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THREE-DIMENSIONAL POINCARÉ COMPLEXES: HOMOTOPY CLASSIFICATION AND SPLITTING

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§1. Introduction

1.1. Poincaré complexes are homotopy analogues of closed manifolds. They were introduced by S. P. Novikov and by W. Browder at the beginning of the 1960's for the requirements of the surgery theory of simply-connected manifolds (see [1]). Later, the range of application of Poincaré complexes widened substantially. In particular, they play an important part in the surgery theory of non-simply-connected manifolds, which was developed by Wall [20].

Let us recall the definition of an n-dimensional Poincaré complex (the term "Poincaré complex of formal dimension n" is also used). Let X be a connected cell complex which is dominated by a finite cell complex. Let $w \in H^1(X; \mathbf{Z}/2)$. We denote the fundamental group of the complex X by π. We denote by $\mathbf{Z}^w$ the group $\mathbf{Z}$ furnished with the structure of a right $\mathbf{Z}[\pi]$-module in the usual way by means of the homomorphism $\alpha \mapsto (-1)^{w(\alpha)}: \pi \to \{\pm 1\}$. Let $[X] \in H_n(X; \mathbf{Z}^w)$. The triple $(X, w, [X])$ is called an *n-dimensional Poincaré complex* if for every left $\mathbf{Z}[\pi]$-module B the homomorphism

$$h \mapsto h \cap [X]: H^*(X; B) \to H_{n-*}(X; \mathbf{Z}^w \otimes_{\mathbf{Z}[\pi]} B)$$

is an isomorphism. For the details, the reader is referred to [19] (see also §4). The Poincaré complex $(X, w, [X])$ is usually denoted simply by X; the class w is denoted $w(X)$. The Poincaré complex X is called *finite* if the space X is homotopy equivalent to a finite cell complex. It is known that every closed topological manifold of dimension n has the homotopy type of a finite n-dimensional Poincaré complex.

In the topology of three-dimensional manifolds, Poincaré complexes do not play the same role as they do in high-dimensional topology. The point is that the methods of high-dimensional surgery theory are not applicable in dimension 3. On the other hand, as is clear from the results stated below, the class of three-dimensional Poincaré complexes can be described in purely algebraic terms. In view of this, the comparison of the class of three-dimensional Poincaré complexes with the class of homotopy types of closed three-dimensional manifolds is of particular interest.

It is not difficult to construct three-dimensional Poincaré complexes with finite fundamental groups which are not homotopy equivalent to three-dimensional manifolds. The existence of such complexes follows already from the fact that the class of

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finite groups which are fundamental groups of three-dimensional Poincaré complexes coincides with the class of finite groups with periodic cohomology of period 4 (see [19]). By results of Milnor [11] there exist finite groups with periodic cohomology of period 4 which are not realizable as the fundamental groups of three-dimensional manifolds. The simplest such group is the group of permutations of a set of three elements.

The supply of examples of three-dimensional Poincaré complexes (up to homotopy equivalence) at the present time runs to no more than the closed three-dimensional manifolds, the Poincaré complexes with finite fundamental group, and the connected sums of these. In fact, nothing contradicts the following conjecture: every finite three-dimensional Poincaré complex whose fundamental group has no torsion (that is, has no nontrivial elements of finite order) is homotopy equivalent to a closed three-dimensional manifold. It is even possible that the hypothesis of finiteness is superfluous here: there are at present no examples of nonfinite Poincaré complexes of dimension 3 with torsion-free fundamental groups. Note that even the analogous problems in two-dimensional topology proved to be very difficult. They were solved comparatively recently by Eckmann, Müller and Linnell, who proved that all two-dimensional Poincaré complexes are homotopy equivalent to closed two-dimensional manifolds (see [3] and [4]). Relying upon this theorem, Hillman [6], [7] and Thomas [14] obtained a series of results towards the conjecture formulated above.

1.2. The homotopy classification problem for three-dimensional Poincaré complexes was considered by Hendriks [5]. He constructed a complete system of homotopy invariants of a three-dimensional Poincaré complex X. This system consists of the fundamental group $\pi = \pi_1(X, x)$, where $x \in X$, the class $w(X) \in H^1(\pi; \mathbf{Z}/2)$, and the class $\mu(X) \in H_3(\pi; \mathbf{Z}^w)$ representing the image of the fundamental class $[X] \in H_3(X; \mathbf{Z}^w)$ under the homomorphism $H_3(X; \mathbf{Z}^w) \to H_3(\pi; \mathbf{Z}^w)$ induced by the natural map $X \to K(\pi, 1)$. We call the triple $(X, w(X), \mu(X))$ the *fundamental triple of the Poincaré complex* X, and say that X *realizes* $(\pi, w(X), \mu(X))$.

In the class of triples (group π, class $w \in H^1(\pi; \mathbf{Z}/2)$, class $\mu \in H_3(\pi; \mathbf{Z}^w)$) the relation of isomorphism is defined in the natural way: two triples (π, w, μ) and (G, v, η) are isomorphic if there exists an isomorphism $\varphi: \pi \to G$ such that the homomorphism $\varphi^*: H^1(G; \mathbf{Z}/2) \to H^1(\pi; \mathbf{Z}/2)$ carries v into w, and the homomorphism $\varphi_*: H_3(\pi; \mathbf{Z}^w) \to H_3(G; \mathbf{Z}^v)$ carries μ into η. It is easy to see that the fundamental triple of a three-dimensional Poincaré complex X does not depend, up to isomorphism, upon the choice of the distinguished point $x \in X$.

THEOREM 0 (Hendriks [5]). *Two three-dimensional Poincaré complexes are homotopy equivalent if and only if their fundamental triples are isomorphic.*

A homotopy equivalence of Poincaré complexes is understood to mean a homotopy equivalence which preserves the class w and the fundamental class.

For the convenience of the reader, a proof of this theorem (different from the proof given in [5]) is detailed in the Appendix to this paper. In the case of three-dimensional closed manifolds, Theorem 0 was first proved by Swarup [13].

In the author's paper [15] an algebraic characterization was given of the pairs (π, w) which correspond to three-dimensional Poincaré complexes. In the present article, that result is strengthened to give a characterization of the triples (π, w, μ) which correspond to three-dimensional Poincaré complexes. More precisely, we state algebraic conditions on the triple (group π, class $w \in H^1(\pi; \mathbf{Z}/2)$, class $\mu \in H_3(\pi; \mathbf{Z}^w)$) which are necessary and sufficient for its realizability as the fundamental triple of a three-dimensional Poincaré complex. The formulation of these realizability conditions is rather complex and requires the algebraic apparatus developed to frame its introduction (see §§2 and 3). The realization theorem is complementary to Theorem 0, and completely reduces the problem of homotopy classification of three-dimensional Poincaré complexes to a purely algebraic problem. This reduction allows one in particular to answer a series of questions concerning splittings of Poincaré complexes into connected sums. (The connected sum operation for Poincaré complexes was defined in [19]; this definition is reproduced in §3.)

THEOREM 1. *If the fundamental group of a three-dimensional Poincaré complex X is isomorphic to the free product of the groups G_1 and G_2, then X is homotopy equivalent to the connected sum of two three-dimensional Poincaré complexes with fundamental groups G_1 and G_2.*

This theorem represents a translation into the category of three-dimensional Poincaré complexes of the classical splitting theorem of Kneser and Stallings for three-dimensional manifolds. The question of whether this translation was possible was posed by Wall ([19], p. 235).

THEOREM 2. *If the fundamental group of a three-dimensional Poincaré complex X has no torsion, then X is homotopy equivalent to a connected sum of aspherical three-dimensional Poincaré complexes and a number of copies of the manifolds $S^1 \times S^2$ and $S^1 \tilde{\times} S^2$.*

Here $S^1 \tilde{\times} S^2$ is the total space of the nonorientable fibration over S^1 with fiber S^2. Recall that a cell complex is called *aspherical* if its universal cover is contractible.

The converse of Theorem 2 is also true (and well known):

If a Poincaré complex X is homotopy equivalent to a connected sum of aspherical three-dimensional Poincaré complexes and a number of copies of the manifolds $S^1 \times S^2$ and $S^1 \tilde{\times} S^2$, then the group $\pi_1(X)$ has no torsion.

We shall denote the relation of homotopy equivalence of topological spaces by the symbol $\simeq$. A three-dimensional Poincaré complex X is called *prime* if it is homotopically distinct from S^3 (i.e. $X \not\simeq S^3$) and if for any three-dimensional Poincaré complexes X_1 and X_2 such that $X \simeq X_1 \# X_2$, either $X_1 \simeq S^3$ or $X_2 \simeq S^3$. It follows from Theorem 1 that X is prime if and only if its fundamental group is nontrivial and does not split into a free product of nontrivial subgroups. (Note that for a three-dimensional Poincaré complex X the conditions $X \simeq S^3$ and $\pi_1(X) = 1$ are equivalent.)

Elementary group-theoretic considerations show that every three-dimensional Poincaré complex decomposes into a connected sum of prime three-dimensional Poincaré complexes. The following theorem shows that this decomposition is unique except for the known (and obvious) relation

$$X \# S^1 \times S^2 \simeq X \# S^1 \tilde{\times} S^2 \tag{1.2.1}$$

where X is a three-dimensional Poincaré complex with $w(X) \neq 0$.

THEOREM 3. *The decomposition of a three-dimensional Poincaré complex into a connected sum of prime three-dimensional Poincaré complexes is unique up to permutation of the summands, replacement of them by their homotopy equivalents, and modulo relation* (1.2.1), *where $w(X) \neq 0$.*

Theorems 1, 2, and 3 remain true if the words "Poincaré complex" in their statements are everywhere replaced by the words "finite Poincaré complex". This follows from Theorems 1, 2, 3, and the following fact: the connected sum of two Poincaré complexes X and Y is a finite Poincaré complex if and only if X and Y are finite Poincaré complexes. This fact is well known. It follows immediately from the additivity of the Wall invariant σ with respect to connected sum. (Recall that a Poincaré complex X is finite if and only if the invariant $\sigma(X) \in \overline{K}_0(\mathbf{Z}[\pi])$ introduced by Wall [17] takes the value zero, where π is the fundamental group of the space X and $\overline{K}_0(\mathbf{Z}[\pi])$ is the factor group of $K_0(\mathbf{Z}[\pi])$ by the infinite cyclic subgroup generated by the class of the module $\mathbf{Z}[\pi]$. Recall also that σ is invariant under homotopy equivalences.)

The results of this paper can be generalized to higher dimensions: more precisely, to the class of n-dimensional Poincaré complexes with $(n-2)$-connected universal coverings, where $n \geq 3$. This allows one to prove the analogues of Theorems 1, 2, and 3 for n-dimensional manifolds with $(n-2)$-connected universal coverings, when $n \geq 5$. The author intends to discuss related results elsewhere.

It would be interesting to generalize the results of this paper to the case of three-dimensional Poincaré pairs.

Theorems 1, 2, 3, and the realization theorem for fundamental triples were announced by the author in [16].

1.3. Plan of the paper. The necessary algebraic apparatus is introduced in §2. The realization theorem for fundamental triples of three-dimensional Poincaré complexes (Theorem 4) is stated in §3. Theorems 1, 2, and 3, are also deduced there from Theorem 4, as is the theorem on characteristic pairs (π, w) mentioned above. In §4, Theorem 4 is proved. In §5 a formula is given which permits one to calculate the Wall invariant of a three-dimensional Poincaré complex from the fundamental triple of the complex. In the Appendix Theorem 0 is proved.

§2. Homotopies of homomorphisms, the module $F^r(C)$, and the homomorphism ν_C

Until the end of the section, we fix an associative ring Λ with unit.

2.1. Homotopies of Λ-homomorphisms. In this subsection we review the elements we shall need later on from the homotopy theory of homomorphisms, which was developed by Hilton and Eckmann in the 1950's (see [8]).

Let A and B be (left) Λ-modules. The Λ-homomorphisms $f, g: A \to B$ are called *homotopic* if their difference $f-g$ can be represented as a composition $A \to \Lambda^m \to B$, where m is a natural number or zero. (As usual, Λ^m is the m-fold sum $\Lambda \oplus \Lambda \oplus \cdots \oplus \Lambda$.) One verifies without difficulty that homotopy is an equivalence relation on the set of homomorphisms. It is denoted by the symbol $\sim$. The set of homotopy classes of Λ-homomorphisms $A \to B$ is denoted by $[A, B]$. It is evident that addition of homomorphisms defines the structure of an abelian group on $[A, B]$.

A Λ-homomorphism $f: A \to B$ is called a Λ-*homotopy equivalence* if it possesses a homotopy inverse, i.e. if there exists a Λ-homomorphism $g: B \to A$ such that $gf \sim \mathrm{id}_A$ and $fg \sim \mathrm{id}_B$. Clearly, a homomorphism which is homotopic to a homotopy equivalence is also a homotopy equivalence. The (possibly empty) subset of $[A, B]$ consisting of the homotopy classes of homotopy equivalences is denoted by $\mathrm{Equi}(A, B)$.

An ample supply of Λ-homotopy equivalences is provided by the following simple observation: if A is an arbitrary Λ-module and P is a finitely generated projective Λ-module, then the inclusion $A \hookrightarrow A \oplus P$ and the projection $A \oplus P \to A$ are mutually homotopy inverse and therefore are homotopy equivalences. (Note that the class of finitely generated projective Λ-modules coincides with the class of direct summands of the modules Λ^m with $m = 1, 2, \ldots$.) The following lemma shows that every Λ-homotopy equivalence can be represented as a composite of isomorphisms and homotopy equivalences of the above form. (This lemma is dual to Theorem 13.7 of Hilton's book [8].)

2.2. Lemma. *Let A and B be Λ-modules, and let $f\colon A \to B$ be a homotopy equivalence. Then f can be represented in the form of a composite*

$$A \overset{i}{\hookrightarrow} A \oplus P \underset{\approx}{\overset{j}{\to}} B \oplus \Lambda^m \overset{\mathrm{pr}}{\to} B, \tag{2.2.1}$$

where P is a finitely generated projective Λ-module, i is the canonical inclusion, j an isomorphism over Λ, $m \in \{0, 1, 2, \ldots\}$, and pr *the projection.*

Proof. Let g be a homomorphism $B \to A$ such that $fg \sim \mathrm{id}_B$ and $gf \sim \mathrm{id}_A$. Suppose that $\mathrm{id}_A - gf = \beta\alpha$, where α and β are homomorphisms $A \to \Lambda^m$ and $\Lambda^m \to B$ respectively, and $m \geq 0$. We set $B' = B \oplus \Lambda^m$. We denote the homomorphisms $(f, \alpha)\colon A \to B'$ and $(g, \beta)\colon B' \to A$ by f' and g' respectively. We set $P = \operatorname{Ker} g'$. It is clear that $g'f' = \mathrm{id}_A$. Therefore f' is an injection and $B' = f'(A) \oplus P$.

From the fact that $fg \sim \mathrm{id}_B$, it follows that $f'g' \sim \mathrm{id}_{B'}$. Therefore $\mathrm{id}_{B'} - f'g' = \delta\gamma$, where γ and δ are homomorphisms $B' \to \Lambda^n$ and $\Lambda^n \to B'$ respectively, with $n \geq 0$. On the other hand, $f'g'|_P = 0$ and $f'g'|_{f'(A)} = \mathrm{id}_{f'(A)}$. Hence $\operatorname{Im}(\mathrm{id}_{B'} - f'g') \subset P$. Therefore the relation $\mathrm{id}_{B'} - f'g' = \delta\gamma$ remains true if δ is replaced by its composite with the projection $B' = f'(A) \oplus P \to P$. Thus we may assume that $\operatorname{Im}\delta \subset P$. In that case $\delta\gamma|_P = (\mathrm{id}_{B'} - f'g')|_P = \mathrm{id}_P$. That means that $\gamma|_P$ is an embedding $P \to \Lambda^n$, and $\Lambda^n = \gamma(P) \oplus \operatorname{Ker}\delta$. Therefore P is a finitely generated projective Λ-module.

Clearly f is the composition of the embedding $A \hookrightarrow A \oplus P$, the isomorphism $f' \oplus \mathrm{id}\colon A \oplus P \to f'(A) \oplus P = B \oplus \Lambda^m$, and the projection $B \oplus \Lambda^m \to B$. The lemma follows from this.

2.3. The module $F^r(C)$. We shall suppose that the ring Λ is provided with an antiautomorphism $\lambda \mapsto \bar{\lambda}\colon \Lambda \to \Lambda$. To every Λ-module A there corresponds a dual module $A^* = \operatorname{Hom}_\Lambda(A, \Lambda)$, where the Λ-module structure on A^* is defined by the rule that if $a \in A$, $x \in A^*$, and $\lambda \in \Lambda$, then $(\lambda x)(a) = x(a)\bar{\lambda}$. To every chain complex $C = (\cdots \to C_{r+1} \overset{\partial_r}{\to} C_r \to \cdots)$ over Λ, there corresponds a dual complex $C^* = (\cdots \leftarrow C^*_{r+1} \overset{\partial^*_r}{\leftarrow} C^*_r \leftarrow \cdots)$, where the homomorphism ∂^*_r is defined by the rule that if $a \in C_{r+1}$ and $x \in C^*_r$, then $\partial^*_r(x)(a) = (-1)^r \times (\partial_r(a))$. For integral r, we denote by $F^r(C)$ the Λ-module $\operatorname{Coker}\partial^*_{r-1} = C^*_r / \operatorname{Im}\partial^*_{r-1}$.

2.4. The homomorphism ν_C. We assume that the ring Λ is provided with an antiautomorphism $\lambda \mapsto \bar{\lambda}$ and an augmentation (ring homomorphism) aug: $\Lambda \to Z$. We denote by I the ideal Ker(aug) of the ring Λ, and we regard it as a left Λ-module. We denote by $\mathbf{Z}^t$ the group $\mathbf{Z}$ provided with a right Λ-module structure by the formula $z\lambda = \operatorname{aug}(\bar{\lambda})z$, where $z \in \mathbf{Z}$ and $\lambda \in \Lambda$.

Let $C = (\cdots \to C_{r+1} \to C_r \to \cdots)$ be a free (left) chain complex over Λ. The aim of this subsection is the construction of an additive homomorphism

$$\nu_{C,r}\colon H_{r+1}(\mathbf{Z}^t \otimes_\Lambda C) \to [F^r(C), I] \tag{2.4.1}$$

for every $r \in \mathbf{Z}$. In fact, we only need the homomorphism $\nu_{C,2}$. It will also be denoted by ν_C.

We define $\nu_{C,r}$. It is evident that the kernel of the homomorphism $\lambda \mapsto 1\otimes\lambda\colon \Lambda \to \mathbf{Z}^t \otimes_\Lambda \Lambda = \mathbf{Z}^t$ is equal to $\overline{I}$. Since the complex C is free, there exists an exact sequence

$$0 \to \overline{I}C \hookrightarrow C \overset{C\mapsto 1\otimes C}{\to} \mathbf{Z}^t \otimes_\Lambda C \to 0. \tag{2.4.2}$$

To this sequence there corresponds a boundary homomorphism

$$H_{r+1}(\mathbf{Z}^t \otimes_\Lambda C) \to H_r(\overline{I}C). \tag{2.4.3}$$

We shall define an additive homomorphism

$$H_r(\overline{I}C) \to [F^r(C), I]. \tag{2.4.4}$$

We note that every element c of the module C_r defines a Λ-homomorphism $x \mapsto \overline{x(c)}\colon C_r^* \to \Lambda$. If $\partial_{r-1}(c) = 0$, then this homomorphism can be factored through the projection $C_r^* \to F^r(C)$. We denote the resulting homomorphism $F^r(C) \to \Lambda$ by $\tilde{c}$. Now suppose $h \in H_r(\overline{I}C)$, and that $c \in \overline{I}C_r \subset C_r$ is a cycle representing h. Since $\partial_{r-1}(c) = 0$, the homomorphism $\tilde{c}\colon F^r(C) \to \Lambda$ is defined, and its image is contained in I. We denote the resulting homomorphism $F^r(C) \to I$ by c^v. We now show that the homotopy class of the homomorphism c^v does not depend upon the choice of the cycle c representing h. It suffices to show that if $d = c + \partial_r(\overline{\alpha}e)$, where $\alpha \in I$ and $e \in C_{r+1}$, then $d^v \sim c^v$. The latter follows from the fact that the difference $d^v - c^v$ is equal to the composition

$$F^r(C) \overset{\widetilde{\partial_r(e)}}{\to} \Lambda \overset{\lambda\mapsto\lambda\alpha}{\to} I.$$

The homomorphism (2.4.4) is defined by assigning to the class h the homotopy class of the homomorphism c^v.

The composition of the homomorphism (2.4.3) and (2.4.4) gives the homomorphism (2.4.1).

2.5. Lemma. *Suppose that $C = (\cdots \to C_{r+1} \to C_r \to \cdots)$ is a free chain complex over Λ. If the module C_r is finitely generated and $H_r(C) = H_{r+1}(C) = 0$, then the homomorphism $\nu_{C,r}$ is an isomorphism.*

Proof. From the exactness of the homology sequence associated with the exact sequence (2.4.2), it follows that the homomorphism (2.4.3) is an isomorphism. We shall show that the homomorphism (2.4.4) is an isomorphism. From this will follow the statement of the lemma. We first prove surjectivity. If φ is a homomorphism $F^r(C) \to I$, then the composition

$$C_r^* \overset{\mathrm{pr}}{\to} F^r(C) \overset{\varphi}{\to} I \hookrightarrow \Lambda$$

is given by a formula $x \mapsto \overline{x(c)}$, where c is some element of the module C_r. Since the homomorphism $x \mapsto \overline{x(c)}\colon C_r^* \to \Lambda$ factors through the projection $C_r^* \mapsto F^r(C)$, it follows that $\partial_{r-1}(c) = 0$. Since $\overline{x(c)} \in I$ for all $x \in C_r^*$, it follows that $c \in \overline{I}C_r$. It is apparent directly from the definition that the homotopy class of the homomorphism φ corresponds to the class $[c] \in H_r(\overline{I}C)$ under the homomorphism (2.4.4).

We now prove that the homomorphism (2.4.4) is injective. Let h be an element of the kernel of that homomorphism. Let $c \in \overline{I}C_r$ be a cycle representing h. Then $c^v \sim 0$, i.e. there exist homomorphisms $\alpha\colon F^r(C) \to \Lambda^m$ and $\beta\colon \Lambda^m \to I$ (where $0 \le m < \infty$) such that $c^v = \beta \circ \alpha$. We denote by β_i the image of the ith term of the canonical basis in Λ^m under the homomorphism β, so that $\beta_1, \dots, \beta_m \in I$. As above, there exist cycles $c_1, \dots, c_m \in C_r$ such that the composition

$$C_r^* \xrightarrow{\text{pr}} F^r(C) \xrightarrow{\alpha} \Lambda^m$$

is given by the formula $x \mapsto (\overline{c_1(x)}, \dots, \overline{c_m(x)})$. It follows from the equation $c^v = \beta \circ \alpha$ that $c = \sum_1^m \overline{\beta}_i c_i$. Since $H_r(C) = 0$ and $\partial_{r-1}(c_i) = 0$ for all i, there exist chains $d_1, \dots, d_m \in C_{r+1}$ such that $\partial_r(d_i) = c_i$ for all i. Then

$$c = \partial_r\left(\sum_{i=1}^{m} \overline{\beta}_i d_i\right) \in \partial_r(\overline{I}C_{r+1}).$$

This means that $h = 0$.

§3. Statement of the realization theorem. Applications

3.1. Statement of the realization theorem. Let π be a group, $w \in H^1(\pi; \mathbf{Z}/2)$, and $\mu \in H_3(\pi; \mathbf{Z}^w)$.

It is known (see [17]) that the fundamental group of a cell complex dominated by a finite cell complex is finitely presentable, i.e. can be given by a finite number of generators and relations. Therefore without loss of generality we shall assume that the group π is finitely presentable.

We set $\Lambda = \mathbf{Z}[\pi]$. We denote by aug the standard augmentation (summation of coefficients) $\Lambda \to \mathbf{Z}$. We denote by $I(\pi)$ the ideal Ker(aug) of the ring Λ, regarded as a left Λ-module.

Let $C = (\cdots \to C_2 \to C_1 \to C_0)$ be a free (left) chain complex over Λ having the following properties: $H_i(C) = 0$ for $i > 0$; $H_0(C) = \mathbf{Z}$, where $\mathbf{Z}$ is given the trivial Λ-module structure; and C_0, C_1, and C_2 are finitely generated (free) Λ-modules. Such a complex can be constructed, for example, from any cell complex of type $K(\pi, 1)$ which has a finite two-dimensional skeleton. It is evident that the complex C can be completed into a resolution of the trivial Λ-module $\mathbf{Z}$ by means of a homomorphism $C_0 \to \mathbf{Z}$. Applying the results of §2.4 to the augmentation aug and the involution of the ring Λ which takes $\alpha \in \pi$ into $(-1)^{w(\alpha)}\alpha^{-1}$, one defines a homomorphism

$$\nu_C\colon H_3(\pi; \mathbf{Z}^w) = H_3(\mathbf{Z}^w \otimes_\Lambda C) \to [F^2(C), I(\pi)].$$

Though it is not essential at the moment, we note that ν_C is an isomorphism because of Lemma 2.4.5.

THEOREM 4. *The triple* (π, w, μ) *is isomorphic to the fundamental triple of some three-dimensional Poincaré complex if and only if*

$$\nu_C(\mu) \in \mathrm{Equi}(F^2(C), I(\pi)). \tag{3.1.1}$$

(*For the definition of the set* Equi, *see* §2.1.)

Theorem 4 is proved in §4. In particular, it follows from Theorem 4 that the validity of the inclusion (3.3.1) does not depend upon the choice of the complex C. In addition, this fact is proved algebraically in the course of the proof of Theorem 4.

3.2. PROOF OF THEOREM 1. We recall first of all the definition of the connected sum of three-dimensional Poincaré complexes. According to Corollary 2.3.1 in [19] (see also [18], p. 137) every three-dimensional Poincaré complex can be obtained, to within homotopy equivalence, by gluing a three-dimensional cell e (along its boundary) to a "homologically two-dimensional" cell complex. Here a cell complex L is called *homologically two-dimensional* if the cellular chain $\mathbf{Z}[\pi_1(L)]$-complex of the universal covering $\widetilde{L}$ of the space L is chain homotopy equivalent to a two-dimensional chain complex. If $X = L \cup_f e$ and $X' = L' \cup_{f'} e$ are representations of this type for the three-dimensional Poincaré complexes X and X', then $X \# X' = (L \vee L') \cup_g e$, where the homotopy class of the map $g\colon \partial e \to L \vee L'$ is equal to the sum of the homotopy classes of the maps $f\colon \partial e \to L$ and $f'\colon \partial e \to L'$. In this situation $w(x \# X') = w(X) + w(X')$ and $[X \# X'] = [X] + [X']$. According to [19], Corollary 2.4.1, the operation $\#$ is well-defined on the set of homotopy types of three-dimensional Poincaré complexes.

Let (π, w, μ) be the fundamental triple of a Poincaré complex X. Suppose that $\pi = G_1 * G_2$. We denote by w_i the restriction of the class w in $H^1(G_i; \mathbf{Z}/2)$. The splitting $\pi = G_1 * G_2$ induces a splitting

$$H_3(\pi; \mathbf{Z}^w) = H_3(G_1; \mathbf{Z}^{w_1}) \oplus H_3(G_2; \mathbf{Z}^{w_2}).$$

Let $\mu = \mu_1 + \mu_2$, where $\mu_i \in H_3(G_i; \mathbf{Z}^{w_i})$ for $i = 1, 2$. We shall show that the triples (G_1, w_1, μ_1) and (G_2, w_2, μ_2) are realized by three-dimensional Poincaré complexes, let us say X_1 and X_2. From this the assertion of the theorem will follow: the connected sum $X_1 \# X_2$ realizes the triple $(G_1 * G_2, w_1 + w_2, \mu_1 + \mu_2) = (\pi, w, \mu)$, and this means, by Hendriks theorem, that the complex X is homotopy equivalent to $X_1 \# X_2$.

Since $G_1 * G_2 = \pi$, the groups G_1 and G_2 are finitely presentable. Let K_i be a cell complex, having one zero-dimensional cell and a finite two-dimensional skeleton, which is an Eilenberg-Mac Lane space of type $K(G_i, 1)$, where $i = 1, 2$. The cellular wedge $K = K_1 \vee K_2$ is an Eilenberg-Mac Lane space $K(\pi, 1)$. We denote by C^i the cellular chain $\mathbf{Z}[G_i]$-complex of the universal covering of the space K_i, where $i = 1, 2$. We denote by C the cellular chain $\mathbf{Z}[\pi]$-complex of the universal covering of the space K. It is evident that for $j \geq 1$

$$C_j = \alpha^1(C_j^1) \oplus \alpha^2(C_j^2),$$

where α^i is the change of rings functor corresponding to the inclusion $\mathbf{Z}[G_i] \to \mathbf{Z}[\pi]$. The boundary homomorphism $\partial_j\colon C_{j+1} \to C_j$ for $j \geq 1$ is here the direct sum of the homomorphisms $\alpha^1(\partial_j^1)$ and $\alpha^2(\partial_j^2)$. Hence

$$F^2(C) = \alpha^1(F^2(C^1)) \oplus \alpha^2(F^2(C^2)).$$

We note that

$$I(\pi) = \alpha^1(I(G_1)) \oplus \alpha^2(I(G_2)), \tag{3.2.1}$$

where the (canonical) inclusion

$$\alpha^i(I(G_i)) = \mathbf{Z}[\pi] \otimes_{\mathbf{Z}[G_i]} I(G_i) \to I(\pi)$$

is defined by the formula $z \otimes \lambda \mapsto z\lambda$, where $z \in \mathbf{Z}[\pi]$ and $\lambda \in I(G_i) \subset I(\pi)$. The equality (3.2.1) is well known; one proves it most simply by noting that

$$I(\pi) = \operatorname{Im}(\partial_0 \colon C_1 \to C_0) = \operatorname{Coker} \partial_1$$

and that, analogously, $I(G_i) = \operatorname{Coker} \partial_1^i$, $i = 1, 2$.

Suppose that $f_i \colon F^2(C_i) \to I(G_i)$ is a homomorphism representing the class $\nu_{C^i}(\mu_i)$, where $i = 1, 2$. It is immediately verified that the class $\nu_C(\mu) = \nu_C(\mu_1 + \mu_2)$ is represented by the homomorphism

$$\begin{array}{ccc} \alpha^1(F^2(C^1)) \oplus \alpha^2(F^2(C^2)) & \xrightarrow{\alpha^1(f_1) \oplus \alpha^2(f_2)} & \alpha^1(I(G_1)) \oplus \alpha^2(I(G_2)) \\ \| & & \| \\ F^2(C) & & I(\pi). \end{array}$$

In view of Theorem 4, it suffices, for the proof of realizability of the triple (G_i, w_i, μ_i) by a three-dimensional Poincaré complex, to show that f_i is a homotopy equivalence. For definiteness we shall assume that $i = 1$. Since the triple (π, w, μ) is realized by a Poincaré complex, it follows from Theorem 4 that the homomorphism $\alpha^1(f_1) \oplus \alpha^2(f_2)$ is a $\mathbf{Z}[\pi]$-homotopy equivalence. On account of Lemma 2.2, it then follows that for any finitely generated projective $\mathbf{Z}[\pi]$-modules P and Q there exists an isomorphism j which completes the following diagram and makes it commutative:

$$\begin{array}{ccc} \alpha^1(F^2(C^1)) \oplus \alpha^2(F^2(C^2)) & \xrightarrow{\alpha^1(f_1) \oplus \alpha^2(f_2)} & \alpha^1(I(G_1)) \oplus \alpha^2(I(G_2)) \\ \downarrow & & \nearrow \mathrm{pr} \\ \alpha^1(F^2(C^1)) \oplus \alpha^2(F^2(C^2)) \oplus P & \underset{\approx}{\xrightarrow{j}} & \alpha^1(I(G_1)) \oplus \alpha^2(I(G_2)) \oplus Q. \end{array} \tag{3.2.2}$$

To this diagram we apply the change of rings functor β which corresponds to the natural projection $\mathbf{Z}[\pi] = \mathbf{Z}[G_1 * G_2] \to \mathbf{Z}[G_1]$. It is clear that $\beta(P)$ and $\beta(Q)$ are finitely generated projective $\mathbf{Z}[G_1]$-modules. It is easy to verify that $\beta \circ \alpha^1 = \mathrm{id}$ and that the application of the functor $\beta \circ \alpha^2$ to a finitely generated $\mathbf{Z}[G_2]$-module gives a module of the form $H \otimes_{\mathbf{Z}} \mathbf{Z}[G_1]$, where H is a finitely generated abelian group. This last module is isomorphic to the direct sum of $\operatorname{rk} H$ copies of the module $\mathbf{Z}[G_1]$ and the module $\operatorname{Tors} H \otimes_{\mathbf{Z}} \mathbf{Z}[G_1]$, all elements of which have finite order over $\mathbf{Z}$. Thus the diagram (3.2.2) gives a commutative diagram of $\mathbf{Z}[G_1]$-modules and homomorphisms

$$\begin{array}{ccc} F^2(C^1) \oplus A \oplus B & \xrightarrow{f_1 \oplus (\beta \circ \alpha^2)(f_2)} & I(G_1) \oplus A' \oplus B' \\ \downarrow & & \nearrow \mathrm{pr} \\ F^2(C^1) \oplus A \oplus B \oplus \beta(P) & \underset{\approx}{\xrightarrow{\beta(j)}} & I(G_1) \oplus A' \oplus B' \oplus \beta(Q), \end{array} \tag{3.2.3}$$

where A and A' are finitely generated free $\mathbf{Z}[G_1]$-modules, and all the elements of the modules B and B' have finite order over $\mathbf{Z}$. It is obvious that the modules $I(G_1)$, $\beta(P)$, and $\beta(Q)$ have no elements of finite order over $\mathbf{Z}$. From the commutativity of the diagram (3.2.3) it follows that

$$\beta(j)(F^2(C^1)) \subset I(G_1) \oplus \beta(Q).$$

Since $\beta(j)$ is an isomorphism, it therefore follows that $\mathrm{Tors}_{\mathbf{Z}} F^2(C^1) = 0$. Thus the **Z**-torsion of the modules which enter into diagram (3.2.3) lies entirely in the modules B and B'. Therefore $\beta(j)|_B$ is an isomorphism $B \to B'$. On factoring out by B and B', we obtain the commutative diagram

$$\begin{tikzcd}
F^2(C^1) \oplus A \arrow[r, "f_1 \oplus f"] \arrow[d, hook] & I(G_1) \oplus A' \\
F^2(C^1) \oplus A \oplus \beta(P) \arrow[r, "\approx"] & I(G_1) \oplus A' \oplus \beta(Q), \arrow[u, "\mathrm{pr}"']
\end{tikzcd}$$

where f is some homomorphism $A \to A'$. It follows from the commutativity of the latter diagram that f_1 is the composite of the homomorphisms

$$F^2(C^1) \hookrightarrow F^2(C^1) \oplus A \oplus \beta(P) \xrightarrow{\approx} I(G_1) \oplus A' \oplus \beta(Q) \xrightarrow{\mathrm{pr}} I(G_1).$$

Therefore f_1 is a homotopy equivalence.

3.3. Proof of Theorem 2. We represent the group $\pi_1(X)$ in the form of a free product $G_1 * \cdots * G_n$, where each of the groups $G_1, \ldots, G_n$ is nontrivial and does not decompose as a free product. By Theorem 1, the complex X is homotopy equivalent to a connected sum $X_1 \# \cdots \# X_n$, where X_i is a three-dimensional Poincaré complex with $\pi_1(X_i) = G_i$ for $i = 1, \ldots, n$. If $G_i = \mathbf{Z}$, then X_i is homotopy equivalent to $S^1 \times S^2$ or to $S^1 \tilde{\times} S^2$. (This follows from Theorem 0, since there are two possibilities for the class $w \in H^1(\mathbf{Z}; \mathbf{Z}/2) = \mathbf{Z}/2$, and only one possibility for the class $\mu \in H_3(\mathbf{Z}; \mathbf{Z}^w) = 0$.) Suppose that $G_i \neq \mathbf{Z}$. We shall show that $\pi_j(X_i) = 0$ for $j \geq 2$. Let $\widetilde{X}_i \to X_i$ be the universal covering. By the Hurewicz and Poincaré duality theorems, $\pi_2(X_i) = H_2(\widetilde{X}_i) = H^1_f(\widetilde{X}_i)$, where H^1_f is the one-dimensional cohomology with compact supports. If $H^1_f(\widetilde{X}_i) \neq 0$, then the group G_i has not fewer than two ends (see [12], §1.A). By Stalling's theorem (see [12], §5.A), either G_i has nontrivial elements of finite order, or $G_i = \mathbf{Z}$, or G_i is a free product of nontrivial groups. The first, the second, and the third possibility all contradict the assumptions. Thus $\pi_2(X_i) = 0$. An inductive application of Poincaré duality shows that $\pi_j(X_i) = 0$ for all $j \geq 2$.

3.4. Proof of Theorem 3. Suppose that $X \simeq X_1 \# \cdots \# X_m \simeq Y_1 \# \cdots \# Y_n$ are two decompositions of the three-dimensional Poincaré complex X into a connected sum of prime three-dimensional Poincaré complexes. We shall show that $m = n$ and that the family $X_1, \ldots, X_m$ is obtained from $Y_1, \ldots, Y_n$ by the following operations: replacement of a space by a homotopy equivalent one; replacement of a pair X_i, $S^1 \times S^2$ with $w(X_i) \neq 0$ by the pair X_i, $S^1 \tilde{\times} S^2$; and replacement of a pair X_i, $S^1 \tilde{\times} S^2$ with $w(X_i) \neq 0$ by the pair X_i, $S^1 \times S^2$.

We shall assume that the complexes $X_1, \ldots, X_m$ are numbered in such a way that for some $k \leq m$ the fundamental groups of the complexes $X_1, \ldots, X_k$ are different from $\mathbf{Z}$, and the fundamental groups of $X_{k+1}, \ldots, X_m$ are isomorphic to $\mathbf{Z}$. Let (π, w, μ) be the fundamental triple of the complex X. The splitting $X \simeq X_1 \# \cdots \# X_m$ induces a decomposition $\pi = F_1 * \cdots * F_m$, where $F_i = \pi_1(X_i)$, so that $F_1, \ldots, F_k \neq \mathbf{Z}$ and $F_{k+1} = \cdots = F_m = \mathbf{Z}$.

In an analogous fashion the splitting $X \simeq Y_1 \# \cdots \# Y_n$ induces a decomposition $\pi = G_1 * \cdots * G_n$, where $G_i = \pi_1(Y_i)$ and it can be assumed that for some $l \leq n$ the groups $G_1, \ldots, G_l$ are different from $\mathbf{Z}$, and the groups $G_{l+1}, \ldots, G_n$ are isomorphic to $\mathbf{Z}$. Since the Poincaré complexes $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ are prime, it follows from Theorem 1 that the groups $F_1, \ldots, F_m$ and $G_1, \ldots, G_n$ cannot be decomposed as free products. By a famous theorem of Kurosh ([10], §35) it then follows that, first, $k = l$ and $m - k = n - l$ (so that $m = n$) and, second, one can set up a bijective correspondence between the families $F_1, \ldots, F_k$ and $G_1, \ldots, G_l$ so that corresponding groups are conjugate in π. We assume that the enumeration is chosen so that the group F_i is conjugate to G_i for $i = 1, \ldots, k$. Suppose that $G_i = a_i^{-1} F_i a_i$, where $a_i \in \pi$, $i = 1, \ldots, k$.

We shall show that for $i = 1, \ldots, k$ the Poincaré complexes X_i and Y_i are homotopy equivalent. We set $u_i = w|_{F_i} \in H^1(F_i; \mathbf{Z}/2)$. The splitting $\pi = F_1 * \cdots * F_m$ induces a direct sum decomposition

$$H_3(\pi; \mathbf{Z}^w) = g_1(H_3(F_1; \mathbf{Z}^{u_1})) \oplus \cdots \oplus g_m(H_3(F_m; \mathbf{Z}^{u_m})), \tag{3.4.1}$$

where g_i is the (injective) inclusion homomorphism $H_3(F_i; \mathbf{Z}^{u_i}) \mapsto H_3(\pi; \mathbf{Z}^w)$.

Let $\mu = g_i(\mu_1) + \cdots + g_m(\mu_m)$, where $\mu_i \in H_3(F_i; \mathbf{Z}^{u_i})$. It is obvious that the Poincaré complex X realizes the triple (F_i, u_i, μ_i). In analogous fashion, the splitting $\pi = G_1 * \cdots * G_m$ induces a decomposition

$$H_3(\pi; \mathbf{Z}^w) = h_1(H_3(G_1; \mathbf{Z}^{v_1})) \oplus \cdots \oplus h_m(H_3(G_m; \mathbf{Z}^{v_m})), \tag{3.4.2}$$

where $v_i = w|_{G_i} \in H^1(G_i; \mathbf{Z}/2)$ and h_i is the (injective) inclusion homomorphism $H_3(G_i; \mathbf{Z}^{v_i}) \mapsto H_3(\pi; \mathbf{Z}^w)$. Let $\mu = h_1(\xi_1) + \cdots + h_m(\xi_m)$, where $\xi_i \in H_3(G_i; \mathbf{Z}^{v_i})$. As above, the Poincaré complex Y_i realizes the triple (G_i, v_i, ξ_i).

In order to prove that $X_i \simeq Y_i$, it suffices to show that the isomorphisms

$$H^1(F_i; \mathbf{Z}/2) \to H^1(G_i; \mathbf{Z}/2), \tag{3.4.3}$$

$$H_3(F_i; \mathbf{Z}^{u_i}) \to H_3(G_i; \mathbf{Z}^{v_i}), \tag{3.4.4}$$

induced by the conjugation $x \mapsto a_i^{-1} x a_i \colon F_i \to G_i$, carry u_i into v_i and μ_i into ξ_i.

From the commutativity of the diagram

$$\begin{tikzcd} F_i \arrow[r, "x \mapsto a_i^{-1} x a_i"] \arrow[d, hook] & G_i \arrow[d, hook] \\ \pi \arrow[r, "x \mapsto a_i^{-1} x a_i"] & \pi \end{tikzcd} \tag{3.4.5}$$

and from the fact that inner automorphisms of the group π act as the identity on $H^1(\pi; \mathbf{Z}/2)$, it follows that the isomorphism (3.4.3) carries u_i into v_i. Analogous considerations show that

$$g_i(H_3(F_i; \mathbf{Z}^{u_i})) = h_i(H_3(G_i; \mathbf{Z}^{v_i})) \tag{3.4.6}$$

for $i \leq k$. If $i > k$, then $H_3(F_i; \mathbf{Z}^{u_i}) = H_3(G_i; \mathbf{Z}^{v_i}) = 0$. This means that (3.4.6) holds for all i, so that the decompositions (3.4.1) and (3.4.2) coincide. Hence $g_i(\mu_i) = h_i(\xi_i)$ for all i. On using once more the commutativity of diagram (3.4.5) and the fact that inner automorphisms of the group π act identically on $H_3(\pi; \mathbf{Z}^w)$, we therefore deduce that the isomorphism (3.4.4) takes μ_i into ξ_i. Therefore $X_i \simeq Y_i$ for $i \le k$.

Each of the complexes $X_{k+1}, \dots, X_m$ and $Y_{k+1}, \dots, Y_m$ is homotopy equivalent either to $S^1 \times S^2$ or to $S^1 \tilde{\times} S^2$. Therefore if $w(X) = 0$, then $X_i \simeq S^1 \times S^2 \simeq Y_i$ for all $i > k$. If $w(X) \ne 0$, then $w(X_i) \ne 0$ and $w(Y_j) \ne 0$ for some $i, j \ge 1$. Therefore, applying the operations referred to in the statement of the theorem, we can arrange that $X_i \simeq S^1 \tilde{\times} S^2 \simeq Y_i$ for all $i > k$. Thus $X_i \simeq Y_i$ for all i.

3.5. A characterization of the pairs (π, w) which correspond to Poincaré complexes. One says that an $m \times n$ matrix M over a ring Λ is a *matrix of relations* for the (left) Λ-module I, or that M *presents* I, if the module I is isomorphic to the cokernel of the homomorphism $\Lambda^m \to \Lambda^n$ defined by the matrix M. Two Λ-modules I and J are called *projectively equivalent* if for some finitely generated Λ-modules P and Q the modules $I \oplus P$ and $J \oplus Q$ are isomorphic.

THEOREM 5 ([15], Appendix 1, Theorem 2). *Let π be a group and $w \in H^1(\pi; \mathbf{Z}/2)$. Let $\lambda \mapsto \overline{\lambda}$ be the involution of the ring $\Lambda = \mathbf{Z}[\pi]$ which takes $\alpha \in \pi$ into $(-1)^{w(\alpha)}\alpha^{-1}$. The following two assertions are equivalent*:

(i) *There exists a three-dimensional Poincaré complex X such that the pair $(\pi_1(X), w(X))$ is isomorphic to (π, w).*

(ii) *The group π is finitely presentable, and for some* (*hence also for any*) *matrix M of relations for the Λ-module $I(\pi)$* (*introduced in* §3.1) *the matrix $\overline{M}^T$ presents a module which is projectively equivalent to the module $I(\pi)$.*

(Here $\overline{M}^T$ is the matrix obtained from M by transposing and elementwise application of the involution $\lambda \mapsto \overline{\lambda}$.)

PROOF. Let M be a matrix of relations for the module $I(\pi)$, and suppose it has m rows and n columns. The matrix M gives a Λ-homomorphism $\partial\colon \Lambda^m \to \Lambda^n$ with $\operatorname{Coker}\partial = I(\pi)$. The homomorphism ∂ determines the initial terms of a resolution of the trivial Λ-module $\mathbf{Z}$:

$$\begin{array}{ccccccccc} \cdots \longrightarrow C_3 & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & \mathbf{Z} \\ & & \| & & \| & & \| & & \| \\ & & \Lambda^m & \xrightarrow{\partial} & \Lambda^n & \longrightarrow & \Lambda & \xrightarrow{\text{aug}} & \mathbf{Z}. \end{array}$$

We denote the chain complex $(\cdots \to C_2 \to C_2 \to C_0)$ by C. It is evident that the module $F^2(C)$ is presented by the matrix $\overline{M}^T$.

If the pair (π, w) is realized by a three-dimensional Poincaré complex, then by Theorem 4 the set $\operatorname{Equi}(F^2(C), I(\pi))$ is nonempty. It follows from Lemma 2.2 that the Λ-module $F^2(C)$ and $I(\pi)$ are projectively equivalent.

Conversely, if $F^2(C)$ and $I(\pi)$ are projectively equivalent, then there exists a Λ-homotopy equivalence $F^2(C) \to I(\pi)$. From Theorem 4 and the surjectivity of the homomorphism ν_C (Lemma 2.4.5) it follows that for some $\mu \in H_3(\pi; \mathbf{Z}^w)$ the triple (π, w, μ) is realized by a three-dimensional Poincaré complex. In particular, this complex realizes (π, w).

§4. Proof of Theorem 4

4.1. Preparation for the proof. In this section, Λ denotes an associative ring with 1 equipped with an involutory antiautomorphism. The mapping constructed in §2.3, which assigns to a chain complex over Λ the module $F^r(C)$, extends in a natural fashion to a contravariant functor from the category of chain complexes over Λ and chain homomorphisms into the category of Λ-modules. This functor is also denoted by F^r. Its value on the chain homomorphism

$$\begin{array}{ccccccccc} C = (\cdots & \longrightarrow & C_{r+1} & \longrightarrow & C_r & \longrightarrow & C_{r-1} & \longrightarrow & \cdots) \\ f\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ C' = (\cdots & \longrightarrow & C_{r+1} & \longrightarrow & C'_r & \longrightarrow & C'_{r-1} & \longrightarrow & \cdots) \end{array}$$

is the homomorphism $F^r(C') \to F^r(C)$ induced by the homomorphism f.

4.1.1. Lemma. *Suppose that f, $g\colon C \to C'$ are chain homotopic homomorphisms of chain complexes over Λ. If the Λ-module C_r is finitely generated and free, then the homomorphisms $F^r(f)$ and $F^r(g)$ are homotopic.*

Proof. One can see directly from the definition of chain homotopy that the difference $F^r(f) - F^r(g)$ can be represented as a composition of the homomorphism $F^r(C') \to (C'_{r+1})^*$ induced by the boundary homomorphism $(C'_r)^* \to (C'_{r+1})^*$, some homomorphism $(C'_{r+1})^* \to C^*_r$, and the projection $C^*_r \to F^r(C)$. If $C_r = \Lambda^m$, then $C^*_r = \Lambda^m$, so that, by the definition of homotopy, $F^r(f) \sim F^r(g)$.

4.1.2. Lemma. *Suppose that $C = (\cdots \to C_1 \to C_0)$ and $C' = (\cdots \to C'_1 \to C'_0)$ are free chain complexes over Λ. Suppose that $r \geq 0$ and that $f\colon C \to C'$ is a chain homomorphism over Λ. If the modules C_i and C'_i are finitely generated for all $i \leq r$ and $f_*\colon H_i(C) \to H_i(C')$ is an isomorphism for all $i \leq r-1$, then $F^r(f)$ is a homotopy equivalence.*

Proof. We shall prove the lemma in the special case where $f_i\colon C_i \to C'_i$ is an isomorphism for all $i \leq r-1$. From the fact that $f_*\colon H_{r-1}(C) \to H_{r-1}(C')$ is an isomorphism, it is easy to deduce that the isomorphism $f_{r-1}\colon C_{r-1} \to C'_{r-1}$ carries $\partial_{r-1}(C_r)$ onto $\partial'_{r-1}(C'_r)$. Since the module C'_r is free, this implies the existence of a homomorphism $g\colon C'_r \to C_r$ such that $\partial'_{r-1} f_r g = \partial'_{r-1}$. We consider the diagram

$$\begin{array}{ccccccc} (C'_{r-1})^* & \xrightarrow{\mathrm{id}} & (C'_{r-1})^* & \underset{\approx}{\xrightarrow{f^*_{r-1}}} & C^*_{r-1} & \xrightarrow{\mathrm{id}} & C^*_{r-1} \\ \big\downarrow{\scriptstyle (\partial'_{r-1})^*} & & \big\downarrow{\scriptstyle ((\partial'_{r-1})^*,0)} & & \big\downarrow{\scriptstyle (\partial^*_{r-1},0)} & & \big\downarrow{\scriptstyle \partial^*_{r-1}} \\ (C'_r)^* & \hookrightarrow & (C'_r)^* \oplus C^*_r & \underset{\approx}{\xrightarrow{j}} & C^*_r \oplus (C'_r)^* & \xrightarrow{\mathrm{pr}} & C^*_r, \end{array}$$

where $j(x, y) = (f^*_r(x) + y,\ x - g^*(f^*_r(x) + y))$ for $x \in (C'_r)^*$ and $y \in C^*_r$. It is easy to verify that j is an isomorphism and that the diagram is commutative. We consider the cokernels of the homomorphisms corresponding to the vertical arrows of this diagram. The horizontal arrows induce homomorphisms of the cokernels

$$\mathrm{Coker}(\partial'_{r-1})^* \hookrightarrow \mathrm{Coker}(\partial'_{r-1})^* \oplus C'_r \xrightarrow{\approx} \mathrm{Coker}\,\partial^*_{r-1} \oplus (C'_r)^* \xrightarrow{\mathrm{pr}} \mathrm{Coker}\,\partial^*_{r-1}.$$

The composition of these three homomorphisms is equal to $F^r(f)$, as is easy to verify. Hence it follows that $F^r(f)$ is a homotopy equivalence.

The general case of the lemma is reduced to the one considered above by means of Schanuel's construction (see, for example, [2]). Specifically, we assume that for some number $n \leq r-1$ and for all $i < n$ the map $f_i\colon C_i \to C_i'$ is an isomorphism, and that $f_*\colon H_n(C) \to H_n(C')$ is an isomorphism. It is easy to see that the homomorphism $C_n/\operatorname{Im}\partial_n \to C_n'/\operatorname{Im}\partial_n'$ induced by the homomorphism f_n is an isomorphism. Since the module C_n' is free, there exists a homomorphism $g\colon C_n' \to C_n$ such that $\mathrm{id}_{C_n'} - fg = \partial_n' \circ l$, where l is some Λ-homomorphism $C_n' \to C_{n+1}'$. We consider the following chain complexes D and D' and the chain homomorphism $\overline{f}\colon D \to D'$:

$$\begin{array}{ccccccccccc}
D = \cdots & \longrightarrow & C_{n+2} & \xrightarrow{\partial_{n+1},0} & C_{n+1}\oplus C_n' & \xrightarrow{\partial_n,\mathrm{id}} & C_n \oplus C_n' & \xrightarrow{\partial_{n-1},0} & C_{n-1} & \longrightarrow & \cdots \\
\big\downarrow \overline{f} & & \big\downarrow f_{n+2} & & \big\downarrow h & & \big\downarrow j & & \big\downarrow f_{n-1} & & \\
D' = \cdots & \longrightarrow & C_{n+2}' & \xrightarrow{\partial_{n+1}',0} & C_{n+1}'\oplus C_n & \xrightarrow{\partial_n',\mathrm{id}} & C_n' \oplus C_n & \xrightarrow{\partial_{n-1}',0} & C_{n+1}' & \longrightarrow & \cdots,
\end{array}$$

where $\overline{f}_i = f_i$ for $i \neq n$, $n+1$, $j(c, c') = (c' + f_n(c - g(c')), c - g(c'))$, and $h(c, c') = (f_{n+1}(c) + l(c'), \partial_n(c) - g(c'))$. The complexes C and C' are regarded as direct summands in D and D' respectively in the obvious fashion. The following diagram is then commutative:

$$\begin{array}{ccc}
C & \hookrightarrow & D \\
\big\downarrow f & & \big\downarrow \overline{f} \\
C' & \xleftarrow{\mathrm{pr}} & D'.
\end{array} \tag{4.1.3}$$

By using the fact that C_n and C_n' are finitely generated free modules, it is easy to verify that the inclusion $C \to 0$ and the projection $D' \to C'$ induce homotopy equivalences of the modules F^i for all i. Therefore, in view of the commutativity of diagram (4.1.3), the assertion of the lemma for f is equivalent to the analogous assertion for $\overline{f}$. The homomorphism $\overline{f}_i\colon D_i \to D_i'$ is an isomorphism for $i < n+1$. In this way, beginning with $n = 0$, we can reduce the assertion of the lemma to the case examined above.

4.2. Proof of the theorem. We shall first prove that the validity of the inclusion relation (3.1.1) does not depend upon the choice of the complex C. In fact, any two resolutions of the module $\mathbf{Z}$ are connected by a chain homomorphism

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & \mathbf{Z} \\
& & \big\downarrow f_2 & & \big\downarrow f_1 & & \big\downarrow f_0 & & \big\downarrow \mathrm{id} \\
\cdots & \longrightarrow & C_2' & \longrightarrow & C_1' & \longrightarrow & C_0' & \longrightarrow & \mathbf{Z}.
\end{array}$$

It follows immediately from the definition of the homomorphism $\nu = \nu_C$ and $\nu' = \nu_{C'}$ that $\nu'(\mu) = \nu(\mu) \cdot F^2(f)\colon F^2(C') \to I(\pi)$. If the modules C_i and C_i' are finitely generated for $i = 0, 1, 2$, then $F^2(f)$ is a homotopy equivalence by virtue of Lemma 4.1.2. Therefore $\nu'(\mu)$ is a homotopy equivalence if and only if $\nu(\mu)$ has this property.

We now prove the necessity of the condition in the theorem. On account of what was said at the outset of the proof, it is sufficient to prove (3.1.1) for any one complex C which satisfies the conditions of §3.1. We assume that (π, w, μ) is the fundamental triple of a three-dimensional Poincaré complex X. Since X is dominated by a finite cell complex, the space X is homotopy equivalent to a cell complex with finite skeleta of all dimensions (see [17], Theorem A). Therefore we may assume immediately that X has a finite two-dimensional skeleton. We shall also assume that X has exactly one zero-dimensional cell.

By gluing cells of dimension ≥ 3 to X, we can obtain an Eilenberg-Mac Lane space of type $K(\pi, 1)$. We denote this space by K. We denote the cellular chain $\mathbf{Z}[\pi]$-complexes of the universal coverings of the spaces X and K by $D = (\cdots \to D_1 \to D_0)$ and $C = (\cdots \to C_1 \to C_0)$ respectively. Clearly D is a subcomplex of C and the "two-dimensional skeleta" of the complexes C and D coincide. In particular, $F^2(C) = F^2(D)$. We shall show that $\nu_C(\mu) \in \mathrm{Equi}(F^2(C), I(\pi))$.

We select a zero-dimensional cell pt in the universal covering space of K, and we identify the module C_0 with $\Lambda = \mathbf{Z}[\pi]$ by identifying $[pt]$ with 1. We set $I = \mathrm{Coker}(\partial_1 : C_2 \to C_1)$. Since $H_1(C) = 0$, the boundary homomorphism $\partial_0 : C_1 \to C_0$ induces an isomorphism $I \to I(\pi)$. We denote this isomorphism by Δ.

Let x be an element of the Λ-module D_3 whose image under the homomorphism

$$a \mapsto 1 \otimes a : D_3 \to \mathbf{Z}^w \otimes_\Lambda D_3 \tag{4.2.1}$$

is a cycle representing $[X]$. We denote by $\cap x$ the chain homomorphism $d \mapsto d \cap x : D^* \to D$. We recall the definition of this homomorphism. Let $g : D \to D \otimes_{\mathbf{Z}} D$ be an approximation to the diagonal and let $g(x) = \sum_1^m \alpha_i \otimes \beta_i$, where α_i and β_i are chains of the complex D. Then $d \cap x = \sum_1^m \overline{d(\alpha_i)} \beta_i$, where the bar denotes the involution of the ring Λ which corresponds to the class w (see, for instance, 3.1), and where $d(\alpha_i) = 0$ if $\dim \alpha_i \neq \dim d$. As usual, the chain homotopy class of the homomorphism $\cap x$ does not depend upon the choice of x and g.

Since X is a Poincaré complex, the homomorphism $\cap x$ is a chain homotopy equivalence (see [19], p. 215). The homomorphism $\cap x$ induces a Λ-homomorphism $(\cap x)_* : F^2(D) \to I$. It is easy to deduce from Lemma 4.1.1 and the finite generation of the modules D_0, D_1, and D_2 that $(\cap x)_*$ is a homotopy equivalence (see §2.1): a homotopy inverse for this homomorphism is induced by a chain homomorphism $D \to D^*$ which is a homotopy inverse for $\cap x$.

We shall show that the following diagram is homotopy commutative:

$$\begin{array}{ccc} F^2(D) & \xrightarrow[=]{\mathrm{id}} & F^2(C) \\ \big\downarrow {\scriptstyle (\cap x)_*} & & \big\downarrow {\scriptstyle \nu_C(\mu)} \\ I & \xrightarrow[\approx]{\Delta} & I(\pi). \end{array} \tag{4.2.2}$$

Let $[d]$ be the element of the module $F^2(D)$ represented by the cochain $d \in D_2^*$. Since $\cap x$ is a chain homomorphism,

$$(\Delta \circ (\cap x)_*)([d]) = \partial_0(d \cap x) = \partial_2^*(d) \cap x.$$

Let $g(x) = y \otimes [pt] + z$, where $y \in D_3$ and $z \in D_2 \otimes D_1 + D_1 \otimes D_2 + D_0 \otimes D_3$. By replacing the approximation to the diagonal g by a chain homotopic homomorphism if necessary, we may assume that the image of the chain y under the homomorphism (4.2.1) is a cycle representing $[X]$. (For the singular chain complex and the Alexander-Whitney diagonal approximation, the corresponding assertion is obvious. The transition from the singular complex into the cellular one is standard.) Thus

$$\partial_2^*(d) \cap x = \overline{\partial_2^*(d)(y)} = \overline{d(\partial_2(y))}.$$

According to the definitions, the homomorphism $d \mapsto \overline{d(\partial_2(y))}: F^2(D) \to I(\pi)$ represents the homotopy class $\nu_C(\mu)$. This proves the homotopy commutativity of diagram (4.2.2). Since $(\cap x)_*$ is a homotopy equivalence, $\nu_C(\mu)$ is also a homotopy equivalence. Thus the necessity of the condition in the theorem is proved.

We prove sufficiency. Let K be a cell complex which is an Eilenberg-Mac Lane space of type $K(\pi, 1)$ and has one zero-dimensional cell and a finite two-dimensional skeleton $K^{(2)}$. We denote by $p: \widetilde{K} \to K$ the universal covering of the space K, and by $E = (\cdots \to E_1 \to E_0)$ the cellular chain Λ-complex of the space $\widetilde{K}$. As above, the augmentation completes the complex E to a resolution of the module $\mathbf{Z}$. By reason of what was said at the beginning of the proof of the theorem, it follows from the inclusion (3.1.1) that $\nu_E(\mu) \in \operatorname{Equi}(F^2(E), I(\pi))$. Let $h: F^2(E) \to I(\pi)$ be a homomorphism representing $\nu_E(\mu)$. It follows from Lemma 2.2 that h can be represented as a composition

$$F^2(E) \hookrightarrow F^2(E) \oplus \Lambda^m \overset{\approx}{\to} I(\pi) \oplus P \overset{\mathrm{pr}}{\to} I(\pi), \tag{4.2.3}$$

where $0 \le m < \infty$ and P is a finitely generated projective Λ-module. We replace the space K by a wedge of that space and m three-dimensional balls, where the latter have a cell decomposition consisting of one zero-dimensional cell, m two-dimensional cells and m three-dimensional ones. Then $K^{(2)}$ is replaced by a wedge of $K^{(2)}$ and m two-dimensional spheres, and the module F^2 of the corresponding chain complex of the wedge is equal to $F^2(E) \oplus \Lambda^m$. Therefore we may assume straightaway that h is the composition of some isomorphism $j: F^2(E) \to I(\pi) \oplus P$ and the projection $I(\pi) \oplus P \to I(\pi)$.

To begin with we consider the case when P is a free module: $P = \Lambda^n$ with $0 \le n < \infty$. We denote by φ the composition of the projection $E_2^* \to F^2(E)$, the isomorphism j, and the inclusion $I(\pi) \oplus P \hookrightarrow \Lambda \oplus P$. We consider the dual homomorphism $\varphi^*: (\Lambda \oplus P)^* \to E_2$. (As usual, $(E_2^*)^*$ is identified with E_2 by means of the isomorphism taking $e \in E_2$ into the homomorphism $a \mapsto \overline{a(e)}: E_2^* \to \Lambda$.) From the equation $\varphi \circ \partial_1^* = 0$ it follows that $\partial_1 \circ \varphi^* = 0$. This means that

$$\operatorname{Im} \varphi^* \subset \operatorname{Ker} \partial_1 = H_2(p^{-1}(K^{(2)})) = \pi_2(K^{(2)}).$$

Therefore we can glue $n+1$ three-dimensional cells to $K^{(2)}$ in such a way that the cellular chain complex of the universal covering of the cell complex so obtained is equal to

$$\begin{array}{c} (\Lambda \oplus P)^* \xrightarrow{\varphi^*} E_2 \xrightarrow{\partial_1} E_1 \xrightarrow{\partial_0} E_0 . \\ \| \qquad\qquad\qquad\qquad\qquad\qquad \\ \Lambda^{n+1} \qquad\qquad\qquad\qquad\qquad\qquad \end{array} \tag{4.2.4}$$

We denote by X the (connected finite three-dimensional) cell complex so obtained. Since $\pi_2(K) = 0$, the inclusion $K^{(2)} \hookrightarrow K$ extends to a continuous map $X \to K$. We denote it by f. It is clear that f induces an isomorphism of fundamental groups. We identify these groups using this isomorphism. Then

$$w \in H^1(\pi; \mathbf{Z}/2) = H^1(X; \mathbf{Z}/2).$$

We shall show that $H_3(X; \mathbf{Z}^w) = \mathbf{Z}$. We denote the chain complex (4.2.4) by D. It is easy to verify that

$$\mathbf{Z}^w \otimes_\Lambda D = \operatorname{Hom}_\Lambda(D^*, \mathbf{Z}).$$

From this equality and the definitions of the homomorphism φ and of the complex D, it follows that
$$H_3(X;\mathbf{Z}^w) = H_3(\mathrm{Hom}_\Lambda(D^*, \mathbf{Z})) = \mathbf{Z}.$$

Let $[X]$ be a generator of the group $H_3(X;\mathbf{Z}^w)$. We show that $(X, w, [X])$ is a three-dimensional Poincaré complex, and that $f_*([X]) = \pm\mu \in H_3(K;\mathbf{Z}^w)$. This will signify that the triple (π, w, μ) is realized either by the Poincaré complex $(X, w, [X])$ or by the Poincaré complex $(X, w, -[X])$.

We now verify that the homomorphism $\cap[X]\colon H^i(D^*) \to H_{3-i}(D)$, defined by $a \mapsto a\cap[X]$, is an isomorphism for all i. The following argument is analogous to the one given in [16]. If $a \in H^3(D^*) = \mathbf{Z}$, then $a\cap[X] = \langle a, [X]\rangle \in H_0(D) = \mathbf{Z}$, where the brackets $\langle\ ,\ \rangle$ denote the Kronecker index. It is clear that if a is a generator of the group $H^3(D^*)$, then $\langle a, [X]\rangle = \pm 1$. Therefore $\cap[X]\colon H^3(D^*) \to H_0(D)$ is an isomorphism. That the homomorphism $\cap[X]\colon H^2(D^*) \to H_1(D)$ is an isomorphism follows from the equations $H^2(D^*) = H_1(D) = 0$. Since the chain homomorphism $\cap[X]\colon D^* \to D$ induces isomorphisms $H^3(D^*) \to H_0(D)$ and $H^2(D^*) \to H_1(D)$, the dual chain homomorphism $(\cap[X])^*\colon D^* \to D$ induces isomorphisms $H^0(D^*) \to H_3(D)$ and $H^1(D^*) \to H_2(D)$. As we know, the homomorphisms $\cap[X]$ and $(\cap[X])^*$ are chain homotopic. Therefore $\cap[X]\colon H^i(D^*) \to H_{3-i}(D)$ is an isomorphism for all i. Hence X is a Poincaré complex (see [19], p. 215).

From the functoriality of the construction of the homomorphism ν (see §2.4) it follows that the diagram
$$\begin{array}{ccc} F^2(E) & \xrightarrow{\nu_E(f_*([X]))} & I(\pi) \\ \Big\downarrow{\scriptstyle\mathrm{id}} & & \Big\downarrow{\scriptstyle\mathrm{id}} \\ F^2(D) & \xrightarrow{\nu_D([X])} & I(\pi) \end{array}$$
is homotopy commutative. It is easy to see that the class $[X] \in H_3(\mathbf{Z}^w \otimes D)$ is represented by the chain $\pm 1 \otimes x$, where
$$x = (1, 0) \in \Lambda \oplus P^* = (\Lambda \oplus P)^* = D_3.$$
From this it follows immediately that $\nu_D([X])$ is the homotopy class of the homomorphism $\pm h$. This means that
$$\nu_E(f_*([X])) = \nu_D([X]) = \pm\nu_E(\mu).$$
By Lemma 2.4.5, ν_E is an isomorphism. Hence $f_*([X]) = \pm\mu$.

In the general case, when P is a finitely generated projective module, the arguments are analogous, with the following modification: instead of the complex (4.2.4) it behooves us to consider the chain complex
$$(\Lambda \oplus P)^* \oplus A^\infty \xrightarrow{\varphi^*\oplus\mathrm{id}} E_2 \oplus \Lambda^\infty \xrightarrow{\partial_1, 0} E_1 \to E_0. \tag{4.2.5}$$
The chain modules of this complex are free: if Q is a module such that $P^* \oplus Q = \Lambda^n$ with $0 \le n < \infty$, then $P^* \oplus \Lambda^\infty = P^* \oplus (Q \oplus P^* \oplus Q \oplus P^* \oplus \cdots) = \Lambda^\infty$. The only additional point in which our arguments are lacking in the general case concerns the necessity of verifying that X is dominated by a finite cell complex. Since X is a finite-dimensional cell complex ($\dim X = 3$), it follows from results of Wall [17] that, in order to prove X is dominated by a finite cell complex, one need only show

that X is homotopy equivalent to a cell complex all of whose skeleta are finite. This last property is a consequence of Theorem 2 of Wall's paper [18] and the fact that the chain complex (4.2.5) is homotopy equivalent to the chain complex

$$\cdots \to \Lambda^n \xrightarrow{q} \Lambda^n \xrightarrow{p} \Lambda^n \xrightarrow{0,0,q} \Lambda \oplus P^* \oplus Q \xrightarrow{\varphi^*,0} E_2 \xrightarrow{\partial_1} E_2 \xrightarrow{\partial_0} E_0,$$

where p and q are the projections on P^* and Q in the decomposition $\Lambda^n = P^* \oplus Q$.

§5. Calculation of the Wall invariant

5.1. LEMMA. *Under the hypotheses of Lemma 2.2, the class* $[P]$ *of the module* P *in* $\overline{K}_0(\Lambda)$ *depends only upon the homotopy class of the homomorphism* f.

From this lemma and Lemma 2.2 it follows that assigning to a homotopy equivalence $f\colon A \to B$ the class $[P] \in \overline{K}_0(\Lambda)$ defines a map $\mathrm{Equi}(A, B) \to \overline{K}_0(\Lambda)$. This map is denoted by θ. It is easy to see that $\theta(f) = 0$ if and only if f can be factored into a composition (2.2.1) where P is a free finitely generated module.

PROOF OF THE LEMMA. It suffices to prove the following assertion:

If P and P' are finitely generated projective Λ-modules and if $g\colon A \oplus P \to A \oplus P'$ is a Λ-isomorphism such that the composition

$$A \overset{i}{\hookrightarrow} A \oplus P \underset{\approx}{\xrightarrow{g}} A \oplus P' \xrightarrow{\mathrm{pr}} A$$

is homotopic to id_A, *then* $[P] = [P']$.

We shall assume that $\mathrm{id}_A - \mathrm{pr} \circ g \circ i = \beta \circ \alpha$, where α and β are homomorphisms $A \to \Lambda^n$ and $\Lambda^n \to A$ respectively, with $0 \le n < \infty$. We consider the homomorphism

$$h\colon A \oplus P \oplus \Lambda^n \to A \oplus P' \oplus \Lambda^n,$$

defined by

$$h(a, x, \lambda) = (g(a, x), 0) + (\beta(\alpha(a) + \lambda), 0, \alpha(a) + \lambda).$$

We shall show that h is an isomorphism. If $(a, x, \lambda) \in \operatorname{Ker} h$, then $\alpha(a) + \lambda = 0$, so that $h(a, x, \lambda) = (g(a, x), 0)$. Since g is an isomorphism, it follows from this that $a = 0$, $x = 0$, and $\lambda = -\alpha(a) = 0$. Thus h is injective. If $(b, y, \lambda) \in A \oplus P' \oplus \Lambda^n$, then $(b - \beta(\lambda), y) = g(a, x)$ for some $a \in A$ and $x \in P$. Then

$$(b, y, \lambda) = (g(a, x), 0) + (\beta(\lambda), 0, \lambda) = h(a, x, \lambda - \alpha(a)).$$

Thus h is surjective.

We note that $h(a, 0, 0) = (g(a, 0), 0) + (\beta\alpha(a), 0, \alpha(a))$. Thus by the choice of the homomorphisms α and β it follows that the homomorphism $h|_A$ is a section of the projection $A \oplus P' \oplus \Lambda^n \to A$. Therefore

$$P \oplus \Lambda^n = (A \oplus P \oplus \Lambda^n)/A \approx (A \oplus P' \oplus \Lambda^n)/h(A) \approx P' \oplus \Lambda^n.$$

It follows from this that $[P] = [P']$.

5.2. THEOREM 6. *Let* (π, w, μ) *be the fundamental triple of a three-dimensional Poincaré complex* X. *Let* $\Lambda = \mathbf{Z}[\pi]$. *If* C *is a chain complex satisfying the conditions of* §3.1, *then*

$$\sigma(X) = (\theta(\nu_C(\mu)))^*,$$

where * *is the involution* $[P] \mapsto [P^*]$ *of* $\overline{K}_0(\Lambda)$.

This theorem is proved in §5.4.

5.3. COROLLARY. *Let π, w, μ, and C be the same as in* §3.1. *The triple (π, w, μ) is isomorphic to the fundamental triple of a finite three-dimensional Poincaré complex if and only if the inclusion* (3.1.1) *is satisfied and* $\theta(\nu_C(\mu)) = 0$.

From this corollary it is easy to deduce an analogue of Theorem 5 (see §3.5) for finite three-dimensional Poincaré complexes. In the corresponding variant of condition (ii) of Theorem 5, one has to require that the matrix $\overline{M}^T$ presents a Λ-module J such that for some natural numbers m and n the modules $I(\pi) \oplus \Lambda^m$ and $J \oplus \Lambda^n$ are isomorphic (cf. [15], Theorem C).

5.4. PROOF OF THEOREM 6. We recall the definition of the Wall invariant $\sigma(X)$ (see [17]). Let $D = (\cdots \to D_2 \to D_1 \to D_0)$ be the cellular chain Λ-complex of the universal covering of the space X. From the fact that X is dominated by a finite cell complex, it follows that the complex D is homotopy equivalent to some chain Λ-complex of finite length

$$\cdots \to 0 \to P_n \to P_{n-1} \to \cdots \to P_0, \tag{5.4.1}$$

where $P_0, P_1, \dots, P_n$ are finitely generated projective Λ-modules. The class

$$\sum_0^n (-1)^i [P_i] \in \overline{K}_0(\Lambda)$$

does not depend on the choice of the complex (5.4.1) within its homotopy class. This class is $\sigma(X)$.

Replacing X by a homotopy equivalent space if necessary, we may assume that X has a finite three-dimensional skeleton and exactly one zero-dimensional cell (cf. the proof of Theorem 4). In particular, $D_0 = \Lambda$. The following argument, due to Wall [18], allows us to compute $\sigma(X)$ explicitly. From the equality $H^4(X; B) = 0$, where $B = D_4/\operatorname{Im}(\partial_4 \colon D_5 \to D_4)$, it follows that the projection $D_4 \to B$ factors through the boundary homomorphism $\partial_3 \colon D_4 \to D_3$; that is, it is equal to $s \circ \partial_3$, where s is some homomorphism $D_3 \to B$. Then $D_3 \approx B \oplus \operatorname{Ker} s$. Since D_3 is a finitely generated free module, B and $\operatorname{Ker} s$ are finitely generated projective modules. It is clear that the chain complex D splits into the direct sum of two complexes

$$\cdots \to D_5 \to D_4 \xrightarrow{\text{pr}} B \to 0 \to 0 \to 0 \quad \text{and} \quad 0 \to \operatorname{Ker} s \to D_2 \to D_1 \to D_0 .$$

Since the space X is homotopy equivalent to a three-dimensional cell complex (see [19], Theorem 2.2), the first of these chain complexes is acyclic and therefore contractible. Therefore D is homotopy equivalent to the second of the given complexes. Hence $\sigma(X) = -[\operatorname{Ker} s]$.

Let I, Δ, and $\cap x \colon D^* \to D$ be the objects introduced in §4.2 in the course of proving the inclusion (3.1.1). We set $J = \operatorname{Ker}(\partial_3^* \colon D_3^* \to D_4^*) = (\operatorname{Ker} s)^*$. Since $H^2(D^*) = H_1(D) = 0$, the homomorphism ∂_2^* induces an inclusion $F^2(D) \hookrightarrow J$. Since the homomorphism $\cap x$ induces an isomorphism $H^3(D^*) \to H_0(D) = \mathbf{Z}$, the following diagram is commutative, and its rows are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^2(D) & \hookrightarrow & J & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\cap x)_*} & & \downarrow{\scriptstyle (\cap x)_j} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & I & \xrightarrow{\Delta} & \Lambda & \longrightarrow & \mathbf{Z} & \longrightarrow & 0.
\end{array}$$

The construction of Schanuel allows us to represent the homomorphism $(\cap x)_*: F^2(D) \to I$ as a composition

$$F^2(D) \hookrightarrow F^2(D) \oplus \Lambda \xrightarrow{\approx} I \oplus J \xrightarrow{\mathrm{pr}} I$$

(cf. the proof of Lemma 4.1.2). Hence $\theta((\cap x)_*) = -[J]$. From the homotopy commutativity of diagram (4.2.2) it follows that $\theta((\cap x)_*) = \theta(\nu_C(\mu))$, where C is the chain complex constructed in the course of proving (3.1.1). Thus

$$\sigma(X) = -[\operatorname{Ker} s] = -[J^*] = (\theta(\nu_C(\mu)))^*.$$

By using the argument explained in the course of proving Lemma 4.1.2, one can easily verify that $\theta(\nu_C(\mu))$ does not depend upon the choice for the chain complex C satisfying the conditions of §3.1. Therefore the equality $\sigma(X) = (\theta(\nu_C(\mu)))^*$ holds for any such complex.

Appendix. Proof of Theorem 0

1. LEMMA. *Let X and Y be three-dimensional Poincaré complexes. Let f be a map $X \to Y$ which induces an isomorphism of fundamental groups and is such that $f^*(w(Y)) = w(X)$ and $f_*([X]) = [Y]$. Then f is a homotopy equivalence.*

This assertion is well known. I shall state the basic considerations which are needed for the proof. Since f induces an isomorphism of fundamental groups, f also induces an isomorphism of one-dimensional cohomology of the spaces X and Y with arbitrary twisted coefficients. By applying duality it is easy to verify that

$$f_*: H_*(X; \mathbf{Z}[\pi_1(X)]) \to H_*(Y; \mathbf{Z}[\pi_1(Y)])$$

is an isomorphism. By Whitehead's theorem, f is a homotopy equivalence.

2. PROOF OF THE THEOREM. Suppose that (π, w, μ) and (G, ν, η) are the fundamental triples respectively of the Poincaré complexes X and Y with distinguished points $x \in X$ and $y \in Y$. Let φ be an isomorphism $\pi \to G$ inducing an isomorphism of these triples. We shall prove the existence of a homotopy equivalence $(X, x) \to (Y, y)$ which induces the isomorphism φ on π_1.

According to the argument in §3.2, we may assume that X is obtained as a result of gluing a three-dimensional cell e to a homologically two-dimensional cell complex X' by means of some map $h: \partial e \to X'$. Let $p: \widetilde{X} \to X$ be the universal covering. Let D be the cellular chain Λ-complex of the space $\widetilde{X}$, where $\Lambda = \mathbf{Z}[\pi]$. Let D' be the subcomplex of D generated by the cells lying in $\widetilde{X}' = p^{-1}(X')$. Since the space X' is homologically two-dimensional, D' can be decomposed into a direct sum of two chain subcomplexes

$$\cdots \to D_4 \to D_3' \to \operatorname{Im}(\partial_2': D_3' \to D_2) \to 0 \to 0 \quad \text{and} \quad 0 \to S \to D_1 \to D_0,$$

where S is a submodule of D_2 (cf. the argument in §5.4).

Let $\tilde{e}$ be an oriented three-dimensional cell of the space $\widetilde{X}$ lying above e. According to the definitions, the module $\operatorname{Im} \partial_2'$ is generated by chains which are represented by the gluing maps of three-dimensional cells of the space $\widetilde{X}'$. Therefore, replacing h by a homotopic map if necessary, we may assume that $\partial_2([\tilde{e}]) \in S$, where $[\tilde{e}]$ is the element of the module D_3 represented by the cell $\tilde{e}$, and where ∂_2 is the boundary homomorphism $D_3 \to D_2$. Thus the complex D is the direct sum of two complexes

$$\cdots \to D_4 \to D_3 \to \operatorname{Im} \partial_2' \to 0 \to 0 \quad \text{and} \quad 0 \to \Lambda[\tilde{e}] \to S \to D_1 \to D_0,$$

where $\Lambda[\tilde{e}]$ is the free Λ-module with generator $[\tilde{e}]$.

Let $\{e_i\}$ be a collection of oriented two-dimensional cells of the space $\widetilde{X}$ such that over every two-dimensional cell of X there lies exactly one cell from this collection. The chains $[e_i] \in D_2$ represented by these cells comprise a basis of the Λ-module D_2. Let $\partial_2([\tilde{e}]) = \sum_i a_i[e_i]$, where $a_i \in \Lambda$.

We use an overbar to denote the involution of the ring Λ taking $\alpha \in \pi$ into $(-1)^{w(\alpha)}\alpha^{-1}$. We set $I = I(\pi) = \mathrm{Ker}(\mathrm{aug}\colon \Lambda \to \mathbf{Z})$. Using the above-indicated decomposition of the complex D as a direct sum, and the equality $H_3(X; \mathbf{Z}^w) = \mathbf{Z}$, we easily see that $\overline{a}_i \in I$ for all i and that the chain $1 \otimes [\tilde{e}] \in \mathbf{Z}^w \otimes_\Lambda D$ is a cycle representing a generator of the group $H_3(X; \mathbf{Z}^w)$. We assume that the cell $\tilde{e}$ is oriented so that the chain $1 \otimes [\tilde{e}]$ represents the homology class $[X]$. It is easy to deduce from the equality $H^3(D^*) = H_0(D) = \mathbf{Z}$ that the set $\{\overline{a}_i\}$ generates I as a (left) Λ-module.

Similarly we assume that the Poincaré complex Y is obtained by gluing an oriented three-dimensional cell e' to a homologically two-dimensional cell complex Y', the chain $[e']$ of the cellular chain complex $C_*(Y; \mathbf{Z}^v)$ being a cycle representing the class $[Y]$.

We denote by K an Eilenberg-Mac Lane space of type $K(G, 1)$ obtained from Y by attaching cells of dimension ≥ 3. Let $q\colon \widetilde{K} \to K$ be the universal covering. We denote by C the cellular chain $\mathbf{Z}[G]$-complex of the space $\widetilde{K}$. We fix a three-dimensional cell $\tilde{e}'$ of $\widetilde{K}$ which lies over e'.

We assume that $x \in X'$ and $y \in Y'$. Since the space X' is homologically two-dimensional, there exists a cellular map $g'\colon (X', x) \to (Y', y)$ inducing the given isomorphism of fundamental groups $\varphi\colon \pi \to G$. (All the obstructions to the existence of such a map lie in zero groups.) Since $\pi_2(K) = 0$, the map g' extends to a cellular map $g\colon (X, x) \to (K, y)$. Let g_* be the chain homomorphism $D \to C$ induced by g. It follows from the equation $\varphi_*(\mu) = \eta$ that

$$g_*([\tilde{e}]) - [\tilde{e}'] \in \mathrm{Im}(\partial\colon C_4 \to C_3) = \overline{I(G)} \cdot C_3,$$

where the bar denotes the involution in $\mathbf{Z}[G]$ taking $\alpha \in G$ to $(-1)^{v(\alpha)}\alpha^{-1}$. Thus

$$g_*(\partial([\tilde{e}])) - \partial([\tilde{e}']) \in \overline{I(G)} \cdot \partial(C_3). \tag{3}$$

We denote the ring homomorphism $\Lambda = \mathbf{Z}[\pi] \to \mathbf{Z}[G]$, induced by the isomorphism φ, by the same symbol φ. Since $\varphi^*(v) = w$, $\varphi(\overline{\lambda}) = \overline{\varphi(\lambda)}$ for any $\lambda \in \Lambda$. From this it follows that $\overline{I(G)} = \varphi(\overline{I})$. Since the set $\{a_i\}$ generates $\overline{I}$ as a right Λ-module, it follows from (3) that

$$g_*(\partial([\tilde{e}])) - \partial([\tilde{e}']) = \sum_i \varphi(a_i)d_i,$$

where d_i is a chain in C_2 which is represented by a certain two-dimensional spheroid in $\widetilde{Y}' = q^{-1}(Y')$. Thus

$$\partial([\tilde{e}]) = \sum_i \varphi(a_i) \cdot (g_*([e_i]) - d_i).$$

By modifying the map $g'\colon X' \to Y'$ inside the two-dimensional cells $\{e_i\}$ by the corresponding spheroids d_i, we obtain a new map which we call $k\colon (X', x) \to (Y', y)$ and which has the property that

$$k_*(\partial([\tilde{e}])) = k_*\left(\sum_i a_i[e_i]\right) = \sum_i \varphi(a_i)\cdot(g_*([e_i]) - d_i) = \partial([\tilde{e}']).$$

On account of the equality $\pi_2(Y') = H_2(Y')$, it follows from this that the map $k \circ h\colon \partial e \to Y'$ is homotopic to the attaching map of the cell e'. Therefore k extends to a map $X \to Y$ satisfying the conditions of Lemma 1. By that lemma, the map so constructed is a homotopy equivalence.

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