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FIRST SYMPLECTIC CHERN CLASS AND MASLOV INDICES

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UDC 517.43

An explicit formula is given in this paper for a two-dimensional cocycle in the bar resolution of the group $G = Sp(n, \mathbb{R})$, which represents the first Chern class of the natural n-dimensional complex vector bundle over BG^{δ}. It is shown that this cocycle is closely connected with the Maslov indices of Lagrangian subspaces of $\mathbb{R}^{2n}$.

1. Introduction

The present paper is a continuation of a version of the author's note [17] and contains, in particular, detailed proofs of the theorems announced in [17].

1.1. Nature of the Results. Let $G = Sp(n, R)$ be the symplectic group (where $n \geq 1$) and $\tilde{G}$ be its universal covering with the natural group structure. The fundamental group $\pi_1(G)$ is isomorphic to $\mathbf{Z}$ and in what follows it is identified with $\mathbf{Z}$ (cf. point 1.3). The present paper is devoted to the study of the two-dimensional cohomology class of the group G, corresponding to the exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

This class is denoted by u and is an element of the group $H^2(G^{\delta}; \mathbf{Z})$, where for a topological group L, we denote by L^{δ} the same group with the discrete topology and where the action of G^{δ} on $\mathbf{Z}$ is trivial. It is easy to describe the class u in terms of the standard theory of characteristic classes: it is equal to the first Chern class of the complex vector bundle obtained from the real vector bundle over BG^{δ} $(= K(G^{\delta}, 1))$, associated with the universal principal G^{δ}-bundle and the action of G on R^{2n}, introduced by the natural complex structure (cf. [2, 3], and the Appendix). Up to sign the class u can be described as the image under the canonical homomorphism $H^2(BG; \mathbf{Z}) \longrightarrow H^2(BG^{\delta}; \mathbf{Z})$ of a generator of the group $H^2(BG; \mathbf{Z}) = \mathbf{Z}$.

In the paper we consider the question of finding an explicit formula for a two-dimensional cocycle in the bar resolution of the group G^{δ}, which represents u. This question is not new, cf. point 2, "History of the Question." The interest in it is due, in the first place to interest in the broad scheme of constructing explicit cocycles representing nontrivial cohomology classes of Lie groups and algebras, and secondly, to the specific role which the class u and its reduction mod2 play in the theory of representations of Lie groups and in the theory of symplectic and metaplectic structures.

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR, Vol. 143, pp. 110-129, 1985.

Even for well-studied groups and their cohomology classes the construction of explicit cocycles representing these classes is not a mechanical matter and usually requires additional considerations. Here the role of such considerations is played by the following observation concerning elementary cobordism theory. Let V be a closed orientable smooth manifold of dimension $4m+2$ with $m>0$. Let Ω be the group of orientation-preserving diffeomorphisms $V \longrightarrow V$. The torus of the diffeomorphism $f: V \longrightarrow V$ [the manifold $V \times [0,1]/a \times 0 = f(a) \times 1$ for $a \in V$] is denoted by $V(f)$. If $f, g \in \Omega$, then $N(f, g)$ denotes the result of gluing the lower bases of the cylinders $V(f) \times [0,1]$ and $V(g) \times [0,1]$ to the upper base of the cylinder $V(fg) \times [0,1]$ according to the following rule: we identify $a \times t \times 0 \in V(f) \times 0$ with $a \times \frac{t+1}{2} \times 1 \in V(fg) \times 1$ and $a \times t \times 0 \in V(g) \times 0$ with $a \times (t/2) \times 1 \in V(fg) \times 1$, where $a \in V$ and $t \in [0,1]$. It is easy to verify that: (i) $N(f,g)$ is a compact orientable $(4m+4)$-dimensional manifold, whose boundary is equal to the disjoint union of $V(f)$, $V(g)$, and $V(fg)$; (ii) if $h \in \Omega$, then the result of gluing $N(f,g)$ with $N(fg, h)$ along the common component of the boundary to the torus $V(fg)$ is homeomorphic with the result of gluing $N(g,h)$ with $N(f, gh)$ along $V(gh)$. In view of (ii) and the additivity of the signature, the function $\sigma: \Omega^2 \longrightarrow \mathbf{Z}$, assigning to the pair f, g the signature of the appropriately oriented manifold $N(f,g)$, satisfies the relation $\sigma(f,g)+\sigma(fg,h)=\sigma(f,gh)+\sigma(g,h)$, i.e., σ is a two-dimensional cocycle (cf. [12]). It turns out that the construction made can be modeled algebraically. This leads to a map $\varphi: G^2 \longrightarrow \mathbf{Z}$, where $G=Sp(n,\mathbb{R})$ and $n=1,2,\ldots$. Modeling the proof of the additivity of the signature one proves that φ is a cocycle. As it turns out, φ represents $4u \in H^2(G^\delta; \mathbf{Z})$. Modifying φ somewhat, one can also construct a cocycle representing u.

Along with the cocycle φ, in this paper we study its primitive $\Phi: \tilde{G} \longrightarrow \mathbf{Z}$, which is a one-dimensional cochain, whose coboundary is equal to the lift of φ to $\tilde{G}$, so that if q is the projection $\tilde{G} \longrightarrow G$ and $F_1, F_2 \in \tilde{G}$, then

$$\varphi(q(F_1), q(F_2)) = \Phi(F_1)+\Phi(F_2)-\Phi(F_1F_2). \tag{1}$$

The existence of the cochain Φ follows from the equation $q^*(u)=0$, and its uniqueness from the equation $H^1(G^\delta; \mathbf{Z})=0$ (this Lie group G is simple). It turn out that φ and Φ are closely connected with the Maslov indices of Lagrangian spaces.

We recall that if there is fixed in a 2n-dimensional real vector space H a nonsingular skew-symmetric form, then a subspace of the space H, which coincides with its annihilator with respect to this form, is called Lagrangian. The Lagrangian subspaces constitute a closed submanifold Λ of the Grassmanian of n-dimensional subspaces of the space H (cf. [7]). The Maslov indices of Lagrangian subspaces are three interconnected objects (cf. [7, 9, 10]): the function on the set of curves in Λ with values in the group of half-integers, corresponding to a fixed Lagrangian space $k \in \Lambda$, is the Maslov index of curves with respect to k (the index of a curve in Λ, whose ends coincide or are transverse to k, is an integer); the ternary Maslov index (index of inertia) $\tau: \Lambda \times \Lambda \times \Lambda \longrightarrow \mathbf{Z}$; the binary Maslov index $m: \tilde{\Lambda} \times \tilde{\Lambda} \longrightarrow \mathbf{Z}$, where $\tilde{\Lambda}$ is the universal covering manifold of Λ [we recall that $\pi_1(\Lambda)=\mathbf{Z}$]. We shall follow the definitions of the binary and ternary Maslov indices used in [10]; the corresponding functions in [7, 9] are obtained from these by linear transformations. The first two of the functions listed can be calculated in terms of the third. Namely, if q is the covering $\tilde{\Lambda} \longrightarrow \Lambda$, then: the Maslov index of a curve in Λ with respect to $k \in \Lambda$ is equal to $(1/2)[m(K_1, K)-m(K_0,K)]$, where K_0 and K_1, respectively, are the beginning and end of an arbitrary lift of the curve to $\tilde{\Lambda}$ and K is any element of the set $q^{-1}(k)$ (cf. [9, 16]); for any $K_1, K_2, K_3 \in \tilde{\Lambda}$

$$\tau(q(K_1), q(K_2), q(K_3)) = m(K_1, K_2) - m(K_1, K_3) + m(K_2, K_3) \tag{2}$$

(Leray's formula [9]).

Arnol'd [1] proved that the Maslov index of curves relative to any $k \in \Lambda$, considered as a singular cocycle on Λ, represents a generator of the group $H^1(\Lambda; \mathbf{Z}) = \mathbf{Z}$. It turns out that τ and m also admit interpretations in terms of homology theory. Namely, we consider maps $\Psi: G^3 \longrightarrow \mathbb{Z}$ and $\tilde{\Psi}: \tilde{G}^2 \longrightarrow \mathbf{Z}$, defined, respectively, by the formulas $\Psi(f_1, f_2, f_3) = \varphi(f_1^{-1} f_2, f_2^{-1} f_3)$ and $\tilde{\Psi}(F_1, F_2) = \Phi(F_1^{-1} F_2)$, where $f_1, f_2, f_3 \in G$ and $F_1, F_2 \in \tilde{G}$. According to [12, p. 158], Ψ is the two-dimensional cocycle in the homogeneous generators of the bar resolution of the group G^δ, corresponding to φ, and $\tilde{\Psi}$ is a primitive of it. Thus, for any $f_1, f_2, f_3, f_4 \in G$, and $F_1, F_2, F_3 \in \tilde{G}$

$$\Psi(f_1, f_2, f_3) - \Psi(f_1, f_2, f_4) + \Psi(f_1, f_3, f_4) - \Psi(f_2, f_3, f_4) = 0, \tag{3}$$

$$\Psi(q(F_1), q(F_2), q(F_3)) = \tilde{\Psi}(F_1, F_2) - \tilde{\Psi}(F_1, F_3) + \tilde{\Psi}(F_2, F_3), \tag{4}$$

where q is the projection $\tilde{G} \longrightarrow G$. In [9, 16] a family of smooth imbeddings $\Lambda \rightarrow G$ was constructed. It is proved here that if γ is any of these imbeddings, then the function $\Psi \circ (\gamma \times \gamma \times \gamma): \Lambda^3 \longrightarrow \mathbf{Z}$ is independent of the choice of γ and is equal to 2τ. If $\Gamma: \tilde{\Lambda} \longrightarrow \tilde{G}$ is the imbedding induced by γ, then the function $\tilde{\Psi} \circ (\Gamma \times \Gamma): \tilde{\Lambda}^2 \longrightarrow \mathbf{Z}$ is equal to $2m$. Thus, up to multiplication by a constant, the cochains Ψ and $\tilde{\Psi}$ are, respectively, extensions of the binary and ternary Maslov indices to the symplectic group and its universal covering. This point of view reveals the nature of Leray's formula (2), which turns out to be a specialization of the assertion "the coboundary of the cochain $\tilde{\Psi}$ is equal to the lift of Ψ to $\tilde{G}$." The familiar formula

$$\tau(k_1, k_2, k_3) - \tau(k_1, k_2, k_4) + \tau(k_1, k_3, k_4) - \tau(k_2, k_3, k_4) = 0 \tag{5}$$

for any $k_1, k_2, k_3, k_4 \in \Lambda$ is a specialization of the assertion "Ψ is a cocycle." Another well-known formula in the theory of Maslov indices is the Souriau formula, which is a consequence of the calculation of the values of the cochain $\tilde{\Psi}$ given below.

1.2. History of the Question. The question of finding an explicit cocycle representing u has been studied by a number of authors. Dupont, and independently Guechardet and Wigner, constructed the same cocycle, which represents the image of the class u under the natural homomorphism $H^2(G^\delta; \mathbb{Z}) \rightarrow H^2(G^\delta; \mathbb{R})$ (cf. [3, 5, 6]). An explicit formula for this cocycle is given in [3, p. 152]. This formula is rather complicated and includes taking the logarithm of matrices of operators, raising them to real powers, and integration. If n = 1, Guechardet and Wigner gave a simple explicit formula for an integral cocycle representing u (cf. [6, p. 289]).

For any n, a cocycle $G^2 \longrightarrow \mathbb{Z}$ representing 4u was constructed by A. Weil in the course of constructing the Shale–Weil representation of the metaplectic group. Later, Lions proved that this cocycle is defined by the rule $(f, g) \mapsto \tau(k, f(k), fg(k))$, where k is a fixed Lagrangian space (cf. [10]). To different k there correspond cocycles obtained from one another by transport by inner automorphisms of the group G. As is evident from what was said in point 1.1 (cf. also point 1.4), the Weil–Lions cocycles can be expressed in terms of the cocycle φ constructed in this paper by means of a simple formula. The author does not know whether φ can be calculated from the Weil–Lions cocycles (although the invariance of φ with respect to inner automorphisms of the group G suggests that φ can be calculated in terms of the Weil–Lions cocycles over Λ). In particular, knowledge of only one of the Weil–Lions cocycles does not permit one to see the phenomenon described at the end of point 1.1 directly.

The interrelations of the Dupont and Guechardet–Wigner cocycles on the one hand, and the Weil–Lions cocycles and the cocycle φ on the other hand are unclear to the author.

1.3. The Cocycle φ. Let H be a finite-dimensional real vector space, let $B : H \times H \longrightarrow \mathbb{R}$ be a nonsingular skew-symmetric form, and let G a group of linear automorphisms of the space H, containing B. The presence of the form B permits one to fix an isomorphism $\mathbb{Z} \longrightarrow \pi_1(G)$, which is necessary for defining the class u (cf. point 1.1; $-u$ corresponds to the other isomorphism). If $a, b \in H$ with $B(a, b) = 1$, then the isomorphism with which we are concerned takes 1 into the homotopy class of the loop $[0, 2\pi] \longrightarrow G$, which makes correspond to the number t the homomorphism $H \longrightarrow H$, which is the identity on the B-annihilator of the plane $\mathbb{R}a + \mathbb{R}b$ and rotates this plane clockwise through the angle t [so that a goes into $\cos(t)a - \sin(t)b$ and b into $\sin(t)a + \cos(t)b$].

For $f, g \in G$ we define a binary real-valued form on the space $(f-1)(H) \cap (g-1)(H)$ (where 1 denotes the identity homomorphism $H \longrightarrow H$) by the rule

$$(a, b) \longmapsto B((f-1)^{-1}(a) + (g-1)^{-1}(a) + a, b). \tag{6}$$

It is proved below that this is a well-defined symmetric biliear form. Generally it is degenerate. We denote by φ the map $G^2 \longrightarrow \mathbb{Z}$, which associates with the pair f, g, the signature of the form (6) (i.e., the signature of its quotient by the annihilator).

THEOEM 1. The map φ is a cocycle and represents 4u.

It follows from Theorem 1 that the cocycle $\varphi/4$ with values in $\mathbb{Q}$ represent the image of u under the inclusion homomorphism $H^2(G^\delta; \mathbb{Z}) \longrightarrow H^2(G^\delta; \mathbb{Q})$. We construct an integer-valued cocycle which represents u.

It is proved below that for $f \in G$, the annihilator with respect to the form B of the space $(f-1)(H)$ is equal to $\mathrm{Ker}(f-1)$. Hence, if $f \neq 1$ and $x_1, \ldots, x_d$ are elements of the space H, whose images under the homomorphism $f-1$ form a basis for the space $(f-1)(H)$, then the determinant of the matrix

$$\{B(x_i, (f-1)(x_j))\}, \quad i, j = 1, \ldots, d$$

is different from zero, and its sign is consequently independent of the choice of $x_1, \ldots, x_d$. Let $\varepsilon(f) = 1$, if this determinant is positive, and $\varepsilon(f) = -1$ otherwise. Let $\varepsilon(1) = 1$. We denote by φ' the cocycle $\varphi - \rho : G^2 \longrightarrow \mathbb{Z}$, where $\rho : G^2 \longrightarrow \mathbb{Z}$ is the coboundary of the cochain

$$f \longmapsto \dim(f-1)(H) + \varepsilon(f) - 1 : G \longrightarrow \mathbb{Z}.$$

THEOREM 2. The cocycle $\varphi'/4$ assumes integral values and represent u.

COROLLARY. For any $m > 2$ the cocycle $\varphi'/4 \pmod m$ represents $u \pmod m \in H^2(G^\delta; \mathbb{Z}/m\mathbb{Z})$.

We note that $u \pmod 2 = w_2(\xi)$, where ξ is the vector bundle over BG^δ with fiber H, associated with the universal principal G^δ-bundle and the natural action of G on H. Apparently explicit cocycles representing $w_2(\xi)$ were not known. It is interesting to note that the author has been unable to find an explicit formula in the literature for a cocycle representing the class w_2 of the universal flat n-dimensional vector bundle, or, what is the same, the class in $H^2(SO(n)^\delta; \mathbb{Z}/2\mathbb{Z})$, corresponding to the extension $\mathrm{Spin}(n)$ of the group $SO(n)$.

1.4. Maslov Indices and the Cochains φ and Φ. Let H, B, and G be the same objects as in point 1.3. Let Λ be the manifold of Lagrangian subspaces of the space H. If k_1, k_2, $k_3 \in \Lambda$, then $\tau(k_1, k_2, k_3)$ is the signature of the symmetric bilinear form A on $(k_1+k_2)\cap k_3$, defined by the following formula: if $a, b \in (k_1+k_2)\cap k_3$ and x is an element of the space k_2, such that $a - x \in k_1$, then $A(a,b) = B(x,b)$. That the form A is well-defined, symmetric, and biliear, follow from the fact that if $y \in k_2$ and $b-y \in k_1$, then

$$B(x,b)-B(y,a)=B(x-a,b-y)=0.$$

If k_1 and k_2 are transverse, the definition of the Maslov index given coincides with the ordinary one (cf. [7, 9, 10]). In general it is equivalent with the definition of Kashiwara (cf. [10]), but seems more convenient to the author.

Let $q:\widetilde{\Lambda} \longrightarrow \Lambda$ be the universal covering. It is easier to define the binary Maslov index $m:\widetilde{\Lambda}^2 \longrightarrow \mathbb{Z}$ axiomatically with the help of the following theorem.

THEOREM 3. There exists a unique function $m:\widetilde{\Lambda}^2 \longrightarrow \mathbb{Z}$, satisfying (2) for any $K_1, K_2, K_3 \in \widetilde{\Lambda}$, which is locally constant on the set of pairs $K_1, K_2 \in \widetilde{\Lambda}$ such that $q(K_1)$ is transverse to $q(K_2)$.

This theorem was proved by Leray in somewhat restricted form [9]: he considered the Maslov index only for pairs K_1, K_2, such that $q(K_1)$ is transverse to $q(K_2)$.

We define the natural imbedding $\Lambda \longrightarrow G$, mentioned in point 1.1. For this we fix a complex structure on H, compatible with B, i.e., a homomorphism $i: H \longrightarrow H$, preserving B, such that $i^2 = -1$ and $B(ia, a) > 0$ for any nonzero $a \in H$. As is easy to verify, the form

$$(a,b) \longmapsto B(ia,b) + iB(a,b) : H^2 \longrightarrow \mathbb{C} \tag{7}$$

is Hermitian. We fix a basis $a_1, \ldots, a_n$ of the space H over $\mathbb{C}$, which is orthonormal with respect to this form. According to [9], for any $k \in \Lambda$ there exists a unitary operator $f: H \longrightarrow H$ [i.e., one which preserves the form (7)], such that $k = f(\mathbb{R}a_1 + \ldots + \mathbb{R}a_n)$. Here, if $g: H \longrightarrow H$ is a linear operator over $\mathbb{C}$, defined with respect to the basis $a_1, \ldots, a_n$ by the matrix which is the transpose of the matrix of the operator f with respect to the same basis, then the composition $f \circ g$ is independent of the choice of f, and the rule $k \longmapsto f \circ g$ defines an imbedding $\Lambda \longrightarrow G$. We denote it by γ. [The image $\gamma(\Lambda)$ is the set of unitary operators, defined with respect to the basis $a_1, \ldots, a_n$ by symmetric matrices.] Since γ induces an isomorphism $\pi_1(\Lambda) \longrightarrow \pi_1(G)$ (cf. [1]), the composition $\gamma \circ q : \widetilde{\Lambda} \longrightarrow G$ lifts to an imbedding $\widetilde{\Lambda} \longrightarrow \widetilde{G}$. We denote such a lift by Γ. Let Ψ and $\widetilde{\Psi}$, respectively, be the maps $(f_1, f_2, f_3) \longmapsto \varphi(f_1^{-1} f_2, f_2^{-1} f_3) : G^3 \longrightarrow \mathbb{Z}$ and $(F_1, F_2) \longmapsto \Phi(F_1^{-1} F_2) : \widetilde{G}^2 \longrightarrow \mathbb{Z}$.

THEOREM 4. For any $k_1, k_2, k_3 \in \Lambda$ and $K_1, K_2 \in \widetilde{\Lambda}$

$$\tau(k_1,k_2,k_3) = \frac{1}{2}\Psi(\gamma(k_1), \gamma(k_2), \gamma(k_3)), \tag{8}$$

$$m(K_1,K_2) = \frac{1}{2}\widetilde{\Psi}(\Gamma(K_1), \Gamma(K_2)). \tag{9}$$

It is interesting to note that like τ and m, the functions Ψ and $\widetilde{\Psi}$ are skew-symmetric (change sign under the interchange of two variables).

1.5. Calculation of the Cochains φ and Φ on the Unitary Group. Let G_0 be the subgroup of the group G consisting of automorphisms of the form (7) (the unitary group). We denote by $\widetilde{G}_0$ the subgroup of the group $G_0 \times \mathbb{R}$ consisting of those pairs (h, d), such that

$\det h = e^{id}$. The projection $\widetilde{G}_0 \longrightarrow G_0$ is the universal covering and hence lifts in a natural way to a monomorphism $\widetilde{G}_0 \longrightarrow \widetilde{G}$. We denote it by η. We define the function $\mu : \mathbb{R} \longrightarrow \mathbb{Z}$ by the formula: if $m \in \mathbb{Z}$, then $\mu(m\pi) = 2m$ and $\mu((m\pi, (m+1)\pi)) = 2m+1$.

THEOREM 5. If $(h, d) \in \widetilde{G}_0$ and if $\theta_1, \ldots, \theta_n$ are real numbers such that $e^{i\theta_1}, \ldots, e^{i\theta_n}$ are the eigenvalues of the operator h (counting multiplicities), then

$$\Phi(\eta(h,d)) = \frac{2}{\pi}\Big(\sum_{\tau=1}^{n}\theta_\tau - d\Big) - 2\sum_{\tau=1}^{n}\mu(\theta_\tau/2). \tag{10}$$

COROLLARY. Let $f, g \in G_0$ and let $\theta_1, \ldots, \theta_{3n}$ be real numbers such that $\theta_1 + \ldots + \theta_{2n} = \theta_{2n+1} + \ldots + \theta_{3n}$ and that the collections

$$\{e^{i\theta_1}, \ldots, e^{i\theta_n}\}, \{e^{i\theta_{n+1}}, \ldots, e^{i\theta_{2n}}\}, \{e^{i\theta_{2n+1}}, \ldots, e^{i\theta_{3n}}\}$$

are, respectively, collections of eigenvalues of the operators f, g, and fg. Then

$$\varphi(f,g) = 2\Big[\sum_{\tau=2n+1}^{3n}\mu(\theta_\tau/2) - \sum_{\tau=1}^{2n}\mu(\theta_\tau/2)\Big].$$

COROLLARY (Souriau's formula, cf. [10, 16]). Let $K_1, K_2 \in \widetilde{\Lambda}$; $\Gamma(K_\tau) = \eta(h_\tau, d_\tau)$ c $\tau = 1, 2$; $\theta_1, \ldots, \theta_n$ be real numbers such that $e^{i\theta_1}, \ldots, e^{i\theta_n}$ are eigenvalues of the operator $h_1^{-1} h_2$ and $\theta_1 + \ldots + \theta_n = d_2 - d_1$. Then $m(K_1, K_2) = -\sum_{\tau=1}^{n}\mu(\theta_\tau/2)$.

1.6. Remarks. 1. The function $\Phi : \widetilde{G} \longrightarrow \mathbb{Z}$ constructed here has a number of remarkable properties: one can recover the Maslov indices from Φ; Φ is a Borel function; Φ is invariant with respect to inner automorphisms of the group $\widetilde{G}$ (cf. point 3.3 also). One can hope that Φ is connected with the generalized character of the Shale–Weil representation (cf. [8]).

2. Since the Weil–Lions cocycles (cf. point 1.2) represent 4u and the values of these cocycles do not exceed $n = \frac{\dim H}{2}$ in modulus, the real class u is represented by cocycles whose values do not exceed n/4 in modulus. It follows from this that the norm of the class u in the sense of the theory of bounded cohomology does not exceed n/4. The estimate $\|u\| < (2^n - 1)/2$ was noted previously (cf. [4]). It seems likely that $\|u\| = n/4$ (for n = 1 this is so, cf. [14]).

3. In [13] the Maslov index of a triple of positive Lagrangian subspaces of a complexified symplectic vector space is defined entirely implicitly. It satisfies (5) and for real Lagrangian spaces it coincides with the ordinary Maslov index. It would be interesting to construct a cocycle related to this generalized index in the same way that φ is related to τ.

4. We note the definite parallelism of the present paper with Novikov [15], where, like here, considerations which relate to cobordism theory were reduced to a purely algebraic construction, which turned out to be connected with the Maslov indices.

1.7. When this paper was already finished, it became known to the author that the cocycle φ was considered previously by Meyer [18]. In [18] an assertion was also formulated which is equivalent with our Theorem 1 (this assertion is unproved in [18], where Meyer refers the reader to his dissertion for a proof). The cocycle φ and the relation with the Maslov indices are missing from [18].

2. Proof of Theorems 1 and 2

2.1. LEMMA. For any $f, g \in G$ the form (6) in $(f-1)(H) \cap (g-1)(H)$ is well-defined, symmetric, and bilinear, so that φ is a well-defined map.

Proof. It suffices to prove that for any $f\in G$ the form

$$(a,b)\longmapsto B((f-1)^{-1}(a)+a/2,\,b)$$

in $(f-1)(H)$ is well-defined, symmetric, and bilinear. These properties follow from the fact that $a=f(x)-x$ and $b=f(y)-y$, with $x,y\in H$, then

$$B(x+a/2,b)=\tfrac{1}{2}B(f(x)+x,f(y)-y)=\tfrac{1}{2}B(x,f(y))+$$
$$+\tfrac{1}{2}B(y,f(x))=\tfrac{1}{2}B(f(y)+y,f(x)-x)=B(y+b/2,a).$$

2.2. LEMMA. If $f\in G$, then the annihilator of the space $\operatorname{Ker}(f-1)$ with respect to the form B is equal to $(f-1)(H)$.

Proof. If $x,y\in H$, then

$$B(f(x)-x,y)=B(f(x),y-f(y)).$$

Hence $(f-1)(H)$ is contained in the space $\operatorname{Ann}_B(\operatorname{Ker}(f-1))$ and for dimensional reasons coincides with it.

2.3. LEMMA. If $f,g,h\in G$, then

$$\varphi(f,g)+\varphi(fg,h)=\varphi(f,gh)+\varphi(g,h). \tag{11}$$

Proof. We denote by E the subspace of the space $H\oplus H\oplus H$, consisting of triples (a_1,a_2,a_3), such that $a_1+a_2+a_3=0$, $a_1\in(f-1)(H)$, $a_2\in(g-1)(H)$, $a_3\in(h-1)(H)$. We denote by E_1 and E_2 the subspaces of the space E defined, respectively, by the equations $a_1=0$ and $a_3=0$. We denote by E_3 (respectively, by E_4) the subspace of the space E, consisting of triples $(a_1,a_2,a_3)\in E$, such that $a_2=(gh-h)(x)$ and $a_3=(h-1)(x)$ for some $x\in H$ [respectively, such that $a_1=(fg-g)(x)$ and $a_2=(g-1)(x)$ for some $x\in H$]. We define the map $\mathfrak{D}:E\times E\longrightarrow\mathbb{R}$ by the rule

$$\mathfrak{D}((a_1,a_2,a_3),(b_1,b_2,b_3))=B((f-1)^{-1}(a_1),b_1)+$$
$$+B((g-1)^{-1}(a_2),b_2)+B((h-1)(a_3),b_3)+B(a_3,b_3)-B(a_2,b_1).$$

As is easy to verify

$$B(a_3,b_3)-B(a_2,b_1)=\tfrac{1}{2}\left[B(a_1,b_1)+B(a_2,b_2)+B(a_3,b_3)+B(a_1,b_2)+B(b_1,a_2)\right].$$

Hence, as is clear from the proof of Lemma 2.1, $\mathfrak{D}$ is a well-defined symmetric form. It is proved below that: (i) the signatures of the restrictions of $\mathfrak{D}$ to E_1, E_2, E_3, and E_4 are equal, respectively, to $\varphi(g,h)$, $\varphi(f,g)$, $\varphi(f,gh)$, and $\varphi(fg,h)$; (ii) $\operatorname{Ann}_{\mathfrak{D}}(E_1\cap E_3)=E_1+E_3$ and $\operatorname{Ann}_{\mathfrak{D}}(E_2\cap E_4)=E_2+E_4$. It follows from (ii) that the signature of the form $\mathfrak{D}$ is equal to the sum of the signatures of the forms $\mathfrak{D}|E_1$ and $\mathfrak{D}|E_3$ and is also equal to the sum of the signatures of the forms $\mathfrak{D}|E_2$ and $\mathfrak{D}|E_4$.

Hence (11) follows from (i). We prove (i). We denote the form (6) in $(f-1)(H)\cap(g-1)(H)$ by $C(f,g)$. Obviously the map

$$a\longmapsto(a,-a,0):(f-1)(H)\cap(g-1)(H)\longrightarrow E_2$$

is an isomorphism, carrying $C(f,g)$ and $\mathfrak{D}|E_2$. Hence the signature of the form $\mathfrak{D}|E_2$ is equal to $\varphi(f,g)$. One proves analogously that the signature of the form $\mathfrak{D}|E_1$ is equal to $\varphi(g,h)$.

Obviously the image of the space E_3 under the projection $(a_1,a_2,a_3)\longmapsto a_1:E\longrightarrow H$ is equal to $(f-1)(H)\cap(gh-1)(H)$. If $(a_1,a_2,a_3),(b_1,b_2,b_3)\in E_3$, where $a_1=(f-1)(x)$, $a_2=(gh-h)(y)$,

$a_3=(h-1)(y)$ with $x, y \in H$, then

$$\mathfrak{D}((a_1,a_2,a_3),(b_1,b_2,b_3)) = B(x,b_1)+B(h(y),b_2) + B(y,b_3)+B(h(y)-y,b_3)-B(a_2,b_1) =$$
$$-B(x,b_1)+B(h(y),b_2+b_3)-B(a_2,b_1) = B(x-h(y)-a_2,b_1) - B(x-y+a_1,b_1) = C(f,gh)(a_1,b_1).$$

It follows from this that the form $C(f,gh)$ is isomorphic with the quotient of the form $\mathfrak{D}|E_3$ by some subspace of the annihilator. Hence the signature of the form $\mathfrak{D}|E_3$ is equal to $\varphi(f,gh)$. Analogously one proves that the signature of the form $\mathfrak{D}|E_4$ is equal to $\varphi(fg,h)$.

We prove (ii). As is easy to verify, $E_1\cap E_3=\{(0,(gh-h)(x),(h-1)(x)) \mid x\in \mathrm{Ker}(gh-1)\}$, and $E_1+E_3=\{(b_1,b_2,b_3)\in E \mid b_1\in(gh-1)(H)\}$. If $x\in \mathrm{Ker}(gh-1)$ and $b=(b_1,b_2,b_3)\in E$, then

$$\mathfrak{D}((0,(gh-h)(x),(h-1)(x)),b) = B(h(x),b_2)+B(x,b_3)+$$
$$+B(h(x)-x,b_3)-B(gh(x)-h(x),b_1) = B(h(x),b_2+b_3)+$$
$$+B(h(x)-gh(x),b_1) = -B(gh(x),b_1) = -B(x,b_1).$$

Hence the inclusion $b\in \mathrm{Ann}_{\mathfrak{D}}(E_1\cap E_3)$ is equivalent with the inclusion $b_1\in \mathrm{Ann}_B(\mathrm{Ker}(gh-1))$ and by Lemma 2.2, equivalent with the inclusion $b_1\in E_1+E_3$. The equation $\mathrm{Ann}_{\mathfrak{D}}(E_2\cap E_4)=E_2+E_4$ is proved analogously.

2.4. LEMMA. If $f, g\in G$, then the annihilator of the form (6) is equal to $(g-1)(\mathrm{Ker}\times(fg-1))$.

Proof. The fact that the space $(g-1)(\mathrm{Ker}(fg-1))$ is contained in the annihilator of the form (6), is verified without difficult. We prove the opposite inclusion. Let z be an element of this annihilator and let $x\in(f-1)^{-1}(z)$ and $y\in(g-1)^{-1}(z)$. Then $x+y+z\in \mathrm{Ann}_B((f-1)(H)\cap(g-1)(H))$. By Lemma 2.2 the latter space is equal to $\mathrm{Ker}(f-1)+\mathrm{Ker}(g-1)$, so that $x+y+z=a+b$ for some $a\in \mathrm{Ker}(f-1)$ and $b\in \mathrm{Ker}(g-1)$. Here $z=(g-1)(y-b)$ and $y-b\in \mathrm{Ker}\times(fg-1)$, since $fg(y-b)=f(g(y)-b)=f(y+z-b)=f(a-x)=a-f(x)=a-x-z=y-b$.

2.5. LEMMA. (i) For any integers p, q, r, and s, the function $\varphi: G^2\longrightarrow \mathbb{Z}$ is constant on path connected components of the set

$$\{(f,g)\in G^2 \mid \dim(f-1)(H)=p,\ \dim(g-1)(H)=q,$$
$$\dim(fg-1)(H)=r,\ \dim((f-1)(H)\cap(g-1)(H))=s\}.$$

(ii) The function φ is a Borel function.

Proof. Point (i) follows from Lemma 2.4, point (ii) from (i).

2.6. LEMMA. Let: $H=\mathbb{C}$, $B: H^2\longrightarrow\mathbb{R}$ be the form $(a,b)\longmapsto \mathrm{Im}(a\bar{b})$; G be the group of automorphisms of the form B. Then the group of rotations of S^1 is contained in G and the restriction of φ to S^1 represents the image of the class 4u under the restriction homomorphism $H^2(G^\delta;\mathbb{Z})\longrightarrow H^2((S^1)^\delta;\mathbb{Z})$. This image is an element of infinite order.

Proof. Since the inclusion $S^1\longrightarrow G$ is a homotopy equivalence, the restriction of u to S^1 represents its class, corresponding to the exact sequence $0\longrightarrow\mathbb{Z}\longrightarrow\mathbb{R}\longrightarrow S^1\longrightarrow 0$, in which the projection $\mathbb{R}\longrightarrow S^1$ is defined by the rule $t\longmapsto e^{2\pi i t}$. This class is represented by the cocycle $v: S^1\times S^1\longrightarrow\mathbb{Z}$, defined by the formula $v(f,g)=s(f)+s(g)-s(fg)$, where $s(e^{i\alpha})=\alpha/2\pi$ for $\alpha\in[0,2\pi)$. Thus, for $\alpha,\beta\in[0,2\pi)$,

$$r(e^{i\alpha}, e^{i\beta}) = \begin{cases} 0, & \text{if } \alpha+\beta < 2\pi \\ 1, & \text{if } \alpha+\beta \geqslant 2\pi \end{cases}$$

Obviously if $\alpha = 0$ or $\beta = 0$, then $\varphi(e^{i\alpha}, e^{i\beta}) = 0$. Let $0 < \alpha, \beta < 2\pi$. Then the form (6) with $f = e^{i\alpha}$ and $g = e^{i\beta}$ is defined on the whole space $H = \mathbb{C}$ by the formula

$$(a, b) \longmapsto \mathrm{Im}\left[((e^{i\alpha}-1)^{-1} + (e^{i\beta}-1)^{-1} + 1)\, a\bar{b}\right].$$

As is easy to verify, $(e^{i\alpha}-1)^{-1} = -(1/2) - (i/2)\,\mathrm{ctg}\,(\alpha/2)$ for any $\alpha \in \mathbb{R}$. Hence the values of the form (6) for $a = 1$, $b = i$, and $a = i$, $b = 1$, are equal to 0, and for $a = b = 1$ and $a = b = i$ are equal to $-\frac{1}{2}(\mathrm{ctg}\,(\alpha/2) + \mathrm{ctg}\,(\beta/2))$. The last number is positive, if $\alpha + \beta > 2\pi$, equal to 0, if $\alpha + \beta = 2\pi$, and negative, if $\alpha + \beta < 2\pi$. Thus, for $\alpha, \beta \in [0, 2\pi)$

$$\varphi(e^{i\alpha}, e^{i\beta}) = \begin{cases} 0, & \text{if } \alpha = 0 \text{ or } \beta = 0, \text{ or } \\ & \alpha + \beta = 2\pi \\ -2, & \text{if } \alpha > 0, \beta > 0 \text{ and } \alpha + \beta < 2\pi \\ 2, & \text{if } \alpha + \beta > 2\pi \end{cases}$$

It follows from this that for any $\alpha, \beta \in [0, 2\pi)$

$$(\varphi - 4r)(e^{i\alpha}, e^{i\beta}) = \omega(e^{i\alpha}) + \omega(e^{i\beta}) - \omega(e^{i\alpha + i\beta}),$$

where ω is the map of $S^1 \longrightarrow \mathbb{Z}$ which carries 1 to 0 and the other elements of the group S^1 to -2. Hence, φ and 4r represent the same element of the group $H^2((S^1)^\delta; \mathbb{Z})$.

The last assertion of the lemma follows, for example, from the fact that if the element of the group $H^2((S^1)^\delta; \mathbb{Z})$ with which we are concerned had finite order m, then its restriction to a cyclic subgroup of the group S^1 of order relatively prime to m, would equal 0, i.e., would correspond to a split extension, which is obviously not so.

2.7. Proof of Theorem 1. It follows from the results of [11] that the two-dimensional Borel cohomology group $H^2_\beta(G; \mathbb{Z})$ is isomorphic to $\mathrm{Hom}\,(\pi_1(G), \mathbb{Z}) = \mathbb{Z}$, where the image of the natural homomorphism $H^2_\beta(G; \mathbb{Z}) \longrightarrow H^2(G^\delta; \mathbb{Z})$ is generated by the class u. Since by Lemma 2.3 and point (ii) of Lemma 2.5, φ is a Borel cocycle, φ represents mu for some integer m. It follows from Lemma 2.6 that m = 4.

2.8. Proof of Theorem 2. By Theorem 1, φ' is a cocycle which represents 4u. Since G is a simple Lie group, $H_1(G^\delta; \mathbb{Z}) = 0$, and consequently the group $H^2(G^\delta; \mathbb{Z})$ is torsion-free. Hence to prove the theorem it suffices to prove that $\varphi'(G^2) \subset 4\mathbb{Z}$. We prove that if $f, g \in G$ and $\mathrm{Ker}\,(g-1) = \mathrm{Ker}\,(fg-1) = 0$, then $\varphi'(f, g) \in 4\mathbb{Z}$. The general case will follow from this: for any $f, g \in G$ one can find an element h of the group G such that 1 is not an eigenvalue of the operators h, gh, and fgh, and hence

$$\varphi'(f, g) = \varphi'(g, h) + \varphi'(f, gh) - \varphi'(fg, h) \in 4\mathbb{Z}.$$

Let $f, g \in G$ and let $\mathrm{Ker}(g-1) = \mathrm{Ker}(fg-1) = 0$. We denote by C the form (6) in $(f-1)(H)$. By Lemma 2.4, the form C is nonsingular. We calculate its determinant (which is an element of the set $\{-1, 1\}$). Let $x_1, \dots, x_l$ be elements of the space H, whose images under the homomorphism $f-1$ constitute a basis of the space $(f-1)(H)$. Obviously

$$C((f-1)(x_i), (f-1)(x_j)) = B([(f-1)^{-1} + (g-1)^{-1} + 1](f-1)(x_i), (f-1)(x_j)) =$$

$$= B((g-1)^{-1}(gf-1)(x_i), (f-1)(x_j)).$$

Since the homomorphism $(g-1)^{-1}(gf-1)$ is the identity on $\mathrm{Ker}(f-1)$, it follows from this that the determinant of the form C is equal to the product of the determinant of the homomorphisms $g-1$, $gf-1$, and the number $\varepsilon(f)$. This product is equal to $\varepsilon(f)\varepsilon(g)\varepsilon(fg)$, since $\det(g-1) = \varepsilon(g)$ and $\det(gf-1)=\det(fg-1)=\varepsilon(fg)$. Obviously, if $\det C = 1$, then the number of negative squares in the diagonal representation of the form C is even, and if $\det C = -1$, then this number is odd. Hence the signature $\varphi(f,g)$ of the form C is congruent modulo 4 with $\dim(f-1)(H) + \det C - 1$. From this, in view of the evenness of the numbers $\varepsilon(f)-1$, $\varepsilon(g)-1$, and $\varepsilon(fg)-1$, the following congruences modulo 4 follow:

$$\varphi(f,g) \equiv \dim(f-1)(H)+\varepsilon(f)\varepsilon(g)\varepsilon(fg)-1 \equiv$$

$$\equiv \dim(f-1)(H)+\dim(g-1)(H)-\dim(fg-1)(H) +\varepsilon(f)-1+\varepsilon(g)-1-(\varepsilon(fg)-1).$$

Hence $\varphi'(f,g)\in 4\mathbf{Z}$.

3. Proof of Theorems 3, 4, and 5

3.1. LEMMA. If under the conditions of point 1.4, $k,k_1,\ldots,k_m\in\Lambda$ with $m\geq 1$, then

$$\dim \bigcap_{i=1}^{m} \mathrm{Ker}(\gamma(k)^{-1}\gamma(k_i)-1) = 2\dim(k\cap k_1\cap\ldots\cap k_m).$$

Proof: cf. [9, p. 35].

3.2. Proof of (8). It follows from Lemmas 2.5 and 3.1 that the number $\Psi(\gamma(k_1),\gamma(k_2),\gamma(k_3))$ does not vary under continuous deformation of the complex structure and basis in H, used in defining the imbedding γ. Since the space of complex structures on H, compatible with B, is path connected (cf. [7]), and since for a fixed complex structure the space of orthonormal bases is aso path connected, it suffices to prove (8) for any one complex structure and one basis.

We set $m=\dim(k_1\cap k_2)$ and $n=(\dim H)/2$. It is easy to construct a symplectic basis $x_1,y_1,\ldots,x_n,y_n$ for the space H, such that $x_1,\ldots,x_n$ is a basis for k_1 and $x_1,\ldots,x_m$, $y_{m+1},\ldots,y_n$ is a basis for k_2. We introduce a complex structure on H by the formula $ix_i=-y_i$ and $iy_i=x_i$, where $i=1,\ldots,n$. Obviously this complex structure is compatible with B and $x_1,\ldots,x_n$ is a basis for H over $\mathbb{C}$, which is orthonormal with respect to the form (7). Let $\gamma:\Lambda\longrightarrow G$ be the imbedding corresponding to this complex structure and this basis, We set $f_i=\gamma(k_i)$ with $i=1,2,3$, and we calculate $\Psi(f_2,f_1,f_3)$ [in our notation it is easier to calculate this number than $\Psi(f_1,f_2,f_3)$]. As a direct calculation shows, $f_1=1$ and f_2 is given in the basis $x_1,\ldots,x_n$ be the diagonal matrix in which the first m diagonal elements are equal to 1 and the other $n-m$ are equal to −1. Let Z be a unitary operator carrying k_1 into k_3. We represent Z in the form $X+iY$, where X and Y are operators $H\longrightarrow H$ which are linear over $\mathbb{C}$, whose matrices in the basis $x_1,\ldots,x_n$ are real (or, equivalently, $X(k_1)\subset k_1$, $Y(k_1)\subset k_1$). Then

$$f_3=ZZ^t=Z\bar{Z}^{-1}=(Z-\bar{Z})\bar{Z}^{-1}+1=2iY(X-iY)^{-1}+1$$

We denote the space $(f_2-1)(H)\cap(f_3-1)(H)$ by E. Obviously $E=\bigoplus_{i>m}\mathbb{C}x_i\cap Y(H)$ and hence $E=(E\cap k_1)\oplus(E\cap ik_1)$. By definition $\Psi(f_2,f_1,f_3)$ is the signature of the form

$$(a,b)\longmapsto B((f_2^{-1}-1)^{-1}(a)+(f_3-1)^{-1}(a)+a,b):E^2\longrightarrow\mathbf{R}.$$

We denote this form by C. Obviously if $a \in E$ and $d \in (f_3-1)^{-1}(a)$, then $-a/2 \in (f_2^{-1}-1)^{-1}(a)$ and $-(a/2)+d+a = d+iY(X-iY)^{-1}(d) = X(X-iY)^{-1}(d)$. Thus, if $a, b \in E$ and $a = 2iY(x)$ with $x \in H$, then $C(a,b) = B(X(x),b)$. It follows directly from this that the spaces $E \cap k_1$ and $E \cap ik_1$ are orthogonal with respect to C. Since multiplication by i leaves the form C invariant, its restrictions to $E \cap k_1$ and $E \cap ik_1$ are isomorphic. Hence the number $\Psi(f_2, f_1, f_3)$ is equal to twice the signature of the restriction of C to $E \cap ik_1$. We show that this signature is equal to $\tau(k_2, k_1, k_3)$.

By definition, $\tau(k_2, k_1, k_3)$ is the signature of the form A on $(k_1+k_2) \cap k_3$, defined by the rule: if $a, b \in (k_1+k_2) \cap k_3$ and x is an element of the space k_1 such that $a - x \in k_2$, then $A(a,b) = B(x,b)$. We denote by p the projection $H = k_1 \oplus ik_1 \longrightarrow ik_1$. Since $k_3 = (X+iY)(k_1)$, $p((k_1+k_2) \cap k_3) = E \cap ik_1$. If $a, b \in (k_1+k_2) \cap k_3$ and $a = (X+iY)(x)$ with $x \in k_1$, then $C(p(a),p(b)) = C(iY(x), p(b)) = \frac{1}{2}B(X(x), p(b)) = \frac{1}{2}A(a,b)$. The assertion required follows from this.

3.3. LEMMA. Let $q: \tilde{G} \longrightarrow G$ be the projection and f be an element of the group G, such that $\operatorname{Ker}(f-1) = 0$. Then: (i) in some neighborhood of 1 in $\tilde{G}$ the map $F \longmapsto \varphi(f, q(F))$ is equal to $\Phi^{\cdot}$; (ii) if $F_0 \in q^{-1}(f)$, then Φ is constant in some neighborhood of the element F_0.

Proof. If $g, h \in G$ are sufficiently close to 1, then by Lemma 2.5, $\varphi(fg, h) = \varphi(f, h)$. Hence,

$$\varphi(g,h) = \varphi(f,g) + \varphi(f,h) - \varphi(f, gh).$$

Comparison of this formula with (1) shows that the map $F \longmapsto \Phi(F) - \varphi(f, q(F))$ is a local homomorphism of a neighborhood of 1 to $\mathbf{Z}$. Since $\tilde{G}$ is semisimple, this homomorphism is zero. (i) follows from this. Point (ii) follows from (i) and (ii).

3.4. Proof of Theorem 3 and (9). If two functions satisfy the conditions of the theorem, then their difference, say χ, is locally constant on the set of transverse pairs, and satisfies the relation $\chi(K_1, K_2) = \chi(K_1, K_3) - \chi(K_2, K_3)$. Since for two Lagrangian spaces one can find a third, transverse to both of them, it follows from this that χ is locally constant everywhere, and hence, constant and equal to zero.

To prove (9) and the existence of a function m, it suffices to prove that the map $\tilde{\Psi} \circ (\Gamma \times \Gamma): \tilde{\Lambda}^2 \longrightarrow \mathbf{Z}$ satisfies the conditions of Theorem 3. (2) follows from (4) and (8). The local constancy follows from Lemma 3.1 and point (ii) of Lemma 3.3.

3.5. Proof of Theorem 5. By Lemma 3.3, $\Phi(\gamma(h,d)) = \varphi(-1, h)$ for (h, d) in some neighborhood of the identity in $\tilde{G}_0$. By definition, $\varphi(-1, h)$ is equal to the signature of the form $(a,b) \longmapsto B((h-1)^{-1}(a) + a/2, b)$ in $(h-1)(H)$. The calculation of the values of this form on eigenvectors over $\mathbb{C}$ of the operators h, analogous to that done in point 2.6, shows that in some basis over $\mathbb{R}$ of the space H this form is given by a diagonal matrix, whose diagonal elements run twice over the set $\{-\operatorname{ctg}(\theta_r/2) \mid r = 1, \ldots, n;\ \theta_r \notin \pi\mathbf{Z}\}$. Hence, if $|\theta_1|, \ldots, |\theta_n| < \pi$ and if s of the numbers $\theta_1, \ldots, \theta_n$ are positive, and ζ are negative, then $\varphi(-1, h) = 2(b-s)$. It is easy to verify that if $|d|, |\theta_1|, \ldots, |\theta_n| < 1/n$, then the right side of (10) is also equal to $2(b-s)$. Thus, (10) is valid for (h, d) in some neighborhood of the identity. To prove (10) in complete generality it suffices to prove that if this formula is true for (h, d), then it is also true for $(h^2, 2d)$.

As follows directly from the definitions, $\varphi(h,h) = \varphi(-1,h)$. Hence $\varphi(h,h) = 2(\nu_1 + \ldots + \nu_n)$, where: $\nu_r = 0$, if the number $\operatorname{ctg}(\theta_r/2)$ is undefined or equal to zero; $\nu_r = 1$, if $\operatorname{ctg}(\theta_r/2) < 0$;

$\nu_t = -1$, if $\operatorname{ctg}(\theta_t/2) > 0$. It is easy to verify that $\nu_t = \mu(\theta_t) - 2\mu(\theta_t/2)$. By (1),

$$\Phi(\eta(h^2, 2d)) = 2\Phi(\eta(h,d)) - \varphi(h,h) - 2\left[\frac{2}{\pi}\left(\sum_{t=1}^{n}\theta_t - d\right) - 2\sum_{t=1}^{n}\mu(\theta_t/2)\right] - 2\sum_{t=1}^{n}\nu_t =$$
$$= \frac{2}{\pi}\left(\sum_{t=1}^{n} 2\theta_t - 2d\right) - 2\sum_{t=1}^{n}\mu(\theta_t).$$

APPENDIX. THE CLASS u IS THE CHERN CLASS

Let G be the group of real-linear automorphisms of the form $\operatorname{Im}\left(\sum_{t=1}^{n} z_t \bar{z}_t\right)$ in $\mathbb{C}^n$. Let $u \in H^2(G^\delta; \mathbb{Z})$ be the class corresponding to the universal covering $\tilde{G} \to G$ and the isomorphism $\mathbb{Z} \to \pi_1(G)$, defined by the form $\operatorname{Im}\left(\sum_{t=1}^{n} z_t \bar{z}_t\right)$ (cf. points 1.1 and 1.3). We consider the maps $j: BG^\delta \to BG$ and $e: BU(n) \to BG$, induced, respectively, by the identity map of the group G and the inclusion of the unitary group in G. Since $U(n)$ is a maximal compact subgroup of the group G, e is a homotopy equivalence. Let $p: BG \to BU(n)$ be a homotopy inverse.

<u>THEOREM.</u> If $c_1 \in H^2(BU(n); \mathbb{Z})$ is the first universal Chern class, then $(p \circ j)^*(c_1) = u$.

This theorem is well known (cf., e.g., [2]). Here is the outline of one possible proof. Since $H^2(B\tilde{U}(n); \mathbb{Z}) = 0$, where $\tilde{U}(n)$ is the universal covering of the group $U(n)$, the lift of the class $(p \circ j)^*(c_1)$ to $H^2(\tilde{G}^\delta; \mathbb{Z})$ is equal to 0. Hence, $(p \circ j)^*(c_1) = mu$, where $m \in \mathbb{Z}$. The equation $m = 1$ is proved with the help of reduction to S^1 (cf., points 2.6 and 2.7), considering the standard model of the space $B(S^1)^\delta$ and taking into account the fact that if $n = 1$ the class c_1 is the Euler class.

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CLASSIFICATION OF ORIENTED MONTESINOS LINKS BY INVARIANTS OF SPIN STRUCTURES

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UDC 515.162

In this paper we give the isotopy classification of oriented Montesinos links. The definition of the invariants of links needed for this and the proof of the classification theorem are based on a new construction, which establishes a correspondence between orientations of a link $\ell \subset S^n$ on the one hand, and spin structures on the two-sheeted branched covering of the sphere, branched over ℓ, on the other. New numerical invariants of spin structures on three-dimensional Seifert manifolds are introduced in the paper; these invariants are used to classify the Montesinos links.

The Montesinos links constitute an extensive class of links in the three-dimensional sphere, including, in particular, all pretzel links and links with two bridges. This class of links was introduced by Montesinos [11] in 1973 and was subjected to intensive investigation in the following decade (cf. [1-4, 12, 14]). In his unpublished paper [4], Bonahon gave the complete isotopy classification of Montesinos links in the category of nonoriented links. In the present paper we establish a theorem of isotopy classification for oriented Montesinos links. The definition of the invariants of oriented links necessary for this classification (and also the proof of the classification theorem) are based on a new construction, which leaves the realms of the theory of Montesinos links far behind. This construction establishes a correspondence between orientations of links, on the one hand, and spin structures on the two-sheeted branched covering of the sphere, branched over this link, on the other.

1. Montesinos Links

1. Terminology and Notation. We recall the definitions we need from knot theory. Let α and β be relatively prime integers with $\alpha > 1$ and let $a_1, \ldots, a_n$ be integers such that

$$\frac{\beta}{\alpha} = \cfrac{1}{a_1 + \cfrac{1}{-a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{(-1)^{n+1} a_n}}}}$$

A rational tangle of type (α, β) [rational tangle of type (α, β)] is the following one-dimensional submanifold of the three-dimensional ball (see Fig. 1). (In Fig. 1, $n=7$, $a_1=3$, $a_2=-2$, $a_3=-2$, $a_4=1$, $a_5=3$, $a_6=-3$, $a_7=-4$, $\alpha=545$, $\beta=152$.) It is known that a rational tangle of type

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR, Vol. 143, pp. 130-146, 1985.

