K f_K & \mu'_K\end{pmatrix}\right)\right),\begin{pmatrix}\theta & \gamma^*_K g^J_K & 0\\ 0 & -\chi_K & \mp f^*_K g'^K_{J'}\gamma'_{J'}\\ 0 & 0 & \theta'\end{pmatrix}\Bigg)$$

$$G\oplus Q_{J,K}\oplus G')$$

defined in the proof of Theorem 3.3 (on page III 3.30, to be precise), and the isomorphisms of $\pm$ forms (defined for $L\subseteq I''-K$)

$$\left(\left[\begin{pmatrix}f_{K,L} & h_{K,L}[f_{K\cup L}]^* e'_{K,L} h'_{K,J'}\\ 0 & h'_{K,J'}\end{pmatrix}\right],\begin{pmatrix}0 & f^*_{K,L}\phi_{K,J\cup L} h_{K,L}[f_{K\cup L}]^* e'_{K,L} h'_{K,J'}\\ 0 & 0\end{pmatrix}\right)$$

$$:(Q_{K,L}\oplus F'^*_{K,L,J'},\phi)$$ (ϕ defined as on page III 3.44 with K replaced by L etc.)

$$\longrightarrow\left(\frac{\ker((e_{K,L}\ -[f_{K\cup L}]^* e'_{K,L}):Q_{K,J\cup L}\oplus Q'_{K,J'\cup L}\to F^*_{K,J,L})}{\operatorname{im}(\begin{pmatrix}f_{K,J}\\ f'_{K,J'}f_K\end{pmatrix}:Q_{K,J}\longrightarrow Q_{K,J\cup L}\oplus Q'_{K,J'\cup L})},\right.$$

$$\left.\left[\begin{pmatrix}\phi_{K,J\cup L} & 0\\ 0 & \phi'_{K,J'\cup L}\end{pmatrix}\right]\right)$$

together define an isomorphism

$$e^K:\partial^K(y*_f y')\longrightarrow \partial^K y *_{f_1}\partial^K y'$$

of n''-ads of $\pm$ forms. $\square$

The isomorphisms given in Theorems 3.3, 3.4, 3.5 enjoy the following obvious naturality properties:

Corollary 3.6 i) Let $g:y\to y'$ be an isomorphism of n-ads, and let J,K be such that $\tau^{J,K},\sigma^{J,K}$ are defined for y (and hence y').

Then the squares

$$\begin{array}{ccc} -\partial^K\partial^J y & \xrightarrow{\tau^{J,K}} & \partial^J\partial^K y\\ {\scriptstyle -\partial^K\partial^J g}\downarrow & & \downarrow{\scriptstyle \partial^J\partial^K g}\\ -\partial^K\partial^J y' & \xrightarrow[\tau^{J,K}]{} & \partial^J\partial^K y' \end{array}\qquad \begin{array}{ccc} \partial^J y *_{\tau^{J,K}}\partial^K y & \xrightarrow{\sigma^{J,K}} & \partial^{J\cup K} y\\ {\scriptstyle \partial^J g*\partial^K g}\downarrow & & \downarrow{\scriptstyle \partial^{J\cup K} g}\\ \partial^J y' *_{\tau^{J,K}}\partial^K y' & \xrightarrow[\sigma^{J,K}]{} & \partial^{J\cup K} y' \end{array}$$

commute.

ii) Let $g:y\longrightarrow\hat{y}$, $g':y'\longrightarrow\hat{y}'$ be isomorphisms of n (resp n')-ads such that

$$g*g':y*_f y'\longrightarrow\hat{y}*_{\hat{f}}\hat{y}'$$

is defined (for some $f:-\partial^J y\to\partial^{J'}y'$, $\hat{f}:-\partial^J\hat{y}\to\partial^{J'}\hat{y}'$), and let K be such that ρ^K is defined for $y *_f y'$ (and hence for $\hat{y} *_{\hat{f}} \hat{y}'$). Then the square

$$\begin{tikzcd}
\partial^K(y *_f y') \arrow[r, "\rho^K"] \arrow[d, "\partial^K(g * g')"'] & \partial^K y *_{f_1} \partial^K y' \arrow[d, "\partial^K g * \partial^K g'"] \\
\partial^K(\hat{y} *_{\hat{f}} \hat{y}') \arrow[r, "\rho^K"'] & \partial^K \hat{y} *_{\hat{f}_1} \partial^K \hat{y}'
\end{tikzcd}$$

commutes.

□

The join operations are commutative and associative in the following sense:

<u>Theorem 3.7</u> i) Let y, y' be n (resp n')-ads and let $f : -\partial^J y \longrightarrow \partial^{J'} y'$ be an isomorphism of n''-ads. Then there is defined a natural isomorphism

$$\kappa : y *_f y' \longrightarrow y' *_{-f^{-1}} y$$

of n''-ads.

ii) Let x, y, z be m (resp n, p)-ads, and suppose given isomorphisms

$$f : -\partial^K x \longrightarrow \partial^J y, \quad g : -\partial^L y \longrightarrow \partial^K z, \quad h : -\partial^L x \longrightarrow \partial^J z$$

such that the composites

$$e : -\partial^L \partial^K x \xrightarrow{\partial^L f} \partial^L \partial^J y \xrightarrow{-\tau^{J,L}} -\partial^J \partial^L y \xrightarrow{\partial^J g} \partial^J \partial^K z$$

$$\tilde{e} : -\partial^L \partial^K x \xrightarrow{\tau^{K,L}} \partial^K \partial^L x \xrightarrow{-\partial^K h} -\partial^K \partial^J z \xrightarrow{\tau^{J,K}} \partial^J \partial^K z$$

are coherent (and defined). Then there is defined a natural isomorphism

$$\alpha : (x *_f y) *_{(h*g)_2} z \longrightarrow x *_{(f*h)_3} (y *_g z)$$

of q-ads, where $(h*g)_2$ is the composite

$$: -\partial^L(x *_f y) \xrightarrow{-\rho^L} -\partial^L x *_{-f_1} -\partial^L y \xrightarrow{h*g} \partial^J z *_{\tau^{J,K}} \partial^K z \xrightarrow{\sigma^{J,K}} \partial^{J \cup K} z$$

and $(f*h)_3$ is the composite

$$: -\partial^{K \cup L} x \xrightarrow{(-\sigma^{K,L})^{-1}} -\partial^K x *_{-\tau^{K,L}} -\partial^L x \xrightarrow{f*h} \partial^J y *_{g_1} \partial^J z \xrightarrow{(\rho^J)^{-1}} \partial^J(y *_g z)$$

<u>Proof</u>: i) If y, y' are n-ads of $\left\{\begin{array}{l} \pm \text{ forms} \\ \pm \text{ formations} \end{array}\right.$, we can take for κ the $\left\{\begin{array}{l} \text{identity isomorphism} \\ \text{isomorphism } \{(\delta,\xi), [\delta_K^{(*)}]\} : y'' \longrightarrow y''' \text{ defined on p. III 3.} \end{array}\right.$

ii) Excersize for the reader. $\square$

§4. The Δ-sets $\mathbb{L}_f(A)$

Recall from §3 of I the abelian monoids $X_n(A)$ of $\begin{cases}\text{isomorphism}\\ \text{stable isomorphism}\end{cases}$ classes of $\begin{cases}\pm\text{forms}\\ \pm\text{formations}\end{cases}$, where $\pm = (-)^i$ if $n = \begin{cases}2i\\ 2i+1\end{cases}$. An n-ad

$$y = \{x; \{x_J\}_{J \subseteq I}\}$$

will be said to be of fibre dimension f if $x \in X_{n+f}(A)$ so that we are dealing with an n-ad of $\begin{cases}\pm\text{forms}\\ \pm\text{formations}\end{cases}$ if $n+f-1 = \begin{cases}2i\\ 2i+1\end{cases}$, with $\pm = (-)^i$, and $f \pmod 4$.

An n-ad $y = \{x; \{x_J\}_{J \subseteq I}\}$ is connected if $\partial^J y$ is defined for all $J \subseteq I$, and highly-connected if $\partial^{J_1}\partial^{J_2}\ldots\partial^{J_r} y$ is defined for all disjoint $J_1, J_2, \ldots, J_r \subseteq I$. By Theorem 3.4 y is highly-connected iff $\partial^{\{j_1\}} \ldots \partial^{\{j_r\}} y$ is defined for all distinct $j_1, j_2, \ldots, j_r \in I$. By Theorem 3.5 the join of highly-connected n-ads is again highly connected.

An <u>oriented</u> n-ad

$$y = \{x ; \{x_J\}_{J \subseteq I}\}$$

is one for which $I = \{0,1,\dots,n\}$.

An <u>orientation</u> of an n-ad

$$y = \{x ; \{x_J\}_{J \subseteq I}\}$$

is the oriented n-ad

$$y_{\#} = \{x ; \{x_{\omega(J)}\}_{J \subseteq \{0,1,\dots,n\}}\}$$

defined by relabelling, according to some bijection

$$\omega : \{0,1,\dots,n\} \longrightarrow I .$$

In applications, it will always be the case that $I \subset \{0,1,2,\dots\dots\}$, with

$$\omega = \omega_I : \{0,1,\dots,n\} \longrightarrow I$$

the unique order-preserving bijection.

Define <u>face operations</u>

$$\partial_j : \{\text{oriented connected } n\text{-ads of fibre dimension } f\} \longrightarrow \{\text{oriented } (n-1)\text{-ads of fibre dimension } f\}$$

for $0 \leqslant j \leqslant n$, $n \geqslant 1$ by

$$\partial_j y = (-)^j (\partial^{\{j\}} y)_{\#} .$$

Theorem 3.4 gives isomorphisms

$$\tau_{j,k} = (-)^{j+k} \tau^{\{j\},\{k\}} : \partial_{k-1}\partial_j y \longrightarrow \partial_j \partial_k y$$

for $0 \leqslant j < k \leqslant n$, whenever $\partial_j \partial_k y , \partial_{k-1}\partial_j y$ are defined.

For $f \pmod 4$, let $\mathcal{L}_f(A)$ be the Δ-set defined by:

An n-simplex, $s \in \mathcal{L}_f(A)^{(n)}$, is a collection

$$s = \{ (y_J , h_{J,k} : y_{J-\{\omega_J(k)\}} \longrightarrow \partial_k y_J)_{J \subseteq \{0,1,\dots,n\}, 0 \leqslant k \leqslant |J|-1} \}$$

of oriented highly connected $(|J|-1)$-ads y_J, of fibre dimension f, one for each non-empty subset J of $\{0,1,\dots,n\}$, and $(|J|-2)$-ad isomorphisms $h_{J,k}$ defined for $0 \leqslant k \leqslant |J|-1$ with $|J| \geqslant 2$, such that

i) the $(|J|-2)$-ad isomorphisms

$$\partial_{k-1} y_{J-\{\omega_J(j)\}} \rightrightarrows \partial_j y_{J-\{\omega_J(k)\}} \quad (0 \leqslant j < k \leqslant |J|-1)$$

defined by requiring the diagrams

$$\begin{tikzcd}
 & \partial_{k-1} y_{J-\{\omega_J(j)\}} \arrow[dd] \\
y_{J-\{\omega_J(j),\omega_J(k)\}} \arrow[ur, "h_{J-\{\omega_J(j)\},k-1}"] \arrow[dr, "h_{J-\{\omega_J(k)\},j}"'] & \\
 & \partial_j y_{J-\{\omega_J(k)\}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial_{k-1} y_{J-\{\omega_J(j)\}} \arrow[r, "\partial_{k-1} h_{J,j}"] \arrow[d] & \partial_{k-1}\partial_j y_J \arrow[d, "\tau_{j,k}"] \\
\partial_j y_{J-\{\omega_J(k)\}} \arrow[r, "\partial_j h_{J,k}"'] & \partial_j \partial_k y_J
\end{tikzcd}$$

to commute, are coherent.

ii) the 0-ads $y_{\{j\}} = \{x_j , \{x_{j,\emptyset}\}\}$ have x_j non-singular, that is

$$x_j \in \ker(\partial : X_f(A) \longrightarrow X_{f-1}(A)) \quad (0 \leqslant j \leqslant n)$$

The base n-simplex, $0 \in \mathcal{L}_f(A)^{(n)}$, is the unique such collection with all the A-modules 0.

Denoting $\omega_{\{0,1,\ldots,n\}-\{j\}}$ by ∂_j, so that

$$\partial_j : \{0,1,\ldots,n-1\} \longrightarrow \{0,1,\ldots,n\}-\{j\} \,;\; i \longmapsto \begin{cases} i & i<j \\ i+1 & i \geqslant j \end{cases},$$

define face maps

$$\partial_j : \mathcal{L}_f(A)^{(n)} \longrightarrow \mathcal{L}_f(A)^{(n-1)} \,;$$

$$\{(y_J, h_{J,k})_{J\subseteq\{0,1,\ldots,n\},\, 0\leqslant k<|J|}\} \longmapsto \{(y_{\partial_j J}, h_{\partial_j J, \partial_j k})_{J\subseteq\{0,1,\ldots,n-1\},\, 0\leqslant k<|J|}\}$$

for $0 \leqslant j \leqslant n$, $n \geqslant 1$. Then

$$\partial_j \partial_k = \partial_{k-1}\partial_j : \mathcal{L}_f(A)^{(n)} \longrightarrow \mathcal{L}_f(A)^{(n-2)}$$
$$(0 \leqslant j < k \leqslant n,\ n \geqslant 2)$$

by construction.

—

We shall establish that

i) the Kan extension condition holds for $\mathcal{L}_f(A)$

ii) $\pi_n(\mathcal{L}_f(A)) = U_{n+f}(A)$ $(n \geqslant 0)$

iii) there are defined homotopy equivalences

$$\Omega\mathcal{L}_f(A) \longrightarrow \mathcal{L}_{f+1}(A) \qquad (f(\mathrm{mod}\,4))$$

in a natural fashion.

The Kan extension condition is that every n-tuple of $(n-1)$-simplexes

$$s = \{s_j \in \mathcal{L}_f(A)^{(n-1)} \mid 0 \leqslant j \leqslant n,\ j \neq i\} \qquad (0 \leqslant i \leqslant n,\ n \geqslant 1$$

with

$$\partial_j s_k = \partial_{k-1} s_j \in \mathcal{L}_f(A)^{(n-2)} \qquad (\text{if } n \geqslant 2)$$

is such that

$$s_j = \partial_j \tilde{s} \in \mathcal{L}_f(A)^{(n-1)} \qquad (0 \leqslant j \leqslant n,\ j \neq i)$$

for some $\tilde{s} \in \mathcal{L}_f(A)^{(n)}$.

In order to verify it for $\mathcal{L}_f(A)$ we have to show that for every collection

$$\{(y_J, h_{J,k} : y_{J-\{\omega_J(k)\}} \longrightarrow \partial_k y_J) \mid J \subseteq \{0,1,\ldots,n\},\ 0 \leqslant k < |J|-1,\ J \neq \{0,1,\ldots,n\}, \{0,1,\ldots,n\}-\{i\}$$

of oriented highly connected $(|J|-1)$-ads y_J, and $(|J|-2)$-ad isomorphisms $h_{J,k}$, such that

$$s_j = \{(y_{\partial_j J}, h_{\partial_j J, \partial_j k})\}_{J\subseteq\{0,1,\ldots,n-1\},\, 0\leqslant k<|J|-1} \in \mathcal{L}_f(A)^{(n-1)}$$
$$(0 \leqslant j \leqslant n,\ j \neq i)$$

there is defined an oriented highly connected n-ad $y_{\{0,1,\ldots,n\}}$ of fibre dimension f, with $(n-1)$-ad isomorphisms

$$h_{\{0,1,\ldots,n\},j} : y_{\{0,1,\ldots,n\}-\{j\}} \longrightarrow \partial_j y_{\{0,1,\ldots,n\}} \quad (0 \leqslant j$$

such that

$$s = \{(y_J, h_{J,k})\}_{J\subseteq\{0,1,\dots,n\},\, 0\leq k<|J|} \in \mathcal{L}_f(A)^{(n)},$$

with

$$h_{\{0,1,\dots,n\},i} : y_{\{0,1,\dots,n\}-\{i\}} \longrightarrow \partial_i y_{\{0,1,\dots,n\}}$$

the identity.

—

For the cases $n=1,2$ this programme was carried out in §§1,2 - as will be detailed below. The general case proceeds by analogy.

—

A 0-simplex of $\mathcal{L}_{2i}(A)$ is a non-singular $\pm$form (Q,ϕ) together with a stable isomorphism $(F_0,(\binom{\gamma_0}{\mu_0}),\theta_0 G_0) \longrightarrow \partial(Q,\phi)$ of trivial $\mp$ formations. A 1-simplex of $\mathcal{L}_{2i}(A)$ is a cobordism of non-singular $\pm$forms, together with such stable isomorphisms.

A 0-simplex of $\mathcal{L}_{2i+1}(A)$ is a non-singular $\pm$formation, and a 1-simplex is a cobordism of such.

In verifying that cobordism of $\left\{\begin{matrix}\pm\text{forms}\\ \pm\text{formations}\end{matrix}\right.$ is an equivalence relation, in Theorem $\left\{\begin{matrix}1.1\\ 2.1\end{matrix}\right.$, we constructed cobordisms

$$\left\{\begin{matrix}\{(Q,0);(Q,\phi),(Q,\phi),((\begin{smallmatrix}1\\ \phi\end{smallmatrix}),0),((\begin{smallmatrix}1\\ \mp\phi^*\end{smallmatrix}),\phi)\}\\ \{(G,\theta);(F,(\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta G),(F^*,(\begin{smallmatrix}\pm\mu\\ \gamma\end{smallmatrix}),-\theta G),[1],[1]\}\end{matrix}\right. \quad \text{for every} \left\{\begin{matrix}\pm\text{form } (Q,\phi)\\ \pm\text{formation } (F,\dots)\end{matrix}\right.$$

In particular, for non-singular $\left\{\begin{matrix}\pm\text{forms}\\ \pm\text{formations}\end{matrix}\right.$, we have the extension condition for $n=1$, for each $\mathcal{L}_f(A)$. Moreover, the 0-simplexes of $\mathcal{L}_{2i}(A)$ defined by the same non-singular $\pm$ form (Q,ϕ) are all seen to lie in the same path-component of $\mathcal{L}_{2i}(A)$. Corollary $\left\{\begin{matrix}1.2\\ 2.2\end{matrix}\right.$ now shows that

$$\pi_0(\mathcal{L}_f(A)) = U_f(A)$$

for all $f \pmod 4$.

The above construction of cobordisms generalizes to:

<u>Lemma 4.1</u> A lattice of $\left\{\begin{matrix}\pm\text{forms}\\ \pm\text{formations}\end{matrix}\right.$ $y=\{x_J\}_{J\subseteq I}$ can be embedded in an n-ad of $\left\{\begin{matrix}\pm\text{forms}\\ \pm\text{formations}\end{matrix}\right.$

$$\tilde{y} = \{x, \{x_J\}_{J\subseteq I\cup\{\infty\}}\} \qquad (\infty\notin I).$$

<u>Proof</u>: i) Let $y=\{(Q_J,\phi_J)\}_{J\subseteq I}$ be a lattice of $\pm$ forms. For $J\subseteq I$ set $J_\infty = J\cup\{\infty\}$ and define a $\pm$form

$$(Q_{J_\infty},\phi_{J_\infty}) = \left(\ker\left(\pm f_I^{I-J*}\phi_I^* \quad f_I^{I-J*}\right) ; Q_I\oplus Q_I^* \to Q_{I-J}^*\right), \begin{pmatrix}0&1\\0&0\end{pmatrix}\Big| \Big)$$

In particular, we have

$$(Q_{\{\infty\}},\phi_{\{\infty\}}) = \operatorname{im}\left(\left(\begin{pmatrix}1\\ \mp\phi_I^*\end{pmatrix}, \phi_I\right) : (Q_I,-\phi_I) \to H_\pm(Q_I)\right)$$

$$(Q_{I_\infty},\phi_{I_\infty}) = H_\pm(Q_I).$$

For $J \subseteq I$ let

$$(f^J_{I_\infty}, \chi^J_{I_\infty}) = \left(\begin{pmatrix} f^J_I \\ \phi_I f^J_I \end{pmatrix}, \chi^J_I\right) : (Q_J, \phi_J) \longrightarrow (Q_{I_\infty}, \phi_{I_\infty})$$

and

$$(f^J_{J_\infty}, \chi^J_{J_\infty}) = (f^J_{I_\infty}| \ , \chi^J_I) : (Q_J, \phi_J) \longrightarrow (Q_{J_\infty}, \phi_{J_\infty}) .$$

For $J \subseteq J' \subset I$ let

$$(f^{J_\infty}_{J'_\infty}, 0) : (Q_{J_\infty}, \phi_{J_\infty}) \longrightarrow (Q_{J'_\infty}, \phi_{J'_\infty})$$

be the subform inclusion defined by the A-module inclusion $f^{J_\infty}_{J'_\infty} \in \mathrm{Hom}_A(Q_{J_\infty}, Q_{J'_\infty})$, and let

$$(f^J_{J'_\infty}, \chi^J_{J'_\infty}) = (f^{J'}_{J'_\infty} f^J_{J'}, \chi^J_{J'}) : (Q_J, \phi_J) \longrightarrow (Q_{J'_\infty}, \phi_{J'_\infty}) .$$

Now, for each $J \subseteq I$,

$$(f^{J_\infty}_{I_\infty}, 0) : (Q_{J_\infty}, \phi_{J_\infty}) \longrightarrow (Q_{I_\infty}, \phi_{I_\infty}), \ (f^{I_\infty - J_\infty}_{I_\infty}, \chi^{I_\infty - J_\infty}_{I_\infty}) : (Q_{I_\infty - J_\infty}, \phi_{I_\infty - J_\infty}) \longrightarrow (Q_{I_\infty}, \phi_{I_\infty})$$

are the inclusions of maximally orthogonal subforms (by construction). Therefore

$$\tilde{y} = \{ (Q_I, 0) ; \{ (Q_J, \phi_J) \}_{J \subseteq I_\infty} \}$$

is an n-ad of $\pm$ forms containing y.

ii) Let $y = \{ (F_J, (\begin{pmatrix} \gamma_J \\ \mu_J \end{pmatrix}, \theta) G) \}_{J \subseteq I}$ be a lattice of $\pm$ formations.

For $J \subseteq I$ set $J_\infty = J \cup \{\infty\}$, and define a $\pm$ formation

$$(F_{J_\infty}, (\begin{pmatrix} \gamma_{J_\infty} \\ \mu_{J_\infty} \end{pmatrix}, \theta) G) = (F^*_{I-J}, (\begin{pmatrix} \mu_{I-J} \\ \pm\gamma_{I-J} \end{pmatrix}, \theta) G)$$

and A-module morphisms

$$\lambda^{J'_\infty}_{J_\infty} = \lambda^{I-J\,*}_{I-J'} \in \mathrm{Hom}_A(F^*_{I-J'}, F^*_{I-J})$$

$$\lambda^{J'_\infty}_{J} = \nu^{I-J'}_J \in \mathrm{Hom}_A(F^*_{I-J'}, F_J) \qquad (J \subseteq J' \subseteq I)$$

$$\nu^J_{K_\infty} = \lambda^{I-K*}_J \in \mathrm{Hom}_A(F^*_J, F^*_{I-K})$$

$$\nu^{J_\infty}_K = \mp \lambda^{I-J}_K \in \mathrm{Hom}_A(F_{I-J}, F_K) \qquad (J, K \subseteq I \text{ disjoint})$$

In particular, $\nu^J_{I_\infty - J} = 1 \in \mathrm{Hom}_A(F^*_J, F_{I_\infty - J})$ is an isomorphism for all $J \subseteq I$.

Therefore

$$\tilde{y} = \{ (G, \theta) ; \{ (F_J, (\begin{pmatrix} \gamma_J \\ \mu_J \end{pmatrix}, \theta) G) \}_{J \subseteq I_\infty} \}$$

is an n-ad of $\pm$formations containing y.

A lattice of $\pm$ forms $\{(Q_J,\phi_J)\}_{J\subseteq I}$ is <u>connected</u> if

$$\begin{pmatrix} (\phi^*_{J\cup K}\pm\phi_{J\cup K})f^J_{J\cup K} & 0 \\ f^J_{J\cup K} & f^K_{J\cup K}\end{pmatrix} : Q_J\oplus Q_K \longrightarrow Q^*_{J\cup K}\oplus Q_{J\cup K}$$

is a split mono for disjoint $J,K\subseteq I$.

A lattice of $\pm$ formations $\{(F_J,(\binom{\gamma_J}{\mu_J}),\Theta G)\}_{J\subseteq I}$ is <u>connected</u> if

$$\begin{pmatrix}\lambda_J^{J\cup K *} & -\lambda_K^{J\cup K *}\end{pmatrix} : F_J^*\oplus F_K^* \longrightarrow F^*_{J\cup K}$$

is onto for disjoint $J,K\subseteq I$.

Lemma 4.1 now specializes to:

<u>Lemma 4.2</u> A connected lattice of $\begin{cases}\pm\text{forms}\\ \pm\text{formations}\end{cases}$ can be embedded in a connected n-ad of $\begin{cases}\pm\text{forms}\\ \pm\text{formations}\end{cases}$.

<u>Proof</u>: Let $y = \begin{cases}\{(Q_J,\phi_J)\}_{J\subseteq I}\\ \{(F_J,(\binom{\gamma_J}{\mu_J}),\Theta G)\}_{J\subseteq I}\end{cases}$ be a connected lattice of $\begin{cases}\pm\text{forms}\\ \pm\text{formations}\end{cases}$, and let $\tilde{y} = \begin{cases}\{(Q_I,0);\{(Q_J,\phi_J)\}_{J\subset I_\infty}\}\\ \{(G,\Theta);\{(F_J,(\binom{\gamma_J}{\mu_J}),\Theta G)\}_{J\subset I_\infty}\}\end{cases}$ be the n-ad defined in the proof of Lemma 4.1.

By Lemma $\begin{cases}3.1\\3.2\end{cases}$, the morphism

$$\begin{cases}\begin{pmatrix} (\phi^*_{J\cup K}\pm\phi_{J\cup K})f^J_{J\cup K} & 0 \\ f^J_{J\cup K} & f^K_{J\cup K}\end{pmatrix} : Q_J\oplus Q_K \longrightarrow Q^*_{J\cup K}\oplus Q_{J\cup K}\\ \begin{pmatrix}\lambda_J^{J\cup K *} & -\lambda_K^{J\cup K *}\end{pmatrix} : F_J^*\oplus F_K^* \longrightarrow F^*_{J\cup K}\end{cases}$$

is $\begin{cases}\text{split mono}\\ \text{onto}\end{cases}$, for some disjoint $J,K\subseteq I_\infty$, iff the same is true of the corresponding morphism with J or K replaced by $L = I_\infty - (J\cup K)$. Therefore it is sufficient to consider the cases with $J,K\subseteq I$, where this is true by hypothesis. Thus $\partial^J\tilde{y}$ is defined for all $J\subseteq I_\infty$.

$\square$

Given $i \in \{0,1,2\}$ and a collection of oriented highly connected 0-ads and 1-ads and isomorphisms

$$S = \{(y_J, h_{J,k} : y_{J-\{\omega_J(k)\}} \to \partial_k y_J) \mid {J \subseteq \{0,1,2\} \atop 0 \leqslant k < |J|} \;\; {J \neq \{0,1,2\}, \atop \{0,1,2\}-\{i\}}\}$$

such that for both $j \in \{0,1,2\} - \{i\}$

$$S_j = \{(y_{\partial_j J}, h_{\partial_j J, \partial_j k})\}_{J \subseteq \{0,1\}, 0 \leqslant k < |J|} \in \mathcal{L}_f(A)^{(1)}$$

write

$$(-)^j y_{\{0,1,2\}-\{j\}} = \{z_j, \{z_{j,k}\}_{k \subseteq \{0,1,2\}-\{j\}}\}$$

and let

$$g_{j,k} : -z_{j,k} \longrightarrow z_{k,j} \quad (0 \leqslant j < k \leqslant 2,\ j \neq i)$$

be the (stable) isomorphism defined by the composite

$$\partial_{k-1} y_{\{i,k\}} \xrightarrow{h_{\{i,k\},k-1}} y_{\{i\}} \xrightarrow{(h_{\{i,j\},j})^{-1}} \partial_j y_{\{i,j\}} .$$

The 1-ads $(-)^j y_{\{i,k\}}$, $(-)^k y_{\{i,j\}}$ can be then regarded as cobordisms (of $\begin{cases} \pm\text{forms} \\ \pm\text{formations} \end{cases}$ if $f = \begin{cases} 2i \\ 2i+1 \end{cases}$)

$$c \equiv (-)^j y_{\{i,k\}} = \{z_j \,;\, z_{j,i}, -z_{j,k}, [1], [1]\}$$

$$(-)^k y_{\{i,j\}} = \{z_k \,;\, z_{k,j}, -z_{k,i}, [1], [1]\}$$

and $g_{j,k} : -z_{j,k} \longrightarrow z_{k,j}$ defines an isomorphism

of cobordisms

$$g_{j,k} : c' \equiv \{z_k \,;\, -z_{j,k}, -z_{k,i}, g_{j,k}, 1\} \longrightarrow (-)^k y_{\{i,j\}} .$$

Now c, c' are adjoining cobordisms of $\begin{cases} \pm\text{forms} \\ \pm\text{formations} \end{cases}$ and Theorem $\begin{cases} 1.4 \\ 2.4 \end{cases}$ shows how such a pair determines a connected lattice $W = \{W_J\}_{J \subseteq \{j,k\}}$ of $\begin{cases} \pm\text{formations} \\ \mp\text{forms} \end{cases}$, as is considered in Theorem $\begin{cases} 1.3 \\ 2.3 \end{cases}$ with 1-ad (alias cobordism) isomorphisms

$$f : c \longrightarrow \partial^{\{j\}} \hat{W} \qquad f' : c' \longrightarrow \partial^{\{k\}} \hat{W}$$

where $\hat{W}$ is the connected 2-ad given by Lemma 4.2. Moreover, the 0-ad isomorphisms

$$g_{j,k}, \tilde{g}_{j,k} : \partial_j y_{\{i,j\}} \longrightarrow \partial_{k-1} y_{\{i,k\}}$$

defined by commutative diagrams

$$\begin{tikzcd}
 & \partial_{k-1} y_{\{i,k\}} \arrow[dd, "g_{j,k}"] & \partial_{k-1} y_{\{i,k\}} \arrow[r, "\partial_{k-1}((-)^j f.((-)^j g_{j,k})^{-1})"] \arrow[dd, "\tilde{g}_{j,k}"'] & \partial_{k-1}\partial_j \tilde{w} \arrow[dd, "\tau_{j,k}"] \\
y_{\{i\}} \arrow[ur, "h_{\{i,k\},k-1}"] \arrow[dr, "h_{\{i,j\},j}"'] & & & \\
 & \partial_j y_{\{i,j\}} & \partial_j y_{\{i,j\}} \arrow[r, "\partial_j((-)^k f')"'] & \partial_j \partial_k \tilde{w}
\end{tikzcd}$$

are coherent : this was the point of the remarks made at the end of § $\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right.$.

In short, setting

$$y_{\{0,1,2\}} = \tilde{w} \qquad y_{\{j,k\}} = \partial_i \tilde{w}$$

$$h_{\{0,1,2\},i} = 1 : y_{\{j,k\}} \longrightarrow \partial_i \tilde{w}$$

$$h_{\{0,1,2\},j} = ((-)^j f).((-)^j g_{j,k})^{-1} : y_{\{i,k\}} \longrightarrow \partial_j \tilde{w}$$

$$h_{\{0,1,2\},k} = (-)^k f' : y_{\{i,j\}} \longrightarrow \partial_k \tilde{w}$$

we have defined a 2-simplex,

$$\tilde{s} = \{(y_J, h_{J,\ell})\}_{J\subseteq\{0,1,2\},\, 0\leqslant l<|J|} \in \mathcal{L}_f(A)^{(2)}$$

containing the horn s.

This verifies the Kan condition for $n=2$.

In order to establish it for $n\geqslant 3$ we shall need the following generalization of Theorem $\left\{\begin{smallmatrix}1.4\\2.4\end{smallmatrix}\right.$:

<u>Theorem 4.3</u> Let I be a set of n elements, $n\geqslant 3$, and let $\infty\notin I$. Suppose given connected $(n-1)$-ads

$$y_i = \{x_i\,;\, \{x_{i,J}\}_{J\subseteq I_\infty - \{i\}}\} \quad (i\in I,\ I_\infty = I\cup\{\infty\})$$

and $(n-2)$-ad isomorphisms

$$g_{j,k} : -\partial^k y_j \longrightarrow \partial^j y_k \quad (j,k\in I \quad j\neq k)$$

(writing $\partial^k y_j$ for $\partial^{\{k\}} y_j$) such that

$$g_{k,j} = -\,g_{j,k}^{-1} : -\partial^j y_k \longrightarrow \partial^k y_j\,,$$

with the $(n-3)$-ad isomorphisms

$$e_{j,k,l} : -\partial^l\partial^k y_j \xrightarrow{\partial^l g_{j,k}} \partial^l\partial^j y_k \xrightarrow{-\tau^{j,l}} -\partial^j\partial^l y_k \xrightarrow{\partial^j g_{k,l}} \partial^j\partial^k y_l$$

$$\tilde{e}_{j,k,l} : -\partial^l\partial^k y_j \xrightarrow{\tau^{k,l}} \partial^k\partial^l y_j \xrightarrow{-\partial^k g_{j,l}} -\partial^k\partial^j y_l \xrightarrow{\tau^{j,k}} \partial^j\partial^k y_l$$

defined and coherent for distinct $j,k,l\in I$

(writing $\tau^{j,k}$ for $\tau^{\{j\},\{k\}}$).

Then there is defined a connected lattice $w = \{w_J\}_{J \subseteq I}$ with $(n-1)$-ad isomorphisms

$$f_j : y_j \longrightarrow \partial^j \tilde{w} \quad (j \in I)$$

where $\tilde{w}$ is the connected n-ad given by Lemma 4.2, such that the $(n-2)$-ad isomorphisms

$$g_{j,k} : -\partial^k y_j \longrightarrow \partial^j y_k$$

$$\tilde{g}_{j,k} : -\partial^k y_j \xrightarrow{-\partial^k f_j} -\partial^k \partial^j \tilde{w} \xrightarrow{\tau^{j,k}} \partial^j \partial^k \tilde{w} \xrightarrow{\partial^j f_k^{-1}} \partial^j y_k$$

are coherent, for distinct $j,k \in I$.

<u>Proof</u>: For any ordering of I define m-ads

$$y_J = \{x_J ; \{x_{J,K}\}_{K \subseteq I_\infty - J}\}$$

for all $J \subseteq I$, and isomorphisms of p-ads

$$g_{J,K} : -\partial^K y_J \longrightarrow \partial^J y_K \quad \left(\begin{array}{r} J,K \subseteq I \text{ disjoint} \\ j<k \ \forall j \in J, k \in K \end{array} \right)$$

such that the composites

$$e_{J,K,L} : -\partial^L \partial^K y_J \xrightarrow{\partial^L g_{J,K}} \partial^L \partial^J y_K \xrightarrow{-\tau^{J,L}} -\partial^J \partial^L y_K \xrightarrow{\partial^J g_{K,L}} \partial^J \partial^K y_L$$

$$\tilde{e}_{J,K,L} : -\partial^L \partial^K y_J \xrightarrow{\tau^{K,L}} \partial^K \partial^L y_J \xrightarrow{-\partial^K g_{J,L}} -\partial^K \partial^J y_L \xrightarrow{\tau^{J,K}} \partial^J \partial^K y_L$$

are defined and coherent for disjoint $J,K,L \subseteq I$ with $j<k<l$ for all $j \in J$, $k \in K$, $l \in L$.

Set

$$y_{\{j\}} = y_j \quad g_{\{j\},k} = g_{j,k} \quad (j,k \in I,\ j<k).$$

Assume inductively that y_J, $g_{J,k}$ have already been defined for some $J \subseteq I$, and all $k \in I$ s.t. $k > \max_{j \in J} j$, with $e_{J,k,l}$, $\tilde{e}_{J,k,l}$ defined and coherent for distinct $k,l \in I-J$. Now set

$$y_{J \cup \{k\}} = y_J *_{g_{J,k}} y_k \quad (k \in I-J \quad \max_{j \in J} j < k)$$

$$g_{J\cup\{k\},l} : -\partial^l y_{J\cup\{k\}} \xrightarrow{-e^i} -\partial^l y_J *_{-(g_{J,k})_1} -\partial^l y_k \xrightarrow{g_{J,l} * g_{k,l}} \partial^J y_l *_{\tau^{j,k}} \partial^k y_l$$

$$\xrightarrow{\sigma^{J,k}} \partial^{J\cup\{k\}} y_l ,$$

$$(k,l \in I-J \quad \max_{j \in J} j < k < l)$$

where we are successively applying Theorems 3.5, 3.4, 3.3 to define $g_{J\cup\{k\},l}$, and $(g_{J,k})_1$ stands for the composite

$$: -\partial^k \partial^l y_J \xrightarrow{\tau^{l,k}} \partial^l \partial^k y_J \xrightarrow{-\partial^l g_{J,k}} -\partial^l \partial^J y_k \xrightarrow{\tau^{J,l}} \partial^J \partial^l y_k$$

(exactly as in the statement of Theorem 3.5) and the diagram

$$\begin{tikzcd}
\partial^k\partial^l y_J \arrow[r, "-(g_{J,k})_l"] \arrow[d, "-\partial^k g_{J,l}"'] & -\partial^J\partial^l y_k \arrow[d, "\partial^J g_{k,l}"] \\
-\partial^k\partial^J y_l \arrow[r, "\tau^{J,k}"] & \partial^J\partial^k y_l
\end{tikzcd}$$

commutes up to coherence (as required for $g_{J,l} * g_{k,l}$ to be defined), because the composites

$$e_{J,k,l} : -\partial^l\partial^k y_J \xrightarrow{\tau^{k,l} = -(\tau^{l,k})^{-1}} \partial^k\partial^l y_J \xrightarrow{-(g_{J,k})_l} -\partial^J\partial^l y_k \xrightarrow{\partial^J g_{k,l}} \partial^J\partial^k y_l$$

$$\tilde{e}_{J,k,l} : -\partial^l\partial^k y_J \xrightarrow{\tau^{k,l}} \partial^k\partial^l y_J \xrightarrow{-\partial^k g_{J,l}} -\partial^k\partial^J y_l \xrightarrow{\tau^{J,k}} \partial^J\partial^k y_l$$

are coherent.

The isomorphisms defined by the composites

$$e_{J\cup\{k\},l,m} : -\partial^m\partial^l y_{J\cup\{k\}} \xrightarrow{\partial^m g_{J\cup\{k\},l}} \partial^m\partial^{J\cup\{k\}} y_l \xrightarrow{-\tau^{J\cup\{k\},m}} -\partial^{J\cup\{k\}}\partial^m y_l \xrightarrow{\partial^{J\cup\{k\}} g_{l,m}} \partial^{J\cup\{k\}}\partial^l y_m$$

$$\tilde{e}_{J\cup\{k\},l,m} : -\partial^m\partial^l y_{J\cup\{k\}} \xrightarrow{\tau^{l,m}} \partial^l\partial^m y_{J\cup\{k\}} \xrightarrow{-\partial^l g_{J\cup\{k\},m}} -\partial^l\partial^{J\cup\{k\}} y_m \xrightarrow{\tau^{J\cup\{k\},l}} \partial^{J\cup\{k\}}\partial^l y_m$$

are coherent, being both coherent with

$$: -\partial^m\partial^l y_{J\cup\{k\}} \xrightarrow{-\rho^m(\rho^l)} -\partial^m\partial^l y_J * -\partial^m\partial^l y_k \xrightarrow{e_{J,l,m} * e_{k,l,m}} \partial^J\partial^l y_m * \partial^k\partial^l y_m \xrightarrow{\sigma^{J,k}} \partial^{J\cup\{k\}}\partial^l y_m$$

writing $\rho^m(\rho^l)$ for the isomorphism given by a double application of Theorem 3.5, and using the naturality properties given in Corollary 3.6 to ensure that $e_{J,l,m} * e_{k,l,m}$ is defined, where $k,l,m \in I-J$ are such that

$$\max_{j\in J} j < k < l < m.$$

This completes the induction step, so we have defined $y_J, g_{J,k}, e_{J,k,l}$. There is then a similar inductive construction for $g_{J,K}$, given by

$$g_{J,K\cup\{l\}} : -\partial^{K\cup\{l\}} y_J \xrightarrow{-\sigma^{K,l}} -\partial^K y_J * -\partial^l y_J \xrightarrow{g_{J,K} * g_{J,l}} \partial^J y_K * \partial^J y_l \xrightarrow{(\rho^J)^{-1}} \partial^J y_{K\cup\{l\}},$$

with $e_{J,K,L}$, $\tilde{e}_{J,K,L}$ defined and coherent.

Theorem 3.7 now shows that a different ordering of I leads to essentially the same $y_J, g_{J,K}, e_{J,K,L}$ whenever there is overlap in the range of definitions. These are now defined for all disjoint $J,K,L \subseteq I$.

For disjoint $J,K \subseteq I$ we have defined cobordisms

$$C = \{x_J \,;\, x_{J,L}, -x_{J,K}, 1, 1\}$$

$$C' = \{x_K \,;\, -x_{J,K}, x_{K,L}, g_{J,K}, 1\}$$

$$C'' = \{x_{J\cup K} \,;\, x_{J,L}, x_{K,L}, ?, ?\}$$

such that $C * C' = C''$ (by construction).

Theorem $\begin{cases} 1.4 \\ 2.4 \end{cases}$ now shows how to consider $x_J, x_K \subseteq x_{J\cup K}$ as orthogonal $\begin{cases} \text{subforms} \\ \text{subformations} \end{cases}$ satisfying the condition for $\{x_\phi, x_J, x_K, x_{J\cup K}\}$ to be a connected lattice of $\begin{cases} \pm \text{ form} \\ \pm \text{ formations} \end{cases}$, with x_ϕ $\begin{cases} \text{the zero } \pm \text{ form} \\ \text{a trivial } \pm \text{ formation} \end{cases}$, where $x_{J\cup K}$ ha

been also used to denote a (stable) isomorph. Therefore $W = \{x_J\}_{J\subseteq I}$ is a connected lattice.

Isomorphisms

$$f_i : y_i \longrightarrow \partial^i \tilde{w} \quad (i \in I)$$

now follow on noting that

$$\partial^i \tilde{W} = \{x_i \,;\, \{x_{i,J}\}_{J\subseteq I_\infty - \{i\}}\} \quad (i \in I)$$

$$(= y_i)$$

from the construction of Lemma 4.1.

The coherence of

$$g_{j,k} : -\partial^k y_i \longrightarrow \partial^j y_k \quad (j,k \in I \quad j \neq k)$$

$$\tilde{g}_{j,k} : -\partial^k y_j \xrightarrow{-\partial^k f_j} -\partial^k \partial^j \tilde{w} \xrightarrow{\tau^{jk}} \partial^j \partial^k \tilde{w} \xrightarrow{\partial^j f_k^{-1}} \partial^j y_k$$

follows from the identity

$$y_{\{j,k\}} = y_j *_{g_{j,k}} y_k = \partial^{\{j,k\}} \tilde{w}$$

$\square$

Let $i\in I=\{0,1,\dots,n\}$, $n\geqslant 3$.

Suppose given a collection

$$s=\left\{\left(y_J,\ h_{J,k}: y_{J-\{\omega_J(k)\}}\longrightarrow \partial_k y_J\right) \,\middle|\, \begin{matrix} J\subseteq I\ \ J\neq I, I-\{i\} \\ 0\leqslant k<|J| \end{matrix}\right\}$$

of oriented highly connected m-ads y_J of fibre dimension f, and isomorphisms $h_{J,k}$, such that

$$s_j=\{(y_{\partial_j J},\ h_{\partial_j J,\partial_j k})\}_{J\subseteq\{0,1,\dots,n-1\},\,0\leqslant k<|J|}\in \mathcal{L}_f(A)^{(n-1)}$$

$$(j\in I-\{i\})$$

Define isomorphisms of $(n-2)$-ads

$$g_{j,k}:\partial_{k-1}y_{I-\{j\}}\longrightarrow \partial_j y_{I-\{k\}}\quad \begin{pmatrix} j,k\in I-\{i\} \\ j<k \end{pmatrix}$$

by requiring the commutativity of

$$\begin{tikzcd}
 & \partial_{k-1}y_{I-\{j\}} \arrow[d, "g_{j,k}"] \\
y_{I-\{j,k\}} \arrow[ur, "h_{I-\{j\},k-1}"] \arrow[r, "h_{I-\{k\},j}"'] & \partial_j y_{I-\{k\}}
\end{tikzcd}$$

The diagrams

$$\begin{tikzcd}
 & & \partial_{l-2}\partial_{k-1}y_{I-\{j\}} \arrow[d, "\partial_{l-2}g_{j,k}"] \arrow[dddddd, bend left=60, "\begin{matrix} e_{j,k,l} \\ (\mathrm{defn.}) \end{matrix}"] \\
 & \partial_{l-2}y_{I-\{j,k\}} \arrow[ur, "\partial_{l-2}h_{I-\{j\},k-1}"] \arrow[r, "\partial_{l-2}h_{I-\{k\},j}"'] & \partial_{l-2}\partial_j y_{I-\{k\}} \arrow[dd, "\tau_{j,l-1}"] \\
 & & \\
y_{I-\{j,k,l\}} \arrow[uur, "h_{I-\{j,k\},l-2}"] \arrow[ddr, "h_{I-\{k,l\},j}"'] & & \partial_j\partial_{l-1}y_{I-\{k\}} \arrow[dd, "\partial_j g_{k,l}"] \\
 & & \\
 & \partial_j y_{I-\{k,l\}} \arrow[uur, "\partial_j h_{I-\{k\},l-1}"] \arrow[r, "\partial_j h_{I-\{l\},k}"'] & \partial_j\partial_k y_{I-\{l\}}
\end{tikzcd}$$

$j,k,l\in I-\{i\}$

$j<k<l$

$$\begin{tikzcd}
 & & \partial_{l-2}\partial_{k-1}y_{I-\{j\}} \arrow[dd, "\tau_{k-1,l-1}"] \arrow[dddddd, bend left=60, "\begin{matrix} \tilde{e}_{j,k,l} \\ (\mathrm{defn.}) \end{matrix}"] \\
 & \partial_{l-2}y_{I-\{j,k\}} \arrow[ur, "\partial_{l-2}h_{I-\{j\},k-1}"] & \\
 & & \partial_{k-1}\partial_{l-1}y_{I-\{j\}} \arrow[dd, "\partial_{k-1}g_{j,l}"] \\
y_{I-\{j,k,l\}} \arrow[uur, "h_{I-\{j,k\},l-2}"] \arrow[r, "h_{I-\{j,l\},k-1}"'] \arrow[ddr, "h_{I-\{k,l\},j}"'] & \partial_{k-1}y_{I-\{j,l\}} \arrow[ur, "\partial_{k-1}h_{I-\{j\},l-1}"] \arrow[dr, "\partial_{k-1}h_{I-\{l\},j}"'] & \\
 & & \partial_{k-1}\partial_j y_{I-\{l\}} \arrow[dd, "\tau_{j,k}"] \\
 & \partial_j y_{I-\{k,l\}} \arrow[dr, "\partial_j h_{I-\{l\},k}"'] & \\
 & & \partial_j\partial_k y_{I-\{l\}}
\end{tikzcd}$$

commute up to coherence, showing that

$$e_{j,k,l}\,,\tilde{e}_{j,k,l} : \partial_{l-2}\partial_{k-1}\, y_{I-\{j\}} \longrightarrow \partial_j\partial_k\, y_{I-\{i\}}$$

are well-defined coherent isomorphisms.

Therefore we can apply Theorem 4.3, to obtain a connected lattice $W = \{W_J\}_{J\subseteq I-\{i\}}$, and isomorphisms

$$f_j : y_{I-\{j\}} \longrightarrow \partial_j\tilde{W} \quad (j\in I-\{i\})$$

(setting $\infty = i$), such that the composite

$$\tilde{g}_{j,k} : \partial_{k-1}y_{I-\{j\}} \xrightarrow{\partial_{k-1}f_j} \partial_{k-1}\partial_j\tilde{W} \xrightarrow{\tau_{j,k}} \partial_j\partial_k\tilde{W} \xrightarrow{\partial_j f_k^{-1}} \partial_j\, y_{I-\{k\}}$$

is coherent with $g_{j,k}$ (for $j,k\in I-\{i\}$ with $j<k$).

Setting

$$y_{\{0,1,\dots,n\}} = \tilde{W} \qquad y_{\{0,1,\dots,n\}-\{i\}} = \partial_i\tilde{W}$$

$$h_{\{0,1,\dots,n\},i} = 1 : y_{\{0,1,\dots,n\}-\{i\}} \longrightarrow \partial_i\tilde{W}$$

$$h_{\{0,1,\dots,n\},j} = f_j : y_{\{0,1,\dots,n\}-\{j\}} \longrightarrow \partial_j\tilde{W} \quad (j\in I-\{i\})$$

we have defined an n-simplex

$$\bar{s} = \{(y_J, h_{J,k})\}_{J\subseteq I,\,0\leq k<|J|} \in \mathcal{L}_f(A)^{(n)}$$

containing the horn s.

This completes the verification of the Kan condition.

We can now consider the homotopy theory of $\mathcal{L}_f(A)$.

- The homotopy groups are defined, as usual, by

$$\pi_n(\mathcal{L}_f(A)) = \{s\in\mathcal{L}_f(A)^{(n)} \mid \partial_i s = 0 \in \mathcal{L}_f(A)^{(n-1)},\, 0\leq i\leq n\}/\simeq \quad (n\geq 1)$$

where

$$s\simeq s' \iff \exists\, t\in\mathcal{L}_f(A)^{(n+1)} \text{ s.t. } \partial_i t = \begin{cases} s & i=0\\ s' & i=1 \\ 0 & \text{otherwise}\end{cases}$$

with the group law given by

$$[s] * [s'] = [s''] \in \pi_n(\mathcal{L}_f(A))$$

$$\iff \exists\, t\in\mathcal{L}_f(A) \text{ s.t. } \partial_i t = \begin{cases} s & i=0\\ s'' & i=1\\ s' & i=2\\ 0 & \text{otherwise}\end{cases}$$

Recalling the definition of the groups $U_*(A)$ from §3 of I, we have:

Theorem 4.4 $\pi_n(\mathcal{L}_f(A)) = U_{n+f}(A) \quad (n \geqslant 0)$.

Proof: The case $n=0$ was considered on III 4.7, so assume $n \geqslant 1$, and let $I = \{0,1,\ldots,n\}$.

An n-simplex

$$s = \{(y_J, h_{J,k} : y_{J-\{\omega_J(k)\}} \longrightarrow \partial_k y_J) \,|\, J \subseteq I,\ 0 \leqslant k < |J|\} \in \mathcal{L}_f(A)^{(n)}$$

such that $\partial_k s = 0 \in \mathcal{L}_f(A)^{(n-1)}$ for all $k \in I$, is determined by an oriented highly connected n-ad of fibre dimension f

$$y \equiv y_I = \{x;\{x_J\}_{J \subseteq I}\}$$

and $(n-1)$-ad isomorphisms

$$h_k \equiv h_{I,k} : 0 \longrightarrow \partial_k y \quad (k \in I).$$

Each

$$\partial^j y = \{x_j;\{x_{j,K}\}_{K \subseteq I-\{j\}}\} \quad (j \in I)$$

is isomorphic to the zero $(n-1)$-ad of $\left\{\begin{matrix}\mp \text{ forms} \\ \mp \text{ formations}\end{matrix}\right.$, if $n+f = \left\{\begin{matrix}2i \\ 2i+1\end{matrix}\right.$, $\pm = (-)^i$. Thus each x_j is $\left\{\begin{matrix}\text{a trivial } \mp \text{ formation} \\ \text{the zero } \pm \text{ form}\end{matrix}\right.$, with each $x_{j,k}$ the zero $\left\{\begin{matrix}\mp \text{ form} \\ \mp \text{ formation}\end{matrix}\right.$. It now follows from Theorem 3.4 that each x_J is $\left\{\begin{matrix}\text{a trivial } \mp \text{ formation} \\ \text{the zero } \pm \text{ form}\end{matrix}\right.$.

In particular, this is true of x_I, an isomorph of ∂x, so that x is a non-singular $\left\{\begin{matrix}\pm \text{ form} \\ \pm \text{ formation}\end{matrix}\right.$.

If $n+f = 2i$, $s \in \bigcap_{k \in I} \ker(\partial_k : \mathcal{L}_f(A)^{(n)} \longrightarrow \mathcal{L}_f(A)^{(n-1)})$ is determined not only by the non-singular $\pm$ form x, but also by the $(n-1)$-ad isomorphisms $h_k : 0 \longrightarrow \partial_k y$ $(k \in I)$. We shall now show that in fact this extra structure does not enter the homotopy groups.

Note first that given any trivial $\mp$ formation $(F,(\binom{\gamma}{\mu}),\theta)G)$ there is defined an isomorphism

$$(\mu^*, 1, 0) : (F,(\binom{\gamma}{\mu}),\theta)G) \longrightarrow (G^*,(\binom{\theta^* \pm \theta}{1}),\theta)G)$$

of $\mp$ formations.

Given a stable isomorphism of trivial $\mp$ formations

$$[\alpha,\beta,\psi] : (F,(\binom{\gamma}{\mu}),\theta)G) \longrightarrow (F',(\binom{\gamma'}{\mu'}),\theta')G')$$

we can characterize the coherence class of the composite

$$(G^*,((\binom{\theta^*\pm\theta}{1}),\theta)G) \xrightarrow{(\mu^{*-1},1,0)} (F,(\binom{\gamma}{\mu}),\theta)G)$$

$$\xrightarrow{[\alpha,\beta,\psi]} (F',(\binom{\gamma'}{\mu'}),\theta')G') \xrightarrow{(\mu'^*,1,0)} (G'^*,((\binom{\theta'^*\pm\theta'}{1}),\theta')G')$$

as follows. Let

$$(\alpha,\beta,\psi) = \left(\begin{pmatrix} a & a_1 \\ ? & ? \end{pmatrix}, \begin{pmatrix} b & b_1 \\ ? & ? \end{pmatrix}, \begin{pmatrix} s & ? \\ ? & ? \end{pmatrix}\right)$$

$$: (F,(\binom{\gamma}{\mu}),\theta)G)\oplus(P,P^*) \longrightarrow (F',(\binom{\gamma'}{\mu'}),\theta')G')\oplus(P',P'^*)$$

be any representative of $[\alpha,\beta,\psi]$ (i.e. use the notation of the proof of Theorem 2.1). Then the normalization of the composite above is given by

$$\left(\begin{pmatrix} \mu'^* a \mu^{*-1} & \mu'^* a_1 b_1^* \\ 1 & -b^* \end{pmatrix}, \begin{pmatrix} b & 1 \\ 1-b'b & -b' \end{pmatrix}, \begin{pmatrix} \mu^* s \mu & \pm\mu^* a' \mu'^{*-1}(\theta'^*\pm\theta') \\ 0 & \theta' \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} b'^* & 1-b'^* b^* \\ 1 & -b^* \end{pmatrix}, \begin{pmatrix} b & 1 \\ 1-b'b & -b' \end{pmatrix}, \begin{pmatrix} b^*\theta' b-\theta & b^*(\theta'\pm\theta'^*) \\ 0 & \theta' \end{pmatrix}\right)$$

$$: (G^*,((\binom{\theta^*\pm\theta}{1}),\theta)G)\oplus(G'^*,G')$$

$$\longrightarrow (G'^*,((\binom{\theta'^*\pm\theta'}{1}),\theta')G')\oplus(G^*,G)$$

This is just the isomorphism (and, indeed, the normalization of) given by Theorem 2.3, when applied to the orthogonal subforms

$$(G,-\theta) \underset{((\binom{1}{0}),0)}{\subseteq} (G\oplus G',\begin{pmatrix} -\theta & 0 \\ 0 & \theta' \end{pmatrix}), \quad (G',\theta') \underset{((\binom{0}{1}),0)}{\subseteq} (G\oplus G',\begin{pmatrix} -\theta & 0 \\ 0 & \theta' \end{pmatrix})$$

with the choice of direct complements given by the configuration

$$G\oplus G' \underset{\binom{e}{g'}=\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}}{\overset{(h\ f')=(\binom{1}{b}\ \binom{0}{1})}{\rightleftarrows}} G\oplus G' \qquad G'\oplus G \underset{\binom{e'}{g}=\begin{pmatrix} 0 & 1 \\ 1 & -b' \end{pmatrix}}{\overset{(h'\ f)=(\binom{b'}{1}\ \binom{1}{0})}{\rightleftarrows}} G\oplus G'$$

Returning to the problem in hand, suppose given two simplexes

$$s, s' \in \bigcap_{k \in I} \ker(\partial_k : \mathcal{L}_f(A)^{(n)} \longrightarrow \mathcal{L}_f(A)^{(n-1)})$$

with s (resp. s') defined by the n-ad

$$y = \{(G,\theta)\,;\,\{(F_J, ((\gamma_J, \mu_J), \theta)G)\}_{J \subseteq I}\}$$

(resp. $y' = \{(G',\theta')\,;\,\{(F'_J, ((\gamma'_J, \mu'_J), \theta')G')\}_{J \subseteq I}\}$)

and $(n-1)$-ad isomorphisms $h_k : 0 \longrightarrow \partial_k y$ (resp. $h'_k : 0 \longrightarrow \partial_k y'$) $(k \in I)$, with the non-singular $\pm$ forms (G,θ), (G',θ') isomorphic.

For any isomorphism $(f,\chi) : (G,\theta) \longrightarrow (G',\theta')$,

$$\mathrm{im}\left(\left(\begin{pmatrix}1\\ f\end{pmatrix}, \chi\right) : (G,0) \longrightarrow \left(G \oplus G', \begin{pmatrix}-\theta & 0\\ 0 & \theta'\end{pmatrix}\right)\right)$$

defines a lagrangian of $(G,-\theta) \oplus (G',\theta')$, which is therefore a trivial $\pm$ form, isomorphic to $H_\pm(G)$. We can therefore define an

$(n+1)$-ad of $\pm$ forms

$$z = \{(G,0)\,;\,\{(Q_J, \phi_J)\}_{J \subseteq I_\infty}\} \quad (I_\infty = \{0,1,\ldots,n+1\})$$

by

$$(Q_J, \phi_J) = \begin{cases} 0 & 0,1 \notin J \\ (G,-\theta) & 0 \notin J \quad 1 \in J \\ (G',\theta') & 0 \in J \quad 1 \notin J \\ \left(G \oplus G', \begin{pmatrix}-\theta & 0\\ 0 & \theta'\end{pmatrix}\right) & 0,1 \in J \end{cases} \qquad (J \subseteq I_\infty)$$

with

$$(f^J_{J'}, \chi^J_{J'}) = \begin{cases} 0 & 0,1 \notin J' \\ \left(\begin{pmatrix}1\\0\end{pmatrix}, 0\right) : (G,-\theta) \longrightarrow \left(G \oplus G', \begin{pmatrix}-\theta & 0\\ 0 & \theta'\end{pmatrix}\right) & 0 \in J'-J \quad 1 \in J \\ \left(\begin{pmatrix}0\\1\end{pmatrix}, 0\right) : (G',\theta') \longrightarrow \left(G \oplus G', \begin{pmatrix}-\theta & 0\\ 0 & \theta'\end{pmatrix}\right) & 0 \in J \quad 1 \in J'-J \\ 1 & 0,1 \in J \end{cases}$$

$$(J \subseteq J' \subseteq I_\infty)$$

Defining a collection of oriented m-ads and isomorphisms

$$t = \{(y_J, h_{J,k} : y_{J-\{\omega_J(k)\}} \longrightarrow \partial_k y_J) \,|\, J \subseteq I_\infty, 0 \leq k < |J|\}$$

by

$$y_J = \begin{cases} z & J = I_\infty \\ y' & J = I_\infty - \{0\} \\ y & J = I_\infty - \{1\} \\ 0 & \text{otherwise} \end{cases}$$

$$h_{J,k} = \begin{cases} \{(1,0);\{\mu_k'^*\}_{k\in I}\} : y' \longrightarrow \partial_0 z & J = I_\infty \quad k=0 \\ \{(1,0);\{\mu_k^*\}_{k\in I}\} : y \longrightarrow \partial_1 z & J = I_\infty \quad k=1 \\ h'_k : 0 \longrightarrow \partial_k y' & J = I_\infty - \{0\} \quad k \in I \\ h_k : 0 \longrightarrow \partial_k y & J = I_\infty - \{1\} \quad k \in I \\ 0 & \text{otherwise} \end{cases}$$

note that the diagrams

$$\begin{tikzcd} & \partial_{k-1} y_{J-\{\omega_J(j)\}} \arrow[r, "\partial_{k-1} h_{J,j}"] & \partial_{k-1}\partial_j y_J \arrow[d, "\tau_{j,k}"] \\ y_{J-\{\omega_J(j),\omega_J(k)\}} \arrow[ur, "h_{J-\{\omega_J(j)\},k-1}"] \arrow[dr, "h_{J-\{\omega_J(k)\},j}"'] & & \\ & \partial_j y_{J-\{\omega_J(k)\}} \arrow[r, "\partial_j h_{J,k}"'] & \partial_j \partial_k y_J \end{tikzcd}$$

$(0 \leq j < k < |J|,\ J \subseteq I_\infty)$

commute up to coherence: certainly, this

is clear when $J \neq I_\infty$. For $J = I_\infty$, it follows from the remarks made above, which ensure that for each j,k there is an appropriate choice of direct complement in the definition of $\tau_{j,k}$ (as in the proof of Theorem 3.4, and hence Theorem 2.3) to make the corresponding diagram commute up to coherence.

Now $t \in \bigcap_{k \in I_\infty - \{0,1\}} \ker(\partial_k : \mathcal{L}_f(A)^{(n+1)} \longrightarrow \mathcal{L}_f(A)^{(n)})$

defines (an $(n+1)$-simplex which is) a homotopy between s and s', so that

$$[s] = [s'] \in \pi_n(\mathcal{L}_f(A))$$

and it is only the (isomorphism class) of the $\pm$ form (G,θ) which matters.

Corollary 1.2 now shows that

$$\pi_n(\mathcal{L}_f(A)) = U_{2i}(A)$$

as sets: the group laws coincide by Lemma 1.5 and Theorem 3.4.

If $n+f = 2i+1$, $s \in \bigcap_{k\in I} \ker(\partial_k : \mathcal{L}_f(A)^{(n)} \to \mathcal{L}_f(A)^{(n-1)})$ is determined solely by the non-singular $\pm$ formation x. Corollary 2.2 shows that

$$\pi_n(\mathcal{L}_f(A)) = U_{2i+1}(A)$$

as sets: the group laws coincide by Lemma 2.5 and Theorem 3.4.

$\square$

Given a Kan Δ-set $\mathcal{K}$ we can define its "loop space" $\Omega\mathcal{K}$ to be the Kan Δ-set given by

$$(\Omega\mathcal{K})^{(n)} = \{s \in \mathcal{K}^{(n+1)} \mid \partial_{n+1}s = 0 \in \mathcal{K}^{(n)}\} \quad (n \geq 0)$$

$$\partial_i : (\Omega\mathcal{K})^{(n)} \to (\Omega\mathcal{K})^{(n-1)}\ ;\ s \mapsto \partial_i s \quad (0 \leq i \leq n,\ n \geq 1)$$

Then

$$\pi_n(\Omega\mathcal{K}) = \pi_{n+1}(\mathcal{K}),$$

as usual.

<u>Corollary 4.5</u> There are defined homotopy equivalences

$$\mathcal{L}_{f+1}(A) \longrightarrow \Omega\mathcal{L}_f(A) \quad (f \pmod 4)$$

in a natural fashion.

<u>Proof</u>: Given an oriented n-ad of fibre dimension f

$$y = \{x\,;\,\{x_J\}_{J\subseteq I}\} \quad (I = \{0,1,\ldots,n\})$$

we shall define an oriented $(n+1)$-ad of fibre dimension $f-1$

$$y_\infty = \{x\,;\,\{x_J\}_{J\subseteq I_\infty}\} \quad (I_\infty = \{0,1,\ldots,n+1\})$$

by "introducing an extra vertex", such that

$$\partial_k y_\infty = (\partial_k y)_\infty \quad (0 \leq k \leq n)$$

there is defined an isomorphism of n-ads

$$\omega_{n+1} : 0 \longrightarrow \partial_{n+1} y_\infty .$$

The operation will be natural in that an isomorphism of oriented n-ads

$$h : y \longrightarrow y'$$

will induce an isomorphism

$$h_\infty : y_\infty \longrightarrow y'_\infty$$

of oriented (n+1)-ads.

If

$$y = \{(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G) \,;\, \{Q_J,\phi_J\}_{J\subseteq I}\}$$

is an n-ad of $\pm$ forms, let

$$y_\infty = \{(F,((\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}),\theta)G) \,;\, \{Q_J,\phi_J\}_{J\subseteq I_\infty}\} ,$$

extending the definitions by setting

$$(Q_{J_\infty},\phi_{J_\infty}) = (Q_J,\phi_J) \quad (J\subseteq I,\ J_\infty = J\cup\{n+1\})$$

$$(f^J_{J'_\infty},\chi^J_{J'_\infty}) = (f^{J_\infty}_{J'_\infty},\chi^{J_\infty}_{J'_\infty}) = (f^J_{J'},\chi^J_{J'}) : (Q_J,\phi_J) \longrightarrow (Q_{J'},\phi_{J'})$$

$$(J\subset J'\subseteq I)$$

The definitions of ω, h_∞ are obvious

If

$$y = \{(G,\theta) \,;\, \{(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G)\}_{J\subseteq I}\}$$

is an n-ad of $\pm$ formations, let

$$y_\infty = \{(G,\theta) \,;\, \{(F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G)\}_{J\subseteq I_\infty}\}$$

extending the definitions by setting

$$(F_{J_\infty},((\begin{smallmatrix}\gamma_{J_\infty}\\ \mu_{J_\infty}\end{smallmatrix}),\theta)G) = (F_J,((\begin{smallmatrix}\gamma_J\\ \mu_J\end{smallmatrix}),\theta)G) \quad (J\subseteq I)$$

$$\lambda^{J'_\infty}_J = \lambda^{J'_\infty}_{J_\infty} = \lambda^{J'}_J \in \mathrm{Hom}_A(F_{J'},F_J) \quad (J\subseteq J'\subseteq I)$$

$$\nu^{J_\infty}_K = \nu^J_{K_\infty} = \nu^J_K \in \mathrm{Hom}_A(F_J^*,F_K) \quad (J,K\subseteq I \text{ disjoint})$$

Then

$$\partial_{n+1} y_\infty = \{(F_\infty,((\begin{smallmatrix}\gamma_\infty\\ \mu_\infty\end{smallmatrix}),\theta)G) \,;\, 0\}$$

$$= \{(G^*,((\begin{smallmatrix}\theta^*\mp\theta\\ 1\end{smallmatrix}),\theta)G) \,;\, 0\}$$

and

$$\omega = \{[(1,1,\theta) : (G^*,G) \longrightarrow (G^*,((\begin{smallmatrix}\theta^*\mp\theta\\ 1\end{smallmatrix}),\theta)G)] \,,\, 0\}$$

$$: 0 \longrightarrow \partial_{n+1} y_\infty$$

defines an isomorphism of n-ads.

The definition of h_∞ is obvious.

Define a Δ-map

$$\mathcal{L}_{f+1}(A) \longrightarrow \Omega\mathcal{L}_f(A)$$

for each $f \pmod 4$ by setting

$$\mathcal{L}_{f+1}(A)^{(n)} \longrightarrow (\Omega\mathcal{L}_f(A))^{(n)} ;$$

$$\{(y_J, h_{J,k} : y_{J-\{\omega_J(k)\}} \longrightarrow \partial_k y_J) \mid J \subseteq I, 0 \leqslant k < |J|\}$$

$$\longrightarrow \{(\tilde{y}_{\tilde{J}}, \tilde{h}_{\tilde{J},\tilde{k}} : \tilde{y}_{\tilde{J}-\{\omega_{\tilde{J}}(\tilde{k})\}} \longrightarrow \partial_{\tilde{k}} y_{\tilde{J}}) \mid \tilde{J} \subseteq I_\infty, 0 \leqslant \tilde{k} < |\tilde{J}|\}$$

with

$$\tilde{y}_{\tilde{J}} = \begin{cases} (y_J)_\infty & \text{if } \tilde{J} = J_\infty, J \subseteq I \text{ non-empty} \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{h}_{\tilde{J},\tilde{k}} = \begin{cases} (h_{J,k} : y_{J-\{\omega_J(k)\}} \longrightarrow \partial_k y_J)_\infty & \text{if } \tilde{J} = J_\infty, J \subseteq I\ |J| \geqslant 2,\ \tilde{k} = k, 0 \leqslant k < |J| \\ \omega_{|J|} : 0 \longrightarrow \partial_{|J|}(y_J)_\infty & \text{if } \tilde{J} = J_\infty, J \subseteq I\ |J| \geqslant 2,\ \tilde{k} = |J| \\ 0 & \text{otherwise} \end{cases}$$

Applying Theorem 4.4 it should be clear that these Δ-maps induce the identity isomorphisms

$$U_{n+f+1}(A) \longrightarrow U_{n+f+1}(A)$$

in the homotopy groups

$$\pi_n(\mathcal{L}_{f+1}(A)) \longrightarrow \pi_n(\Omega\mathcal{L}_f(A)).$$

The Kan Δ-set analogue of Whitehead's theorem (appearing as Theorem 6.6 in Rourke & Sanderson "Δ-sets I : Homotopy theory" Quart. J. Math. Oxford (2), 22 (1971), 321-338) now shows that we are dealing with homotopy equivalences

$$\mathcal{L}_{f+1}(A) \longrightarrow \Omega\mathcal{L}_f(A).$$

Naturality (w.r.t. A) obvious. $\square$

Iterating four times, we have natural homotopy equivalences

$$\Omega^4\mathcal{L}_f(A) \simeq \mathcal{L}_f(A) \quad (f \pmod 4)$$

giving a purely algebraic interpretation to the

Given a Δ-map of Kan Δ-sets

$$h : \mathcal{K} \longrightarrow \mathcal{L}$$

we can define its "path space" to be the Kan Δ-set $\mathcal{P}_h$ with

$$\mathcal{P}_h^{(n)} = \{(x,y) \in \mathcal{K}^{(n)} \times \mathcal{L}^{(n+1)} \mid fx = \partial_{n+1} y \in \mathcal{L}^{(n)}\} \quad (n \geq 0)$$

$$\partial_i : \mathcal{P}_h^{(n)} \longrightarrow \mathcal{P}_h^{(n-1)} \; ; \; (x,y) \longmapsto (\partial_i x, \partial_i y) \quad (0 \leq i \leq n).$$

Defining Δ-maps

$$\alpha : \mathcal{P}_h \longrightarrow \mathcal{K} \quad , \beta : \Omega\mathcal{L} \longrightarrow \mathcal{P}_h$$

by

$$\alpha : \mathcal{P}_h^{(n)} \longrightarrow \mathcal{K}^{(n)} \; ; \; (x,y) \longmapsto x$$

$$\beta : \Omega\mathcal{L}^{(n)} \longrightarrow \mathcal{P}_h^{(n)} \; ; \; z \longmapsto (0,z) \quad (z \in \mathcal{L}^{(n+1)}, \partial_{n+1} z = 0 \in \mathcal{L}^{(n)})$$

we have a homotopy exact sequence

$$\ldots \longrightarrow \pi_n(\mathcal{P}_h) \xrightarrow{\alpha} \pi_n(\mathcal{K}) \xrightarrow{h} \pi_n(\mathcal{L}) \xrightarrow{\beta} \pi_{n-1}(\mathcal{P}_h) \xrightarrow{\alpha} \ldots$$

as usual.

Corollary 4.6 Let

$$h : A \longrightarrow B$$

be a 1-preserving morphism of rings with involution. There is defined a Kan Δ-set $\mathcal{L}_f(h)$ for each $f \pmod 4$ with $\Omega\mathcal{L}_f(h) \simeq \mathcal{L}_{f+1}(h)$, and Δ-maps

$$\alpha : \mathcal{L}_f(h) \longrightarrow \mathcal{L}_f(A) \qquad \beta : \Omega\mathcal{L}_f(B) \longrightarrow \mathcal{L}_f(h)$$

inducing homotopy exact sequences

$$\ldots \xrightarrow{\alpha} U_{n+f}(A) \xrightarrow{h} U_{n+f}(B) \xrightarrow{\beta} U_{n+f}(h) \xrightarrow{\alpha} U_{n+f-1}(A) \longrightarrow \ldots$$

where $\pi_n(\mathcal{L}_f(h)) = U_{n+f+1}(h)$ (definition).

Proof. The morphism

$$h : A \longrightarrow B$$

induces a Δ-map

$$h : \mathcal{L}_f(A) \longrightarrow \mathcal{L}_f(B) \quad (f \pmod 4)$$

generalizing the definitions of $\S 6$ in the obvious way in I.

Let $\mathcal{L}_f(h)$ be the path space of this Δ-map, and apply Theorem 4.4 & Corollary 4.5

(Alternatively, we could have defined the relative U-groups by generalizing the formulation of Theorem 3.1 of I. Given a morphism of ground rings

$$h : A \longrightarrow B$$

let $X_n(h)$ be the abelian monoid of isomorphism classes of pairs (x,c) with x a $\begin{cases} \mp\text{formation} \\ \pm\text{form} \end{cases}$ over A if $n = \begin{cases} 2i \\ 2i+1 \end{cases}$, $\pm = (-)^i$ and $c = \{y\,;z, -hx, g_0, g_1\}$ a cobordism of $\begin{cases} \mp\text{formations} \\ \pm\text{forms} \end{cases}$ over B from hx (alias a 1-ad c with $\partial_1 c = hx$), an isomorphism of such pairs

$$(e,f) : (x,c) \longrightarrow (x',c')$$

being defined by isomorphisms $e : x \to x'$, $f : c \to c'$ such that the square

$$\begin{array}{ccc} hx & \xrightarrow{g_1} & \partial_1 c \\ {\scriptstyle h(e)}\downarrow & & \downarrow{\scriptstyle \partial_1 f} \\ hx' & \xrightarrow[g_1']{} & \partial_1 c' \end{array}$$

commutes up to ch[illegible]

The monoid law is the direct sum

$$(x,c) \oplus (x',c') = (x \oplus x', c \oplus c')$$

with $(0,0)$ as zero. Boundary maps

$$\partial : X_n(h) \longrightarrow X_{n-1}(h)\,;$$

$$(x, c = \{y\,;z, hx, g_0, g_1\}) \longmapsto (\partial x, \{z\,; 0, h\partial x, 0, (\partial g_0^{-1})\tau(\partial g_1)\})$$

such that $\partial^2 = 0$ may be defined, with

$$\tau : \partial\partial_0 c \to \partial\partial_1 c$$

as given by Theorem 3.4.

We can then define an abelian group

$$U_n(h) = \ker(\partial : X_n(h) \longrightarrow X_{n-1}(h)) \big/ \operatorname{im}(\partial : X_{n+1}(h) \longrightarrow X_n(h))$$

The exact sequences

$$\ldots \longrightarrow U_n(A) \xrightarrow{h} U_n(B) \xrightarrow{\beta} U_n(h) \xrightarrow{\alpha} U_{n-1}(A) \xrightarrow{h} \ldots$$

are induced by chain maps

$$\alpha : X_*(h) \longrightarrow SX_*(A) \qquad \alpha : X_n(h) \longrightarrow SX_n(A) \equiv X_{n-1}(A)\,;\ (x,c) \longmapsto x$$

with

$$\beta : X_*(B) \longrightarrow X_*(h) \qquad \beta : X_n(B) \longrightarrow X_n(h)\,;\ x \longmapsto (0\,; \{x\,; \partial x, 0, 1, c\})$$

Of course, the two definitions of $U_n(h)$ coincide.)

By considering stably f.g. free (resp. based) A-modules instead of f.g. projective ones we can define functors

$$\mathcal{L}_f^V \ (\text{resp. } \mathcal{L}_f^W) : \text{rings with involution} \longrightarrow \text{Kan } \Delta\text{-sets}$$

for V- (resp. W-) theory, generalizing the definition of $\mathcal{L}_f^U \equiv \mathcal{L}_f$ for U-theory, with

$$\pi_n(\mathcal{L}_f^V(A)) = V_{n+f}(A) \quad (\text{resp. } \pi_n(\mathcal{L}_f^W(A)) = W_{n+f}(A)).$$

Theorem 4.3 of I. can be regarded as a calculation of

$$\pi_n(\mathcal{L}_f^V(A) \longrightarrow \mathcal{L}_f^U(A)) \quad (= \Sigma_{(-)^{n+f}}(A))$$

where $\mathcal{L}_f^V(A) \longrightarrow \mathcal{L}_f^U(A)$ is the Δ-map defined by the inclusion (stably f.g. free A-modules) $\subseteq$ (f.g. projective A-modules)

Theorem 5.7 of I. calculates

$$\pi_n(\mathcal{L}_f^W(A) \longrightarrow \mathcal{L}_f^V(A)) (= \Omega_{(-)^{n+f}}(A))$$

where $\mathcal{L}_f^W(A) \longrightarrow \mathcal{L}_f^V(A)$ is the Δ-map defined by the forgetful map (based A-modules) $\longrightarrow$ (f.g. free A-modules).

Theorem 1.1 of II calculates

$$\pi_n(\mathcal{L}_f^V(A) \xrightarrow{\overline{\varepsilon}} \mathcal{L}_f^V(A_Z)) \quad (= U_{n+f-1}(A))$$

$$\pi_n(\mathcal{L}_f^W(A) \xrightarrow{\overline{\varepsilon}} \mathcal{L}_f^W(A_Z)) \quad (= V_{n+f-1}(A)).$$

An algebraic formulation of surgery

by A. A. Ranicki

Summary

The subject of my dissertation is some pure algebra to which I was led by a study of the invariants of manifolds, which are higher-dimensional analogues of curves and curved surfaces. The surgery referred to in the title is the name of a recently developed technique of cutting up manifolds. In order to make the abstractness of the work more palatable, it should be noted that it was mathematical physics which first motivated mathematicians towards this kind of geometry.

The differential calculus was invented by Newton (ca 1666) as a mathematical

tool with which to handle equations of motion, particularly those of the planets. The formulation of the equations was then generalized and perfected by Lagrange (1787) and Hamilton (1835). It turned out that the set of equations governing the motion of a dynamical system with more than two bodies does not admit a complete solution: while every set of initial conditions does determine a solution, it is not in general possible to work out that solution, except approximately, by numerical methods. Poincaré (1880) developed new ways of obtaining information about the orbits which do not require a complete solution - trying to answer general questions such as

will the planets ever collide?

rather than particular ones such as

where will the planets be tomorrow at noon?

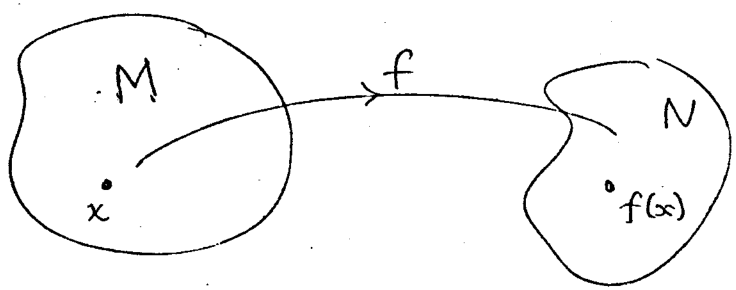
Instead of attempting a quantitative solution

for a particular set of initial data, he described some of the qualitative features of all the solutions for all possible initial data. Essential use was made of the concept of an n-dimensional manifold introduced earlier by Riemann (1854). Supposing that the initial data are given by n numbers (for a least such n), it is possible to consider the corresponding solution as a point of a geometrical object (the manifold in question) in which there is a notion of distance in the neighbourhood of each point, a small change in the initial data leading to only a small shift along the manifold. For example, a 1-dimensional manifold is just a curve, and a 2-dimensional manifold is a curved surface. Poincaré went on (in ca. 1895) to study manifolds for their own sake, thus founding the branch of pure

There are far too many manifolds for it to be feasible to describe them all. As indicated above, this would be the same as asking for the solution of all differential equations – an impossible task. The best one can hope for is a classification according to some scheme under which manifolds sharing certain properties are grouped together. For the purposes of topology two manifolds are held to be the same if they are homeomorphic, that is if one can be distorted into the other by a stretching-cum-compression, but without tearing, and in such a way as to keep distinct points apart – "rubber-sheet geometry" is the traditional description of the subject. For example, any two closed curves (without intersections) are homeomorphic:

as is clear to anyone who has played with a rubber band

A more mathematical description of when two manifolds are homeomorphic runs as follows: a <u>map</u> f from a manifold M to another such, N, is a transformation which assigns to each point x of M its <u>image</u>, $f(x)$, a point of N, a small change in x leading to only a small change in $f(x)$. It is denoted by $f: M \to N$, and may be pictured as



Given maps $f: M \to N$, $g: N \to P$ there is defined a map

$$gf: M \to P$$

by sending each point x of M to the image $g(f(x))$ in P of its image $f(x)$ in N.

Two manifolds M, N are homeomorphic precisely when they are related by maps

$$f: M \to N, \quad g: N \to M$$

such that

$$gf = 1_M : M \to M, \quad fg = 1_N : N \to N$$

where

$$1_M : M \to M$$

is the map from M to M taking each point back to itself, and similarly for $1_N : N \to N$.

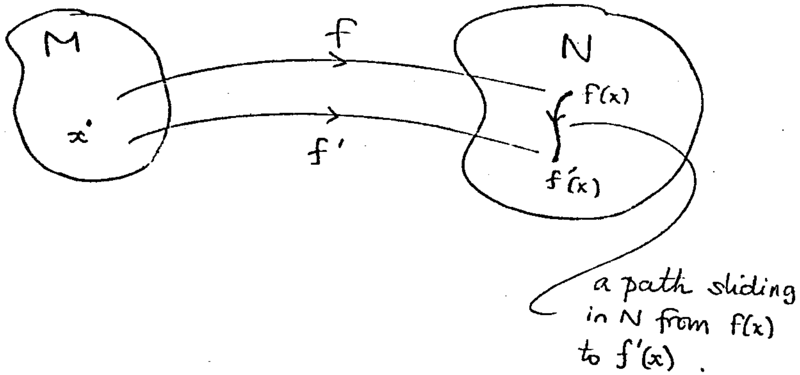
Even when one identifies homeomorphic manifolds, there are still very many of them. It is possible to classify (up to homeomorphism) only 1- and 2-dimensional manifolds. The requirements have to be reduced yet further.

Call two maps

$$f: M \to N, \quad f': M \to N$$

<u>homotopic</u> if it is possible to slide along N from the image $f(x)$ to the image $f'(x)$ of each point

x of M in such a way that the path chosen varies only a little for small changes of x :



Homotopic maps are denoted by

$$f \simeq f' : M \longrightarrow N .$$

Two manifolds are <u>homotopic</u> when they are related by maps (called <u>homotopy equivalences</u>)

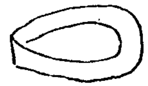
$$f : M \longrightarrow N \quad , \quad g : N \longrightarrow M$$

such that

$$gf \simeq 1_M : M \longrightarrow M \quad , \quad fg \simeq 1_N : N \longrightarrow N .$$

Less mathematically, two manifolds are homotopic when one can be distorted into the other

by a deformation which may be more violent than that allowed for a homeomorphism : it is no longer required that distinct points be kept apart. For example, the Möbius band



(obtained by introducing a twist into an ordinary band) is homotopic to a circle

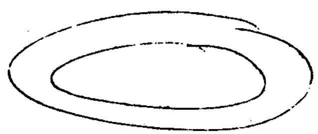
by squeezing it down to one.

Homeomorphic manifolds are homotopic : the last example above shows the converse to be false. This is because the Möbius band is a two-dimensional manifold, whereas the circle is one-dimensional : it is a non-trivial theorem of topology, first proved by Brouwer (1911) that an m-dimensional manifold cannot be homeomorphic to an n-dimensional manifold, if m is different from n.

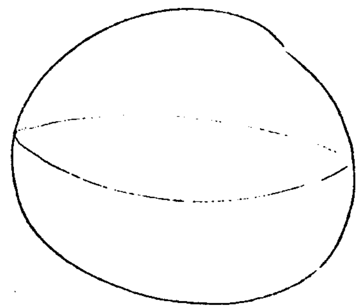
It is the aim of algebraic topology to classify manifolds up to homotopy, by first reducing the problem to algebra, and then doing it.
The first such reduction is due to Poincaré: given a manifold M there is defined its fundamental group, $\pi(M)$, an algebraic entity which measures how complicated M is. It is defined by considering how many different non-homotopic maps

$$(\text{circle}) \longrightarrow M$$

there are. It is a tractable enough quantity to be computable. Homotopic manifolds have the same fundamental groups, so that manifolds with different fundamental groups cannot be homotopic. For example, the surface of a doughnut



is not homotopic to the surface of a sphere



for that reason.
Unfortunately, it is possible to have manifolds with the same fundamental group which are not homotopic.
Carrying on from there, algebraic topologists have produced a host of ever finer such algebraic invariants with which to tackle the homotopy classification of manifolds. Up to 1960 it seemed

that the difficulty of such a classification of n-dimensional manifolds increased with n. In that year came a remarkable discovery of Smale: it is possible to break up manifolds of dimension six and above into <u>handles</u>, which are manifolds of a particularly simple kind. This made the study of high-dimensional manifolds very much easier: the cases $n = 3, 4, 5$ remain difficult. From now on, all manifolds will be assumed to have dimension six and over, and to be "compact", which means not too large, and "framed", which means not too twisted.

<u>Surgery obstruction theory</u> is the investigation of the homotopy properties of manifolds by means of their handle decompositions. It does this by supplying an answer, in every possible case, to the question:

given a map of manifolds of the same dimension

$$f : M \longrightarrow N$$

is it possible to make f into a homotopy equivalence

by performing a sequence of <u>surgeries</u> on M? or (more brutally) can we kill the homotopy of $f : M \longrightarrow N$ by surgery? A surgery on M is the addition or removal of a handle.

A research paper of Kervaire and Milnor (1962) reduced the problem to one in the algebraic theory of quadratic forms, and solved it, in the special case when $\pi(N)$ is trivial. Wall (1965) went on to consider the general case, for any $\pi(N)$. He showed that to answer the question posed it is necessary as well as sufficient to consider the <u>surgery obstruction group</u>, $L_n(\pi)$, which can be defined purely algebraically (also using quadratic forms) with n = dimension of N, and $\pi = \pi(N)$.

Wall gave an account of his surgery obstruction theory and some of its important applications in his book

"Surgery on compact manifolds"

Academic Press (1970).

However, to quote the author,

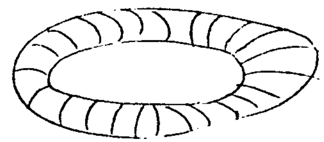
"The algebra in this work, particularly in §6, is complicated, and even so it is not altogether satisfactory: most obviously, in §8."

Further on:

"... an independent algebraic treatment gives a payoff in the topology too: topologically motivated results like (12.6), proved in an algebraic setting, can apply more generally, and lead to new results with a different topological application."

Theorem 12.6 is a description of the groups

$L_n(\pi(N))$ in the special case when N is made up of circles; the surface of a doughnut is such a manifold:



In a recent paper Novikov (1970) proposed an algebraic theory of the L-groups in terms of the formalism of hamiltonian and lagrangian physics, and "algebraic K-theory" (an earlier algebraic offshoot of topology). Unfortunately, Novikov's theory did not quite capture the L-groups: on the other hand, it did give an approximate algebraic proof of a special case of Wall's Theorem 12.6 (including the surface of a doughnut, a type of space which is very important in the topology of manifolds).

Parts I and II of my dissertation arose out of an attempt to understand Novikov's paper. Part I, "Foundations of L-theory", deals with an algebraic theory which does capture the L-groups. Part II, "Algebraic L-theory", goes on to improve Novikov's methods to give an absolutely precise algebraic proof of the case of Wall's Theorem 12.6 he considers.

Part III, "Geometric L-theory", is entirely original. It contains another, more sophisticated, characterization of the L-groups, using the geometrical background for motivation, but purely algebraic methods in execution. Hitherto,

such a characterization had only been possible geometrically, in which form it was first considered in the Princeton PhD thesis "A geometric formulation of surgery" of Quinn (1969). There is a standard technique, "geometrical realization", leading back from my algebra to Quinn's geometry.

The work of Part III resolves the difficulties of §§6, 8 of Wall's book, to do with the definition of $L_n(\pi)$ for n odd. One of the main results of Wall's book is that the group $L_n(\pi)$ is the same as $L_{n+4}(\pi)$. There is a simple algebraic interpretation of this periodicity in my theory.

It is to be hoped, however, that my L-theory does more than give new proofs of old facts, and that there is indeed a payoff in the topology.

August 1972