A COCYCLE FOR THE SYMPLECTIC FIRST CHERN CLASS AND THE MASLOV INDEX

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1. Introduction

Let $G = Sp(n, \mathbf{R})$ be the symplectic group ($n \geqslant 1$) and G^δ be the same group with the discrete topology. The present paper is devoted to the study of the element u of the group $H^2(G^\delta; \mathbf{Z})$, which is determined up to sign in any of the following (equivalent) ways: a) u is the image under the canonical homomorphism $H^2(BG; \mathbf{Z}) \to H^2(BG^\delta; \mathbf{Z})$ of the generator of the group $H^2(BG; \mathbf{Z}) = \mathbf{Z}$; b) u is the cohomology class corresponding to an extension of the group G by its universal covering, provided with the natural group structure (recall that $\pi_1(G) = \mathbf{Z}$); c) u is the first Chern class of the complex vector bundle obtained from the real vector bundle over BG^δ ($= K(G^\delta, 1)$) associated with the universal principal G^δ-bundle and the action of G on $\mathbf{R}^{2n}$ introduced by the natural complex structure (see [2, 3]).

In this paper we consider the question of finding an explicit formula for a two-dimensional cocycle in the bar-resolution of the group G, representing u. This question is not new; see Sec. 2. The interest in it is due to firstly the interest in a broad scheme for constructing explicit cocycles which represent nontrivial cohomology classes of Lie groups and algebras, and secondly to the specific role which the class u and its reduction mod 2 play in the theory of representations of Lie groups and in the theory of symplectic and metaplectic structures.

Even for widely studied groups and their cohomology classes the construction of explicit cocycles representing these classes is not a mechanical matter and usually requires additional arguments. Here the role of such arguments was played by the following observation relating to the elementary theory of cobordism. Let V be an orientable closed smooth manifold of dimension 4m + 2 with $m \geqslant 0$. Let Ω be the group of orientation-preserving diffeomorphisms $V \to V$. The torus of the diffeomorphism $f: V \to V$ (the manifold $V \times [0, 1]/a \times 0 = f(a) \times 1$ for $a \in V$) is denoted by T(f). If $f, g \in \Omega$, then by N(f, g) we denote the result of gluing the lower bases of the cylinders $T(f) \times [0, 1]$ and $T(g) \times [0, 1]$ to the upper base of the cylinder $T(fg) \times [0, 1]$ according to the following rule: we identify $a \times t \times 0 \in T(f) \times 0$ with $a \times (t+1)/2 \times 1 \in T(fg) \times 1$ and $a \times t \times 0 \in T(g) \times 0$ with $a \times (t/2) \times 1 \in T(fg) \times 1$, where $a \in V$ and $t \in [0, 1]$. It is easy to verify that: (i) N(f, g) is a compact orientable (4m + 4)-dimensional manifold, whose boundary is equal to the disjoint union of T(f), T(g), and T(fg); (ii) if $h \in \Omega$, then the result of gluing N(f, g) with N(fg, h) along the common boundary component, the torus T(gf), is homeomorphic with the result of gluing N(g, h) with N(f, gh) along T(gh). In view of (ii) and the additivity of the signature, the function $\alpha: \Omega^2 \to \mathbf{Z}$ (associating the pair f, g with the signature of the suitably oriented manifold N(f, g)) satisfies the relation $\alpha(f, g) + \alpha(fg, h) = \alpha(g, h) + \alpha(f, gh)$, i.e., α is a two-dimensional cocycle (see [12]). It turns out that the construction made can be modeled algebraically. This leads to a cocycle $\varphi: G^2 \to \mathbf{Z}$ representing 4u (see Sec. 3). Adjusting $\varphi/4$ by a coboundary, we construct an integral cocycle representing u. Both these cocycles are invariant with respect to inner automorphisms of the group G.

Another result of the paper is that the cocycle φ considered here is closely connected with the Maslov indices of Lagrangian spaces (see Sec. 4).

2. History of the Question

Dupont and independently Guechardet and Wigner constructed the same real cocycle which represents the image of the class u under the natural homomorphism $H^2(G^\delta; \mathbf{Z}) \to H^2(G^\delta; \mathbf{R})$ (see [3, 5, 6]). An explicit but quite involved formula for this cocycle is given in [3, p. 152]. If n = 1, Guechardet and Wigner found a simple explicit formula for an integral cocycle which represents u (see [6, p. 289]). For any n a cocycle $G^2 \to \mathbf{Z}$, representing 4u was constructed

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by Weil in the course of constructing the Shale–Weil representation of the metaplectic group. Later Lion proved that this cocycle is defined by the rule $(f, g) \mapsto \tau(k, f(k), fg(k))$, where τ is the ternary Maslov index and k is a fixed Lagrangian space (see [10]; different k correspond to cocycles obtained from one another by inner automorphisms of the group G). The Weil–Lion cocycles can be expressed in terms of φ, see Sec. 4. The interconnections of the cocycles of Dupon and Guechardet–Wigner on the one hand, and the cocycles of Weil–Lion and φ, on the other, are not clear to the author.

3. Cocycle φ

Let H be a finite-dimensional real vector space; $B: H \times H \to \mathbf{R}$ be a nonsingular skew-symmetric form; G be the group of linear automorphisms of the space H, preserving B. The existence of the form B allows us to fix an isomorphism $\mathbf{Z} \to \pi_1(G)$, which eliminates the ambiguity in the definition b) given in Sec. 1 of the class u (the other isomorphism corresponds to −u). Namely, if $a, b \in H$ with $B(a, b) = 1$, then one fixes the isomorphism which carries 1 into the homotopy class of the loop $[0, 2\pi] \to G$ which assigns to the number t the homomorphism which is the identity on the B-annihilator of the plane $\mathbf{R}a + \mathbf{R}b$ and which carries a into $a \cos t - b \sin t$ and b into $a \sin t + b \cos t$ (i.e., rotating the plane $\mathbf{R}a + \mathbf{R}b$ by the angle t clockwise).

For $f, g \in G$ in the space $(f - 1)(H) \cap (g - 1)(H)$, where 1 denotes the identity operator, we define a binary real-valued form by

$$(a, b) \mapsto B((f - 1)^{-1}(a) + (g - 1)^{-1}(a) + a, b). \tag{1}$$

It is easy to verify that this is a well-defined symmetric bilinear form. In general, it is degenerate. We denote by φ the map $G^2 \to \mathbf{Z}$ which associates with the pair f, g the signature of the form (1) (factored by the annihilator).

THEOREM 1. The map φ is a cocycle and represents 4u.

It follows from Theorem 1 that the cocycle $\varphi/4$ with values in Q represents the image of u under the inclusion homomorphism $H^2(G^\delta; \mathbf{Z}) \to H^2(G^\delta; \mathbf{Q})$. We shall construct an integral cocycle which represents u. It is easy to verify that if $f \in G$, $f \neq 1$ and $x_1, \ldots, x_d$ is a basis of the space (f − 1) (H), then the determinant of the matrix $\{B((f - 1)^{-1}(x_i), x_j)\}$, $i, j = 1, \ldots, d$ is different from 0, and its sign is consequently independent of the choice of basis. We set $\varepsilon(f) = 1$, if this determinant is positive, and $\varepsilon(f) = -1$ if not. We set $\varepsilon(1) = 1$. We denote by ρ the coboundary $G^2 \to \mathbf{Z}$ of the cochain $f \to \dim(f - 1)(H) + \varepsilon(f) - 1$: $G \to \mathbf{Z}$.

THEOREM 2. The cocycle $(\varphi - \rho)/4$ takes integral values and represents u.

COROLLARY. For any $m \geqslant 2$ the cocycle $1/4(\varphi - \rho) \pmod{m}$ represents $u \pmod{m} \in H^2(G^\delta; \mathbf{Z}/m\mathbf{Z})$.

An interesting consequence of Theorem 1 is the existence of a primitive function for the cocycle φ. Namely, let $p: \tilde{G} \to G$ be the universal covering, where $\tilde{G}$ is provided with the natural group structure. Since $p^*(u) = 0$ and $\tilde{G} = [\tilde{G}, \tilde{G}]$, there exists a unique one-dimensional cochain $\Phi: \tilde{G} \to \mathbf{Z}$ whose coboundary is equal to the lift of φ to $\tilde{G}$; for any $F_1, F_2 \in \tilde{G}$

$$\varphi(p(F_1), p(F_2)) = \Phi(F_1) + \Phi(F_2) - \Phi(F_1F_2). \tag{2}$$

The function Φ has a number of remarkable properties: Φ gives rise to the Maslov index; Φ is a Borel function; Φ is invariant with respect to inner automorphisms of the group $\tilde{G}$. One can expect that Φ is connected with the generalized character of the Shale–Weil representation (see [8]).

The values of the cochains φ and Φ on elements of the unitary group can be calculated from the eigenvalues of the operators. We restrict ourselves to the formulation relating to Φ. In H we fix a complex structure compatible with B, i.e., a homomorphism $J: H \to H$, preserving B, such that $J^2 = -1$ and $B(J(a), a) > 0$ for any nonzero $a \in H$. We denote by U the subgroup of the group G consisting of automorphisms of the Hermitian form

$$(a, b) \to B(J(a), b) + iB(a, b): H^2 \to \mathbf{C}. \tag{3}$$

We denote by $\tilde{U}$ the subgroup of the group $U \times \mathbf{R}$ consisting of pairs (h, d) with $\det h = e^{id}$. The projection $\tilde{U} \to U$ is the universal covering and hence lifts to a monomorphism $\eta: \tilde{U} \to \tilde{G}$.

We define a function μ: $\mathbf{R} \to \mathbf{Z}$ by: if $m \in \mathbf{Z}$, then $\mu(m\pi) = 2m$ and $\mu(]m\pi, (m+1)\pi[) = 2m+1$.

THEOREM 3. If $(h, d) \in \tilde{U}$ and if $\theta_1, \ldots, \theta_n$ are real numbers such that $e^{i\theta_1}, \ldots, e^{i\theta_n}$ are eigenvalues of the operator h (taking multiplicities into account), then

$$\Phi(\eta(h, d)) = \frac{2}{\pi}\Big(\sum_{r=1}^{n} \theta_r - d\Big) - 2\sum_{r=1}^{n} \mu(\theta_r/2). \tag{4}$$

4. Maslov Indices and the Cochains φ and Φ

Let Λ be the manifold of Lagrangian subspaces of the space H (a subspace is Lagrangian if it coincides with its B-annihilator). Let q: $\tilde{\Lambda} \to \Lambda$ be the universal covering ($\pi_1(\Lambda) = \mathbf{Z}$). In symplectic geometry an important role is played by the maps τ: $\Lambda \times \Lambda \times \Lambda \to \mathbf{Z}$ and m: $\tilde{\Lambda} \times \tilde{\Lambda} \to \mathbf{Z}$, called respectively the ternary and binary Maslov indices and defined as follows. If $k_1, k_2, k_3 \in \Lambda$, then $\tau(k_1, k_2, k_3)$ is the signature of the symmetric bilinear form A in $(k_1 + k_2) \cap k_3$ defined by: if $a, b \in (k_1 + k_2) \cap k_3$ and x is an element of k_2 such that $a - x \in k_1$, then $A(a, b) = B(x, b)$. If k_1 and k_2 are transverse, this definition of the index τ coincides with the usual one (up to a linear transformation; see [7, 9, 10]).

In general it is equivalent with the definition of Kashiwara (see [10]).

The binary Maslov index is most easily defined axiomatically.

THEOREM 4. There exists a unique function m: $\tilde{\Lambda}^2 \to \mathbf{Z}$ locally constant on the set of pairs $K_1, K_2 \in \tilde{\Lambda}$, such that $q(K_1)$ is transverse to $q(K_2)$, which for any $K_1, K_2, K_3 \in \tilde{\Lambda}$ satisfies the formula

$$\tau(q(K_1), q(K_2), q(K_3)) = m(K_1, K_2) - m(K_1, K_3) + m(K_2, K_3). \tag{5}$$

This theorem was proved in somewhat restricted form by Leray [9]: he considered the Maslov index only for transverse pairs.

To describe the connection between the Maslov indices and the cochains φ and Φ, we fix in H a complex structure compatible with B (see Sec. 3) and a basis $a_1, \ldots, a_n$ for the space H over $\mathbf{C}$, which is orthonormal with respect to the form (3). According to [10], for any $k \in \Lambda$ there exists a unitary operator f: $H \to H$ such that $k = f(\mathbf{R}a_1 + \ldots + \mathbf{R}a_n)$. Here if g: $H \to H$ is the linear operator over $\mathbf{C}$, defined with respect to $a_1, \ldots, a_n$ by the matrix obtained by transposing the matrix of the operator f with respect to the same basis, then the composition $f \circ g$ is independent of the choice of f, and the rule $k \mapsto f \circ g$ defines an imbedding of $\Lambda \to G$. We denote it by γ. (The image of γ consists of unitary operators, defined by symmetric matrices with respect to the basis $a_1, \ldots, a_n$.) Since γ induces an isomorphism $\pi_1(\Lambda) \to \pi_1(G)$ (see [1]), the composition $\gamma \circ q$: $\tilde{\Lambda} \to G$ lifts to an imbedding $\tilde{\Lambda} \to \tilde{G}$. We denote such a lift by Γ. We denote by ψ and Ψ, respectively, the maps $(f_1, f_2, f_3) \mapsto \varphi(f_1^{-1}f_2, f_2^{-1}f_3)$: $G^3 \to \mathbf{Z}$ and $(F_1, F_2) \mapsto \Phi(F_1^{-1}F_2)$: $\tilde{G}^2 \to \mathbf{Z}$. According to [12], ψ is the "two-dimensional cocycle in homogeneous generators of the bar-resolution" of the group G," corresponding to φ and Ψ is a primitive of it.

THEOREM 5. For any $k_1, k_2, k_3 \in \Lambda$,

$$\tau(k_1, k_2, k_3) = \frac{1}{2}\psi(\gamma(k_1), \gamma(k_2), \gamma(k_3)). \tag{6}$$

For any $K_1, K_2 \in \tilde{\Lambda}$

$$m(K_1, K_2) = \frac{1}{2}\Psi(\Gamma(K_1), \Gamma(K_2)). \tag{7}$$

Thus, the composition $\psi \circ (\gamma \times \gamma \times \gamma)$: $\Lambda^3 \to \mathbf{Z}$ is independent of the choice of complex structure and basis in H, used in constructing the imbedding γ, and is equal to 2τ. The composition $\Psi \circ (\Gamma \times \Gamma) : \tilde{\Lambda}^2 \to \mathbf{Z}$ is also independent of the choice of Γ in the class of imbeddings $\tilde{\Lambda} \to \tilde{G}$ constructed and is equal to 2m. In other words, up to multiplication by a constant the cochains ψ and Ψ are respectively extensions of the ternary and binary Maslov indices to the symplectic group and its universal covering. This point of view allows one to describe the relations between the Maslov indices in terms of homological algebra. We shall say that Leray's formula (5) is a specialization of the assertion "the coboundary of the cochain Ψ is equal to the lift of ψ to $\tilde{G}$." The specialization of the assertion "ψ is a cocycle" is the familiar formula

$$\tau(k_1, k_2, k_3) - \tau(k_1, k_2, k_4) + \tau(k_1, k_3, k_4) - \tau(k_2, k_3, k_4) = 0 \quad (8)$$

for any $k_1, k_2, k_3, k_4 \in \Lambda$.

COROLLARY OF THEOREMS 3 and 5 (Souriau's Formula in Lion–Vergne Form, see [10, 16]). Let $K_1, K_2 \in \tilde{\Lambda}$; $\Gamma(K_r) = \eta(h_r, d_r)$ with $r = 1, 2$; $\theta_1, \ldots, \theta_n$ be real numbers such that $e^{i\theta_1}, \ldots, e^{i\theta_n}$ are eigenvalues of the operator $h_1^{-1}h_2$ and $\sum_{r=1}^{n} \theta_r = d_2 - d_1$. Then $m(K_1, K_2) = -\sum_{r=1}^{n} \mu(\theta_r/2)$.

5. Remarks

a) Since the Weil–Lion cocycles (see Sec. 2) represent 4u and the values of these cocycles do not exceed n = (dim H)/2 in modulus, the real class of u can be represented by cocycles whose values do not exceed n/4 in modulus. Hence the norm of the class u in the sense of the theory of bounded cohomology is not greater than n/4. The estimate $\|u\| < (2^n - 1)/2$ was noted previously (see [4]). It seems likely that $\|u\| = n/4$ (this is true for n = 1; see [14]).

b) In [13] the Maslov index of a triple of positive Lagrangian subspaces of a complexified symplectic vector space is defined. It satisfies (8) and for real Lagrangian spaces is equal to τ. It would be interesting to give it a homological interpretation.

c) We note a definite parallelism between the present paper and Novikov [15], where, just as here, the considerations of relations to cobordism theory led to a purely algebraic construction which turned out to be connected with the Maslov indices.

6. Outline of the Proofs of Theorems 1-5

(1). The fact that φ is a cocycle, i.e., that for any $f, g, h \in G$

$$\varphi(f, g) + \varphi(fg, h) = \varphi(g, h) + \varphi(f, gh) \quad (9)$$

is proved by imitating the proof in linear algebra of the additivity theorem for the signature for the special manifolds considered in Sec. 1. It is easy to see that φ is a Borel cocycle. By results of [11], the image of the Borel cohomology group $H_B^2(G; \mathbf{Z})$ under the natural homomorphism $H_B^2(G; \mathbf{Z}) \to H^2(G^\delta; \mathbf{Z})$ is generated by the class u. Hence φ represents mu, where $m \in \mathbf{Z}$. Calculation of the restrictions of φ and u to S^1 shows that m = 4.

(2). Since $H^2(G^\delta; \mathbf{Z})$ has no torsion, it suffices to show that $\varphi(f, g) \equiv \rho(f, g) \pmod 4$ for any f, g. If Ker (g − 1) = Ker (fg − 1) = 0, then the congruence needed follows from the fact that the determinant of the form (1) is equal to $\varepsilon(f)\varepsilon(g)\varepsilon(fg)$ (the latter is proved with the help of the identity $[(f-1)^{-1} + (g-1)^{-1} + 1](f-1) = (g-1)^{-1}(gf-1)$). From this, using (9), the general case follows.

(3). It follows from (9) that in a neighborhood of $1 \in \tilde{G}$ the map $F \mapsto \varphi(-1, p(F))$ satisfies (2) and hence coincides with Φ. Now (4) is verified in a neighborhood of 1 by direct calculation. It follows from (2) that if (4) is valid for (h, d), then (4) is also valid for $(h^2, 2d)$.

(4 and 5). The proof of Theorems 4 and 5 reduces to the proof of (6): the uniqueness in Theorem 4 is obvious, and the existence and (7) follow from the fact that by (2) and (6) the map $(K_1, K_2) \mapsto \Psi(\Gamma(K_1), \Gamma(K_2))/2$ satisfies the hypotheses of Theorem 4. It follows from Leray's results [9] that the number $\psi(\gamma(k_1), \gamma(k_2), \gamma(k_3))$ does not change under continuous deformation of the complex structure and basis used in defining the imbedding γ. Since the space of these structures and bases is path connected, it suffices to prove (6) for one γ. For suitable choice of the complex structure and basis it turns out that up to factorization by the annihilator, the bilinear form used in the definition of the right side of (6) splits into the direct sum of two forms which are isomorphic with one another and to the form used in the definition of the left side.

The author intends to publish detailed proofs later.

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CLASSIFICATION OF NONSINGULAR SURFACES OF DEGREE 4 IN $\mathbf{R}P^3$ WITH RESPECT TO RIGID ISOTOPIES

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The fundamental objects of the topology of real algebraic varieties are nonsingular curves in $\mathbf{R}P^2$ and nonsingular surfaces in $\mathbf{R}P^3$. There are several classical versions of their division into classes. In all versions the degree of the curve or respectively of the surface is assumed to be given. In the first version surfaces are considered up to a homeomorphism between their sets of real points. In the second up to arbitrary topological isotopies in $\mathbf{R}P^3$ of the set of real points of the surface; such isotopies are called real, following Rokhlin [1]. Finally, in the third version, surfaces of degree m are considered up to real isotopies composed of surfaces of degree m; following Rokhlin [1], such isotopies are called rigid. In the case of curves these versions are analogous — it is only necessary to replace surfaces by curves and $\mathbf{R}P^3$ by $\mathbf{R}P^2$.

The classification with respect to homeomorphisms in the case of nonsingular curves in $\mathbf{R}P^2$ of arbitrary degree is given by Harnack's theorem (see, e.g., [2]), while in the case of nonsingular surfaces in $\mathbf{R}P^3$ it has been finished up to degree 4 (see [2, 3]), and starting with degree 5 it remains open. The classification with respect to real isotopies is completed for nonsingular curves of degree $\leqslant 7$ in $\mathbf{R}P^2$ (see [2, 4]) and for nonsingular surfaces of degree $\leqslant 4$ in $\mathbf{R}P^3$ (see [2, 5]). The classification with respect to rigid iso-

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