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The Neumann-Siebenmann invariant and Seifert surgery

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Dedicated to Professor Yukio Matsumoto for his 60th birthday

Abstract. We discuss some relations between the invariant originated in Fukumoto-Furuta and the Neumann-Siebenmann invariant for the Seifert rational homology 3-spheres. We give certain constraints on Seifert 3-manifolds to be obtained by surgery on knots in homology 3-spheres in terms of these invariants.

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In this paper we consider a Seifert rational homology 3-sphere S equipped with spin structure c and the Neumann-Siebenmann invariant $\overline{\mu}(S, c)$, which is an integral lift of the Rochlin invariant of (S, c) (see §3 for the definition). In case of a Seifert integral homology 3-sphere $\overline{\mu}(S, c)$ is equal to Fukumoto-Furuta invariant up to sign [4], and is a homology cobordism invariant [6]. In case of a spherical 3-manifold we define an analogous invariant $\delta(S, c)$ originated in Fukumoto-Furuta's theorem, which is also equal to $\overline{\mu}(S, c)$ (if S is a lens space $L(p, q)$ then $\delta(S, c)$ is represented by a σ-function $\sigma(q, p, \pm 1)$, [7]). In this paper based on Saveliev's observation on Seifert integral homology 3-spheres [6], we extend the results in [6] and [8] to general Seifert rational homology 3-spheres. In Theorem 1 we show the spin homology cobordism invariance of the Neumann-Siebenmann invariant of Seifert rational homology 3-spheres. In Theorems 2 and 3 we give some constraints on Seifert rational homology 3-spheres to be obtained by surgery on knots in homology 3-spheres in terms of the Neumann-Siebenmann invariants. These theorems generalize the results in [8], where similar constraints on spherical 3-manifolds obtained by surgery on knots are discussed.

A lot of results have been proved about obtaining Seifert 3-manifolds by surgery on knots, in particular, hyperbolic knots in S^3. The cases of integral surgeries are of particular interest since it is conjectured that no non-integral surgery on a hyperbolic knot in S^3 could yield a Seifert 3-manifold. As integral surgeries are concerned, our result (Theorem 2) only gives a constraint for even surgeries, in which case our condition is independent of the knot type. We note that the value of the Neumann-Siebenmann invariant of a Seifert 3-manifold obtained by an odd surgery on a knot in S^3 certainly depends on the knot type. Nevertheless our results lead to some interesting conclusions. For example, Dean [2] proved that for any given integers p, q, r with $\gcd(p, q) = 1$, there exists a Seifert 3-manifold with three singular fibers with multiplicity p, q and r that is obtained by an integral surgery on a hyperbolic knot in S^3 (see also [5]). On the other hand for any integers p, q, r such that $\gcd(p, q, r) = 1$ and the number of even integers among $\{p, q, r\}$ is 0 or 2, Theorem 2 provides infinitely many Seifert 3-manifolds with three singular fibers of multiplicities p, q, r and with even cyclic H_1 that are not obtained by integral surgery on knots in S^3 (see §5). Throughout the paper we denote by $K_{p/q}$ a p/q surgery on a knot K in an integral homology 3-sphere M. We note that the number of the spin structures on $K_{p/q}$ is 1 if p is odd and 2 if p is even. We also assume that a Seifert 3-manifold considered in this paper is orientable and has an orientable base 2-orbifold unless otherwise specified.

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1. Main results

In [6] it is proved that the Neumann-Siebenmann invariant of Seifert integral homology 3-spheres is a homology cobordism invariant. The following result generalizes this to Seifert rational homology 3-spheres.

Theorem 1. *Suppose that for given Seifert rational homology 3-spheres with spin structures* (S_i, c_i) $(i = 1, 2)$ *there exists a spin cobordism* (W, c) *such that* $b_2(W) = 0$ *and* $\partial(W, c) = (S_1, c_1) \cup -(S_2, c_2)$. *Then* $\overline{\mu}(S_1, c_1) = \overline{\mu}(S_2, c_2)$.

In particular for a Seifert $\mathbf{Z}_2$ homology 3-sphere (in this case the spin structure is unique), the Neumann-Siebenmann invariant is a $\mathbf{Z}_2$ homology cobordism invariant. The proof is based on Proposition 3, which was observed by Saveliev in case of Seifert integral homology 3-spheres. This proposition is also used to give some conditions on $\overline{\mu}(S, c)$ for a Seifert 3-manifold S to be obtained by surgery on knots in homology 3-spheres in Theorems 2 and 3 below. In Theorem 2 the condition is given in terms of the σ function $\sigma(q, p, -1)$, whose value is easily computed recursively as is stated in §2, Remark 2 (see also [4]). These results have been proved in [8] for spherical 3-manifolds.

First we note that a spin structure c on $K_{p/q}$ for a knot K in a homology 3-sphere M is determined by giving a $\mathbf{Z}_2$ value $c(\mu)$ for the meridian μ of K, which corresponds to a spin structure on $M \setminus K$. This structure extends to a spin structure on $K_{p/q}$ if and only if $pc(\mu) + pq \equiv 0 \pmod 2$. Hence if p is even $c(\mu)$ may be arbitrary, while $c(\mu) \equiv q \pmod 2$ if p is odd .

Theorem 2. *(1) Let K be a knot in a* $\mathbf{Z}$ *homology 3-sphere M that bounds an acyclic 4-manifold. Suppose that $K_{p/q}$ is Seifert for* $pq \equiv 0 \pmod 2$, $p \neq 0$. *Then* *for a spin structure c characterized by the equation* $c(\mu) \equiv 0$ *for a meridian* μ *of* K,

$$\overline{\mu}(K_{p/q}, c) = -\sigma(q, p, -1).$$

(2) In particular if K_p *is Seifert with* p *even and* $p \neq 0$, *then*

$$\overline{\mu}(K_p, c) = \operatorname{sgn} p \text{ for the same } c.$$

Theorem 3. *Let* K *be a knot in any* $\mathbf{Z}$ *homology 3-sphere* M, *and suppose that both* K_{α_1} *and* K_{α_2} *are Seifert for some nonzero integers* α_i *with* $\alpha_1 < \alpha_2$. *Then*

(1) *For spin structures* c_1, c_2 *on* K_{α_1} *and* K_{α_2} *characterized by* $c_1(\mu) \equiv c_2(\mu) \equiv 1$,

$$\overline{\mu}(K_{\alpha_2}, c_2) - \overline{\mu}(K_{\alpha_1}, c_1) = \begin{cases} \alpha_1 - \alpha_2 \text{ or } \alpha_1 - \alpha_2 - 16 & \text{if } \alpha_1\alpha_2 > 0, \\ \alpha_1 - \alpha_2 + 2 \text{ or } \alpha_1 - \alpha_2 + 18 & \text{if } \alpha_1\alpha_2 < 0. \end{cases}$$

(2) *Moreover if* $\alpha_1 \equiv \alpha_2 \equiv 0 \pmod 2$, *then for* $c_1' \neq c_1, c_2' \neq c_2$ *(characterized by* $c_1'(\mu) \equiv c_2'(\mu) \equiv 0$*)*,

$$\overline{\mu}(K_{\alpha_2}, c_2') - \overline{\mu}(K_{\alpha_1}, c_1') = \begin{cases} 0 \text{ or } -16 & \text{if } \alpha_1\alpha_2 > 0, \\ 2 \text{ or } 18 & \text{if } \alpha_1\alpha_2 < 0. \end{cases}$$

Remark 1. (1) In [8] we showed that if the resulting manifolds are spherical then the values $\alpha_1 - \alpha_2 - 16$, $\alpha_1 - \alpha_2 + 18$ in (1) and -16, 18 in (2) in the statement of Theorem 3 never occur. But these extra values actually occur if the resulting manifold is a non-spherical Seifert 3-manifold (see §5, Remark 5).

(2) The existence of irreducible Seifert integral homology 3-spheres (for example those with exactly three singular fibers) never obtained by surgery on knots in S^3 seems to be still unknown. According to Theorem 2, if $1/q$ surgery on a knot K in S^3 is Seifert then its Neumann-Siebenmann invariant (in this case the spin structure is unique) must be zero if q is even, while we have no such a criterion if q is odd. In fact the $\overline{\mu}$ invariant of $K_{1/q}$ for q odd depends on K (the simplest such examples are given by S^3 and $\Sigma(2, 3, 7)$, which are obtained by 1 surgery on the unknot and the figure-eight knot respectively). But it not clear what kind of knot invariants should be related to the $\overline{\mu}$ invariant.

2. The δ-invariants and the Fukumoto-Furuta invariant

We define an integer invariant $\delta(S, c)$ for a spherical 3-manifold S and its spin structure c as follows.

Definition 1 ([7]).(1) *For a spin 4-manifold* Y *with* $\partial Y = S$, $b_1(Y) = 0$, *a closed 4-orbifold* $Z = cS \cup (-Y)$ *has a spin structure, where* cS *is a cone over* S. *Then the index of the spin Dirac operator on* Z *is given as follows by the V-index theorem.*

$$\operatorname{ind} \mathcal{D}_Z = -(\operatorname{sign} Z + \delta(S, c))/8 = (\operatorname{sign} Y - \delta(S, c))/8.$$

Here sign$Z$ *is the signature of* Z *and* $\delta(S, c)$ *is a contribution from the singularity of* Z *and depends only on* (S, c).

(2) $\delta(S, c)$ (mod 16) *is the Rochlin invariant of* (S, c) *due to the fact that* ind $\mathcal{D}_Z$ *is even.*

Our results in this paper are based on the following "orbifold 10/8-theorem".

Theorem 4 (Fukumoto-Furuta [3]).(1) *For a closed spin 4-orbifold Z with* $b_1(Z) = 0$*, we have either* ind $\mathcal{D}_Z = 0$ *or*

$$1 - b^-(Z) \leq \text{ind } \mathcal{D}_Z \leq b^+(Z) - 1.$$

(2) (*The vanishing theorem*) *If* $b^+(Z) \leq 2$ *and* $b^-(Z) \leq 2$*, then* ind $\mathcal{D}_Z = 0$.

In [7] we have a complete list of the values of $\delta(S, c)$.

Remark 2. A lens space $L(p, q)$ is obtained by $-p/q$-surgery on a trivial knot in S^3 and its spin structure c is represented by assigning a mod 2 value $c(\mu)$ for a meridian μ of the trivial knot satisfying $pc(\mu) \equiv pq \pmod 2$. Then the δ-invariant of $L(p, q)$ is represented by a σ-function $\sigma(q, p, \pm 1)$ so that $\delta(L(p, q), c) = \sigma(q, p, (-1)^{c(\mu)-1})$. Here $\sigma(q, p, \epsilon)$ for $\gcd(p, q) = 1, \epsilon = \pm 1$ is originally defined as a contribution from the cone over $L(p, q)$ to the index of the Dirac operator over a spin 4-orbifold, and is uniquely determined and computed by the following recursive formulae.

(1) $\sigma(q + cp, p, \epsilon) = \sigma(q, p, (-1)^c \epsilon)$.
(2) $\sigma(-q, p, \epsilon) = \sigma(q, -p, \epsilon) = -\sigma(q, p, \epsilon)$.
(3) $\sigma(q, 1, \epsilon) = 0$.
(4) $\sigma(p, q, -1) + \sigma(q, p, -1) = -\text{sgn } pq$ if $p + q \equiv 1 \bmod 2$.

See [4], [8] for the more detailed properties of $\sigma(q, p, \pm 1)$.

Next we consider a rational homology 3-sphere M that bounds a 4-orbifold X with spin structure c whose singularities are all isolated.

Definition 2. *We define* $\delta(X, c)$ *to be a sum of* $\delta(S, c|_S)$*, where S runs over all the links of the isolated singularities of X and* $c|_S$ *is the spin structure on S induced from c.*

Then $\text{sign}(X) + \delta(X, c)$ is considered to be the Fukumoto-Furuta invariant for (M, X, c) since for any spin 4-manifold Y with $\partial Y = M$ whose spin structure restricted on M coincides with that induced from c, we have

$$\text{sign}(X) + \delta(X, c) = -8 \text{ind } \mathcal{D}_{X \cup (-Y)} + \text{sign}(Y).$$

We note that if M is a $\mathbf{Z}$ homology 3-sphere then this invariant is equal to the Fukumoto-Furuta invariant $w(M, X, c)$ originally defined in [4] up to multiplicative constant. The value of the Fukumoto-Furuta invariant depends on the choice of (X, c) in general. But if M is a Seifert 3-manifold this invariant is related to the Neumann-Siebenmann invariant of M defined in the next section by some canonical choices of (X, c) (see Proposition 3 in §3).

3. The Neumann-Siebennmann invariant

Let S be a Seifert 3-manifold over an orientable 2-orbifold with $|H_1(S, \mathbf{Z})| < \infty$. Then the base orbifold of S is a 2-orbifold $S^2(p_1, \ldots, p_n)$ of genus 0 with n singular points whose multiplicities are $p_1, \ldots, p_n$ respectively for some $n > 0$, $p_i > 1$ ($i = 1, \ldots, n$). Then S is represented by (unnormalized) Seifert invariants of the form

$$S = \{(1, b), (p_1, q_1), \ldots, (p_n, q_n)\} \tag{1}$$

with

$$e(S) := -(b + \sum(q_i/p_i)) \neq 0. \tag{2}$$

If $b = 0$ then the term $(1, b)$ is omitted. Here $H_1(S, \mathbf{Z})$ is generated by the general fiber h and the cross sectional curves $g_0, g_1, \ldots, g_n$ with the following relations.

$$\begin{aligned} g_0 + bh = p_i g_i + q_i h = 0 \quad (i = 1, \ldots, n), \\ g_0 + g_1 + \cdots + g_n = 0. \end{aligned} \tag{3}$$

Then S is represented by a framed link $\mathcal{L}(S)$ with $n + 1$ components such that their meridians correspond to $h, g_1, \ldots, g_n$ with framings $-b, p_1/q_1, \ldots, p_n/q_n$ respectively. We describe the spin structure c on S by assigning the $\mathbf{Z}_2$-values $c(h), c(g_i)$ for the meridians h, g_i (which corresponds to the spin structure on the complement of the link $\mathcal{L}(S)$) satisfying the following conditions:

$$\begin{aligned} -\sum_{i=1}^{n} c(g_i) + bc(h) \equiv b \pmod 2, \\ p_i c(g_i) + q_i c(h) \equiv p_i q_i \pmod 2. \end{aligned} \tag{4}$$

If we replace g_i by $g_i' = g_i - a_i h$ with $\sum a_i = 0$ for $i = 0, \ldots, n$, then the corresponding Seifert invariants are given by

$$\{(1, b + a_0), (p_1, q_1 + a_1 p_1), \ldots, (p_n, q_n + a_n p_n)\}.$$

Note that S also bounds a plumbing $P(\Gamma)$ for some integrally weighted tree Γ. We can assume without loss of generality that Γ is star-shaped with n branches, which corresponds to the above Seifert invariants as follows. The central vertex (denoted by v_0) has a weight $-b$, and the weights of the vertices in the ith branch (denoted by $v_1^i, \ldots, v_{k_i}^i$) are $\alpha_1^i, \ldots, \alpha_{k_i}^i$ respectively for some k_i satisfying

$$p_i/q_i = [\alpha_1^i, \ldots, \alpha_{k_i}^i] := \alpha_1^i - \cfrac{1}{\alpha_2^i - \cfrac{1}{\ddots - \cfrac{1}{\alpha_{k_i}^i}}}. \tag{5}$$

Here α_j^i can be chosen so that

$$\begin{aligned} &|\alpha_j^i| \geq 2 \quad \text{for all } j \quad \text{if } p_i > |q_i|, \\ &\alpha_1^i \;= 0, |\alpha_j^i| \geq 2 \quad \text{for } j \geq 2 \quad \text{if } p_i < |q_i|. \end{aligned} \tag{6}$$

Then $S = \partial P(\Gamma)$ is represented by a standard framed link picture $\mathcal{L}(\Gamma)$ corresponding to Γ, where we denote the link component corresponding to the vertex v_j by the same symbol and the meridian of v_j by μ_j. A spin structure c on $S = \partial P(\Gamma)$ is also represented by assigning a $\mathbf{Z}_2$ value $c(\mu_j)$ for each meridian μ_j satisfying the conditions described in (7) below. The two representations of the spin structure c of S associated with $\mathcal{L}(S)$ and $\mathcal{L}(\Gamma)$ are related as follows (see [6]).

Proposition 1. *Let g_i be the meridian of the component with framing p_i/q_i of $\mathcal{L}(S)$ and μ_1^i be the meridian of the component with framing α_1^i in the ith branch of $\mathcal{L}(\Gamma)$. Then for any spin structure c on S, the assignments $c(g_i)$ and $c(\mu_1^i)$ coincide.*

Proof. The meridian of the component corresponding to the central vertex v_0 of $\mathcal{L}(\Gamma)$ is the general fiber h of S. We can see easily that $p_i\mu_1^i + q_i h$ is null homologous in a solid torus in $\partial P(\Gamma)$ and that $\mathcal{L}(S)$ and $\mathcal{L}(\Gamma)$ are identified so that g_i corresponds to μ_1^i. Therefore $c(g_i) = c(\mu_1^i)$. □

For given (S, c) and $P(\Gamma)$ with $\partial P(\Gamma) = S$, there exists a unique characteristic element called a spherical Wu class $w(\Gamma, c) \in H_2(P(\Gamma), \mathbf{Z})$, which is described as follows. If we denote the zero section of the 2-disk bundle over the 2-sphere corresponding to the vertex v_j (with weight n_j) by the same symbol, then $H_2(P(\Gamma), \mathbf{Z})$ is generated by v_j whose intersection form is given by

$$v_i \cdot v_j = \begin{cases} n_i & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and } v_i \text{ and } v_j \text{ are connected by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the spherical Wu class $w = w(\Gamma, c)$ is defined to be

$$w(\Gamma, c) = \sum c(\mu_j) v_j \in H_2(P(\Gamma), \mathbf{Z}),$$

where the coefficient $c(\mu_j)$ is an integer 0 or 1 according to the mod 2 value of $c(\mu_j)$ and satisfies

$$w \cdot v_j \equiv v_j \cdot v_j \pmod 2. \tag{7}$$

The set of v_j with $c(\mu_j) \neq 0$ is called a Wu set. It is easy to see that the Wu set contains no v_j's corresponding to the adjacent vertices simultaneously. Now we can define the Neumann-Siebenmann invariant $\overline{\mu}(S, c)$ as follows.

Definition 3 (The Neumann-Siebenmann invariant)**.**

$$\overline{\mu}(S, c) = \text{sign } P(\Gamma) - w \cdot w \in \mathbf{Z}.$$

It is known that $\overline{\mu}(S, c)$ depends only on (S, c), although the choices of $P(\Gamma)$ with $S = \partial P(\Gamma)$ and $w(\Gamma, c)$ are not unique.

If S is an integral homology 3-sphere, then $\overline{\mu}(S, c)$ is divisible by 8 and usually $\overline{\mu}(S, c)/8$ is called the Neumann-Siebenmann invariant.

Proposition 2 ([6] for lens spaces, [7] for the general cases)**.** *If S is spherical,*

$$\delta(S, c) = \overline{\mu}(S, c).$$

We note that $\overline{\mu}(-S,-c) = -\overline{\mu}(S,c)$ (and that $\delta(S,c)$ has the same property). The following proposition generalizes the observation by Saveliev [6].

Proposition 3. *For a Seifert* $\mathbf{Q}$ *homology 3-sphere* S *and its spin structure* c*, there exist spin 4-orbifolds* (X_+, c_+), (X_-, c_-) *with isolated singularities such that*

(1) $\partial(X_\pm, c_\pm) = (S, c)$,
(2) $\operatorname{sign} X_+ + \delta(X_+, c_+) = \operatorname{sign} X_- + \delta(X_-, c_-) = \overline{\mu}(S,c)$,
(3) $b_1(X_+) = b_1(X_-) = 0$, $b_+(X_+) \leq 1$ *and* $b_-(X_-) \leq 1$.

Here $\delta(X_\pm, c_\pm)$ *is defined in Definition 2.*

Remark 3. If S is spherical, we can put $X_+ = X_- = cS$. If S is not spherical, the above $X_\pm$ can be chosen so that they have only isolated singularities whose links are lens spaces. As is seen in the proof below, we can put $X_+ = X_-$ if either p_i is even for some i or the spin structure c satisfies $c(h) \equiv 0$ for the general fiber h. In this case $b_+(X_\pm) = 1, b_-(X_\pm) = 0$ if $e(S) > 0$, and $b_+(X_\pm) = 0, b_-(X_\pm) = 1$ if $e(S) < 0$.

The rest of this section is devoted to the proof of Proposition 3. Hereafter we assume that $n \geq 3$ since if $n \leq 2$ then S is a lens space. To obtain the orbifolds in Proposition 3 we use the following constructions. Start with the star-shaped weighted tree Γ with $S = \partial P(\Gamma)$. In the rest of this section we always assume that the weights α^i_j of Γ satisfy the conditions (6). Let $\Gamma_0 = \cup \Gamma^i_0$ be a subgraph of Γ consisting of a disjoint union of subtrees Γ^i_0 with no multivalent vertices such that there are no edges that interpolate between them. Suppose that $\partial P(\Gamma^i_0)$ is a lens space for each i (this is the case unless $\partial P(\Gamma^i_0)$ is $S^2 \times S^1$). Then we can construct an 4-orbifold X from $P(\Gamma)$ by replacing $P(\Gamma^i_0)$ naturally embedded in the interior of $P(\Gamma)$ by the cone over $\partial P(\Gamma^i_0)$. Let $I(\Gamma)$ and $I(\Gamma_0)$ be the intersection forms of $P(\Gamma)$ and $P(\Gamma_0)$ respectively.

Proposition 4. *There exists a matrix* $P \in GL(\operatorname{rank} H_2(P(\Gamma)), \mathbf{Q})$ *with* $\det P = 1$ *such that* ${}^tPI(\Gamma)P$ *is a direct sum of* $I(\Gamma_0)$ *and a matrix* B *with rational entries, which represents a rational intersection form of* X*. In particular* $\operatorname{sign} X + \sum \operatorname{sign} P(\Gamma^i_0) = \operatorname{sign} P(\Gamma)$.

Proof. This is proved essentially in the same way as in [6]. The second claim also follows from the Novikov additivity of the signature. See [6] for the details. □

To construct $X_\pm$ more concretely we consider the following two cases.

3.1. The case when p_i is even for some i

We can assume that p_1 is even. Then since $p_1 c(g_1) + q_1 c(h) \equiv p_1 q_1 \pmod 2$ and $\gcd(p_1, q_1) = 1$, we must have $c(h) \equiv 0$. Therefore the Wu set for the corresponding $P(\Gamma)$ does not contain the central vertex v_0. Thus we define Γ_0 to be the disjoint union of n branches Γ^i_0 that is obtained from Γ by removing v_0 and all the adjacent edges. Then $\partial P(\Gamma^i_0)$ is the lens space $L(p_i, -q_i)$ and we can construct X

from $P(\Gamma)$ by deleting $P(\Gamma_0^i)$ and replacing it by the cone over $L(p_i, -q_i)$. Then X has a spin structure since Γ_0 contains the Wu set and hence the spin structure c on $\partial P(\Gamma)$ extends to that on the complement of $P(\Gamma_0)$ and the induced spin structure on $\partial P(\Gamma_0^i)$ (which we denote by c_i) extends to that on the cone over it (see [7]). If we denote by w_i the sum of the elements of the Wu set contained in Γ_0^i, then w_i is a spherical Wu class for $(P(\Gamma_0^i), c_i)$ and w is the sum of w_i. In this case $b_1(X) = 0$ and $b_2(X) = 1$. Moreover we have an element $z \in H_2(X, \mathbf{Q})$ such that $z \cdot z = -b - \sum q_i/p_i = e$, and hence sign $X = \operatorname{sgn} e$. By Proposition 4 sign $X =$ sign $P(\Gamma) - \sum$ sign $P(\Gamma_i^0)$. On the other hand by Saveliev's observation [6] we have $\delta(\partial P(\Gamma_0^i), c_i) =$ sign $P(\Gamma_0^i) - w_i \cdot w_i$. Hence we have

$$\begin{aligned}\text{sign } X &= \text{sign } P(\Gamma) - \sum (w_i \cdot w_i + \delta(L(p_i - q_i), c_i)) \\ &= \text{sign } P(\Gamma) - w \cdot w - \sum \delta(L(p_q, -q_i), c_i)\end{aligned}$$

since $w \cdot w = \sum w_i \cdot w_i$. Therefore we have

$$\text{sign } X + \delta(X) = \overline{\mu}(S, c).$$

Hence in this case it suffices to put $X_+ = X_- = X$.

3.2. p_i is odd for every i

In this case we have $c(g_i) + q_i c(h) \equiv q_i \pmod 2$ and $\sum_{i=1}^n c(g_i) + bc(h) \equiv b \pmod 2$. Suppose that $c(h) \equiv 0$. This is the case when $c(g_i) \equiv q_i$ and $\sum q_i \equiv b$ (and $|H_1(S, \mathbf{Z})|$ is even). Then we can choose the spin orbifold X just as in the first case and put $X_1 = X_2 = X$. Suppose that $c(h) \equiv 1$. Then we must have $c(g_i) \equiv 0$. Thus by Proposition 1 we can see that the Wu set for the corresponding $P(\Gamma)$ contains the central vertex v_0 but does not contain the adjacent vertices whose weights are α_1^i. We define the subtree Γ_0^1 as the union of v_0 and all the vertices in the first branch and the edges connecting them. Also we define the subtree Γ_0^i for $i \geq 2$ as the union of all the vertices in the ith branch except for the first vertex with weight α_1^i and the edges connecting them. Then the disjoint union Γ_0 of Γ_0^i contains the Wu set. Moreover since

$$[-b, \alpha_1^1, \ldots, \alpha_{k_1}^1] = -b - q_1/p_1,$$

we see that $\partial P(\Gamma_0^1)$ is the lens space $L(bp_1 + q_1, p_1)$ (note that $bp_1 + q_1 \neq 0$). Likewise since

$$[\alpha_2^i, \ldots, \alpha_{k_i}^i] = q_i/(\alpha_1^i q_i - p_i)$$

for $i \geq 2$ we see that $\partial P(\Gamma_0^i)$ is the lens space $L(q_i, p_i - \alpha_1^i q_i)$. We denote by X the orbifold obtained from $P(\Gamma)$ by deleting $P(\Gamma_0^i)$ for $i \geq 1$ contained in the interior of $P(\Gamma)$ and attaching the cones over $\partial P(\Gamma_0^i)$. Then as in the first case the spin structure c on S extends to that on X. Moreover $b_1(X) = 0$ and $b_2(X) = n - 1$. We will construct such 4-orbifolds in two ways corresponding to the different choices of the Seifert invariants and will obtain the desired orbifolds $X_\pm$. To this end we first observe sign X. We can see that $I(\Gamma)$ is congruent to a diagonal matrix with diagonal entries

$$\alpha^i_{k_i}, [\alpha^i_{k_i-1}, \alpha^i_{k_i}], \ldots, [\alpha^i_1, \ldots, \alpha^i_{k_i}] = p_i/q_i \ (i = 1, \ldots, n),$$
$$-b - \sum q_i/p_i = e(S).$$

Likewise $I(\Gamma^i_0)$ is congruent to a diagonal matrix with diagonal entries

$$[\alpha^i_j, \alpha^i_{j+1}, \ldots, \alpha^i_{k_i}] \ (j = 2, \ldots, k_i)$$

if $i \geq 2$ and

$$[-b, \alpha^1_1, \ldots, \alpha^1_{k_1}], \quad [\alpha^1_j, \alpha^1_{j+1}, \ldots, \alpha^1_{k_1}] \ (j = 1, \ldots, k_1)$$

if $i = 1$.

Note that by arranging the Seifert invariants so that $|b|$ is sufficiently large relative to q_1/p_1, we can assume that

$$\mathrm{sgn}\ [-b, \alpha^1_1, \ldots, \alpha^1_{k_1}] = \mathrm{sgn} - (b + q_1/p_1) = \mathrm{sgn}\ \ e(S). \tag{8}$$

Thus under this condition we can see by Proposition 4 that $b^+(X)$ (resp. $b^-(X)$) is the number of the positive (resp. negative) values among

$$\{[\alpha^i_1, \alpha^i_2, \ldots, \alpha^i_{k_i}] \mid i \geq 2\}.$$

On the other hand we can replace the Seifert invariants (1) by

$$\{(1, b), (p_1, q'_1), (p_2, q'_2), \ldots, (p_{n-1}, q'_{n-1}), (p_n, q'_n)\}$$

such that

$$q'_1 = q_1, \ q'_i = q_i + Np_i \ (2 \leq i \leq n-1), \ q'_n = q_n - (n-2)Np_n$$

without changing the value of $b + q_1/p_1$. Hence by choosing N appropriately we can arrange the Seifert invariants in two ways so that they satisfy (8) and either $q_i > 0$ for $2 \leq i \leq n-1$ and $q_n < 0$, or $q_i < 0$ for $2 \leq i \leq n-1$ and $q_n > 0$ (note that we have assumed that $n \geq 3$). Now noticing

$$\mathrm{sgn}[\alpha^i_1, \ldots, \alpha^i_{k_i}] = \mathrm{sgn}\ (p_i/q_i) = \mathrm{sgn}\ q_i,$$

we have two spin 4-orbifolds $X_\pm$ corresponding to the above two choices of the invariants satisfying $b^+(X_+) = 1$ and $b^-(X_-) = 1$. As in the first case we can see that

$$\mathrm{sign}\ X_+ + \delta(X_+) = \mathrm{sign}\ X_- + \delta(X_-) = \overline{\mu}(S, c).$$

Thus $X_\pm$ are the desired orbifolds.

4. Proofs of the main theorems

In this section we prove the main theorems.

4.1. Proof of Theorem 1

We can assume that $b_1(W) = 0$ (perform spin surgery on W to kill $b_1(W)$ if necessary). Let $(X^1_\pm, c^1_\pm)$ and $(X^2_\pm, c^2_\pm)$ be spin 4-orbifolds bounded by (S_1, c_1) and (S_2, c_2) respectively provided by Proposition 3. Consider two closed spin 4-orbifolds $\widehat{X}_\pm$ with $b_1(\widehat{X}_\pm) = 0$ defined by

$$(\widehat{X}_+, c_+) = (-X^1_-, -c^1_-) \cup (-W, -c) \cup (X^2_+, c^2_+),$$
$$(\widehat{X}_-, c_-) = (-X^1_+, -c^1_+) \cup (-W, -c) \cup (X^2_-, c^2_-).$$

Then

$$\operatorname{ind} \mathcal{D}_{\widehat{X}_+} = -(\operatorname{sign}(X^2_+) - \operatorname{sign}(X^1_-) + \delta(X^2_+, c^2_+) - \delta(X^1_-, c^1_-))/8$$

is equal to

$$\operatorname{ind} \mathcal{D}_{\widehat{X}_-} = -(\operatorname{sign}(X^2_-) - \operatorname{sign}(X^1_+) + \delta(X^2_-, c^2_-) - \delta(X^1_+, c^1_+))/8.$$

Here we note that $\delta(-X, -c) = -\delta(X, c)$. Moreover by Theorem 4 either $\operatorname{ind} \mathcal{D}_{\widehat{X}_\pm} = 0$ or $1 - b^-(\widehat{X}_\pm) \leq \operatorname{ind} \mathcal{D}_{\widehat{X}_\pm} \leq b^+(\widehat{X}_\pm) - 1$. But

$$b^+(\widehat{X}_+) = b^+(X^2_+) + b^-(X^1_-) \leq 2$$

and

$$b^-(\widehat{X}_-) = b^-(X^2_-) + b^+(X^1_+) \leq 2,$$

and hence we must have $\operatorname{ind} \mathcal{D}_{\widehat{X}_\pm} = 0$ since $\operatorname{ind} \mathcal{D}_{\widehat{X}_\pm}$ is even. This implies that

$$\overline{\mu}(S_1, c_1) = \operatorname{sign}(X^1_\pm) + \delta(X^1_\pm, c^1_\pm) = \operatorname{sign}(X^2_\pm) + \delta(X^2_\pm, c^2_\pm) = \overline{\mu}(S_2, c_2).$$

□

4.2. Proof of Theorem 2

Suppose that M bounds an acyclic 4-manifold W, and $K_{p/q}$ for a knot K in M is a Seifert 3-manifold S. We denote by M_K the complement of the knot K in a homology 3-sphere M and let μ and λ be the meridian and the (preferred) longitude of K. We construct a certain spin 4-orbifold X, which is essentially the same one as is given in the proofs of the main results in [8]. Let $\triangle$ be the 2-simplex with edges e_j $(j = 1, 2, 3)$ and construct a 4-manifold X of the form

$$X = T^2 \times \triangle \cup (M_K \times e_1) \cup (D^2 \times S^1 \times e_2) \cup (D^2 \times S^1 \times e_3). \tag{9}$$

Here $\partial M_K \times e_1$ and $\partial D^2 \times S^1 \times e_j$ are glued with $T^2 \times e_j \subset T^2 \times \partial\triangle (j = 1, 2, 3)$ respectively via the following identifications. Let $\mu_0 = S^1 \times *$ and $\lambda_0 = * \times S^1$ in T^2, and $m_j = \partial D^2 \times *$ and $\ell_j = * \times S^1$ in the $D^2 \times S^1$ factor (which we denote by $(D^2 \times S^1)_j$) of $D^2 \times S^1 \times e_j$ for $j = 2, 3$. Then these curves are identified so that

$$\begin{aligned} \mu &= \mu_0, & \lambda &= \lambda_0, \\ m_2 &= p\mu_0 + q\lambda_0, & \ell_2 &= r\mu_0 + s\lambda_0, \\ m_3 &= \mu_0, & \ell_3 &= \lambda_0, \end{aligned}$$

where r and s are integers with $ps - qr = 1$. Then X is oriented so that ∂X consists of the three components $-M = -M_K \cup -(D^2 \times S^1)_3$, $S = K_{p/q} = M_K \cup (D^2 \times S^1)_2$, and $L(q, -p) = (D^2 \times S^1)_3 \cup -(D^2 \times S^1)_2$. In the third component we identify $(D^2 \times S^1)_3$ with the complement of the trivial knot in S^3 so that (ℓ_3, m_3) correspond to the meridian and the longitude of the trivial knot. Thus the resulting manifold is the q/p-surgery on the trivial knot, which is $L(q, -p)$. Note that the above X is the same as the one constructed in the proof of the main theorem in [8]. It is easy to see that $b_1(X) = 0$, $b_2(X) = 1$, and the self-intersection number of the generator of $H_2(X)$ is pq. We also see that the spin structure on T^2 determined by the values $c(\mu_0)$ and $c(\lambda_0)$ extends to that on X if and only if

$$\begin{aligned} c(\mu_0) \equiv c(\lambda_0) &\equiv 0 \pmod 2, \\ pc(\mu_0) + qc(\lambda_0) + pq &\equiv 0 \pmod 2. \end{aligned}$$

Hence if pq is even (hereafter we assume this condition) then we have a spin structure c on X defined by the above equation. The restriction of c on M induces the unique spin structure on M, which extends uniquely to that on W. Moreover $(S, c|_S)$ bounds two spin 4-orbifolds $(X_\pm, c_\pm)$ provided by Proposition 3. Thus we can construct closed 4-orbifolds of the form

$$\widehat{X}_\pm = X \cup W \cup cL(q, p) \cup (-X_\pm).$$

We can see that c extends to a spin structure on $\widehat{X}_\pm$ and we have

$$\operatorname{ind} \mathcal{D}_{\widehat{X}_\pm} = -(\operatorname{sign} X - \operatorname{sign} X_\pm - \delta(X_\pm, c_\pm) + \delta(L(q, p), c))/8.$$

Again by Proposition 3 and Theorem 4 we have $\operatorname{ind} D_{\widehat{X}_+} = \operatorname{ind} \mathcal{D}_{\widehat{X}_-}$, which is either 0 or satisfies $1 - b^-(\widehat{X}_\pm) \le \operatorname{ind} \mathcal{D}_{\widehat{X}_\pm} \le b^+(\widehat{X}_\pm) - 1$. On the other hand

$$b^+(\widehat{X}_-) - 1 = b^+(X) + b^-(X_-) - 1 \le 1$$

since $b^+(X) \le 1$ and $b^-(X_-) \le 1$, and likewise

$$1 - b^-(\widehat{X}_+) = 1 - b^-(X) - b^+(X_+) \ge -1$$

since $b^-(X) \le 1$ and $b^+(X_+) \le 1$. Therefore $\operatorname{ind} \mathcal{D}_{\widehat{X}_\pm} = 0$ since this value must be even. It follows that

$$\begin{aligned} \overline{\mu}(S, c) &= \operatorname{sign} X_\pm + \delta(X_\pm, c_\pm) = \delta(L(q, p), c) + \operatorname{sign} X \\ &= \sigma(p, q, (-1)^{c(\mu_0)-1}) + \operatorname{sgn} pq = -\sigma(q, p, -1). \end{aligned}$$

The last equality is deduced from the reciprocity of $\sigma(q, p, -1)$ (see [8]). This proves Theorem 2. □

4.3. Proof of Theorem 3

First we construct a 4-manifold X as in (9), where $T^2 \times e_j$ and $\partial M_K \times e_1$, $\partial D^2 \times S^1 \times e_j$ are identified as follows (we use the same notations as above).

$$\begin{aligned} \mu &= \mu_0, & \lambda &= \lambda_0, \\ m_2 &= \alpha_2\mu_0 + \lambda_0, & \ell_2 &= -\mu_0, \\ m_3 &= \alpha_1\mu_0 + \lambda_0, & \ell_3 &= -\mu_0. \end{aligned}$$

Then ∂X consists of $-K_{\alpha_1} = -M_K \cup -(D^2 \times S^1)_3$, $K_{\alpha_2} = M_K \cup (D^2 \times S^1)_2$, and $L(\alpha_2 - \alpha_1, 1) = (D^2 \times S^1)_3 \cup -(D^2 \times S^1)_2$. In the third component we have the identification of the form

$$m_2 = m_3 - (\alpha_2 - \alpha_1)\ell_3,$$

and hence the third one corresponds to the $-(\alpha_2 - \alpha_1)$-surgery on the trivial knot in S^3, which is $L(\alpha_2 - \alpha_1, 1)$. Next we examine the topology of X. Consider the connecting homomorphism

$$\delta : H_2(X, T^2 \times \triangle) \to H_1(T^2 \times \triangle)$$

in the exact sequence for $(X, T^2 \times \triangle)$. Then via the excision $H_2(X, T^2 \times \triangle)$ is isomorphic to $\mathbf{Z}^3$, which is generated by the meridian disks D_i^2 of $(D^2 \times S^1)_i$ and the Seifert surface S_K of K. Thus we have

$$\begin{aligned} H_1(X) &= \operatorname{coker} \delta \\ &= \mathbf{Z}^2\langle \mu_0, \lambda_0 \rangle / \{\lambda_0 = \alpha_1\mu_0 + \lambda_0 = \alpha_2\mu_0 + \lambda_0 = 0\} \cong \mathbf{Z}_{\gcd(\alpha_1,\alpha_2)}. \end{aligned}$$

Likewise we have

$$H_2(X) = \ker \delta \cong \mathbf{Z},$$

which is generated by $\alpha_2'(D_3^2 - S_K) - \alpha_1'(D_2^2 - S_K)$ for $\alpha_i' = \alpha_i / \gcd(\alpha_1, \alpha_2)$. If we consider $z = \alpha_2(D_3^2 - S_K) - \alpha_1(D_2^2 - S_K)$ we have

$$z \cdot z = -\alpha_2^2\alpha_1 + \alpha_1^2\alpha_2 = -\alpha_1\alpha_2(\alpha_2 - \alpha_1),$$

and hence sign $X = -\mathrm{sgn}\alpha_1\alpha_2(\alpha_2 - \alpha_1)$. Next we consider the condition on X to be spin. Start with the spin structure on T^2, which is determined by $c(\mu_0), c(\lambda_0) \in \mathbf{Z}_2$. To extend this spin structure to those on M_K and $(D^2 \times S^1)_i$ $(i = 1, 2)$ we must have

$$\begin{gathered} c(\lambda_0) \equiv 0 \pmod 2, \\ \alpha_i c(\mu_0) + c(\lambda_0) + \alpha_i \equiv 0 \pmod 2 \ (i = 1, 2). \end{gathered}$$

Thus we can see that the spin structure on T^2 given by $c(\mu_0) \equiv 1$ and $c(\lambda_0) \equiv 0$ extends to the spin structure c on X. Now we construct two closed 4-orbifolds with $b_1 = 0$ as follows.

$$\begin{aligned} \widehat{X}_+ &= X \cup X_+^1 \cup (-X_-^2) \cup (-cL(\alpha_2 - \alpha_1, 1)), \\ \widehat{X}_- &= X \cup X_-^1 \cup (-X_+^2) \cup (-cL(\alpha_2 - \alpha_1, 1)), \end{aligned}$$

where $X^j_\pm$ are the spin 4-orbifolds with spin structure $c^j_\pm$ provided by Proposition 3 for K_{α_j}. Then c also extends to the spin structures on $\widehat{X}_\pm$. Moreover we deduce from the equation $\overline{\mu}(K_{\alpha_j}, c) = \text{sign } X^j_\pm + \delta(X^j_\pm, c^j_\pm)$ that

$$\begin{aligned}\text{ind } \mathcal{D}_{\widehat{X}_+} &= \text{ind } \mathcal{D}_{\widehat{X}_-} \\ &= -(\text{sign } X + \overline{\mu}(K_{\alpha_1}, c) - \overline{\mu}(K_{\alpha_2}, c) - \delta(L(\alpha_2 - \alpha_1, 1), 1))/8.\end{aligned}$$

Again either this value is zero or satisfies $1 - b^-(\widehat{X}_\pm) \le \text{ind } \mathcal{D}_{\widehat{X}_\pm} \le b^+(\widehat{X}_\pm) - 1$. We have the following inequality according to the sign of

$$\text{sign } X = -\text{sgn } \alpha_1\alpha_2(\alpha_2 - \alpha_1).$$

(1) The case when $\alpha_1\alpha_2(\alpha_2 - \alpha_1) > 0$. Then since $b^+(X) = 0$ and $b^-(X) = 1$, we have

$$b^+(\widehat{X}_+) - 1 = b^+(X) + b^+(X^1_+) + b^-(X^2_-) - 1 \le 0 + 1 + 1 - 1 = 1,$$

and

$$1 - b^-(\widehat{X}_-) = 1 - b^-(X) - b^-(X^1_-) - b^+(X^2_+) \ge 1 - 1 - 1 - 1 = -2.$$

It follows that either $\text{ind } \mathcal{D}_{\widehat{X}_\pm} = 0$ or

$$-2 \le \text{ind } \mathcal{D}_{\widehat{X}_\pm} \le 1,$$

and hence $\text{ind } \mathcal{D}_{\widehat{X}_\pm}$ is either 0 or -2.

(2) The case when $\alpha_1\alpha_2(\alpha_2 - \alpha_1) < 0$. Then since $b^+(X) = 1$ and $b^-(X) = 0$ we have

$$\begin{aligned}b^+(\widehat{X}_+) - 1 &\le 1 + 1 + 1 - 1 = 2, \\ 1 - b^-(\widehat{X}_-) &\ge 1 - 0 - 1 - 1 = -1.\end{aligned}$$

It follows that either $\text{ind } \mathcal{D}_{\widehat{X}_+} = \text{ind } \mathcal{D}_{\widehat{X}_-} = 0$ or

$$-1 \le \text{ind } \mathcal{D}_{\widehat{X}_\pm} \le 2,$$

and hence $\text{ind } \mathcal{D}_{\widehat{X}_\pm}$ is either 0 or 2.

Now since

$$\begin{aligned}\delta(L(\alpha_2 - \alpha_1, 1), c) &= \sigma(1, \alpha_2 - \alpha_1, (-1)^{(c(\mu)-1)}) \\ &= \sigma(1 - (\alpha_2 - \alpha_1), \alpha_2 - \alpha_1, -1) \\ &= -\sigma(\alpha_2 - \alpha_1 - 1, \alpha_2 - \alpha_1, -1) = \alpha_2 - \alpha_1 - 1,\end{aligned}$$

(see Remark 2 or [4]) and $\overline{\mu}(K_{\alpha_i}, c_i) = \text{sign } X^i_\pm + \delta(X^i_\pm, c^i_\pm)$ we see that

$$\overline{\mu}(K_{\alpha_2}, c_2) - \overline{\mu}(K_{\alpha_1}, c_1) = -(\alpha_2 - \alpha_1) \text{ or } -(\alpha_2 - \alpha_1) - 16$$

if $\alpha_1\alpha_2(\alpha_2 - \alpha_1) > 0$, while

$$\overline{\mu}(K_{\alpha_2}, c_2) - \overline{\mu}(K_{\alpha_1}, c_1) = 2 - (\alpha_2 - \alpha_1) \text{ or } 18 - (\alpha_2 - \alpha_1)$$

if $\alpha_1\alpha_2(\alpha_2-\alpha_1) < 0$. Here the spin structure c_i on K_{α_i} is given by $c_i(\mu) = 1$ for the meridian μ of K and hence is different from that given in Theorem 2 in the case that α_i is even. If both α_1 and α_2 are even then the spin structure c' on T^2 determined by $c'(\mu_0) \equiv c'(\lambda_0) \equiv 0$ also extends to that on $\widehat{X}_\pm$. We denote by c_i' the spin structure induced on K_{α_i} (which satisfies $c_i'(\mu) \equiv 0$ for the meridian μ of K). In this case $\delta(L(\alpha_2-\alpha_1, 1), c') = \sigma(1, \alpha_2-\alpha_1, (-1)^{c'(\mu_0)-1}) = \sigma(1, \alpha_2-\alpha_1, -1) = -\mathrm{sgn}(\alpha_2-\alpha_1) = -1$. It follows that if $\alpha_1\alpha_2 > 0$, we have

$$\overline{\mu}(K_{\alpha_2}, c_2') - \overline{\mu}(K_{\alpha_1} c_1') = 0 \text{ or } -16,$$

while if $\alpha_1\alpha_2 < 0$, we have

$$\overline{\mu}(K_{\alpha_2}, c_2') - \overline{\mu}(K_{\alpha_1}, c_1') = 2 \text{ or } 18.$$

Note that if M bounds an acyclic 4-manifold then the cases with values -16 and 18 do not occur by Theorem 2. This proves Theorem 3. □

5. Some remarks

Now we give some remarks and examples derived from the main theorems.

Corollary 1. *Suppose that k copies of a Seifert 3-manifold with spin structure* (S, c) *bounds a* $\mathbf{Q}$*-acyclic spin 4-manifold for some k. Then* $|\overline{\mu}(S, c)| < 8$.

Proof. Let (W, c) be a $\mathbf{Q}$-acyclic spin 4-manifold with $\partial(W, c) = k(S, c)$. Then for spin 4-orbifolds $(X_\pm, c_\pm)$ with $\partial(X_\pm, c_\pm) = (S, c)$ provided by Proposition 3, we can put $\widehat{X}_\pm = W \cup k(-X_\pm)$, for which we have either ind $D_{\widehat{X}_\pm} = 0$ or $1 - b_-(\widehat{X}_\pm) \le \mathrm{ind} D_{\widehat{X}_\pm} \le b^+(\widehat{X}_\pm) - 1$. Since ind $D_{\widehat{X}_+} =$ ind $D_{\widehat{X}_-} = -(0 - k\overline{\mu}(S, c))/8$, $b^+(\widehat{X}_-) \le k$, and $b^-(\widehat{X}_+) \le k$, we have either $\overline{\mu}(S, c) = 0$ or

$$1 - k \le k\overline{\mu}(S, c)/8 \le k - 1.$$

In either case we have $|\overline{\mu}(S, c)| < 8$. □

Remark 4. Under the same assumption $\overline{\mu}(S, c) = 0$ if either S is a Seifert integral homology 3-sphere (since $\overline{\mu}(S, c)$ is divisible by 8) [4], or S is spherical (since we can put $X_\pm = cS$ in this case) [7].

Corollary 2. *Suppose that* K *is an amphicheiral knot in* S^3*, and* K_p *is a Seifert 3-manifold over an orientable orbifold for* $p > 0$*. Then for a spin structure c on* K_p *determined by* $c(\mu) \equiv 1$ *for the meridian* μ *of* K*, we have*

$$\overline{\mu}(K_p, c) = 1 - p \textit{ or } 9 - p.$$

Moreover if p is even

$$\overline{\mu}(K_p, c') = 1$$

for the spin structure c' *different from the above c.*

Proof. Note that $K_{-p} = -K_p^*$ for the mirror image K^* of K. Hence if K is amphicheiral, then $K_{-p} = -K_p$. Furthermore if we denote by c the spin structure on K_p determined by $c(\mu) \equiv 1 \pmod 2$ for the meridian μ of K, then $-c$ on $-K_p$ is also the spin structure on K_{-p} with the same property. Thus by Theorem 3 (and by its proof) we have

$$2\overline{\mu}(K_p, c) = \overline{\mu}(K_p, c) - \overline{\mu}(K_{-p}, -c) = 2 - 2p \text{ or } 18 - 2p.$$

The second claim in the case that p is even is deduced from Theorem 2 since the spin structure c' different from c is the same as that given in Theorem 2. □

Remark 5. In [8] we showed that if K is amphicheiral and hyperbolic then K_p is never spherical by using the fact that the value $9 - p$ in the first claim does not occur together with Boyer-Zhang's result [1] on finite surgery. But the value $9 - p$ actually appears when the resulting manifold is a non-spherical Seifert 3-manifold. For example, let K be the figure eight knot (which is amphicheiral). Then $K_1 = \Sigma(2, 3, 7)$, and $\overline{\mu}(\Sigma(2, 3, 7), c) = 8$ for the unique spin structure c.

We can show by the main theorems that certain Seifert 3-manifolds are not obtained by integral surgery on knots in S^3. First of all, $H_1(S, \mathbf{Z})$ must be cyclic if S is obtained by surgery on a knot in a homology 3-sphere.

Proposition 5. *Let S be a Seifert 3-manifold over an orientable 2-orbifold $S^2\ (p_1, \dots, p_n)$ with $e(S) \neq 0$. Then $H_1(S, \mathbf{Z})$ is cyclic (of order $|\prod p_i e(S)|$) if and only if*

$$\gcd\Big(\prod_{k\neq i,j} p_k \mid i \neq j\Big) = 1.$$

Proof. We can assume that the Seifert invariants of S are given by (1) with $b = 0$. The presentation matrix of $H_1(S, \mathbf{Z})$ is given by the $(n + 1) \times (n + 1)$ matrix (c_{ij}) satisfying

$$c_{ij} = \begin{cases} p_i & \text{if } i = j \leq n, \\ q_i & \text{if } i \leq n \text{ and } j = n + 1, \\ 1 & \text{if } i = n + 1 \text{ and } j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\det(c_{ij})$ is equal to $|\prod p_i e(S)| = |\sum_{i=1}^n p_1 \dots p_{i-1} q_i p_{i+1} \dots p_n|$. Next let Δ_{ij} be the $n \times n$ minor of (c_{ij}) obtained from (c_{ij}) by deleting the ith row and the jth column. Then $H_1(S, \mathbf{Z})$ is cyclic if and only if

$$\gcd(\Delta_{ij} \mid \Delta_{ij} \neq 0) = \pm 1.$$

It is easy to see that

$$\Delta_{ij} = \begin{cases} \prod p_i & \text{if } i = j = n + 1, \\ \prod_{k\neq i} p_k(e(S) + q_i/p_i) & \text{if } i = j \leq n, \\ \pm \prod_{k\neq i} p_k & \text{if } i \leq n \text{ and } j = n + 1, \\ \pm q_i \prod_{k\neq i} p_k & \text{if } i = n + 1 \text{ and } j \leq n, \\ \pm q_j \prod_{k\neq i,j} p_k & \text{if } i, j \leq n \text{ and } i \neq j. \end{cases}$$

Since $\gcd(p_j, q_j) = 1$ we have

$$\gcd(\Delta_{ij}) = \gcd(\prod_{k\neq i} p_k, q_j \prod_{k\neq i,j} p_k \mid i \neq j) = \gcd(\prod_{k\neq i,j} p_k \mid i \neq j).$$

Thus we obtain the required result. □

In particular if $n = 3$ then $H_1(S, \mathbf{Z})$ is cyclic if and only if $\gcd(p_1, p_2, p_3) = 1$ (see [5]). Next we examine the value of $\overline{\mu}(S, c)$ according to the parity of p_i for a suitable choice of the Seifert invariants of the form (1). We use the notations in the previous sections.

5.1. The case when p_i is even for some i

First note that from the relations (4) on c we deduce that if p_i is even then q_i is odd and hence we must have $c(h) \equiv 0$. We can also arrange the Seifert invariants so that $|q_i| < p_i$ for every i and if p_j is odd then q_j is even (replace q_j by $q_j \pm p_j$ if necessary). Then we must have $c(q_j) \equiv 0$ if p_j is odd. Moreover we can also assume that $c(q_i) \equiv 0$ even if p_i is even since if we replace g_i by $g_i \pm h$ if necessary then $c(g_i \pm h) \equiv c(g_i) + 1$. It follows that $c(g_i) \equiv 0$ for every i and $b \equiv \sum c(g_i)$ is even. Under these conditions we obtain the plumbing $P(\Gamma)$ with spherical Wu class $w = 0$ bounded by (S, c) whose weights of the vertices on the branchs in Γ are given by even nozero integers α^i_j with

$$p_i/q_i = [\alpha^i_1, \ldots, \alpha^i_{k_i}].$$

Note that since $|\alpha^i_j| \geq 2$ we have

$$\operatorname{sgn}[\alpha^i_j, \alpha^i_{j+1}, \ldots, \alpha^i_{k_i}] = \operatorname{sgn} \alpha^i_j \tag{10}$$

for all i, j. Thus as in the proof of Proposition 3 we have

$$\begin{aligned}\overline{\mu}(S, c) &= \operatorname{sign}\ P(\Gamma) = \operatorname{sgn} e(S) + \sum \operatorname{sgn} \alpha^i_j \\ &= \operatorname{sgn} e(S) - \sum \sigma(q_i, p_i, -1).\end{aligned}$$

Note that under the above conditions on the Seifert invariants $|H_1(S, \mathbf{Z})|$ is odd if and only if p_i is even for just one i. We can also see by Proposition 5 that $H_1(S, \mathbf{Z})$ is even cyclic if and only if the number of i with p_i even is 2. In this case we have the representation of (S, c) such that $p_i > |q_i|$, p_i and q_i have opposite parity for all i, p_i is even for $i = 1, 2$, p_i is odd for $i \geq 3$, b is even and $c \equiv 0$. Under these conditions the other spin structure c' on S is given by $c'(g_1) \equiv c'(g_2) \equiv 1$, $c'(g_i) \equiv 0$ for $i \geq 3$ and $c'(h) \equiv 0$. Hence by replacing g_i by $g_i + \operatorname{sgn}(q_i)h$ for $i = 1, 2$, we have another representation of S of the form

$$\{(1, b + \epsilon_1 + \epsilon_2)), (p_1, q_1 - \epsilon_1 p_1), (p_2, q_2 - \epsilon_2 p_2), (p_3, q_3), \ldots, (p_n, q_n)\}$$

for which $c' \equiv 0$, where $\epsilon_i = \operatorname{sgn}(q_i)$. Then as before we have

$$\overline{\mu}(S, c') = \operatorname{sgn} e(S) - \sum \sigma(q_i - \epsilon_i p_i, p_i, -1).$$

5.2. The case when p_i is odd for every i

We can also arrange the Seifert invariants so that $|q_i| < p_i$ and q_i is even. Thus we must have $c(g_i) \equiv 0$ and $b(c(h)+1) \equiv 0$. Under these conditions we have $|H_1(S,\mathbf{Z})| \equiv b \pmod 2$. Moreover $H_1(S,\mathbf{Z}_2)$ is $\mathbf{Z}_2$ if b is even and 0 otherwise.

(1) The case when $c(h) \equiv 0$. This case occurs only when $|H_1(S,\mathbf{Z})|$ is even. Then (S,c) bounds a plumbing $P(\Gamma)$ with spherical Wu class $w = 0$ of exactly the same type as in the first case. The formula for $\overline{\mu}(S,c)$ is also the same.

(2) The case when $c(h) \equiv 1$. This case occurs not depending on the parity of $|H_1(S,\mathbf{Z})|$. In this case we have the continued fraction expansion

$$p_i/q_i = [\alpha^i_1, \alpha^i_2, \ldots, \alpha^i_{k_i}]$$

with α^i_1 odd, α^i_j even and nonzero for $j \geq 2$. (If we write $p_i/q_i = \alpha - p'_i/q_i$ then we can replace it by either $\alpha + 1 - (p'_i + q_i)/q_i$ or $\alpha - 1 - (p'_i - q_i)/q_i$ if necessary). We note that under the above condition we also have the relation (10) even if $\alpha^i_1 = \pm 1$. Thus we can see that (S,c) bounds a plumbing $P(\Gamma)$ such that the spherical Wu class w is given by the central vertex with weight $-b$ and the weights of the other vertices are given by α^i_j above. Therefore

$$\overline{\mu}(S,c) = \operatorname{sgn} e(S) + \sum \operatorname{sgn} \alpha^i_j - w \cdot w = \operatorname{sgn} e(S) + \sum \operatorname{sgn} \alpha^i_j + b.$$

Finally we show the existence of Seifert 3-manifolds not obtained by integral surgery on knots stated in the introduction. That is, we can see that for any (p_1, p_2, p_3) with $\gcd(p_1, p_2, p_3) = 1$ such that the number of i with p_i even is either 0 or 2, there exist infinitely many Seifert 3-manifolds over $S^2(p_1, p_2, p_3)$ with H_1 even cyclic that are not obtained by integral surgery on knots in S^3. We will give the Seifert invariants of such examples as follows. We denote by c the spin structure defined by $c \equiv 0$ with respect to the given representation and the other spin structure by c'.

5.2.1. The case when p_1, p_2 are even and p_3 is odd

(1) The case when $p_3 = 4k - 1$ for some k. Consider

$$(*) \quad S = \{(1,b), (p_1, 1), (p_2, 1), (p_3, 2)\}, \ (b \text{ even}).$$

Then according to the above calculation we have $\overline{\mu}(S,c) = \operatorname{sgn} e(S) + 4$ for c define by $c \equiv 0$. The other spin structure c' is defined by $c' \equiv 0$ with respect to the Seifert invariants of the form

$$(**) \quad \{(1, b+2), (p_1, 1-p_1), (p_2, 1-p_2), (p_3, 2)\}$$

and we have $\overline{\mu}(S,c') = \operatorname{sgn} e(S) + 4 - p_1 - p_2$. Hence unless $p_1 = p_2 = 2$, neither $\overline{\mu}(S,c)$ nor $\overline{\mu}(S,c')$ is ± 1 if $b > 0$ (and $\operatorname{sgn} e(S) = -1$). If $p_1 = p_2 = 2$ we consider

$$S = \{(1,b), (2,1), (2,-1), (4k-1, 2)\}, \ (b \text{ even}).$$

Then we have $\overline{\mu}(S,c) = \overline{\mu}(S,c') = \operatorname{sgn} e(S) + 2$. Thus if $b < 0$, then $\overline{\mu}(S,c) = \overline{\mu}(S,c') = 3$.

(2) The case when $p_3 = 4k + 1$ for some k. Unless $p_1 = p_2 = 2$, consider S represented by $(*)$ with $b < 0$. Then we have $\overline{\mu}(S, c) = \text{sgn } e(S) + 2 = 3$ and $\overline{\mu}(S, c') = \text{sgn } e(S) + 2 - p_1 - p_2 = 3 - p_1 - p_2$. If $p_1 = p_2 = 2$, consider

$$S = \{(1, b), (2, 1), (2, 1), (p_3, p_3 - 1)\}, \ (b \text{ even}, b < 0).$$

Then we have $\overline{\mu}(S, c) = \text{sgn } e(S) + 4k + 2 = 4k + 3$, and $\overline{\mu}(S, c') = \text{sgn } e(S) + 4k - 2 = 4k - 1$.

5.2.2. The case when p_i is odd for every i

(1) The case when $p_i = 4k_i + 1$ for $i = 1, 2, 3$. We consider

$$S = \{(1, b), (p_1, 2), (p_2, 2), (p_3, p_3 - 1)\}, \ (b \text{ even}, b > 0).$$

Then we have $\overline{\mu}(S, c) = \text{sgn } e(S) + 4k_3 = 4k_3 - 1$. As is explained above considering the continued fractions of the form

$$(4k_1 + 1)/2 = [2k_1 + 1, 2], \ (4k_2 + 1)/2 = [2k_2 + 1, 2],$$
$$(4k_3 + 1)/4k_3 = [1, -4k_3],$$

we have $\overline{\mu}(S, c') = \text{sgn} e(S) + b + 4 = b + 3$ for $c' \neq c$.

(2) The case when $p_1 = 4k_1 + 1$, $p_2 = 4k_2 + 1$, and $p_3 = 4k_3 - 1$. We consider

$$(***) \ S = \{(1, b), (p_1, 2), (p_2, 2), (p_3, 2)\}, (b \text{ even}, b < -6).$$

Then we have $\overline{\mu}(S, c) = \text{sgn} e(S) + 2 = 3$. Since

$$(4k_3 - 1)/2 = [2k_3 - 1, -2],$$

we have $\overline{\mu}(S, c') = \text{sgn } e(S) + b + 4 = b + 5$ for $c' \neq c$.

(3) The case when $p_1 = 4k_1 + 1$, $p_2 = 4k_2 - 1$, and $p_3 = 4k_3 - 1$. Again consider S of the form $(***)$ with $b > 2$. Then we have $\overline{\mu}(S, c) = \text{sgn } e(S) + 4 = 3$, and $\overline{\mu}(S, c') = \text{sgn } e(S) + b + 2 = b + 1$.

(4) The case when $p_i = 4k_i - 1$ for $i = 1, 2, 3$. We consider S of the form $(***)$ with $b > 2$. Then we have $\overline{\mu}(S, c) = \text{sgn } e(S) + 6 = 5$, and $\overline{\mu}(S, c') = \text{sgn } e(S) + b = b - 1$.

In either case we have a Seifert 3-manifold S for infinitely many choices of b such that the Neumann-Siebenmann invariant of S is not ± 1 for any choice of spin structures. It follows from Theorem 2 that S is not obtained by an integral surgery on a knot in S^3.

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