INDEX THEORY FOR COVERINGS

BORIS VAILLANT

Contents

1. Introduction	2
2. Operators on Hilbert Γ -modules	4
2.1. Hilbert Γ -modules and the Γ -dimension function	4
2.2. Classes of Γ -operators	5
2.3. The spectrum of Γ -operators	6
2.4. Γ -Fredholm operators and the Γ -index	7
3. Manifolds of bounded geometry	9
3.1. Basics	9
3.2. Smoothing operators	11
3.3. Finite propagation speed estimates for the heat kernel	13
3.4. Families of operators	16
4. Coverings of compact manifolds	17
4.1. Coverings and Hilbert Γ -modules	18
4.2. Elliptic operators	19
4.3. The Γ -Eta-invariant	22
4.4. Residually finite coverings	25
5. Coverings of noncompact manifolds	27
5.1. Decomposition principle	28
6. Manifolds with cylindrical ends	29
6.1. Γ -operators with product structure on the cylinder	30
6.2. The L^2 - Γ -index	33
6.3. The L^2 - Γ - index theorem	37
6.4. The signature operator	42
Appendix A. Clifford algebra conventions	46
References	47

1. INTRODUCTION

This paper is a translated and revised version of the author's Diploma thesis [Va]. The starting point is Atiyah's work [A] on the index theory for elliptic operators on coverings of compact manifolds M.

Every elliptic differential operator $P \in \text{Diff}^k(M; E, F)$ on sections of hermitian vector bundles E, F over a compact manifold is invertible upto a smoothing operator and has a well-defined index.

In the case of a regular covering $\overline{M} \to M$ with an infinite group of deck transformations Γ and lifted vector bundles $\overline{E}, \overline{F}, \overline{P}$, the lifted operator \overline{P} can have essential spectrum down to 0 and is in general not Fredholm. It is however still possible to define the Γ -trace of a Γ -equivariant operator K on \overline{M} as the integral of its local trace over a fundamental domain \mathcal{F}

$$\operatorname{tr}_{\Gamma}(K) := \int_{\mathcal{F}} \operatorname{tr}([K](x,x)) dx.$$

The operator \overline{P} is then ' Γ -Fredholm' and has a well defined Γ -index

$$\operatorname{ind}_{\Gamma}(\overline{P}) = \operatorname{tr}_{\Gamma}(N(\overline{P})) - \operatorname{tr}_{\Gamma}(N(\overline{P^*})).$$

It is shown in [A] that

$$\operatorname{ind}_{\Gamma}(\overline{P}) = \operatorname{ind}(P). \tag{1.1}$$

In this paper, we will look at Γ -index Theorems for coverings $\overline{N} \to N$, where the base manifold N is noncompact but still has bounded geometry.

The simplest case of a noncompact manifold with bounded geometry is a manifold N with cylindrical ends $M \times [0, \infty]$. If we have a Clifford bundle (E, h^E) , also with a product structure over the cylinder, the Dirac operator D on E will in general not be Fredholm but still have finite dimensional null space and co-null space. For dim(N) = 2n the well-known formula of Atiyah, Patodi and Singer describes its L²-index:

$$L^{2}\text{-}\mathrm{ind}(D) + \frac{h_{-} - h_{+}}{2} = \int_{N} \widehat{A}(N)Ch(E/S) + \frac{1}{2}\eta(D^{M})$$

Here, $\eta(D^M)$ is the Eta-invariant of the restriction D^M of D to M. Die LHS is known as the *modified* L^2 -index of D.

The main result shown in this paper is the analogous formula for the (L^2) - Γ -index of the Dirac operator \overline{D} on $\overline{E} \to \overline{N}$. The proof, given in Chapter 6, combines the methods for the proof of the L^2 -index theorem developed in [Me], [Mue2] with the heat kernel methods used in the

proof of the Γ -index theorem in [Ro1]. The terminology is developped in Chapters 2 to 4.

We start in Chapter 2 by restating the purely functional analytic description of operators on Γ -Hilbert modules from [Br], [Sh2]. This is the natural context to develop the notion of the Γ -trace tr_{Γ}. We give the definitions of some classes of Γ -operators on Γ -Hilbert modules such as Γ -trace class- and Γ -compact operators, and describe them in terms of their spectral properties.

Chapter 3 gives an outline of the analysis on manifolds of bounded geometry as found in [Sh1], [Bu1]. Following [Ro1], we prove the central estimates for heat kernels of elliptic operators. A cornerstone of the proof of the L^2 - Γ -index theorem will be the introduction of a spectral modification $\overline{D}^M + f(\overline{D}^M)$ of D^M , and we formulate the theory and the estimates for such 'generalised elliptic' operators whenever possible.

The results of Chapters 2 and 3 will then be applied in Chapter 4 to elliptic Γ -differential operators \overline{P} on coverings of compact manifolds M. The focus will be on a proof of the existence of the Γ -Eta-invariant $\eta_{\Gamma}(\overline{D}^M + f(\overline{D}^M))$ for spectral modifications of the Dirac operator over M.

Finally, Chapter 6 gives the details of the proof of the L^2 - Γ -index theorem 6.11 for Dirac operators over manifolds with cylindrical ends. First, under the condition that

0 is an isolated point in the spectrum of
$$\overline{D}^M$$
 (1.2)

the analysis follows the lines of the classical case described in [Me]. In the general case, a spectral modification $\overline{D}_{\epsilon,u}$ of the Dirac operator \overline{D} is introduced for which (1.2) holds. The main task is then to set up a good book-keeping procedure to compare the Γ -dimensions of the null spaces of $\overline{D}_{\epsilon,u}$ and \overline{D} .

This work is the result of a project to better understand the ubiquituous condition (1.2) that appears in all generalisations of the L^2 index theorem. Using the book-keeping procedure developped in 6.2 the main result would also follow from the similar result for manifolds with boundaries given in [Ra]. In view of the Γ -signature theorem, the approach given here might perhaps be considered to be the more natural one.

Thanks are due to Werner Müller who started me on this project and whose work on Eta-invariants is at the center of many of the developments given here, as well as to Paolo Piazza and Thomas Schick who by their kind interest revived this work. Special thanks also to Mrs. Lütz who reteXed the original manuscript.

2. Operators on Hilbert Γ -modules

This Chapter describes some of the (spectral) theory of operators on Hilbert spaces with an equivariant action of a discrete group Γ . Much of this material can be found in [Sh2] and [Br].

2.1. Hilbert Γ -modules and the Γ -dimension function. We start with some terminology

Definition 2.1. (a) A free Hilbert Γ -module is a unitary right Γ module of the form $\mathcal{V} \otimes L^2(\Gamma)$, where \mathcal{V} is a Hilbert space.

- (b) A (projective) Hilbert Γ-module is a Hilbert space H with a unitary right action of Γ, along with a Γ-equivariant imbedding H → V ⊗ L²(Γ) into a free Hilbert Γ-module.
- (c) A morphism of Hilbert Γ -modules $\mathcal{H}_1, \mathcal{H}_2$ is a bounded Γ -equivariant operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. We denote the space of all such morphisms by $\mathcal{B}_{\Gamma}(\mathcal{H}_1, \mathcal{H}_2)$.

Denote by $tr_{\mathcal{V}} : \mathcal{B}(\mathcal{V})_+ \to [0, \infty]$ the usual trace on the Hilbert space \mathcal{V} . We will now analyse the properties of the trace on the von Neumann Algebra $\mathcal{B}_{\Gamma}(\mathcal{V} \otimes L^2(\Gamma)) \cong \mathcal{B}(\mathcal{V}) \otimes \mathcal{L}(\Gamma)$ of endomorphisms of the free Hilbert Γ -module $\mathcal{V} \otimes L^2(\Gamma)$ that is induced by $tr_{\mathcal{V}}$ and tr_{Γ} .

On $\mathcal{B}_{\Gamma}(\mathcal{V} \otimes L^2(\Gamma))$, we have the unique f.n.s trace $tr_{\mathcal{V}} \otimes tr_{\Gamma}$ which we simply denote by tr_{Γ} Also, for a projective Hilbert Γ -module \mathcal{H} , the Γ -embedding $\mathcal{H} \hookrightarrow \mathcal{V} \otimes L^2(\Gamma)$ gives a Γ -embedding of operators $\mathcal{B}_{\Gamma}(\mathcal{H}) \hookrightarrow \mathcal{B}_{\Gamma}(\mathcal{V} \otimes L^2(\Gamma))$. Through this embedding, we can define a trace tr_{Γ} on $\mathcal{B}_{\Gamma}(\mathcal{H})$. It is shown in [Sh2], that this definition is in fact independent of the projective embedding.

Lemma 2.2. For each (projective)Hilbert Γ -module there is a canonical f.n.s. trace $\operatorname{tr}_{\Gamma}$ on $\mathcal{B}_{\Gamma}(\mathcal{H})$. For a free Hilbert module $\mathcal{H} = \mathcal{V} \otimes L^2(\Gamma)$ with orthonormal basis $(\psi_j \otimes \gamma)_{j \in \mathbb{N}, \gamma \in \Gamma}$, and elements $A \in \mathcal{B}_{\Gamma}(\mathcal{H})_+$ this trace can be calculated by

$$\operatorname{tr}_{\Gamma}(A) = \sum_{j \in \mathbb{N}} \langle A\psi_j \otimes \epsilon, \psi_j \otimes e \rangle$$

Using the tr_{Γ} on \mathcal{H} , a Γ -dimension function on Γ -invariant subspaces $\mathcal{V} \subset \mathcal{H}$ can be defined. First, note that the closure $cl(\mathcal{V})$ is Γ -invariant and denote the orthogonal projection onto $cl(\mathcal{V})$ by $E_{\mathcal{V}} \in \mathcal{B}_{\Gamma}(\mathcal{H})$. The Γ -dimension of \mathcal{V} is then defined by

$$\dim_{\Gamma}(\mathcal{V}) := \operatorname{tr}_{\Gamma}([\operatorname{cl}(\mathcal{V})])$$

The f.n.s.-property of tr_{Γ} implies the following properties for \dim_{Γ} :

Lemma 2.3. (a) $\dim_{\Gamma}(\mathcal{V}) \in [0, \infty]$ and $\dim_{\Gamma}(\mathcal{V}) = 0 \Leftrightarrow \mathcal{V} = \{0\}$

- (b) Let $(\mathcal{V}_i)_{i\in\mathbb{N}}$ be an increasing family of Γ -invariant subspaces of \mathcal{H} . Then $\dim_{\Gamma}(\bigcup \mathcal{V}_i) = \liminf_{\Gamma}(\mathcal{V}_i)$.
- (c) Let $(\mathcal{V}_i)_{i \in \mathbb{N}}$ be a decreasing of Γ invariant subspaces of \mathcal{H} . Then $\dim_{\Gamma}(\bigcap \mathcal{V}_i) = \liminf \dim_{\Gamma}(\mathcal{V}_i).$
- (d) dim_{Γ} is additive: dim_{Γ}($\mathcal{V}_1 \oplus \mathcal{V}_2$) = dim_{Γ}(\mathcal{V}_1) + dim_{Γ}(\mathcal{V}_2).

2.2. Classes of Γ -operators. As in classical Hilbert space theory, we can introduce different sub-classes of the endomorphisms $\mathcal{B}_{\Gamma}(\mathcal{H})$ of a Hilbert Γ -module \mathcal{H} .

Write R(A) for the projection onto the image A

 $R(A) = \inf\{P \mid P \text{ projection in } \mathcal{B}_{\Gamma}(\mathcal{H}) \text{ with } PA = A\},\$

and denote the projection onto the null space of A by N(A). From the polar decomposition of A we obtain

Lemma 2.4. (a) Let ~ denote the equivalence of projections in in $\mathcal{B}_{\Gamma}(\mathcal{H})$. Then for every $A \in \mathcal{B}_{\Gamma}(\mathcal{H})$

$$R(A) \sim R(A^*) = 1 - N(A).$$

(b) Let $A \in \mathcal{B}_{\Gamma}(\mathcal{H}_1, \mathcal{H}_2)$ be a quasiisomorphism of Hilbert Γ -modules, i.e. N(A) = 0 and $R(A) = \mathcal{H}_2$. Then

$$\dim_{\Gamma}(\mathcal{H}_1) = \dim_{\Gamma}(\mathcal{H}_2).$$

We now define as usual

Definition 2.5. Let $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ be Hilbert Γ -modules.

- (a) $\mathcal{B}^{f}_{\Gamma}(\mathcal{H}_{1},\mathcal{H}_{2}) := \{A \in \mathcal{B}_{\Gamma}(\mathcal{H}_{1},\mathcal{H}_{2}) \mid \operatorname{tr}_{\Gamma}(R(A)) < \infty\}$ are the Γ -operators of finite Γ -rank.
- (b) The space $\mathcal{B}^{\infty}_{\Gamma}(\mathcal{H}_1, \mathcal{H}_2) \equiv \mathcal{K}_{\Gamma}(\mathcal{H}_1, \mathcal{H}_2)$ of Γ -compact operators is the norm closure of $\mathcal{B}^f_{\Gamma}(\mathcal{H}_1, \mathcal{H}_2)$.
- (c) $\mathcal{B}^2_{\Gamma}(\mathcal{H}) := \{A \in \mathcal{B}_{\Gamma}(\mathcal{H}) \mid \operatorname{tr}_{\Gamma}(AA^*) < \infty\}$ are the Γ -Hilbert-Schmidt operators.
- (d) $\mathcal{B}^1_{\Gamma}(\mathcal{H}) = \mathcal{B}^2_{\Gamma}(\mathcal{H})\mathcal{B}^2_{\Gamma}(\mathcal{H}^*) \equiv \{A \in \mathcal{B}_{\Gamma}(\mathcal{H}) \mid A = \sum_{i=1}^n S_i T_i^* \text{ with } S_i, T_i \in \mathcal{B}^2_{\Gamma}(\mathcal{H})\}$ are the Γ -trace class operators.

The spaces thus-defined share a number of the properties of their classical counterparts

Lemma 2.6. (a) $\mathcal{B}^{f}_{\Gamma}(\mathcal{H}), \mathcal{B}^{1}_{\Gamma}(\mathcal{H}), \mathcal{B}^{2}_{\Gamma}(\mathcal{H}), \mathcal{B}^{\infty}_{\Gamma}(\mathcal{H})$ are two-sided *-ideals in $\mathcal{B}_{\Gamma}(\mathcal{H})$.

- (b) $\mathcal{B}^{1}_{\Gamma}(\mathcal{H}) = \{A \in \mathcal{B}_{\Gamma}(\mathcal{H}) | \operatorname{tr}_{\Gamma}(|A|) < \infty \}.$
- (c) $\mathcal{B}^f_{\Gamma}(\mathcal{H}) \subset \mathcal{B}^1_{\Gamma}(\mathcal{H}) \subset \mathcal{B}^2_{\Gamma}(\mathcal{H}) \subset \mathcal{B}^{\infty}_{\Gamma}(\mathcal{H}).$

(d) $A \in \mathcal{B}^*_{\Gamma}(\mathcal{H}) \Leftrightarrow |A| \in \mathcal{B}^*_{\Gamma}(\mathcal{H}), \ * = f, 1, 2, \infty.$

Remark 2.7. Let $A, B \in \mathcal{B}_{\Gamma}(\mathcal{H})$ be two self adjoint operators, and let *B* be Γ -trace class. Then *A* has a unique decomposition $A = A^+ - A^$ into a sum of positive operators $A^{\pm}, ||A^{\pm}|| \leq ||A||$, and

$$|\operatorname{tr}_{\Gamma}(BAB^*)| = |\operatorname{tr}_{\Gamma}(BA^*B^*) - \operatorname{tr}_{\Gamma}(BA^-B^*)|$$

$$\leq \sup\{\operatorname{tr}_{\Gamma}(BA^+B^*), \operatorname{tr}_{\Gamma}(BA^-B^*)\} \leq \operatorname{tr}_{\Gamma}(BB^*) \parallel A \parallel.$$

For positive B we note especially $|\operatorname{tr}_{\Gamma}(AB)| \leq \operatorname{tr}_{\Gamma}(B) ||A||$.

2.3. The spectrum of Γ -operators. In the following, let \mathcal{H} be a Hilbert Γ -module, and $T : \mathcal{H} \supset dom(T) \rightarrow \mathcal{H}$ a not necessarily bounded Γ -operator on \mathcal{H} . Thus, the domain of T is Γ -invariant, $\mathbf{r}(\gamma)dom(T) \subset$ dom(T), and $T\mathbf{r}(\gamma)\psi = \mathbf{r}(\gamma)T\psi$ for all $\gamma \in \Gamma, \psi \in dom(T)$ (T is said to be *affiliated* to $\mathcal{B}_{\Gamma}(\mathcal{H})$). For selfadjoint T the projection-valued measure is denoted by $E_T(U) \in \mathcal{B}_{\Gamma}(\mathcal{H})$. For each Borel set $U \subset \mathbb{R}$ we denote the corresponding spectral subspace by $\mathcal{H}_T(U) := \operatorname{im}(E_T(U))$. From the results of Section 2.1 we deduce that

$$\mu_{\Gamma,T}(U) := \operatorname{tr}_{\Gamma}(E_T(U)) = \dim_{\Gamma}(\mathcal{H}_T(U))$$

defines a Borel measure on \mathbb{R} whose support is the spectrum spec(T) of T. If $f : \mathbb{R} \to [0, \infty]$ is a bounded Borel function, we have

$$\int_{\mathbb{R}} f d\mu_{\Gamma,T} = \operatorname{tr}_{\Gamma}(f(T)),$$

where we allow both sides of the equation to equal ∞ .

The Γ -spectral measure $\mu_{\Gamma,T}$ gives a rough but useful classification of the spectrum of T.

Definition 2.8. spec_{Γ,e}(T) := { $\lambda \in \mathbb{R} \mid \forall_{\epsilon>0} \mu_{\Gamma,T}([\lambda - \epsilon, \lambda + \epsilon]) = \infty$ } is called the Γ -essential spectrum of T.

This can be used to obtain the following simple spectral characterisation of Γ -compact operators

Proposition 2.9. Let A, S be selfadjoint Γ -operators on \mathcal{H} . Let S be bounded.

- (a) $S \in \mathcal{B}^f_{\Gamma}(\mathcal{H}) \Rightarrow \operatorname{spec}_{\Gamma,e}(S) \subset \{0\}.$
- (b) S is Γ -compact \Leftrightarrow spec $_{\Gamma,e}(S) \subset \{0\}$.
- (c) Let S be Γ -compact. Then $(A+S \text{ is selfadjoint and}) \operatorname{spec}_{\Gamma,e}(A+S) = \operatorname{spec}_{\Gamma,e}(A).$

Proof. (a) is obvious. (b): '⇒': Choose *S* compact and selfadjoint. First note that *S* can be approximated by *selfadjoint* elements in $\mathcal{B}_{\Gamma}^{f}(\mathcal{H})$. For $\lambda \in \operatorname{spec}_{\Gamma,e}(S)$ and $\epsilon > 0$ the space $\mathcal{H}_{S}(]\lambda - \epsilon, \lambda + \epsilon[)$ is of infinite Γ-dimension. Then choose a selfadjoint *F* of finite Γ-rank, such that $|| S - F || < \epsilon$. For every $\varphi \in \mathcal{H}_{S}(]\lambda - \epsilon, \lambda + \epsilon[)$ we then have

$$\| (F - \lambda)\varphi \| \le \| S - F \| \| \varphi \| + \| (s - \lambda)\varphi \| \le 2\epsilon \| \varphi \|,$$

thus $\mathcal{H}_S(]\lambda - \epsilon, \lambda + \epsilon[) \subset \mathcal{H}_F(]\lambda - 2\epsilon, \lambda + 2\epsilon[)$, so the RHS must have infinite Γ -dimension. Using (a), this implies $\lambda = 0$.

' \Leftarrow ' : Let $\operatorname{spec}_{\Gamma,e}(S) \subset \{0\}$. Since S is bounded, the projections $E_S(\mathbb{R}-]-\epsilon,\epsilon[)$ must be of finite Γ -rank for every $\epsilon > 0$. Thus $SE_S(\mathbb{R}-]-\epsilon,\epsilon[)$ gives a norm-approximation of S by Γ -finite operators for $\epsilon \to 0$. (c): We show $\operatorname{spec}_{\Gamma,e}(A) \subset \operatorname{spec}_{\Gamma,e}(A+S)$. To do this, let $\lambda \in \operatorname{spec}_{\Gamma,e}(A)$, i.e. we have $\dim_{\Gamma}(\mathcal{H}_A(]\lambda - \epsilon, \lambda + \epsilon[)) = \infty$ for all $\epsilon > 0$. Now consider the set

$$G_{\epsilon} := \{ \varphi \in \mathcal{H}_{A}(]\lambda - \epsilon, \lambda + \epsilon[) \mid \| S\varphi \| < \epsilon \| \varphi \| \}$$
$$= \mathcal{H}_{A}([\lambda - \epsilon, \lambda + \epsilon[) \cap \mathcal{H}_{S}(] - \epsilon, \epsilon[)$$

But $\mathcal{H}_{S}(]-\epsilon, \epsilon[)$ is of finite Γ -codimension, and therefore G_{ϵ} is of infinite Γ -Dimension. By construction, $G_{\epsilon} \subset \mathcal{H}_{A+S}(]\lambda - 2\epsilon, \lambda + 2\epsilon[)$ thus $\lambda \in \operatorname{spec}_{\Gamma,e}(A+S)$.

2.4. Γ -Fredholm operators and the Γ -index. Here we look at Γ -Fredholm operators and their properties. Again, this closely follows the lines of the Hilbert space analogue. As usual, denote by $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ Hilbert Γ -modules.

Definition 2.10. An operator $F \in \mathcal{B}_{\Gamma}(\mathcal{H}_1, \mathcal{H}_2)$ is called Γ -Fredholm if there are $G \in \mathcal{B}_{\Gamma}(\mathcal{H}_2, \mathcal{H}_1)$ and $K_1 \in \mathcal{K}_{\Gamma}(\mathcal{H}_1), K_2 \in \mathcal{K}_{\Gamma}(\mathcal{H}_2)$, such that

$$FG = 1 - K_2$$
 $GF = 1 - K_1$.

Denote by $\mathcal{F}_{\Gamma}(\mathcal{H}_1, \mathcal{H}_2)$ the space of Γ -Fredholm operators $\mathcal{H}_1 \to \mathcal{H}_2$.

From this definition and the ideal property of $\mathcal{K}_{\Gamma}(\mathcal{H})$ we deduce that the space of Γ -Fredholm operators $\mathcal{F}_{\Gamma}(\mathcal{H})$ is closed unter the *-operation and under concatenation of operators. It is also easy to see that $\mathcal{F}_{\Gamma}(\mathcal{H})$ is an open subset of $\mathcal{B}_{\Gamma}(\mathcal{H})$.

Contrary to their classical counterparts, Γ -Fredholm operators usually do not have a closed image, their essential spectrum can contain 0. However, a version of the spectral description of Fredholm operators also holds in the Γ -case. **Proposition 2.11.** The following statements are equivalent for a Γ operator $F = \mathcal{B}_{\Gamma}(\mathcal{H}_1, \mathcal{H}_2)$:

- (a) F is Γ -Fredholm.
- (b) $0 \notin \operatorname{spec}_{\Gamma,e}(F^*F)$ and $0 \notin \operatorname{spec}_{\Gamma,e}(FF^*)$.
- (c) $0 \notin \operatorname{spec}_{\Gamma,e} \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}$, where $\begin{pmatrix} 1 & F^* \\ F & 0 \end{pmatrix} \in \mathcal{B}_{\Gamma}(\mathcal{H}_1 \oplus \mathcal{H}_2).$
- (d) N(F) is a projection of finite Γ -rank in \mathcal{H}_1 and there is a projection E in \mathcal{H}_2 of finite Γ -rank, such that $\operatorname{im}(1-E) \subset \operatorname{im}(F)$.

Proof. This follows essentially like in the classical case.

Definition 2.12. Let $F \in \mathcal{B}_{\Gamma}(\mathcal{H}_1, \mathcal{H}_2)$ be Γ -Fredholm, E and as before. Then $\operatorname{tr}_{\Gamma}(N(F)) < \infty$ and $\operatorname{tr}(1 - R(F)) \leq \operatorname{tr}_{\Gamma}(E) < \infty$. We can therefore define the Γ -index of F as

$$ind_{\Gamma}(F) := \operatorname{tr}_{\Gamma}(N(F)) - \operatorname{tr}_{\Gamma}(1 - R(F)).$$

The Γ -index shares the algebraic properties of the classical index:

Proposition 2.13. For operators $S, T \in \mathcal{F}_{\Gamma}(\mathcal{H}), \mathcal{K} \in \mathcal{K}_{\Gamma}(\mathcal{H})$ the following holds true

- (a) $\operatorname{ind}_{\Gamma}(S^*) = \operatorname{ind}_{\Gamma}(S)$.
- (b) $\operatorname{ind}_{\Gamma}(ST) = \operatorname{ind}_{\Gamma}(S) + \operatorname{ind}_{\Gamma}(T)$.
- (c) $\operatorname{ind}_{\Gamma}(S+K) = \operatorname{ind}_{\Gamma}(S)$ especially $\operatorname{ind}_{\Gamma}(1+K) = 0$.
- (d) $\operatorname{ind}_{\Gamma} : \mathcal{F}_{\Gamma}(\mathcal{H}) \to \mathcal{C}$ is locally constant.

Proof The proof of these statements is analogous to the classical proofs and can all be found in [Br].

An unbounded, closed Γ -Operator $T : \mathcal{H}_1 \supset dom(T) \to \mathcal{H}_2$ is called Γ -Fredholm, if the bounded Γ -operator $T : (dom(T), \|\cdot\|_T) \to \mathcal{H}_2$ is Γ -Fredholm. Here, $\|\cdot\|_T$ is the *T*-graph norm. We will frequently use a \mathbb{Z}_2 -graded version of Proposition 2.11(c). An unbounded, closed, odd Γ -Operator on a \mathbb{Z}_2 -graded Hilbert Γ -module $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is called Γ -Fredholm, when the (unbounded) operator $T^+ : \mathcal{H}^+ \to \mathcal{H}^-$ is Γ -Fredholm. The Γ -index of T is then defined as $\operatorname{ind}_{\Gamma}(T) := \operatorname{ind}_{\Gamma}(T^+)$.

Proposition 2.14. Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ be a \mathbb{Z}_2 -graded Hilbert Γ -module, and T an unbounded, closed, odd Γ -operator on \mathcal{H} . Then the following two statements are equivalent

- (a) T is Γ -Fredholm.
- (b) $0 \notin \operatorname{spec}_{\Gamma,e}(T)$.

3. Manifolds of bounded geometry

This Chapter describes some of the methods of the analysis of differential operators on manifolds of bounded geometry. The results in this Chapter will be applied in the next Chapter to differential operators on covering manifolds. Most of the concepts presented here are wellknown and can be found in different variations in [Sh1], [Bu1], [Ro1], [Lo3], [Ei].

3.1. **Basics.** A not necessarily compact riemannian manifold N of dimension n is said to be of *bounded geometry* if the injectivity radius i(N) of N is positive, and the curvature R^N and all of its covariante derivatives are bounded. A hermitian vector bundle $E \to N$ is of *bounded geometry*, if, in addition, the curvature F^E and all of its covariant derivatives are bounded. Manifolds of bounded geometry admit systems of local coordinates that have uniform C^{∞} -estimates [Ei]:

Lemma 3.1. Let $E \to N$ be of bounded geometry. Fix a constant $r_0 < i(N)$ and choose a 'good' trivialisation of E, i.e. E is trivialised via radial parallel transport over each ball $B(x, r_0)$.

- (a) The metric tensor g has bounded C^{∞} -norm w.r.t. all normal coordinate neighborhoods $B(x, r_0)$, independent of x. All derivatives of coordinate change maps Φ_{xy} between such normal neighborhoods $B(x, r_0)$, $B(y, r_0)$ are uniformly bounded independent of x and y.
- (b) In each normal neighborhood B(x,r₀) with a 'good' trivialisation of E, the local connection form has bounded C[∞]-norm independent of x. All derivatives of transition maps Ψ_{xy} between the trivialisations over B(x,r₀), B(y,r₀) are uniformly bounded independent of x and y.

In the following, let N, E be of bounded geometry and choose a fixed $r_0 < i(N)$. We will always use a 'good' trivialisation of E over normal neighborhoods $B(x, r_0)$. Now define

$$UC^{\infty}(N) := \{ f \in C^{\infty}(N) | \| \nabla^{N,k} f \|_{\infty} < C(k) \text{ for all } k \in \mathbb{N} \}$$
$$U\Gamma(N,E) := \{ \xi \in \Gamma(N,E) | \| \nabla^{E,k} \xi \|_{\infty} < C(k) \text{ for all } k \in \mathbb{N} \} \text{ etc.}$$

Equivalently, a section ξ is in $U\Gamma$ if ξ and its derivatives are uniformly bounded in any local normal coordinate neighborhood U (and a corresponding 'good' trivialisation of E) independent of U.

As usual, Sobolev spaces $H^k(N)$, $k \ge 0$, can be defined as the completion of $C_c^{\infty}(N)$ with respect to the norm

$$\parallel f \parallel^2_{H^k(N)} := \sum_{j=0}^k \parallel \nabla^j f \parallel^2_{L^2(N, \ \otimes^j T^*N)}$$

The negative Sobolev space $H^{-k}(N)$ is then just the dual $H^k(N)$. Similar definitions apply for spaces of sections over N.

Using a uniformly bounded partition of unity (within a 'good' trivialisation of E as before), the Sobolev norms for Sections $f \in C_c^{\infty}(N, E)$ can locally be described as follows

$$\parallel f \parallel^2_{H^s(N,E)} \sim \sum_{j \in N} \parallel \phi_i f \parallel^2_{H^s(U_j,\mathbb{C}^N)}.$$

We will use a variety of notations for the same space $H^s(N, E) \triangleq H^s(N)$ $\triangleq H^s(E) \triangleq H^s$ etc., depending on which part of the information is important in the particular context.

The following version of the Sobolev inequalities is now easy to prove along the lines of its classical counterpart.

Proposition 3.2. (Sobolev) Let $k \in \mathbb{N}$ and s > k + n/2. Then there is a continuous embedding $H^s(N, E) \to UC^k(N, E)$.

The algebra $\operatorname{UDiff}^*(N, E)$ of uniform differential oprators is generated by the uniform Sections $\Phi \in U\Gamma(N, End(E))$ and the covariant derivatives ∇_X^E along uniform vectorfields $X \in U\Gamma(N, TN)$. A differential operator P is uniform if and only if its local symbol and all its derivatives are uniformly bounded with respect to a 'good' system of coordinates of N and E. The operator $P \in \operatorname{UDiff}^k(N, E)$, maps $UC^l(N, E)$ continuously to $UC^{l-k}(N, E)$ and maps $H^s(N, E)$ continuously to $H^{s-k}(N, E)$. Note that the uniformity of the estimates for P is essential for this to hold! Uniform pseudodifferential operators can be defined in a similar manner.

A uniform differential operator P of order k on E is uniformly elliptic, if its principal symbol $\sigma(P) \in U\Gamma(T^*N, \pi^*End(E))$ has a uniform inverse outside of an ϵ -neighborhood of the null section in T^*N . The construction of a parametrix for such operators can then also be performed in a uniform manner and one can use this to show

Proposition 3.3. (Garding) Let $T \in \text{UDiff}^k(N, E)$ be a uniformly elliptic differential operator. Then

$$\|\varphi\|_{H^{s+k}(N,E)} \leq C(s,k) (\|\varphi\|_{H^s(N,E)} + \|T\varphi\|_{H^s(N,E)})$$

$$(3.3)$$

for $\varphi \in C_c^{\infty}(N, E), \ s \in \mathbb{R}$.

In Chapter 6 we will be working with spectral modifications of the Dirac operator that are not pseudo-differential operators but which still share a number of their mapping properties. We introduce the according spaces of operators here. A bounded operator $T: C_c^{\infty}(N, E) \rightarrow$ $C_c^{\infty}(N, E)'$ with Schwartz-kernel [T], is called an operator of order $k \in$ \mathbb{Z} , if it has extensions into all spaces $\mathcal{B}(H^s(N, E), H^{s-k}(N, E)), s \in \mathbb{R}$. As a (possibly unbounded) operator on $L^2(N, E)$, T is closable. To simplify things a bit we will always ask that T map components of N_i , of N to themselves: $T(C_c^{\infty}(N_i, E)) \subset C_c^{\infty}(N_i, E)'$. Obviously, all uniform (pseudo-) differential operators of order k are in $Op^k(N, E)$. We put the obvious family of norms onto the space $Op^k(N, E)$ and also write $Op^{-\infty} = \bigcap_{k \in \mathbb{N}} Op^{-k}$. An operator $T \in Op^k(\mathbb{N}, E), k \geq 1$, will be called *elliptic*, simply when it satisfies the Garding-inequality (3.3). Note that if $T \in Op^k(N, E)$ is elliptic and $U \in Op^0(N, E)$, then T + Uis also elliptic. Also, if $T \in Op^k(N, E)$ is elliptic and selfadjoint, then all spectral projections of T are in $Op^0(N, E)$. We note

Proposition 3.4. Let $T \in Op^k(N, E)$ elliptic and formally selfadjoint, $k \ge 1$. Then T is essentially selfadjoint and (without a different notation for the closure of T) $dom(T) = H^k(N, E)$.

3.2. Smoothing operators. Let again $T \in Op^k(N, E)$ be elliptic and formally selfadjoint. Following Proposition 3.4 we can interpret T as a selfadjoint operator with $dom(T) = H^k(N, E)$. In this Section, we analyse the properties of operators of the form f(T) for a sensible choice of function f. The most sensible spaces of such functions are

$$RB(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} \text{ Borelfuntkion } | |(1+x^2)^{k/2} f(x)|_{\infty} < \infty, \ k \in \mathbb{N} \}$$
$$RC(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} \text{ stetig } | |(1+x^2)^{k/2} f(x)|_{\infty} < \infty, \ k \in \mathbb{N} \}$$

The function space $RC(\mathbb{R})$ with the family of seminorms $|(1+x^2)^{k/2}f(x)|_{\infty}$ is Fréchet. For $f \in RB(\mathbb{R})$ and $l \in \mathbb{N}$, the operator $T^lf(T)$ is bounded on $L^2(N, E)$ and we find using the Garding-inequality (3.3)

$$\| f(T)\psi \|_{H^{l}(N)} \leq C(l) \sum_{i=0}^{l} \| T^{i}f(T)\psi \|_{L^{2}(N)} \leq C(l) \| \psi \|_{L^{2}(N)} \sum_{i=0}^{l} |x^{i}f|_{\infty},$$

for any $\Psi \in C_c^{\infty}(N, E)$. Here we have made the simplifying (but by no means essential) assumption that T is of order 1. Using the duality $(H^s)^* = H^{-s}$ one can show for all $k, l \in \mathbb{Z}, l \geq k$:

$$\| f(T)\psi \|_{H^{l}(N)} \leq C(l,k) \sum_{i=0}^{l-k} \| T^{i}f(T)\psi \|_{H^{k}(N)} \leq C(l,k) \| \psi \|_{H^{k}(N)} \sum_{i=0}^{l-k} |x^{i}f|_{\infty},$$

from which it follows that the map

$$RC(\mathbb{R}) \to \mathcal{B}(H^k(N, E), H^l(N, E)) \quad f \longmapsto f(T)$$

is continuous for all $k, l \in \mathbb{Z}$, especially $RC(\mathbb{R}) \to Op^{-\infty}(N, E)$ continuously. The kernels of such operators are smooth.

Proposition 3.5. Set $L := [n/2 + 1], l \in \mathbb{N}$. Then the Schwartz-kernel map

$$Op^{-2L-l}(N, E) \to UC^{l}(N \times N, E \boxtimes E^{*})(T \longmapsto [T])$$

is continuous.

Proof. We simplify notation a bit by forgetting about the coefficient bundle, i.e. $E = \{0\}$. Choose $r, s \in \mathbb{N}$ with $r + s \leq l$, and an elliptic selfadjoint operator $Q \in \text{UDiff}^1(N)$. Then

$$\begin{aligned} |\nabla_x^r \nabla_y^s [T](x_0, y_0)| &\stackrel{3.2}{\leq} C(l) \parallel \nabla_x^r [T](x_0, \bullet) \parallel_{H^{L+s}(N)} \\ &\stackrel{3.3}{\leq} C(l) \sum_{j=0}^{L+s} \parallel \nabla_x^r [TQ^j](x_0, \bullet) \parallel_{L^2(N)} \end{aligned}$$

Setting $\xi_j(y) = \nabla_x^r [TQ^j](x_0, y)$ we can do the same estimate again

$$\| \xi_j \|_{L^2(N)}^2 = \int \nabla_x^r [TQ^j](x_0, y)\xi_j(y)dy = |\nabla_x^r (TQ^j\xi_j)(x_0)|$$

$$\leq C(l) \| TQ^j\xi_j \|_{H^{L+r}(N)}$$

$$\leq C(l) \| TQ^j \|_{\mathcal{B}(L^2, H^{L+r})} \| \xi_j \|_{L^2(N)}$$

$$\leq C(l) \| T \|_{\mathcal{B}(H^{-L-s}, H^{L+r})} \| \xi_j \|_{L^2(N)}$$

Together

$$\begin{aligned} |\nabla_x^r \nabla_y^s[T](x_0, y_0)| &\leq C(l) \sum_{j=0}^{L+s} \| TQ^j \|_{\mathcal{B}(L^2, H^{L+r})} \leq C(l) \| T \|_{\mathcal{B}(H^{-L-s}, H^{L+r})} \end{aligned}$$

This implies that $Op^{-2L-l}(N, E) \to UC^l(N \times E \boxtimes E^*)$ is continuous.

Corollary 3.6. (a) $[f(T)] \in UC^{\infty}(N \times N, E \boxtimes E^*)$ for $f \in RB(\mathbb{R})$.

(b) The corresponding map $RC(\mathbb{R}) \to UC^{\infty}(N \times N, E \boxtimes E^*)$ is continuous.

3.3. Finite propagation speed estimates for the heat kernel. Sofar, for T as in Section 3.2, we have no estimates on [f(T)] at infinity. For differential operators $P \in \text{UDiff}(N, E)$ which are uniformly elliptic and selfadjoint, such estimates can be obtained by using finite propagation speed methods. For this, we again assume that Pis of order 1. From the formal selfadjointness of P we find for any $\xi \in \Gamma_c(N, E)$:

$$\langle P\xi,\xi\rangle_E - \langle\xi,P\xi\rangle_E = d^*\langle\xi,\sigma(P)\xi\rangle_E$$
 (3.4)

The expression $|\sigma(P)|_{2, T^*N\otimes End(E)}(x)$ is known as the propagation speed of P in x. The maximal propagation speed of P on N is then $c = c(P) := \sup\{|\sigma(P)|_2(x) \mid x \in N\}$: Using the spectral theorem, we know that for each $\xi_0 \in C_c^{\infty}(N, E)$ there is a *unique* solution $\xi(t) = e^{itP}\xi_0 \in L^2(N, E)$ of the wave equation $\frac{\partial \xi}{\partial t} - iP\xi = 0, \xi(0) = \xi_0$. Using 3.2 and 3.3 it is easy to see that $\xi(t) \in UC^{\infty}(N, E)$. The following Lemma states, that $\xi(t)$ 'propagates' with finite speed:

Lemma 3.7. ('Energy estimate') For a sufficiently small $\Lambda \in \mathbb{R}$ an all $x \in N$ the norm $|| \xi(t) ||_{L^2(B(x,\Lambda-ct))}$ is monotonously decreasing in t. More specifically, P has propagation speed limited by c, since $\operatorname{supp}(\xi_0) \subset B(x,r)$ implies $\operatorname{supp}(\xi(t)) \subset B(x,r+ct)$.

Proof. This is proved using (3.4) as in [Ro1, Proposition 5.5]. See also the proof of Lemma 6.1

The finite propagation speed of P can be used to obtain estimates for more general f(P) and [f(P)]. Using the above Lemma for $\xi \in L^2(N, E)$ with $\operatorname{supp}(\xi) \subset B(x, r)$ we know

$$\operatorname{supp}(e^{isP}\xi) \subset B(x, r+c|s|).$$

This can then be plugged into the spectral representation

$$f(P)\xi = (2\pi)^{1/2} \int_{\mathbb{R}} \widehat{f}(s) e^{isP} \xi ds,$$

valid for all $f \in \mathcal{S}(\mathbb{R})$. Thus

$$\| f(P)\xi \|_{L^{2}(N-B(x,R))} = \| (2\pi)^{-1/2} \int_{\mathbb{R}} \widehat{f}(s)e^{isP}\xi ds \|_{L^{2}(N-B(x,R))}$$

$$\leq \| (2\pi)^{-1/2} \int_{\mathbb{R}-I_{R}} \widehat{f}(s)e^{isP}\xi ds \|_{L^{2}(N)}$$

$$\leq (2\pi)^{-1/2} \| \xi \|_{L^{2}(N)} \int_{\mathbb{R}-I_{R}} |\widehat{f}(s)| ds.$$
(3.5)

Here, we have set $I_R :=] - \frac{R-r}{c}, \frac{R-r}{c} [$, or $I_R = \emptyset$ if $R \leq r$. Thus, for s in I_R the solution $e^{isP}\xi$ has not yet left B(x, R).

Proposition 3.8. For a sufficiently small r_1 , and all $x, y \in N$ set $R(x, y) := \max\{0, d(x, y) - r_1\}$. Then, writing $L := \lfloor n/2 + 1 \rfloor$, $I(x, y) := \lfloor -\frac{R(x,y)}{c}, \frac{R(x,y)}{c} \rfloor$ for all $f \in \mathcal{S}(\mathbb{R})$, we have the estimate

$$|\nabla_x^l \nabla_y^k [f(P)](x,y)| \le C(P,l,k,r_1) \sum_{j=0}^{2L+l+k} \int_{\mathbb{R}^{-I(x,y)}} |\widehat{f}^{(j)}(s)| ds.$$

Proof. Using the same technique as in the proof of Proposition 3.5, we obtain

$$\begin{aligned} |\nabla_x^l \nabla_y^k [f(P)](x_0, y_0)| &\leq C \sum_{i=0}^{L+l} \sum_{j=0}^{L+k} \parallel P^{j+i} f(P) \parallel_{L^2(B(x_0, r_1/2)), L^2(B(y_0, r_1/2))} \\ &\stackrel{(3.5)}{\leq} C \sum_{j=0}^{2L+l+k} \int_{\mathbb{R}^{-I}(x_0, y_0)} |\widehat{f}^{(j)}(s)| ds. \end{aligned}$$

We now want to use this result to obtain specific estimates for the heat kernel $[f(P)] = [e^{-tP^2}]$. It is well-known that

$$\hat{f}^{(k)}(s) = \frac{1}{(2t)^{1/2}(4t)^{k/2}} \left((4t)^{1/2} \frac{\partial}{\partial s} \right)^k e^{-(s/(4t)^{1/2})^2} \\ = \frac{C(k)}{t^{(k+1)/2}} H_k(s/(4t)^{1/2}) e^{-(s/(4t)^{1/2})^2},$$

where H_k is the kth Hermite polynomial. This is even for even k and odd for odd k, and using

$$\int_{u}^{\infty} e^{-x^2} dx \le e^{-u^2}, \quad y^s e^{-ay^2} \le \left(\frac{s}{2ae}\right)^{s/2}, \quad s, u, y, a \in \mathbb{R}_+,$$

one obtains

$$\int_{u}^{\infty} y^{s} e^{-y^{2}} dy = \int_{u}^{\infty} y^{s} e^{-ey^{2}} e^{-(1-\epsilon)y^{2}} dy \le C(s,\epsilon) e^{-(1-\epsilon)u^{2}}$$

Using Proposition 3.8 and setting $R = R(x, y), \Lambda = 2L + l + k$, we get

$$\begin{aligned} \left| \nabla_x^l \nabla_y^k [P^m e^{-tP^2}](x,y) \right| \\ &\leq C \sum_{j=m}^{\Lambda+n} t^{-j/2} \int_{R/c}^{\infty} |H_j(s/(4t)^{1/2})| e^{-(s/(4t)^{1/2})^2} (4t)^{-1/2} ds \\ &\leq C \sum_{j=m}^{\Lambda+m} t^{-j/2} \int_{R/2c\sqrt{t}}^{\infty} |H_j(x)| e^{-x^2} dx \end{aligned} \tag{3.6}$$
$$&\leq C e^{-R^2/5c^2t} \sum_{j=m}^{\Lambda+m} t^{-j/2} \leq \begin{cases} C(k,l,m,P) t^{-m/2} e^{-R^2/6c^2t}, \quad t > T \\ C(k,l,m,P) e^{-R^2/6c^2t}, \quad d(x,y) > 2r_1, t \in \mathbb{R}_+. \end{cases}$$

This is only useful away from the diagonal. We will obtain estimates for $[e^{-tP^2}]$ in a neighborhood of the diagonal $N \times N$ and for small tfrom the corresponding estimates for operators on compact manifolds. To do this we need a 'relative' version of Proposition 3.8. Let N_1, N_2 be manifolds and $E_1 \to N_1, E_2 \to N_2$ hermitian vector bundles, all of them of bounded geometry. On these we consider as before formally selfadjoint, uniformly elliptic differential operators P_1, P_2 of order 1. We assume that all these structures are isomorphic over an open set $N_1 \supset U \subset N_2$, i.e. there is a commutative diagram of isometries

$$N_1 \supset U_1 \xrightarrow{\phi} U_2 \subset N_2$$

$$\uparrow \qquad \uparrow \qquad \uparrow \qquad ,$$

$$E_1|_{U_1} \xrightarrow{\Phi} E_2|_{U_2}$$

such that $P_2 = \Phi P_1 \Phi^{-1}$ over U_2 . Usually, we will not make these isometries explicit, but simply write $U \equiv U_1 \equiv U_2$, $E_1|_U \equiv E_2|_U$ etc.. Now write $c = c(P_1, P_2)$ for the maximum propagation speed of both operators P_1, P_2 . We then have the following relative variant of Proposition 3.8, see [Bu1]:

Proposition 3.9. Let $r_2 > 0$. For $x, y \in U$ we write $Q(x, y) = \max\{\min\{d(x, \partial U), d(y, \partial U)\} - r_2, 0\}$ and $J(x, y) := \left] - \frac{Q(x,y)}{c}, \frac{Q(x,y)}{c}\right]$. Then for $f \in \mathcal{S}(\mathbb{R})$:

$$\begin{aligned} |\nabla_x^l \nabla_y^k ([f(P_1)](x,y) - [f(P_2)](x,y))| \\ &\leq C(P_1,k,l,r_2) \sum_{j=0}^{2L+l+k} \int_{\mathbb{R}^- J(x,y)} |\widehat{f}^{(j)}(s)| ds. \end{aligned}$$

Proof. The proof is analogous to the one of Proposition 3.8. The only thing to note is that the differences of operators can be estimated by

$$| (g(P_1) - g(P_2))\xi_j ||_{L^2(B(x_0, r_2/2))}$$

= $|| (2\pi)^{-1/2} \int_{\mathbb{R}} \widehat{g}(s)(e^{isP_1} - e^{isP_2})\xi_j ds ||_{L^2(B(x_0, r_2/2))}$
= $|| (2\pi)^{-1/2} \int_{\mathbb{R}^- J(x_0, y_0)[} \widehat{g}(s)(e^{isP_1} - e^{isP_2})\xi_j ds ||_{L^2(B(x_0, r_2/2))}$

The last equality holds, because $d(\operatorname{supp}(\xi_j), \partial U) \ge Q(x, y)$, and therefore uniqueness of the solution of the wave equation implies $e^{isP_1}\xi_j = e^{isP_2}\xi_j$ as long as |s| < Q(x, y)/c.

This result can now again be applied to the heat kernel, yielding in the same manner as in (3.6) for $x, y \in U$ and $d(x, \partial U), d(y, \partial U) > r_2$:

$$\begin{aligned} |\nabla_x^l \nabla_y^k \left([P_1^m e^{-tP_1^2}](x,y) - [P_2^m e^{-tP_2^2}](x,y) \right)| \\ &\leq \begin{cases} C(k,l,m,P_1) t^{-m/2} e^{-Q(x,y)^2/6c^2t}, t > T\\ C(k,l,m,P_1) e^{-Q(x,y)^2/6c^2t}, t \in \mathbb{R}_+ \end{cases} \end{aligned}$$
(3.7)

We finish this Section by adding a result on the $t \to \infty$ asymptotics of the heat kernel. We know from Corollary 3.6 that the projection $N(T) := E_T(0)$ onto the null space of an elliptical and selfadjoint operator $T \in Op^k(N, E)$ has kernel $[T] \in UC^{\infty}$.

Proposition 3.10. For $t \to \infty$ the kernel $[e^{-tT^2}]$ converges in $C^{\infty}(N \times N, E \boxtimes E^*)$ to [N(T)].

Proof. This is shown in [Ro1, Proposition 13.14].

3.4. Families of operators. A uniform family of differential operators on an open subset $U \subset \mathbb{R}$ is a uniform differential operator $\mathbf{P} \stackrel{\wedge}{=} (P_u)_{u \in U} \in U \operatorname{Diff}^k(N \times U, E)$, which is uniformly tangential to N. Locally this means that the symbols of the uniform differential operators P_u on N have derivatives (in N and U) of any order that

can be estimated uniformly in 'good' normal neighborhoods, independent of u and the choice of neighborhood. such uniform families can therefore be seen as UC^{∞} -families $u \longmapsto P_u \in Op^k(N, E)$ with $P'_u \triangleq \mathbf{P}'(u) := [\partial_u, \mathbf{P}]|_u$.

For a given such family of elliptic and selfadjoint operators $\mathbf{T} : u \mapsto T_u \in Op^k(N, E)$ we consider the family $e^{-t\mathbf{T}^2} \stackrel{\wedge}{=} (u \mapsto e^{-tT_u^2})$. Its derivative along u has a well-known representation.

Proposition 3.11. (Duhamel formula) $e^{-t\mathbf{T}^2}$ is a differentiable map $U \subset \mathbb{R}$ nach $Op^{-\infty}(N, E)$, and

$$(e^{-t\mathbf{T}^2})'(u) = -\int_0^t e^{-sT_u^2} (T'_u T_u + T_u T'_u) e^{-(t-s)T_u^2} ds.$$

Corollary 3.6 implies the differentiability of $s \mapsto e^{-sT_u^2}$ as a map from \mathbb{R}_+ to $Op^{-\infty}$. To make sense of the integral at $t \to 0$, we also need

Lemma 3.12. Let $K \in Op^{-\infty}(N, E)$. Then the maps $[0, \infty[\to Op^{-\infty}(N, E), s \mapsto Ke^{-sT_u^2}, and s \mapsto e^{-sT_u^2}K$ are differentiable.

Proof. This is a simple exercise. Just note that $(1 + T^2)^{-1}(e^{-sT^2} - 1)$ converges in norm to 0 for $s \to 0$.

Proof. The proof of 3.11 follows the lines of the classical case by proving first continuity and then differntiability of the family directly from the definitions.

4. Coverings of compact manifolds

Following Atiyah's article [A] we will now apply our methods to Hilbert Γ -modules stemming from coverings of a compact riemannian manifold. Section 3.3 introduces the Γ -Eta-invariant from [ChG], [Ra] and analyses its properties using the methods of [Mue1].

Start with a manifold N and a vector bundle E of bounded geometry. We agree to understand by a covering of N a Γ -principal bundle π : $\overline{N} \to N$. This is what we mean when referring to Γ as the 'covering group'.

Fix a fundamental domain $\mathcal{F} \subset \overline{N}$. For $L \subset N$, we write the lift as $\overline{L} := \pi^{-1}(L)$ qnd define $\mathcal{F}(L)$ to be $\mathcal{F} \bigcap \overline{L}$. The lift of the vector bundle E to \overline{N} is denoted by \overline{E} , and $\overline{\xi} : \overline{N} \to \overline{E}$ is meant to refer to the lift of the section $\xi : N \to E$ etc.. From the bounded geometry of N, E it follows immediately that $\overline{N}, \overline{E}$ also have bounded geometry.

4.1. Coverings and Hilbert Γ -modules. We show that the Sobolev spaces over \overline{N} can be interpreted as Hilbert Γ -modules. First, note that

$$L^2(\overline{N},\overline{E}) \xrightarrow{\sim} L^2(\mathcal{F},\overline{E}|_{\mathcal{F}}) \otimes L^2(\Gamma) \left(\xi \longmapsto \sum_{\gamma \in \Gamma} \xi \gamma|_{\mathcal{F}} \otimes \gamma^{-1} \right)$$

is an isomorphism of (right) Γ -modules (i.e. $(\xi\gamma)(x) = \xi(x\gamma^{-1})$.) Thus, $L^2(\overline{N}, \overline{E})$ is a free Hilbert Γ -module.

For Sobolev spaces other than L^2 the above map is not usually an isomorphism. However, for s > 0 the map

$$H^{s}(\overline{N},\overline{E}) \to H^{s}(\mathcal{F},\overline{E}|_{\mathcal{F}}) \otimes L^{2}(\Gamma)$$

is still an isometric imbedding, which is the requirement for $H^s(\overline{N}, \overline{E})$ to be a Hilbert Γ -module.

We now have the following generalisation of Rellich's Lemma:

Proposition 4.1. (Rellich) Let $f \in C_c^{\infty}(N)$ a compactly supported function. For $s, s' \in \mathbb{R}$ and s > s' the map

$$H^{s}(\overline{N},\overline{E}) \xrightarrow{M_{\overline{f}}} H^{s}(\overline{N},\overline{E}) \xrightarrow{\iota} H^{s'}(\overline{N},\overline{E})$$

is a Γ -compact morphism of Hilbert Γ -modules.

Proof. Instead of $M_{\overline{f}}$ we write \overline{f} , and we drop the explicit mention of the embedding ι . Obviously, the maps \overline{f}, ι are Γ -operators. Now for s, s' > 0, we have a commutative diagram:

$$\begin{array}{cccc} H^{s}(\overline{N},\overline{E}) & & -\overline{f} & & H^{s'}(\overline{N},\overline{E}) \\ & & & & \downarrow \\ & & & & \downarrow \\ H^{s}(\mathcal{F},\overline{E}|_{\mathcal{F}}) \otimes L^{2}(\Gamma) & & \overline{f}|_{\mathcal{F} \otimes I} & H^{s'}(\mathcal{F},\overline{E}|_{\mathcal{F}}) \otimes L^{2}(\Gamma) \end{array}$$

But $\operatorname{supp}(\overline{f}) \cap \mathcal{F}$ is compact and the map $H^s(\mathcal{F}, \overline{E}|_{\mathcal{F}}) \xrightarrow{\overline{f}|_{\mathcal{F}}} H^{s'}(\mathcal{F}, \overline{E}|_{\mathcal{F}})$ is compact in the classical sense due to the classical Rellich-Lemma. Thus, $\overline{f}|_{\mathcal{F}} \otimes I$ is Γ - compact and the assertion follows from the fact that the vertical arrows in the diagram are isometric embeddings.

We will now look into the description of the Schwartz-kernels of the Γ -trace class and Γ -Hilbert-Schmidt operators on $L^2(\overline{N}, \overline{E})$.

Proposition 4.2. Let $A \in \mathcal{B}_{\Gamma}(L^2(\overline{N}, \overline{E}))$. Then

- (a) $A \in \mathcal{B}^{1}_{\Gamma}(L^{2}(\overline{N}, \overline{E})) \Leftrightarrow \chi_{\mathcal{F}}|A|\chi_{\mathcal{F}} \in \mathcal{B}^{1}(L^{2}(\mathcal{F}, \overline{E}|_{\mathcal{F}})).$
- (b) $A \in \mathcal{B}^1_{\Gamma}(L^2(\overline{N}, \overline{E})) \Rightarrow \operatorname{tr}_{\Gamma}(A) = \operatorname{tr}(\chi_{\mathcal{F}} A \chi_{\mathcal{F}})$. If the kernel of A is continuous:

$$\operatorname{tr}_{\Gamma}(A) = \int_{\mathcal{F}} tr_{\overline{E}}([A])(x, x) d\operatorname{vol}_{\mathcal{F}}(x) = \int_{N} \pi_* tr_E([A])(x, x) d\operatorname{vol}_N(x),$$

where we have used $[A](x\gamma, x\gamma) = [A](x, x)$ to push down [A](x, x)on the diagonal to $\pi_*([A](x, x))$ on the basis N. (c) $A \in \mathcal{B}^2_{\Gamma}(L^2(\overline{N}, \overline{E})) \Leftrightarrow [\chi_{\mathcal{F}} A] \in L^2(\mathcal{F} \times \overline{N}, \overline{E}|_{\mathcal{F}} \boxtimes \overline{E}^*).$

Proof. (a) and (b): Let
$$(\psi_j)_{j\in\mathbb{N}}$$
 be an orthonormal basis of $L^2(\mathcal{F}, \overline{E}_{\mathcal{F}}) \subset L^2(\overline{N}, \overline{E})$. Then $(\psi_j \gamma)_{j\in\mathbb{N}, \gamma\in\Gamma} \cong (\psi_j \otimes \gamma)_{j\in\mathbb{N}, \gamma\in\Gamma}$ is an orthonormal basis of $L^2(\overline{N}, \overline{E}) \cong L^2(\mathcal{F}, \overline{E}|_{\mathcal{F}}) \otimes L^2(\Gamma)$ and Lemma 2.2 implies:

$$\operatorname{tr}_{\Gamma}(|A|) \stackrel{\wedge}{=} (\operatorname{tr}_{\Gamma} \otimes tr)(|A|) = \sum_{j \in \mathbb{N}} \langle |A|\psi_{j} \otimes e, \psi_{j} \otimes e \rangle$$
$$= \sum_{j \in \mathbb{N}} \langle |A|\psi_{j}, \psi_{j}\rangle = \operatorname{tr}(\chi_{\mathcal{F}}|A|\chi_{\mathcal{F}}).$$

Thus, the operator A is Γ -trace class, if and only if $\chi_{\mathcal{F}} A_{\chi \mathcal{F}}$ is trace class. The integral representation of A directly follows. (c): AA^* is positive, thus

$$AA^* \in \mathcal{B}^1_{\Gamma}(L^2(\overline{N}, \overline{E})) \stackrel{(a)}{\Leftrightarrow} \chi_{\mathcal{F}} AA^* \chi_{\mathcal{F}} \in \mathcal{B}^1(L^2(\mathcal{F}, \overline{E}|_{\mathcal{F}}))$$
$$\Leftrightarrow [\chi_{\mathcal{F}} A] \in L^2(\mathcal{F} \times \overline{N}, \overline{E}|_{\mathcal{F}} \boxtimes \overline{E}^*).$$

4.2. Elliptic operators. Let now M be a *compact* riemannian manifold and E a \mathbb{Z}_2 -graded vector bundle over M. Denote by $Op_{\Gamma}^k(\overline{M}, \overline{E})$, k > 0, the subspace of Γ -equivariant operators in $Op^k(\overline{M}, \overline{E})$. Of course, the typical example will be the lifts $\overline{P} \in \mathrm{UDiff}^k(\overline{M}, \overline{E})$ of differential operators $P \in \mathrm{Diff}^k(M, E)$.

From Proposition 3.3, for an operator $T \in Op_{\Gamma}^{k}(\overline{M}, \overline{E})$ that is elliptic and selfadjoint, the operator $(T \pm i)^{-1}$ is in $\mathcal{B}_{\Gamma}(H^{s}(\overline{M}, \overline{E}), H^{s+k}(\overline{M}, \overline{E}))$, and thus Γ -compact according to Proposition 4.1:

$$(T \pm i)^{-1} \in \mathcal{K}_{\Gamma}(L^2(\overline{M}, \overline{E})).$$

Especially spec_{Γ,e} $(T^2 + 1) \subset \{0\}$, i.e. T is Γ - Fredholm. More generally, set $n = \dim(M)$, L = [n/2 + 1] and write $Op_{\Gamma}^m(\overline{M}, \overline{E})_+$ for the elements in $Op_{\Gamma}^m(\overline{M}, \overline{E})$, that are positive operators on $L^2(\overline{M}, \overline{E})$.

(a) The elements of $Op_{\Gamma}^{-2L}(\overline{M}, \overline{E})$ are Γ -trace Proposition 4.3. class, and the map $Op_{\Gamma}^{-2L}(\overline{M}, \overline{E})_{+} \to \mathcal{B}_{\Gamma}^{-1}(\overline{M}, \overline{E})$ is continuous. (b) The elements in $Op_{\Gamma}^{-L}(\overline{M}, \overline{E})$ are Γ -Hilbert-Schmidt, and the

- $\begin{array}{c} map \ Op_{\Gamma}^{-L}(\overline{M},\overline{E}) \to \mathcal{B}_{\Gamma}^{2}(L^{2}(\overline{M},\overline{E})) \ is \ continuous. \\ (c) \ RC(\mathbb{R}) \to \mathcal{B}_{\Gamma}^{1}(L^{2}(\overline{M},\overline{E}))(f \longmapsto f(T)) \ is \ continuous. \end{array}$

Proof. (a): From Proposition 3.5 the operator $A \in Op_{\Gamma}^{-2L}(\overline{M}, \overline{E})$ has uniformly continuous Schwartz-kernel [A], which therefore can be integrated over $\mathcal{F} \subset \Delta \subset \overline{M} \times \overline{M}$. Proposition 4.2 then implies that A is Γ -trace class, if A is a *positive* operator. The continuity of the kernel-map then follows from the estimate

$$\operatorname{tr}_{\Gamma}(|A|) = \int_{\mathcal{F}} [A](x, x) d\operatorname{vol}_{\overline{M}}(x) \le |[A]|_{\infty} \operatorname{vol}(\mathcal{F}).$$

This proves the second part of (a). Part (b) follows from the continuity of

 $Op_{\Gamma}^{-L}(\overline{M}, \overline{E}) \to Op_{\Gamma}^{-2L}(\overline{M}, \overline{E})_{+} \quad (A \longmapsto AA^{*})$

Now, for the first part of (a) choose an elliptic differential operator $T \in \text{Diff}^{L}(\overline{M}, E)$ such that ST = 1 - R, with suitable parametrix $S \in Op_{\Gamma}^{-L}$ and error term $R \in Op_{\Gamma}^{-\infty}$. Given a (not necessarily positive) $A \in Op_{\Gamma}^{-2L}$, part (b) then implies that the operators $S, TA, R, A \in \mathcal{B}_{\Gamma}^{2}$, thus $A = (ST + R)A = S(TA) + RA \in \mathcal{B}_{\Gamma}^{1}$. (c) now follows from (a) and the continuity of

$$RC(\mathbb{R}) \xrightarrow{|\cdot|} RC(\mathbb{R})_+ \to Op_{\Gamma}^{-2L}(\overline{M}, \overline{E})_+.$$

Corollary 4.4. As before, let $T \in Op_{\Gamma}^k(\overline{M}, \overline{E})$ elliptic and selfadjoint. Then $spec_{\Gamma,e}(T) = \emptyset$ and the spectral measure $\mu_{\Gamma,T}$ is polynomially bounded. More precisely, the spectral 'counting function' satisfies the estimate $N_{\Gamma,T}(\lambda) := \dim_{\Gamma}(\mathcal{H}_{T}(] - \lambda, \lambda[)) \leq C\lambda^{2L/k}$.

Proof. The operator $(1+T^2)^{-L/k}$ is Γ -trace class, thus

$$\int_{\mathbb{R}} (1+x^2)^{-L/k} d\mu_{\Gamma,T}(x) = \operatorname{tr}_{\Gamma}((1+T^2)^{-L/k}) < \infty.$$

and we can write for the spectral counting function

$$N_{\Gamma,T}(\lambda) = \int_{\mathbb{R}} \chi_{]-\lambda,\lambda[}(x) d\mu_{\Gamma,T}(x) \le (1+\lambda^2)^{L/k} \int_{\mathbb{R}} (1+x^2)^{-L/k} d\mu_{\Gamma,T}(x).$$

In the case that $E \to N$ is \mathbb{Z}_2 -graded, T odd, we can use these results to calculate the index:

Lemma 4.5. $\operatorname{ind}_{\Gamma}(T) = \operatorname{str}_{\Gamma}(e^{-tT^2}).$

Proof. The proof now runs parallel to the proof for compact manifolds. The map $f \mapsto f(T)$ is continuos from $RC(\mathbb{R})$ to $\mathcal{B}^1_{\Gamma}(L^2(\overline{M}, \overline{E}))$. The family $t \mapsto e^{-tx^2}$ is differentiable as a map from \mathbb{R}_+ to $RC(\mathbb{R})$. Therefore $\operatorname{str}_{\Gamma}(e^{-tT^2})$ is differentiable in t > 0 and

$$\frac{d}{dt}\operatorname{str}_{\Gamma}(e^{-tT^{2}}) = -\operatorname{str}_{\Gamma}(T^{2}e^{-tT^{2}}) = -\frac{1}{2}\operatorname{str}_{\Gamma}([T, Te^{-tT^{2}}]) = 0.$$

This shows that $\operatorname{str}_{\Gamma}(e^{-tT^2})$ is independent of t. But according to Proposition 3.10, the heat kernel $[e^{-tT^2}]$ converges in C^{∞} to [N(T)]for $t \to \infty$. Thus

$$\operatorname{str}_{\Gamma}(N(T)) = \int_{\mathcal{F}} \operatorname{str}^{\overline{E}}[N(T)](x, x) d\operatorname{vol}_{\overline{M}}(x)$$
$$= \lim_{t \to \infty} \int_{\mathcal{F}} \operatorname{str}^{\overline{E}}[e^{-tT^{2}}](x, x) d\operatorname{vol}_{\overline{M}}(x) = \operatorname{str}_{\Gamma}(e^{-tT^{2}})$$

where the LHS is of course $\operatorname{ind}_{\Gamma}(T)$.

In the special case that $T = \overline{P}$ is the lift of an odd operator $P \in \text{Diff}^1(M, E)$, the operator \overline{P} can locally be compared to P. For small $\epsilon > 0$ and any $\overline{x} \in \overline{M}$, the ball $B(\overline{x}, \epsilon) \subset \overline{M}$ is isometric to the ball $B(x, \epsilon) \subset M$, $x = \pi(\overline{x})$, and the vector bundles and operators are isometric as well, i.e. $\overline{E}|_{B(\overline{x},\epsilon)} \simeq E|_{B(x,\epsilon)}$ and $\overline{P}|_{B(\overline{x},\epsilon)} \simeq P|_{B(x,\epsilon)}$. Applying the estimate (3.7) then gives for $t \in \mathbb{R}_+$ and suitable constants $C_1, c_2 > 0$

$$|[e^{-t\overline{P}^{2}}](\overline{x},\overline{x}) - [e^{-tP^{2}}](x,x)| \le C_{1}e^{-c_{2}/t}.$$
(4.8)

This allows us to reprove Atiyah's theorem

Theorem 4.6. (Atiyah) $\operatorname{ind}_{\Gamma}(\overline{P}) = \operatorname{ind}(P)$.

Proof. We just need to put together the results obtained thusfar

$$\operatorname{ind}_{\Gamma}(\overline{P}) = \lim_{t \to 0} \operatorname{str}_{\Gamma}(e^{-t\overline{P}^{2}}) = \lim_{t \to 0} \int_{\mathcal{F}} \operatorname{str}^{\overline{E}}[e^{-t\overline{P}^{2}}](\overline{x},\overline{x})d\operatorname{vol}_{\overline{M}}(\overline{x})$$
$$\stackrel{4.8}{=} \lim_{t \to 0} \int_{M} \operatorname{str}^{E}[e^{-tP^{2}}](x,x)d\operatorname{vol}_{M}(x) = \lim_{t \to 0} \operatorname{str}(e^{-tP^{2}}) = \operatorname{ind}(P).$$

4.3. The Γ -Eta-invariant. Let $T \in Op_{\Gamma}^1(\overline{M}, \overline{E})$ elliptic and selfadjoint. We will try to define the Γ -Eta-invariant for T as the value in s = 0 of

$$\eta(T)(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{(s-1)/2} \mathrm{tr}_{\Gamma}(Te^{-tT^2}) dt.$$
(4.9)

This will require an analysis of the expression $\operatorname{tr}_{\Gamma}(Te^{-tT^2})$ for $t \to \infty$ and $t \to 0$.

The first part of the analysis can be done for $T = \overline{P}$, with an elliptic and selfadjoint differential operator $P \in \text{Diff}^1(M, E)$. For $t \to 0$, there is the well-known asymptotic development on M:

$$\operatorname{tr}(Pe^{-tP^2}) \sim \sum_{j=0}^{\infty} b_j(P) t^{(j-n-1)/2}.$$

The coefficients $b_j(P)$ are local integrals $b_j(P) = \int_M \beta_j(P)$, where $\beta_j(P)(x)$ only depends on the symbol of P and its derivatives. As in the proof of Theorem 4.6, the estimate (4.8) implies that $\operatorname{tr}_{\Gamma}(\overline{P}e^{-t\overline{P}^2})$ and $\operatorname{tr}(Pe^{-tP^2})$ have the same asymptotics for $t \to 0$

$$\operatorname{tr}_{\Gamma}(\overline{P}e^{-t\overline{P}^2}) \sim \sum_{j=0}^{\infty} b_j(\overline{P})t^{(j-n-1)/2}, \qquad (4.10)$$

and $b_j(\overline{P}) = b_j(P) = \int_{\mathcal{F}} \beta_j(\overline{P})$. Thus the expression

$$\eta_{\Gamma}(\overline{P})(s)_{\kappa} := \frac{1}{\Gamma(\frac{s+1}{2})} \int_{0}^{\kappa} t^{(s-1)/2} \mathrm{tr}_{\Gamma}(\overline{P}e^{-t\overline{P}^{2}}) dt$$

exists for $\kappa \in \mathbb{R}_+, s > n+1,$ and has an asymptotic development of the form

$$\eta_{\Gamma}(\overline{P})(s)_{\kappa} \sim \frac{1}{\Gamma(\frac{s+1}{2})} \sum_{j=0}^{\infty} b_j(\overline{P}) \frac{\kappa^{(j-n)/2+s/2}}{(j-n)/2+s/2}$$

The expression $\Gamma((s+1)/2)\eta_{\Gamma}(\overline{P})(s)_{\kappa}$ therefore has a meromorphic continuation to \mathbb{C} with singularities of 1st order (at most) in $s \in \{n-j | j \in \mathbb{N}\}$. Thus, $\eta_{\Gamma}(\overline{P})(s)_{\kappa}$ is holomorphic in s = 0, exactly when $b_n(\overline{P}) = 0$. An equivalent formulation is to say that the development

$$\int_{\delta}^{\kappa} t^{-1/2} \operatorname{tr}_{\Gamma}(\overline{P}e^{-t\overline{P}^2}) dt = \sum_{j \neq n} b_j(\overline{P}) \frac{2}{j-n} (\kappa^{\frac{j-n}{2}} - \delta^{\frac{j-n}{2}})$$

in powers of δ exists. For a function $f(\delta)$ permitting such a development in $\delta \to 0$ we denote the constant coefficient in the development by $\operatorname{LIM}_{\delta \to 0} f(\delta)$. Thus, for $b_n(\overline{P}) = 0$

$$\eta_{\Gamma}(\overline{P})(0)_{\kappa} = \frac{1}{\Gamma(\frac{1}{2})} \sum_{j \neq n} b_j(\overline{P}) \frac{2}{j-n} \kappa^{\frac{j-n}{2}} = \text{LIM}_{\delta \to 0} \int_{\delta}^{\kappa} \frac{t^{-1/2}}{\Gamma(\frac{1}{2})} (\overline{P}e^{-t\overline{P}^2}) dt.$$

In the case of the Dirac operator \overline{D} , the coefficients $b_j(\overline{D})$ vanish for all $j \leq n$ and $\eta_{\Gamma}(\overline{D})(s)_{\kappa}$ is holomorphic in 0, i.e. it exists without any regularisation.

The analysis of the term

$$\eta_{\Gamma}(\overline{P})(s)^{\kappa} := \frac{1}{\Gamma(\frac{s+1}{2})} \int_{\kappa}^{\infty} t^{(s-1)/2} \mathrm{tr}_{\Gamma}(\overline{P}e^{-t\overline{P}^2}) dt$$

is easy, because this integral exists for $s \leq 0$ due to the polynomial boundedness of $\mu_{\Gamma,\overline{P}}$.

For the Dirac operator, we can thus define:

$$\eta_{\Gamma}(\overline{D}) = \eta_{\Gamma}(\overline{D})(0)_{\kappa} + \eta_{\Gamma}(\overline{D})(0)^{\kappa}.$$

We have to extend this definition of the Γ -Eta-invariant to modifications $Q = \overline{D} + K$ of \overline{D} with $K \in Op^{-\infty}(\overline{M})$. The main example we have in mind are modifications of \overline{D} where $K = f(\overline{D})$ for a bounded measurable function $f : \mathbb{R} \to \mathbb{R}$. First

$$Qe^{-tQ^2} - \overline{D}e^{-t\overline{D}^2}$$

= $Ke^{-t(\overline{D}+K)^2} - \overline{D}\int_0^t e^{-s(\overline{D}+K)^2}(K\overline{D}+\overline{D}K+K^2)e^{-(t-s)\overline{D}^2}ds.$

Lemma 3.12 implies that this converges for $t \to 0$ to K in $Op^{-\infty}(\overline{M})$. This gives an asymptotic development

$$\operatorname{tr}_{\Gamma}(Qe^{-tQ^2}) \sim \sum_{j=0}^{n+1} b_j(\overline{D}) t^{(j-n-1)/2} + tr_{\Gamma}(K) + g(t), \quad t \to 0,$$
(4.11)

where g is a continuous on $[0, \infty[$ and g(0) = 0. Since $b_j(\overline{D}) = 0$ for $j \leq n$, the expression $\eta_{\Gamma}(Q)(0)_{\kappa}$ exists. Using again the polynomial boundedness of the measure $\mu_{\Gamma,\overline{Q}}$.

To compare the $\Gamma - \eta$ -invariants for \overline{D} and $\overline{D} + K$, consider now the differentiable family $u \mapsto Q_u := \overline{D} + K + u$ of elliptic, selfadjoint

operators in $Op_{\Gamma}^1(\overline{M}, \overline{E})$. From Proposition 3.11, the trace $\operatorname{tr}_{\Gamma}(Q_u e^{-tQ_u^2})$ is differentiable in u and using the trace property one has

$$\frac{\partial}{\partial u} \operatorname{tr}_{\Gamma}(Q_u e^{-tQ_u^2}) = \operatorname{tr}_{\Gamma}\left(Q'_u e^{-tQ_u^2} - t(Q'_u Q_u + Q_u Q'_u)e^{-tQ_u^2}\right)$$
$$= \left(1 + 2t\frac{\partial}{\partial t}\right)\operatorname{tr}_{\Gamma}(Q'_u e^{-tQ^2u}).$$

From the asymptotic development of $\operatorname{tr}_{\Gamma}(Q_u e^{-tQ_u^2})$, one deduces that $\eta_{\Gamma}(Q_u)(s)_{\kappa}$ exists for large s and can be continued meromorphically. Partial integration gives

$$\begin{split} \frac{\partial}{\partial u} \eta_{\Gamma}(Q_u)(s)_{\kappa} &= \frac{\partial}{\partial u} \int_0^{\kappa} \frac{t^{(s-1)/2}}{\Gamma(\frac{s+1}{2})} \mathrm{tr}_{\Gamma}(Q_u e^{-tQ_u^2}) dt \\ &= \int_0^{\kappa} \frac{t^{(s-1)/2}}{\Gamma(\frac{s+1}{2})} (1+2t\frac{\partial}{\partial t}) \mathrm{tr}_{\Gamma}(Q'_u e^{-tQ_u^2}) dt \\ &= \frac{2\kappa^{(s+1)/2}}{\Gamma(\frac{s+1}{2})} \mathrm{tr}_{\Gamma}(Q'_u e^{-\kappa Q_u^2}) - \frac{s}{\Gamma(\frac{s+1}{2})} \int_0^{\kappa} t^{(s-1)/2} \mathrm{tr}_{\Gamma}(Q'_u e^{-tQ_u^2}) dt. \end{split}$$

Here, we have used the identity:

$$t^{(s-1)/2}(1+2t\frac{\partial}{\partial t})h = (2\frac{\partial}{\partial t}t-s)t^{(s-1)/2}h.$$

In order to understand the asymptotic development of $\frac{\partial}{\partial u}\eta_{\Gamma}(Q_u)(s)_{\kappa}$, we need to understand the asymptotic development of the last integral. In the case at hand, Q'_u is the identity, and the asymptotic development of the heat kernel for $\overline{D} + u$ gives for $t \to 0$:

$$\operatorname{tr}_{\Gamma}(Q'_{u}e^{-tQ_{u}^{2}}) = \operatorname{tr}_{\Gamma}(e^{-t(\overline{D}+K+u)^{2}}) \sim \sum_{j=0}^{n} a_{j}(\overline{D}+u)t^{(j-n)/2} + g(t),$$

where again g is continuous on $[0, \infty[$ ith g(0) = 0. The integral therefore has a meromorphic extension to a neighborhood of 0 in \mathbb{C} with a singularity of order at most 1 in 0. But then, $\frac{\partial}{\partial u}\eta_{\Gamma}(Q_u)(s)_{\kappa}$ is holomorphic in 0 and

$$\frac{\partial}{\partial u} \operatorname{Res}_{s=0} \eta_{\Gamma}(Q_u)(s)_{\kappa} = \operatorname{Res}_{s=0} \frac{\partial}{\partial u} \eta_{\Gamma}(Q_u)(s)_{\kappa} = 0,$$

and $Res_{s=0}\eta_{\Gamma}(Q_u)(s)_{\kappa}$ is constant in u. Since $\eta_{\Gamma}(Q_0)(s)_{\kappa}$ is holomorphic in 0 so is $\eta_{\Gamma}(Q_u)(s)_{\kappa}$ holomorphic in 0.

In Chapter 6 we will look at families $Q_u = \overline{D} + u - \Pi \overline{D}$, with $\Pi = E_{\overline{D}}(] - \epsilon, \epsilon[)$. For this special case we note

Lemma 4.7. For $Q_u = \overline{D} + u - \Pi \overline{D}$ the Γ -Eta-invariant

$$\eta_{\Gamma}(Q_u) = LIM_{\delta \to 0} \int_{\delta}^{\kappa} \frac{t^{-1/2}}{\Gamma(\frac{1}{2})} \operatorname{tr}_{\Gamma}(Q_u e^{-tQ_u^2}) dt + \int_{\kappa}^{\infty} \frac{t^{-1/2}}{\Gamma(\frac{1}{2})} \operatorname{tr}_{\Gamma}(Q_u e^{-tQ_u^2}) dt$$

exists for all u, and for $\epsilon > 0$ we have

(a) $\eta_{\Gamma}(Q_u) - \eta_{\Gamma}(Q_0) = \operatorname{sgn}(u)\operatorname{tr}_{\Gamma}(\Pi) \ also \ \eta_{\Gamma}(Q_0) = \frac{1}{2}(\eta_{\Gamma}(Q_u) + \eta_{\Gamma}(Q_{-u})).$ (b) $|\eta_{\Gamma}(\overline{D}) - \eta_{\Gamma}(Q_0)| = |\eta_{\Gamma}(\Pi\overline{D})| \le \mu_{\Gamma,\overline{D}}(] - \epsilon, \epsilon[-\{0\}).$

4.4. Residually finite coverings. As usual, let M be a compact manifold, $\pi : \overline{M} \to M$ a Γ -principal bundle. We assume that the the covering group Γ is *finitely generated*, and fix a word metric $|\bullet|$. The following is a standard result in geometric group theory.

Lemma 4.8. Let \overline{M} be connected. Then \overline{M} and Γ are quasiisometric, *i.e.* there are constants A, B > 0 with

$$B^{-1}|\gamma| - A \le d(x, x\gamma) \le B|\gamma| + A$$
for all $x \in \overline{M}, \ \gamma \in \Gamma$.

The considerations in this Section are based on the following Proposition that shows how the heat kernel of a selfadjoint elliptic differential operator $P \in \text{Diff}^1(M, E)$ can be reconstructed from the heat kernel of its lift \overline{P} .

Proposition 4.9. Choose \overline{x} and write $x = \pi(\overline{x})$. Then

$$[P^{l}e^{-tP^{2}}](x,y) = \sum_{\gamma \in \Gamma} [\overline{P}^{l}e^{-t\overline{P}}](\overline{x},\overline{y}\gamma),$$

and the RHS converges in UC^∞ uniformly in $0 < a \leq t \leq b \leq \infty$.

Proof. The proof can be found in [Lo3]. Here, we only show the absolute convergence of the RHS independent of $\overline{x}, \overline{y}$ and $t \in [a, b]$. It suffices to show this for $\overline{x}, \overline{y} \in \mathcal{F}$:

$$\sum_{\gamma \in \Gamma} |[\overline{P}^l e^{-t\overline{P}^2}](\overline{x}, \overline{y}\gamma)| \stackrel{(3.6)}{\leq} C + C \sum_{e \neq \gamma \in \Gamma} t^{-l/2} e^{-\delta(d(\overline{x}, \overline{y}\gamma) - r_0)_+^2/t}$$
$$\leq C + C(a, b) \sum_{e \neq \gamma \in \Gamma} e^{\delta d(\overline{x}, \overline{x}y)^2/b} \leq C + C(a, b) \sum_{e \neq \gamma \in \Gamma} e^{-\delta B|\gamma|^2/b} < \infty$$

and similarly for the derivatives.

Let now $\overline{M} \to M$ be *residually finite*, i.e. there is a tower of groups

$$\Gamma = \Gamma_0 \vartriangleright \Gamma_1 \vartriangleright \ldots \vartriangleright \Gamma_i \vartriangleright \Gamma_{i+1} \ldots \Gamma_{\infty} = \{e\},\$$

i.e. Γ_{i+1} is normal of finite index in Γ_i . Setting $M_i := \overline{M}/\Gamma_i$ the covering $M_i \to M$ is finite with deck-transformations Γ/Γ_i . If $\overline{M} \to M$ is residually finite then so is $\overline{M}^0 \to M$. Let now E be a over M with Dirac operator D and denote the lifts of the vector bundle E to M_i by E_i and the lifted Dirac operators by D_i . The following is a simple consequence of Proposition 4.9

Lemma 4.10. Write $b_i := \dim(\ker(D_i)), b_{\Gamma} := \dim_{\Gamma}(\ker(D)), d_i := [\Gamma : \Gamma_i]$. Then

(a) $\lim_{i\to\infty} \frac{1}{d_i} \operatorname{tr}(e^{-tD_i^2}) = \operatorname{tr}_{\Gamma}(e^{-t\overline{D}^2})$ (b) $\limsup_{i\to\infty} \frac{b_i}{d_i} \leq b_{\Gamma}$

Choose an increasing sequence $\{e\} = S_0 \subset S_1 \subset S_2 \ldots$ of representant systems S_i of the quotient groups Γ/Γ_i . If $\mathcal{F} = \mathcal{F}(M)$ is the fundamental domain of $\overline{M} \to M$, then $\mathcal{F}(M_i) := \mathcal{F}S_i$ defines fundamental domains for all $\overline{M} \to M_i$. For the classical Eta-invariant we have from Proposition 4.9

$$\begin{split} \eta(D_i) &= \int_0^\infty \frac{1}{\sqrt{\pi t}} \int_{\mathcal{F}(M_i)} \sum_{\gamma \in \Gamma_i} tr_E([\overline{D}e^{-t\overline{D}^2}](x,x\gamma)) dx dt \\ &= d_i \int_0^\infty \frac{1}{\sqrt{\pi t}} \int_{\mathcal{F}(M_i)} \sum_{\gamma \in \Gamma_i} tr_E([\overline{D}e^{-t\overline{D}^2}](x,x\gamma)) dx dt, \end{split}$$

thus

$$\frac{1}{d_i}\eta(D_i) - \eta_{\Gamma}(\overline{D}) = \int_0^\infty \frac{1}{\sqrt{\pi t}} \int_{\mathcal{F}(M)} \sum_{e \neq \gamma \in \Gamma_i} tr_E([\overline{D}e^{-t\overline{D}^2}](x,x\gamma)) dx dt.$$
(4.12)

Formula (4.12) raises the question of the convergence of the $\frac{1}{d_i}\eta(D_i)$ for $i \to \infty$. The following theorem gives a partial answer.

Theorem 4.11. ([ChG])Assume that one of the following conditions hold:

- (a) ker $\overline{D} = \{0\}.$
- (b) \overline{D}^2 is the Laplace operator on $\Lambda T^*\overline{M}$.

Then $\lim_{i\to\infty} \frac{1}{d_i}\eta(D_i) = \eta_{\Gamma}(\overline{D}).$

Proof. We split the integral in (4.12) into integrals over $[0, \kappa]$ and $[\kappa, \infty[$. Then

$$\begin{aligned} \left|\frac{1}{d_{i}}\eta(D_{i})_{\kappa} - \eta_{\Gamma}(\overline{D})_{\kappa}\right| &\leq \int_{0}^{\kappa} \frac{1}{\sqrt{\pi t}} \int_{\mathcal{F}(M)} \sum_{e \neq \gamma \in \Gamma_{i}} \left|tr_{E}([\overline{D}e^{-t\overline{D}^{2}}](x,x\gamma))\right| dxdt \\ &\leq \int_{0}^{\kappa} \frac{C}{\sqrt{\pi t}} \sum_{e \neq \gamma \in \Gamma_{i}} e^{-(d(x_{0},x_{0}\gamma)-r_{0})^{2}/6c^{2}t} dt \\ &\leq C \sum_{e \neq \gamma \in \Gamma_{i}} \kappa e^{-(d(x_{0},x_{0}\gamma)-r_{0})^{2}/6c^{2}\kappa} \\ &\leq C(\kappa) \sum_{e \neq \gamma \in \Gamma_{i}} e^{-\delta|\gamma|^{2}/\kappa}. \end{aligned}$$

This series converges absolutely, such that the RHS converges to 0 for $i \to \infty$. Using the definition of $\eta_{\Gamma}(\overline{D})^{\kappa}$ we have

$$|\eta_{\Gamma}(\overline{D})^{\kappa}| \leq \operatorname{tr}_{\Gamma}(e^{-\kappa \overline{D}^2}) - b_{\Gamma} \text{ and } \frac{1}{d_i}\eta(D_i)^{\kappa} \leq \frac{1}{d_i}(\operatorname{tr}(e^{-\kappa D_i^2}) - b_i).$$

Under condition (a) Lemma 4.10 implies $\lim_{i\to\infty} b_i/d_i = b_{\Gamma}$. The same holds true under condition (b) as is shown in [Lue]. We now have

$$\left|\frac{\eta(D)_{i}}{d_{i}} - \eta_{\Gamma}(\overline{D})\right| \leq C(\kappa) \sum_{e \neq \gamma \in \Gamma_{i}} e^{-\delta|\gamma|^{2}/\kappa} + \left|\frac{\operatorname{tr}(e^{-tD_{i}^{2}}) - b_{i}}{d_{i}} - \operatorname{tr}_{\Gamma}(e^{-t\overline{D}^{2}}) + b_{\Gamma}\right| + 2(\operatorname{tr}_{\Gamma}(e^{-\kappa\overline{D}^{2}}) - b_{\Gamma})$$

The last summand can be made arbitrarily small by choosing an appropriate κ . The two remaining summands can be made arbitrarily small by letting $i \to \infty$.

5. Coverings of noncompact manifolds

This Chapter contains some Γ -analogues of classical Theorems from the index theory on noncompact manifolds.

5.1. **Decomposition principle.** According to the classical decomposition principle (cf. [DoLi]), the essential spectrum of a differential operator on a noncompact manifold is determined by the operator 'at infinity'. This Section proves the analogous assertion for coverings. For our variant of the decomposition principle, we look at hermitian vector bundles $E_i \to N_i$ of bounded geometry and selfadjoint operators $P_i \in \text{UDiff}^1(N_i, E_i)$. We assume that there is a decomposition

$$N_1 = U \cup K_1, \quad N_2 = U \cup K_2$$

with K_i compact, and that all structures on N_1, N_2 are isometric over U. We also have to assume that the pre-image \overline{U} of U under $\overline{N}_1 \to N_1$ is Γ -isometric to the pre-image of U under $\overline{N}_2 \to N_2$.

Proposition 5.1. (Decomposition principle) $\operatorname{spec}_{\Gamma,e}(\overline{P}_1) = \operatorname{spec}_{\Gamma,e}(\overline{P}_2)$.

Proof. Let $\lambda \in \operatorname{spec}_{\Gamma,e}(\overline{P}_1)$. Then for all $\epsilon > 0$ the space $G_{\epsilon} := \mathcal{H}_{\overline{P}_1}(]\lambda - \epsilon, \lambda + \epsilon[)$ has infinite Γ -dimension. We write $E(\epsilon) := E_{\overline{P}_1}(]\lambda - \epsilon, \lambda + \epsilon[)$. All Sobolev-norms are equivalent on G_{ϵ} . This follows from the estimate for $f \in G_{\epsilon}$ and $k \in \mathbb{N}$

$$\| f \|_{H^{2k}(\overline{N}_1)} \leq C \left(\| f \|_{L^2(\overline{N}_1)} + \| (\overline{P}_1 - \lambda)^k f \|_{L^2(\overline{N}_1)} \right)$$

$$\leq (C + \epsilon^{\kappa}) \| f \|_{L^2(\overline{N}_1)} .$$

The proof now proceeds to show that a Γ -infinite dimensional space of sections f in G_{ϵ} essentially lives in \overline{U} . For this, choose cut-off functions $\phi, \psi \in C_c^{\infty}(N_1)$ with $\phi|_{K_1} \equiv 1$ and $\psi|_{\mathrm{supp}(\phi)} \equiv 1$. According to 4.1 the concatenation

$$C_{\overline{\psi}} : (G_{\epsilon}, \|\cdot\|_{L^2}) \xrightarrow{Id} (G_{\epsilon}, \|\cdot\|_{H^2}) \xrightarrow{\overline{\psi}} H^1(\overline{N}_1, \overline{E}_1)$$

is Γ -compact. Then the map $C^*_{\overline{\psi}}C_{\overline{\psi}} \ (\neq \overline{\psi}^2)$ is Γ -compact, selfadjoint on $(G_{\epsilon}, \|\cdot\|_{L^2})$, and

$$\widetilde{G}_{\epsilon} := \mathcal{H}_{C^*_{\overline{\psi}}C_{\overline{\psi}}}(] - \epsilon^2, \epsilon^2[) \subset G_{\epsilon}$$

is of infinite Γ -dimension. For $f \in \widetilde{G}_{\epsilon}$ we then find

$$\| \overline{\psi}f \|_{H^1(\overline{N}_1)}^2 = \langle C_{\overline{\psi}}f, C_{\overline{\psi}}f \rangle_{H^1(\overline{N}_1)} = \langle C_{\overline{\psi}}^*C_{\overline{\psi}}f, f \rangle_{L^2(\overline{N}_1)} \le \epsilon^2 \| f \|_{L^2(\overline{N}_1)}^2$$

$$\begin{split} \| (\overline{P}_2 - \lambda)(1 - \overline{\phi}) f \|_{L^2(\overline{N}_2)} \\ \leq & \| [\overline{P}_1, \overline{\phi}] f \|_{L^2(\overline{N}_1)} + \| (1 - \overline{\phi})(\overline{P}_1 - \lambda) f \|_{L^2(\overline{N}_1)} \\ \leq & C \| \overline{\psi} f \|_{H^1(\overline{N}_1)} + \| (\overline{P}_1 - \lambda) f \|_{L^2(\overline{N}_1)} \leq \epsilon (1 + C) \| f \|_{L^2(\overline{N}_1)} . \end{split}$$

This shows that

$$(1-\overline{\phi})\widetilde{G}_{\epsilon} \subset \mathcal{H}_{\overline{P}_2}(]\lambda - \epsilon(1+C), \lambda + \epsilon(1+C)[).$$

But, again using Proposition 4.1, the multiplication operator $1 - \overline{\phi}$ on \widetilde{G}_{ϵ} is Γ -Fredholm in L^2 . This means that the LHS is of infinite Γ -dimension, i.e. $\lambda \in \operatorname{spec}_{\Gamma,e}(\overline{P}_2)$.

6. Manifolds with cylindrical ends

In this Chapter, (X, g) is a connected (oriented, riemannian) manifold with cylindrical ends of even dimension n. Thus, X is of the form

$$X = X_0 \cup Z$$
 with $Z = M \times [0, \infty]$, $M = \partial X_0$ compact.

On X we have the Clifford bundle (E, ∇^E, h^E) . We assume that all these structures are of product form over the cylinder:

 $g|_Z = g^M + \langle \cdot, \cdot \rangle_{[0,\infty[}, E_Z \cong \rho^*(E|_M), h^E|_Z = \rho^*(h^E|_M), \nabla^E|_Z = \rho^*\nabla^{E|_M}$ etc.. Here $\rho : Z = M \times [0, \infty[\to M \text{ denotes the projection onto the basis of the cylinder, where we have identified <math>M$ with $M \times \{0\} \subset X$. We will often write $X = X_{\kappa} \cup Z_{\kappa}$, where $X_{\kappa} = X_0 \cup (M \times [0, \kappa])$, and $Z_{\kappa} = M \times [\kappa, \infty]$.

Choose the orientation on (all components of) M such that $E_1, \ldots, E_{2n-1}, \frac{\partial}{\partial r}$ is a positively oriented local frame in TZ, whenever $E_1 \ldots, E_{2n-1}$ is a positively oriented local frame in TM. Then Clifford multiplication with $\frac{\partial}{\partial r}$ gives an isomorphism of C(M)-modules

$$\mathbf{c}(\frac{\partial}{\partial r}): E^+|_M \xrightarrow{\sim} E^-|_M$$

Writing F for the C(M)-module $E^+|_M$ we can thus identify $E|_Z$ and $\rho^*(F \oplus F)$ as Clifford modules via

$$\mathbf{c}^{E}(X) \stackrel{\wedge}{=} \begin{pmatrix} 0 & \mathbf{c}^{F}(W) \\ \mathbf{c}^{F}(W) & 0 \end{pmatrix}, \text{ for } W \in TM, \quad \mathbf{c}^{E}(\frac{\partial}{\partial r}) \stackrel{\wedge}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(Cf. Annex A). In this representation the Dirac operator D on $\rho^*(F \oplus F)$ over Z has the form

$$D = \mathbf{c}(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + \mathbf{c}|_{M} \circ \nabla^{E|_{M}} = \mathbf{c}(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + \Omega D^{F}, \ \Omega = \begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix}_{(6.13)}^{E}$$

Here and in the future, we drop mention of the identifying map ρ^* and just write $E|_Z \cong F \oplus F$.

In this Chapter, we consider the Γ -index theory of the lifted Dirac operator \overline{D} on a regular covering \overline{X} of X, with lifted structures $\overline{Z}, \overline{E}$ etc.. Section 6.3 contains the proof of the L²- Γ -index theorem for \overline{D} using methods from [Me], [Mue2]. This will require the introduction of a spectral modification of \overline{D}^F which effaces the spectrum of this operator around 0. The resulting modification $\overline{D}_{\epsilon,u}$ of \overline{D} then is Γ -Fredholm, but not a differential operator. In the first two Sections of we therefore go into some of the details of the analysis of such operators.

6.1. Γ -operators with product structure on the cylinder. Let $T \in Op_{\Gamma}^1(\overline{X}, \overline{E})$ be elliptic and selfadjoint. T is said to have product structure, if its restriction to \overline{X}_0 is a differential operator $\overline{P}|_{\overline{X}_0}$, lifted from a uniformly elliptic and formally selfadjoint differential operator $\overline{P} \in \text{UDiff}^1(\overline{X}, \overline{E})$, and its restriction to Z looks like

$$T = \mathbf{c}\left(\frac{\partial}{\partial r}\right)\frac{\partial}{\partial r} + \Omega B(r) = \begin{pmatrix} 0 & B(r) - \frac{\partial}{\partial r} \\ B(r) + \frac{\partial}{\partial r} & 0 \end{pmatrix}.$$
 (6.14)

Here, $[0, \infty[\ni r \longmapsto B(r) \in Op_{\Gamma}^{1}(\overline{M}, \overline{F})$ is a differentiable family of elliptic and selfadjoint operators. We assume that $B(r) \equiv \overline{B}_{0}$ for r < 1 and $B(r) \equiv B$ for r > 2. We recall that we allow $\overline{M}, \overline{Z}$ to have countably many components.

Let's start the analysis of these operators by noting that the results of Section 3.3, especially (3.6) and (3.7) remain true for T on \overline{X}_0 , and that for each $\xi_0 \in C_c^{\infty}(\overline{X}, \overline{E})$ a unique solution $\xi(t) := e^{itT}\xi_0$ of the wave equation for T exists. The following Lemma shows that we still have an energy estimate for $\xi(t)$, though only along the cylinder:

Lemma 6.1. Choose $U = \overline{M} \times]a, b[\subset \overline{Z} \text{ with } 0 < a < b$. For $\Lambda < a$ the norm $|| \xi(t) ||_{L^2(B(U,\Lambda-t))}$ is then monotonously decreasing in t. Especially, e^{itT} has propagation speed along the cylinder ≤ 1 , since $\operatorname{supp}(\xi_0) \subset U$ implies $\operatorname{supp}(\xi(t)) \subset B(U, t)$.

Proof. The proof follows the proof of Proposition 5.5 in [Ro1]:

$$\begin{aligned} &\frac{\partial}{\partial t} \parallel \xi(t) \parallel^{2}_{L^{2}(B(U,\Lambda-t))} = \frac{\partial}{\partial t} \int_{B(U,\Lambda-t)} |\xi(t)|^{2}(z)dz \\ &\leq |\int_{B(U,\Lambda-t)} (\langle \xi(t), iT\xi(t) \rangle + \langle iT\xi(t), \xi(t) \rangle)(z)dz| - \int_{\partial B(U,\Lambda-t)} |\xi(t)|^{2}(z)dz \end{aligned}$$

Now $T = \mathbf{c}(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + \Omega B(r)$, and the domain of integration $B(U, \Lambda - t)$ is the product of \overline{M} and an intervall. From the selfadjointness of B(r) over M we can deduce

$$\int_{\overline{M}} (\langle i\Omega B(r)\xi(t),\xi(t)\rangle(x) + \langle \xi(t),i\Omega B(r)\xi(t)\rangle(x))dx = 0.$$

The estimation of the derivative above can therefore be continued as follows:

$$\begin{aligned} \frac{\partial}{\partial t} &\| \xi(t) \|_{L^{2}(B(U,\Lambda-t))}^{2} \\ \leq &| \int_{B(U,\Lambda-t)} \frac{\partial}{\partial r} \langle \xi(t), \mathbf{c}(\frac{\partial}{\partial r}) \xi(t) \rangle(z) dz | - \int_{\partial B(U,\Lambda-t)} |\xi(t)|^{2}(z) dz. \end{aligned}$$

Since $|\mathbf{c}(\frac{\partial}{\partial r})| = 1$ this must be less than or equal 0.

This Lemma can be used to apply the methods from Section 2.3 for the operator family T over \overline{Z} . For simplicity, continue the $r \mapsto B(r)$ by the constant operator B_0 on $r \in \mathbb{R}_-$, thus obtaining a family over all of \mathbb{R} . Now define the reference operator $\mathcal{S} = \mathbf{c}(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + \Omega B(r)$ on the cylinder $\overline{F} \oplus \overline{F} \to \overline{M} \times \mathbb{R}$.

Proposition 6.2. Choose $z_i = (x_i, s_i) \in \overline{Z} = \overline{M} \times [0, \infty[, s_1, s_2 > r_1, r_1 = r_1(\overline{M} \times \mathbb{R}) \text{ as in Propositions 3.8 and 3.9.}$

(a) If $|s_1 - s_2| > 2r_1$ and $t \in \mathbb{R}_+$, then

$$|\nabla_{z_1}^l \nabla_{z_2}^k [T^m e^{-tT^2}](z_1, z_2)| \le C(k, l, m, T) e^{-(|s_1 - s_2| - r_1)^2/6t}.$$

(b) Choose cut-off functions $\psi_1, \psi_2 \in C^{\infty}(\overline{Z})^{\Gamma}$ whose supports in the r-direction have distance d. Then for $t \in \mathbb{R}_+$ the following estimate of the operator norms holds:

$$\| \psi_1 T^m e^{-tT^2} \psi_2 \| \le C(m, \psi_i) e^{-d^2/6t}$$

Note that this estimate that the constants are independent of T as long as T has propagation speed ≤ 1 along the cylinder.

(c) The relative version of b) also holds for $t \in \mathbb{R}_+$, $s_1, s_2 > r_1$:

$$\begin{aligned} |\nabla_{z_1}^l \nabla_{z_2}^k ([T^m e^{-tT^2}](z_1, z_2) - [\mathcal{S}^m e^{-t\mathcal{S}^2}](z_1, z_2))| \\ &\leq C(k, l, m, T) e^{-((s_1 \wedge s_2) - r_1)^2/6t}. \end{aligned}$$

Using a variant of the Decomposition Principle 5.1, we now describe the Γ -essential spectrum of T^2 on \overline{X} . The reference operator at infinity on $\overline{M} \times \mathbb{R}$ is $S = \mathbf{c}(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + \Omega B$. This will be compared with $T \sqcup -T$ on $\overline{X} \sqcup -\overline{X}$:

Lemma 6.3. $\operatorname{spec}_{\Gamma,e}(T^2) = \operatorname{spec}_{\Gamma,e}(S^2).$

To calculate the Γ -essential spectrum of T, it is therefore enough to calculate the Γ -essential spectrum of $S^2 = B^2 - (\frac{\partial}{\partial r})^2$. This can obviously be done on just one 'half' $L^2(\overline{M} \times \mathbb{R}, \overline{F})$. Thus choose a spectral resolution V of B, i.e. $V : L^2(\overline{M}, \overline{F}) \xrightarrow{\sim} L^2(\mathbb{R}_\lambda \times \mathbb{N}, \mu_B(\lambda, j))$ is a unitary equivalence such that $Vf(B)V^{-1}$ is the multiplication operator with the function f on $L^2(\mathbb{R} \times \mathbb{N}, \mu_B(\lambda, j))$. Write $U : L^2(\mathbb{R}_r) \to L^2(\mathbb{R}_y)$ for Fourier transformation along r, to obtain the unitary equivalence:

$$W = V \otimes U : L^2(\overline{M}, \overline{F}) \otimes L^2(\mathbb{R}_r) \to L^2((\mathbb{R}_\lambda \times \mathbb{N}) \times \mathbb{R}_y, \mu_B \times dy),$$

which transforms the action of S into a multiplication operator:

 $WSW^{-1}\cong \lambda^2+y^2.$

The Γ -trace on the Hilbert Γ -module $L^2(\overline{M} \times \mathbb{R}, \overline{F})$ can then be described as the product of $tr_{\Gamma}^{\overline{M}}$ on $L^2(\overline{M}, \overline{F})$ with the usual trace $tr^{\mathbb{R}}$ on $L^2(\mathbb{R})$. Writing $\omega := \inf(\operatorname{supp}(\mu_{\Gamma,B^2}))$ we thus find for $0 \le a < b$ with $\omega < b$ and $\epsilon < (b - \omega)/2$

$$\mu_{\Gamma,S^{2}}(]a,b[) = \dim_{\Gamma}^{\overline{M}} \otimes \dim^{\mathbb{R}}[\mathcal{H}_{S^{2}}(]a,b[)]$$

$$= \dim_{\Gamma}^{\overline{M}} \otimes \dim^{\mathbb{R}} W^{-1} \left[L^{2}(\{(\lambda,j,y) | a < \lambda^{2} + y^{2} < b\}, \mu_{B}(\lambda,j) \times dy) \right]$$

$$\geq \dim_{\Gamma}^{\overline{M}} V^{-1} \left[L^{2}(\{(\lambda,j) | a < \lambda^{2} < \omega + \epsilon\}, \mu_{B}(\lambda,j)) \right]$$

$$\cdot \dim^{\mathbb{R}} \left[L^{2}(\{y | 0 < y^{2} < \epsilon\}, dy) \right]$$

which equals ∞ since the first factor is non-zero and the second factor is infinite. In the same manner, one shows that $\mu_{\Gamma,S^2}(]a,b[) = 0$, if $\omega \geq b$.

We have thus shown

Proposition 6.4. spec_{Γ,e} $(T^2) = [inf(supp(\mu_{\Gamma,B^2})), \infty[.$

6.2. The L^2 - Γ -index. Proposition 6.4 implies (together with Proposition 2.14) that $T \in Op_{\Gamma}^1(\overline{X}, \overline{E})$, with product structure (6.14), is Γ -Fredholm if and only if B is invertible. In this Section, we describe the structure of the null space of such operators without the condition of invertibility on B.

The analysis of the asymptotics at infinity of the sections in (T^{\pm}) can be done most naturally in the context of weighted L^2 - and Sobolevspaces on X. To introduce these concepts, let $\vartheta \in C^{\infty}(\overline{X})$ be a weight function with

$$\theta = \theta(r) = r \text{ on } \overline{Z}_3, \ \theta|_{\overline{X}_2} \equiv 0, \quad \vartheta = \theta'(r) \text{ on } \overline{Z}_2, \ \vartheta|_{\overline{X}_2} \equiv 0.$$

Then, the operator T is closed as an operator on $e^{u\theta}L^2$, $u \in \mathbb{R}$ with domain of definition $e^{u\theta}H^1$ and we have a commutative diagram:

$$e^{u\theta}H^{1}(\overline{X},\overline{E}^{\pm}) \xrightarrow{T^{\pm}} e^{u\theta}L^{2}(\overline{X},\overline{E}^{\mp})$$

$$\uparrow e^{u\theta} \qquad \uparrow e^{u\theta}$$

$$H^{1}(\overline{X},\overline{E}^{\pm}) \xrightarrow{T^{\pm}\pm u\vartheta} L^{2}(\overline{X},\overline{E}^{\mp})$$
(6.15)

The vertical maps in this diagram are isomorphisms of Hilbert Γ modules. The following Lemma states the most important properties of these weighted Sobolev spaces.

Lemma 6.5. Choose $\delta' < \delta \in \mathbb{R}$. Then

- (a) The map $\iota : e^{-\delta\theta} L^2(\overline{X}) \hookrightarrow e^{-\delta'\theta} L^2(\overline{X})$ is a continuous embedding.
- (b) The map $\iota: e^{-\delta\theta} H^1(\overline{X}) \hookrightarrow e^{-\delta'\theta} L^2(\overline{X})$ is Γ -compact.
- (c) The subspace \mathcal{W} is Γ finite-dimensional in $e^{-\delta'\theta}L^2$, if and only if $\iota(\mathcal{W})$ is of finite Γ -dimension in $e^{-\delta\theta}L^2$. In that case

$$\dim_{\Gamma}(\mathcal{W} \subset e^{-\delta'\theta}L^2) = \dim_{\Gamma}(\iota(\mathcal{W}) \subset e^{-\delta\theta}L^2)$$

Proof. (a) is clear, for (b) set $\delta' = 0$ wlog. First note that the restriction operator

$$\Lambda_{\kappa}: e^{-\delta\theta} H^1(\overline{X}) \stackrel{\cdot \chi_{\overline{X}_{\kappa}}}{\longrightarrow} L^2(\overline{X})$$

is Γ -compact according to Rellich's Theorem 4.1. We proceed to show that the operators Λ_{κ} converge to 1 in norm. For $\xi \in e^{-\delta\theta} H^1$:

$$\| (1 - \Lambda_{\kappa})\xi \|_{L^{2}(\overline{X})}^{2} = \int_{\kappa}^{\infty} \| \xi(\bullet, r) \|_{L^{2}(\overline{M})}^{2} dr$$

$$\leq e^{-2\delta\kappa} \int_{\kappa}^{\infty} e^{2\delta r} \| \xi(\bullet, r) \|_{L^{2}(\overline{M})}^{2} dr \leq e^{-2\delta\kappa} \| \xi \|_{e^{-\delta\theta}L^{2}}^{2} \leq e^{-2\delta\kappa} \| \xi \|_{e^{-\delta\theta}H^{1}}^{2}$$

This converges to 0 for $\kappa \to \infty$ proving (b). For (c) it suffices to note that

$$\iota: (\mathcal{W} \subset e^{-\delta\theta}L^2) \to (\iota(\mathcal{W}) \subset e^{-\delta'\theta}L^2)$$

is a quasiisomorphism and to recall Lemma 2.4.

Part (c) of this Lemma enables us to indiscriminately use the notation \dim_{Γ} on all weighted Sobolev spaces. The Lemma also implies

Corollary 6.6. As before let $T \in Op_{\Gamma}^{1}(\overline{X}, \overline{E})$, elliptic, selfadjoint and with product structure (6.14) and let $u \in \mathbb{R}$. Then $e^{u\theta}L^2$ -null(T) is of finite Γ -dimension.

Proof. Again wlog set u = 0. As all Sobolev-norms are equivalent on null(T), Lemma 6.5(b) implies that the map

$$\iota: (\ker(T), L^2) \longrightarrow (\ker(T), H^1) \longrightarrow e^{\delta \theta} L^2$$

is Γ -compact for all $\delta > 0$. Hence $\iota(\ker(T), L^2) \subset e^{\delta\theta}L^2$ is of finite Γ -dimension. Part (c) of Lemma 6.5 then implies that $\operatorname{null}(T) \subset L^2$ is of finite Γ -dimension.

Definition 6.7. Let $T \in Op_{\Gamma}^1(\overline{X}, \overline{E})$ elliptic, selfadjoint and with product structure (23). The L^2 - Γ -index of T is defined as

$$L^{2}\operatorname{-ind}_{\Gamma}(T) := \dim_{\Gamma}(\ker(T^{+})) - \dim(\ker(T^{-})).$$

We now introduce the following modification of the Dirac operator on X:

$$\overline{D}_{\epsilon,u} := \overline{D} + \vartheta \Omega(u - A\Pi_{\epsilon}), \quad A = \overline{D}^F, \quad \Pi_{\epsilon} = E_A(] - \epsilon, \epsilon[).$$

We also agree to write \overline{D}_{ϵ} for $\overline{D}_{\epsilon,0}$. The operator $\overline{D}_{\epsilon,u}$ is our prototype of an eliptic selfadjoint operator in $Op_{\Gamma}^1(\overline{X}, \overline{E})$ with product structure over the cylinder that we have considered above. Its restriction to the basis M of the cylinder is $A_{\epsilon,u} = A(1 - \Pi_{\epsilon}) + u$. For $0 < \epsilon$, 0 is an isolated point in the spectrum of $A_{\epsilon} := A_{\epsilon,0}$, and $A_{\epsilon,u}$ is invertible for $0 < |u| < \epsilon$. In this case the operator $\overline{D}_{\epsilon,u}$ is Γ - Fredholm according to the results in Section 6.1. In the remainder of this Section, we establish

the relationship between the Γ -index of the operators $\overline{D}_{\epsilon,u}$ and the L²- Γ -index of \overline{D} .

On the cylinder, $\overline{D}_{\epsilon,u}$ can be written as

$$\overline{D}_{\epsilon,u} = \mathbf{c}(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + \Omega A + \vartheta \Omega(u - A\Pi_{\epsilon}),$$

i.e. the Sections $\xi^{\pm} \in e^{\infty \theta} L^2 := \bigcup_{\delta > 0} e^{\delta \theta} L^2$ in the C^{∞} -null space of $\overline{D}_{\epsilon,u}^{\pm}$ satisfy the following equation over \overline{Z}

$$\left(\pm\frac{\partial}{\partial r} + \lambda + \vartheta(r)(u - \chi_{\epsilon}(\lambda)\lambda)\right)V\xi^{\pm} = 0, \quad \chi_{\epsilon} := \chi_{]-\epsilon,\epsilon[}.$$

Here, $V : L^2(\overline{M}, \overline{F}) \to L^2(\mu_A)$ is again the spectral resolution of A. We thus have

$$V\xi^{\pm}(\lambda, r) = \zeta^{\pm}(\lambda, i)e^{\mp u\theta(r)}e^{\mp\lambda(r-\theta(r)\chi_{\epsilon}(\lambda))}$$
(6.16)

for suitably chosen $\chi^{\pm}(\lambda, i) \in L^2(\mu_A)$.

All solutions of $\overline{D}_{\epsilon,u}^{\pm}\xi = 0$ are thus exponentially decreasing, constant or increasing along the cylinder. For $\epsilon \geq 0$, define the space of extended L^2 -Sections in the null space of \overline{D}_{ϵ} and its Γ -dimension $h_{\Gamma,\epsilon}^{\pm}$ by

$$\operatorname{Ext}(\overline{D}_{\epsilon}^{\pm}) := \bigcap_{u>0} e^{u\theta} L^{2} \operatorname{-ker}(\overline{D}_{\epsilon}^{\pm}),$$
$$h_{\Gamma,\epsilon}^{\pm} := \dim_{\Gamma}(\operatorname{Ext}(\overline{D}_{\epsilon}^{\pm})) - \dim_{\Gamma}(L^{2} \operatorname{-ker}(\overline{D}_{\epsilon}^{\pm})). \quad (6.17)$$

From (6.16), (6.15) one deduces for $0 < u < \epsilon$

$$L^{2}-\ker(\overline{D}_{\epsilon}^{\pm}) = e^{-u\theta}L^{2}-\ker(\overline{D}_{\epsilon}^{\pm}) = L^{2}-\ker(\overline{D}_{\epsilon,\pm u}^{\pm})$$

$$\operatorname{Ext}(\overline{D}_{\epsilon}^{\pm}) = e^{u\theta}L^{2}-\ker(\overline{D}_{\epsilon}^{\pm}) = L^{2}-\ker(\overline{D}_{\epsilon,\pm u}^{\pm})$$
(6.18)

Taking the limit $u \to 0$ is thus harmless:

Lemma 6.8. Let
$$\epsilon > 0$$
. Then
(a) $\dim_{\Gamma}(\ker(\overline{D}_{\epsilon}^{\pm})) = \lim_{u \searrow 0} \dim_{\Gamma}(\ker(\overline{D}_{\epsilon,\mp u}^{\pm}))$
 $= \lim_{u \searrow 0} \dim_{\Gamma}(\ker(\overline{D}_{\epsilon,\pm u}^{\pm})) - h_{\Gamma,\epsilon}^{\pm}$
(b) $L^{2}\text{-ind}_{\Gamma}(\overline{D}_{\epsilon}) = \lim_{u \searrow 0} \operatorname{ind}_{\Gamma}(\overline{D}_{\epsilon,u}) - h_{\Gamma,\epsilon}^{+}$
 $= \lim_{u \searrow 0} \operatorname{ind}_{\Gamma}(\overline{D}_{\epsilon,-u}) + h_{\Gamma,\epsilon}^{-}$

Proof. These claims follow from (6.18), diagram (6.15) and Lemma 6.5.

The description of the null spaces of \overline{D}_{ϵ} for $\epsilon \searrow 0$ is a little bit more subtle. Let's start by collecting some of the consequences of (6.16):

Lemma 6.9. Choose $\epsilon > \delta > 0$, $\delta' \in \mathbb{R}$. Then

(a) $\xi \in e^{\delta' \theta} L^2 \operatorname{-ker}(\overline{D}^+) \Rightarrow \xi|_{\overline{Z}} = e^{-rA}\zeta, \quad \zeta \in \mathcal{H}_A(] - \delta', \infty[)$ (b) $\xi \in L^2 \operatorname{-ker}(\overline{D}_{\epsilon}^+) \Rightarrow \xi|_{\overline{Z}} = e^{-rA + \theta(r)A\Pi_{\epsilon}}\zeta, \quad \zeta \in \mathcal{H}_A(]\epsilon, \infty[)$ (c) $\xi \in e^{\delta\theta} L^2 \operatorname{-ker}(\overline{D}_{\epsilon}^+) \Rightarrow \xi|_{\overline{Z}} = e^{-rA + \theta(r)A\Pi_{\epsilon}}\zeta, \quad \zeta \in \mathcal{H}_A(] - \epsilon, \infty[)$ (d) The operator $e^{\pm \theta(r)A\Pi_{\epsilon}} : L^2(\overline{X}, \overline{F}) \to e^{2\epsilon\theta} L^2(\overline{X}, \overline{F})$

is quasiisometric onto its image.
(e)
$$D^{\pm}e^{\mp\theta(r)A\Pi_{\epsilon}} = e^{\mp\theta(r)A\Pi_{\epsilon}}D^{\pm}_{\epsilon}$$

Analogous statements can be made for \overline{D}^- .

Proof. These claims all follow from the representation

$$\overline{D}_{\epsilon}^{\pm} = A \pm \frac{\partial}{\partial r} - \vartheta(r) A \Pi_{\epsilon} \quad \text{on} \quad \overline{Z},$$

and the description of solutions in (6.16). For (d), note in addition that the operator $e^{\pm\theta(r)A\Pi_{\epsilon}}$ has no null space.

Lemma 6.10. (a) $\lim_{\epsilon \searrow 0} \dim_{\Gamma}(\ker(\overline{D}_{\epsilon}^{\pm})) = \dim_{\Gamma}(\ker(\overline{D}^{\pm})).$ (b) $\lim_{\epsilon \searrow 0} L^2 \operatorname{-ind}_{\Gamma}(\overline{D}_{\epsilon}) = L^2 \operatorname{-ind}_{\Gamma}(\overline{D}).$

(c) $\lim_{\epsilon \searrow 0} \dim_{\Gamma}(\operatorname{Ext}(\overline{D}_{\epsilon}^{\pm})) = \dim_{\Gamma}(\operatorname{Ext}(\overline{D}^{\pm})).$

Proof. For (a), let $\xi \in L^2$ -ker $(\overline{D}_{\epsilon}^+)$, thus on the cylinder $\xi|_{\overline{Z}} = e^{-rA + \theta A \Pi_{\epsilon}} \zeta$ with $\zeta \in \mathcal{H}_A([\epsilon, \infty[))$. Here we then also have

$$\overline{D}^{+}\xi|_{\overline{Z}} = (\overline{D}_{\epsilon}^{+} + \vartheta(r)A\Pi_{\epsilon})\xi|_{\overline{Z}} = \vartheta(r)A\Pi_{\epsilon}(\xi|_{\overline{Z}})$$
$$= \vartheta(r)A\Pi_{\epsilon}(e^{-rA + \theta A\Pi_{\epsilon}}\zeta) = 0, \quad \text{da} \quad \Pi_{\epsilon}\zeta = 0,$$

thus L^2 -ker $(\overline{D}_{\epsilon}^+) \hookrightarrow L^2$ -ker (\overline{D}^+) . The operator $\overline{D}_{\epsilon}^+$, when restricted to L^2 -ker (\overline{D}^+) , has null space L^2 -ker $(\overline{D}_{\epsilon}^+)$ and its image satisfies

 $\overline{D}_{\epsilon}^{+}(L^{2}-\mathrm{null}(\overline{D}^{+})) = -\vartheta(r)A\Pi_{\epsilon}(L^{2}-\ker(\overline{D}^{+})) \subset -\vartheta(r)Ae^{-rA}\mathcal{H}_{A}(]0,\epsilon[),$ hence $\dim_{\Gamma}(\overline{D}_{\epsilon}^{+}(L^{2}-\ker(\overline{D}^{+}))) \to 0$ for $\epsilon \searrow 0$. This and Lemma 2.4

hence $\dim_{\Gamma}(D_{\epsilon}(L^{2}-\ker(D^{-}))) \to 0$ for $\epsilon \searrow 0$. This and Lemma 2.4 prove the result.

(c): Let $0 < \delta < \epsilon$. Using the description of Sections in the null space over the cylinder in 6.9, one shows that the operator $\Psi_{\epsilon}^{\pm} = e^{\pm \theta(r)A\Pi_{\epsilon}}$ satisfies:

$$e^{\theta\delta}L^2$$
-ker $(\overline{D}^+) \xrightarrow{\Psi_{\epsilon}^+} e^{\theta\delta}L^2$ -ker $(\overline{D}_{\epsilon}^+) \xrightarrow{\Psi_{\epsilon}^-} e^{\theta(\delta+\epsilon)}L^2$ -ker (\overline{D}^+) .

These maps are injective, and when restricted (e.g.) to

$$e^{\theta\delta}L^2(\overline{X},\overline{F}) \xrightarrow{\Psi_{\epsilon}^+} e^{2\theta\epsilon}L^2(\overline{X},\overline{F}) \xrightarrow{\Psi_{\epsilon}^-} e^{4\theta\epsilon}L^2(\overline{X},\overline{F})$$

also continuous. Taking $\epsilon\searrow 0$ and using Lemma 6.5 then shows the result.

6.3. The L²- Γ - index theorem. In this Section we show (cf. [BGV], Chapter 3 for notation):

Theorem 6.11. (L^2 - Γ -index theorem)

$$L^{2}\operatorname{-ind}_{\Gamma}(\overline{D}) = \int_{X} \widehat{A}(X)Ch(E/S) + \frac{1}{2}(\eta_{\Gamma}(A) - h_{\Gamma}^{-} + h_{\Gamma}^{+})$$

Using Lemma 6.8 and Lemma 6.10 we will reduce the calculation of the L^2 - Γ -index of \overline{D} to the calculation of the Γ - index of $\overline{D}_{\epsilon,u}$, for small $0 < |u| < \epsilon$.

According to Proposition 3.10 for $t \to \infty$, the operator $[e^{-t\overline{D}_{\epsilon,u}^2}]$ converges in C^{∞} to the Schwartz-kernel $[N(\overline{D}_{\epsilon,u})]$ of the projection onto the null space of $\overline{D}_{\epsilon,u}$. We will use cut-off functions $\phi_{\kappa} \in C^{\infty}(\overline{X})^{\Gamma}$ with $\phi_{\kappa}|\overline{X}_{\kappa} = 1$ and $\phi_{\kappa}|_{\overline{Z}_{\kappa+1}} = 0$, such that the operators $\phi_{\kappa}e^{-s\overline{D}_{\epsilon,u}^2}\phi_{\kappa}$ are Γ -trace class. The Γ -index of $\overline{D}_{\epsilon,u}$ can then be calculated as

$$\operatorname{ind}_{\Gamma}(\overline{D}_{\epsilon,u}) = \operatorname{str}_{\Gamma}(N(\overline{D}_{\epsilon,u})) = \lim_{\kappa \to \infty} \lim_{t \to \infty} \operatorname{str}_{\Gamma}(\phi_{\kappa}e^{-t\overline{D}_{\epsilon,u}^{2}}\phi_{\kappa})$$
$$= \lim_{\kappa \to \infty} \left(\operatorname{str}_{\Gamma}(\phi_{\kappa}e^{-s\overline{D}_{\epsilon,u}^{2}}\phi_{\kappa}) - \int_{s}^{\infty} \operatorname{str}_{\Gamma}(\phi_{\kappa}\overline{D}_{\epsilon,u}^{2}e^{-t\overline{D}_{\epsilon,u}^{2}}\phi_{\kappa})dt\right) (6.19)$$

Of course, the RHS is independent of s > 0. The integral can be split as follows

$$\int_{s}^{\kappa} \operatorname{str}_{\Gamma}(\phi_{\kappa}\overline{D}_{\epsilon,u}^{2}e^{-t\overline{D}_{\epsilon,u}^{2}}\phi_{\kappa})dt + \int_{\kappa}^{\infty} \operatorname{str}_{\Gamma}(\phi_{\kappa}\overline{D}_{\epsilon,u}^{2}e^{-t\overline{D}_{\epsilon,u}^{2}}\phi_{\kappa})dt.$$
(6.20)

We first show that the second integral in 6.20 vanishes for $\kappa \to \infty$: Since $\overline{D}_{\epsilon,u}$ is Γ -Fredholm ist, there is $\alpha = \alpha(u) > 0$ (e.g. $\alpha(u) = |u|/2$), such that the projection $H_{\alpha} = E_{\overline{D}_{\epsilon,u}}([-\alpha, \alpha])$ has finite Γ -trace. This implies (Cf. Remark 2.7:

$$\begin{split} |\int_{\kappa}^{\infty} \mathrm{str}_{\Gamma}(\phi_{\kappa}\overline{D}_{\epsilon,u}^{2}e^{-t\overline{D}_{\epsilon,u}^{2}}\phi_{\kappa})dt| \\ &\leq \int_{\kappa}^{\infty} |\mathrm{str}_{\Gamma}\left(\phi_{\kappa}\overline{D}_{\epsilon,u}^{2}e^{-\overline{D}_{\epsilon,u}^{2}/2}(1-H_{\alpha})e^{-(t-1)\overline{D}_{\epsilon,u}^{2}}e^{-\overline{D}_{\epsilon,u}^{2}/2}\overline{D}_{\epsilon,u}\phi_{\kappa}\right)|dt \\ &\quad + \int_{\kappa}^{\infty} |\mathrm{str}_{\Gamma}\left(e^{-t\overline{D}_{\epsilon,u}^{2}/2}H_{\alpha}\overline{D}_{\epsilon,u}\phi_{\kappa}^{2}\overline{D}_{\epsilon,u}H_{\alpha}e^{-t\overline{D}_{\epsilon,u}^{2}/2}\right)|dt. \\ &\leq \int_{\kappa}^{\infty} e^{-(t-1)\alpha}|\mathrm{str}_{\Gamma}\left(\phi_{\kappa}\overline{D}_{\epsilon,u}^{2}e^{-\overline{D}_{\epsilon,u}^{2}}\phi_{\kappa}\right)|dt \\ &\quad + \int_{\kappa}^{\infty}|\mathrm{str}_{\Gamma}\left(\overline{D}_{\epsilon,u}^{2}e^{-t\overline{D}_{\epsilon,u}^{2}}H_{\alpha}\right)|dt \end{split}$$

The Schwartz-kernel of $\overline{D}_{\epsilon,u}^2 e^{-\overline{D}_{\epsilon,u}^2}$ is UC^{∞} . Thus the trace in the first integral is majored by $c_1 + c_2 \kappa$, and the integral converges to 0 for $\kappa \to \infty$. Using the Γ -Fredholm property of $\overline{D}_{\epsilon,u}$ we get for the second integral

$$\int_{\kappa}^{\infty} |\operatorname{str}_{\Gamma} \left(\overline{D}_{\epsilon,u}^{2} e^{-t\overline{D}_{\epsilon,u}^{2}} H_{\alpha} \right) | dt \leq \int_{\kappa}^{\infty} \int_{-\alpha}^{\alpha} x^{2} e^{-tx^{2}} d\mu_{\Gamma,\overline{D}_{\epsilon,u}}(x) dt$$
$$= \int_{-\alpha}^{\alpha} e^{-\kappa x^{2}} \int_{0}^{\infty} x^{2} e^{-tx^{2}} dt \, d\mu_{\Gamma,\overline{D}_{\epsilon,u}}(x) \leq C \int_{-\alpha}^{\alpha} e^{-\kappa x^{2}} d\mu_{\Gamma,\overline{D}_{\epsilon,u}}(x)$$
$$\leq C \mu_{\Gamma,\overline{D}_{\epsilon,u}}([-\alpha,\alpha]).$$

Thus, the penultimate term also converges to 0 for $\kappa \to \infty$ from Levi's theorem.

We now turn our attention to the first integral in in (6.20) zu. The integrand can be written as follows

$$\operatorname{str}_{\Gamma}(\phi_{\kappa}\overline{D}_{\epsilon,u}^{2}e^{-t\overline{D}_{\epsilon,u}^{2}}\phi_{\kappa}) = \frac{1}{2}\operatorname{str}_{\Gamma}(\phi_{\kappa}[\overline{D}_{\epsilon,u},\overline{D}_{\epsilon,u}e^{-t\overline{D}_{\epsilon,u}^{2}}]\phi_{\kappa})$$
$$= \frac{1}{2}\operatorname{str}_{\Gamma}([\overline{D}_{\epsilon,u},\phi_{\kappa}\overline{D}_{\epsilon,u}e^{-t\overline{D}_{\epsilon,u}^{2}}\phi_{\kappa}] - [\overline{D}_{\epsilon,u},\phi_{\kappa}^{2}]\overline{D}_{\epsilon,u}e^{-t\overline{D}_{\epsilon,u}^{2}})$$
$$= \frac{1}{2}\operatorname{str}_{\Gamma}(-[\overline{D}_{\epsilon,u},\phi_{\kappa}^{2}]\overline{D}_{\epsilon,u}e^{-t\overline{D}_{\epsilon,u}^{2}})$$
$$= -\frac{1}{2}\operatorname{str}_{\Gamma}(\mathbf{c}(\frac{\partial}{\partial r})\frac{\partial(\phi_{\kappa}^{2})}{\partial r}\overline{D}_{\epsilon,u}e^{-t\overline{D}_{\epsilon,u}^{2}}).$$
(6.21)

The derivatives of ϕ_{κ} are supported in $\overline{M} \times [\kappa, \kappa + 1]$. Thus expression (6.21) can be compared with the operator $S_{\epsilon,u} = \mathbf{c}(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + (A(1-\Pi_{\epsilon}) + u)\Omega$ on the cylinder $\overline{Y} = \overline{M} \times \mathbb{R}$. Using Proposition 6.2 (c) we get for $(x, r) \in \overline{M} \times [3, \infty] \cong \overline{Z}_3$

$$\left(\left[\overline{D}_{\epsilon,u} e^{-t\overline{D}_{\epsilon,u}^2} \right] - \left[S_{\epsilon,u} e^{-tS_{\epsilon,u}^2} \right] \right) (x,r;x,r) \le C e^{-(r-3)^2/6t}$$

i.e. both Schwartz-kernels have the same $t \to 0\text{-asymptotics}$ for large r. Hence

$$\begin{split} &\int_{s}^{\kappa} |\operatorname{str}_{\Gamma}(\mathbf{c}(\frac{\partial}{\partial r})\frac{\partial(\phi_{\kappa}^{2})}{\partial r}\overline{D}_{\epsilon,u}e^{-t\overline{D}_{\epsilon,u}^{2}}) - \operatorname{str}_{\Gamma}(\mathbf{c}(\frac{\partial}{\partial r})\frac{\partial(\phi_{\kappa}^{2})}{\partial r}S_{\epsilon,u}e^{-tS_{\epsilon,u}^{2}})|dt \\ &= \int_{s}^{\kappa} |\int_{M\times[\kappa,\kappa+1]} \pi_{*}\operatorname{str}_{E}\left(\mathbf{c}(\frac{\partial}{\partial r})\frac{\partial(\phi_{\kappa}^{2})}{\partial r}[\overline{D}_{\epsilon,u}e^{-t\overline{D}_{\epsilon,u}^{2}} - S_{\epsilon,u}e^{-tS_{\epsilon,u}^{2}}]\right)dxdr|dt \\ &\leq C\int_{s}^{\kappa}\int_{\kappa}^{\kappa+1}e^{-(r-3)^{2}/6t}drdt \leq C\int_{s}^{\kappa}e^{-(\kappa-3)^{2}/6t}dt \\ &\leq C\int_{1/\kappa}^{1/s}a^{-2}e^{-(\kappa-3)^{2}a/6}da \leq C(\kappa^{2}e^{-\kappa/c_{1}} + e^{-c_{2}/s}) \end{split}$$

for large κ and small s. Thus in the first integral in (6.20), when looking at the asymptotics $s \to 0$, and the limit $\kappa \to \infty$, the operator $\overline{D}_{\epsilon,u}$ can be replaced by $S_{\epsilon,u}$.

But the integral for $S_{\epsilon,u}$ on the cylinder can be calculated. The Schwartzkernel of $S_{\epsilon,u}e^{-tS_{\epsilon,u}^2}$, restricted to the diagonal in $\overline{Y} \times \overline{Y}$, is of the form

$$[S_{\epsilon,u}e^{-tS_{\epsilon,u}^2}](x,r) = (A_{\epsilon,u}\Omega + \mathbf{c}(\frac{\partial}{\partial r})\frac{\partial}{\partial r})\left(\left[e^{-t(A_{\epsilon,u}\Omega)^2}\right](x,y)\frac{e^{-\frac{(r-s)^2}{4t}}}{\sqrt{4\pi t}}\right)|_{\substack{y=x\\s=r}}\right)$$
$$= \frac{1}{\sqrt{4\pi t}}\Omega[A_{\epsilon,u}e^{-tA_{\epsilon,u}^2}](x,x).$$

Now (Cf. Appendix A) $str^{E}(\mathbf{c}(\frac{\partial}{\partial r})\Omega \bullet) = -2tr^{F}(\bullet)$, and thus

$$\begin{split} &\int_{s}^{\kappa} \operatorname{str}_{\Gamma}(\mathbf{c}(\frac{\partial}{\partial r}) \frac{\partial(\phi_{\kappa}^{2})}{\partial r} S_{\epsilon,u} e^{-tS_{\epsilon,u}^{2}}) dt \\ &= \int_{\kappa}^{\kappa+1} \frac{\partial(\phi_{\kappa}^{2})}{\partial r} dr \int_{s}^{\kappa} \frac{1}{\sqrt{4\pi t}} \int_{\mathcal{F}(M)} -2 tr^{F}([A_{\epsilon,u} e^{-tA_{\epsilon,u}^{2}}](x,x)) dx dt, \\ &= \int_{s}^{\kappa} \frac{1}{\sqrt{\pi t}} \int_{\mathcal{F}(M)} tr^{F}([A_{\epsilon,u} e^{-tA_{\epsilon,u}^{2}}](x,x)) dx dt \end{split}$$

Taking the limit $\kappa \to \infty$ and selecting the constant term LIM in the asymptotic development for $s \to 0$, this just gives the Γ -Eta-invariant $\eta_{\Gamma}(A_{\epsilon,u})$ as described in Section 4.3. Using (6.19), (6.21) our present knowledge can be summarised like this

$$\operatorname{ind}_{\Gamma}(\overline{D}_{\epsilon,u}) = \lim_{\kappa \to \infty} \operatorname{LIM}_{s \to 0} \operatorname{str}_{\Gamma}\left(\phi_{\kappa} e^{-s\overline{D}_{\epsilon,u}^{2}}\phi_{\kappa}\right) + \frac{1}{2}\eta_{\Gamma}(A_{\epsilon,u}),$$
(6.22)

especially the limit in the first term exists.

It remains to analyse the asymptotics of the local trace $\operatorname{str}^{E}([e^{-s\overline{D}_{\epsilon,u}}])$ (x,x) for $s \to 0$. For $x \in \overline{X}_1$ the asymptotic development of $\operatorname{str}^{E}([e^{-sD^2}])$ on X gives

$$LIM_{s\to 0} \operatorname{str}^{E}([e^{-s\overline{D}_{\epsilon,u}^{2}}])(x,x)\operatorname{dvol}_{\overline{X}} = \widehat{A}(X)Ch(E/S)(x), \quad x \in \overline{X}_{1}.$$

'Far out' on the cylinder, for instance for $(y,r) \in \overline{Z}$ with r > 4, Proposition 6.2 allows us to look at the kernel $[e^{-sS_{\epsilon,u}^2}](y,r)$ instead of $[e^{-s\overline{D}_{\epsilon,u}^2}](y,r)$. Since $A_{\epsilon,u}$ is invertible, we have on \overline{M} (Cf. Lemma 4.5)

$$0 = \operatorname{ind}_{\Gamma}(A_{\epsilon,u}) = \operatorname{str}_{\Gamma}(e^{-sA_{\epsilon,u}^2}),$$

independent of s. Thus for 4 < a < b

$$\begin{split} & \underset{s \to 0}{\text{LIM}} \int_{\overline{M} \times [a,b]} \operatorname{str}^{E}([e^{-s\overline{D}_{\epsilon,u}^{2}}](y,r)) dy \, dr \\ &= \underset{s \to 0}{\text{LIM}} \int_{\overline{M} \times [a,b]} \operatorname{str}^{E}([e^{-s\overline{S}_{\epsilon,u}^{2}}](y,r)) dy \, dr = \underset{s \to 0}{\text{LIM}} \frac{b-a}{\sqrt{4\pi s}} \operatorname{str}_{\Gamma}(e^{-s\overline{A}_{\epsilon,u}^{2}}) = 0. \end{split}$$

The part $\overline{M} \times [1, 4]$ is a little bit more complicated, as ϑ is not constant here. Using Proposition 6.2 we can replace $\overline{D}_{\epsilon,u}$ by the operator $S_{\epsilon,u} := \mathbf{c}(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + \Omega(A + \vartheta(u - \Pi_{\epsilon}A))$ in that area. For a cut-off function $\psi_1 \in C^{\infty}(\overline{Y})^{\Gamma}$ with $\psi_1|_{\overline{M} \times [1,4]} \equiv 1$ and support in $\overline{M} \times [0,5]$, we already know from the above and (6.22) that $\mathrm{LIM}_{s \to 0} \mathrm{str}_{\Gamma}(\psi_1 e^{-sS_{\epsilon,u}^2}\psi_1)$ exists. We now show

Lemma 6.12. $\lim_{s\to 0} \operatorname{str}_{\Gamma}(\psi_1(e^{-sS_{\epsilon,u}^2} - e^{-sS_u^2})\psi_1) = 0, \quad S_u := S_{0,u}.$

Proof. First,

$$S_{\epsilon,u}^2 - S_u^2 = (\vartheta \Omega \Pi_{\epsilon} A)^2 - \mathbf{c} (\frac{\partial}{\partial r}) \Omega \vartheta' \Pi_{\epsilon} A - 2 \vartheta \Omega \Pi_{\epsilon} A S_u$$

This is an operator of finite Γ -rank in the \overline{M} -direction. A little care is needed as it has a first dervative acting in the \mathbb{R}_r -direction. We apply the Duhamel-method (Proposition 3.11):

$$|\operatorname{str}_{\Gamma}(\psi_{1}(e^{-sS_{u}^{2}}-e^{-sS_{\epsilon,u}^{2}})\psi_{1})| = |\operatorname{str}_{\Gamma}(\psi_{1}e^{-\delta S_{u}^{2}}e^{-(s-\delta)S_{\epsilon,u}^{2}}\psi_{1})|_{0}^{s}|$$

$$= |\int_{0}^{s} \operatorname{str}_{\Gamma}(\psi_{1}^{2}\Pi_{\epsilon}e^{-\delta S_{u}^{2}}(S_{\epsilon,u}^{2}-S_{u}^{2})\Pi_{\epsilon}e^{-(s-\delta)S_{\epsilon,u}^{2}})d\delta|$$

$$\leq C\int_{0}^{s} |\operatorname{tr}_{\Gamma}(\psi_{1}e^{-\delta S_{u}^{2}}\Pi_{\epsilon}\psi_{1})| \parallel (S_{\epsilon,u}^{2}-S_{u}^{2})\Pi_{\epsilon}e^{-(s-\delta)S_{\epsilon,u}^{2}} \parallel d\delta$$

(6.23)

Using the results of Section 3.3 one finds

$$|\operatorname{tr}_{\Gamma}(\psi_{1}e^{-\delta S_{u}^{2}}\Pi_{\epsilon}\psi_{1})| \leq C \ \delta^{-1/2} \| (S_{\epsilon,u}^{2} - S_{u}^{2})\Pi_{\epsilon}e^{-(s-\delta)S_{\epsilon,u}^{2}} \| \leq C \ (s-\delta)^{-1/2}$$
(6.24)

where the constants are independent of any small $|u| < \epsilon$. Thus (6.23) can be majored by

$$C \int_{0}^{s/2} |\operatorname{tr}_{\Gamma}(\psi_{1}e^{-\delta S_{u}^{2}}\Pi_{\epsilon}\psi_{1})| d\delta + C \int_{s/2}^{s} || (S_{\epsilon,u}^{2} - S_{u}^{2})\Pi_{\epsilon}e^{-(s-\delta)S_{\epsilon,u}^{2}} || d\delta$$
$$\leq C \int_{0}^{s/2} \delta^{-1/2} d\delta + C \int_{s/2}^{s} (s-\delta)^{-1/2} d\delta \leq C s^{1/2}$$
(6.25)

But this converges to 0 for $s \to 0$.

We thus have brought back the asymptotic development of the Γ -trace of the heat kernel of $\overline{D}_{\epsilon,u}$ over the critical area $\overline{M} \times [1,4]$ to the asymptotic development

$$\operatorname{str}^{E}([e^{-sS_{u}^{2}}])(z,z) \sim \sum_{j \in \mathbb{N}} a_{j}(S_{u})(z)t^{(j-n)/2},$$

with coefficients $a_j(S_u)$ differentiable in u. Since S_0 is the Dirac operator, we have $a_j(S_0) = 0$ for $j \leq n/2$. Using Lemma 6.12 and (6.22), this implies

Proposition 6.13. The Γ -index dof the Γ -Fredholm operator $\overline{D}_{\epsilon,u}$ is given by

$$\operatorname{ind}_{\Gamma}(\overline{D}_{\epsilon,u}) = \int_{X} \widehat{A}(X) Ch(E/S) + \frac{1}{2} \eta_{\Gamma}(A_{\epsilon,u}) + g(u), \text{ with } \lim_{u \to 0} g(u) = 0.$$

Now, from Lemma 4.7 the Γ - Eta-invariant with $0 < |u| < \epsilon$ satisfies

$$\eta_{\Gamma}(A_{\epsilon}) = \frac{1}{2}(\eta_{\Gamma}(A_{\epsilon,u}) + \eta_{\Gamma}(A_{\epsilon,-u})),$$

which with Lemma 6.8 and the definition (6.17) of $h_{\Gamma,\epsilon}^{\pm}$ implies

$$L^{2}\operatorname{-ind}_{\Gamma}(\overline{D}_{\epsilon}) = \lim_{u \searrow 0} \frac{1}{2} \left[\operatorname{ind}_{\Gamma}(\overline{D}_{\epsilon,u}) + \operatorname{ind}_{\Gamma}(\overline{D}_{\epsilon,-u}) + h_{\Gamma,\epsilon}^{-} - h_{\Gamma,\epsilon}^{+} \right]$$
$$= \int_{X} \widehat{A}(X)Ch(E/S) + \frac{1}{2}(\eta_{\Gamma}(A_{\epsilon}) + h_{\Gamma,\epsilon}^{-} - h_{\Gamma,\epsilon}^{+}).$$

Theorem 6.11 follows from this and the observations (Cf. Lemma 6.10)

• $|\eta_{\Gamma}(A_{\epsilon}) - \eta_{\Gamma}(A)| \leq \operatorname{tr}_{\Gamma}(E_{A}(] - \epsilon, \epsilon[-\{0\})) \to 0 \text{ for } \epsilon \searrow 0.$

•
$$h_{\Gamma,\epsilon}^{\pm} \to h_{\Gamma}^{\pm}$$
 for $\epsilon \searrow 0$.

• L^2 -ind_{Γ} $(\overline{D}_{\epsilon}) \to L^2$ -ind_{Γ} (\overline{D}) for $\epsilon \searrow 0$.

As a first simple application of Theorems 4.11 and 6.11 consider a residually finite covering

$$\overline{X} \cdots \to X_{i+1} \to X_i \cdots \to X$$

of X with lifted bundles E_i , Dirac operators D_i etc., such that \overline{D}^F satisfies conditions (a) or (b) in Theorem 4.11:

Proposition 6.14. (Convergence of the modified L^2 -Index)

$$\lim_{i \to \infty} d_i^{-1} (L^2 \operatorname{-ind}(D_i) - \frac{1}{2} (h_i^+ - h_i^-)) = L^2 \operatorname{-ind}_{\Gamma}(\overline{D}) - \frac{1}{2} (h_{\Gamma}^+ - h_{\Gamma}^-).$$

6.4. The signature operator. In this Section we specialise the L²- Γ -index Theorem 6.11 to the case of the signature operator. The underlying Clifford bundle is $E \otimes W = \Lambda T^* X \otimes W$, with W a flat bundle with product structure on the cylinder of X. Clifford multiplication is given by $\mathbf{c}(v) = \epsilon(v) - \iota(v)$, for $v \in T^* X$. The grading operator $\tau \equiv \tau_X = i^{n/2} *_X (-1)^{|\cdot|(|\cdot|-1)/2}$ gives a \mathbb{Z}_2 -grading on E and the Dirac operator $D = S_X = d_X + d_X^*$ is also called the signature operator in this case.

We have $S_{\overline{X}}^2 = \Delta_{\overline{X}}$ and from Hodge's theorem for \overline{X} , the map

$$\mathcal{H}^*(\overline{X};\overline{W}) := L^2 \operatorname{-ker}(\Delta_{\overline{X}}) = L^2 \operatorname{-ker}(S_{\overline{X}}) \to H^*_{(2)}(\overline{X};\overline{W}),$$

from the null space of $\Delta_{\overline{X}}$ into the reduced L^2 -Cohomology of \overline{X} with coefficients \overline{W} , is an isomorphism.

For dim(X) = n = 4k we have $*_X^2 = 1$ and the Γ -signature $\sigma_{\Gamma}(\overline{X}; \overline{W})$ of \overline{X} with coefficients \overline{W} is defined as the difference of the Γ -dimensions of the +1- and -1-Eigenspaces of the quadratic form $\alpha \mapsto (\alpha, *_{\overline{X}}\alpha)$ on $\mathcal{H}^{2k}(\overline{X}; \overline{W})$. The Γ -signature is then just the L^2 - Γ -index of $S_{\overline{X}}$.

Using the identification $\overline{E} = \Lambda T^* \overline{M} \oplus \Lambda T^* \overline{M}$ via

$$\overline{F} := \Lambda T^* \overline{M} \stackrel{1+\tau_X}{\to} (1+\tau_X) \Lambda T^* \overline{X} = \overline{E}^+$$
(6.26)

over the cylinder \overline{Z} , $S_{\overline{X}}$ can be written

$$S_{\overline{X}} = \mathbf{c}(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + \Omega(*_{\overline{M}}d_{\overline{M}} - d_{\overline{M}}*_{\overline{M}})(-1)^{|\cdot|(|\cdot|-i)|\cdot|/2} \equiv \mathbf{c}(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + \Omega A.$$

In this Section, we prove

Proposition 6.15. $h_{\Gamma,\epsilon}^+(S_{\overline{X}}) = h_{\Gamma,\epsilon}^-(S_{\overline{X}})$ for all $\epsilon \ge 0$.

This immediately implies

Theorem 6.16. (Γ -Signature theorem)

$$\sigma_{\Gamma}(\overline{X};\overline{W}) = rk(W) \int_{X} L(X) + \frac{1}{2}\eta_{\Gamma}(A).$$

Proof. Apply Theorem 6.11 and Proposition 6.15, recalling that $\widehat{A}(X)Ch(\Lambda T^*X \otimes W/S) = rk(W)L(X)$, (Cf. [BGV])

For residually finite coverings this then implies

Corollary 6.17. Let the conditions of Proposition 6.14 hold.

- (a) $\lim_{i\to\infty} d_i^{-1}\sigma(X_i; W_i) = \sigma_{\Gamma}(\overline{X}; \overline{W}).$
- (b) If the universal covering \widetilde{X} of X is residually finite then $\sigma_{\Gamma}(\widetilde{X})$ is a proper homotopy-invariant of X.

Proof. (a) follows from 6.16 and Proposition 6.14, (b) then follows from the homotopy invariance of the signatures $\sigma(X_i)$.

The proof of Proposition 6.15 is an adaptation to the Γ -case of the proof for the classical case given in [Me]. Thus, consider the operator $S_{\overline{X},\epsilon}$ for $\epsilon > 0$. If, instead of the identification (6.26) given above, one uses the identification $\Lambda T^*\overline{X} = \Lambda T^*\overline{M} \oplus \Lambda T^*\overline{M} \wedge dr$, for the Clifford bundle over \overline{Z} , then

$$S_{\overline{X},\epsilon} = c(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + S_{\overline{M}} \oplus S_{\overline{M}} - \vartheta \widehat{\Pi}_{\epsilon} S_{\overline{M}} \oplus S_{\overline{M}}$$
(6.27)

where $S_{\overline{M}} = d_{\overline{M}} + d_{\overline{M}}^*$ and $\widehat{\Pi}_{\epsilon} := E_{S_{\overline{M}} \oplus S_{\overline{M}}}(] - \epsilon, \epsilon[)$. This modification of $S_{\overline{X}}$ is indeed the same as the one introduced in Section 6.2, as the intervall $] - \epsilon, \epsilon[$ cut out of the spectrum is symmetric around 0. For simplicity we forget about the coefficient bundle W.

With respect to the identification (6.26) Sections $\xi^{\pm} \in \operatorname{Ext}(S_{\overline{X}_{\epsilon}}^{\pm})$ can be written on \overline{Z}_3 :

 $V\xi^{\pm}(\lambda, r) = \zeta^{\pm}(\lambda, i)(\chi_{]-\epsilon,\epsilon}(\lambda) + (1 - \chi_{]-\epsilon,\epsilon}(\lambda))e^{\pm\lambda r})$ with suitable $\zeta^{\pm}(\lambda, i) \in L^2((\pm [0, \infty[) \times \mathbb{N}, \mu_A))$. The coefficient of the component of ξ^{\pm} that is constant in r is $\zeta^{\pm}(\lambda, i)\chi_{]-\epsilon,\epsilon[}(\lambda)$ and

$$V^{-1}(\zeta^{\pm}(\lambda, i)(\chi_{]-\epsilon,\epsilon[}(\lambda)) \in \mathcal{H}_{S_{\overline{M}} \oplus S_{\overline{M}}}(]-\epsilon,\epsilon[)^{\pm},$$

Here, we use that $\tau_{\overline{X}}$ anticommutes with $S_{\overline{M}} \oplus S_{\overline{M}}$ and thus induces a grading on $\mathcal{H}_{S_{\overline{M}} \oplus S_{\overline{M}}}(] - \epsilon, \epsilon[).$

Using the decomposition $\Lambda T^*\overline{X} \cong \Lambda T^*\overline{M} \oplus \Lambda T^*\overline{M} \wedge dr$ into forms with or without a dr-component, we find

$$\mathcal{H}_{S_{\overline{M}} \oplus S_{\overline{M}}}(] - \epsilon, \epsilon[) = \mathcal{H}_{S_{\overline{M}}}(] - \epsilon, \epsilon[) \oplus \mathcal{H}_{S_{\overline{M}}}(] - \epsilon, \epsilon[),$$

and the operation of $\tau_{\overline{X}}$ on the RHS is just $\begin{pmatrix} 0 & \tau_{\overline{M}}(-1)^{n-1-|\cdot|} \\ \tau_{\overline{M}}(-1)^{|\cdot|} & 0 \end{pmatrix}$
Concatenation of these isomorphisms gives

$$\operatorname{Ext}(S_{\overline{X},\epsilon}^{\pm}) \xrightarrow{\mathcal{J}^{\pm}} \left[\mathcal{H}_{S_{\overline{M}}}(] - \epsilon, \epsilon[) \oplus \mathcal{H}_{S_{\overline{M}}}(] - \epsilon, \epsilon[) \right]^{\pm} \\ \xi^{\pm} \longmapsto V^{-1}(\xi^{\pm}(\lambda, i)\chi_{]-\epsilon,\epsilon[}(\lambda)).$$

Following (6.18), $\ker(S_{\overline{X},\epsilon})$ is closed in $e^{-\delta\theta}L^2$, for $0 < \delta < \epsilon$. Also, $\operatorname{Ext}(S_{\overline{X},\epsilon})$ is closed in $e^{\delta\theta}L^2$. It is than straightforward to verify that

Lemma 6.18. (a) The sequence $(\tau^{2} \tau (\alpha^{+}) \parallel \parallel) = (\tau^{2} \tau (\alpha^{+}) \parallel \parallel)$

$$(L^{2}-\ker(S^{\pm}_{\overline{X},\epsilon}), \|\cdot\|_{e^{-\delta\theta}L^{2}}) \longrightarrow (\operatorname{Ext}(S^{\pm}_{\overline{X},\epsilon}), \|\cdot\|_{e^{\delta\theta}L^{2}})$$
$$\xrightarrow{\mathcal{I}^{\pm}} \left[\mathcal{H}_{S_{\overline{M}}}(]-\epsilon, \epsilon[) \oplus \mathcal{H}_{S_{\overline{M}}}(]-\epsilon, \epsilon[)\right]^{\pm}$$

is continuous and exact in the middle. (b) $h_{\Gamma,\epsilon}^{\pm} = \dim_{\Gamma}(\operatorname{im}(\mathcal{J}^{\pm})).$

Let $\mathcal{J} := \mathcal{J}^+ + \mathcal{J}^-$. We now want to show that $\operatorname{im}(\mathcal{J}) = \mathcal{V} \oplus \mathcal{W}$ in $\mathcal{H}_{S_{\overline{M}}}(]-\epsilon,\epsilon[)\oplus \mathcal{H}_{S_{\overline{M}}}(]-\epsilon,\epsilon[)$. As $\tau_{\overline{X}}$ operates on $\operatorname{im}(\mathcal{J})$ and maps \mathcal{V} and \mathcal{W} to each other, the ± 1 -Eigenspaces im (\mathcal{J}^{\pm}) must be isomorphic. With a view of Lemma 6.10 and 6.18 this then implies the above Proposition.

To describe the structure of the image of \mathcal{J} , we consider the complement of L^2 -ker $(S_{\overline{X},\epsilon})$ in $\operatorname{Ext}(S_{\overline{X},\epsilon})$. Denoting by \perp the ortho-complement with regard to the nondegenerate pairing $e^{-\delta\theta}L^2 \times e^{\delta\theta}L^2 \to \mathbb{C}$, we see that

and

 $(L^2 \operatorname{-ker}(S_{\overline{X},\epsilon}), \|\cdot\|_{e^{-\delta\theta}})^{\perp} = (e^{-\delta\theta}L^2 \operatorname{-ker}(S_{\overline{X},\epsilon}))^{\perp} = \operatorname{cl}(e^{\delta\theta}L^2 \operatorname{-im}(S_{\overline{X},\epsilon})).$

Writing $\mathcal{K} := \operatorname{cl}(e^{\delta\theta}L^2 \operatorname{-im}(S_{\overline{X},\epsilon})) \cap \operatorname{Ext}(S_{\overline{X},\epsilon})$, it follows that $\operatorname{im}(\mathcal{J}) = \mathcal{J}(\mathcal{K})$. The following Lemma is the reason why we have to look at the modification $S_{\overline{X},\epsilon}$ instead of $S_{\overline{X}}$ also in this Section.

Lemma 6.19. Let $0 < \delta < \epsilon$.

- (a) e^{δθ}L²-im(S_{X,ϵ}) is Γ-dense in cl(e^{δθ}L²-im(S_{X,ϵ})), i.e. for every κ > 0 there is a closed subspace M ⊂ e^{δθ}L²-im(S_{X,ϵ}) such that dim_Γ(cl(e^{δθ}L²-im(S_{X,ϵ}))) dim_Γ(M) < κ.
 (b) K⁰ := e^{δθ}L²-im(S_{X,ϵ}) ∩ Ext(S_{X,ϵ}) is e^{δθ}L²-dense in K.
- $(0) \ \mathcal{H} = \mathcal$

Proof. (a) is a direct consequence of the Γ-Fredholm-property (Cf. (6.15) and Section 6.2) of the operator $T = S_{\overline{X},\epsilon}$ on $e^{\delta\theta}L^2(\overline{E})$. It implies that 0 is not in the Γ-essential part of the spectrum of TT^* (where T^* denotes the adjoint of T on $e^{\delta\theta}L^2$) and the spaces $\mathcal{M}_{\kappa} := \mathcal{H}_{TT^*}(\mathbb{R}-] - \kappa, \kappa[)$ are of finite Γ-codimension dim_Γ($\mathcal{H}_{TT^*}(]-\kappa,\kappa[-\{0\}))$ in L^2 -im(T). (b) is a simple consequence of (a), cf. [Sh2].

To show that $\mathcal{J}(\mathcal{K})$ can be decomposed into a direct sum as claimed, this Lemma and the continuity of the map \mathcal{J} allow to restrict our considerations to the dense subspace $\mathcal{J}(\mathcal{K}^0)$.

Thus, let $\xi \in \mathcal{K}^0$. According to the definition of \mathcal{K}^0 there exists $\alpha \in e^{\delta\theta} L^2(\Lambda T^*\overline{X})$ with $\xi = S_{\overline{X},\epsilon}\alpha$ and $S^2_{\overline{X},\epsilon}\alpha = 0$. On the cylinder we can then write $\alpha = \alpha_0 + \alpha_1 \wedge dr$ with $\alpha_0, \alpha_1 \in e^{\delta\theta} H^\infty(\overline{Z}, \Lambda T^*\overline{M})$. Using the spectral resolution $V : L^2(\Lambda T^*\overline{M}) \to L^2(\mathbb{R}_\lambda \times \mathbb{N}, \mu_{S_M})$ of

 $S_{\overline{M}}$, the α_l satisfy the following equation for r > 3

$$-(\frac{\partial}{\partial r})^2 V \alpha_l + (1 - \chi_{\epsilon}) \lambda^2 V \alpha_l = 0.$$

The solution is of the general form

$$\alpha_l(x,r) = r\beta_{l,1}(x)r + \beta_{l,2}(x) + O(e^{-\epsilon r}),$$

for suitable $\beta_{l,1}, \beta_{l,2} \in \mathcal{H}_{S_{\overline{M}}}(] - \epsilon, \epsilon[)$. We can now write

 $S_{\overline{X},\epsilon} = d_{\overline{X},\epsilon} + d_{\overline{X},\epsilon}^* \text{ with } d_{\overline{X},\epsilon} = d_{\overline{X}} - d_{\overline{M}} \vartheta \widehat{\Pi}_{\epsilon} \text{ iund } d_{\overline{X},\epsilon}^* = d_{\overline{X}}^* - d_{\overline{M}}^* \vartheta \widehat{\Pi}_{\epsilon}.$ Then, in the area r > 3

$$d_{\overline{X},\epsilon}\alpha_0(x,r) = (\epsilon(dr)\frac{\partial}{\partial r} + d_{\overline{M}}(1-\widehat{\Pi}_{\epsilon}))(r\beta_{0,1}(x) + \beta_{0,2}(x) + O(e^{-\epsilon r}))$$

= $dr \wedge \beta_{0,1}(x) + O(e^{-\epsilon r}),$

because $(1 - \widehat{\Pi}_{\epsilon})\beta_{0,j} = 0$. In the same manner one shows $d_{\overline{X},\epsilon}\alpha_1 \wedge dr(x,r) = O(e^{-\epsilon r})$. The analogous calculation for $d_{\overline{X},\epsilon}^* \alpha$ gives

$$\begin{aligned} d^*_{\overline{X},\epsilon} \alpha_1(x,r) \wedge dr \\ &= (-\iota(\frac{\partial}{\partial r})\frac{\partial}{\partial r} + d^*_{\overline{M}}(1-\widehat{\Pi}_{\epsilon}))(r\beta_{1,1}(x) \wedge dr + \beta_{1,2}(x) + O(e^{-\epsilon r})) \\ &= -(-1)^{|\beta_{1,1}|}\beta_{1,1}(x) + O(e^{-\epsilon r}), \end{aligned}$$

and $d_{\overline{X},\epsilon}\alpha_0(x,r) = O(e^{-\epsilon r})$. We have shown

$$\mathcal{J}(\xi) = \mathcal{J}(d_{\overline{X},\epsilon}\alpha + d_{\overline{X},\epsilon}^*\alpha) = 0 \oplus (-1)^{|\beta_{0,1}|}\beta_{0,1} - (-1)^{|\beta_{1,1}|}\beta_{1,1} \oplus 0,$$

that is, the image of \mathcal{J} can be decomposed into a direct sum as claimed. This finishes the proof of Proposition 6.15.

Appendix A. Clifford Algebra conventions

Denote by $\mathbb{C}(k)$ the (complex) Clifford algebra over the euclidean space \mathbb{R}^k , with generators $\mathbf{c}_1, \ldots, \mathbf{c}_k$ satisfying $\mathbf{c}_i \mathbf{c}_j + \mathbf{c}_j \mathbf{c}_i = -2\delta_{ij}$. The algebra $\mathbb{C}(k)$ is \mathbb{Z}_2 -graded: $\mathbb{C}(k) = \mathbb{C}^+(k) \oplus \mathbb{C}^-(k)$, and the map $\mathbf{c}_i \longmapsto \mathbf{c}_i \mathbf{c}_{k+1}$ defines an isomorphism $\mathbb{C}(k) \xrightarrow{\sim} \mathbb{C}^+(k+1)$. The volume element $\tau_k := i^{[(k+1)/2]} \mathbf{c}_1 \ldots \mathbf{c}_k \in \mathbb{C}(k)$ satisfies $\tau_k^2 = 1$ and thus induces a \mathbb{Z}_2 -grading on the representations of $\mathbb{C}(k)$. Note however, that $\tau_k \mathbf{c} = -(-1)^k \mathbf{c} \tau_k$ for $\mathbf{c} \in \mathbb{R}^k \subset \mathbb{C}(k)$ implies that this grading is trivial on irreducible representations $\mathbb{C}(k)$, when k is odd.

 $\mathbb{C}(2l)$ has a unique irreducible representation, called its spinor space and denoted by S(2l). Its dimension is dim $S(2l) = 2^l$. Decomposing into the ± 1 -Eigenspaces of τ_{2l} we write $S(2l) = S^+(2l) \oplus S^-(2l)$. Via the identification $\mathbb{C}(2l-1) \cong \mathbb{C}^+(2l)$ the spaces $S^+(2l)$, $S^-(2l)$ are non-equivalent irreducible representations of $\mathbb{C}(2l-1)$, which can be considered as being isomorphic representations of $\mathbb{C}(2l-2) \cong \mathbb{C}^+(2l-1)$ 1) via the map $S^+(2l) \xrightarrow{c_{2l}} S^-(2l)$. This of course is then just the representation S(2l-2) of $\mathbb{C}(2l-2)$. For $S^{\pm}(2l)$ we also write $S^{\pm}(2l-1)$ when these spaces are seen as representations of $\mathbb{C}(2l-1)$. It is easy to see that $\mathbb{C}(2l)$ acts injectively on S(2l). Comparison

of dimensions then yields $\mathbb{C}(2l) \cong End(S(2l))$, and, using $\mathbb{C}(2l - 1) \cong \mathbb{C}^+(2l)$ also $\mathbb{C}(2l - 1) \cong End^+(S(2l))$. The identification $\mathbb{C}(2l - 1) \to End(S^{\pm}(2l - 1))$ maps τ_{2l-1} to ± 1 and thus has null space $(1 \mp \tau_{2l-1})\mathbb{C}(2l - 1)$.

The traces tr^{\pm} on $End(S^{\pm}(2l-1))$ and the graded trace str on End(S(2l))then induce traces on $\mathbb{C}(2l-1)$ and $\mathbb{C}(2l)$. On elements of the form

 $\mathbf{c}_{I} := \mathbf{c}_{i1} \dots \mathbf{c}_{i|I|}$ where $I = \{i_{1} \leq \dots \leq i_{|I|}\} \subset \{1, \dots, k\}$ these are calculated as follows

Lemma A.1. (a) In
$$\mathbb{C}(2l)$$
 we have $\operatorname{str}(\tau_{2l}) = 2^l$ and $\operatorname{str}(1) = \operatorname{str}(\mathbf{c}_I) = 0$ for $I \neq \{1, \ldots, k\}$.

(b) In $\mathbb{C}(2l-1)$ we have $str^+(\tau_{2l-1}) = -tr^-(\tau_{2l-1}) = tr^{\pm}(1) = 2^{l-1}$

and for $I \neq \{1, \ldots, k\}$ we have $tr^{\pm}(\mathbf{c}_1) = 0$. On $(\mathbb{C}(2l-1) - \mathbb{C}) \subset \mathbb{C}(2l)$ therefore $tr^{\pm}(\bullet) = \mp \frac{1}{2} \operatorname{str}(\mathbf{c}_{2l}\bullet)$ and on $(\mathbb{C}(2l) \subset \mathbb{C}(2l+1) \text{ we have } \operatorname{str}(\bullet) = \pm itr^{\pm}(\mathbf{c}_{2l+1}\bullet)$

Proof. Cf. [BGV], Proposition 3.21

The map $S^+(2l) \xrightarrow{c_{2l}} S^-(2l)$ gives an identification $S(2l) \cong S^{\pm}(2l-1) \oplus$ $S^{\pm}(2l-1)$. In this representation, $\mathbb{C}(2l)$ acts on S(2l) as follows

$$\mathbf{c}_{i} \in \mathbb{C}(2l-1) \stackrel{\wedge}{=} \begin{pmatrix} 0 & \pm \mathbf{c}_{i} \\ \pm \mathbf{c}_{i} & 0 \end{pmatrix} \mathbf{c}_{2l} \stackrel{\wedge}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $str\begin{pmatrix} \phi_{1} & \phi_{2} \\ \phi_{3} & \phi_{4} \end{pmatrix} = tr^{\pm}(\phi_{1}) - tr^{\pm}(\phi_{4})$

References

- [An] N. Anghel: "On the Index of Callias-Type Operators", Geom. Funct. Anal. 3 (1993), 431-438
- M.F. Atiyah: "Elliptic Operators, Discrete Groups and Von Neumann [A] *Algebras* ", Asterisque **32** – **33** (1976), 43-72
- [APS] M. F. Atiyah, V.K. Patodi, I.M. Singer: "Spectral Asymmetry in Riemannian Geometry I", Math. Proc. Camb. Phil. Soc. 77 (1975), 43-69
- N. Berline, E. Getzler, M. Vergne: "Heat Kernels and Dirac Operators", [BGV] Grundl. Math. Wiss. 297, Springer (1992)
- M. Breuer: "Fredholm Theories in von Neumann Algebras I,II", Math. [Br] Ann. 178 (1968), 243-254, and Math. Ann. 180 (1969), 313-325
- U. Bunke: "Spektraltheorie von Dirac operatoren auf offenen Mannig-[Bu1] faltigkeiten", Doktorarbeit, Uni Greifswald (1991)
- U. Bunke: "Relative Index Theory", Jour. Funct. Anal. 105 (1992), 63-[Bu2] 76
- [Bu3] U. Bunke: "Comparison of Dirac operators on manifolds with Boundary", Rend. Circ. Mat. Palermo (Suppl.) **30** (1993, 191-203)
- [Bu4] U. Bunke: "A K-Theoretic Relative Index Theorem and Callias-Type Dirac Operators", Math. Ann. 303 (1995), 241-279
- [ChG] J. Cheeger, M. Gromov: "On the Characteristic Numbers of Complete Manifolds of Bounded Curvature and Finite Volume", Rauch Memorial Volume: Differential Geometry and Complex Analysis, eds. I. Chavel, H.M. Fakras, Springer (1985)
- A. Cones: "Noncommutative Geometry", Academic Press (1994) [C1]
- A. Connes, H. Moscovici: "Cyclic Cohomology, the Novikov Conjecture [CM]and Hyperbolic Groups", Topology 29 (1990), 345-388

- [DoLi] H. Donnelly, P. Li: "Pure Point Spectrum and Negative Curvature for Noncompact Manifolds", Duke Math. Jour. 46 (1979), 497-503
- [Ei] J. Eichhorn: "The Boundedness of Connection Coefficients and Their Derivatives", Math. Nachr. 152 (1991), 145-158
- [Li] Li Bing-Ren: "Introduction to Operator Algebras", Word Scientific (1992)
- [Lo1] J. Lott: "Superconnections and Higher Index Theory", Geom. Funct. Anal. 2 (1992), 421-453
- [Lo2] J. Lott: "Higher Eta Invariants", K-Theory 6 (1992), 191-233
- [Lo3] J. Lott: "Heat Kernels on Covering Spaces", Jour. Diff. Geom. 35 (1992), 471-510
- [Lue] W. Lück: "Approximating L²-Invariants by Their Finite-Dimensional Analogues", Geom. Funct. Anal. 4 (1994), 455-481
- [Me] R.B. Melrose: "The Atiyah-Patodi-Singer Index Theorem", Research Notes in Mathematics vol. 4, A.K. Peters
- [Mue1] W. Müller: Eta Invariants and Manifolds with Boundary", Jour. Diff. Geom. 40 (1994), 311-377
- [Mue2] W. Müller: "On the Index of Dirac Operators on Manifolds with Corners of Codimension Two I.", MPI-Preprint **95** – **23** (1995)
- [Ra] M. Ramachandran: "von Neumann Index Theorems for Manifolds with Boundary", Jour. Diff. Geom. 38 (1993), 315-349
- [Ro1] J. Roe: "Elliptic operators, topology and asymptotic mehtods", Pitman Research Notes in Mathematics 179 (1988)
- [Ro2] J. Roe: "An Index Theorem on Open Manifolds. I", Jour. Diff. Geom. 27 (1988), 87-113
- [Sch] T. Schick: "Analysis on ∂-Manifolds of Bounded Geometry and Hodge-De Rham Isomorphism", Preprint Uni Mainz (1995)
- [Sh1] M.A. Shubin: "Spectral Theory of Elliptic Operators on Non-Compact Manifolds", Asterisque 207 (1992), 37-108
- [Sh2] M.A. Shubin: L^2 Riemann-Roch Theorem for Elliptic Operators ", Geom. Func. Anal 5 (1995), 482-527
- [Va] B. Vaillant: "Indextheorie für Überlagerungen", Diploma thesis (1996), University of Bonn, unpublished
- [Wu] F. Wu: "The Chern-Connes Character for the Dirac Operator on Manifolds with Boundary", K-Theory 7 (1993), 145-174

BVAILLANT@QUANTITATIVE-CONSULTING.EU