Uspekhi Mat. Nauk 41:1 (1986), 97-147 Russian Math. Surveys 41:1 (1986), 119-182

Reidemeister torsion in knot theory

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CONTENTS

Introduction

1. Torsion invariants were introduced by Reidemeister [32], [33] about 50 years ago, and were historically the first non-homotopy invariants for manifolds. Reidemeister defined the torsion invariants for closed three-dimensional pl-manifolds. As an application, he obtained the complete piecewise-linear classification of the three-dimensional lens spaces. Franz [11] transferred the definition of torsion to the many-dimensional situation, and classified lens spaces in all dimensions. In the category of smooth manifolds, torsions were defined by Whitehead [48] (using C^1-triangulations) and de Rham [34] (using the nerves of covers by geodesically convex sets). Whitehead also developed a new, deeper, viewpoint on the Reidemeister torsions, indicating their place in his theory of simple homotopy types, and in particular, comparing them with the Whitehead torsions, which are defined for homotopy equivalences. We note that as well as the term "Reidemeister torsion", the terms "Reidemeister-Franz torsion" and "Reidemeister-Franz-de Rham torsion" are also used.

Reidemeister torsions are by nature (dimensionally) global invariants: the definition of the torsion of a CW-complex (or pl-manifold) X requires the consideration of cell-chains (and boundary homomorphisms) of X in all dimensions. In this respect, the torsions are similar to the Euler characteristic. The torsions have particular beautiful properties, including a multiplicativity recalling the additivity of the Euler characteristic. At the same time, in contrast to the Euler characteristic, the torsions are well-suited for the study of odd-dimensional manifolds and are usually of little interest in the four-dimensional case. Poincaré duality is reflected in torsion theory: as Franz [12] and Milnor [26] showed, the torsions of manifolds are symmetric in the appropriate sense.

2. In 1928, 7 years before Reidemeister first considered the torsion invariants, Alexander [1] introduced a new link invariant in the three-dimensional sphere, the so-called Alexander polynomial. Subsequently it was established that the Alexander polynomial is a homology invariant computable from the 1-dimensional homology group of the exterior of the link with the appropriate twisted coefficients. Such an approach makes it possible to generalize the definition of the Alexander polynomial, and to consider Alexander polynomials of many-dimensional links and compact manifolds. These polynomials can be viewed as slightly modified "modular" versions of the oldest homology invariants—the torsion coefficients.

The methods of algebraic topology connected with the study of Alexander polynomials play a fundamental role in knot theory (see [6], [17], [35]). One can say that these methods form the nucleus of the apparatus of knot theory.

3. As Milnor [26] first noticed, there is a close connection between Alexander polynomials and Reidemeister torsion. Namely, Milnor showed that the Alexander polynomial of a link in S^3 is equal (up to a standard factor) to a certain Reidemeister torsion of the exterior of the link. From the theorems formulated below one can see that this equality is analogous in nature to the following obvious formula of the topology of surfaces: the one-dimensional Betti number of a connected compact surface F is equal, up to a standard summand, to $-\chi(F)$, where χ is the Euler characteristic.

The kind of Reidemiester torsion considered by Milnor in [26] is called Milnor torsion below. As an application of his interpretation of the Alexander polynomial as a torsion, Milnor showed that the classical Seifert-Torres theorem that the Alexander polynomial of a link in S^3 is palindromic is a very special case of the Franz-Milnor theorem on the symmetry of torsions. Using torsion techniques, Fox and Milnor [9] showed that the Alexander polynomial of a knot in S^3, considered modulo a suitable equivalence relation, is a cobordism invariant. Kervaire [22], also using torsions, carried over the Fox-Milnor theorem to the case of many-dimensional knots. In the paper [44] the author computed the Milnor torsion of a closed three-dimensional manifold. It turned out that, as in the case of the exterior of links, this torsion, up to a standard factor, is equal to the Alexander polynomial of the manifold in question.

4. The main purpose of the present paper is to study systematically the connections between Alexander polynomials and torsions, and relying on

these connections, to give a survey of the main properties of Alexander polynomials of links in the three-dimensional sphere. This approach makes it possible to represent the numerous and diverse properties of Alexander polynomials of links as consequences of several universal torsion properties. Regarding the Alexander polynomial as a torsion not infrequently sheds new light on the essence of the problem and is, in the author's opinion, the most correct viewpoint.

It is natural that the organic merging of two extensive deeply developed theories such as torsion theory and the theory of Alexander polynomials leads to the enrichment of these theories by new results. Some of these results are expounded here for the first time. We also remark that, in our approach, many well-known theorems naturally receive new and wider formulations and new proofs.

The methods of torsion theory and the theory of Alexander polynomials developed in this paper transcend the framework of knot theory and links in S^3, and can be applied (and are applied in the paper) also to the study of many-dimensional knots and links, and to the study of closed manifolds.

The circle of problems connected with the study of Alexander polynomials and torsions in the theory of dynamical systems is beyond the scope of this paper. For this the reader is referred to [10], [13], [27]. This article also excludes the application of torsions to the computation of elementary ideals and Fox-Brody invariants of three-dimensional manifolds considered by the author (see [45], [46]).

5. We briefly describe the contents of the article by sections. In §0 we recall the necessary definitions and results of torsion theory, and also introduce the notation used later. In §1 we first formulate Theorems 1.1.1, 1.1.2, and 1.1.3, which establish the connection between torsions and Alexander polynomials, and then we formulate and (using Theorems 1.1.1, 1.1.2, and 1.1.3) prove the main properties of the Alexander polynomials of links in S^3. In §2 we carry out the proof of Theorems 1.1.1, 1.1.2, and 1.1.3.

In §3 we introduce and study a new modification of Reidemeister torsion, namely, the refined (or sign-determined) torsion. In §4 we consider the Conway function of a link in S^3. Here we adopt an axiomatic approach to the definition of the Conway function. A model satisfying the axioms is constructed using refined torsions.

In §5 we consider a polynomial invariant of manifolds and links close to the Alexander polynomial (and in some cases coinciding with it), the so-called polynomial δ. In particular, the polynomial δ of a link in S^3 is the first non-zero term in the sequence of Alexander polynomials of the link group. Our approach to the study of the polynomial δ is founded on the exploitation of the connection between this polynomial and yet another modification of Reidemeister torsion, namely the torsion ω. Theorem 5.1.1, formulated in §5, which establishes the connection between δ and ω, is proved in §6. In addition, we formulate and prove the duality theorems for torsions that are used in the main text.

6. The present paper grew out of a series of lectures I gave at the Seminar of V.A. Rokhlin in 1974–1984. My long contact with Rokhlin has exerted a decisive influence on my development as a topologist. I am deeply grateful to Vladimir Abramovich for his benevolent interest in my scientific work, and for his great labour in reading and editing my papers.

I should also like to use this opportunity to express my gratitude to O.Ya. Viro, who in the early seventies, during my study at the Leningrad State University, aroused my interest in torsion theory and its applications in low-dimensional topology.

§0. Preliminary material

0.1. The torsion of a chain complex.

In this paper, a ring is a ring with identity, $1 \neq 0$. The word "module" means a left module. A chain complex is a finitely generated chain complex of finite length.

We say that the chain complex $C = (C_m \xrightarrow{\partial_{m-1}} C_{m-1} \to \dots \to C_1 \xrightarrow{\partial_0} C_0)$ over a ring K is *free* if the K-modules $C_0, C_1, \dots, C_m$ are free. The complex C is called *acyclic* if $H_*(C) = 0$, where $H_*(C) = \bigoplus_{i=0}^{m} H_i(C)$, and $H_0(C) = \operatorname{Coker} \partial_0$ and $H_m(C) = \operatorname{Ker} \partial_{m-1}$. If $b = (b^1, \dots, b^r)$ and $c = (c^1, \dots, c^r)$ are two bases for one and the same vector space over the field F, then $[b/c]$ denotes the determinant of the matrix taking c to b, that is, the determinant of the $r \times r$ matrix $(a_{i,j})$ over F for which $b^i = \sum_{j=1}^{r} a_{i,j} c^j$ for $i = 1, \dots, r$.

Let $C = (C_m \to \dots \to C_0)$ be a chain complex over F. We suppose that for each $i = 0, \dots, m$ a (finite, ordered) basis c_i is fixed in the vector space C_i. (The case $C_i = 0$ is not excluded, of course. By definition, the zero module has a unique basis.) In this situation, we define an element $\tau(C)$ of F, called the *torsion* τ of the complex C corresponding to the given system of bases. If C is not acyclic, then $\tau(C) = 0$. We suppose that C is acyclic. We consider for $i = 1, \dots, m$ a sequence of vectors $b_i = (b_i^1, \dots, b_i^{r_i})$ in C_i for which $\partial_{i-1}(b_i) = (\partial_{i-1}(b_i^1), \dots, \partial_{i-1}(b_i^{r_i}))$ is a basis of $\operatorname{Im} \partial_{i-1}$. It is obvious that for every $i = 0, 1, \dots, m$ the sequence $\partial_i(b_{i+1})b_i = (\partial_i(b_{i+1}^1), \dots, \partial_i(b_{i+1}^{r_{i+1}}), b_i^1, \dots, b_i^{r_i})$ is a basis in C_i (it is understood that b_0 and b_{m+1} are empty sequences). We put $\varepsilon(i) = (-1)^{i+1}$. The product

$$\prod_{i=0}^{m} [\partial_i (b_{i+1}) b_i / c_i]^{\varepsilon(i)}$$

is a non-zero element of the field F which, as can easily be verified, does not depend on the choice of the sequences $b_1, \ldots, b_m$. This element is $\tau(C)$.

We require a simple generalization of the torsion τ. Let $C = (C_m \to \ldots \to C_0)$ be a free chain complex over an integral domain K. (By an integral domain we understand a commutative ring without zero divisors.) Let the K-modules $C_0, C_1, \ldots, C_m$ have given bases. The field of quotients of the ring K will be denoted by $Q(K)$. The basis in C_i determines in an obvious way a basis in the $Q(K)$-module $Q(K) \otimes_K C_i$. The torsion τ of the chain complex $Q(K) \otimes_K C$ corresponding to the given system of bases is denoted by $\tau(C)$ and called the *torsion* τ of the complex C. It is clear that $\tau(C) \in Q(K)$. From the equality $H_*(Q(K) \otimes_K C) = Q(K) \otimes_K H_*(C)$ it becomes apparent that the following four conditions are mutually equivalent: $\tau(C) \neq 0$; the complex $Q(K) \otimes_K C$ is acyclic; $\operatorname{rg} H_*(G) = 0$; $H_*(C) = \operatorname{Tors} H_*(C)$. Here for a K-module H, $\operatorname{rg} H$ denotes its rank, that is, the dimension of the vector space $Q(K) \otimes_K H$, and $\operatorname{Tors} H$ denotes the submodule of H consisting of those $h \in H$ such that $kh = 0$ for some non-zero $k \in K$. It is clear that if $\tau(C) \neq 0$, then $\chi(C) = 0$, where χ is the Euler characteristic.

We note, although we do not require this later, that for an acyclic chain complex equipped with bases the torsion can also be defined in a more general situation, without assuming that the ground ring is a field or integral domain. The definition of this generalized torsion is given, for example, in the well-known survey by Milnor on torsion theory [28]. It should be said that for an acyclic chain complex C over a field the torsion defined in [28] is the inverse (in the multiplicative group of non-zero elements of the field) of the torsion $\tau(C)$ defined above.

0.1.1. ***Theorem*** (the multiplicativity of torsion; see Whitehead [48], Theorem 6). *Let $C = (C_m \to \ldots \to C_0)$ be a chain complex over an integral domain, let $C' = (C'_m \to \ldots \to C'_0)$ be a subcomplex of it, and let $C'' = (C''_m \to \ldots \to C''_0)$ be the factor complex C/C'. We suppose that for each $i = 0, 1, \ldots, m$ the modules C_i, C'_i, and $C''_i = C_i/C'_i$ are free and are equipped with bases, and moreover that the chosen basis in C_i is obtained by writing down successively the elements of the chosen basis in C'_i and elements of C_i whose images under the projection $C_i \to C''_i$ give the chosen basis for C''_i. If $\tau(C') \neq 0$ or $\tau(C'') \neq 0$, then $\tau(C) = \pm\tau(C')\tau(C'')$.*

For completeness, we give a proof of this theorem, taken from [28].

Proof of Theorem 0.1.1. By replacing the ground ring by its quotient ring, if necessary, we may assume that it is a field. If the complex C is not acyclic, then by the exactness of the homology sequence of the pair (C, C'), at least one of the complexes C', C'' is also not acyclic, so that both sides of the desired formula vanish. If C is acyclic, and $\tau(C') \neq 0$ or $\tau(C'') \neq 0$, then all three complexes in question are acyclic. Let c_i, c'_i, and c''_i be the fixed bases for the modules C_i, C'_i, and C''_i respectively. By the condition, $[c_i/c'_ic''_i] = 1$. Let b_i, b'_i, and b''_i be the sequences of vectors in C_i, C'_i, and C''_i, respectively, which were discussed in the definition of the torsion. By changing b_i if necessary, we can suppose that $[b_i/b_i'b_i''] = 1$ for all i. Then

$$\tau(C) = \prod_i [\partial_i(b_{i+1})\, b_i/c_i]^{\varepsilon(i)} = \prod_i [\partial_i'(b_{i+1}')\, \partial_i''(b_{i+1}'') b_i' b_i''/c_i' c_i'']^{\varepsilon(i)} =$$

$$= \pm \prod_i [\partial_i'(b_{i+1}')\, b_i'/c_i']^{\varepsilon(i)} \prod_i [\partial_i''(b_{i+1}'')\, b_i''/c_i'']^{\varepsilon(i)} = \pm\, \tau(C')\, \tau(C'').$$

0.2. The torsion of a CW-pair.

Up to the end of this section we fix a finite CW-pair (X, Y), an integral domain K, and a ring homomorphism $\varphi\colon \mathbf{Z}[H_1(X;\ \mathbf{Z})] \to K$. Below, we omit the coefficient group $\mathbf{Z}$ in the notation for integral homology. In accordance with the accepted practice in torsion theory (and the theory of Alexander polynomials) the group operation in $H_1(X)$, namely the addition of homology classes, is written multiplicatively and is called multiplication.

The *Reidemeister torsion* $\tau^\varphi(X, Y)$ is a subset of the field $Q(K)$ defined as follows. We put $H = H_1(X)$. We consider a maximal Abelian cover $p : \widetilde{X} \to X$ of the space X. This is a regular cover, whose translation group is H. (If X is not connected, and X_0 is a connected component of it, then $H_1(X_0)$ preserves the connected components of the space $p^{-1}(X_0)$, while $H/H_1(X_0)$ permutes these components.) A CW-decomposition of X can be lifted in an obvious way to an equivariant decomposition of $\widetilde{X}$. We consider the integral-valued cellular complex $C_*(\widetilde{X},\ p^{-1}(Y))$ of the pair $(\widetilde{X},\ p^{-1}(Y))$. The action of H on $\widetilde{X}$ gives this complex the structure of a $\mathbf{Z}[H]$-chain complex. It is clear that the $\mathbf{Z}[H]$-modules of chains are free, and moreover the number of free generators for the module of i-dimensional chains is the number of i-dimensional cells in $X \setminus Y$. We denote the K-chain complex $K \otimes_{\mathbf{Z}[H]} C_*(\widetilde{X},\ p^{-1}(Y))$ by $C_*^\varphi(X, Y)$ (the $\mathbf{Z}[H]$-module structure in K is given by the formula $zk = \varphi(z)k$, where $z \in \mathbf{Z}[H]$ and $k \in K$). Let e be a sequence of oriented cells of $\widetilde{X}$ with the property that over every cell of X there lies exactly one cell of the sequence e. (Such sequences of cells are called *base* sequences.) The cells of e that lie in $\widetilde{X} \setminus p^{-1}(Y)$, written out in the order in which they occur in e, define the "natural" basis for the K-chain complex $C_*^\varphi(X, Y)$. The torsion τ of the complex $C_*^\varphi(X, Y)$ corresponding to this basis is denoted by $\tau^\varphi(X, Y, e)$. The totality of torsions $\tau^\varphi(X, Y, e)$ corresponding to all possible base sequences e is $\tau^\varphi(X, Y)$.

It is not hard to understand how the set $\tau^\varphi(X, Y)$ can be completely recovered from any of its elements with the homomorphism φ: if $a \in \tau^\varphi(X, Y)$, then $\tau^\varphi(X, Y) = \{\pm\varphi(h)a \mid h \in H\}$. We say that $\tau^\varphi(X, Y)$ is "an element of $Q(K)$, defined up to a factor $\pm\varphi(h)$, with $h \in H$". The elements of the set $\tau^\varphi(X, Y)$ are called the representatives of the torsion $\tau^\varphi(X, Y)$. The torsion $\tau^\varphi(X, \emptyset)$ is denoted by $\tau^\varphi(X)$.

By the results of Section 0.1 the torsion $\tau^\varphi(X, Y)$ is non-zero if and only if rg $H_i^\varphi(X, Y) = 0$ for all i, where $H_i^\varphi(X, Y)$ denotes the K-module $H_i(C_*^\varphi(X, Y))$.

The torsion $\tau^{\varphi}(X, Y)$ is preserved under subdivision of the cell complexes of the spaces X and Y, and is, moreover, an invariant under simple homotopy equivalences. More precisely, we have the following theorem.

0.2.1. ***Theorem*** (see [48]). *Let (X', Y') be a finite CW pair, and let $(X, Y) \to (X', Y')$ be a simple homotopy equivalence induced by the homotopy equivalence $X \to X'$. Let the homomorphism $\varphi: \mathbf{Z}[H_1(X)] \to K$ be the composition of the isomorphism $\mathbf{Z}[H_1(X)] \to \mathbf{Z}[H_1(X')]$ induced by the latter equivalence and the ring homomorphism $\varphi': \mathbf{Z}[H_1(X')] \to K$. Then $\tau^{\varphi'}(X', Y') = \tau^{\varphi}(X, Y)$.*

Theorem 0.2.1 is easily deduced from Theorem 0.1.1 using well-known arguments in terms of the cylinder of a map f. Below (in §3) we formulate and give a detailed proof of a more precise version of Theorem 0.2.1.

The following two theorems represent geometric versions of the theorem on the multiplicativity of torsion.

0.2.2. ***Theorem.*** *Let $j: \mathbf{Z}[H_1(Y)] \to \mathbf{Z}[H_1(X)]$ be the inclusion homomorphism. If $\tau^{\varphi\circ j}(Y) \neq 0$ or $\tau^{\varphi}(X, Y) \neq 0$, then $\tau^{\varphi}(X) = \tau^{\varphi}(X, Y)\tau^{\varphi\circ j}(Y)$.*

The product $\tau^{\varphi}(X, Y)\tau^{\varphi\circ j}(Y)$ is here understood as the product of sets: the product AB of subsets A and B of K is the set $\{ab \mid a \in A,\ b \in B\}$. Theorem 0.2.2 is obtained by direct application of Theorem 0.1.1 to the canonical embedding $C_*^{\varphi\circ j}(Y) \to C_*^{\varphi}(X)$.

0.2.3. ***Theorem.*** *Let X_1 and X_2 be subcomplexes of X whose union is X, and whose intersection is Y. Let $j: \mathbf{Z}[H_1(Y)] \to \mathbf{Z}[H_1(X)]$ and $j_r: \mathbf{Z}[H_1(X_r)] \to \mathbf{Z}[H_1(X)]$ with $r = 1, 2$ be the inclusion homomorphisms. If $\tau^{\varphi\circ j}(Y) \neq 0$, then $\tau^{\varphi}(X) = \tau^{\varphi\circ j_1}(X_1)\tau^{\varphi\circ j_2}(X_2)[\tau^{\varphi\circ j}(Y)]^{-1}$.*

This theorem is obtained by applying Theorem 0.1.1 to the embedding

$$C_*^{\varphi\circ j}(Y) \to C_*^{\varphi\circ j_1}(X_1) \oplus C_*^{\varphi\circ j_2}(X_2).$$

0.3. The torsion of a manifold.

All manifolds and maps between them are assumed to be piecewise linear in this paper; submanifolds are assumed to be locally flat. The fact that the torsion of a CW complex is preserved under subdivision enables us to define in an obvious way (using piecewise linear triangulations) the torsion of a compact manifold. It is clear that to compute such a torsion one can use a CW-decomposition of the manifold, some subdivision of which is a pl-triangulation. (Indeed, to compute the torsion one can use an arbitrary decomposition of the manifold: as is well known, the homeomorphisms of CW-complexes are simple homotopy equivalences, and hence preserve torsions. We do not require these facts.)

0.4. The order of a module.

Let H be a finitely generated module over a commutative ring K. We represent H as the cokernel of a K-linear homomorphism $f: K^m \to K^n$ with

$n = 1, 2, \ldots$ and with $\infty \geqslant m \geqslant n$. It is well known that the ideal of a ring K generated by $(n \times n)$-minors of the matrix of the homomorphism f depends only on H. If K is a factorial ring, then the set of greatest common divisors of the elements of this ideal is called the *order* of the module H, written ord H, and is "an element of the ring K defined up to multiplication by invertible elements of K". It is easy to see that ord $H \neq 0$ if and only if rg $H = 0$. In particular, if $K = \mathbf{Z}$ and H is a finitely generated Abelian group, then ord $H = 0$ in the case when H is infinite, and ord $H = \pm\mathrm{card}(H)$ when H is finite.

0.5. Notations.
If H is an Abelian group, then $H^\#$ denotes the quotient group $H/\mathrm{Tors}\, H$. If G is a free Abelian group, then $Q(G)$ denotes the field of fractions of the integral domain $\mathbf{Z}[G]$. The ring $\mathbf{Z}[G]$ and its extension $Q(G)$ are equipped with the canonical involution taking $g \in G$ to g^{-1}. The image of an element $a \in Q(G)$ under this involution is denoted by $\bar{a}$.

The map assigning to orientation-reversing loops in a manifold M the number -1, and to other loops the number 1, is denoted by $w_1(M)$, or more briefly, w_1. The induced map $H_1(M) \to \{1, -1\}$ is also denoted by w_1.

§1. Milnor torsion and the Alexander polynomial

1.1. Computation of Milnor torsion.
Let (X, Y) be a finite m-dimensional CW-pair. We denote the group $H_1(X)^\#$ $(=H_1(X)/\mathrm{Tors}\, H_1(X))$ by G. It is clear that G is a free Abelian group of rank rg $H_1(X)$. Let θ be the ring homomorphism $\mathbf{Z}[H_1(X)] \to \mathbf{Z}[G]$ induced by the projection $H_1(X) \to G$. The torsion $\tau^\theta(X, Y)$ is called the *Milnor torsion* of the pair (X, Y), and is denoted by $\tau(X, Y)$. By the remarks in §1, $\tau(X, Y)$ is an element of the field $Q(G)$ defined up to multiplication by $\pm g$ with $g \in G$.

The module $H_i^\theta(X, Y)$ over the ring $\mathbf{Z}[G]$ is called the *i-dimensional Alexander module* of the pair (X, Y). Obviously if $q : \hat{X} \to X$ is a maximal free Abelian cover, then

$$H_i^\theta(X, Y) = H_i(C_*^\theta(X, Y)) = H_i(\hat{X}, q^{-1}(Y)).$$

Since $\mathbf{Z}[G]$ is factorial and Noetherian, we can consider the orders of Alexander modules of the pair (X, Y). The *Alexander function* of the pair (X, Y) is an element of the field $Q(G)$ defined up to multiplication by $\pm g$ with $g \in G$ whose value is $\prod_{i=0}^{m} [\mathrm{ord}\, H_i^\theta(X, Y)]^{\varepsilon(i)}$ (where $\varepsilon(i) = (-1)^{i+1}$) if ord $H_i^\theta(X, Y) \neq 0$ for all i, and is zero otherwise. The Alexander function of the pair (X, Y) is denoted by $A(X, Y)$.

1.1.1. ***Theorem.*** *If (X, Y) is a finite CW pair, then*

$$\tau(X, Y) = A(X, Y).$$

Theorem 1.1.1, as well as Theorems 1.1.2 and 1.1.3 stated later in this section, are proved in §2. In the case when rg $H_1(X) = 1$, the equality $\tau(X, Y) = A(X, Y)$ (in a somewhat weaker form: up to multiplication by a non-zero rational number) was proved by Milnor [27].

Theorem 1.1.1 often enables us to compute the Milnor torsion explicitly. It also follows from this theorem that Milnor torsion is invariant under homotopy equivalences (not necessarily simple). However, the latter statement can also be easily verified directly. For the present paper, it is most important that Theorem 1.1.1 makes it possible to apply the techniques of torsion theory to the study of the Alexander function.

If M is a compact m-dimensional manifold, then the order of its $[m/2]$-dimensional Alexander module is called the *Alexander polynomial* of M, and is denoted by $\Delta(M)$. (Here the square brackets denote integer part.) Thus $\Delta(M)$ is an element of the ring $\mathbf{Z}[H_1(M)^{\#}]$ defined up to multiplication by $\pm g$ with $g \in H_1(M)^{\#}$. In the case when $m = 3$, the polynomial $\Delta(M)$ is the first Alexander polynomial of the group $\pi_1(M)$ in the sense of Fox (see [6]). As the following theorem shows, for a wide class of three-dimensional manifolds the Alexander function and the Alexander polynomial represent (up to a standard factor) one and the same invariant.

1.1.2. ***Theorem.*** *Let M be a connected compact three-dimensional manifold with $\chi(M) \leqslant 0$, and let $H = H_1(M)$. If* rg $H \geqslant 2$, *then $A(M) = \Delta(M)$. If* rg $H = 1$, *and t is a generator for the infinite cyclic group $H^{\#}$, then*

$$A(M) = \begin{cases} \Delta(M)(t-1)^{-1} & \text{if } \partial M \neq \emptyset \text{ or } w_1(\operatorname{Tors} H) \neq 1, \\ \Delta(M)(t-1)^{-2} & \text{if } \partial M = \emptyset \text{ and } w_1(H) = 1, \\ \Delta(M)(t^2-1)^{-1} & \text{if } \partial M = \emptyset,\ w_1(\operatorname{Tors} H) = 1 \text{ and } w_1(H) \neq 1. \end{cases}$$

Theorem 1.1.2 is apparently new: the Alexander function of three-dimensional manifolds has not been particularly considered. The condition $\chi(M) \leqslant 0$ in Theorem 1.1.2 is already satisfied in the two most interesting cases: when M is a closed manifold, and when M is the exterior of a link in a closed manifold. In both cases $\chi(M) = 0$.

The combination of Theorems 1.1.1 and 1.1.2 yields formulae expressing the Milnor torsion of a connected compact three-dimensional manifold M with $\chi(M) \leqslant 0$ in terms of $\Delta(M)$. In the case when $\partial M \neq \emptyset$, these formulae were obtained by Milnor [26]. (Milnor considered only the exteriors of links in S^3 but his arguments generalize directly to the situation we describe.) For closed M these formulae were obtained by the author (see [44]).

The following Theorem 1.1.3 shows that from the Alexander function of a three-dimensional manifold one can compute not only the Milnor torsion but also some other torsions. When formulating Theorem 1.1.3 we use the fact, which follows from Theorem 1.1.2, that if (under the conditions of Theorem 1.1.2) rg $H \geqslant 2$, then $A(M) \subset \mathbf{Z}[H^{\#}]$.

1.1.3. *Theorem.* *Under the conditions of Theorem* 1.1.2, *suppose that* Π *is a free Abelian group and let* $\varphi: \mathbf{Z}[H] \to \mathbf{Z}[\Pi]$ *be the composition of the projection* $\theta: \mathbf{Z}[H] \to \mathbf{Z}[H^{\#}]$ *and the ring homomorphism* $\psi: \mathbf{Z}[H^{\#}] \to \mathbf{Z}[\Pi]$ *induced by a non-trivial group homomorphism* $H^{\#} \to \Pi$. *If* $\operatorname{rg} H \geqslant 2$, *then* $\tau^{\varphi}(M) = \psi(A(M))$. *If* $\operatorname{rg} H = 1$, *then* ψ *can be uniquely extended to a ring homomorphism* $\tilde{\psi}: Q(H^{\#}) \to Q(\Pi)$ *and* $\tau^{\varphi}(M) = \tilde{\psi}(A(M))$.

1.2. The Alexander invariants of links.

A *link* in the m-dimensional manifold M is an oriented submanifold of M whose components are homeomorphic to the sphere S^{m-2}. A link is called *ordered* if its components are numbered. The *exterior* of a link in M is the complement in M of an open regular neighbourhood of it.

We consider an ordered link $l = l_1 \cup \ldots \cup l_n$ in S^m, where $l_1, \ldots, l_n$ are the components of l. The sphere S^m is supposed to be oriented once and for all for each m. If V is the exterior of l, then the group $H_1(V)$ is canonically isomorphic to a free Abelian (multiplicative) group with n free generators $t_1, \ldots, t_n$: the generator t_i corresponds to the homology class of a meridian of the component l_i. (If l is a knot, that is, if $n = 1$, then instead of t_1 we simply write t.) The ring $\mathbf{Z}[H_1(V_1)]$ is identified via this correspondence with the Laurent polynomial ring $\mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$.

The *Alexander polynomial* (respectively, *function*) of the link l is the Alexander polynomial (respectively, function) of its exterior. These invariants are denoted by $\Delta_l(t_1, \ldots, t_n)$ and $A_l(t_1, \ldots, t_n)$ respectively, or, more briefly, by Δ_l and A_l. Thus Δ_l is a Laurent polynomial in the variables $t_1, \ldots, t_n$ determined up to multiplication by polynomials of the form $\pm t_1^{r_1} \ldots t_n^{r_n}$ with integral $r_1, \ldots, r_n$. The Alexander function A_l is an element of the field $\mathbf{Q}(t_1, \ldots, t_n)$ of rational functions in $t_1, \ldots, t_n$ with rational coefficients, determined up to the same accuracy as Δ_l.

The invariant A_l is meaningful only when $m = 3$ or when l is a knot: if $m > 3$ and $n > 1$, then $A_l = 0$. By Theorem 1.1.2, if $m = 3$, then $A_l = \Delta_l$ for $n \geqslant 2$, and $A_l = \Delta_l(t - 1)^{-1}$ for $n = 1$. Although the Alexander function and the Alexander polynomial of a link in S^3 are essentially one and the same invariant, we find it convenient to use both terms. In particular, we shall formulate some classical theorems about Alexander polynomials of links in S^3 in terms of Alexander functions, leaving to the reader their translation into the standard terminology.

1.3. The Alexander polynomial of an iterated link. The monodromy theorem.

The following theorem was proved in several special cases by Burau [3], [4], and Seifert [39], and in complete generality by Sumners and Woods [40]. In what follows, $\mu(k, l)$ denotes the linking number of the knots k and l in S^3.

1.3.1. ***Theorem*** (see [40]). *Let $l = l_1 \cup \ldots \cup l_n$ be a link in S^3, and let k be a knot in $S^3 \setminus l$ located at the boundary of a closed regular neighbourhood of the knot l_n in $S^3 \setminus (l_1 \cup \ldots \cup l_{n-1})$. We suppose that the knot k is homological in this neighbourhood (which is a solid torus) to the p-th power of the loop l_n, with $p \neq 0$. Let $A(t_1, \ldots, t_n)$, $A'(t_1, \ldots, t_n)$, and $A''(t_1, \ldots, t_n, t_{n+1})$ be the Alexander functions of the ordered links l, $l' = l_1 \cup \ldots \cup l_{n-1} \cup k$, and $l'' = l' \cup l_n = l_1 \cup \ldots \cup l_{n-1} \cup k \cup l_n$, respectively. We put $T = t_1^{\mu_1} t_2^{\mu_2} \ldots t_{n-1}^{\mu_{n-1}} t_n^q$, where $\mu_i = \mu(l_i, l_n)$ and $q = \mu(k, l_n)$. Then*

$$A'(t_1, \ldots, t_n) = A(t_1, \ldots, t_{n-1}, t_n^p) \frac{T^p - 1}{T - 1}, \tag{1}$$

$$A''(t_1, \ldots, t_{n+1}) = A(t_1, \ldots, t_{n-1}, t_n^p t_{n+1}) (T^p t_{n+1}^q - 1). \tag{2}$$

Here the expression $A(t_1, \ldots, t_{n-1}, t_n^p)$ denotes the rational function in $t_1, \ldots, t_n$ defined with the same accuracy as $A(t_1, \ldots, t_n)$, and obtained from it by the substitution $t_n \mapsto t_n^p$. The expression $A(t_1, \ldots, t_{n-1}, t_n^p t_{n+1})$ is to be understood analogously. In the case when $T = 1$, the fraction $(T^p - 1)/(T - 1)$ is regarded as unity.

The knot k discussed in Theorem 1.3.1 is called a (p, q)-cable of the knot l_n. One says of the links l' and l'' that they are obtained from l by iterating the component l_n. The links in S^3 that can be obtained by iteration from the trivial knot are called iterated torus links. Theorem 1.3.1 enables us to compute the Alexander polynomials of such links inductively. For example, from this theorem it follows immediately that the Alexander polynomial of the torus knot of type (p, q) is $(t^{pq} - 1)(t - 1)(t^p - 1)^{-1}(t^q - 1)^{-1}$. From Theorem 1.3.1 it also follows that the Alexander polynomial of any (ordered n-component) iterated torus link can be represented as the product of polynomials of the form $t_1^{r_1} t_2^{r_2} \ldots t_n^{r_n} - 1$ and their divisors.

As is known, a link in S^3 that is algebraic in the sense of Brauner is an iterated torus link (see for example [40]). Theorem 1.3.1 makes it possible to compute without difficulty the Alexander polynomial of an algebraic link from the Puiseux numbers of the equation $f(z_1, z_2) = 0$ determining this link. Relying on Theorem 1.3.1, Yamamoto [52] recently proved that two ordered algebraic links are isotopic if and only if their Alexander polynomials are equal. (In the case of knots and two-component links, this result was already obtained in the thirties by Burau [3], [4].) We remark that the first results in the direction of Theorem 1.3.1 were obtained by Burau [3], [4] exactly for the purpose of computing the Alexander polynomials of algebraic links. The computational methods used by Burau were based on a detailed study of the presentations of the groups of links by generators and relations. Fox writes (we quote from [7]): "the calculations in the two Burau papers are almost too painful to contemplate, but I am sure that the results are correct ...". The proof of Theorem 1.3.1 given by Sumners and Woods [40] uses modern homology techniques, but is also rather complicated.

The proof of Theorem 1.3.1 given below demonstrates to its full extent the efficiency of applying torsions to such problems.

We mention an important consequence of Theorem 1.3.1. Since the characteristic polynomial of the monodromy homomorphism of an algebraic link l in S^3 is $A_l(t, t, \ldots, t)(t-1)$ (see [29], Lemma 10.1), Theorem 1.3.1 implies the case $r = 1$ of the following theorem of Grothendieck [15].

1.3.2. ***Theorem*** ([15], see also [40], [49] and their references). *The characteristic polynomial of the monodromy homomorphism of an algebraic link in S^{2r+1} is the product of cyclotomic polynomials.*

We prove Theorem 1.3.1.

1.3.3. ***Lemma.*** *Let X be a finite CW-complex, and let φ be a ring homomorphism from the ring $\mathbf{Z}[H_1(X)]$ to an integral domain. If X is simply homotopically equivalent to a circle, t is a generator for $H_1(X)$, and $\varphi(t) \neq 1$, then $(\varphi(t)-1)^{-1} \in \tau^{\varphi}(X)$. If X is simply homotopically equivalent to a 2-dimensional torus and $\varphi(H_1(X)) \neq 1$, then $1 \in \tau^{\varphi}(X)$.*

Proof. We may assume that the integral domain in question is a field, and that in the first case $X = S^1$, and in the second case $X = S^1 \times S^1$.

Let $C = (C_1 \to C_0)$ be the chain complex $C_*^{\varphi}(S^1)$ corresponding to the decomposition of the circle consisting of one zero-dimensional cell and one one-dimensional cell. With a suitable choice of natural bases, the boundary homomorphism $C_1 \to C_0$ is given by the 1×1 matrix $(\varphi(t)-1)$ for which, as follows immediately from the definition of the torsion, $\tau(C) = (\varphi(t)-1)^{-1}$.

Let $C = (C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0)$ be the chain complex $C_*^{\varphi}(S^1 \times S^1)$ corresponding to the standard decomposition of the torus, consisting of one zero-dimensional cell, two one-dimensional cells, and one two-dimensional cell. Let g and h be generators of the group $H_1(S^1 \times S^1)$ representable by the one-dimensional cells. With a suitable choice of natural bases c_0^1, (c_1^1, c_1^2), and c_2^1 of the modules C_0, C_1, and C_2 respectively, the boundary homomorphism ∂_0 is given by the column $\begin{bmatrix}\varphi(g)-1\\ \varphi(h)-1\end{bmatrix}$, and the homomorphism ∂_1 by the row $(\varphi(h)-1, 1-\varphi(g))$. By hypothesis, $\varphi(g) \neq 1$ or $\varphi(h) \neq 1$. For definiteness we suppose that $\varphi(g) \neq 1$. Then the vector $\partial_1(c_2^1)$ generates $\operatorname{Im} \partial_1$, while the vector $\partial_0(c_1^1)$ generates $\operatorname{Im} \partial_0$. Therefore

$$\tau(C) = [\partial_0(c_1^1)/c_0^1]^{-1}[\partial_1(c_2^1),\ c_1^1/c_1^1,\ c_1^2][c_2^1/c_2^1]^{-1} =$$
$$= (\varphi(g)-1)^{-1} \det \begin{bmatrix}\varphi(h)-1 & 1-\varphi(g)\\ 1 & 0\end{bmatrix} = 1.$$

1.3.4. *Proof of Theorem* 1.3.1. We prove the equality (1). If $p = \pm 1$, then the links l' and l are isotopic (with a change in the orientation of the knot k in the case $p = -1$), and (1) holds. We assume that $p \neq \pm 1$. Then $q \neq 0$, and hence $T \neq 1$. The exterior V of the link l' can be represented in a natural way as the union of two manifolds V_1 and V_2, where V_1 is a solid torus containing l_n as axis, and V_2 is the exterior of the original link l; the intersection $V_1 \cap V_2$ coincides with $\partial V_1 \cap \partial V_2$ and is homeomorphic to the two-dimensional cylinder $S^1 \times [0, 1]$. We put $N = V_1 \cap V_2$. We denote the inclusion homomorphisms $\mathbf{Z}[H_1(V_r)] \to \mathbf{Z}[H_1(V)]$ and $\mathbf{Z}[H_1(N)] \to \mathbf{Z}[H_1(V)]$ by j_r and j respectively, where $r = 1, 2$. It is easy to see that j_1 takes the generator $[l_n]$ of $H_1(V_1) = \mathbf{Z}$ to T, while j takes some generator of $H_1(N) = \mathbf{Z}$ to T^p. By Lemma 1.3.3, the torsions $\tau^{j_1}(V_1)$ and $\tau^j(N)$ are $(T-1)^{-1}$ and $(T^p - 1)^{-1}$ respectively. The homomorphism j_2 takes the canonical generators $t_1, \ldots, t_n$ of $H_1(V_2)$ to $t_1, \ldots, t_{n-1}, t_n^p$ respectively. From Theorem 1.1.3 it follows that $\tau^{j_2}(V_2) = A(t_1, \ldots, t_{n-1}, t_n^p)$. Thus, by Theorem 0.2.3, the torsion $\tau(V)$ is equal to the right hand side of (1). Hence the truth of this formula follows from the equalities $A'(t_1, \ldots, t_n) = A(V) = \tau(V)$.

Formula (2) is proved analogously, using the second statement of Lemma 1.3.3.

1.4. The Torres formula and its generalizations.

1.4.1. ***Theorem*** (Torres [41]). *Let $l = l_1 \cup l_2 \cup \ldots \cup l_n$ be a link in S^3 with $n \geqslant 2$. Let $k = l_1 \cup l_2 \cup \ldots \cup l_{n-1}$, and let $\mu_i = \mu(l_i, l_n)$ for $i = 1, \ldots, n-1$. Then*

$$(3) \quad A_l(t_1, \ldots, t_{n-1}, 1) = A_k(t_1, \ldots, t_{n-1})(t_1^{\mu_1} t_2^{\mu_2} \ldots t_{n-1}^{\mu_{n-1}} - 1).$$

The formula (3) plays an important role in link theory. In particular, in the case when at least one of the numbers $\mu_1, \mu_2, \ldots, \mu_{n-1}$ is non-zero, (3) makes it possible to compute the Alexander polynomial of the sublink k from that of the link l. The formula (3) also places considerable restrictions on the form of polynomials that can be realized as Alexander polynomials of links.

1.4.2. ***Theorem*** (generalization of Theorem 1.4.1). *Let M be a connected compact three-dimensional manifold, with $\chi(M) \leqslant 0$, and with* rg $H_1(M) \geqslant 1$. *Let $G = H_1(M)^{\#}$. Let $l_1 \cup \ldots \cup l_n$ be a link in* Int M *whose components represent the elements $g_1, \ldots, g_n$, respectively, of G. Let V be the exterior of this link and let ψ be the inclusion homomorphism $\mathbf{Z}[H_1(V)^{\#}] \to \mathbf{Z}[G]$. If* rg $H_1(V) \geqslant 2$, *then* $\psi(A(V)) = A(M) \prod_{i=1}^{m} (g_i - w_1(l_i))$. *If* rg $H_1(V) = 1$, *then ψ can be extended to an isomorphism $\tilde{\psi}$: $Q(H_1(V)^{\#}) \to Q(G)$ and* $\tilde{\psi}(A(V)) = A(M) \prod_{i=1}^{m} (g_i - w_1(l_i))$.

Theorem 1.4.2 follows from Theorems 1.1.1, 1.1.3, and the following lemma.

1.4.3. ***Lemma.*** *If, under the conditions of Theorem 1.4.2, $\varphi: \mathbf{Z}[H_1(V)] \to \mathbf{Z}[G]$ is the composition of the projection $\mathbf{Z}[H_1(V)] \to \mathbf{Z}[H_1(V)^{\#}]$ and the homomorphism ψ, then*

$$(4) \qquad \tau^{\varphi}(V) = \tau(M) \prod_{i=1}^{n} (g_i - w_1(l_i)).$$

Proof. We restrict ourselves to the case $n = 1$. We put $d = g_1 - w_1(l_1)$. We denote the projection $\mathbf{Z}[H_1(M)] \to \mathbf{Z}[G]$ by θ. The pair (M, V) has a CW-decomposition consisting of cells lying in V, one two-dimensional cell (of a meridional disc of the curve l_1) and one three-dimensional cell. The non-zero part of the complex $C_*^\theta(M, V)$ is thereby reduced to the homomorphism of one-dimensional $\mathbf{Z}[G]$-modules $C_3 \to C_2$ defined by the 1×1 matrix (d). Hence if $d \neq 0$, then $d^{-1} \in \tau(M, V)$. Obviously, the homomorphism φ is the composition of the inclusion homomorphism $\mathbf{Z}[H_1(V)] \to \mathbf{Z}[H_1(M)]$ and the projection θ. By Theorem 0.2.2, if $d \neq 0$, then $\tau(M) = \tau(M,V)\tau^\varphi(V) = d^{-1}\tau^\varphi(V)$. Thence follows (4). If $d = 0$, then $H_3^\theta(M, V) = \mathbf{Z}[G]$. It is easy to see that $H_3^\theta(M) = 0$. From the exactness of the homology sequence of the pair (M, V) with (twisted) coefficients in $\mathbf{Z}[G]$ it follows that rg $H_2^\varphi(V) \geqslant 1$. So $\tau^\varphi(V) = 0 = \tau(M)d$.

1.5. The Seifert-Torres formula.

The following theorem, proved by Seifert [39] in the case of knots, and Torres [41] for links with an arbitrary number of components, shows that for a link l situated in a solid torus $U \subset S^3$ the Alexander polynomial Δ_l is completely determined by the disposition of l in U and the knotted character of the axis of U in S^3.

1.5.1. ***Theorem*** (see [39], [41]). *Let k be a knot in S^3 and U a closed regular neighbourhood of it. Let f be a homeomorphism from the solid torus U to a standard (unknotted) solid torus in S^3; let f preserve the orientation (inherited from S^3) and take the canonical (that is, homologous to zero in $S^3 \backslash k$) latitude of the knot k to the canonical latitude of the trivial knot $f(k)$. If $l = l_1 \cup \ldots \cup l_n$ is a link in* Int *U whose components are homologous in U to the μ_1-th, ..., μ_n-th power of the loop k respectively, then*

$$\Delta_l(t_1, \ldots, t_n) = \Delta_{f(l)}(t_1, \ldots, t_n)\,\Delta_k(t_1^{\mu_1} \ldots t_n^{\mu_n}). \tag{5}$$

Proof. By Theorem 1.1.2, the formula (5) is equivalent to the formula

$$A_l(t_1, \ldots, t_n) = A_{f(l)}(t_1, \ldots, t_n)\,A_k(t_1^{\mu_1} \ldots t_n^{\mu_n})\,(t_1^{\mu_1} \ldots t_n^{\mu_n} - 1). \tag{6}$$

We prove (6). We suppose that at least one of the numbers $\mu_1, \ldots, \mu_n$ is non-zero. The torus ∂U divides the exterior V of the link l into two manifolds V_1 and V_2, where V_1 is the exterior of the knot k, and V_2 is the exterior of l in U. From Lemma 1.3.3 and Theorems 0.2.3, 1.1.1, and 1.1.3 it follows that

$$A_l = \tau(V) = \tau^{j_1}(V_1)\,\tau^{j_2}(V_2) = A_k(t_1^{\mu_1} \ldots t_n^{\mu_n})\,\tau^{j_2}(V_2),$$

where j_r is the inclusion homomorphism $\mathbf{Z}[H_1(V_r)] \to \mathbf{Z}[H_1(V)]$. An analogous argument applied to the partition of the exterior of the link $f(l)$ in S^3 by the torus $f(\partial U)$ shows that

$$\tau^{j_2}(V_2) = A_{f(l)}(t_1, \ldots, t_n)\,(t_1^{\mu_1} \ldots t_n^{\mu_n} - 1).$$

The required equality now follows from this.

If $\mu_1 = \mu_2 = \ldots = \mu_n = 0$, then the statement of the theorem can be derived from the case already handled, using the following trick. We add to l one component that has non-zero linking numbers with the meridian of the solid torus U and with the knots $l_1, \ldots, l_n$; we apply the formula (6) to this $(n+1)$-component link; in the resulting formula we substitute $t_{n+1} = 1$ and use the Torres formula (3).

1.6. The Alexander formula and its generalizations.
As Alexander [1] showed, the sum of the coefficients of the Alexander polynomial of a knot in S^3 is ± 1. This equality can be generalized to the following theorem.

1.6.1. ***Theorem.*** *Let M be a connected compact three-dimensional manifold with* rg $H_1(M) = 1$ *and* $\chi(M) = 0$. *We put* $r =$ ord(Tors $H_1(M)$). *The sum of the coefficients of the polynomial* $\Delta(M)$ *is* r *if* $\partial M \neq \emptyset$ *or* $\partial M = \emptyset$ *and* w_1(Tors $H_1(M)$) $= 1$, *and is* $r/2$ *in the remaining cases.*

Proof. The case $\partial M \neq \emptyset$. The manifold M can be compressed (or even collapsed) into a two-dimensional subcomplex of it, say X. One can assume that X has one zero-dimensional cell and that the closure of one of the one-dimensional cells of X is a circle which represents a generator t of the group $H_1(M)^\# = \mathbf{Z}$. We denote this circle by Y. By Theorems 1.1.1, 1.1.2, the theorem on the multiplicativity of the torsion, and Lemma 1.3.3, $\Delta(M) = A(M)(t-1) = A(X)(t-1) = \tau(X)(t-1) = \tau(X, Y)$. If θ is the projection $\mathbf{Z}[H_1(X)] \to \mathbf{Z}[H_1(X)^\#]$, then the non-zero part of the complex $C^\theta_*(X, Y)$ is exhausted by the boundary homomorphism $C_2 \to C_1$. Let B be the matrix of this homomorphism with respect to the natural bases. Since $\chi(X, Y) = 0$, B is a square matrix and its determinant represents $\tau(X, Y)$. The integral matrix B^0 obtained from B by replacing the entries by their coefficient-sums is the matrix of the boundary homomorphism of the chain complex of the pair (X, Y) with coefficients in $\mathbf{Z}$. Hence $r =$ ord $H_1(X, Y) =$ $= \pm\det B^0$ and aug($\Delta(M)$) $=$ aug($\tau(X, Y)$) $= r$, where aug is the augmentation (summation of coefficients).

The case $\partial M = \emptyset$.
Let k be a knot in M that represents a generator t of the group $H_1(M)^\# = \mathbf{Z}$. Let V be its exterior. We shall assume that in the case of non-orientable M the loop k is orientation-reversing. We remark that the inclusion homomorphism $H_1(V) \to H_1(M)$ is an isomorphism. If M is orientable, then this follows from Poincaré duality; if M is non-orientable, it follows from the equalities $H_i(M, V) = 0$ for $i \neq 2$, $H_2(M, V) = \mathbf{Z}/2$, Tors $H_2(V) = 0$, and Tors $H_2(M) \neq 0$. (The last two formulae follow from the universal coefficient formula and the equalities $H^3(V) = 0$, $H^3(M) = \mathbf{Z}/2$.) From Theorems 1.1.2 and 1.4.2 it follows that the inclusion homomorphism $\mathbf{Z}[H_1(V)^\#] \to \mathbf{Z}[H_1(M)^\#]$ takes $\Delta(V)$ into $\Delta(M)$ if w_1(Tors $H_1(M)$) $= 1$, and into $(1+t)\Delta(M)$ in the remaining cases. From the above, aug($\Delta(V)$) $=$ ord(Tors $H_1(V)$) $= r$. Thence the desired statement follows.

1.7. Symmetry of the Alexander function.

As Seifert [38] showed, the Alexander polynomial of an arbitrary knot k in S^3 is invariant under the canonical involution in $\mathbf{Z}[t, t^{-1}]$ (see §0.5; the canonical involution $a \mapsto \bar{a}$ of the ring $\mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ takes the polynomial $a(t_1, \ldots, t_n)$ into $a(t_1^{-1}, \ldots, t_n^{-1})$). In more detail, Seifert's theorem states that if $\Delta \in \Delta_k$, then $\bar{\Delta} = t^\nu \Delta$ with even ν. The Alexander function has an analogous property: if $A \in A_k$, then $\bar{A} = -t^\nu A$ with odd ν. The following theorem, due to Torres and Fox (see [41], [42]), generalizes these results to the case of links.

1.7.1. ***Theorem*** (Torres-Fox). *Let $l = l_1 \cup \ldots \cup l_n$ be a link in S^3. If $A \in A_l$, then $\bar{A} = (-1)^n t_1^{\nu_1} t_2^{\nu_2} \ldots t_n^{\nu_n} A$ with integral $\nu_1, \ldots, \nu_n$. Here if $A \neq 0$, then for $i = 1, \ldots, n$*

$$\nu_i \equiv 1 + \sum_{j \neq i} \mu(l_i, l_j) \pmod 2. \tag{7}$$

We prove this theorem.

1.7.2. ***Lemma.*** *Let M be an orientable compact three-dimensional manifold, whose boundary is empty or consists of tori. Then $\overline{A(M)} = A(M)$.*

Here $\overline{A(M)} = \{\bar{A} \mid A \in A(M)\}$. The equality of the sets $A(M)$ and $\overline{A(M)}$ is of course equivalent to the fact that for some (and so for every) $A \in A(M)$ there is an element g of $H_1(M)^\#$ for which either $\bar{A} = gA$ or $\bar{A} = -gA$.

Proof of the Lemma. By Theorem 1.1.1 the equality $\overline{A(M)} = A(M)$ is equivalent to the equality $\overline{\tau(M)} = \tau(M)$. According to the duality theorem for torsions (see [26] or the Appendix, Theorem 2), $\overline{\tau(M)} = \tau(M, \partial M)$. The equality $\tau(M, \partial M) = \tau(M)$ follows from the multiplicativity of torsion, Lemma 1.3.3, and the well-known fact that if R is a component of the boundary ∂M then the inclusion homomorphism $H_1(R) \to H_1(M)^\#$ is non-trivial.

1.7.3. *Proof of Theorem* 1.7.1. By Lemma 1.7.2 we have $\bar{A} = \varepsilon t_1^{\nu_1} \ldots t_n^{\nu_n} A$ with integral $\nu_1, \ldots, \nu_n$ and $\varepsilon = \pm 1$. If $n = 1$, then the product $\Delta(t) = (t-1)A(t)$ represents Δ_l. Here $\Delta(1) = \pm 1$, $\Delta(-1) \equiv \Delta(1) \equiv 1 \pmod 2$ and $\bar{\Delta} = -\varepsilon t^{\nu_1 - 1}\Delta$. Hence it follows that $\varepsilon = -1$ and ν_1 is odd. If $n = 2$ and the linking number $\mu = \mu(l_1, l_2)$ is non-zero, then by the Torres theorem the fraction $a = A(t, 1)(t^\mu - 1)^{-1}$ represents A_{l_1}; in particular, $a \neq 0$. Here $\bar{a} = -\varepsilon t^{\nu_1 - \mu} a$. By the above, $\varepsilon = 1$ and $\nu_1 - \mu \equiv 1 \pmod 2$. The same argument also proves the congruence $\nu_2 \equiv 1 + \mu \pmod 2$. In an analogous way, by induction on n, we can prove the validity of the statement of the theorem for all links $l_1 \cup \ldots \cup l_n$ such that all the numbers $\mu(l_i, l_n)$ with $i = 1, \ldots, n-1$ are non-zero. The general case can be reduced to this by means of the trick used at the end of the proof of Theorem 1.5.

1.8. The Hosokawa polynomial.

The *Hosokawa polynomial* of an n-component link l in S^3 is the rational function $A_l(t, t, \ldots, t)(t-1)^{-(n-2)}$. The Hosokawa polynomial is denoted by $h_l(t)$, or more briefly by h_l. Like the Alexander invariants, the Hosokawa polynomial is defined only up to multiplication by -1 and powers of the variable t. If l is a knot, then $h_l = \Delta_l$. The main algebraic properties of the Hosokawa polynomial are described in the following theorem.

1.8.1. ***Theorem*** (Hosokawa [18]). *Let $l = l_1 \cup \ldots \cup l_n$ be a link in S^3. Then h_l is a Laurent polynomial, that is, $h_l \subset \mathbf{Z}[t, t^{-1}]$; if $h \in h_l$, then $\bar{h} = t^\nu h$ with even ν; the number $h_l(1) = \operatorname{aug} h_l$ is equal, up to sign, to an arbitrary minor of order $n-1$ of the matrix $a = (a_{i,j})$, where $i, j = 1, \ldots, n$; $a_{i,j} = \mu(l_i, l_j)$ for $i \neq j$ and $a_{i,i} = -\sum_{j \neq i} \mu(l_i, l_j)$.*

Proof. We denote by M the result of surgery on the sphere S^3 along l, with the framing defined by assigning to the component l_i the number $a_{i,i}$. We put $H = H_1(M)$. It is clear that in terms of the generators represented by the meridians of the components $l_1, \ldots, l_n$, the group H is given by the relation matrix equal to a. Since the sum of the columns of the matrix a is zero, the assignment of the variable t to the indicated generators defines a ring homomorphism $\mathbf{Z}[H^\#] \to \mathbf{Z}[t, t^{-1}]$. We denote it by η. We denote the exterior of the link l in S^3 by V, and the inclusion homomorphism $\mathbf{Z}[H_1(V)] \to \mathbf{Z}[H^\#]$ by ψ.

To prove the theorem, it suffices to consider the case $n \geqslant 2$. By Theorem 1.4.2, $\psi(A_l) = A(M)\prod_{i=1}^{n}(\psi(t_i) - 1)$. If the minors of order $n-1$ of the matrix a are zero, then $\operatorname{rg} H \geqslant 2$, $A(M) \subset \mathbf{Z}[H^\#]$ and

$$h_l = (\eta \circ \psi)(A_l) \times (t-1)^{2-n} = \eta(A(M))(t-1)^2.$$

Thus in this case h_l is a Laurent polynomial and $h_l(1) = 0$. If the indicated minors are non-zero, then $\operatorname{rg} H = 1$ and as in the previous case $h_l = \eta(\Delta(M)) \subset \mathbf{Z}[t, t^{-1}]$. It is easy to see that if $\operatorname{rg} H = 1$, then the inidcated minors are, up to sign, equal to the order of the group Tors H. By Theorem 1.6.1, this order is aug $\Delta(M)$, and hence is $h_l(1)$. The equality $\bar{h} = t^\nu h$ with even ν follows from Theorem 1.7.1.

1.9. The Fox formulae and their generalizations.

A traditional problem in knot theory is the computation of the homology invariants of branched covering spaces of the sphere S^3 with branching at a given knot or link. One of the deepest results in this direction was obtained by Fox [8]. (Fox's proof contained inaccuracies which were removed in [47].) Up to the end of this section we fix a natural number r. We denote by $\omega_1, \ldots, \omega_r$ the complex r-th roots of unity.

1.9.1. *Theorem* (see [8]). *If $N \to S^3$ is a branched cyclic r-fold cover with branching at a knot $k \subset S^3$, then for any $\Delta(t) \in \Delta_k$*

$$\text{(8)} \qquad \operatorname{ord} H_1(N) = \pm \prod_{i=1}^{r} \Delta(\omega_i).$$

Associated closely with (8) is another formula, also pointed out by Fox.

1.9.2. *Theorem* (see [8]). *Let $\widetilde{V} \to V$ be a cyclic r-fold (non-branched) cover of the exterior V of a knot $k \subset S^3$. Let ψ be the ring homomorphism $\mathbf{Z}[H_1(\widetilde{V})^{\#}] \to \mathbf{Z}[H_1(V)] = \mathbf{Z}[t, t^{-1}]$ induced by this cover. If* $\operatorname{rg} H_1(\widetilde{V}) = 1$, *then*

$$\psi(\Delta(\widetilde{V})) = \{\pm \prod_{i=1}^{r} \Delta(\omega_i t) \mid \Delta(t) \in \Delta_k\}.$$

It should be explained that $\psi(\Delta(\widetilde{V}))$ is a subset of the ring $\mathbf{Z}[t, t^{-1}]$, any two elements of which are obtained from each other by multiplication by ± 1 and a monomial power t^r.

We remark that if under the conditions of Theorem 1.9.1 the group $H_1(N)$ is finite, then (8) follows from Theorem 1.9.2: as is well known, $H_1(\widetilde{V}) = \mathbf{Z} \times H_1(N)$, so that $\operatorname{rg} H_1(\widetilde{V}) = 1$ and by Theorems 1.9.2 and 1.6.1 $\operatorname{ord} H_1(N) = \operatorname{aug} \Delta(\widetilde{V}) = \operatorname{aug} \psi(\Delta(\widetilde{V})) = \pm \prod_{i=1}^{r} \Delta(\omega_i)$. In the case when the group $H_1(N)$ is infinite, the left hand side of (8) vanishes. The fact that the right hand side also vanishes is easily observed from the proof of the following theorem.

1.9.3. *Theorem* (generalization of Theorem 1.9.2). *Let M be an orientable connected compact three-dimensional manifold with $\chi(M) = 0$. Let $G = H_1(M)^{\#}$; let $t, t_1, \ldots, t_n$ be free generators of the (free Abelian) group G, with $n \geq 0$. Let $\widetilde{M} \to M$ be a cyclic r-fold cover corresponding to the kernel of the composition of the natural homomorphism $\pi_1(M) \to G$ and the homomorphism $G \to \mathbf{Z}/r\mathbf{Z}$ taking $t_1, \ldots, t_n$ into zero and t into* 1 (mod r). *Let ψ be the ring homomorphism $\mathbf{Z}[H_1(\widetilde{M})^{\#}] \to \mathbf{Z}[G]$ induced by the cover $\widetilde{M} \to M$. If $n \geq 1$, or $n = 0$ and* $\operatorname{rg} H_1(\widetilde{M}) = 1$, *then*

$$\text{(9)} \qquad \psi(\Delta(\widetilde{M})) = \{\pm \prod_{i=1}^{r} \Delta(t_1, \ldots, t_n, \omega_i t) \mid \Delta \in \Delta(M)\}.$$

If $n = 0$ and $\operatorname{rg} H_1(\widetilde{M}) > 1$, *then*

$$\text{(9')} \qquad \psi(\Delta(\widetilde{M})) = \{\pm (t^r - 1)^{-1} \prod_{i=1}^{r} \Delta(\omega_i t) \mid \Delta \in \Delta(M)\}.$$

Theorem 1.9.3 can be easily generalized to the case of finite Abelian covers. Theorem 1.9.1 can also be generalized to the case of Abelian branched covers of a sphere with branching at links—see [19], [25]. We emphasize that the results of [19], [25] do not follow directly from the generalization of Theorem 1.9.3 mentioned above; to obtain these results by the methods developed here, one requires a number of additional considerations which go beyond the scope of this paper.

We prove Theorem 1.9.3.

1.9.4. ***Lemma.*** *Let $B_1, B_2, \ldots, B_r$ be square matrices of the same order over a commutative ring. Let $b(t)$ be the matrix polynomial $B_1 + tB_2 + \ldots + t^{r-1}B_r$. Then the determinant of the matrix*

$$B = \begin{bmatrix} B_1 & B_2 & B_3 & \ldots & B_r \\ t^rB_r & B_1 & B_2 & \ldots & B_{r-1} \\ t^rB_{r-1} & t^rB_r & B_1 & \ldots & B_{r-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ t^rB_2 & t^rB_3 & t^rB_4 & \ldots & B_1 \end{bmatrix}$$

is $\prod_{i=1}^{r} \det b(\omega_i t)$.

Proof. Let E be the identity matrix of the same order as $B_1, \ldots, B_r$. We put $D = tE = \mathrm{diag}(t, t, \ldots, t)$. We denote the following two square matrices by α and β respectively:

$$\begin{bmatrix} E & E & \ldots & E \\ \omega_1 D & \omega_2 D & \ldots & \omega_r D \\ (\omega_1 D)^2 & (\omega_2 D)^2 & \ldots & (\omega_r D)^2 \\ \ldots & \ldots & \ldots & \ldots \\ (\omega_1 D)^{r-1} & (\omega_2 D)^{r-1} & \ldots & (\omega_r D)^{r-1} \end{bmatrix}, \quad \frac{1}{r}\begin{bmatrix} E & (\omega_1 D)^{-1} & (\omega_1 D)^{-2} & \ldots & (\omega_1 D)^{-(r-1)} \\ E & (\omega_2 D)^{-1} & (\omega_2 D)^{-2} & \ldots & (\omega_2 D)^{-(r-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ E & (\omega_r D)^{-1} & (\omega_r D)^{-2} & \ldots & (\omega_r D)^{-(r-1)} \end{bmatrix}.$$

Direct computation shows that $\beta\alpha$ is the identity matrix, and that $\beta B\alpha$ is the block-diagonal matrix $\mathrm{diag}(b(\omega_1 t), \ldots, b(\omega_r t))$. Thence follows the statement of the lemma.

1.9.5. *Proof of Theorem* 1.9.3. We consider the case $\partial M \neq \emptyset$. Let Y be a simple closed curve in M representing t. Since $\partial M \neq \emptyset$ and $\chi(M) = 0$, the manifold M can be collapsed onto one of its subcomplexes X, which contains Y and is obtained from Y by gluing s one-dimensional cells and s two dimensional cells with $s \geqslant 0$. We denote by $\widetilde{X}$ and $\widetilde{Y}$ the respective inverse images of X and Y in $\widetilde{M}$. (Clearly $\widetilde{Y}$ is a simple closed curve in $\widetilde{M}$.) We identify $H_1(X)$ with $H_1(M)$ and $H_1(\widetilde{X})$ with $H_1(\widetilde{M})$ via the inclusion isomorphisms. It is obvious that $\tau(M, Y) = \tau(X, Y)$ and $\tau^{\psi}(\widetilde{M}, \widetilde{Y}) = \tau^{\psi}(\widetilde{X}, \widetilde{Y})$. The non-zero part of the complex $C_*^{\psi}(\widetilde{X}, \widetilde{Y})$ is reduced to the boundary homomorphism $C_2 \to C_1$. With a suitable choice of natural bases the matrix of this homomorphism has the form of the matrix B from the formulation of Lemma 1.9.4, where $B_1, \ldots, B_r$ are square matrices of order s over the Laurent polynomial ring in the variables $t_1, \ldots, t_n, t^r$; the determinants $\det B$ and $\det(B_1 + tB_2 + \ldots + t^{r-1}B_r)$ represent respectively the torsions $\tau^{\psi}(\widetilde{X}, \widetilde{Y})$ and $\tau(X, Y)$. By Lemma 1.9.4 it follows that

$$(10) \qquad \tau^{\psi}(\widetilde{M}, \widetilde{Y}) = \prod_{i=1}^{r} \tau(M, Y)(t_1, \ldots, t_n, \omega_i t).$$

If $n \geqslant 1$, then $\operatorname{rg} H_1(\widetilde{M}) \geqslant \operatorname{rg} H_1(M) \geqslant 2$ and hence $\tau(M, Y) = \tau(M)(t-1) = \Delta(M)(t-1)$ and $\tau^{\psi}(\widetilde{M}, \widetilde{Y}) = \tau^{\psi}(\widetilde{M})(t^r-1) = \psi(\Delta(\widetilde{M}))(t^r-1)$. Hence we see that in the case $n \geqslant 1$ (10) implies (9). Let $n = 0$. Similar calculations show that $\tau(M, Y) = \Delta(M)$; if $\operatorname{rg} H_1(\widetilde{M}) = 1$, then $\tau^{\psi}(\widetilde{M}, \widetilde{Y}) = \psi(\Delta(\widetilde{M}))$; if $\operatorname{rg} H_1(\widetilde{M}) > 1$, then $\tau^{\psi}(\widetilde{M}, \widetilde{Y}) = \psi(\Delta(\widetilde{M}))(t^r-1)$. Therefore in the case $n = 0$ the statement of the theory also follows from (10).

The case of closed M reduces to the case of a manifold with boundary by cutting out a solid torus and applying Theorem 1.4.2.

1.10. The Alexander polynomial of a periodic link.
Let M be an oriented three-dimensional homology sphere (over $\mathbf{Z}$) equipped with an orientation-preserving homomorphism $f: M \to M$ of period r. We suppose that the set of f-periodic points of period less than r coincides with the set $\operatorname{Fix}(f)$ of fixed points and is a knot. We denote the knot $\operatorname{Fix}(f)$ by k and the projection $M \to M/f$ by p. It is easily verified that the factor space M/f is a manifold, and a homology sphere.

The definition of the Alexander polynomial of a link in S^3 given in §1.2 can be carried over in an obvious way to the case of links in M and M/f. Murasugi [30], in the case of knots, and Sakuma [37], in the general case, proved the following theorem, which makes it possible to calculate the Alexander polynomial of a periodic link l in M from the Alexander polynomials of the links $p(l)$ and $p(l \cup k)$. (For applications of this theorem, see [30].)

1.10.1. ***Theorem*** (see [30], [37]). *Let $l = l_1 \cup \ldots \cup l_n$ be a periodic link in M (that is, $l \subset M \setminus k$ and $f(l_i) = l_i$ for all $i = 1, \ldots, n$). Then*

$$\Delta_l(t_1, \ldots, t_n) = \Delta_{p(l)}(t_1, \ldots, t_n) \prod_{i=1}^{r-1} \Delta_{p(l \cup k)}(t_1, \ldots, t_n, \omega_i),$$

where $\omega_1, \ldots, \omega_{r-1}$ are the complex r-th roots of unity other than 1.

Proof. Let μ_i be the linking number of the knots $p(l_i)$ and $p(k)$ in M/f. It is easily seen that $\mu_i \neq 0$ and that the linking number of the knots l_i and k in M is also μ_i. By the Torres theorem

$$\Delta_{l \cup k}(t_1, \ldots, t_n, 1) = \Delta_l(t_1, \ldots, t_n)(t_1^{\mu_1} t_2^{\mu_2} \ldots t_n^{\mu_n} - 1),$$

$$\Delta_{p(l \cup k)}(t_1, \ldots, t_n, 1) = \Delta_{p(l)}(t_1, \ldots, t_n)(t_1^{\mu_1} t_2^{\mu_2} \ldots t_n^{\mu_n} - 1).$$

It is obvious that the projection $M \setminus (l \cup k) \to p[M \setminus (l \cup k)]$ is an r-fold cyclic cover, which induces in the one-dimensional homology the homomorphism $t_i \mapsto t_i$ for $i = 1, \ldots, n$ and $t_{n+1} \mapsto t_{n+1}^r$. By Theorem 1.9.3

$$\Delta_{l \cup k}(t_1, \ldots, t_n, t_{n+1}^r) = \prod_{i=1}^{r} \Delta_{p(l \cup k)}(t_1, \ldots, t_n, \omega_i t_{n+1}),$$

where $\omega_r = 1$. Substituting $t_{n+1} = 1$ here and comparing the resulting formula with the formulae mentioned above, we obtain the statement of the theorem.

1.11. The Fox-Milnor theorem and its generalization.
We recall that the ordered n-component links l and l' in S^m are called *cobordant* if there is an n-component submanifold of the cylinder $S^m \times [0, 1]$ each component of which is homeomorphic to $S^{m-2} \times [0, 1]$, and moreover the boundary of the i-th component is $(l_i \times 0) \cup (l_i' \times 1)$ for $i = 1, ..., n$. The following theorem of Fox and Milnor gave the historically first non-trivial obstruction to the cobordancy of knots and links in S^3. (This theorem was announced by Fox and Milnor in 1957; they published a detailed proof in 1966 [9].) We say that two elements a and a' of the field $\mathbf{Q}(t_1, ..., t_n)$ (or more generally two elements of the field $Q(G)$, where G is a free Abelian group) are *c-equivalent* if there are elements $b, b' \in \mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ (respectively $b, b' \in \mathbf{Z}[G]$) for which $ab\bar{b} = a'b'\overline{b'}$ and $\mathrm{aug}(b) = \mathrm{aug}(b') = 1$.

1.11.1. ***Theorem*** (Fox-Milnor). *The Alexander polynomials of cobordant links in S^3 have c-equivalent representatives.*

In fact Fox and Milnor [9] considered only knots, but their arguments generalize directly to the case of links (although the condition $\mathrm{aug}(b) = \mathrm{aug}(b') = 1$ requires additional consideration in this case). Kervaire [22] proved an analogue of Theorem 1.11.1 for knots in S^m with odd m. The following theorem generalizes the results of Fox-Milnor and Kervaire.

1.11.2. ***Theorem.*** *Let (M, V, V') be an orientable compact even-dimensional cobordism with $H_*(M, V) = 0$, whose boundary $(\partial M\backslash \mathrm{Int}(V \cup V'), \partial V, \partial V')$ is either empty or homeomorphic to the cylinder $\partial V \times [0, 1]$. We identify the groups $H_1(V)$, $H_1(V')$, and $H_1(M)$ using the isomorphisms induced by the inclusions $V \to M$, $V' \to M$. Then the Alexander functions $A(V)$ and $A(V')$ have c-equivalent representatives.*

We prove this theorem.

1.11.3. ***Lemma.*** *Let $C = (C_m \to \ldots \to C_0)$ be a free chain complex over a factorial Noetherian (commutative) ring. Let A be the matrix of the boundary homomorphism $C_{i+1} \to C_i$ (relative to certain bases). Then the greatest common divisor of the minors of order* $\mathrm{rg}\, A$ *of the matrix A is* $\mathrm{ord}(\mathrm{Tors}\, H_i(C))$.

Proof. We denote by J the cokernel of the boundary homomorphism $C_{i+1} \to C_i$. Since A is the relation matrix of the module J, the greatest common divisor in question is $\mathrm{ord}(\mathrm{Tors}\, J)$ (see [17], 31 and [2], Lemma 4.10). From the exact sequence $0 \to H_i(C) \to J \to C_{i-1}$ it follows that $\mathrm{Tors}\, J = \mathrm{Tors}\, H_i(C)$.

1.11.4. ***Lemma.*** *Let (X, Y) be a finite CW-pair, and let E_i be its i-dimensional Alexander module (see* §1.1). *If $H_i(X, Y) = 0$, then* $\mathrm{rg}\, E_i = 0$ *and* $\mathrm{aug}(\mathrm{ord}\, E_i) = \pm 1$.

Proof. Let θ be the projection $\mathbf{Z}[H_1(X)] \to \mathbf{Z}[H_1(X)^{\#}]$. Let A_j be the matrix of the boundary homomorphism $C^{\theta}_{j+1}(X, Y) \to C^{\theta}_j(X, Y)$ relative to the natural bases. We denote by A^0_j the integral matrix obtained from A_j by replacing the entries by their coefficient sums. We denote the number of j-dimensional cells in $X \setminus Y$ by r_j. The arguments in the proof of Theorem 1.6.1 and the condition $H_i(X, Y) = 0$ show that $\operatorname{rg} A^0_{i-1} + \operatorname{rg} A^0_i = \operatorname{rg} C_i(X, Y) = r_i$. Hence it follows that the obvious inequalities $\operatorname{rg} A^0_j \leqslant \operatorname{rg} A_j$ and $\operatorname{rg} A_{j-1} + \operatorname{rg} A_j \leqslant \operatorname{rg} C^{\theta}_j(X, Y) = r_j$ become equalities for $j = i$. Hence it follows in turn that $\operatorname{rg} E_i = 0$. By Lemma 1.11.3 the number $\operatorname{aug}(\operatorname{ord} E_i)$ divides all the minors of order $\operatorname{rg} A_i = \operatorname{rg} A^0_i$ of the matrix A^0_i and so by the same lemma divides

$$\operatorname{ord}(\operatorname{Tors} H_i(X, Y)) = \pm 1.$$

1.11.5. *Proof of Theorem* 1.11.2. We denote the i-dimensional Alexander module of the pair (M, V) by E_i. We put $r = (\dim M)/2$, $b = \prod_{i=0}^{r} \operatorname{ord} E_{2i+1}$ and $b' = \prod_{i=0}^{r} \operatorname{ord} E_{2i}$. By Lemma 1.11.4, $\operatorname{aug}(b) = \operatorname{aug}(b') = \pm 1$. By Theorem 1.1.1 $\tau(M, V) = b(b')^{-1}$. According to the duality theorem for torsions (see Appendix, Theorem 3) $\tau(M, V') = \overline{\tau(M, V)}^{-1}$. Therefore

$$A(V) = \tau(V) = \tau(M)\,\tau(M, V)^{-1} = \tau(M, V')\,\tau(V')\,\tau(M, V)^{-1} =$$
$$= \overline{\tau(M, V)}^{-1}\,\tau(M, V)^{-1}\tau(V') = (b\bar{b})^{-1}(b'\bar{b}')\,A(V').$$

Remark 1. In the study of the Alexander polynomials of links in S^3, there arises naturally the problem of the algebraic characterization of these polynomials, that is, the problem of finding algebraic conditions on a polynomial that are necessary and sufficient for its realizability as the Alexander polynomial of a link. The first such result was obtained by Seifert [38], who proved that a polynomial $\Delta \in \mathbf{Z}[t, t^{-1}]$ can be realized as the Alexander polynomial of a knot in S^3 if and only if $\operatorname{aug}(\Delta) = \pm 1$ and $\bar{\Delta} = t^{\nu}\Delta$ with even ν. For the Hosokawa polynomial, the algebraic characterization was obtained by Hosokawa himself [18]: for every $n \geqslant 2$ a polynomial $h \in \mathbf{Z}[t, t^{-1}]$ can be realized as the Hosokawa polynomial of an n-component link if and only if $\bar{h} = t^{\nu}h$ with even ν. The problem of characterizing the Alexander polynomials of links with at least two components is considerably more complicated (some partial results can be found in [17]).

Remark 2. The problem of characterizing the Alexander polynomials of links is closely related to the analogous problem concerning the Alexander polynomials of orientable connected closed three-dimensional manifolds. We denote this class of manifolds by $\mathfrak{M}$. The Alexander polynomial of a manifold $M \in \mathfrak{M}$ has the following properties: (i) if $\operatorname{rg} H_1(M) = 1$, then

$\operatorname{aug}(\Delta(M)) \neq 0$; (ii) if $\Delta \in \Delta(M)$, then $\bar{\Delta} = g^2\Delta$ with $g \in H_1(M)^{\#}$; (iii) if $n = \operatorname{rg} H_1(M)$ and I is the kernel of the augmentation $\mathbf{Z}[H_1(M)^{\#}] \to \mathbf{Z}$, then $\Delta(M) \subset I^{n-3}$; here if n is odd and $n \geqslant 3$, then for some $\Delta \in \Delta(M)$, $r \in \mathbf{Z}$, and $a \in I^{(n-3)/2}$ we have the inclusion $r\Delta - a^2 \in I^{n-2}$. Here statement (i) follows from Theorem 1.6.1; (ii) is easily deduced from Theorems 1.4.2 and 1.7.1, representing M as the result of surgery on the sphere along the framed link with even framing; statement (iii) is deduced similarly from the results of Traldi [43], appropriately generalized to the case of links in $\mathbf{Q}$-homology spheres. It is possible that the listed conditions exhaust all algebraic conditions on Alexander polynomials of manifolds in $\mathfrak{M}$. This is confirmed by the following facts. From the characterization theorem for the Hosokawa polynomial and from the proof of Theorem 1.8.1 it is not hard to deduce that every polynomial $\Delta \in \mathbf{Z}[t, t^{-1}]$ with $\operatorname{aug}(\Delta) \neq 0$ and with $\bar{\Delta} = t^{\nu}\Delta$, where ν is even, is realizable as the Alexander polynomial of a manifold $M \in \mathfrak{M}$ with $\operatorname{rg} H_1(M) = 1$. If Δ is a Laurent polynomial in the variables t_1, t_2 for which $\bar{\Delta} = t_1^{\nu_1}t_2^{\nu_2}\Delta$ with even ν_1, ν_2, then Δ can be realized as the Alexander polynomial of a manifold $M \in \mathfrak{M}$ with $\operatorname{rg} H_1(M) = 2$. This follows from Theorem 1.4.2 and a theorem of Bailey, according to which the product $\Delta(t_1 - 1)(t_2 - 1)$ can be realized as the Alexander polynomial of some link $l_1 \cup l_2 \subset S^3$ with $\mu(l_1, l_2) = 0$ (see [17]).

§2. Proof of Theorems 1.1.1, 1.1.2, and 1.1.3

2.1. *Proof of Theorem* 1.1.1. Theorem 1.1.1 follows from the following algebraic lemma.

2.1.1. ***Lemma.*** *Let $C = (C_m \to \dots \to C_0)$ be a free chain complex over an integral domain K with* $\operatorname{rg} H_i(C) = 0$ *for all* $i = 0, \dots, m$. *Let the modules* $C_0, C_1, \dots, C_m$ *be equipped with bases over* K. *If K is factorial and Noetherian, then*

$$\tau(C) = \prod_{i=0}^{m} [\operatorname{ord} H_i(C)]^{\varepsilon(i)} \quad (\textit{where } \varepsilon(i) = (-1)^{i+1}).$$

This lemma is proved in §2.1.4 using the results of §§2.1.2 and 2.1.3.

2.1.2. Auxiliary definition: matrix τ-chains.
Let C be the K-chain complex discussed in the formulation of Lemma 2.1.1. A *matrix chain* of the complex C is an arbitrary collection $\{a_0, a_1, \dots, a_m; B_0, B_1, \dots, B_{m-1}\}$ where a_i is a subset (possibly empty) of the set $\{1, 2, \dots, \operatorname{rg} C_i\}$; B_i is the matrix obtained from the matrix of the boundary homomorphism $C_{i+1} \to C_i$ (relative to the chosen bases) by crossing out the rows whose numbers are in a_{i+1} and the columns whose numbers are not in a_i. It is clear that B_i is a matrix over K of size $(\operatorname{rg} C_{i+1} - \operatorname{card} a_{i+1}) \times \operatorname{card} a_i$. A matrix chain

$$\{a_0, a_1, \dots, a_m; B_0, B_1, \dots, B_{m-1}\}$$

is called a *τ-chain* if $a_0 = \{1, 2, \ldots, \operatorname{rg} C_0\}$ and $\operatorname{rg} C_{i+1} - \operatorname{card} a_{i+1} = \operatorname{card} a_i$ for $i = 0, 1, \ldots, m-1$. Here $B_0, B_1, \ldots, B_{m-1}$ are square matrices and $\operatorname{card} a_i = \alpha_i(C)$, where

$$\alpha_i(C) = \operatorname{rg} C_i - \operatorname{rg} C_{i-1} + \ldots + (-1)^i \operatorname{rg} C_0 \tag{11}$$

and $i = 0, 1, \ldots, m$. It is easy to see that C has a matrix τ-chain if and only if $\alpha_i(C) \geqslant 0$ for all $i = 0, 1, \ldots, m$.

2.1.3. ***Lemma.*** *Under the hypotheses of Lemma* 2.1.1, *let A_i be the matrix of the boundary homomorphism $C_{i+1} \to C_i$ with respect to the chosen bases. Let $\{a_0, a_1, \ldots, a_m;\ B_0, \ldots, B_{m-1}\}$ be a matrix τ-chain of the complex C. Then* $\operatorname{rg} A_i = \alpha_i(C)$ *and*

$$\tau(C) \prod_{i=0}^{[(m-1)/2]} \det B_{2i} = (-1)^W \prod_{i=0}^{[m/2]-1} \det B_{2i+1}, \tag{12}$$

where $W = \sum_{i=0}^{m-1} \operatorname{card}\{(x, y) \in \mathbf{Z} \times \mathbf{Z} \mid 1 \leqslant x < y,\ x \notin a_i,\ y \in a_i\}$.

Proof. Replacing the ground ring K by its quotient field (see §0.1) if necessary, we can assume that K is a field. We put $r_i = \dim C_i$ and denote the boundary homomorphism $C_{i+1} \to C_i$ by ∂_i. The first statement of the lemma follows from the equality $\operatorname{rg} A_i = \dim(\operatorname{Im} \partial_i)$ and from the exact sequence $\operatorname{Im} \partial_i \to C_i \to C_{i-1} \to \ldots \to C_0 \to 0$.

We prove formula (12).

Case 1. $\det B_i = 0$ for some $i = 0, 1, \ldots, m-1$. We show that in this case either $\det B_{i-1} = 0$ or $\det B_{i+1} = 0$, so that both sides of (12) vanish. We suppose that the matrices B_{i-1} are B_{i+1} are non-singular and reduce this to a contradiction. From the equality $A_{i+1}A_i = 0$ and the fact that B_{i+1} is non-singular it follows that the subspace of the space K^{r_i} generated by the rows of A_i is already generated by the rows of this matrix whose numbers are not in a_{i+1}. Similarly, from the fact that B_{i-1} is non-singular it follows that the subspace of $K^{r_{i+1}}$ generated by the columns of A_i is already generated by the columns whose numbers are in a_i. Thus, deleting from A_i the columns with numbers not in a_i and rows whose numbers are in a_{i+1} does not affect the rank of A_i. In other words, $\operatorname{rg} A_i = \operatorname{rg} B_i$. Since B_i is a square matrix of order $\alpha_i(C)$ and $\det B_i = 0$, it follows that $\operatorname{rg} A_i < \alpha_i(C)$. This inequality contradicts the equality $\operatorname{rg} A_i = \alpha_i(C)$ established above.

Case 2. $\det B_i \neq 0$ for all i. We denote the chosen base of the module C_i by $c_i = (c, \ldots, c_i^{r_i})$. Let b_i be a subsequence of the sequence c_i consisting of the vectors whose numbers are not in a_i, where $i = 1, \ldots, m$. Since $\operatorname{rg} B_i = \alpha_i(C) = \operatorname{rg} A_i$, the sequence $\partial_i(b_{i+1})$ is a base for the space $\operatorname{Im} \partial_i$. Direct calculation shows that $[\partial_i(b_{i+1})b_i/c_i] = (-1)^{n_i} \det B_i$, where

$$n_i = \operatorname{card}\{(x, y) \in \mathbf{Z} \times \mathbf{Z} \mid 1 \leqslant x < y,\ x \notin a_i,\ y \in a_i\}.$$

Hence, taking into account the definition of torsion, we have (12).

2.1.4. *Proof of Lemma* 2.1.1. We denote the matrix of the boundary homomorphism $C_{i+1} \to C_i$ by A_i. We denote by I_i the ideal of the ring K generated by the minors of order $\operatorname{rg} A_i = \alpha_i(C)$ of the matrix A_i (see §§2.1.2 and 2.1.3). We prove the equality

$$\tau(C) \prod_{i=0}^{[(m-1)/2]} I_{2i} = \prod_{i=0}^{[m/2]-1} I_{2i+1}. \tag{13}$$

If B_{2i} is a square submatrix of A_{2i} of order $\alpha_{2i}(C)$ with $i = 0, 1, \ldots, [(m-1)/2]$, then, as is easily seen, there is a (unique) matrix τ-chain

$$\{a_0, a_1, \ldots, a_m;\ B_0', B_1', \ldots, B_{m-1}'\}$$

of the complex C such that $B_{2i}' = B_{2i}$ for all i. Hence the inclusion of the left hand side of (13) in the right hand side follows from Lemma 2.1.3. The reverse inclusion is proved similarly.

If $\tau(C) = xy^{-1}$, where x and y are non-zero elements of K, then by multiplying both sides of the equality (13) by y, taking greatest common divisors of the elements of the resulting ideals of K, and applying the result of Lemma 1.11.3, we obtain the equality

$$x \prod_{i=0}^{[(m-1)/2]} \operatorname{ord} H_{2i}(C) = y \prod_{i=0}^{[m/2]-1} \operatorname{ord} H_{2i+1}(C).$$

Hence we have the statement of the lemma.

2.2. *Proof of Theorem* 1.1.2.

2.2.1. ***Lemma.*** *Let M be a compact m-dimensional manifold; let σ be the involution of the ring $\mathbf{Z}[H_1(M)]$ taking $h \in H_1(M)$ into $w_1(h)h^{-1}$; and let φ be a ring homomorphism of $\mathbf{Z}[H_1(M)]$ into a factorial Noetherian ring. Then for any i*

$$\operatorname{ord}(\operatorname{Tors} H_i^{\varphi}(M)) = \operatorname{ord}(\operatorname{Tors} H_{m-i-1}^{\varphi\circ\sigma}(M, \partial M)).$$

This lemma is proved in §2.2.3. For its proof we require the (well-known) constructions of dual chain complexes and dual CW-decompositions of manifolds. These constructions are reproduced in §2.2.2.

We prove Theorem 1.1.2. If $\Delta(M) = 0$, then from the definitions $A(M) = 0$ and the statement is obvious. We suppose that $\Delta(M) \neq 0$. We denote the i-dimensional Alexander module of the manifold M by E_i. It is clear that $E_0 = \mathbf{Z}$ and $E_3 = 0$. Hence $\operatorname{rg} E_0 = \operatorname{rg} E_3 = 0$. Since $\operatorname{ord} E_1 = \Delta(M) \neq 0$, $\operatorname{rg} E_1 = 0$. Hence by the equality $\chi(M) = \sum_{i=0}^{3} (-1)^i \operatorname{rg} E_i$ and the condition $\chi(M) \leqslant 0$, it follows that $\operatorname{rg} E_2 = 0$. Thus $\operatorname{ord} E_i \neq 0$ for all i, and so $A(M) = \prod_{i=0}^{3} [\operatorname{ord} E_i]^{\varepsilon(i)}$. The order of the module E_3 is represented by the identity $1 \in \mathbf{Z}[H^{\#}]$. The order of the module E_0 is represented by the identity if rg $H \geqslant 2$, and is represented by the difference $t-1$ if rg $H = 1$. So to complete the proof it suffices to calculate ord E_2 = ord(Tors E_2), which is not hard to do by using Lemma 2.2.1.

2.2.2. Auxiliary definitions.

(i) Let $C = (C_m \to \dots \to C_0)$ be a free chain complex over a commutative ring K. We say that the chain complex $C^* = (C'_m \to C'_{m-1} \to \dots \to C'_0)$ over K is *dual* to the complex C if, firstly, $C'_i = \operatorname{Hom}_K(C_{m-i}, K)$ for all i, and secondly, for $i < m$ the boundary homomorphism $\partial'_i \colon C'_{i+1} \to C'_i$ is $(-1)^{m-i}\, \partial^*_{m-i-1}$ (that is, for any $c \in C_{m-i}$ and $d \in C'_{i+1}$ we have $\partial'_i(d)(c) = (-1)^{m-i} d(\partial_{m-i-1}(c))$. It is clear that if the complex C is equipped with a distinguished basis, then the complex C^* is naturally equipped with the "dual" basis. Here the matrix of the homomorphism ∂'_i is obtained from that of the homomorphism ∂_{m-i-1} by transposition and multiplication by $(-1)^{m-i}$. If the ring K is factorial and Noetherian, then by Lemma 1.11.3 ord (Tors $H_i(C^*)$) = ord (Tors $H_{m-i-1}(C)$) for all i.

(ii) Let X be a piecewise-linear triangulation of the compact m-dimensional manifold M. It is well known (see, for example, [26]) that if a is a simplex of the triangulation X, then the union of all simplexes of the first barycentric subdivision of X which have as a vertex the barycentre of the simplex a, and which have no other points in common with a, is a cell of dimension $m - \dim a$. This cell is called the *dual* of a, and is denoted by a^*. If $a \subset \partial M$, then to the simplex a there corresponds, in addition to a^*, the cell a^*_∂, which is the dual of a in ∂M. It is clear that $a^*_\partial \subset \partial a^*$. The cells of type a^* and a^*_∂ form a CW-decomposition of M which is "dual" to the triangulation X. This decomposition is denoted by X^*. The cells of type a^*_∂ constitute a decomposition ∂X^* of the manifold ∂M.

2.2.3. *Proof of Lemma* 2.2.1. The lemma follows from the results of §2.2.2 and the well-known fact that if X is a piecewise-linear triangulation of the manifold M, then the chain complex $C^{\varphi\circ\sigma}_*(X^*, \partial X^*)$ is dual to the complex $C^{\varphi}_*(X)$ (see, for example, [26]).

2.3. ***Lemma*** (obvious). *Let K and K' be integral domains, and let ψ be a ring homomorphism $K \to K'$. Let $Q(\psi)$ be the subring of $Q(K)$ consisting of elements of the form xy^{-1} with $x, y \in K$, $\psi(y) \neq 0$. Then $K \subset Q(\psi)$ and the formula $\psi_\Box(xy^{-1}) = \psi(x)\psi(y)^{-1}$ defines a ring homomorphism $\psi_\Box \colon Q(\psi) \to K'$, extending ψ.*

2.4. ***Lemma.*** *Under the conditions of Lemma 2.3, let $C = (C_m \to \dots \to C_0)$ be a free chain complex over K; let the modules $C_0, C_1, \dots, C_m$ be equipped with the bases $c_0, c_1, \dots, c_m$ respectively; let C' be the K'-chain complex $K' \otimes_K C$; and for each i let the module $C'_i = K' \otimes_K C_i$ be given the basis c'_i induced by the basis c_i (that is, $c'_i = 1 \otimes c_i$). Then, if $\tau(C') \neq 0$, $\tau(C) \in Q(\psi)$ and $\psi_\Box(\tau(C)) = \tau(C')$.*

Proof. We denote the boundary homomorphisms $C_{i+1} \to C_i$ and $C'_{i+1} \to C'_i$ by ∂_i and ∂'_i respectively. Let b'_i be a subsequence of the sequence c'_i for which $\partial'_{i-1}(b'_i)$ is a basis for the $Q(K')$-module $Q(K') \otimes_{K'} \operatorname{Im} \partial'_{i-1}$. We denote by b_i the inverse image of the sequence b'_i under the natural bijection $c_i \to c'_i$ of the bases. Let D_i (respectively D'_i) be the transition matrix from the basis c_i to the sequence $\partial_i(b_{i+1})b_i$ (respectively from c'_i to $\partial'_i(b'_{i+1})b'_i$). It is obvious that D_i is a matrix over K and that $D'_i = \psi(D_i)$ (elementwise). If $\tau(C') \neq 0$, then the complex $Q(K') \otimes C'$ is acyclic, D'_i is a square matrix, and $\det D'_i \neq 0$ for all i. So $\det D_i \neq 0$ for all i, $\tau(C) = \prod_i (\det D_i)^{\varepsilon(i)} \in Q(\psi)$ and

$$\psi_\square(\tau(C)) = \prod_i (\psi(\det D_i))^{\varepsilon(i)} = \prod_i (\det D'_i)^{\varepsilon(i)} = \tau(C').$$

2.5. *Proof of Theorem* 1.1.3. Let $\psi_\square\colon Q(\psi) \to Q(\Pi)$ be the ring homomorphism extending ψ afforded by Lemma 2.3. By the equality $A(M) = \tau(M)$, to prove the theorem it suffices to establish the inclusion $\tau(M) \subset Q(\psi)$ and the equality $\psi_\square(\tau(M)) = \tau^\varphi(M)$. If $\tau^\varphi(M) \neq 0$, then this inclusion and equality follow from Lemma 2.4. We suppose that $\tau^\varphi(M) = 0$, and show that $\tau(M) \subset Q(\psi)$ and $\psi_\square(\tau(M)) = 0$.

We consider the case $\partial M \neq \emptyset$. If $\chi(M) \neq 0$, then $\tau(M) = 0$, and the required assertion is obvious. Let $\chi(M) = 0$. It is clear that M can be collapsed onto a finite two-dimensional subcomplex of itself, say X, which has one zero-dimensional cell, s one-dimensional cells, and $(s-1)$ two-dimensional cells, with $s \geqslant 1$. We identify the groups $H_1(X)$ and $H = H_1(M)$ via the inclusion isomorphism. Here $\tau(M) = \tau(X)$. We denote by A_i the matrix of the boundary homomorphism $C^\theta_{i+1}(X) \to C^\theta_i(X)$ with $i = 0, 1$. It is obvious that A_1 is a matrix of size $(s-1) \times s$ and that A_0 is a column vector whose elements, with a suitable choice of natural bases, have the form $h_1 - 1, \ldots, h_s - 1$, where $h_1, \ldots, h_s$ are the generators of $H^\#$ represented by the one-dimensional cells of X. Since $\psi(H^\#) \neq 1$, there is some j for which $\psi(h_j) \neq 1$. If B is the matrix obtained from A_1 by deleting the j-th column, then by formula (12) $(\det B)[\psi(h_j) - 1]^{-1} \in \tau(X)$. Hence we have the inclusion $\tau(X) \subset Q(\psi)$. If $\psi(\det B) \neq 0$, then, as is easily seen, the complex $Q(\Pi) \otimes C^\theta_*(X)$ is acyclic, in contradiction to the assumption that $\tau^\varphi(X) = 0$. Therefore $\psi(\det B) = 0$ and hence $\psi_\square(\tau(X)) = 0$.

The case of closed M is considered similarly, the role of the space X being played by a suitable CW-decomposition of M.

Remark 1. For chain complexes over principal ideal rings Lemma 2.1.1 was first proved by Milnor [27], using the structure theorem for modules over such rings. A formula analogous to (13) was obtained by Buchsbaum and Eisenbud [50] for acyclic free chain complexes over commutative Noetherian rings.

Remark 2. Using Lemma 2.4, it is not hard to generalize the Alexander formula $\operatorname{aug}(\Delta_k) = \pm 1$, where k is a knot in S^3, to the following assertion:

if X is a finite CW-complex with the rational homology of a circle, and if t is a generator of the group $H_1(X)^\#$, then $A(X)(t-1) \subset Q(\text{aug})$, (where aug is the augmentation $\mathbf{Z}\, H_1(X)^\#] \to \mathbf{Z}$) and

$$\text{aug}_\square\,(A(X) \times (t-1)) = \prod_i [\text{ord}(\text{Tors } H_i(X))]^{\varepsilon(i)}.$$

Similarly, one can also carry over Theorems 1.9.1, 1.9.2, and 1.9.3 to the many-dimensional situation. We state here a many-dimensional version of Theorem 1.9.1. Let $N \to S^m$ be a branched cyclic r-fold cover, with branching at the knot $k \subset S^m$, where $m \geqslant 3$. Let $\omega_1, \ldots, \omega_{r-1}$ be ring homomorphisms $\mathbf{Z}[t, t^{-1}] \to \mathbf{C}$ for which $\omega_1(t), \ldots, \omega_{r-1}(t)$ are the r-th roots of unity other than 1. If N is a rational homology sphere, then $A_k \subset Q(\omega_i)$ for all i, and for any $A \in A_k$ we have

$$\prod_{i=1}^{m-1} [\text{ord}\, H_i(N)]^{\varepsilon(i)} = \pm r \prod_{i=1}^{r-1} (\omega_i)_\square (A).$$

This formula is significant only for odd m: if m is even, then, as is easily verified by using Lemma 2.2.1, both the left and right hand sides of the above formula are ± 1.

§3. Refined torsion and the refined Alexander function

3.1. Preliminary definitions.

3.1.1. The torsion $\check{\tau}$.

Let $C = (C_m \to \ldots \to C_0)$ be a chain complex over a field F. We suppose that for each $i = 0, 1, \ldots, m$ a basis is fixed in each of the vector spaces C_i and $H_i(C)$. In this situation, we define the torsion $\check{\tau}(C) \in F$ in the following way. Let c_i be the fixed basis in C_i; let h_i be a sequence of vectors in the space $\text{Ker}(\partial_{i-1}\colon C_i \to C_{i-1})$ whose images under the projection $\text{Ker}\, \partial_{i-1} \to H_i(C)$ form the given basis in $H_i(C)$; and let b_i be a sequence of vectors in C_i for which $\partial_{i-1}(b_i)$ is the basis in $\text{Im}\, \partial_{i-1}$. We put $N(C) = \sum_{i=0}^{m} \alpha_i(C)\beta_i(C)$, where $\alpha_i(C)$ is defined by (11) and

$$\beta_i(C) = \dim H_i(C) - \dim H_{i-1}(C) + \ldots + (-1)^i \dim H_0(C).$$

For every i the sequence $\partial_i(b_{i+1})h_i b_i$ is a basis in C_i; the product

$$(-1)^{N(C)} \prod_{i=0}^{m} [\partial_i(b_{i+1})\, h_i b_i / c_i]^{\varepsilon(i)} \in F$$

does not depend on the choices made (see [28]). This product is $\check{\tau}(C)$. The definition here differs from that given in [28] by the presence of the factor $(-1)^{N(C)}$, which enables us to simplify the later statements slightly. It is clear that $\check{\tau}(C) \neq 0$, and that if C is acyclic, then $\check{\tau}(C) = \tau(C)$. When it is necessary to emphasize the dependence of $\check{\tau}(C)$ on the chosen bases $h_0, h_1, \ldots, h_m$, say, for the spaces $H_0(C), H_1(C), \ldots, H_m(C)$, this torsion is denoted by $\check{\tau}(C;\ h_0, h_1, \ldots, h_m)$.

3.1.2. Homological orientation.

A *homological orientation*, or, more briefly, *h-orientation*, of the finite *CW*-pair (X, Y) is an arbitrary orientation of the vector space $H_*(X, Y; \mathbf{R}) = \bigoplus_{i \geqslant 0} H_i(X, Y; \mathbf{R})$. If $H_*(X, Y; \mathbf{R}) = 0$, then the pair (X, Y) has a single h-orientation, in the remaining cases (X, Y) has two opposite h-orientations.

3.2. Refined torsion.

Refined (or sign-determined) torsion is defined in the same situation as the usual Reidemeister torsion, but with one additional condition: the *CW*-complex (*CW*-pair, *pl*-manifold) in question is supposed to be equipped with an h-orientation. The presence of the h-orientation makes it possible, having slightly improved the definition of the torsion, to remove the indeterminacy of its sign; the result is the refined torsion. If the h-orientation is replaced by the opposite one (when this is possible), the refined torsion is multiplied by -1. The refined torsion becomes the usual Reidemeister torsion when it is considered only up to multiplication by -1.

Now we pass to the precise definitions. Let (X, Y) be a homologically oriented finite *CW*-pair, and let φ be a ring homomorphism from the ring $\mathbf{Z}[H_1(X)]$ into an integral domain K. We define the refined torsion $\tau_0^\varphi(X, Y)$. We choose bases $h_0, h_1, \ldots$ for the spaces $H_0(X, Y; \mathbf{R})$, $H_1(X, Y; \mathbf{R})$, ... respectively, so that the basis $h_0, h_1, \ldots$, for the space $H_*(X, Y; \mathbf{R})$ determines the chosen orientation for this space. We consider a base sequence e of oriented cells of a maximal Abelian covering space of X (see Section 0.2). By what was said in §0.2, the sequence e determines bases in the chain modules of the complexes $C_*^\varphi(X, Y)$ and $C_*(X, Y; \mathbf{R})$. We consider the torsions $\tau = \tau(C_*^\varphi(X, Y))$ and $\xi = \check{\tau}(C_*(X, Y; \mathbf{R}); h_0, h_1, \ldots)$ corresponding to these bases. Here $\tau \in Q(K)$ and $\xi \in \mathbf{R}$, $\xi \neq 0$. We put $\tau_0^\varphi(X, Y, e) =$ $= \operatorname{sign}(\xi)\tau$, where sign denotes the sign of a number ($\operatorname{sign}(\xi) = \pm 1$). The totality of the torsions $\tau_0^\varphi(X, Y, e)$ corresponding to all possible sequences e is $\tau_0^\varphi(X, Y)$. It is easily verified that on replacing the bases $h_0, h_1, \ldots$ by other bases giving the chosen homological orientation, the number $\operatorname{sign}(\xi)$, and consequently the torsion $\tau_0^\varphi(X, Y, e)$, are unchanged. The torsion $\tau_0^\varphi(X, Y, e)$ is also unchanged by a change in the orientation of the cells of e, and by renumbering these cells: in the given operations the torsions τ and ξ change sign simultaneously. Hence we can see that $\tau_0^\varphi(X, Y)$ is an "element of $Q(K)$ determined up to multiplication by $\varphi(g)$ with $g \in H_1(X)$". It is obvious that $\tau^\varphi(X, Y) = \pm\tau_0^\varphi(X, Y)$.

3.2.1. ***Theorem.*** *Under the conditions of Theorem* 0.2.1, *if the pairs* (X, Y) *and* (X', Y') *are homologically oriented, and if the isomorphism* $H_*(X, Y; \mathbf{R}) \to H_*(X', Y'; \mathbf{R})$ *induced by the given simple homotopy equivalence* $(X, Y) \to (X', Y')$ *preserves the homological orientation, then* $\tau_0^{\varphi'}(X', Y') = \tau_0^\varphi(X, Y)$.

This theorem enables us to study in particular the refined torsions of h-oriented compact manifolds (see §0.3).

Proof of Theorem 3.2.1. We restrict ourselves to the case $Y = Y' = \emptyset$. As is known (see, for example, [36] or [48]), every simple homotopy equivalence can be written as the composition of elementary (cell) expansions and elementary collapses. We recall that the map $f: X \to X'$ is called an elementary cell expansion if f is a cell embedding; $X' = f(X) \cup B$, where B is a closed j-dimensional ball intersecting $f(X)$ in a $(j-1)$-dimensional ball $D \subset \partial B$; and the CW-decomposition of X' is obtained from the decomposition of $f(X)$ by adding two cells: $\operatorname{Int} B$ and $\partial B \setminus D$. Elementary collapses are maps homotopically inverse to elementary cell expansions. So it suffices to prove the theorem for an elementary cell expansion $f: X \to X'$.

We identify X and $f(X)$ by means of f. We denote the chain complexes $C_*^{\varphi}(X)$ and $C_*^{\varphi'}(X)$ by C_* and C_*' respectively. Let e_{j-1} and e_j be elements of the modules C_{j-1}' and C_j', respectively, that can be represented by oriented cells in the maximal Abelian cover of X' situated over $\partial B \setminus D$ and $\operatorname{Int} B$. It may be supposed, by transferring one of these cells by a sliding homomorphism if necessary, and changing its orientation, that these two cells are incident with incidence coefficient 1. In addition, we can assume that the domain of values of the homomorphisms φ and φ' is a field.

If the complex C_* is not acyclic, then C_*' is also not acyclic, and both the torsions in the formulation of the theorem are zero. We suppose that C_* and C_*' are acyclic. Let c_i be the natural basis in C_i, and let b_i be a sequence of elements of the vector space C_i for which $\partial_{i-1}(b_i)$ is a basis for $\operatorname{Im}(\partial_{i-1}: C_i \to C_{i-1})$. It is obvious that $C_i' = C_i$ for $i \neq j-1, j$, and that c_i, e_i is the natural basis for C_i' for $i = j-1, j$. It is not hard to verify that $\operatorname{Im} \partial_i' = \operatorname{Im} \partial_i$ for $i \neq j-1$ and that $\partial_{j-1}(b_j), \partial_{j-1}'(e_j)$ is a basis for $\operatorname{Im} \partial_{j-1}'$. It follows immediately from the definition of the torsion that the ratio $\tau(C_*')/\tau(C_*)$ is

$$\left(\frac{[\partial_j(b_{j+1})\, b_j e_j / c_j e_j]}{[\partial_j(b_{j+1})\, b_j / c_j]}\right)^{\varepsilon(j)} \times \left(\frac{[\partial_{j-1}(b_j)\, \partial_{j-1}'(e_j)\, b_{j-1}/c_{j-1} e_{j-1}]}{[\partial_{j-1}(b_j) b_{j-1}/c_{j-1}]}\right)^{\varepsilon(j-1)}.$$

It is easily verified that the first factor is equal to 1, while the second one is equal to $(-1)^{\alpha}$, where α is the number of terms of the sequence b_{j-1}. Since C_* is acyclic, $\alpha = \alpha_{j-2}(C_*)$.

We put $E = C_*(X;\ \mathbf{R})$ and $E' = C_*(X';\ \mathbf{R})$. We equip E and E' with bases corresponding to the same sequences of cells as the bases considered above of the complexes C_* and C_*'. We equip the $\mathbf{R}$-modules $H_i(E)$ and $H_i(E')$, where $i = 0, 1, \ldots$, with bases defining the given orientation in $H_*(X;\ \mathbf{R})$. Arguments analogous to the ones above show that $\check{\tau}(E')/\check{\tau}(E) = (-1)^{\beta}$, where $\beta = \alpha_{j-2}(E) + \beta_{j-1}(E) + N(E) + N(E')$. It is obvious that $\alpha_{j-2}(E) = \alpha_{j-2}(C_*)$. As is easily verified, $N(E') = N(E) + \beta_{j-1}(E)$. Hence $\beta \equiv \alpha \pmod 2$, and so $\tau_{\mathfrak{o}}^{\varphi}(X) = \tau_{\mathfrak{o}}^{\varphi'}(X')$.

3.3. The refined Alexander function.

The *refined Alexander function* $A_0(X, Y)$ of a homologically oriented finite CW-pair (X, Y) is the refined torsion $\tau_0^\theta(X, Y)$, where θ is the projection $\mathbf{Z}[H_1(X)] \to \mathbf{Z}[G]$, $G = H_1(X)^\#$. From the results of §3.2 and Theorem 1.1.1 it follows that $A_0(X, Y)$ is an element of the field $Q(G)$ defined up to multiplication by elements of the group G; $A(X, Y) = \pm A_0(X, Y)$; when the h-orientation is replaced by the opposite one (which is possible if $H_*(X, Y;\ \mathbf{R}) \neq 0$), the function $A_0(X, Y)$ is multiplied by -1. If $H_*(X, Y;\ \mathbf{R}) = 0$, then $A_0(X, Y)$ is calculated from $A(X, Y)$; by Lemma 1.11.4, $A(X, Y)$ can be represented as the fraction ab^{-1} with $a, b \in \mathbf{Z}[G]$, $\mathrm{aug}(a) > 0$ and $\mathrm{aug}(b) > 0$; here, as follows from Lemma 2.4, $ab^{-1} \in A_0(X, Y)$. It is not hard to show that A_0 is an invariant of homotopy equivalences preserving h-orientation (but not necessarily simple).

3.4. Properties of the refined torsion and refined Alexander function.

Most of the properties of torsion and the Alexander function discussed in §§0, 1, and 2 can be sharpened to properties of the corresponding refined invariants. Here we restrict ourselves to those properties of the refined torsion and refined Alexander function that are needed below.

3.4.1. ***Theorem.*** *Under the conditions of Theorem* 0.2.2, *let the spaces X, Y and the pair (X, Y) be homologically oriented. Let these orientations be coordinated in the following way: for some (and then for any) choice of bases over* $\mathbf{R}$ *for the real homology groups of X, Y and the pair (X, Y) that determine the given orientations, the torsion of the homology sequence with coefficients in* $\mathbf{R}$ *of the pair (X, Y), considered as a chain complex over* $\mathbf{R}$, *is positive. If $\tau^\varphi(X, Y) \neq 0$, or $\tau^{\varphi\circ j}(Y) \neq 0$, then $\tau_0^\varphi(X) = (-1)^\mu \tau_0^{\varphi\circ j}(Y)\tau_0^\varphi(X, Y)$ where* $\mu = \sum_{i=0}^{\dim X} [(\beta_i+1)(\beta_i'+\beta_i'') + \beta_{i-1}'\beta_i'']$ *with* $\beta_i = \sum_{r=0}^{i} \mathrm{rg}\, H_r(X)$

$\beta_i' = \sum_{r=0}^{i} \mathrm{rg}\, H_r(Y)$, *and* $\beta_i'' = \sum_{r=0}^{i} \mathrm{rg}\, H_r(X, Y)$.

We remark that if $H_*(X, Y;\ \mathbf{R}) = 0$, then μ is even, and that if the h-orientations of X, Y, and the pair (X, Y) are not coordinated, then by replacing one of these orientations by its opposite we obtain a coordinated triple of orientations.

It is not hard to deduce Theorem 3.4.1 from the following lemma, which refines and generalizes Theorem 0.1.1.

3.4.2. ***Lemma.*** *Under the conditions of Theorem* 0.1.1, *let the ground ring be a field; let the homology modules of the complexes C, C', and C'' be equipped with bases; and let $\mathscr{H}$ be the homology sequence of the pair (C, C'): $H_m(C') \to H_m(C) \to H_m(C'') \to \ldots \to H_0(C') \to H_0(C'')$ considered as an (acyclic) chain complex (here $m = \dim C$). Then*

$$\check{\tau}(C) = (-1)^{\mu+\nu}\check{\tau}(C') \times \check{\tau}(C'')\tau(\mathscr{H})$$

where $\nu = \sum_{i=0}^{m} \alpha_i(C'')\alpha_{i-1}(C')$ *and*

$$\mu = \sum_{i=0}^{m} [(\beta_i(C) + 1)(\beta_i(C') + \beta_i(C'')) + \beta_{i-1}(C')\beta_i(C'')]$$

(*the definition of the numbers* α_i *and* β_i *can be found in* §§2.1.2 *and* 3.1.1). *In particular, if* C, C', *and* C'' *are acyclic, then* $\tau(C) = (-1)^{\nu}\tau(C')\tau(C'')$.

Proof. According to [28], $\check{\tau}(C) = (-1)^{\chi}\check{\tau}(C')\check{\tau}(C'')\tau(\mathscr{H})$, where χ is an integer computed as follows. We denote by x_i and x_i'' the dimensions of the images of the inclusion homomorphisms $H_i(C') \to H_i(C)$ and $H_i(C) \to H_i(C'')$ respectively. We denote by d_i' and d_i'' the dimensions of the images of the boundary homomorphisms $C'_{i+1} \to C'_i$ and $C''_{i+1} \to C''_i$ respectively. Then

$$\chi = N(C) - N(C') + N(C'') + \sum_{i=0}^{m} (x_i d_i'' + x_i'' d_{i-1}' + d_{i-1}' d_i'').$$

All the numbers on the right hand side of the last formula can be expressed in terms of $\alpha_i(C)$, $\alpha_i(C'')$, $\beta_i(C)$, $\beta_i(C')$, $\beta_i(C'')$ with $i = 0, 1, \ldots, m$. For example, $d_i' = \alpha_i(C') - \beta_i(C')$ and $x_i = \beta_i(C) + \beta_{i-1}(C') - \beta_i(C'')$. Substituting these expressions into the formula for χ, and cancelling similar terms (taking into account the equalities $\alpha_i(C) = \alpha_i(C') + \alpha_i(C'')$ and $\alpha_m = \beta_m$), we obtain $\chi \equiv \mu + \nu \pmod 2$.

3.4.3. Theorem. *Under the conditions of Theorem* 1.1.3, *let the manifold* M *be homologically oriented. If* $\operatorname{rg} H \geqslant 2$, *then* $\tau_0^{\mathscr{E}}(M) = \psi(A_0(M))$. *If* $\operatorname{rg} H = 1$, *then* $\tau_0^{\mathscr{E}}(M) = \widetilde{\psi}(A_0(M))$.

The proof of this theorem is obtained by a simple modification of the proof of Theorem 1.1.3, and is therefore omitted. For similar reasons we also omit the proof of the following theorem.

3.4.4. Theorem. *Under the conditions of Theorem* 1.11.2, *let the manifolds* M, V, *and* V' *be homologically oriented, and let the inclusion isomorphisms* $H_*(V;\ \mathbf{R}) \to H_*(M;\ \mathbf{R})$ *and* $H_*(V';\ \mathbf{R}) \to H_*(M;\ \mathbf{R})$ *preserve the orientations. Then the functions* $A_0(V)$ *and* $A_0(V')$ *have c-equivalent representatives.*

Remark 1. Every oriented (in the usual sense of the term) closed odd-dimensional manifold M has a canonical h-orientation. Namely, we put $m = \dim M$; for $i = 0, 1, \ldots, (m-1)/2$ we fix an arbitrary orientation in $H_i(M;\ \mathbf{R})$ and the orientation in $H_{m-i}(M;\ \mathbf{R})$ that is dual to it with respect to the intersection form $H_i(M;\mathbf{R}) \times H_{m-i}(M;\ \mathbf{R}) \to \mathbf{R}$; the direct sum of these orientations is the canonical orientation of the space $H_*(M;\ \mathbf{R})$. The refined Alexander function of M corresponding to this orientation is denoted by $A_0(M)$. On replacing the given orientation of M by the opposite orientation, the canonical h-orientation is also replaced by its opposite one if $s = \sum_{i=0}^{(m-1)/2} \operatorname{rg} H_i(M)$ is odd, and is unchanged if s is even. So $A_0(-M) = (-1)^s A_0(M)$. Hence it can be seen that in the case when $A(M) \neq 0$ and s

is odd the invariant $A_0(M)$ cannot be calculated from $A(M)$. From Theorem 5 of the Appendix it is not hard to deduce that if $w_{m-1}(M) = 0$, then the function $A_0(M)$ has a canonical representative: there also exists a unique element $A \in A_0(M)$ such that $\bar{A} = \pm A$. Here if $m \equiv 3 \pmod 4$, then $\bar{A} = A$.

An oriented four-dimensional manifold has in general no natural h-orientation. This can be seen for example from the fact that complex conjugation in CP^2 preserves the usual orientation, and inverts the h-orientation.

Remark 2. The problems associated with distinguishing the orientations, and in particular the problem of the existence of an orientation-reversing automorphism of a manifold, form the most natural area for the application of refined torsions and refined Alexander functions. For example, from the results of Remark 1 it follows that if M is an orientable connected closed three-dimensional manifold with an even one-dimensional Betti number, and if $\operatorname{aug}(\Delta(M)) \neq 0$ or $\Delta(M)$ can be represented as the square of a non-zero polynomial, then M does not admit orientation-reversing automorphisms. The Alexander polynomial can also carry other useful information on the automorphisms of a manifold. We consider the following example. Let M be the result of surgery on the sphere S^3 along the two-component link l depicted in Fig. 1, equipped with zero framing.

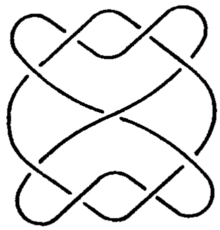

Fig. 1.

It is obvious that $H_1(M) = \mathbf{Z} \times \mathbf{Z}$. By [35], 419, we have

$$(t_1 - 1)(t_2 - 1)(t_1 + t_2 + 1 + t_1^{-1} + t_2^{-1}) \in \Delta_l.$$

By Theorem 1.4.2, $t_1 + t_2 + 1 + t_1^{-1} + t_2^{-1} \in \Delta(M)$. Hence we can see that M does not admit orientation-reversing automorphisms, and that the automorphisms of the group $H_1(M)$ induced by homeomorphisms $M \to M$ leave the set $\{t_1, t_2, t_1^{-1}, t_2^{-1}\}$ invariant. Hence in turn it follows that the image of the natural homomorphism $\operatorname{Aut}(M) \to \operatorname{Aut}(H_1(M))$ consists of at most eight elements. Using the symmetry of the link l, it can easily be shown that this image contains the homomorphisms $(t_1, t_2) \mapsto (t_1^{-1}, t_2)$ and $(t_1, t_2) \mapsto (t_1, t_2^{-1})$.

Remark 3. Using refined torsions, we can prove without difficulty the following theorem announced by Conway [5]: if an ordered link l in S^3

with an even number of components is isotopic (in the category of ordered links) to its mirror image, then $\Delta_l = 0$. For let V and n be the exterior and the number of components of l respectively. We give V a homological orientation. The composition of the mirror symmetry and the isotopy defines a homeomorphism $f\colon V \to V$ that induces multiplication by -1 in the vector space $H_1(V;\ \mathbf{R}) \oplus H_2(V;\ \mathbf{R})$. Since the dimension of this space is $2n-1$, f reverses the homological orientation. Consequently $f_*(A_0(V)) = -A_0(V)$. On the other hand, if $A = A(t_1, \ldots, t_n) \in A_0(V)$, then by Theorem 1.7.1 and the fact that n is even we have

$$f_*(A) = A(t_1^{-1}, \ldots, t_n^{-1}) = t_1^{\nu_1} \ldots t_n^{\nu_n} A$$

with integral $\nu_1, \ldots, \nu_n$. Hence it follows that $f_*(A_0(V)) = A_0(V)$. Therefore $\Delta_l = \pm A_0(V) = 0$. (Another proof can be found in [16]; see also §4.)

§4. The Conway link function

4.1. History of the question.

In 1970 Conway [5] proposed a new method for computing the Alexander polynomials of links in S^3, based on a bright and completely unexpected idea. The cornerstone of this method is the link invariant introduced by Conway, which he called the *potential function* of the link. (The Conway function has no relation to the potentials studied in mathematical physics.) The potential function is defined for an arbitrary ordered (oriented) n-component link $l \subset S^3$. It is denoted by $\nabla_l(t_1, \ldots, t_n)$, or more briefly by ∇_l, and represents a uniquely defined rational function of the variables $t_1, \ldots, t_n$. If $n \geqslant 2$, then ∇_l is a Laurent polynomial: $\nabla_l \in \mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. If $n = 1$, then the potential function ∇_l is a priori not a Laurent polynomial, but nevertheless has a fairly simple form. Namely, it can be written as a fraction whose numerator is a Laurent polynomial in t, and whose denominator is $t - t^{-1}$. The potential function is symmetric ($\overline{\nabla}_l = (-1)^n \nabla_l$) and is related to the Alexander function by the formula

$$\nabla_l(t_1, \ldots, t_n) \doteq A_l(t_1^2, \ldots, t_n^2),$$

where the symbol $\doteq$ means equality up to multiplication by -1 and powers of the variables. From these formulae it can be seen that the function ∇_l, considered up to multiplication by -1, is the result of symmetrization of the function $A_l(t_1^2, \ldots, t_n^2)$. In particular, the potential function considered with this accuracy can be computed from the Alexander function. The potential function cannot be completely restored from the Alexander function. For example, the Alexander functions of a link l and its mirror image l' are equal, whereas $\nabla_{l'} = (-1)^{n+1} \nabla_l$.

Conway showed that the potential functions of different links are interconnected by additive relations which make it possible to compute the potential functions (and with them the Alexander polynomials) recursively, by successive simplification of links (see [5] or §4.2).

The precise definition of the potential function, which in essence reduces to the simultaneous and consistent attaching of signs to the Alexander polynomials of links, by the traditional methods of defining the Alexander polynomials runs into considerable difficulties. The fundamental paper of Conway [5] does not contain a definition of the potential function. In 1981 Kauffman [20] gave a simple definition of the function $\tilde{\nabla}_l$ obtained from ∇_l by the substitution $t_1 = t_2 = \dots = t_n = t$. Namely: $\tilde{\nabla}_l = (t - t^{-1})^{-1} \det (tS - t^{-1}S^T)$, where S is the Seifert matrix of the link l. The function $(t - t^{-1})\tilde{\nabla}_l = \det (tS - t^{-1}S^T)$ is called the *reduced potential function*[1] of the link l. In 1983 Hartley [16], on the basis of Conway's ideas, gave a definition of the Conway function in full generality. This definition is formulated in terms of the diagrams of links and associated notions—Wirtinger presentations, Seifert circles, and so on. The paper of Hartley also contains the first published proofs of the properties of the potential function announced by Conway.

The approach of Conway and Hartley to the definition of the potential function, using the diagrams of links, has its advantages and disadvantages. The main advantage of this approach is that it enables one to compute the potential function of a link directly from the diagram, in which the amount of computation is not much larger than that in the well-known method of computing the Alexander polynomial from the Wirtinger presentation of the link group using the free differential calculus of Fox. The disadvantages of the definition of Conway and Hartley include the technical complexity of this definition, which makes it not very transparent, and also the need to carry out a (not at all obvious) verification of the invariance of the potential function under Reidemeister transformations of link diagrams. In addition, this definition cannot be generalized to the case of links in three-dimensional homology spheres.

We give two (new) definitions of the potential function: an axiomatic one, which is an extension of Kauffman's axiomatics for the reduced potential functions, and a constructive one, based on the use of refined torsions. The constructive definition can be generalized verbatim to the case of links in three-dimensional homology spheres.

4.2. Axioms for the Conway function.

We use the term *Conway map* for an arbitrary map ∇ which assigns to each ordered link l in S^3 an element of the field $\mathbf{Q}(t_1, \dots, t_n)$, where n is the number of components of l, and which posesses the following properties:

---

[1] It is well known that the reduced potential function of any link can be uniquely represented as a polynomial (in the usual sense of this word) in $t - t^{-1}$. The one-variable polynomial arising in this way is called the Conway polynomial of the link. (In some papers this polynomial is also called the potential function, which is of course unfortunate, and may lead to confusion.) We do not require the Conway polynomial.

4.2.1. $\nabla(l)$ is unchanged under (ambient) isotopy of the link l.
4.2.2. If l is the trivial knot, then $\nabla(l) = (t - t^{-1})^{-1}$.
4.2.3. If $n \geqslant 2$, then $\nabla(l) \in \mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$.
4.2.4. The one-variable function $\tilde{\nabla}(l) = \nabla(l)(t, t, \ldots, t)$ is unchanged by a renumbering of the components of l (in other words, $\tilde{\nabla}(l)$ is an invariant of the unordered link).
4.2.5 (Conway identity). If l_+, l_-, and l_0 are links coinciding (except possibly for the numbering of components) outside a certain ball, and inside this ball having the form depicted in Fig. 2, then

$$\tilde{\nabla}(l_+) = \tilde{\nabla}(l_-) + (t - t^{-1})\tilde{\nabla}(l_0).$$

4.2.6 (Doubling Axiom). If the link l' is obtained from the link $l = l_1 \cup \ldots \cup l_n$ by replacing the component l_i by its (2, 1)-cable (see §1.3), then

$$\nabla(l')(t_1, \ldots, t_n) = (T + T^{-1}) \times \nabla(l)(t_1, \ldots, t_{i-1}, t_i^2, t_{i+1}, \ldots, t_n),$$

where $T = t_i \prod_{j \neq i} t_j^{\mu(l_i, l_j)}$.

The conditions 4.2.1–4.2.5 appear among the properties announced by Conway [5] and proved by Hartley [16] of the map $l \mapsto \nabla_l$ assigning to each link its potential function. The condition 4.2.6 was not considered by these authors, however, it is not hard to prove that their map $l \mapsto \nabla_l$ satisfies 4.2.6 and so is a Conway map in the above sense.

4.2.7. ***Theorem.*** *There exists at most one Conway map.*

Proof. Let ∇ be the difference of two Conway maps. Clearly ∇ satisfies Axioms 4.2.1 and 4.2.3–4.2.6. If l is the trivial knot, then $\nabla(l) = 0$. For each link l we put $\eta(l) = \nabla(l)(t, t, \ldots, t)$. We show that $\eta = 0$. Let l be a trivial n-component link with $n \geqslant 2$; let l_+ and l_- be trivial $(n-1)$-component links, as shown in Fig. 3.

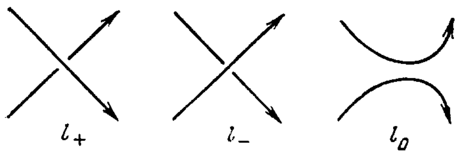

Fig. 2

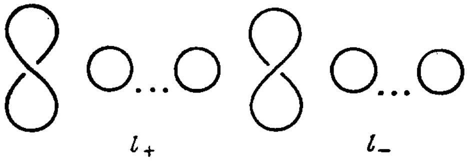

Fig. 3

By Axioms 4.2.1 and 4.2.5, $(t - t^{-1})\eta(l) = \eta(l_+) - \eta(l_-) = 0$, that is, $\eta(l) = 0$. We suppose that the equality $\eta(l) = 0$ has been proved for links having diagrams with at most m crossings. Let l be a link given by a diagram with $m + 1$ crossings. In a certain number of steps, replacing underpasses by overpasses in such a diagram, we can obtain the diagram of an unlink. By Axiom 4.2.5 and the inductive assumption, the value of the function η is unchanged during these operations. Therefore $\eta(l) = 0$.

We show that $\nabla = 0$. If l is a knot in S^3, then $\nabla(l) = \eta(l) = 0$. Let l be a link with $n \geqslant 2$ components. We suppose that $\nabla(l) \neq 0$. Then $\nabla(l)$ is a non-zero Laurent polynomial, and obviously we can choose natural numbers $a_1, \ldots, a_n$ that are powers of two for which $\nabla(l)\,(t^{a_1}, \ldots, t^{a_n}) \neq 0$. Let $a_i = 2^{b_i}$ and let k be a link obtained from l by successive b_i-fold replacement of its i-th component by its (2,1)-cable for all $i = 1, \ldots, n$. By Axiom 4.2.6 the polynomial $\nabla(l)\,(t_1^{a_1}, \ldots, t_n^{a_n})$ divides $\nabla(k)(t_1, \ldots, t_n)$, the quotient being a product of polynomials of the form $T + T^{-1}$, where T is a monomial. By what we have proved above, $\nabla(k)(t, t, \ldots, t) = 0$. Hence it follows that $\nabla(l)(t^{a_1}, \ldots, t^{a_n}) = 0$. The resulting contradiction shows that $\nabla(l) = 0$.

4.2.8. ***Corollary.*** *Let ∇ be a Conway map, and let l be an ordered n-component link in S^3. Then*: (i) $\overline{\nabla(l)} = (-1)^n\nabla(l)$; (ii) *if k is the mirror image of the link l, then* $\nabla(k) = (-1)^{n+1}\nabla(l)$; (iii) *if π is a permutation $\{1, \ldots, n\} \to \{1, \ldots, n\}$, and if l' is the link $l_{\pi(1)} \cup \ldots \cup l_{\pi(n)}$, then*

$$\nabla(l')(t_1, \ldots, t_n) = \nabla(l)(t_{\pi(1)}, \ldots, t_{\pi(n)}).$$

Proof. It can easily be verified that the functional $l \mapsto (-1)^n \times \overline{\nabla(l)}$ satisfies the axioms for a Conway functional. By Theorem 4.2.7, (i) follows. Points (ii) and (iii) can be verified similarly.

4.2.9. *Remark.* When considering the Conway identity 4.2.5, several authors use the mirror image of Fig. 2 (or, what is equivalent, use the same figure as we do, but use l_- for the left link, and l_+ for the middle link). It is easily seen that if ∇ is a Conway map in our sense, then the map assigning to an n-component link l the rational function $(-1)^{n+1}\nabla(l)$ satisfies the similarly modified identity 4.2.5.

4.3. The construction of the Conway function.

Let l be an ordered n-component link in S^3. We fix a "canonical" homological orientation of the exterior V of the link l, defined by the basis $([pt], t_1, \ldots, t_n, q_1, \ldots, q_{n-1})$ in $H_*(V;\ \mathbf{R})$, where $[pt]$ is the homology class of a point; $t_1, \ldots, t_n$ are meridional generators of the group $H_1(V)$; $q_1, \ldots, q_{n-1}$ are the generators of $H_2(V)$ represented by (oriented) boundaries of regular neighbourhoods of the knots $l_1, \ldots, l_{n-1}$ respectively. (We adhere to the convention on the orientation of the boundary of an oriented manifold under which the sequence (an outward directed vector; a positively oriented basis of the tangent space at the boundary) gives a positive orientation of the whole manifold.)

Let $A = A(t_1, \ldots, t_n)$ be a representative of the refined Alexander function $A_0(V)$ (see §3.3). By Theorem 1.7.1 we have $\overline{A} = (-1)^n t_1^{\nu_1} \ldots t_n^{\nu_n} A$ with integral $\nu_1, \ldots, \nu_n$. We put

$$\nabla(l) = -t_1^{\nu_1} \ldots t_n^{\nu_n} A\,(t_1^2, \ldots, t_n^2).$$

As we can verify directly, $\nabla(l)$ is independent of the choice of representative A of the function $A_0(V)$. It is obvious that $\overline{\nabla(l)} = (-1)^n\nabla(l)$, and that the function $l \mapsto \nabla(l)$ satisfies Axioms 4.2.1 and 4.2.3. The verification of Axiom 4.2.2 reduces to the calculation of the refined Alexander function of an oriented circle. This function is $-(t-1)^{-1}$(see §1.3.3; in the calculation one should not forget the factor $(-1)^{N(C)}$, where $C = C_*(S^1;\ \mathbf{R})$, see §3.1.1). The result of symmetrization of the function $-(t^2-1)^{-1}$ is $-(t-t^{-1})^{-1}$. In fact, the presence here of an "extra" minus sign leads to the necessity of introducing the same sign in the definition of the invariant $\nabla(l)$.

4.3.1. The verification of Axiom 4.2.4.
Let ψ be the ring homomorphism $\mathbf{Z}[H_1(V)] \to \mathbf{Z}[t, t^{-1}]$ taking $t_1, \ldots, t_n$ to t. Both the homomorphism ψ and, as we can verify directly, the canonical h-orientation of the manifold V are unchanged under a renumbering of the components of l. Hence the refined torsion $\tau_0^\psi(V)$ is also unchanged under this. By Theorem 3.4.3, if $A(t_1, \ldots, t_n) \in A_0(V)$, then $A(t, t, \ldots, t) \in \tau_0^\psi(V)$. Hence 4.2.4 follows.

4.3.2. The verification of Axiom 4.2.6.
It is obvious that the exterior V' of the link l' is obtained from that of the link l by gluing a Möbius strip to its boundary. A CW-decomposition of the manifold V' can be obtained from the decomposition of V by adding a one-dimensional cell e_1 and a two-dimensional cell e_2 (Fig. 4).

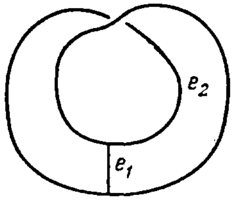

Fig. 4.

It is easy to see that with a suitable choice of liftings $\tilde{e}_1$ and $\tilde{e}_2$ of these cells to the maximal Abelian cover of V' the boundary of the cell chain $[\tilde{e}_2]$ is equal to the sum of the chain $(1+T)[\tilde{e}_1]$ and the chain generated by the cells situated over V. We remark that the inclusion homomorphism $H_*(V;\ \mathbf{R}) \to H_*(V';\ \mathbf{R})$ is an orientation-preserving isomorphism. Let $\psi : \mathbf{Z}[H_1(V)] \to \mathbf{Z}[H_1(V')]$ be the inclusion homomorphism: $\psi(t_i) = t_i^2$ and $\psi(t_j) = t_j$ for $j \neq i$. Arguments analogous to those used in the proof of Theorem 3.2.1 show that $A_0(V') = \tau_0(V') = (1+T)\tau_0^\psi(V)$. By Theorem 3.4.3, $\tau_0^\psi(V) = A_0(V)(t_1, \ldots, t_{i-1}, t_i^2, t_{i+1}, \ldots, t_n)$. Hence we have 4.2.6.

4.3.3. Preparation for the verification of Axiom 4.2.5.
(i) ***Lemma*** (Milnor [26]). *Let C and C^* be dual acyclic m-dimensional chain complexes over a field, equipped with dual bases* (*see* §2.2.2). *Then* $\tau(C^*) = \pm[\tau(C)]^{\varepsilon(m)}$, *where* $\varepsilon(m) = (-1)^{m+1}$.

Proof. Let b_i be a sequence of vectors in the space C_i for which $\partial_{i-1}(b_i)$ is a basis in $\operatorname{Im}(\partial_{i-1}\colon C_i \to C_{i-1})$. Let b'_{m-i+1} be a sequence of vectors in C'_{m-i+1} whose matrix of values on the terms of $\partial_{i-1}(b_i)$ is a unit square matrix. It is easy to see that $\partial'_{m-i}(b'_{m-i+1})$ is a basis in $\operatorname{Im} \partial'_{m-i}$ and that if the bases c_i and C'_{m-i} are dual, then $[\partial'_{m-i}(b'_{m-i+1})b'_{m-i}/c'_{m-i}] = \pm[\partial_i(b_{i+1})b_i/c_i]^{-1}$. Hence we have the statement of the lemma.

(ii) We say that a CW complex is *regular* if for every (open) cell a of it the closure $\bar{a}$ is the union of cells and a homeomorphism of an open ball $\mathring{D}$ onto a can be extended to a homeomorphism of the closed ball D onto $\bar{a}$. For example, all triangulations are regular.

The next lemma refines the theorem on the invariance of the torsion of a CW-complex under subdivision. (It is possible that the condition of regularity in this lemma is superfluous.)

(iii) ***Lemma.*** *Let X be a finite regular cell complex; let $\widetilde{X} \to X$ be a maximal Abelian cover; let X' be an (arbitrary) subdivision of the space X, and let $\widetilde{X}'$ be the induced subdivision of the space $\widetilde{X}$; let e be a base sequence of oriented cells of $\widetilde{X}$, and let e' be a base sequence of cells of $\widetilde{X}'$ consisting of all the cells of this space contained in cells of the sequence e. If φ is a ring homomorphism from the ring $\mathbf{Z}[H_1(X)]$ into a field, then $\tau^{\varphi}(X, e) = \pm\tau^{\varphi}(X', e')$.*

Proof. If $H^{\varphi}_*(X) \neq 0$, then both sides of the equation to be proved become zero. We prove the lemma in the following strengthened form, meaningful also in the case $H^{\varphi}_*(X) \neq 0$: *if in the vector spaces $H^{\varphi}_0(X) = H^{\varphi}_0(X')$, $H^{\varphi}_1(X) = H^{\varphi}_1(X')$, ... we fix bases g_0, g_1, ..., then (in the obvious notations)*

$$\check{\tau}(C^{\varphi}_*(X), e, g_0, g_1, \ldots) = \pm\check{\tau}(C^{\varphi}_*(X'), e', g_0, g_1, \ldots).$$

In the course of the proof, for brevity the chain complexes $C^{\varphi}_*(X)$ and $C^{\varphi}_*(X')$ are denoted by $C(X)$ and $C(X')$ respectively.

Let a be an (open) cell of the space X, of maximal dimension. It is clear that $X\backslash a$ is a regular cell complex. We denote it by Y. We denote by Y' the decomposition of the space Y induced by X'. Let b be a cell of X' contained in a, whose dimension is $\dim a$. We put $Z = X'\backslash b$. It is obvious that Y' is a subcomplex, and a deformation retract of the space Z. We denote by $C(Y)$ the subcomplex of $C(X)$ generated by the cells situated over Y. Similarly we define subcomplexes $C(Y')$ and $C(Z)$ of $C(X')$. The cells of the sequence e (respectively e') that lie over Y (respectively over Y', Z) give bases of the three subcomplexes mentioned, which we denote by u, u', and v respectively. For every $i \geqslant 0$ we fix a basis h_i in the vector space $H_i(C(Y)) = H_i(C(Y')) = H_i(C(Z))$. We put $g = (g_0, g_1, \ldots)$, and $h = (h_0, h_1, \ldots)$. Below we prove two formulae

$$\check{\tau}(C(Z), v, h) = \pm\check{\tau}(C(Y'), u', h) \text{ and } \frac{\check{\tau}(C(X), e, g)}{\check{\tau}(C(Y), u, h)} = \pm\frac{\check{\tau}(C(X'), e', g)}{\check{\tau}(C(Z), v, h)}.$$

These formulae show that the equality $\check{\tau}(C(X),\ e,\ g) = \pm\check{\tau}(C(X'),\ e',\ g)$ is equivalent to the equality $\check{\tau}(C(Y),\ u,\ h) = \pm\check{\tau}(C(Y'),\ u',\ h)$. Thus, by successive removal of the cells of X we can reduce the statement of the lemma to the case $\dim X = 0$. In this case, $X' = X$, $e' = e$, and the desired statement is obvious.

Since $\bar{a}\backslash a$ is a sphere, the inclusion $\bar{a}\backslash a \to \bar{a}\backslash b$ is a simple homotopy equivalence, and hence $\bar{a}\backslash b$ can be deformed into $\bar{a}\backslash a$ by elementary cell expansions and collapses. The pairs of base cells arising in elementary expansions or destroyed by collapses can be assumed to be incident, with incidence coefficient 1. Hence the arguments used in the proof of Theorem 3.2.1 show that $\check{\tau}(C(Z),\ v,\ h) = \pm\check{\tau}(C(Y'), u', h)$. The second of the formulae given above follows from 3.4.2 and the obvious fact that both the factor complexes $C(X)/C(Y)$ and $C(X')/C(Z)$ and the homology sequences of the pairs $C(Y) \subset C(X)$ and $C(Z) \subset C(X')$ are constructed in exactly the same way.

4.3.4. The verification of Axiom 4.2.5.

This verification is the most difficult part of the construction of the Conway map, and is carried out in three steps.

Step 1. We assume that the common part of the links l_+, l_-, and l_0 is located in a closed three-dimensional ball $D \subset S^3$, and intersects ∂D transversely in four points (see Fig. 5, in which D is represented as the closure in S^3 of the half-space below the plane of the figure).

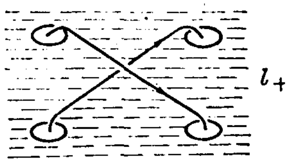

Fig. 5.

We denote by V the complement in D of an open regular neighbourhood of the manifold $l_+ \cap D = l_- \cap D = l_0 \cap D$. Clearly $V \cap \partial D$ is the complement in ∂D of four open discs. We denote by X_α, where $\alpha \in \{+,\ -,\ 0\}$, the result of gluing to V a two-dimensional disc B_α by a homeomorphism from the boundary ∂B_α onto the circle $X_\alpha \subset V \cap \partial D$ depicted in Fig. 6.

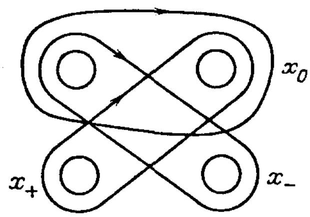

Fig. 6.

It is obvious that X_α is a deformation retract of the exterior of the link l_α. Let $\widetilde{X}_\alpha \to X_\alpha$ be an infinite cyclic cover whose group of covering transformations $\{t^m\}_{m\in\mathbf{Z}}$ corresponds to the kernel of the homomorphism $\pi_1(X_\alpha) \to \{t^m\}_{m\in\mathbf{Z}}$, which takes the homotopy classes of meridians into t. Let $\widetilde{V}$ be the inverse image of V under this covering. Clearly $\widetilde{X}_\alpha = \widetilde{V} \cup \bigcup_{m\in\mathbf{Z}} t^m(\widetilde{B}_\alpha)$, where $\widetilde{B}_\alpha$ is a lifting of the disc B_α to $\widetilde{X}_\alpha$. Replacing the discs $\widetilde{B}_+$ and $\widetilde{B}_-$ by their images under covering homeomorphisms if necessary, we can assume that the curves $\partial\widetilde{B}_+$, $\partial\widetilde{B}_-$, $\partial\widetilde{B}_0$ have a common point (situated over the unique common point of the curves x_+, x_-, x_0). In B_α and $\widetilde{B}_\alpha$ we fix compatible orientations, which induce the orientation of the curve x_α given in Fig. 6. We fix a triangulation of V, and the equivariant triangulation of $\widetilde{V}$ induced by it. We fix in $\widetilde{V}$ a base sequence of simplexes (that is, a sequence of oriented simplexes of the chosen triangulation of $\widetilde{V}$ such that over every simplex of the chosen triangulation for V there lies exactly one simplex of this sequence). We add to this sequence the cell $\widetilde{B}_\alpha$ (as the last counted cell), and so obtain a base sequence of cells in $\widetilde{X}_\alpha$. We denote the $\mathbf{Q}(t)$-chain complex $\mathbf{Q}(t) \otimes_{\mathbf{Z}[t,t^{-1}]} C_*(\widetilde{X}_\alpha; \mathbf{Z})$ by C_α. We fix in C_α a basis corresponding to the indicated base sequence of cells in $\widetilde{X}_\alpha$. We put $\tau_\alpha = \tau(C_\alpha) \in \mathbf{Q}(t)$. We prove that

$$\tau_+ = -t\tau_- + (t-1)\tau_0. \tag{14}$$

We denote the $\mathbf{Q}(t)$-chain complex $\mathbf{Q}(t) \otimes_{\mathbf{Z}[t,t^{-1}]} C_*(\widetilde{V}; \mathbf{Z})$ by C. It is obvious that for every $\alpha = +, -, 0$ the complex C is a subcomplex of C_α, and moreover the space of two-dimensional chains of the factor complex C_α/C exhausts the non-zero part of this complex and is equal to $\mathbf{Q}(t)[\widetilde{B}_\alpha]$. We fix in $H_2(C_\alpha/C)$ a generator $[\widetilde{B}_\alpha]$. Clearly $\check{\tau}(C_\alpha/C) = (-1)^{N(C_\alpha/C)} = -1$.

If $\tau_\alpha = 0$ for all α, then (14) holds. We assume that $\tau_\alpha \neq 0$ for some $\alpha \in \{+, -, 0\}$. Then the complex C_α is acyclic, and hence from the exactness of the homology sequence of the pair $C \subset C_\alpha$ it follows that $H_1(C) = \mathbf{Q}(t)$ and $H_i(C) = 0$ for $i \neq 1$. We fix an arbitrary generator y in $H_1(C)$. By Lemma 3.4.2

$$\tau_\alpha = \tau(C_\alpha) = \check{\tau}(C_\alpha) = \lambda\check{\tau}(C)\check{\tau}(C_\alpha/C)\tau(\mathcal{H}_\alpha),$$

where $\mathcal{H}_\alpha$ is an acyclic chain complex with two non-zero terms $H_2(C_\alpha/C)$ and $H_1(C)$; $\lambda = \pm 1$ and λ, as can easily be seen, is independent of the choice of $\alpha \in \{+, -, 0\}$ with $\tau_\alpha \neq 0$. The boundary homomorphism $H_2(C_\alpha/C) \to H_1(C)$ takes $[\widetilde{B}_\alpha]$ to $\gamma_\alpha y$ with $\gamma_\alpha \in \mathbf{Q}(t)$; here $\tau(\mathcal{H}_\alpha) = \gamma_\alpha$. Thus $\tau_\alpha = \pm\check{\tau}(C)\gamma_\alpha$, where the sign $\pm$ is independent of the choice of α with $\tau_\alpha \neq 0$. If $\tau_\alpha = 0$ for some α, then the equality $\tau_\alpha = \pm\check{\tau}(C)\gamma_\alpha$ with the same sign is also valid: in this case the complex C_α is not acyclic and the exactness of the homology sequence of the pair $C \subset C_\alpha$ implies that $\gamma_\alpha = 0$. Hence we can see that for the proof of (14) it suffices to show that $\gamma_+ = -t\gamma_- + (t-1)\gamma_0$.

We prove the equivalent equality $[\gamma_+ + t\gamma_- - (t-1)\gamma_0]y = 0$. Let a, b, c, and d be the loops in $V \cap \partial D$ starting at the common point of the curves x_+, x_-, and x_0, depicted in Fig. 7.

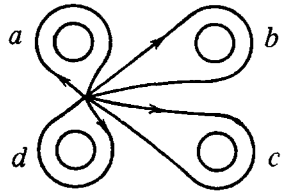

Fig. 7.

The homology class of $\gamma_+ y$ is represented by the curve $\partial\widetilde{B}_+$, and so it can be represented by a loop in $\widetilde{V}$ starting at the common point of the curves ∂B_+, ∂B_-, and ∂B_0 and covering the loop bd. Similarly, the classes of $\gamma_0 y$, $t\gamma_0 y$, and $t\gamma_- y$ can be represented by loops in $\widetilde{V}$ starting at the same point covering the loops ab, $a^{-1}(ab)a \sim ba$, and $a^{-1}(ac)a \sim ca$ respectively. (The loop a^{-1} is a meridian of the corresponding component, and represents in the covering group the element t.) Hence the class $[\gamma_+ + t\gamma_- - (t-1)\gamma_0]y$ can be represented by a loop covering $(bd)(ab)(ca)(ba)^{-1} \sim b(dabc)b^{-1} \sim 1$. The required equality follows from this.

Step 2. We prove the analogue of (14) for refined torsions. For $\alpha = +, -, 0$ we fix a basis of the $\mathbf{R}$-chain complex $C_*(X_\alpha;\ \mathbf{R})$ corresponding to the base sequence of cells in $\widetilde{X}_\alpha$ fixed in step 1. We equip the homology groups of this complex with bases over $\mathbf{R}$ which determine in $H_*(X_\alpha,\ \mathbf{R})$ the orientation induced by the canonical homological orientation of the exterior of the link l_α (we recall that X_α is a deformation retract of this exterior). We denote the sign of the torsion $\check{\tau}(C_*(X_\alpha;\ \mathbf{R})) \in \mathbf{R}\backslash 0$ by λ_α. We put $\Delta_\alpha = \lambda_\alpha \tau_\alpha$, and show that

$$(15) \qquad \Delta_+ = t\Delta_- + (t-1)\Delta_0.$$

By (14) it suffices to show that $\lambda_+ = -\lambda_- = \lambda_0$. We restrict ourselves to the case when the parts of the link l_+ that are being rearranged lie in the same component. (The opposite case is considered similarly.)

We number the common components of the links l_+, l_-, and l_0 by the numbers $1, 2, \ldots, n-1$. We assign the number n to the rearranged component of the links l_+ and l_-. To the upper (in Fig. 2) component of l_0 we assign the number n, and to the lower one the number $n+1$. For $\alpha = +, -$ the chosen orientation in $H_+(X_\alpha;\ \mathbf{R})$ is given by the basis $[pt], t_1, \ldots, t_n, q_1, \ldots, q_{n-1}$ (in the notation introduced at the start of §4.3). The chosen orientation in $H_*(X_0;\ \mathbf{R})$ is given by the basis $[pt], t_1, \ldots, t_{n+1}, q_1, \ldots, q_n$. We fix in $H_*(V;\ \mathbf{R})$ the basis $[pt], t_1, \ldots, t_{n+1}, q_1, \ldots, q_{n-1}$, and we fix in $H_*(X_\alpha,\ V;\ \mathbf{R})$ the generator $[B_\alpha]$ (for $\alpha = +, -, 0$). By applying Lemma 3.4.2 to the exact sequence of three-dimensional chain complexes

$$0 \to C_*(V;\ \mathbf{R}) \to C_*(X_\alpha;\ \mathbf{R}) \to C_*(X_\alpha,\ V;\ \mathbf{R}) \to 0,$$

we obtain the equality $\lambda_\alpha = (-1)^{(\mu_\alpha+\nu_\alpha)} \times \lambda\lambda'_\alpha$, where μ_α and ν_α are the integers defined by the formulae in the statement of Lemma 3.4.2; $\lambda = \operatorname{sign}[\check{\tau}(C_*(V;\ \mathbf{R}))\check{\tau}(C_*(X_\alpha,\ V;\ \mathbf{R}))]$ and λ'_α is the sign of the torsion τ of the homology sequence of the pair $(X_\alpha,\ V)$ with coefficients in $\mathbf{R}$. It is obvious that λ and ν_α are independent of the choice of $\alpha \in \{+,\ -,\ 0\}$. Direct calculation shows that $\mu_+ = \mu_- \equiv 1 \pmod 2$, $\mu_0 \equiv n+1 \pmod 2$, $\lambda'_+ = -\lambda'_- = (-1)^{n+1}$, and $\lambda'_0 = -1$. (The equalities $\mu_+ = \mu_-$ and $\lambda'_+ = -\lambda'_-$ are easily seen without calculation: the homology sequences of the pairs $(X_+,\ V)$ and $X_-,\ V)$ differ from each other only in that $\partial([B_+]) = t_n - t_{n+1}$ whereas $\partial([B_-]) = t_{n+1} - t_n$.) Hence we have the equalities $\lambda_+ = -\lambda_- = \lambda_0$, and together with these the equality (15).

Step 3. It follows from the definitions and Theorem 3.4.3 that the function $\widetilde{\nabla}_\alpha = \widetilde{\nabla}(l_\alpha)$ is obtained by symmetrization of the rational function $-\Delta_\alpha(t^2) \in \mathbf{Q}(t)$. More precisely: for $\alpha = +, -, 0$ there is an integer r_α for which $\Delta_\alpha(t^{-1}) = \pm t^{r_\alpha}\Delta_\alpha(t)$. Then $\widetilde{\nabla}_\alpha(t) = -t^{r_\alpha}\Delta_\alpha(t^2)$. By formula (15)

$$\widetilde{\nabla}_+ = -t^{r_+}\Delta_+(t^2) = -t^{r_+}[t^2\Delta_-(t^2) + (t^2-1)\Delta_0(t^2)] = t^{(r_+ + 2 - r_-)}\widetilde{\nabla}_- + \\ + t^{(r_+ + 1 - r_0}(t - t^{-1})\widetilde{\nabla}_0.$$

We remark that the number r_α is uniquely determined if $\Delta_\alpha \neq 0$, and can be chosen arbitrarily if $\Delta_\alpha = 0$.

We denote by V_α the exterior of the link l_α viewed as the result of attaching to V a handle of index 2 with axis B_α (where $\alpha = +, -, 0$). We denote by $\widetilde{V}_\alpha$ the infinite cyclic cover of V_α resulting from the attachment to $\widetilde{V}$ of handles of index 2 with axes $t^m(\widetilde{B}_\alpha)$, $m \in \mathbf{Z}$. We suppose that the equivariant triangulation of $\widetilde{V}$ fixed in step 1 can be extended to an equivariant triangulation, say Y_α, of $\widetilde{V}_\alpha$. We extend the base sequence of simplexes in $\widetilde{V}$ fixed in step 1 to a base sequence e_α of simplexes of Y_α as follows: we fix in $\widetilde{V}_\alpha \setminus \widetilde{V}$ those simplexes that lie in the handle with axis $\widetilde{B}_\alpha$. (The order in this set of simplexes, and their orientations, are arbitrary.) We denote by E_α, F_α, and ∂F_α the $\mathbf{Q}(t)$-chain complexes obtained as the result of tensor multiplication of the field $\mathbf{Q}(t)$ by the $\mathbf{Z}[t,\ t^{-1}]$-chain complexes of cell chains of the decompositions Y_α, Y^*_α, and ∂Y^*_α respectively (see §2.2.2 (ii)). We fix the basis of the complex E_α defined by the sequence e_α, and the bases of the complexes F_α and ∂F_α defined by the sequence e^*_α consisting of the cells of the decomposition Y^*_α dual to the simplexes of the sequence e_α. We denote by R the union of the toral components of the boundary ∂V. It is clear that $R \subset \partial V_\alpha$, and here the difference $\partial V_\alpha \setminus R$ is either a torus or the disjoint union of two tori. The complex ∂F_α is the direct sum of its two subcomplexes generated by the cells lying over R and over $\partial V_\alpha \setminus R$ respectively. By Lemma 1.3.3 these complexes are acyclic, and their torsions are $\pm t^r$ and $\pm t^{s_\alpha}$ respectively, where r and s_α are integers and r is independent of the choice of α.

We prove that with a suitable choice of the numbers r_α corresponding to $\alpha \in \{+, -, 0\}$ with $\Delta_\alpha = 0$ we have the equalities

$$(16) \quad r_+ + 2 - r_- = s_- + 2 - s_+; \; r_+ + 1 - r_0 = s_0 + 1 - s_+.$$

From Lemma 4.3.3 (iii) one can easily deduce that $\tau(E_\alpha) = \pm\tau_\alpha = \pm\Delta_\alpha$ for all $\alpha = +, -, 0$. Since the CW-decompositions Y_α and Y_α^* of the manifold $\widetilde{V}_\alpha$ have a common subdivision, we have $\tau(F_\alpha) = \pm\, t^{\beta_\alpha}\tau(E_\alpha)$ with $\beta_\alpha \in \mathbf{Z}$. We compute β_α. By Lemma 4.3.3 (iii) the torsion $\tau(F_\alpha)$ is equal, up to sign, to the torsion of the complex $G_\alpha = \mathbf{Q}(t) \otimes_{\mathbf{Z}[t,\, t^{-1}]} C_*(Y'_\alpha;\ \mathbf{Z})$ where Y'_α is the first barycentric subdivision of the triangulation Y_α and where in Y'_α there is chosen a base sequence of simplexes, say $a_1, \ldots, a_N$, consisting of simplexes contained in the cells of the sequence e_α^*. Similarly, the torsion $\tau(E_\alpha)$ is equal, up to sign, to the torsion of the same complex G_α with another basis, namely the basis corresponding to a base sequence, say $b_1, \ldots, b_N$, consisting of simplexes of the triangulation Y'_α contained in simplexes of the sequence e_α. Every simplex b_i can be uniquely represented as $t^{m(i)}\, a_{j(i)}$ with integral $m(i)$ and with $j(i) \in \{1, 2, \ldots, N\}$. Therefore $\beta_\alpha = -\sum_{i=1}^{N} \varepsilon(\dim a_{j(i)})m(i)$. We remark that if the simplex b_i does not lie in $\widetilde{V}$, then by the construction of the sequence e_α it lies in the handle with axis $\widetilde{B}_\alpha$ attached to $\widetilde{V}$. By similar arguments $a_{j(i)}$ lies in the same handle, so that $a_{j(i)} = b_i$ and $m(i) = 0$. Hence it follows that $\beta_+ = \beta_- = \beta_0$. Thus $\tau(F_\alpha) = \pm t^{\beta_0}\Delta_\alpha$. On the other hand, since the torsion is multiplicative, we have $\tau(F_\alpha) = \pm\tau(\partial F_\alpha)\tau(F_\alpha/\partial F_\alpha) = \pm\, t^{(r+s_\alpha)}\, \tau(F_\alpha/\partial F_\alpha)$. As is easily verified, the factor complex $F_\alpha/\partial F_\alpha$ is obtained from a complex dual to the complex E_α by the replacement of rings $f(t) \mapsto f(t^{-1})$: $\mathbf{Z}[t, t^{-1}] \to \mathbf{Z}[t, t^{-1}]$ (see §2.2.3). Hence by Lemma 4.3.3 (i), if E_α is acyclic, that is, if $\Delta_\alpha \neq 0$, then $\tau(F_\alpha/\partial F_\alpha) = \pm\overline{\Delta}_\alpha$. Thus if $\Delta_\alpha \neq 0$, then $t^{\beta_0}\Delta_\alpha = \pm\, t^{(r+s_\alpha)}\, \overline{\Delta}_\alpha$ and so $r_\alpha = \beta_0 - r - s_\alpha$ for all α. Hence we have (16) (in the case $\Delta_\alpha = 0$, we should put $r_\alpha = \beta_0 - r - s_\alpha$).

We consider the case when the parts of the link l_+ to be reorganized lie in the same component. We put $u = s_- + 2 - s_+$ and $v = s_0 + 1 - s_+$. By what we proved above,

$$(17) \qquad \widetilde{\nabla}(l_+) = t^u\widetilde{\nabla}(l_-) + t^v(t - t^{-1})\widetilde{\nabla}(l_0).$$

We put $S = \partial V \setminus R$. It is obvious that the numbers s_+, s_-, and s_0, and with them the numbers u and v, are completely determined by the following data: the triangulation of the two-handled sphere S; the infinite cyclic cover $\widetilde{S} \to S$; the embedding $\partial B_\alpha \times [0, 1] \to S$ by which the handle of index 2 with axis B_α is attached, where $\alpha = +, -, 0$; the triangulations of these three handles, made compatible with the triangulation in S; the base sequence of simplexes in $\widetilde{S}$. All these data are in fact independent of the links l_+, l_-, and l_0. Hence we easily conclude that k_+, k_-, and k_0 is another triple of links satisfying the conditions of Axiom 4.2.5, and if the portion of k_+ to be reorganized lies on one component, then $\tilde{\nabla}(k_+) = t^u\tilde{\nabla}(k_-) + t^v(t - t^{-1})\tilde{\nabla}(k_0)$ with the same u and v as in (17). So to complete the verification of the axiom it suffices to point out a model triple l_+, l_-, l_0 for which (17) holds if and only if $u = v = 0$. Such a triple is depicted in Fig. 8.

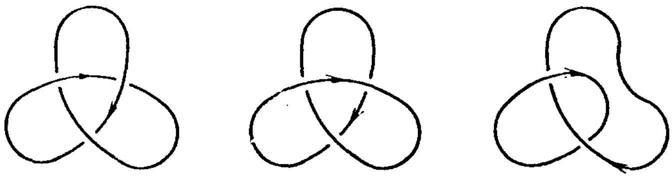

Fig. 8.

Here $\tilde{\nabla}(l_-) = 1/(t - t^{-1})$; $\tilde{\nabla}(l_+) = \pm(t^2 - 1 + t^{-2})/(t - t^{-1})$ (since the Alexander polynomial of the trefoil is $t^2 - t + 1$); $\tilde{\nabla}(l_0) = \pm 1$ (by similar arguments; in fact the signs here are +, but for us this is unimportant). It can easily be verified that these three rational functions have the desired property.

In the case when the portions of l_+ to be reorganized lie on different components, the proof is carried out in a similar way. As a model example, we can take the triple depicted in Fig. 9. Here $\tilde{\nabla}(l_+) = -1$ and $\tilde{\nabla}(l_-) = -(t^2 + t^{-2})$.

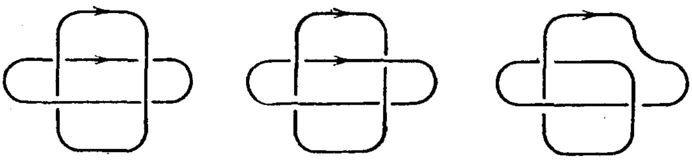

Fig. 9.

Remark 1. Further properties of the potential function can be found in [5], [14], [16]. These properties can be proved by the methods of the present paper; it would be instructive to deduce them directly from the axioms for a Conway map. We point out one new property of the potential function, which follows directly from Theorem 3.4.4; the potential functions of cobordant links are c-equivalent. This statement is stronger than the corresponding statement for Alexander polynomials. It implies, for example, that if a link l in S^3 with an even number of components is cobordant to its mirror image or to its mirror image with the opposite orientation, then $\Delta_l = 0$ (compare with Remark 3 to §3).

Remark 2. From the potential function of a link l it is possible in many cases to compute the refined Alexander function of the closed manifold M

obtained by surgery of a sphere along l and given the orientation extending the canonical orientation in $S^3\backslash l$. In the simplest case, when all the linking coefficients of the components and the framing numbers of the components are equal to zero, the function $A_0(M)(t_1^2, \ldots, t_n^2)$ is represented by the product $\nabla_l(t_1^2-1)^{-1}\ldots(t_n^2-1)^{-1}$.

Remark 3. It would be interesting to carry over the above axiomatic definition of the potential function to the case of links in homology spheres so that the uniqueness property is satisfied. It is possible that for this it might be necessary to modify or extend the list of axioms.

§5. The torsion ω and the polynomial δ

5.1. The torsion ω.

In contrast to the torsions considered above, the torsion ω is defined only for odd-dimensional compact manifolds (possibly with boundary). The definition of the torsion ω is based on a modification of the standard constructions of the theory of torsions, taking into account the Poincaré duality. The main advantage of the torsion ω is that this torsion never vanishes. However, this is achieved at the expense of increasing the indeterminacy.

Now we move to the precise definitions. Up to the end of this section we fix an orientable compact manifold M of odd dimension $m=2r+1$, an integral domain K with an involution $a\mapsto \bar a\colon K\to K$, and a ring homomorphism $\varphi\colon \mathbf{Z}[H_1(M)]\to K$ for which $\overline{\varphi(h)}=\varphi(h^{-1})$ for any $h\in H_1(M)$. We denote the inclusion homomorphism $\mathbf{Z}[H_1(\partial M)]\to\mathbf{Z}[H_1(M)]$ by in (possibly $\partial M=\varnothing$). We suppose that $\operatorname{rg} H_*^{\varphi\circ\mathrm{in}}(\partial M)=0$.

We define the torsion $\omega^\varphi(M)$. First we consider the case when K is a field. Then the condition $\operatorname{rg} H_*^{\varphi\circ\mathrm{in}}(\partial M)=0$ means that $H_*^{\varphi\circ\mathrm{in}}(\partial M)=0$. Hence it follows that the sesquilinear form of the intersection numbers $H_i^\varphi(M)\times H_{m-i}^\varphi(M)\to K$ corresponding to some orientation of M is nondegenerate for any i. We fix a triangulation of M. We consider the product

$$\check\tau(C_*^\varphi(M), g)\times\prod_{i=0}^{r}(\det u_i)^{-\varepsilon(i)},$$

where $g=(g_0, g_1, \ldots, g_m)$ is a sequence of bases of the K-modules $H_0^\varphi(M), H_1^\varphi(M), \ldots, H_m^\varphi(M)$, and u_i is the matrix of the above sesquilinear form with respect to the bases g_i and g_{m-i} (we recall that $-\varepsilon(i)=(-1)^i$). The collection of such products corresponding to all the possible sequences g and to all natural bases of the complex $C_*^\varphi(M)$ is $\omega^\varphi(M)$. Equivalently, $\omega^\varphi(M)$ can be defined as the collection of the torsions $\check\tau(C_*^\varphi(M), g)$ corresponding to all possible natural bases of the complex $C_*^\varphi(M)$ and sequences $g=(g_0, g_1, \ldots, g_m)$ for which g_i and g_{m-i} are dual for all i. In the case when K is not a field, the torsion $\omega^\varphi(M)$ is defined as $\omega^{j\circ\varphi}(M)$, where j is the inclusion of the ring K in its quotient field $Q(K)$.

It is not hard to verify that the torsion $\omega^{\varphi}(M)$ is independent of the choice of triangulation of M, and is an "element of $Q(K)$ defined up to multiplication by $\pm\varphi(h)f\bar{f}$ with $h \in H_1(M)$ and $f \in Q(K), f \neq 0$". It is clear that $\omega^{\varphi}(M) \neq 0$ and that if $\tau^{\varphi}(M) \neq 0$, then $\tau^{\varphi}(M) \subset \omega^{\varphi}(M)$.

5.1.1. ***Theorem.*** *Let J be the image of the inclusion homomorphism $H_r^{\varphi\circ\mathrm{in}}(\partial M) \to H_r^{\varphi}(M)$. If the ring K is factorial and Noetherian, then up to multiplication by $kf\bar{f}$, where k is an invertible element of K and f is a non-zero element of the field $Q(K)$, we have*

$$(18) \quad \omega^{\varphi}(M) = \operatorname{ord}(\operatorname{Tors} H_r^{\varphi}(M, \partial M))(\operatorname{ord} J)^{\varepsilon(r)} \times \prod_{i=0}^{r-1} [\operatorname{ord} H_i^{\varphi\circ\mathrm{in}}(\partial M)]^{\varepsilon(i)}.$$

In particular, if M is closed, then (up to the accuracy mentioned)

$$\omega^{\varphi}(M) = \operatorname{ord}(\operatorname{Tors} H_r^{\varphi}(M)).$$

This theorem is proved in §6. In the case when K is a factorial Noetherian ring whose invertible elements are exhaused by the elements of the form $\pm\varphi(h)$ with $h \in H_1(M)$, Theorem 5.1.1 enables us to calculate the entire torsion $\omega^{\varphi}(M)$ from the homology invariants of M and its boundary.

If M is oriented and homologically oriented, then in the same way as in §3.2 we can define a refined torsion $\omega_0^{\varphi}(M)$, which is "an element of $Q(K)$ defined up to multiplication by $\varphi(h)f\bar{f}$ with $h \in H_1(M)$ and $f \in Q(K), f \neq 0$". It is obvious that $\omega^{\varphi}(M) = \pm\omega_0^{\varphi}(M)$. When the homological orientation of M is changed, the torsion $\omega_0^{\varphi}(M)$ is multiplied by -1; under change of the (ordinary) orientation this torsion is multiplied by $(-1)^a$, where

$$a = \sum_{i=0}^{r} \operatorname{rg} H_i^{\varphi}(M).$$

5.2. The polynomial δ.

As in the previous subsection, let M be an orientable compact $(2r+1)$-dimensional manifold. We denote by E its r-dimensional Alexander module, that is, the $\mathbf{Z}[H_1(M)^{\#}]$-module $H_r^{\theta}(M)$, where θ is the projection $\mathbf{Z}[H_1(M)] \to \mathbf{Z}[H_1(M)^{\#}]$. By $\delta(M)$ we denote the order of the $\mathbf{Z}[H_1(M)^{\#}]$-module Tors E. By what was said in §0.4, $\delta(M) \neq 0$. If the Alexander polynomial $\Delta(M) = \operatorname{ord} E$ is different from zero (and this happens if and only if $\operatorname{rg} E = 0$), then $\delta(M) = \Delta(M)$. The number $\operatorname{rg} E$ is denoted below by $\gamma(M)$.

As can be seen from Theorem 5.1.1, the polynomial $\delta(M)$ is closely associated with the torsion $\omega^{\theta}(M)$. This association is analogous to the connection between the Alexander polynomial and Milnor torsion considered in §1, but has a more complicated character. The torsion $\omega^{\theta}(M)$ is denoted below by $\omega(M)$ (this torsion is defined if and only if $\operatorname{rg} H_*^{\theta\circ\mathrm{in}}(\partial M) = 0$). In particular, if M is a closed manifold, then the torsion $\omega(M)$ is defined, and $\delta(M) \subset \omega(M)$. Thus in the case of closed M the torsion $\omega(M)$ can be completely calculated from the polynomial $\delta(M)$; this polynomial in turn can be calculated from $\omega(M)$ up to factors of the form $f\bar{f}$.

We remark, although we do not require this below, that if M is closed and oriented, then we can define the refined polynomial $\delta_0(M) = \{\delta \mid \delta \in \delta(M)$ and $\delta \in \omega_0(M)\}$. Here the refined torsion $\omega_0(M)$ corresponds to the canonical homological orientation of M (see Remark 1 of §3). If $A_0(M) \neq 0$, then, as is easily deducible from Lemma 2.1.1,

$$A_0(M) = \delta_0(M)f\bar{f}, \quad \text{where } f = \prod_{i=0}^{r-1} (\operatorname{ord} H_i^\theta(M))^{\varepsilon(i)}.$$

5.3. The invariants δ and ω for links.
Let $l = l_1 \cup \ldots \cup l_n$ be a link in S^{2r+1} with exterior V. Let δ_l, $\gamma(l)$, $\omega(l)$, and $\omega_0(l)$ denote the invariants $\delta(V)$, $\gamma(V)$, $\omega(V)$, and $\omega_0(V)$ respectively (see §5.2; the torsion $\omega(V)$, as is easily verified, is defined; the refined torsion $\omega_0(V)$ corresponds to the homological orientation of V defined as in §4.3). If $r \geqslant 2$, then the invariants introduced here give nothing new as compared with the Alexander polynomial Δ_l. For by Lemma 1.11.4, $\gamma(l) = 0$ so that $\delta_l = \Delta_l$; as is easily deduced from Theorem 5.1.1 and the results of §3, $\Delta_l(t_1 - 1) \ldots (t_n - 1) \subset \omega(l)$ and

$$\{\Delta(t_1 - 1) \ldots (t_n - 1) \mid \Delta \in \Delta_l, \ \operatorname{aug}(\Delta) = -1\} \subset \omega_0(l).$$

We consider the case $r = 1$. It can easily be seen that δ_l is the first non-zero term in the sequence of Alexander polynomials of the link l. Its number in this sequence is $\gamma(l) + 1$. Here $0 \leqslant \gamma(l) \leqslant n - 1$; this can be proved by applying Lemma 1.11.4 to the pair (V, the meridian of the component l_1).

5.3.1. ***Theorem.*** *Let $l = l_1 \cup \ldots \cup l_n$ be a link in S^3 with $n \geqslant 2$. Then:* (i) *there is a unique subset $\alpha = \alpha(l)$ of the set $\{1, 2, \ldots, n\}$ for which $\delta_l \prod_{i\in\alpha} (t_i - 1) \subset \omega(l)$;* (ii) *if $\delta \in \delta_l$, then $\bar{\delta} = (-1)^m t_1^{\nu_1} \ldots t_n^{\nu_n}\delta$, where $m = n - \operatorname{card}(\alpha)$, and where ν_i is an integer congruent modulo 2 to $\sum_{j\neq i} \mu(l_i, l_j)$ in the case $i \in \alpha$, and to $1 + \sum_{j\neq i} \mu(l_i, l_j)$ in the case $i \notin \alpha$;* (iii) *if $i \in \alpha$, then $\mu(l_i, l_j) = 0$ for all $j \neq i$.*

Points (i) and (ii) show that $\alpha(l)$ and $\omega(l)$ can be calculated from δ_l and the linking numbers of the components. Point (ii) also gives the symmetry relation for δ_l.

Theorem 5.3.1 makes it possible in some cases to describe explicitly the set $\alpha(l)$. For example, if every component of l has a non-zero linking number with some other component, then $\alpha(l) = \emptyset$. If $\Delta_l \neq 0$, then $\delta_l = \Delta_l$ and comparison of Theorems 1.7.1 and 5.3.1 shows that $\alpha(l) = \emptyset$. If $\gamma(l) = n - 1$, then $\mu(l_i, l_j) = 0$ for all $i \neq j$ and $\operatorname{aug}(\delta_l) = \pm 1$ (see for example [17]). From the last equality it follows purely algebraically that the numbers $r, \nu_1, \ldots, \nu_n$ in the formula $\bar{\delta} = (-1)^r t_1^{\nu_1} \ldots t_n^{\nu_n} \delta$ are even. Therefore, if $\gamma(l) = n - 1$, then $\alpha(l) = \{1, 2, \ldots, n\}$. For example, if l is a

2-component link with $\Delta_l = 0$, then $\gamma(l) = 1$ and $\alpha(l) = \{1, 2\}$. From Theorems 5.4.1 and 5.4.2 stated below it is easy to deduce that the set $\alpha(l)$ is an invariant for cobordism of links.

We prove Theorem 5.3.1.

5.3.2. ***Lemma.*** *Under the conditions of Theorem* 5.3.1, *if* $\omega \in \omega(l)$, *then* $\overline{\omega} = (-1)^n t_1^{\chi_1} \dots t_n^{\chi_n}\omega$ *with* $\chi_i \equiv 1 + \sum_{j\neq i} \mu(l_i, l_j) \pmod 2$ *for all* $i = 1, \dots, n$.

Proof. As is well known (see for example [24]), there is a link $l' = l'_1 \cup \dots \cup l'_n$ in S^3 for which $\Delta_{l'} \neq 0$ and $\mu(l'_i, l'_j) = \mu(l_i, l_j)$ for all $i \neq j$. We denote by M the closed manifold obtained as the result of gluing the exteriors V and V' of the links l and l' by a homeomorphism of the boundaries taking the meridian and parallel of the component l_i to the meridian and parallel of the component l'_i respectively, for all i. It is obvious that the canonical isomorphisms from the groups $H_1(V)$ and $H_1(V')$ to the free Abelian group with free generators $t_1, \dots, t_n$ induce a ring homomorphism $\mathbf{Z}[H_1(M)] \to \mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. We denote this by φ. From the multiplicativity of torsion and Lemma 1.3.3 it can easily be deduced that $\omega(l)\omega(l') \subset \omega^{\varphi}(M)$. From the results of §1 and §5.1 it follows that $\Delta_{l'} \subset \omega(l')$. Thus if $\Delta \in \Delta_{l'}$ then $\omega\Delta \in \omega^{\varphi}(M)$. By the duality theorem for the torsion ω of a closed manifold (see Appendix, Theorem 6), we have $\overline{\omega\Delta} = t_1^{\mu_1} \dots t_n^{\mu_n}\omega\Delta$ with even $\mu_1, \dots, \mu_n$. Hence by Theorem 1.7.1 we have the statement of the Lemma.

5.3.3. *Proof of Theorem* 5.3.1. Let $\widetilde{V}$ be a maximal Abelian cover of the exterior V of the link l. We denote the ring $\mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ by K. We denote the order of the K-module $H_i(\partial\widetilde{V})$ by a_i (where $i = 0, 1$). By Theorem 5.1.1, the torsion $\omega(l) = \omega(V)$ can be represented by the product $\mathrm{ord}(\mathrm{Tors}\, H_1(\widetilde{V}, \partial\widetilde{V})) \times (\mathrm{ord}\, J)a_0^{-1}$ where J is the image of the inclusion homomorphism $H_1(\partial\widetilde{V}) \to H_1(\widetilde{V})$. From Lemma 2.2.1 it follows that

$$\mathrm{ord}\,(\mathrm{Tors}\, H_1(\widetilde{V}, \partial\widetilde{V})) = \overline{\mathrm{ord}\,(\mathrm{Tors}\, H_1(\widetilde{V})) = \delta_l}.$$

Since by Lemma 5.3.2, $\overline{\omega(l)} = \omega(l)$ it follows from this that $\omega(l)$ can be represented as the product $\delta \times \overline{\mathrm{ord}\, J} \times \overline{a_0^{-1}}$. We denote the set $\{i \mid 1 \leqslant i \leqslant n,\ \mu(l_i, l_j) = 0 \text{ for all } j \neq i\}$ by I. We put $T_i = \prod_{j\neq i} t_j^{\mu(l_i, l_j)}$ for $i = 1, \dots, n$. It can easily be verified that $H_0(\partial\widetilde{V}) = \bigoplus_{i=1}^{n} K/(t_i - 1, T_i - 1)K$ and $H_1(\partial\widetilde{V}) = \bigoplus_{i\in I} K/(t_i - 1)K$. If $i \notin I$, then the polynomials $t_i - 1$ and $T_i - 1$ are mutually coprime; if $i \in I$ then $T_i = 1$. Hence the orders a_0 and a_1 are equal, and are represented by the product $\prod_{i\in I}(t_i - 1)$. Since the order of the module J divides a_1, the fraction $\overline{\mathrm{ord}\, J/a_0}$ is represented by the product $\prod_{i\in\alpha} \overline{(t_i - 1)^{-1}}$, where α is some subset of I. Hence it follows that

$\delta \prod_{i \in \alpha} (t_i - 1) \in \omega(l)$. From this inclusion and Lemma 5.3.2 there follow all the statements of the theorem.

5.4. The invariants ω, δ and cobordisms.

5.4.1. ***Theorem.*** *Under the conditions of Theorem* 1.11.2, $\omega(V_1) = \omega(V_2)$. *Under the conditions of Theorem* 3.4.4, $\omega_0(V_1) = \omega_0(V_2)$.

The proof of this theorem is similar to the proof of Theorem 1.11.2 and is therefore omitted.

5.4.2. ***Theorem.*** *If l and l' are cobordant links in an odd-dimensional sphere, then the polynomials δ_l and $\delta_{l'}$ have c-equivalent representatives* (*see* § 1.11).

In the case of links in S^3, this theorem was proved in 1978 by Kawauchi [21] and independently by Nakagawa [31] (these authors did not use torsions). From Theorem 5.4.2 one can easily deduce the existence (which was problematic for some time) for $r \geqslant 2$ of links in S^{2r+1} not cobordant to split links. We consider for example the link l in S^{2r+1} composed of $n \geqslant 2$ parallels of a knot $k \subset S^{2r+1}$. It is easily seen that the polynomial δ_l can be obtained from Δ_k by the substitution $t \mapsto t_1 t_2 \ldots t_n$. From Theorem 5.4.2 it follows that if l is cobordant to a split link, then the polynomials $\Delta_k(t_1 t_2 \ldots t_n)$ and $\Delta_k(t_1)\Delta_k(t_2) \ldots \Delta_k(t_n)$ have c-equivalent representatives. The latter happens if and only if the polynomial Δ_k has representatives of the form $f\bar{f}$, where $f \in \mathbf{Z}[t, t^{-1}]$ holds.[(1)]

Proof of Theorem 5.4.2. From Theorem 5.4.1 and the results of §5.3 we have the existence of non-zero Laurent polynomials f, f' and integers $r_1, \ldots, r_n$ such that

$$\delta_l f\bar{f} = \delta_{l'} f'\bar{f}' (t_1 - 1)^{r_1} \ldots (t_n - 1)^{r_n}, \tag{19}$$

where n is the number of components of the links l and l'. By considering the homology sequences relating the homology groups of the exteriors of l and l' and the exterior of the cobordism between l and l', and using the multiplicativity of the order and Lemma 1.11.4, it is not hard to prove the existence of $g, g' \in \mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ with $\operatorname{aug}(g) = \operatorname{aug}(g') = 1$ such that $\delta_l g = \delta_{l'} g'$ (see [21]). From this the statement of the theorem follows purely algebraically in view of (19).

5.4.3. ***Theorem.*** *If the n-component link l in S^3 is cobordant* (*in the category of ordered links*) *to its mirror image or to its mirror image with reversed orientation, then* $\gamma(l) \equiv n + 1 \pmod 2$.

This theorem strengthens Conway's theorem discussed in Remark 3 to §3: if the numbers n and $\gamma(l) - n - 1$ are even, then $\gamma(l) \neq 0$, and hence $\Delta_l = 0$.

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(1) As Kawauchi [53] showed, if $r \geqslant 2$, then the link l in question is cobordant to a split link if and only if k is a truncated knot.

Proof of Theorem 5.4.3. Let k be the image of the link l under mirror symmetry, and let k' be the link k with the reversed orientation. The natural homeomorphism of the exteriors of the links l and k changes the (canonical) orientation and h-orientation and takes the generators $t_1, \ldots, t_n$ of the one-dimensional homology group to $t_1^{-1}, \ldots, t_n^{-1}$ respectively. From the results of §5.1 and Lemma 5.3.2 it follows that

$$\omega_0(k) = -(-1)^{\gamma(l)+1}\overline{\omega_0(l)} = (-1)^{\gamma(l)+n+1}\omega_0(l).$$

It is not hard to show that $\omega_0(k') = \omega_0(k)$. Hence if l is cobordant to k or k', then by Theorem 5.4.1 the sum $\gamma(l)+n+1$ is even.

5.5. The analogue of the Torres formula for the polynomial δ.

5.5.1. ***Theorem.*** *Let $l = l_1 \cup \ldots \cup l_n$ be a link in S^3 with $n \geqslant 3$, and let k be the link $l_1 \cup l_2 \cup \ldots \cup l_{n-1}$. Let at least one of the numbers $\mu(l_1, l_n), \mu(l_2, l_n), \ldots, \mu(l_{n-1}, l_n)$ be non-zero and let $T = \prod_{i=1}^{n-1} t_i^{\mu(l_i, l_n)}$. Then $\gamma(k) \geqslant \gamma(l)$; if $\gamma(k) > \gamma(l)$, then $\delta_l(t_1, \ldots, t_{n-1}, 1) = 0$; if $\gamma(k) = \gamma(l)$, then there is a polynomial $h \in \mathbf{Z}[t_1, t_1^{-1}, \ldots, t_{n-1}, t_{n-1}^{-1}]$ for which*

$$\delta_k \times (T-1) = \delta_l(t_1, \ldots, t_{n-1}, 1) \times h\bar{h} \prod_{i\in\beta}(t_i - 1), \tag{20}$$

where $\beta = (\alpha(k) \setminus \alpha(l)) \cup (\alpha(l) \setminus \alpha(k))$.

Outwardly, formula (20) differs from the Torres formula (3) by the appearance of two additional factors $h\bar{h}$ and $\Pi(t_i - 1)$. These factors have different nature. The factor $\Pi(t_i - 1)$ arises from the presence of a non-empty boundary in the exteriors of the links. In particular, this factor compensates for the difference between the symmetry relations satisfied by the polynomials $\delta_k \times (T-1)$ and $\delta_l(t_1, \ldots, t_{n-1}, 1)$. The presence of the factor $h\bar{h}$ reflects the fact that in the transition from the $\mathbf{Z}^{n-1}$-fold cover of the exterior of an n-component link to the $\mathbf{Z}^n$-fold cover, part of the information about the order of the homology modules is lost. The special form of the factor $h\bar{h}$ indicates that these losses occur within certain limits.

In the case when $\gamma(l) = 0$, the polynomial δ_l coincides with the Alexander polynomial Δ_l, and the statement of Theorem 5.5.1 follows directly from the Torres theorem 1.4.1 (for h one should take 1). We consider (20) in greater detail in the case when $\gamma(l) = \gamma(k) = n-2$. Then $\alpha(k) = \{1, 2, \ldots, n-1\}$ and if $\mu(l_j, l_n) \neq 0$, then by Theorem 5.3.1, $j \notin \alpha(l)$, so that $j \in \beta$. Since $\operatorname{aug}(\delta_k) = \pm 1$, and $\delta_k \times (T-1)$ is divisible by $\prod_{i\in\beta}(t_i - 1)$, the set β consists of the single element j. From (20) it follows that

$$\delta_k \times (1 + t_j + t_j^2 + \ldots + t_j^{|\mu|-1}) = \delta_l(t_1, \ldots, t_{n-1}, 1) \times h\bar{h},$$

where $\mu = \mu(l_j, l_n)$. In the case when k is the trivial knot, from this it follows that $\delta_l(t_1, \ldots, t_{n-1}, 1) \doteq 1 + t_j + \ldots + t_j^{|\mu|-1}$. The last example may give the impression that we always have $h\bar{h} = 1$. That this is not so is shown by the examples given in §5.6.

For the proof of Theorem 5.5.1 we require the following lemma, which is a version of Lemma 2.4 for the torsion ω.

5.5.2. ***Lemma.*** *Let M be an orientable connected compact manifold of odd dimension, and let $G = H_1(M)^{\#}$. Let φ be the composition of the projection $\theta : \mathbf{Z}[H_1(M)] \to \mathbf{Z}[G]$ and the projection ψ from the ring $\mathbf{Z}[G]$ into the group ring of a free Abelian factor group of G. Let the torsions $\omega(M)$ and $\omega^{\varphi}(M)$ be defined, and let* rg $H_i^{\varphi}(M) =$ rg $H_i^{\theta}(M)$ *for all i. If ω is an element of the ring $\mathbf{Z}[G]$ representing $\omega(M)$, then either $\psi(\omega) = 0$ or $\psi(\omega) \in \omega^{\varphi}(M)$.*

Proof. An argument similar to that given in the proof of Lemma 2.4 shows that there are elements a and b of the ring $\mathbf{Z}[G]$ with the properties: $ab^{-1} \in \omega(M)$, $\psi(a) \neq 0$, $\psi(b) \neq 0$, $\psi(a)\psi(b)^{-1} \in \omega^{\varphi}(M)$. Then there are elements $f, g \in \mathbf{Z}[G]$ for which $f\bar{f}$ and $g\bar{g}$ are mutually coprime and $\omega f\bar{f} = ab^{-1}g\bar{g}$. Here a is divisible by f and hence $\psi(f) \neq 0$. Therefore $\psi(\omega) = \psi(a)\psi(b)^{-1}\psi(g)\overline{\psi(g)}\psi(f)^{-1}\overline{\psi(f)}^{-1}$. Hence one can see that either $\psi(\omega) = \psi(g) = 0$ or $\psi(\omega) \in \omega^{\varphi}(M)$.

5.5.3. *Proof of Theorem* 5.5.1. Let U be the exterior of the link k. Let V be the exterior of the knot l_n in U. We put $K = \mathbf{Z}[t_1, t_1^{-1}, \ldots, t_{n-1}, t_{n-1}^{-1}]$ and $L = \mathbf{Z}[t_1, t^{-}, \ldots, t_n, t_n^{-1}]$. We denote the canonical isomorphisms $\mathbf{Z}[H_1(U)] \to K$ and $\mathbf{Z}[H_1(V)] \to L$ by η and θ respectively. We denote by ψ the ring homomorphism $f(t_1, \ldots, t_{n-1}, t_n) \mapsto f(t_1, \ldots, t_{n-1}, 1): L \to K$. We put $\varphi = \psi \circ \theta : \mathbf{Z}[H_1(V)] \to K$.

We consider a segment of the homology sequence of the pair (U, V) with twisted coefficients in K:

$$H_2^{\eta}(U, V) \to H_1^{\varphi}(V) \to H_1^{\eta}(U) \to 0.$$

According to the definitions, $\delta_k =$ ord(Tors $H_1^{\eta}(U)$). It is obvious that the K-module $H_2^{\eta}(U, V)$ is isomorphic to $K/(T-1)K$, and in particular, it is a periodic module. Hence it follows that $\gamma(k) =$ rg $H_1^{\eta}(U) =$ rg $H_1^{\varphi}(V)$ and that the polynomial ord(Tors $H_1^{\varphi}(V)$) divides the product $(T-1)\delta_k$.

Let v be a point of the manifold V. From the exactness of the homology sequence of the pair (V, v) with coefficients in L, it follows that $\gamma(l) =$ rg $H_1^{\theta}(V) = 1 +$ rg $H_1^{\theta}(V, v)$ and that Tors $H_1^{\theta}(V) =$ Tors $H_1^{\theta}(V, v)$. Similarly, Tors $H_1^{\varphi}(V) =$ Tors $H_1^{\varphi}(V, v)$ and $\gamma(k) = 1 +$ rg $H_1^{\varphi}(V, v)$. It is obvious that $H_1^{\varphi}(V, v) = K \otimes_L H_1^{\theta}(V, v)$.

Let A be the relation matrix of the L-module $H_1^{\theta}(V, v)$. Let m be the number of columns of A. We denote by A' the matrix obtained from A by replacing the entries by their images under the homomorphism ψ. It is clear that A' is the relation matrix of the K-module $H_1^{\varphi}(V, v)$. Therefore

$\gamma(l) = 1+m-\mathrm{rg}\,A \leqslant 1+m-\mathrm{rg}\,A' = \gamma(k)$. If $\gamma(l) < \gamma(k)$, then $\mathrm{rg}\,A > \mathrm{rg}\,A'$ and hence all the minors of order $\mathrm{rg}\,A$ of A belong to Ker ψ. Since Ker ψ is a principal ideal of the ring L, the greatest common divisor of these minors, which is equal to δ_l by Lemma 1.11.3, also belongs to Ker ψ. From Lemma 1.11.3 it also follows that if $\gamma(l) = \gamma(k)$, then $\psi(\delta_l)$ divides ord(Tors $H_1^\varphi(V)$) and *a fortiori* divides $(T-1)\delta_k$. In particular, $\psi(\delta_l) \neq 0$.

We suppose that $\gamma(l) = \gamma(k)$. We put

$$a = \prod_{i\in\alpha(k)} (t_i-1),\quad b = \prod_{i\in\alpha(l)} (t_i-1) \text{ and } c = \prod_{i\in\alpha(k)\setminus\alpha(l)} (t_i-1).$$

By Theorem 5.3.1, $a\delta_k \subset \omega(k) = \omega^\eta(U)$ and $b\delta_l \subset \omega(l) = \omega^\theta(V)$. It is obvious that $\mathrm{rg}\,H_i^\varphi(V) = \mathrm{rg}\,H_i^\theta(V)$ for all i (for $i = 0$, both ranks are zero, for $i = 1$, the ranks are $\gamma(k)$ and $\gamma(l)$, for $i = 2$ the equality of the ranks follows by consideration of the Euler characteristic). Since $\mu(l_j, l_n) \neq 0$ for some j, the number n does not appear in $\alpha(l)$ and so $\psi(b\delta_l) \neq 0$. By Lemma 5.5.2, $\psi(b\delta_l) \subset \omega^\varphi(V)$. From the multiplicativity of the torsion it follows that $\omega^\varphi(V) = (T-1)\omega^\eta(U)$ (see § 1.4.3). Hence there exist non-zero $f, g \in L$ for which the polynomials $f\bar{f}$ and $g\bar{g}$ are coprime and

$$\psi(\delta_l)bf\bar{f} = \delta_k a g\bar{g}(T-1). \tag{21}$$

Since $\psi(\delta_l)$ divides $(T-1)\delta_k$, from (21) it follows that $g\bar{g}$ divides b. Hence $g\bar{g} = 1$. Since a divides $bf\bar{f}$, c divides f. Thus from (21) we have (20) with $h = fc^{-1}$.

5.6. *Examples.* To illustrate Theorem 5.5.1 we formulate a realization theorem for the Alexander modules of three-component links. We recall that the Alexander module of a link in S^3 is the one-dimensional Alexander module of the exterior of this link.

5.6.1. ***Theorem.*** *A module over the ring* $L = \mathbf{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}, t_3, t_3^{-1}]$ *can be realized as the Alexander module of a three-component link* $l_1 \cup l_2 \cup l_3 \subset S^3$ *with* $\mu(l_1, l_2) = \mu(l_1, l_3) = 0$ *and* $\mu(l_2, l_3) = 1$ *if and only if it has a relation matrix*

$$\begin{bmatrix} t_3-1 & 1-t_2 & 0 \\ 0 & 0 & (t_1-1)(t_2-1)\bar{f}^t+(t_1-1)(t_3-1)\bar{g}^t \\ f & g & B \end{bmatrix},$$

where t denotes transposition; f and g are columns, and B is a square matrix over L with $\bar{B} = B^t$ *and* $\mathrm{aug}\,B = \mathrm{diag}(\pm1, \pm1, \ldots, \pm1)$. *Here the Alexander modules of the links* $l_1 \cup l_2$ *and* $l_1 \cup l_3$ *are given respectively by the matrices*

$$\begin{bmatrix} 0 & (t_1-1)(t_2-1)\overline{f(t_1, t_2, 1)^t} \\ f(t_1, t_2, 1) & B(t_1, t_2, 1) \end{bmatrix}, \begin{bmatrix} 0 & (t_1-1)(t_3-1)\overline{g(t_1, 1, t_3)^t} \\ g(t_1, 1, t_3) & B(t_1, 1, t_3) \end{bmatrix}.$$

This theorem is analogous to Bailey's theorem on the Alexander modules of two-component links, and is proved similarly (see for example [23]).

We consider a concrete example. Let $a \in \mathbf{Z}[t_1, t_2]$ with $\operatorname{aug}(a) = 1$. We put $b = a + t_3 - 1$. By Theorem 5.6.1 there is a link $l = l_1 \cup l_2 \cup l_3$ whose Alexander module is given by the matrix

$$\begin{bmatrix} t_3-1 & 1-t_3 & 0 & 0 \\ 0 & 0 & (t_1-1)(t_2-1)\bar{a} & (t_1-1)(t_2-1)\bar{a} \\ a & 0 & b\bar{b} & 0 \\ a & 0 & 0 & -b\bar{b} \end{bmatrix}.$$

We put $k = l_1 \cup l_2$. Direct calculation shows that $\gamma(l) = \gamma(k) = 1$, $1 \in \delta_l$ and $a\bar{a} \in \delta_k$. Formula (20) in this case reduces to the equality $\delta_k(t_2-1) = \delta_l(t_1, t_2, 1) \times a\bar{a}(t_2-1)$. Here $\alpha(l) = \{1\}$ and $\alpha(k) = \{1, 2\}$.

§6. Proof of Theorem 5.1.1

6.1. Auxiliary construction: the torsion ρ.

Let $C = (C_m \to \dots \to C_0)$ be a chain complex over the principal ideal ring K. The modules $C_i/\operatorname{Tors} C_i$ and $H_i(C)/\operatorname{Tors} H_i(C)$ are free; we equip them with certain bases over K (for all i). These bases are simultaneously bases for the vector spaces $Q(K) \otimes_K C_i$ and $H_i(Q(K) \otimes_K C)$. To these bases there corresponds the torsion $\check{\tau}(Q(K) \otimes_K C) \in Q(K)$. It is easy to see that this torsion, considered up to multiplication by invertible elements of K, is independent of the choice of bases, and hence is completely determined by the complex C. This torsion, considered up to the accuracy mentioned, is denoted by $\rho(C)$. We show that

$$\rho(C) = \prod_{i=0}^{m} [\operatorname{ord}(\operatorname{Tors} C_i)]^{-\varepsilon(i)} [\operatorname{ord}(\operatorname{Tors} H_i(C))]^{\varepsilon(i)}. \tag{22}$$

We put $Z_i = \operatorname{Ker}(\partial_{i-1}\colon C_i \to C_{i-1})$ and $G_i = H_i(C)/\operatorname{Tors} H_i(C)$. We denote by s_i a cross-section $G_i \to Z_i$ of the projection $Z_i \to G_i$. We consider the chain complex $E = (E_m \to \dots \to E_0)$, where $E_i = C_i \oplus G_{i-1}$, and where the boundary homomorphism $E_i \to E_{i-1}$ takes the pair (c, g) with $c \in C_i$ and $g \in G_{i-1}$ to $\partial_{i-1}(c) + s_{i-1}(g) \in C_{i-1}$. It follows immediately from the definitions that $H_i(E) = \operatorname{Tors} H_i(C)$ for all i, and $\rho(E) = \rho(C)$. Hence we can see that it suffices to establish (22) in the case when $H_*(C)$ is a periodic module (that is, when $H_i(C) = \operatorname{Tors} H_i(C)$ for all i).

We denote by C' the subcomplex $\operatorname{Tors} C_m \to \operatorname{Tors} C_{m-1} \to \dots \to \operatorname{Tors} C_0$ of the complex C. We denote the factor complex C/C' by D. It is obvious that the projection $C \to D$ induces an isomorphism of the acyclic chain complexes $Q(K) \otimes_K C \to Q(K) \otimes_K D$. Hence $\rho(C) = \rho(D)$. By Lemma 2.1.1, $\rho(D) = \prod_{i=0}^{m} [\operatorname{ord} H_i(D)]^{\varepsilon(i)}$. We recall that the order ord is multiplicative:

if A is a periodic K-module and B a submodule of it, then $\operatorname{ord} A = \operatorname{ord} B \times \times \operatorname{ord}(A/B)$. From the multiplicativity of order and the exactness of the homology sequence of the pair (C, C') it is easily deduced that

$$\prod_{i=0}^{m} [\operatorname{ord} H_i(D)]^{\varepsilon(i)} = \prod_{i=0}^{m} [\operatorname{ord} H_i(C)]^{\varepsilon(i)} \prod_{i=0}^{m} [\operatorname{ord} H_i(C')]^{-\varepsilon(i)}.$$

Similar considerations show that

$$\prod_{i=0}^{m} [\operatorname{ord} H_i(C')]^{-\varepsilon(i)} = \prod_{i=0}^{m} [\operatorname{ord} C'_i]^{-\varepsilon(i)}.$$

Hence we have (22).

6.2. *Proof of the theorem.* We fix a triangulation of the manifold M. We denote the chain complex $Q(K) \otimes_K C_*^{\varphi}(M)$ by C. We fix a sequence $g = (g_0, g_1, \ldots, g_m)$ of bases for the $Q(K)$-modules $H_0(C), H_1(C), \ldots, H_m(C)$. We denote by u_i the matrix of the sesquilinear form of the intersection numbers $H_i(C) \times H_{m-i}(C) \to Q(K)$ with respect to the bases g_i and g_{m-i}. By the definitions, the torsion $\omega^{\varphi}(M)$ can be represented as the product $\check{\tau}(C, g) \prod_{i=0}^{r} (\det u_i)^{-\varepsilon(i)}$. We denote this product by ω. We denote by ω' the product of the right hand side of (18) by

$$\prod_{i=r+1}^{2r} [\operatorname{ord}(\operatorname{Tors} H_i^{\varphi}(M)) \times \overline{\operatorname{ord}(\operatorname{Tors} H_i^{\varphi}(M))}]^{\varepsilon(i)}.$$

We say that the elements a and a' of the field $Q(K)$ are equivalent, and write $a \sim a'$, if there are invertible elements λ, λ' of the ring K for which $\lambda a = \lambda' a'$. We represent $\omega(\omega')^{-1}$ as the product of integral powers of pairwise inequivalent irreducible elements of K. We denote the degree to which an irreducible element π appears in this product by $s(\pi)$. To prove the theorem, it suffices to verify that $s(\overline{\pi}) = s(\pi)$ and that if $\overline{\pi} \sim \pi$, then $s(\pi)$ is even.

We denote by K_π the result of localizing the ring K by the multiplicative system consisting of the elements prime to π. We denote the composition of the homomorphism $\varphi: \mathbf{Z}[H_1(M)] \to K$ and the inclusion $K \to K_\pi$ by $\psi(\pi)$, or, more briefly, by ψ. It is clear that K_π is a local ring, and in particular, a principal ideal ring. We fix a basis h_i of the free K_π-module $H_i^{\psi}(M)/\operatorname{Tors} H_i^{\psi}(M)$. The inclusion $K_\pi \to Q(K)$ induces an embedding of the latter module in $H_i(C)$; here the image of the basis h_i is a basis for the $Q(K)$-module $H_i(C)$. Let $[h_i/g_i] \doteq \pi^{p_i}$, where the sign $\doteq$ denotes equality up to multiplication by an invertible element of the ring K_π, and where $p_i = p_i(\pi) \in \mathbf{Z}$. We denote by g_i^* the basis of the $Q(K)$-module $H_{m-i}(Q(K) \otimes_K C_*^{\varphi}(M, \partial M))$ dual to the basis g_i relative to the sesquilinear form of the intersection numbers

$$(23) \qquad H_i(C) \times H_{m-i}(Q(K) \otimes_K C_*^{\varphi}(M, \partial M)) \to Q(K).$$

Let h_i' be a basis for the free K_π-module $H^\psi_{m-i}(M, \partial M)/\mathrm{Tors}\, H^\psi_{m-i}(M, \partial M)$, and let $[h_i'/g_i^*] \doteq \pi^{q_i}$, where $q_i = q_i(\pi) \in \mathbf{Z}$. We remark that the numbers p_i and q_i are independent of the choice of the bases h_i and h_i'. Below we prove two assertions: (i) $s(\pi) = \sum_{i=r+1}^{m} \varepsilon(i)\,(p_i - q_i)$; (ii) $p_i(\pi) = -q_i(\overline{\pi})$ for any i. Hence it follows that $s(\overline{\pi}) = s(\pi)$, and that if $\overline{\pi} \sim \pi$, then $s(\pi)$ is even.

We prove (i). It is obvious that $\check{\tau}(C, g) \doteq \check{\tau}(C, h) \times \prod_{i=0}^{m} \pi^{\varepsilon(i)p_i}$, where $h = (h_0, h_1, \dots, h_m)$. The fact that the complex $C^\psi_*(M)$ is free implies that $\mathrm{Tors}\, H^\psi_m(M) = 0$, and by the results of §6.1, also $\check{\tau}(C, h) = \prod_{i=0}^{m-1} [\mathrm{ord}(\mathrm{Tors}\, H^\psi_i(M))]^{\varepsilon(i)}$. Since the localization function is exact, we have $H^\psi_i(M) = K_\pi \otimes_K H^\varphi_i(M)$, so that

$$\mathrm{ord}\,(\mathrm{Tors}\, H^\psi_i(M)) \doteq \mathrm{ord}\,(\mathrm{Tors}\, H^\varphi_i(M)).$$

Thus

$$\check{\tau}(C, g) \doteq \prod_{i=0}^{m-1} [\mathrm{ord}\,(\mathrm{Tors}\, H^\varphi_i(M))]^{\varepsilon(i)} \prod_{i=0}^{m} \pi^{\varepsilon(i)s_i}.$$

It is obvious that the matrix u_i is equal to the matrix of the inclusion homomorphism $H_i(C) \to H_i(Q(K) \otimes_K C^\varphi_*(M, \partial M))$ with respect to the bases g_i and g^*_{m-i}. Let v_i be the matrix of this homomorphism with respect to h_i and h'_{m-i}. It is clear that $\det u_i \doteq \det v_i \times \pi^{p_i - q_{m-i}}$. Hence

$$\prod_{i=0}^{r} (\det u_i)^{-\varepsilon(i)} = \prod_{i=0}^{r} (\det v_i)^{-\varepsilon(i)} \prod_{i=0}^{r} \pi^{-\varepsilon(i)(p_i - q_{m-i})}.$$

It follows immediately from the definitions that the product $\prod_{i=0}^{r} (\det v_i)^{-\varepsilon(i)}$ is equal to the torsion ρ of the following acyclic chain complex over K_π:

$$K_\pi \otimes_K J \to H^\psi_r(M) \to H^\psi_r(M, \partial M) \to H^{\psi\circ\mathrm{in}}_{r-1}(\partial M) \to \dots \to H^{\psi\circ\mathrm{in}}_0(\partial M) \to$$
$$\to H^\psi_0(M) \to H^\psi_0(M, \partial M) \to 0.$$

(We remark that $\mathrm{rg}\, J = \mathrm{rg}\, H^{\psi\circ\mathrm{in}}_*(\partial M) = 0$.) By formula (22) and the exactness of the localization functor

$$\prod_{i=0}^{r} (\det v_i)^{-\varepsilon(i)} \doteq \prod_{i=0}^{r} [\mathrm{ord}\,(\mathrm{Tors}\, H^\varphi_i(M))]^{-\varepsilon(i)} \times$$
$$\times \prod_{i=0}^{r} [\mathrm{ord}\,(\mathrm{Tors}\, H^\varphi_i(M, \partial M))]^{\varepsilon(i)} \times \prod_{i=0}^{r-1} [\mathrm{ord}\, H^{\varphi\circ\mathrm{in}}_i(\partial M)]^{\varepsilon(i)} (\mathrm{ord}\, J)^{\varepsilon(r)}.$$

We remark that $\mathrm{ord}\,(\mathrm{Tors}\ H^\varphi_i(M, \partial M)) = \overline{\mathrm{ord}\,(\mathrm{Tors}\, H^\varphi_{2r-i}(M))}$ (see §2.2.1). Combining all these formulae, we obtain

$$\omega \doteq \omega' \prod_{i=r+1}^{m} \pi^{\varepsilon(i)(p_i - q_i)}.$$

From this (i) follows.

We show that $p_i(\pi) = -q_i(\bar{\pi})$. We denote the $K_{\bar{\pi}}$-module $H^{\psi(\bar{\pi})}_{m-i}(M, \partial M)$ by E. It suffices to prove that some basis h_i^* of the free $K_{\bar{\pi}}$-module $E/\mathrm{Tors}\, E$ is dual to the basis h_i of the module $H_i^{\psi}(M)/\mathrm{Tors}\, H_i^{\psi}(M)$ (where $\psi = \psi(\pi)$) with respect to the sesquilinear form

$$(24) \qquad H_i^{\psi}(M)/\mathrm{Tors}\, H_i^{\psi}(M) \times E/\mathrm{Tors}\, E \to Q(K),$$

obtained by restriction of the form (23). Hence it follows that

$$\pi^{p_i(\pi)} \doteq [h_i/g_i] = \overline{[h_i^*/g_i^*]^{-1}} \doteq \overline{\bar{\pi}^{-q_i(\bar{\pi})}} = \pi^{-q_i(\bar{\pi})},$$

that is, that $p_i(\pi) = -q_i(\bar{\pi})$.

If $\bar{\pi} \sim \pi$, then $K_{\bar{\pi}} = K_{\pi}$, and the existence of the basis h_i^* dual to the basis h_i is a well-known corollary of the duality theorem and the universal coefficient formula (for principal ideal rings). The following argument goes through for any π. We denote by σ the canonical involution of the ring $\mathbf{Z}[H_1(M)]$. Since $\bar{\varphi} = \varphi \circ \sigma$, the change of rings $a \mapsto \bar{a}$: $K_{\bar{\pi}} \to K_{\pi}$ yields an isomorphism $E \to H^{\psi\circ\sigma}_{m-i}(M, \partial M)$. This isomorphism takes (24) to the form, bilinear over K_{π}:

$$H_i^{\psi}(M)/\mathrm{Tors}\, H_i^{\psi}(M) \times H^{\psi\circ\sigma}_{m-i}(M, \partial M)/\mathrm{Tors}\, H^{\psi\circ\sigma}_{m-i}(M, \partial M) \to K_{\pi}.$$

The latter form is non-singular (see §§2.2.2 and 2.2.3). Hence we have the existence of the h_i^* with the desired property.

Appendix

Duality theorems for torsions

1. Description of the situation.
Up to the end of this appendix, we suppose that we are given a compact m-dimensional manifold M, an integral domain K, and a ring homomorphism $\varphi : \mathbf{Z}[H_1(M)] \to K$. We assume that the ring K is equipped with an involution $a \mapsto \bar{a}$: $K \to K$ such that $\overline{\varphi(h)} = w_1(h)\varphi(h^{-1})$ for any $h \in H_1(M)$. The induced involution $Q(K) \to Q(K)$ will also be denoted by a bar.

2. ***Theorem*** (Franz [12], Milnor [26]). *If m is odd, and if $\tau^{\varphi}(M) \neq 0$, then*

$$\tau^{\varphi}(M, \partial M) = \overline{[\tau^{\varphi}(M)]}^{\varepsilon(m)} \quad (\textit{where } \varepsilon(m) = (-1)^{m+1}).$$

Proof. We denote by σ the involution of $\mathbf{Z}[H_1(M)]$ taking $h \in H_1(M)$ to $w_1(h)h^{-1}$. It is obvious that $\bar{\varphi} = \varphi \circ \sigma$, and $\overline{\tau^{\varphi}(M)} = \tau^{\bar{\varphi}}(M)$. Hence the equality in question is equivalent to the equality $\tau^{\varphi}(M, \partial M) = [\tau^{\varphi\circ\sigma}(M)]^{\varepsilon(m)}$, which follows from Lemma 4.3.3 (i) and the fact, which has already been used above, that if X is a piecewise-linear transformation of M, then the chain complex $C_*^{\varphi\circ\sigma}(X^*, \partial X^*)$ is dual to the complex $C_*^{\varphi}(X)$ (see §2.2).

3. ***Theorem*** (a generalization of Theorem 2). *Let V and V' be disjoint compact $(m-1)$-dimensional submanifolds (possibly with boundary) of ∂M for which $\partial M \setminus \mathrm{Int}(V \cup V')$ is a cylinder with the bases ∂V and $\partial V'$. If m is odd, and if $\tau^\varphi(M, V) \neq 0$, then*

$$\tau^\varphi(M, V') = [\overline{\tau^\varphi(M, V)}]^{\varepsilon(m)}.$$

The proof of this theorem is similar to the proof of Theorem 2, with the difference that instead of triangulations and the CW decompositions dual to them we use dual decompositions into handles (see [36]).

4. **The Stiefel-Whitney class $v_1(M)$.**
We recall that in the category of *pl*-manifolds we can define the so-called *Stiefel-Whitney homology classes* (see for example [51]). In particular, if the manifold M is closed, then its first Stiefel-Whitney homology class $v_1(M)$ is defined as the element of the group $H_1(M)$ if m is even, and the element of the group $H_1(M;\ \mathbf{Z}/2\mathbf{Z})$ if m is odd, that can be represented by the cycle

$$\sum_{a<b} (-1)^{\dim b - \dim a} \langle \underline{a}, \underline{b} \rangle, \tag{25}$$

where: a and b are simplexes of some piecewise-linear triangulation X of M; the notation $a < b$ means that a is a proper face of b; $\underline{a}$ is the barycentre of the simplex a; $\langle \underline{a}, \underline{b} \rangle$ is the one-dimensional simplex of the first barycentric subdivision of X with vertices $\underline{a}$ and $\underline{b}$. (It is known that the class $v_1(M)$ is independent of the choice of triangulation X.) According to the classical theorem of Whitney, if the piecewise-linear structure on M is induced by a smooth structure, then the class $v_1(M)$ is the Poincaré dual to the $(m-1)$-dimensional Stiefel-Whitney cohomology class of M (a detailed formulation and proof of this theorem can be found in [51]). In particular, if M is an orientable closed three-dimensional manifold, then $w_2(M) = 0$ and so $v_1(M) = 0$.

5. ***Theorem*** (refinement of Theorem 2 in the case $\partial M = \emptyset$). *Let M be closed and orientable. We put*

$$z = \begin{cases} 0 \ \text{if } m \equiv 2 \pmod 4 \text{ or } m \equiv 3 \pmod 4, \\ \sum_{i=0}^{[m/2]} \operatorname{rg} H_i(M) \ \text{if } m \equiv 1 \pmod 4, \\ z' + \sum_{i=0}^{m/2} \operatorname{rg} H_i(M), \text{ where } z' \text{ is the number of negative squares in the} \\ \text{diagonal representation of the form of intersection numbers in} \\ H_{m/2}(M;\mathbf{R}), \text{ if } m \equiv 0 \pmod 4. \end{cases}$$

If m is even and $\tau^\varphi(M) \neq 0$, then $\tau\overline{\tau}\varphi(v_1(M)) = (-1)^z$ for any $\tau \in \tau^\varphi(M)$. If m is odd, then for any $\tau \in \tau^\varphi(M)$ there is an element g of the group $H_1(M)$ for which $g \pmod 2 = v_1(M)$ and $\overline{\tau} = (-1)^z\varphi(g)\tau$.

We remark that if $m \equiv 0 \pmod 4$ and $\tau^{\varphi}(M) \neq 0$, then the residue z' (mod 2) is independent of the choice of orientation of M, since then $\operatorname{rg} H_{m/2}(M) \equiv \chi(M) \pmod 2$ and $\chi(M) = 0$.

Theorem 5 is proved in §9, using Lemmas 7 and 8. The given proof of Theorem 5 can be carried over to the case of a non-orientable closed M, and gives in this case a weaker form of the statement of Theorem 5: $\tau\bar{\tau}\varphi(v_1(M)) = \pm 1$ if m is even, and $\bar{\tau} = \pm\varphi(g)\tau$ with $g \pmod 2 = v_1(M)$ if m is odd.

6. ***Theorem*** (duality for the torsion ω). *Let M be closed, orientable, and odd-dimensional*; $\dim M = 2r+1$. *Then for any $\omega \in \omega^{\varphi}(M)$ there is an element g of the group $H_1(M)$ for which $g \pmod 2 = v_1(M)$ and $\bar{\omega} = (-1)^y\varphi(g)\omega$, where $y = 0$ if r is odd, and $y = \sum_{i=0}^{r} (\operatorname{rg} H_i(M) + \operatorname{rg} H_i^{\varphi}(M))$ if r is even.*

The proof of this theorem is similar to the proof of Theorem 5 given below, and is therefore omitted.

7. ***Lemma*** (variant of Lemma 4.3.3 (i) for the torsion $\check{\tau}$). *Let $C = (C_m \to \dots \to C_0)$ and $C^* = (C'_m \to \dots \to C'_0)$ be dual chain complexes over a field F, with $\chi(C) = 0$. For each $i = 0, 1, \dots, m$, let the bases c'_{m-i} and h'_{m-i} of the vector spaces C'_{m-i} and $H_{m-i}(C^*)$ be dual to the bases c_i and h_i of the vector spaces C_i and $H_i(C)$ with respect to the Kronecker pairings $C_i \times C'_{m-i} \to F$ and $H_i(C) \times H_{m-i}(C^*) \to F$. Then the torsions $\check{\tau}$ of the complexes C and C^* corresponding to these bases are related by the equality $\check{\tau}(C^*) = (-1)^n \check{\tau}(C)^{\varepsilon(m)}$, where*

$$n = \sum_{i=0}^{m} [\alpha_i(C)\alpha_{i-1}(C) + \beta_i(C)\beta_{i-1}(C)] + \sum_{i=0}^{[m/2]} [\alpha_{2i}(C) + \beta_{2i}(C)].$$

Proof. We use the notation introduced in the course of the proof of Lemma 4.3.3 (i). We put $x_i = \dim \operatorname{Im} \partial_i$ and $y_i = x_i \dim H_i(C) + x_{i-1}(\dim H_i(C) + x_i + i)$. It can be verified directly that

$$[\partial'_{m-i}(b'_{m-i+1})h'_{m-i}b'_{m-i}/c'_{m-i}] = (-1)^{y_i}[\partial_i(b_{i+1})h_ib_i/c_i]^{-1}.$$

(We remark that the appearance of the term ix_{i-1} in y_i is connected with the presence of the sign in the equality $\partial'_{m-i} = (-1)^i\partial^*_{i-1}$; see §2.2.2.) Using the condition $\chi(C) = 0$, it can easily be verified that $N(C^*) = N(C)$. Hence the statement of the lemma follows from the congruence $y_0 + y_1 + \dots + y_m \equiv n \pmod 2$. This congruence can be verified directly by considering the equalities $x_i = \alpha_i(C) - \beta_i(C)$, $\dim H_i(C) = \beta_i(C) - \beta_{i-1}(C)$, and $\alpha_m(C) = \beta_m(C)$.

8. ***Lemma*** (refinement of Lemma 4.3.3 (iii)). *Under the conditions of Lemma 4.3.3 (iii), if the space X is homologically oriented, then $\tau_{\mathfrak{F}}^{\varphi}(X, e) = \tau_{\mathfrak{F}}^{\varphi}(X', e')$.*

The proof of this lemma is obtained by a simple modification of the proof of Lemma 4.3.3 (iii), and is therefore omitted. (We remark that if $\varphi(g) \neq -1$ for all $g \in H_1(X)$, then the statement of Lemma 8 follows from Lemma 4.3.3 (iii) and Theorem 3.2.1.)

9. *Proof of Theorem* 5. We assume that $\tau^{\varphi}(M) \neq 0$. Here $\chi(M) = 0$. By replacing the ring K by its quotient field if necessary, we can assume that K is a field.

Let X be a piecwise-linear triangulation of M. Let $p : \widetilde{M} \to M$ be a maximal Abelian cover. Let $e = (a_1, \ldots, a_N)$ be a base sequence of oriented simplexes induced by the triangulation $\widetilde{X}$ of the manifold $\widetilde{M}$. It is obvious that the sequence $e^* = (a_1^*, \ldots, a_N^*)$ of dual cells is a base sequence of cells of the CW decomposition $\widetilde{X}^*$ of $\widetilde{M}$ dual to $\widetilde{X}$. We orient M and the cells $a_1^*, \ldots, a_N^*$ so that for each $i = 1, \ldots, N$ the orientations of the cells a_i and a_i^* define the chosen orientation in M. The sequences e and e^* define dual bases of the dual chain complexes $C_*^{\varphi}(X)$ and $C_*^{\bar{\varphi}}(X^*)$ (compare with the proof of Theorem 2). By Lemma 7 it then follows that

$$(26) \qquad \tau(C_*^{\varphi}(X^*), e^*) = (-1)^n \overline{\tau(C_*^{\varphi}(X), e)}^{\varepsilon(m)},$$

where $n = \sum_{i=0}^{m} \alpha_i \alpha_{i-1} + \sum_{i=0}^{[m/2]} \alpha_{2i}$; α_i is the number of simplexes of X of dimension $\leqslant i$.

We consider an arbitrary basis h_i in $H_i(M;\ \mathbf{R})$, and denote by h_i^* the basis of $H_{m-i}(M;\ \mathbf{R})$ dual to it with respect to the form of the intersection numbers $H_i(M;\ \mathbf{R}) \times H_{m-i}(M;\ \mathbf{R}) \to \mathbf{R}$ (where $i = 0, 1, \ldots, m$). It is obvious that the chain complexes $C_*(X;\ \mathbf{R})$ and $C_*(X^*;\ \mathbf{R})$ over $\mathbf{R}$, when equipped with the bases of the spaces of chains corresponding to the sequences e and e^* respectively, and equipped with the respective bases $h_0, h_1, \ldots, h_m$ and $h_m^*, \ldots, h_0^*$ in the homology, are dual. By Lemma 7

$$(27) \qquad \check{\tau}(C_*(X^*; \mathbf{R})) = (-1)^{n'} \check{\tau}(C_*(X; \mathbf{R}))^{\varepsilon(m)},$$

where $n' = n + \sum_{i=0}^{m} \beta_i \beta_{i-1} + \sum_{i=0}^{[m/2]} \beta_{2i}$; $\beta_i = \sum_{j=0}^{i} \operatorname{rg} H_j(M)$.

We fix the homological orientation in M given by the basis $h_0, h_1, \ldots, h_m$ in $H_*(M;\ \mathbf{R})$. We put $\lambda = 1$ if the basis $h_m^*, h_{m-1}^*, \ldots, h_0^*$ gives the same orientation, and $\lambda = -1$ otherwise. By multiplying the left and right hand sides of (26) by the signs of the corresponding sides of (27), we obtain

$$(28) \qquad \tau_0^{\varphi}(X^*, e^*) = (-1)^{n'-n} \lambda [\overline{\tau_0^{\varphi}(X, e)}]^{\varepsilon(m)}.$$

We remark that if a and b are simplexes in $\widetilde{M}$ with $p(a) \subset p(b)$, then there is a unique covering transformation $\widetilde{M} \to \widetilde{M}$ taking a to a subset of b. We denote the element of the group $H_1(M)$ corresponding to this covering transformation by $g(a, b)$. We put

$$g = \prod_{\substack{1 \leq i, j \leq N \\ p(a_i) \subset p(a_j)}} g(a_i, a_j)^{\varepsilon(\dim a_i - \dim a_j)}.$$

The following statements hold:

(i) $(-1)^{n'-n}\lambda = (-1)^z$;

(ii) $\tau_0^\varphi(X^*, e^*) = \varphi(g)\tau_0^\varphi(X, e)$;

(iii) *if m is even, then* $v_1(M) = g$; *if m is odd, then* $v_1(M) \equiv g \pmod 2$.

Hence by (28) it follows that the statement of the theorem holds for $\tau = \tau_0^\varphi(X, e)$. Since $\tau^\varphi(M) = \{\pm\varphi(h)\tau_0^\varphi(X, e) \mid h \in H_1(M)\}$, the statement of the theorem also holds for arbitrary $\tau \in \tau^\varphi(M)$.

The proof of statement (i) consists in applying Poincaré duality (taking into account the equality $\chi(M) = 0$), and is omitted.

We prove (ii). Let X' be the first barycentric subdivision of the triangulation X of M, and let $\widetilde{X}'$ be the induced triangulation of $\widetilde{M}$. Let e' (respectively e'') be the base sequence of simplexes of the triangulation $\widetilde{X}'$ consisting of the simplexes lying in simplexes of the sequence e (respectively in cells of the sequence e^*). By Lemma 8, $\tau_0^\varphi(X, e) = \tau_0^\varphi(X', e')$ and $\tau_0^\varphi(X^*, e^*) = \tau_0^\varphi(X', e'')$. Each simplex a of the triangulation X' can be lifted uniquely to simplexes $a' \in e'$ and $a'' \in e''$. It can easily be verified that

$$\tau_0^\varphi(X', e'') = \tau_0^\varphi(X', e') \times \varphi\Big(\prod_{a \in X'} g(a', a'')^{-\varepsilon(\dim a)}\Big).$$

If $a = \langle \underline{b_0}, \underline{b_1}, \ldots, \underline{b_q} \rangle$, where $b_0 \subset b_1 \subset \ldots \subset b_q$ are simplexes of the triangulation X, then $a' \subset \widetilde{b}_q$ and $a'' \subset (\widetilde{b}_0)^*$, where $\widetilde{b}$ denotes the unique simplex of the sequence e situated over b. Hence $g(a', a'') = g(\widetilde{b}_0, \widetilde{b}_q)^{-1}$, and

$$\prod_{a \in X'} g(a', a'')^{-\varepsilon(\dim a)} = \prod_{b < c} g(\widetilde{b}, \widetilde{c})^{-f(b, c)},$$

where b and c are simplexes of X, and $f(b, c)$ is the Euler characteristic of the complex whose q-dimensional simplexes are all possible sequences $b = b_0 \subset b_1 \subset \ldots \subset b_q = c$ (with $q = 1, 2, \ldots$). If d is the simplex spanned by the vertices of c that do not appear in b, then the q-dimensional simplexes mentioned are in bijective correspondence with the $(q-1)$-dimensional simplexes of the first barycentric subdivision of d that contain the barycentre $\underline{d}$ (as a vertex). Hence

$$f(b, c) = -[\chi(d) - \chi(\partial d)] = -[1 - \chi(S^{\dim d - 1})] = -\varepsilon(\dim b - \dim c).$$

From this we have (ii).

We prove (iii). We fix a point x in $\widetilde{M}$ and for every simplex a of the triangulation $\widetilde{X}$ we fix a path l_a in $\widetilde{M}$, going from x to $\underline{a}$. If $p(a_i) \subset p(a_j)$, then the class $g(a_i, a_j) \in H_1(M)$ can be represented by the singular cycle $(p\circ l_{a_j}) - (p\circ l_{a_i}) - \langle \underline{a_i}, \underline{a_j}\rangle$. Hence g can be represented as the sum of the cycles (25) and the cycle $\sum k_a(p\circ l_a)$, where

$$k_a = \sum_{b<a} \varepsilon(\dim b - \dim a) - \sum_{a<b} \varepsilon(\dim b - \dim a).$$

The boundary of the cycle (25), as is easily verified, is equal to $-\sum_a k_a a$. So if m is even, then $k_a = 0$ for all a, and if m is odd, then $k_a \equiv 0 \pmod 2$ for all a. Hence we have (iii).

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Translated by L. Queen

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Received by the Editors 22 February 1985