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Reciprocity for Gauss sums on finite abelian groups

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1. *Introduction*

1·0. *Outline*

A classical reciprocity formula for Gauss sums due to Cauchy, Dirichlet, and Kronecker states that

$$|b|^{-1/2} \sum_{x \in \mathbb{Z}/b\mathbb{Z}} e^{\frac{\pi i a}{b}(x+w)^2} = e^{\frac{\pi i}{4}\operatorname{sign}(ab)} |a|^{-1/2} \sum_{x \in \mathbb{Z}/a\mathbb{Z}} e^{-\frac{\pi i b}{a}x^2 - 2\pi i w x} \qquad (1)$$

where a, b are nonzero integers and $w \in \mathbb{Q}$ such that $ab + 2aw \in 2\mathbb{Z}$. For a detailed proof and historical background, see [**4**, chapter IX]. Various versions of formula (1) have been extensively studied in connection with transformation properties of theta functions. A version of (1) for multivariate Gauss sums was first obtained by A. Krazer [**10**] in 1912, see also [**2**, **5**, **11**, **14**]. Krazer's formula generalizes the case $w = 0$ of (1) via replacing one of the numbers a, b by an integer quadratic form of several variables.

Recently, Florian Deloup [**6**] found a new and most beautiful reciprocity formula for multivariate Gauss sums. His formula is a far reaching generalization of Krazer's result. Roughly speaking, Deloup replaces both numbers a, b with quadratic forms. Deloup involves Wu classes of quadratic forms which allows him to remove the evenness condition appearing in Krazer's formulation. However, Deloup's formula covers only the cases $w = 0$ and $w = b/2$ of (1).

In this paper we establish a more general reciprocity for Gauss sums including the Krazer and Deloup formulas and formula (1) in its full generality. Our reciprocity law involves two quadratic forms and a so-called rational Wu class, as defined below.

The original proofs of Cauchy, Dirichlet, Kronecker and Krazer are analytical and involve a study of a limit of a transformation formula for theta-functions. Deloup's proof goes by a reduction to the case $w = b/2$ of (1) based on a careful study of Witt groups of quadratic forms. Our proof is more direct and uses only (a generalization of) the van der Blij computation of Gauss sums via signatures of integer quadratic forms. In particular, our argument provides a new proof of (1).

1·1. *Quadratic forms on finite abelian groups and Gauss sums*

By a quadratic form on a finite abelian group G, we mean a function $q: G \to \mathbb{Q}/\mathbb{Z}$ such that the associated pairing $b^q: G \times G \to \mathbb{Q}/\mathbb{Z}$ defined by $b^q(x, y) = q(x + y) - q(x) - q(y)$, for $x, y \in G$, is bilinear. In generalization of the standard definition, we do not require q to be homogeneous, i.e., we do not require that $q(nx) = n^2 q(x)$, for $n \in \mathbb{Z}, x \in G$. For instance, a sum of a linear homomorphism $G \to \mathbb{Q}/\mathbb{Z}$ with a quadratic form $G \to \mathbb{Q}/\mathbb{Z}$ is considered to be a quadratic form on G. Note that the 'free term' $q(0)$ of a quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$ is always zero:

$$q(0) = q(0) + q(0) - q(0) = -b^q(0,0) = 0.$$

The Gauss sum (or the Gaussian sum) associated with a quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$ is defined by

$$\Gamma(G,q) = |G|^{-1/2} \sum_{x \in G} e^{2\pi i q(x)} \in \mathbb{C}$$

(cf. [**13**, chapter 5]). The factor $|G|^{-1/2}$ is convenient for normalization purposes. For instance, if $q = 0$ then $\Gamma(G,q) = |G|^{1/2}$; if G is the trivial group then $\Gamma(G,q) = 1$.

The standard properties of the quadratic Gauss sums generalize to our setting. It is obvious that the invariant $\Gamma(G,q)$ is multiplicative with respect to the direct sum of quadratic forms. The absolute value of $\Gamma(G,q)$ can be explicitly computed (see Lemma 1 below). For example, if b^q is non-degenerate then $|\Gamma(G,q)| = 1$. Note also that if $\Gamma(G,q) \neq 0$ then $\Gamma(G,q)/|\Gamma(G,q)|$ is a root of unity.

1·2. *A computation of the Gauss sums*

Van der Blij [**1**] computed the Gauss sums in terms of signatures of symmetric bilinear forms on lattices. This generalizes Gauss' original evaluation of the Gauss sums. We give here a slightly generalized version of the van der Blij formula. By a lattice we shall mean a free abelian group of finite rank.

We first introduce the notion of a rational Wu class. Let $f: L \times L \to \mathbb{Z}$ be a symmetric bilinear form on a lattice L. Consider the vector space $\mathbb{Q} \otimes L = \mathbb{Q} \otimes_{\mathbb{Z}} L$ and denote by $f_{\mathbb{Q}}$ the $\mathbb{Q}$-linear extension $(\mathbb{Q} \otimes L) \times (\mathbb{Q} \otimes L) \to \mathbb{Q}$ of f. An element $v \in \mathbb{Q} \otimes L$ is said to be a rational Wu class for f if $f_{\mathbb{Q}}(v,x) - f(x,x) \in 2\mathbb{Z}$ for any $x \in L$. This definition generalizes the usual one where $v \in L$. Note that for any rational Wu class $v \in \mathbb{Q} \otimes L$ and any $x \in L$, we have $f_{\mathbb{Q}}(v,x) = f_{\mathbb{Q}}(x,v) \in \mathbb{Z}$.

A standard construction produces from f and a rational Wu class $v \in \mathbb{Q} \otimes L$ a quadratic form on a finite abelian group (cf., for instance, [**7**]). Let $\hat{f}$ denote the homomorphism $L \to L^* = \mathrm{Hom}\,(L,\mathbb{Z})$ adjoint to f. Set

$$G_f = \mathrm{Tors}\,(\mathrm{Coker}\hat{f}) = \mathrm{Tors}\,(L^*/\hat{f}(L))$$

and define a mapping $q_{f,v}: G_f \to \mathbb{Q}/\mathbb{Z}$ by

$$q_{f,v}(x \bmod \hat{f}(L)) = \frac{1}{2}\left(\frac{x(x')}{n} - \tilde{x}(v)\right) \bmod \mathbb{Z}$$

where: $x \in L^*$; n is a nonzero integer such that $nx \in \hat{f}(L)$; x' is an arbitrary element of $\hat{f}^{-1}(nx)$; and $\tilde{x}$ is the $\mathbb{Q}$-linear extension $\mathbb{Q} \otimes L \to \mathbb{Q}$ of $x: L \to \mathbb{Z}$. An easy verification shows that $q_{f,v}$ is a well defined quadratic form on the finite abelian group G_f. It is instructive to note that this form is homogeneous if and only if $v \in L$. The bilinear form $b_f: G_f \otimes G_f \to \mathbb{Q}/\mathbb{Z}$ associated with $q_{f,v}$ does not depend on v; it is given by

$$b_f(x \bmod \hat{f}(L), y \bmod \hat{f}(L)) = \frac{y(x')}{n} \bmod \mathbb{Z},$$

where $x, y \in L^*$ and n, x' are as above. The form b_f is nondegenerate in the sense that its annihilator is trivial. Indeed, if $x \bmod \hat{f}(L) \in G_f$ lies in the annihilator of b_f then $y(x') \in n\mathbb{Z}$, for any $y \in L^*$. Hence x' is divisible by n in L and $x = \hat{f}(x'/n) \in \hat{f}(L)$.

The Gauss sum corresponding to $q_{f,v}$ can be computed in terms of the signature $\sigma(f) \in \mathbb{Z}$ of f:

$$\Gamma(G_f, q_{f,v}) = e^{\frac{\pi i}{4}(\sigma(f) - f_{\mathbb{Q}}(v,v))}. \tag{2}$$

In the case $v \in L$ this formula is due to van der Blij [**1**]; his argument applies to the rational Wu classes word for word.

1·3. *The reciprocity formula*

Now we can formulate our reciprocity law for the Gauss sums. Consider symmetric bilinear forms $f: L \times L \to \mathbb{Z}$ and $g: M \times M \to \mathbb{Z}$ on lattices L, M. Consider the lattice $K = L \otimes M$ and the symmetric bilinear form $e: K \times K \to \mathbb{Z}$ defined by $e(x \otimes y, x' \otimes y') = f(x, x')\, g(y, y')$, for $x, x' \in L, y, y' \in M$. Clearly, the adjoint homomorphism $\hat{e}: K \to K^* = L^* \otimes M^*$ is equal to $\hat{f} \otimes \hat{g}$. Consider the finite abelian groups

$$G_f = \mathrm{Tors}(\mathrm{Coker}\,\hat{f}), \quad G_g = \mathrm{Tors}(\mathrm{Coker}\,\hat{g}), \quad G_e = \mathrm{Tors}(\mathrm{Coker}\,\hat{e}).$$

There is a homomorphism $\alpha_{f,g}: G_f \otimes M \to G_e$ defined by

$$\alpha_{f,g}(x \bmod \hat{f}(L) \otimes y) = x \otimes \hat{g}(y) \bmod \hat{e}(K),$$

where $x \in L^*, y \in M$. Similarly, there is a homomorphism $\beta_{f,g}: L \otimes G_g \to G_e$ defined by

$$\beta_{f,g}(z \otimes t \bmod \hat{g}(M)) = \hat{f}(z) \otimes t \bmod \hat{e}(K),$$

where $z \in L, t \in M^*$.

Let $u \in \mathbb{Q} \otimes K$ be a rational Wu class for e. The quadratic form $q_{e,u}: G_e \to \mathbb{Q}/\mathbb{Z}$ induces quadratic forms $q_{e,u} \circ \alpha_{f,g}$ and $q_{e,u} \circ \beta_{f,g}$ on $G_f \otimes M$ and $L \otimes G_g$, respectively. Here is the main reciprocity formula obtained in this paper:

$$\frac{\Gamma(G_f \otimes M, q_{e,u} \circ \alpha_{f,g})}{|G_f|^{\mathrm{corank}(g)/2}} = e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - e_{\mathbb{Q}}(u,u))} \, \frac{\overline{\Gamma(L \otimes G_g, q_{e,u} \circ \beta_{f,g})}}{|G_g|^{\mathrm{corank}(f)/2}}, \tag{3}$$

where corank denotes the rank of the annihilator of a symmetric bilinear form and the overline on the right-hand side denotes the complex conjugation. Formula (3) will be proven in Section 3 using the lemmas established in Section 2.

Note the symmetry of (3) in f and g: exchanging f and g and replacing u with its image under the permutation $\mathbb{Q} \otimes L \otimes M \to \mathbb{Q} \otimes M \otimes L$ we get an equivalent formula.

Under further assumptions on f and g formula (3) simplifies. If f and g are non-degenerate then $\mathrm{corank}(f) = \mathrm{corank}(g) = 0$ and we obtain

$$\Gamma(G_f \otimes M, q_{e,u} \circ \alpha_{f,g}) = e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - e_{\mathbb{Q}}(u,u))} \, \overline{\Gamma(L \otimes G_g, q_{e,u} \circ \beta_{f,g})}.$$

If at least one of the forms f, g is even then e is even and we can take $u = 0$. This yields

$$\frac{\Gamma(G_f \otimes M, q_{e,0} \circ \alpha_{f,g})}{|G_f|^{\mathrm{corank}(g)/2}} = e^{\frac{\pi i}{4}\sigma(f)\sigma(g)} \, \frac{\overline{\Gamma(L \otimes G_g, q_{e,0} \circ \beta_{f,g})}}{|G_g|^{\mathrm{corank}(f)/2}}.$$

If g is unimodular then $G_g = 0$, $\mathrm{corank}(g) = 0$ and formula (3) yields

$$\Gamma(G_f \otimes M, q_{e,u} \circ \alpha_{f,g}) = e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - e_{\mathbb{Q}}(u,u))}.$$

To end this subsection we give an explicit formula for the quadratic form $q_{e,u} \circ \alpha_{f,g}: G_f \otimes M \to \mathbb{Q}/\mathbb{Z}$:

$$(q_{e,u} \circ \alpha_{f,g})(x \bmod \hat{f}(L) \otimes y) = \frac{1}{2}\left(\frac{x(x')}{n} g(y,y) - (x \otimes \hat{g}(y))(u)\right) \bmod \mathbb{Z},$$

where $x \in L^*, y \in M$, n is a nonzero integer such that $nx \in \hat{f}(L)$ and x' is an arbitrary element of $\hat{f}^{-1}(nx)$. The bilinear form on $G_f \otimes M$ associated with $q_{e,u} \circ \alpha_{f,g}$ is nothing but the tensor product of $b_f: G_f \times G_f \to \mathbb{Q}/\mathbb{Z}$ and $g: M \times M \to \mathbb{Z}$. Similar results hold for $q_{e,u} \circ \beta_{f,g}$.

1·4. *Special cases and corollaries*

We formulate a few important cases of formula (3).

1·4·1. *The Cauchy–Dirichlet–Kronecker formula*

We shall deduce formula (1) from (3). To this end, take $L = M = \mathbb{Z}$. Let $f : L \times L \to \mathbb{Z}$ and $g : M \times M \to \mathbb{Z}$ be the bilinear forms defined by $f(x,y) = axy, g(x,y) = bxy$, for $x, y \in \mathbb{Z}$. Clearly, $K = L \otimes M = \mathbb{Z}$ and the bilinear form $e = f \otimes g$ on K is given by $e(x,y) = abxy$. For any $r \in \mathbb{Z}$, the number $u_r = 1 - 2r/ab \in \mathbb{Q}$ is a rational Wu class for e. The form $q_{e,u} \circ \alpha_{f,g}$ on $G_f \otimes M = \mathbb{Z}/a\mathbb{Z}$ sends $x \bmod a$ with $x \in \mathbb{Z}$ to $\frac{1}{2}((b/a)x^2 + ((2r/a) - b)x)$. Similarly, the form $q_{e,u} \circ \beta_{f,g}$ on $L \otimes G_g = \mathbb{Z}/b\mathbb{Z}$ sends $x \bmod b$ with $x \in \mathbb{Z}$ to $\frac{1}{2}((a/b)x^2 + ((2r/b) - a)x)$. Substituting these values in (3) and applying the complex conjugation we obtain (1) with $w = r/a - b/2$.

1·4·2. *The generalized van der Blij formula*

The generalized van der Blij formula (2) is a special case of (3). To see this, set $M = \mathbb{Z}$ and take g to be the bilinear form $g: M \times M \to \mathbb{Z}$ defined by $g(1,1) = 1$. Clearly, $G_g = 0$ so that the Gauss sum on the right hand side of (3) equals 1. It is clear that $G_f \otimes M = G_f$ and $q_{e,u} \circ \alpha_{f,g} = q_{f,u}$. Therefore in this case formula (3) yields (2).

1·4·3. *The Deloup formula*

We state here a reciprocity formula due to Deloup [**6**] and deduce it from (3). We need the following construction. Given a quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$ on a finite abelian group G and a symmetric bilinear form $g: M \times M \to \mathbb{Z}$ on a lattice M there is a unique quadratic form $q \otimes g: G \otimes M \to \mathbb{Q}/\mathbb{Z}$ such that $(q \otimes g)(x \otimes y) = q(x)\, g(y,y)$ for any $x \in G, y \in M$ (see [**8**, **12**]). The associated bilinear form on $G \otimes M$ is just the tensor product of $b^q: G \times G \to \mathbb{Q}/\mathbb{Z}$ and g. Similarly, we can consider the quadratic form $g \otimes q: M \otimes G \to \mathbb{Q}/\mathbb{Z}$, it is isomorphic to $q \otimes g$ via the permutation $M \otimes G \approx G \otimes M$. Hence $\Gamma(G \otimes M, q \otimes g) = \Gamma(M \otimes G, g \otimes q)$.

Let $f: L \times L \to \mathbb{Z}$ and $g: M \times M \to \mathbb{Z}$ be symmetric bilinear forms on lattices L, M. Let $v \in L$ and $w \in M$ be Wu classes for f and g, respectively. Here is a slightly modified version of the Deloup reciprocity formula:

$$\frac{\Gamma(G_f \otimes M, q_{f,v} \otimes g)}{|G_f|^{\operatorname{corank}(g)/2}} = e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - f(v,v)\, g(w,w))}\, \frac{\overline{\Gamma(G_g \otimes L, q_{g,w} \otimes f)}}{|G_g|^{\operatorname{corank}(f)/2}}. \tag{4}$$

Formula (4) is obtained from (3) by setting $u = v \otimes w$. Indeed, it is easy to deduce from the inclusions $v \in L, w \in M$ that $q_{e,u} \circ \alpha_{f,g} = q_{f,v} \otimes g$ and $q_{e,u} \circ \beta_{f,g} = f \otimes q_{g,w}$.

Note a few special cases of (4). If g is even and $w = 0$ then we have

$$\frac{\Gamma(G_f \otimes M, q_{f,v} \otimes g)}{|G_f|^{\mathrm{corank}(g)/2}} = e^{\frac{\pi i}{4}\sigma(f)\sigma(g)} \frac{\overline{\Gamma(G_g \otimes L, q_{g,0} \otimes f)}}{|G_g|^{\mathrm{corank}(f)/2}}. \tag{5}$$

Observe that the quadratic form $q_{f,v} \otimes g = q_{e,0} \circ \alpha_{f,g}$ does not depend on the choice of v.

If g is unimodular then $g(w,w) = \sigma(g) \bmod 8$ (this well known fact follows from (2)) and (4) yields

$$\Gamma(G_f \otimes M, q_{f,v} \otimes g) = e^{\frac{\pi i}{4}(\sigma(f) - f(v,v))\sigma(g)} = (\Gamma(G_f, q_{f,v}))^{\sigma(g)}.$$

If g is both unimodular and even then $\sigma(g) \in 8\mathbb{Z}$ and we obtain $\Gamma(G_f \otimes M, q_{f,v} \otimes g) = 1$.

1·4·4. *The Krazer formula*

We formulate the Krazer formula [**10**] following H. Braun [**2**] and deduce it from (5).

Let A be a nondegenerate symmetric integer $(m \times m)$-matrix and let a, b be positive integers such that either ab is even or all the diagonal entries of A are even. Set $s = |\det(A)|$ and denote by $\sigma(A)$ the signature of A. Then

$$b^{-m/2} \sum_{x \in (\mathbb{Z}/b\mathbb{Z})^m} e^{\frac{\pi i a}{b}\, x^t A x} = e^{\frac{\pi i \sigma(A)}{4}} a^{-m/2} s^{1/2-m} \sum_{y \in (\mathbb{Z}/sa\mathbb{Z})^m} e^{-\frac{\pi i b}{a}\, y^t A^{-1} y}. \tag{6}$$

It is easy to see that

$$\sum_{y \in (\mathbb{Z}/sa\mathbb{Z})^m} e^{-\frac{\pi i b}{a}\, y^t A^{-1} y} = s^{m-1} \sum_{y \in \mathbb{Z}^m/aA\mathbb{Z}^m} e^{-\frac{\pi i b}{a}\, y^t A^{-1} y}.$$

Therefore setting $A' = aA$ we can rewrite (6) in the following equivalent way:

$$b^{-m/2} \sum_{x \in (\mathbb{Z}/b\mathbb{Z})^m} e^{\frac{\pi i}{b}\, x^t A' x} = \frac{e^{\frac{\pi i \sigma(A')}{4}}}{|\det(A')|^{1/2}} \sum_{y \in \mathbb{Z}^m/A'\mathbb{Z}^m} e^{-\pi i b\, y^t (A')^{-1} y}.$$

This formula is obtained from (5) by the following substitution. If b is even then we take f to be the bilinear form $\mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$ defined by A' and take g to be the form $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $g(1,1) = b$. If b is odd then we make the opposite choice.

In her study of genera of integer quadratic forms H. Braun [**2**] used a special case of (6) where b is divisible by $2as$. In this case the sum on the right-hand side is equal to the number of summands and (6) yields

$$b^{-m/2} \sum_{x \in (\mathbb{Z}/b\mathbb{Z})^m} e^{\frac{\pi i a}{b}\, x^t A x} = e^{\frac{\pi i \sigma(A)}{4}} a^{m/2} s^{1/2}.$$

1·5. *Remarks*

1. The Gauss sum $\Gamma(G, q)$ is often considered in the case where G is a direct sum of a finite number of copies of $\mathbb{Z}/2\mathbb{Z}$. In this case any quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$ takes values in $\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ (this follows from the bilinearity of the pairing $b^q: G \times G \to \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ and the identity $2q(x) = 2q(x) - q(2x) = -b^q(x,x)$). If the pairing b^q is nondegenerate and $q(G) \subset \mathbb{Z}/2\mathbb{Z}$ then $\Gamma(G, q)$ is the classical Arf invariant of q. It is equal to 1 if q takes value 0 more often than 1 and is equal to -1 otherwise. This allows us to apply the formulas stated above to the Arf invariant.

2. In order to apply formula (2) to a quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$ on a finite

abelian group G, one should present q as $q_{f,v}$ with f, v as above. This is possible if and only if the associated bilinear form $b^q: G \times G \to \mathbb{Q}/\mathbb{Z}$ is nondegenerate. The necessity of this condition follows from the discussion in Section 1.2. Conversely, if b^q is nondegenerate then it can be realized as the form b_f associated with an even symmetric form $f: L \times L \to \mathbb{Z}$ on a lattice L (see [**16**]). When $v \in \mathbb{Q} \otimes L$ varies over the rational Wu classes for f, the form $q_{f,v}: G = G_f \to \mathbb{Q}/\mathbb{Z}$ runs over all quadratic forms on G with associated bilinear form $b_f = b^q$. Therefore $q = q_{f,v}$, for a certain v.

3. The Gauss sums appear systematically in the topology of manifolds, see [**3**, **9**, **15**]. This often encourages the algebraic study of Gauss sums. In particular, the Deloup reciprocity formula discussed above has been inspired by his study of quantum-like invariants of 3-manifolds. The proof of formula (3) given below was suggested by the following topological construction. One can realize the bilinear form $f: L \times L \to \mathbb{Z}$ as the intersection form $H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$ of a compact oriented 4-dimensional manifold (with boundary) X. Moreover, one can assume that X is simply connected or at least $H_1(X;\mathbb{Z}) = 0$. Clearly, ∂X is a closed orientable 3-dimensional manifold. It is easy to see from the homological sequence of the pair $(X, \partial X)$ that $G_f = \operatorname{Tors} H_1(\partial X;\mathbb{Z})$. The bilinear form $b_f: G_f \times G_f \to \mathbb{Q}/\mathbb{Z}$ can be interpreted as the homological linking form of the 3-manifold ∂X (provided with the orientation opposite to the one induced from X). Similarly, the bilinear form $g: M \times M \to \mathbb{Z}$ can be realized as the intersection form of a compact oriented 4-manifold Y with $H_1(Y;\mathbb{Z}) = 0$. Then $b_g: G_g \times G_g \to \mathbb{Q}/\mathbb{Z}$ is the linking form of the 3-manifold ∂Y. It is clear that $X \times Y$ is a compact oriented 8-manifold. The form $e = f \otimes g$ appears as the homological intersection form on $H_4(X \times Y;\mathbb{Z}) = L \otimes M$. As above, $G_e = \operatorname{Tors} H_3(\partial(X \times Y);\mathbb{Z})$ and the bilinear form $b_e: G_e \times G_e \to \mathbb{Q}/\mathbb{Z}$ is the linking form of the 7-manifold $\partial(X \times Y)$. The decomposition $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$ induces the embeddings

$$\alpha_{f,g}: G_f \otimes M = \operatorname{Tors} H_1(\partial X;\mathbb{Z}) \otimes H_2(Y;\mathbb{Z}) \to \operatorname{Tors} H_3(\partial(X \times Y);\mathbb{Z}) = G_e$$

and

$$\beta_{f,g}: L \otimes G_g = H_2(X;\mathbb{Z}) \otimes \operatorname{Tors} H_1(\partial Y;\mathbb{Z}) \to \operatorname{Tors} H_3(\partial(X \times Y);\mathbb{Z}) = G_e.$$

A further analysis of this geometric situation suggested a proof of (3).

2. *Lemmas*

In the following lemmas $q: G \to \mathbb{Q}/\mathbb{Z}$ is an arbitrary quadratic form on a finite abelian group G. The symbol b denotes the associated symmetric bilinear form $b^q: G \times G \to \mathbb{Q}/\mathbb{Z}$. For a subgroup H of G set $H^\perp = \{x \in G \mid b(x, H) = 0\}$.

LEMMA 1. *Let B be the kernel of the homomorphism $G \to \operatorname{Hom}(G, \mathbb{Q}/\mathbb{Z})$ adjoint to the pairing b. If $q(B) \neq 0$, then $\Gamma(G, q) = 0$. If $q(B) = 0$, then $|\Gamma(G, q)| = |B|^{1/2}$.*

Proof. We have

$$|\Gamma(G,q)|^2 = \frac{1}{|G|} \Big| \sum_{x \in G} e^{2\pi i q(x)} \Big|^2 = \frac{1}{|G|} \sum_{x \in G} e^{2\pi i q(x)} \sum_{y \in G} e^{-2\pi i q(y)}$$

$$= \frac{1}{|G|} \sum_{x,y \in G} e^{2\pi i q(x+y) - 2\pi i q(y)} = \frac{1}{|G|} \sum_{x \in G} \left(\sum_{y \in G} e^{2\pi i b(x,y)} \right) e^{2\pi i q(x)}.$$

Observe that when y runs over G the complex number $e^{2\pi i b(x,y)}$ runs over a finite subgroup of the unit circle. We have $\sum_{y\in G} e^{2\pi i b(x,y)} = 0$ unless this subgroup is trivial. The latter holds iff $x \in B$ and in this case $\sum_{y\in G} e^{2\pi i b(x,y)} = |G|$. Therefore

$$|\Gamma(G,q)|^2 = \sum_{x\in B} e^{2\pi i q(x)}.$$

The restriction of q to B is a linear homomorphism $B \to \mathbb{Q}/\mathbb{Z}$. If $q(B) \neq 0$, then $\sum_{x\in B} e^{2\pi i q(x)} = 0$. If $q(B) = 0$, then $\sum_{x\in B} e^{2\pi i q(x)} = |B|$. This implies the claim of the lemma.

Lemma 2. *Let H be a subgroup of G. If $q(H) = 0$, then $H \subset H^\perp$, the form q induces a quadratic form $q' : H^\perp/H \to \mathbb{Q}/\mathbb{Z}$, and*

$$\Gamma(H^\perp/H, q') = \left(\frac{|G|}{|H|\,|H^\perp|}\right)^{1/2} \Gamma(G,q). \tag{7}$$

In particular, if $q(H) = 0$ and $b(G,H) = 0$, then $H^\perp = G$ and $\Gamma(G/H,q') = |H|^{-1/2}\,\Gamma(G,q)$.

Proof. The first two claims of the lemma are obvious because $q(H) = 0$ implies that $b(H,H) = 0$. Let us prove formula (7). Let $s\colon G/H \to G$ be a set-theoretic section of the projection $G \to G/H$. Every element of G can be uniquely presented as a sum $s(y) + z$ with $y \in G/H$ and $z \in H$. Therefore

$$\sum_{x\in G} e^{2\pi i q(x)} = \sum_{y\in G/H}\left(\sum_{z\in H} e^{2\pi i q(s(y)+z)}\right).$$

Note that $q(s(y)+z) = q(s(y)) + b(s(y),z)$ for $z \in H$. Hence

$$\sum_{z\in H} e^{2\pi i q(s(y)+z)} = e^{2\pi i q(s(y))}\sum_{z\in H} e^{2\pi i b(s(y),z)}.$$

The same argument as in Lemma 1 shows that the right hand side equals $e^{2\pi i q(s(y))}|H|$ if $s(y) \in H^\perp$ and equals 0 otherwise. Therefore

$$\sum_{x\in G} e^{2\pi i q(x)} = \sum_{y\in H^\perp/H} e^{2\pi i q(s(y))}|H| = |H| \sum_{y\in H^\perp/H} e^{2\pi i q'(y)}.$$

This formula is equivalent to (7).

Lemma 3. *If the form $b\colon G\times G \to \mathbb{Q}/\mathbb{Z}$ is nondegenerate, then $|H|\,|H^\perp| = |G|$, for any subgroup H of G.*

Proof. By definition, $H^\perp$ is the kernel of the homomorphism $G \to \mathrm{Hom}(H,\mathbb{Q}/\mathbb{Z})$ induced by b. Therefore $|G/H^\perp| \leqslant |H|$.

Consider the homomorphism $H \to \mathrm{Hom}(G/H^\perp,\mathbb{Q}/\mathbb{Z})$ induced by b. Since b is nondegenerate, this homomorphism is injective. Therefore $|H| \leqslant |G/H^\perp|$. Hence $|G/H^\perp| = |H|$ or, equivalently, $|H|\,|H^\perp| = |G|$.

Remark. If there is a subgroup $H \subset G$ such that $q(H) = 0$ and $H^\perp = H$, then the quadratic form $q\colon G \to \mathbb{Q}/\mathbb{Z}$ is said to be metabolic. Lemmas 2 and 3 imply that for a metabolic q with nondegenerate associated bilinear form, $\Gamma(G,q) = 1$.

3. *Proof of formula (3)*

3·1. *The nondegenerate case*

Assume up to the end of Section 3·1 that both forms f and g are nondegenerate. Then the adjoint homomorphisms $\hat{f}: L \to L^*$ and $\hat{g}: M \to M^*$ are isomorphisms over $\mathbb{Q}$. Therefore the adjoint homomorphism $\hat{e} = \hat{f} \otimes \hat{g}: K \to K^*$ is also an isomorphism over $\mathbb{Q}$ and the pairing $e: K \times K \to \mathbb{Z}$ is nondegenerate. It follows from definitions that

$$G_f = \operatorname{Coker} \hat{f} = L^*/\hat{f}(L) \quad \text{and} \quad G_g = \operatorname{Coker} \hat{g} = M^*/\hat{g}(M).$$

Similarly $G_e = \operatorname{Coker} \hat{e} = K^*/\hat{e}(K)$. It is useful to note that the homomorphisms $\hat{f}: L \to L^*, \hat{g} : M \to M^*, \hat{e}: K \to K^*$ are injective.

We claim that the homomorphism $\alpha_{f,g}: G_f \otimes M \to G_e$ is injective. Indeed, we can present $\hat{e} = \hat{f} \otimes \hat{g}$ as the composition of $\hat{f} \otimes \mathrm{id}_M: L \otimes M \to L^* \otimes M$ and $\mathrm{id}_{L^*} \otimes \hat{g}: L^* \otimes M \to L^* \otimes M^*$. Since the latter homomorphism is injective, it induces a monomorphism of the cokernel of $\hat{f} \otimes \mathrm{id}_M$ into the cokernel of $\hat{e}$. This monomorphism is nothing but $\alpha_{f,g}: G_f \otimes M \to G_e$. Clearly,

$$\operatorname{Coker} \alpha_{f,g} = \operatorname{Coker}(\mathrm{id}_{L^*} \otimes \hat{g}) = L^* \otimes G_g$$

which gives an exact sequence

$$0 \longrightarrow G_f \otimes M \xrightarrow{\alpha_{f,g}} G_e \longrightarrow L^* \otimes G_g \longrightarrow 0. \tag{8}$$

A similar argument shows that the homomorphism $\beta_{f,g}: L \otimes G_g \to G_e$ is injective with cokernel $G_f \otimes M^*$.

In the remaining part of the proof we treat $G_f \otimes M$ and $L \otimes G_g$ as subgroups of G_e via the embeddings $\alpha_{f,g}$ and $\beta_{f,g}$, respectively. Denote their intersection by H. We can describe H more explicitly as a subgroup of $G_f \otimes M$ and $L \otimes G_g$. Composing the inclusion $G_f \otimes M \hookrightarrow G_e$ with the projection $G_e \to G_f \otimes M^*$ we obtain the homomorphism $\mathrm{id}_{G_f} \otimes \hat{g}: G_f \otimes M \to G_f \otimes M^*$. Therefore $H = \operatorname{Ker}(\mathrm{id}_{G_f} \otimes \hat{g}) \subset G_f \otimes M$. Similarly, $H = \operatorname{Ker}(\hat{f} \otimes \mathrm{id}_{G_g}) \subset L \otimes G_g$.

Let us check that the subgroups $G_f \otimes M$ and $L \otimes G_g$ of G_e are orthogonal to each other with respect to the bilinear form $b_e: G_e \times G_e \to \mathbb{Q}/\mathbb{Z}$. The first subgroup is generated by elements $(x \otimes \hat{g}(y)) \bmod \hat{e}(K)$ where $x \in L^*, y \in M$. The second subgroup is generated by elements $(\hat{f}(z) \otimes t) \bmod \hat{e}(K)$ where $z \in L, t \in M^*$. It suffices to verify that such elements are orthogonal. Choose a nonzero integer n such that $nx = \hat{f}(x')$ with $x' \in L$. Then $nx \otimes \hat{g}(y) = \hat{e}(x' \otimes y)$ and

$$\begin{aligned} b_e((x \otimes \hat{g}(y)) \bmod \hat{e}(K), (\hat{f}(z) \otimes t) \bmod \hat{e}(K)) &= \frac{(\hat{f}(z) \otimes t)(x' \otimes y)}{n} \\ &= \frac{f(x', z)\, t(y)}{n} = \frac{nx(z)\, t(y)}{n} = x(z)\, t(y) = 0 \bmod \mathbb{Z}. \end{aligned}$$

Let $q_1 = q_{e,u} \circ \alpha_{f,g}$ and $q_2 = q_{e,u} \circ \beta_{f,g}$ be the reductions of the quadratic form $q_{e,u}: G_e \to \mathbb{Q}/\mathbb{Z}$ to $G_f \otimes M \subset G_e$ and $L \otimes G_g \subset G_e$, respectively. We should prove that

$$\Gamma(G_f \otimes M, q_1) = e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - e_{\mathbb{Q}}(u,u))}\, \overline{\Gamma(L \otimes G_g, q_2)}. \tag{9}$$

Consider first the case when $q_{e,u}(H) \neq 0$. We claim that $\Gamma(G_f \otimes M, q_1) = 0$. Denote by h the homomorphism

$$G_f \otimes M \to \mathrm{Hom}\,(G_f \otimes M, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}\,(G_f, \mathbb{Q}/\mathbb{Z}) \otimes M^*$$

adjoint to the bilinear form associated with q_1. This bilinear form is the tensor product of the bilinear forms $b_f: G_f \times G_f \to \mathbb{Q}/\mathbb{Z}$ and $g: M \times M \to \mathbb{Z}$. Therefore

$$h = \mathrm{ad}(b_f) \otimes \hat{g} = (\mathrm{ad}(b_f) \otimes \mathrm{id}_M)(\mathrm{id}_{G_f} \otimes \hat{g}),$$

where $\mathrm{ad}(b_f): G_f \to \mathrm{Hom}\,(G_f, \mathbb{Q}/\mathbb{Z})$ is the homomorphism adjoint to b_f. It follows from the discussion in Section 1·2 that $\mathrm{ad}(b_f)$ is an isomorphism. Hence $\mathrm{Ker}\, h = \mathrm{Ker}\,(\mathrm{id}_{G_f} \otimes \hat{g}) = H$. By assumption, $q_1(H) = q_{e,u}(H) \neq 0$. By Lemma 1, $\Gamma(G_f \otimes M, q_1) = 0$. A similar argument shows that the kernel of the homomorphism $L \otimes G_g \to \mathrm{Hom}(L \otimes G_g, \mathbb{Q}/\mathbb{Z})$ adjoint to $f \otimes b_g$ is equal also to H. Therefore $\Gamma(L \otimes G_g, q_2) = 0$. This implies (9).

Assume that $q_{e,u}(H) = 0$. Then $q_1(H) = q_2(H) = 0$. Since the subgroups $G_f \otimes M$ and $L \otimes G_g$ of G_e are mutually orthogonal, their sum in G_e, say H', is orthogonal to their intersection H. Thus, $H' \subset H^{\perp}$. Note that

$$|H'| = |G_f \otimes M|\,|L \otimes G_g|/|H| = |G_e|/|H|,$$

where the last equality follows from (8). Comparing this formula with Lemma 3 (and using the nondegeneracy of the form b_e) we obtain that $|H'| = |H^{\perp}|$ and hence $H' = H^{\perp}$.

Denote by q' the quadratic form on H'/H induced by $q_{e,u}: G_e \to \mathbb{Q}/\mathbb{Z}$. By formulas (7) and (2),

$$\Gamma(H'/H, q') = \Gamma(G_e, q_{e,u}) = e^{\frac{\pi i}{4}(\sigma(e) - e_{\mathbb{Q}}(u,u))} = e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - e_{\mathbb{Q}}(u,u))}.$$

(The equality $\sigma(e) = \sigma(f)\,\sigma(g)$ is easily obtained by considering the diagonal forms of f and g.) On the other hand,

$$(H'/H, q') = ((G_f \otimes M)/H, q_1') \oplus ((L \otimes G_g)/H, q_2'),$$

where q_1' and q_2' are the quadratic forms induced by q_1 and q_2, respectively. Therefore

$$\begin{aligned}\Gamma(H'/H, q') &= \Gamma((G_f \otimes M)/H, q_1')\; \Gamma((L \otimes G_g)/H, q_2') \\ &= |H|^{-1}\, \Gamma(G_f \otimes M, q_1)\; \Gamma(L \otimes G_g, q_2),\end{aligned}$$

where we use the multiplicativity of Γ with respect to direct sum of quadratic forms and the last claim of Lemma 2. By Lemma 1 and the equality $\mathrm{Ker}\, h = H$ obtained above,

$$|H| = |\mathrm{Ker}\, h| = |\Gamma(G_f \otimes M, q_1)|^2 = \Gamma(G_f \otimes M, q_1)\,\overline{\Gamma(G_f \otimes M, q_1)}.$$

Combining these formulas we obtain

$$e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - e_{\mathbb{Q}}(u,u))} = \Gamma(L \otimes G_g, q_2)/\,\overline{\Gamma(G_f \otimes M, q_1)}.$$

Applying the complex conjugation we obtain (9).

3·2. *The general case*

Note that f (resp. g) is a direct sum of a 0-form and a non-degenerate form $f': L' \times L' \to \mathbb{Z}$ (resp. $g': M' \times M' \to \mathbb{Z}$) where L', M' are direct summands of the lattices L, M, respectively. Projecting u into $\mathbb{Q} \otimes L' \otimes M'$ we obtain a rational Wu class, u', for $e' = f' \otimes g'$. Clearly, $G_f = G_{f'}$. We have

$$\Gamma(G_f \otimes M, q_{e,u} \circ \alpha_{f,g}) = \Gamma(G_f \otimes M', q_{e',u'} \circ \alpha_{f',g'})\,\Gamma(G_f \otimes (M/M'), 0)$$
$$= |G_f|^{\operatorname{corank}(g)/2}\,\Gamma(G_{f'} \otimes M', q_{e',u'} \circ \alpha_{f',g'}).$$

Similarly,

$$\Gamma(L \otimes G_g, q_{e,u} \circ \beta_{f,g}) = |G_g|^{\operatorname{corank}(f)/2}\Gamma(L' \otimes G_{g'}, q_{e',u'} \circ \beta_{f',g'}).$$

Note that $\sigma(f) = \sigma(f'), \sigma(g) = \sigma(g')$ and $e(u,u) = e'(u',u')$. Applying the results of Section 3.1 to f', g', u' and using the formulas above we obtain (3).

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