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h-COBORDISMS AND MAPPING CYLINDER OBSTRUCTIONS

BOGDAN VAJIAC

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Abstract

In this paper we prove a realizability theorem for Quinn's mapping cylinder obstructions for stratified spaces. We prove a continuously controlled version of the s-cobordism theorem which we further use to prove the relation between the torsion of an h-cobordism and the mapping cylinder obstructions. This states that the image of the torsion of an h-cobordism is the mapping cylinder obstruction of the lower stratum of one end of the h-cobordism in the top filtration. These results are further used to prove a theorem about the realizability of end obstructions.

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1. Introduction

In [19, p. 405], and [21, p. 447], Quinn introduces the notion of the mapping cylinder neighborhood as a homotopical substitute for the normal bundle. He uses controlled algebra to prove that, for a locally compact absolute neighborhood retract pair (X, A), such that A is closed and tame in X and $X - A$ is a manifold, there is a single obstruction for A having a mapping cylinder neighborhood in X; it is denoted $q_0(X, A) \in \tilde{K}_0^{\mathrm{lf}}(A, p_X)$. We will review the terminology in more detail below.

Connolly and Vajiac [8] proved an end theorem for stratified spaces. The main result of [8] states that there is a single K-theoretical obstruction to completing a tame-ended stratified space. We will state the main result below. For more details on the terminology and notation we refer the reader to [8].

THEOREM 1.1 (Main theorem of [8]). *Let X be a tame-ended stratified space. Define $\gamma_*(X)$, the obstruction to completing, as a direct sum of obstructions (one for each stratum) by localizing Quinn's mapping cylinder obstruction near infinity:*

$$\gamma_*(X) = \bigoplus_m \gamma_m(X) \in \bigoplus_m K_0^{\mathrm{lf}}((\hat{X}^{m-1}, p_{\hat{X}^m})_{(\infty)}).$$

Assume that $\gamma_*(X) = 0$. Let A be any pure subset of X, containing X^5, such that A is the interior of a compact stratified space $\bar{A}$. Then X is the interior of a compact stratified space $\bar{X}$ such that $Cl_{\bar{X}}(A) = \bar{A}$.

The main result of this paper is Theorem 6.1 below. It proves a formula that relates the torsion of an h-cobordism to the mapping cylinder obstructions.

THEOREM (Theorem 6.1). *Let (X, B) be relative manifold (see Definition 3.1). Let W be an h-cobordism of relative manifolds, from (X, B) to (Y, B). Let $\tau \in \mathrm{Wh}(X, B)$ be the element represented by W. Then*

$$\Delta(\tau) = r_* q_0(W, B) - q_0(X, B). \tag{$*$}$$

The map $r_* : \tilde{K}_0(B, p_W) \to K_0(B, p_X)$ is the isomorphism induced by the retraction r from W to X.

We would like to point out that this formula is also given in the work of Quinn [21]. We hope that our approach, which is different than the one just mentioned, makes this formula more transparent, and thus can be readily used in applications. An immediate corollary of this result is a realizability theorem for mapping cylinder obstructions (see the corollary below and also Section 6). It realizes every element of $\tilde{K}_0(B, p)$, which is in the kernel of the assembly map.

COROLLARY (Corollary 6.7). *Let X be a compact stratified space, so that its singular stratum, B, has a mapping cylinder neighborhood in X. Let $x \in \mathrm{Ker}(\mathfrak{a})$, where $\mathfrak{a}$ is the forget-control assembly map*

$$\tilde{K}_0(B, p_X) \xrightarrow{\mathfrak{a}} \tilde{K}_0(\mathbb{Z}\pi_1(X - B)).$$

Then there exists a stratified space Y, so that the singular stratum of Y is homeomorphic to B and the mapping cylinder obstruction of B in Y is given by $q_0(Y, B) = i(x)$. The map $i : \tilde{K}_0(B, p_X) \to \tilde{K}_0(B, p_Y)$ is an inclusion-induced isomorphism.

Another important result that follows from the proof of Theorem 6.1 is the fact that the mapping cylinder obstruction is a simple homotopy invariant, in the following sense.

COROLLARY 1.2. *Let X and Y be n-dimensional (compact) stratified spaces, $n \geqslant 6$, and let $f : X \to Y$ be a stratified simple homotopy equivalence. Then*

$$q_0(Y, \sigma Y) = f_*(q_0(X, \sigma X)),$$

where $f_* : \tilde{K}_0(\sigma X, p_X) \to \tilde{K}_0(\sigma Y, p_Y)$ is the isomorphism induced by f.

It also proves the simple homotopy invariance of the obstruction to completing a stratified space, $\gamma_*(X)$ (see [8] for details on the obstruction $\gamma_*(X)$).

COROLLARY 1.3. *Let X and Y be n-dimensional stratified spaces, $n \geqslant 6$, and let $f : X \to Y$ be a stratified proper homotopy equivalence. Assume that the torsion of f is compact, that is to say $\tau(f) \in \mathrm{Im}(\iota)$, where $\iota : \mathrm{Wh}(X) \to \mathrm{Wh}^{\mathrm{lf}}(X)$ is the map induced by inclusion. Then*

$$\gamma_*(Y) = f_*(\gamma_*(X)),$$

where $f_* : \bigoplus_m K_0^{\mathrm{lf}}((\hat{X}^{m-1}, p_{\hat{X}^m})_{(\infty)}) \to \bigoplus_m K_0^{\mathrm{lf}}((\hat{Y}^{m-1}, p_{\hat{Y}^m})_{(\infty)})$ is the isomorphism induced by f.

The structure of this paper is as follows. In Section 2 we give the basic definitions of the theory of geometric modules. We recall the basic properties and provide a treatment of the continuously controlled K-theory, which will be used later. In Section 3 we prove a continuously controlled s-cobordism theorem.

THEOREM 1.4 (Theorem 3.2). *Let (W, A) be an h-cobordism of relative manifolds from (X, A) to (Y, A), of $\dim(W - A) \geqslant 5$. Then the triple (W, X, A) specifies an element $\tau(W, X, A) \in \mathrm{Wh}(X, A)$ such that the map $\psi : \mathrm{hCob}(X; A) \to \mathrm{Wh}(X, A)$ given by $\psi((W, X, A)) = \tau(W, X, A)$ is a bijection.*

Note that there are different 'controlled' versions of the s-cobordism theorem, depending on the type of control used. See, for example, the work of Pedersen [15] for a bounded version.

In Section 4 we recall the definitions of the controlled obstructions and the controlled $\tilde{K}_0$-groups of Quinn. Our definitions are very explicit, and are based on the algebraic approach of Ranicki and Yamasaki [24].

Section 5 describes the construction of the boundary map Δ in the exact sequence of [1]:

$$\cdots \longrightarrow \mathrm{Wh}(X, B) \xrightarrow{\Delta} \tilde{K}_0(B, p) \xrightarrow{\alpha} \tilde{K}_0(\mathbb{Z}\pi_1(X - B)) \longrightarrow \cdots.$$

Here X is a topological space and B is a metric space which is closed and forward tame in X. We do not use the full sequence; we only need exactness at $\tilde{K}_0(B, p)$. We give an argument for this fact in Proposition 5.8. The analogy between the boundary map from [1] and our description is not pursued here.

2. On geometric modules and K-theory

The goal of this section is to introduce the reader to the theory of geometric modules. Geometric modules were first introduced by Connell and Hollingsworth in 1969 [7] and extensively developed and used by Quinn in his work on the ends of maps [18–20]. Since the literature is not standard, we present in this paper a self-contained account.

We define the homotopy category of finite geometric modules on a topological space X as a certain additive category, $\mathrm{Ad}\,\mathbb{Z}[\Pi(X)]$, where $\Pi(X)$ denotes the fundamental groupoid of the space X.

2.1. The categorical constructions $\mathbb{Z}[\mathbb{C}]$, Ad $\mathbb{C}$ In any category $\mathbb{C}$, we denote by $|\mathbb{C}|$ the objects of $\mathbb{C}$ and by $\mathbb{C}(y, x)$ the set of morphisms *from x to y*. An *Ab-category* is a category $\mathbb{C}$ in which, for any two objects, $\mathbb{C}(y, x)$ forms an abelian group, and composition is a $\mathbb{Z}$ bilinear map. If in addition $\mathbb{C}$ has a zero object, and if the direct sum and product of any two objects of $\mathbb{C}$ exist and coincide in $\mathbb{C}$, then $\mathbb{C}$ is called an *additive category*.

Any category $\mathbb{C}$ embeds in an Ab-category, $\mathbb{Z}[\mathbb{C}]$, with the same objects as $\mathbb{C}$, so that the functor $\mathbb{C} \to \mathbb{Z}[\mathbb{C}]$ is universal with respect to functors from $\mathbb{C}$ to Ab-categories. The morphism group, $\mathbb{Z}[\mathbb{C}](y, x)$, for any two objects x, y of $\mathbb{C}$ is defined to be $\mathbb{Z}[\mathbb{C}](y, x) = \mathbb{Z}[\mathbb{C}(y, x)]$, the free abelian group on $\mathbb{C}(y, x)$. The composition on $\mathbb{C}$ obviously extends to a bilinear composition on $\mathbb{Z}[\mathbb{C}]$.

Any Ab-category $\mathbb{C}$ embeds in an additive category, denoted Ad $\mathbb{C}$, the *additive closure* of $\mathbb{C}$. An object of Ad $\mathbb{C}$ is a pair (S, j), where S is a finite set and $j : S \to |\mathbb{C}|$ is a function. A morphism f in Ad $\mathbb{C}$, from (T, k) to (S, j), is an $S \times T$ matrix $(f_s^t)_{(s,t)\in S\times T}$ such that $f_s^t \in \mathbb{C}(j(s), k(t))$ for every $(s, t) \in S \times T$. Composition is defined via matrix multiplication. Any one-point set provides a categorical embedding, $\mathbb{C} \to$ Ad $\mathbb{C}$. The empty set gives a zero object for Ad $\mathbb{C}$. The disjoint union of two finite sets yields a product of any two objects of Ad $\mathbb{C}$.

For more details on these constructions, see [12, p. 194].

2.2. The category $\mathcal{G}(X)$ of geometric modules on a space X Let X be a topological space. Denote the set of Moore paths from a point x_0 to a point x_1 by $P(X)(x_0, x_1)$. The points of X can be considered as the objects of a category, $P(X)$, in which $P(X)(x_0, x_1)$ is the set of morphisms *from x_1 to x_0*. Multiplication of paths provides the composition law.

Let $\Pi(X)$ denote the fundamental groupoid of X. More precisely, objects of $\Pi(X)$ are points of X; the morphisms from x_1 to x_0 are the homotopy classes of paths from x_0 to x_1. There is a quotient functor $j : P(X) \to \Pi(X)$ sending each object to itself.

Let $H(X)$ denote the category whose objects are the points of X, and in which a morphism from x_1 to x_0 is a homotopy with endpoints fixed between two Moore paths from x_0 to x_1. The category $H(X)$ comes equipped with two functors, $\partial_0, \partial_1 : H(X) \to P(X)$, sending a homotopy to its initial and terminal paths, respectively. Note that $j : P(X) \to \Pi(X)$ is the coequalizer of the two functors $\partial_0, \partial_1 : H(X) \to P(X)$ (according to [12, p. 64]).

DEFINITION 2.1. The *category of finite geometric modules on X* is defined as

$$GM(X) = \mathrm{Ad}\, \mathbb{Z}[P(X)].$$

The *homotopy category of finite geometric modules on X* is defined as

$$\mathcal{G}(X) = \mathrm{Ad}\, \mathbb{Z}[\Pi(X)].$$

The *category of geometric homotopy relations on X* is defined as

$$\mathrm{GH}(X) = \mathrm{Ad}\, \mathbb{Z}[H(X)].$$

These three categories all have the same objects $M = (S, j)$; we call them *(finite) geometric modules on* X. For any two geometric modules M, N on X, we obtain an exact sequence of morphism groups:

$$\mathrm{GH}(X)(N, M) \xrightarrow{\partial_1 - \partial_0} GM(X)(N, M) \xrightarrow{j} \mathcal{G}(X)(N, M) \to 0.$$

So $GM(X) \xrightarrow{j} \mathcal{G}(X)$ is the coequalizer of the two functors:

$$\mathrm{GH}(X) \xrightarrow{\partial_1, \partial_0} GM(X).$$

PROPOSITION 2.2. *For any path-connected pointed space (X, x_0), $\mathcal{G}(X)$ is equivalent to the category $\mathcal{F}_{\mathbb{Z}[\pi_1(X,x_0)]}$ of finitely generated free modules over $\mathbb{Z}[\pi_1(X, x_0)]$. To see this, let $\mathbb{C}$ be any small category in which each morphism is an isomorphism, and any two objects are isomorphic. Then $\mathbb{C}$ is equivalent to the full subcategory, say π, spanned by one object x_0 of $\mathbb{C}$. This equivalence induces an equivalence* $\mathrm{Ad}\,\mathbb{Z}[\mathbb{C}] \approx \mathrm{Ad}\,\mathbb{Z}[\pi]$. *When $\mathbb{C} = \Pi(X)$, X path-connected, we get $\mathbb{Z}[\pi](x_0, x_0) = \mathbb{Z}[\pi_1(X, x_0)]$, so* $\mathrm{Ad}(\mathbb{Z}[\pi]) = \mathcal{F}_{\mathbb{Z}[\pi_1(X,x_0)]}$. *Therefore, $\mathcal{GM}(X) \approx \mathcal{F}_{\mathbb{Z}[\pi_1(X,x_0)]}$.*

(Quinn [22] proves this when X is semilocally 1-connected.)

2.3. Continuously controlled K-theory We define the K-theory of a pair (X, B) using the continuously controlled theory of [2].

A *category over a topological space* X, is a category $\mathbb{C}$, together with a function, supp, assigning to each morphism, σ, in $\mathbb{C}$ a compact set, $\mathrm{supp}(\sigma) \subset X$, such that $\mathrm{supp}(\sigma \circ \tau) \subset \mathrm{supp}(\sigma) \cup \mathrm{supp}(\tau)$.

If also $\mathbb{C}$ is an Ab-category, and $\mathrm{supp}(\sigma + \tau) \subset \mathrm{supp}(\sigma) \cup \mathrm{supp}(\tau)$ and $\mathrm{supp}(\sigma) = \emptyset$ if and only if $\sigma = 0$, we call $\mathbb{C}$ an Ab-category over X. The support of an object $x \in |\mathbb{C}|$ is defined as $\mathrm{supp}(x) = \mathrm{supp}(1_x)$.

If $\mathbb{C}$ is a category over X, and $\mathrm{supp}(\sigma) \neq \emptyset$ then $\mathbb{Z}[\mathbb{C}]$ is an Ab-category over X in a natural way: the support of any morphism in $\mathbb{Z}[\mathbb{C}]$, say $\sigma = \sum_{i=1}^{k} n_i \sigma_i$, where each σ_i is in $\mathbb{C}$, is defined to be the union of the supports of those σ_i for which $n_i \neq 0$.

DEFINITION 2.3. Let B be a subspace of X. A collection, $\{S_\lambda \mid \lambda \in \Lambda\}$, of compact subsets of $X - B$ is *continuously controlled* (cc) over (X, B), if:

(1) for each X-neighborhood U of B, $\{\lambda \in \Lambda \mid S_\lambda \not\subset U\}$ is a finite subset of Λ;
(2) for each point $p \in B$ and each X-neighborhood $U(p)$, there is an X-neighborhood $V(p)$ so that any set S_λ meeting V must lie in U.

EXAMPLES 2.4.

(i) A cc-collection over $(X, \emptyset)$ is a finite collection.

(ii) Let $\hat{X}$ be the one-point compactification of an arbitrary topological space X. A collection $\mathcal{S} = \{S_\lambda \mid \lambda \in \Lambda\}$ of compact subsets of X is a cc-collection over $(\hat{X}, \infty)$ if and only if $\mathcal{S}$ is a locally finite collection on X (that is, each compact set of X meets only finitely many members of $\mathcal{S}$).

(iii) Let B be a closed subset of a space X. Then $\hat{B} \subset \hat{X}$ and $X - B = \hat{X} - \hat{B}$. Let $\mathcal{S} = \{S_\lambda \mid \lambda \in \Lambda\}$ be a collection of compact subsets of $X - B$. For each compact set $K \subset X$, write Λ^K for $\{\lambda \in \Lambda \mid S_\lambda \cap K \neq \emptyset\}$, and $S^K = \{S_\lambda \mid \lambda \in \Lambda^K\}$. Then we see that $\mathcal{S}$ is a cc-collection over $(\hat{X}, \hat{B})$ if and only if for each compact set $K \subset X$, $\mathcal{S}^K$ is a cc-collection over $(X, B \cap K)$.

We will now define the category of locally finite geometric modules, and the geometric modules of a pair (X, A). The constructions are similar to those for finite geometric modules.

Let B be a subset of the space X, and $\mathbb{C}$ an Ab-category over $X - B$. We define an additive category, $\mathrm{Ad}_{(X,B)}\mathbb{C}$, containing $\mathrm{Ad}\,\mathbb{C}$, called the *additive closure of* $\mathbb{C}$ *over* (X, B). An object of $\mathrm{Ad}_{(X,B)}\mathbb{C}$ is a pair (S, j) where S is a set, and $j : S \to |\mathbb{C}|$ is a function for which the indexed collection $\{\mathrm{supp}(j(s)) \mid s \in S\}$ is a cc-collection over (X, B). A morphism $f = (f_t^s)_{(s,t)\in S\times T}$ is defined as for $\mathrm{Ad}\,\mathbb{C}$ except that we require $\{\mathrm{supp}(f_t^s) \mid (s, t) \in S \times T\}$ to be a cc-collection over (X, B). In particular, then, the matrix is row and column finite.

If $\mathbb{C}$ is an Ab-category over X, the *locally finite additive closure*, $\mathrm{Ad}^{\mathrm{lf}}\,\mathbb{C}$, is defined as $\mathrm{Ad}_{(\hat{X},\infty)}\,\mathbb{C}$. The objects are pairs (S, j), where S is a set, and $j : S \to |\mathbb{C}|$ is a function such that $\{\mathrm{supp}(j(s)) \mid s \in S\}$ is a locally finite collection.

Note that $\mathrm{Ad}\,\mathbb{C}$, $\mathrm{Ad}^{\mathrm{lf}}\,\mathbb{C}$, and $\mathrm{Ad}_{(X,B)}\,\mathbb{C}$ are functorial in $\mathbb{C}$ and also that $\mathrm{Ad}_{(X,\emptyset)}\,\mathbb{C} = \mathrm{Ad}\,\mathbb{C}$.

DEFINITION 2.5. The *category of locally finite geometric modules on* X is

$$GM^{\mathrm{lf}}(X) = \mathrm{Ad}^{\mathrm{lf}}\,\mathbb{Z}[P(X)].$$

An object $M = (S, j)$ of $GM(X)$ is called a locally finite geometric module on X. We set $\mathrm{GH}^{\mathrm{lf}}(X) = \mathrm{Ad}^{\mathrm{lf}}\,\mathbb{Z}[H(X)]$. We again get two functors $\mathrm{GH}^{\mathrm{lf}}(X) \xrightarrow{\partial_1,\partial_0} GM^{\mathrm{lf}}(X)$.

More generally, let B be a subset of X. The *category of cc-geometric modules and morphisms over* (X, B) and the *category of cc-geometric homotopy relations on* (X, B) are defined as

$$GM(X, B) = \mathrm{Ad}_{(X,B)}\,\mathbb{Z}[P(X - B)], \quad \mathrm{GH}(X, B) = \mathrm{Ad}_{(X,B)}\,\mathbb{Z}[H(X - B)].$$

An object $M = (S, j)$ of $GM(X, B)$ is called a geometric module on (X, B). We define the *homotopy category of cc-geometric modules and morphisms* denoted $\mathcal{G}(X, B)$, and a functor $j : GM(X, B) \to \mathcal{G}(X, B)$ to be the equalizer of the two functors: $\mathrm{GH}(X, B) \xrightarrow{\partial_1-\partial_0} GM(X, B)$.

The categories $GM(X, B)$, $\mathcal{G}(X, B)$, $\mathrm{GH}(X, B)$ have the same objects, and for any cc-geometric modules M, N, we get an exact sequence:

$$\mathrm{GH}(X, B)(N, M) \xrightarrow{\partial_1-\partial_0} GM(X, B)(N, M) \xrightarrow{j} \mathcal{G}(X, B)(N, M) \to 0.$$

The *homotopy category of locally finite geometric modules on* X, is defined as

$$\mathcal{G}^{\mathrm{lf}}(X) = \mathcal{G}(\hat{X}, \infty).$$

Notice that $\mathcal{G}(X, \emptyset) = \mathcal{G}(X)$.

For two morphisms in $GM(X, B)$, say $f_0, f_1 : N \to M$, we write $f_0 \sim f_1$ if $f_1 - f_0 = (\partial_1 - \partial_0)g$, for some $g : N \to M$ in $\mathrm{GH}(X, B)$.

DEFINITION 2.6. Let B be a subset of the topological space X. We define the groups $K_i(X, B) = K_i(\mathcal{G}(X, B))$, $i \leqslant 1$, in the sense of Quillen (see [17, 26]).

If B is a closed set of X, we define the locally finite K-theory by

$$K_i^{\mathrm{lf}}(X, B) = K_i(\mathcal{G}(\hat{X}, \hat{B})), \qquad K_i^{\mathrm{lf}}(X) = K_i(\mathcal{G}(\hat{X}, \infty)) = K_i(\mathcal{G}^{\mathrm{lf}}(X)).$$

These additive categories are given the 'semisimple' exact structure: sequences $0 \to M_1 \xrightarrow{i_1} M_1 \oplus M_2 \xrightarrow{p_2} M_2 \to 0$ given by the natural direct sum and product, and only sequences isomorphic to these, are decreed to be exact.

The reduced K_0-groups for these categories are defined as usual: for any additive category, $\tilde{K}_0(\mathbb{C})$ is defined as the cokernel of the natural map, $K_0(\mathbb{C}) \to K_0(\hat{\mathbb{C}})$, where $\hat{\mathbb{C}}$ is the idempotent completion of $\mathbb{C}$ (see [23] for more details).

We now define the *Whitehead group*, $\mathrm{Wh}(X, B)$. A cc-morphism α of geometric modules over (X, B) is a *basis change* if its matrix $(\alpha_t^s)_{(s,t)\in(S\times T)}$ has a single nonzero entry of the form $\pm\sigma_t^s$ in each row and column (σ_t^s is a path from $k(t)$ to $j(s)$). Such a matrix is obviously invertible. The adjoint of α is then α^*, where $(\alpha^*)_s^t = (\alpha_t^s)^{-1}$. Note that α^{-1} is obviously a cc-morphism, and $j(\alpha^*) = j(\alpha)^{-1}$. Let H be the subgroup of $K_1(X, B)$ generated by basis change matrices. Then we define

$$\mathrm{Wh}(X, B) = K_1(X, B)/H, \qquad \mathrm{Wh}(X) = \mathrm{Wh}(X, \emptyset) = K_1(\mathcal{GM}(X))/H.$$

When X is path-connected, it is easy to see that $\mathrm{Wh}(X) \approx \mathrm{Wh}(\pi_1(X, x_0))$, because $K_1(\mathcal{G}(X)) \approx K_1(\mathbb{Z}[\pi_1(X, x_0)])$. If B is a closed subset of X we write $\mathrm{Wh}^{\mathrm{lf}}(X, B)$ instead of $\mathrm{Wh}(\hat{X}, \hat{B})$, and $\mathrm{Wh}^{\mathrm{lf}}(X)$ instead of $\mathrm{Wh}(\hat{X}, \infty)$. If X is a locally compact polyhedron, then one can prove $\mathrm{Wh}^{\mathrm{lf}}(X) = \mathrm{Wh}_\infty(X)$, Siebenmann's locally finite Whitehead group [25].

The *Whitehead torsion* of an isomorphism in $\mathcal{G}(X, B)$ is defined as follows. Note that for any two isomorphic objects M, N in $\mathcal{G}(X, B)$, there is a basis change isomorphism, $b : N \to M$. For any isomorphism $f : M \to N$, the Whitehead torsion of f is defined as $\tau(f) = [b \circ f] \in K_1(X, B)/H = \mathrm{Wh}(X, B)$; it does not depend on the choice of b. It is immediate that $\tau(f \circ g) = \tau(f) + \tau(g)$. By mimicking [24] one can also define the group $\mathrm{Wh}(X, B, n)$, where n is a positive integer, as the equivalence class of contractible continuously controlled chain complexes over $\mathcal{G}(X, B)$. The equivalence relation is generated by stable Σ-equivalences, in the sense of [24]. As explained in [24], there is an isomorphism between $\mathrm{Wh}(X, B, n)$ and $\mathrm{Wh}(X, B)$, so we will regard the class of a contractible continuously controlled chain complexes over $\mathcal{G}(X, B)$ as an element of $\mathrm{Wh}(X, B)$.

2.4. Functorial properties The constructions $\mathcal{G}(\ ,\ )$, $GM(\ ,\ )$, $\mathrm{GH}(\ ,\ )$ are functors on the category of topological pairs (X, B) and *stratum-preserving* continuous maps of these. A continuous map $f : (X, B) \to (X', B')$ is *stratum-preserving* if $f^{-1}(B') = B$. Further, $\mathcal{G}(\ ,\ )$ is also a homotopy functor, in the sense we now explain. A *stratum-preserving homotopy*, $F : X \times I \to X'$ between $f^0, f^1 : (X, B) \to (X', B')$, is a homotopy for which $F^{-1}(B') = B \times I$, and, for each $a \in B$, $F(a \times I)$ is one point. If $\{S_\lambda \mid \lambda \in \Lambda\}$ is a cc-family over (X, B), and $F : (X, B) \times I \to (X', B')$ is a stratum-preserving homotopy, then $\{F(S_\lambda \times I) \mid \lambda \in \Lambda\}$ is a cc-family over (X', B'). This means that, for each geometric module M over (X, B), F defines a morphism $F(M) : f^0_*(M) \to f^1_*(M)$ in $GM(X', B')$ such that $j(F(M))$ is an isomorphism in $\mathcal{G}(X', B')$. In addition, if $g : N \to M$ is a morphism in $GM(X, B)$, F defines a morphism $F(g) : f^0_*(N) \to f^1_*(M)$ in $\mathrm{GH}(X', B')$, so that $f^1_*(g) \circ F(N) - F(M) \circ f^0_*(g) = (\partial_1 - \partial_0)(F(g))$. This says that the correspondence $M \to j(F(M))$ is a natural equivalence of functors:

$$\mathcal{G}(F) : \mathcal{G}(f^0) \to \mathcal{G}(f^1).$$

So $\mathcal{G}(\ ,\ )$ is a functor on the category of stratum-preserving *homotopy classes* of stratum-preserving maps.

Now $K_i(\ ,\ )$ is an abelian group valued functor from the category of topological pairs and stratum-preserving homotopy classes of continuous maps. Furthermore, $\mathrm{Wh}(\ ,\ )$ is a functor on the same category because the map $\mathcal{G}(f) : \mathcal{G}(X, B) \to \mathcal{G}(X', B')$, induced by a stratum-preserving map $f : (X, B) \to (X', B')$, sends basis change isomorphisms to basis change isomorphisms.

Suppose that $f : (X, B) \to (X', B')$ is stratum-preserving, B, B' are closed sets, and $f : X \to X'$ is proper. Then f induces a map $f_* : K_i^{\mathrm{lf}}(X, B) \to K_i^{\mathrm{lf}}(X', B')$. Moreover, $f^0_* = f^1_*$ if there is a proper stratum-preserving homotopy from f^0 to f^1. For the reader's convenience, we will recall some definitions from [21].

DEFINITION 2.7 (Holink). Let A be a subspace of a topological space X. The holink of A in X is

$$\mathrm{Holink}(X, A) = \{\sigma \in \mathrm{Map}([0, 1], X) \mid \sigma^{-1}(A) = 0\}.$$

It is given the compact-open topology. It comes with a projection map, $p_X : \mathrm{Holink}(X, A) \to A$; $p_X(\sigma) = \sigma(0)$. It is also equipped with a map $j_X : \mathrm{Holink}(X, A) \to (X - A)$; $j_X(\sigma) = \sigma(1)$.

DEFINITION 2.8 (Forward tame). A closed set A in a topological space X is forward tame in X if there exists a neighborhood U of A in X and a map $F : U \times [0, \infty] \to X$ such that $F^{-1}(A) = A \times [0, \infty] \cup U \times \{\infty\}$ and $F(a, t) = a$, for all $(a, t) \in A \times [0, \infty]$.

DEFINITION 2.9 (Reverse tame). A closed set A in a topological space X is reverse tame in X if there is a map $R : (X - A) \times [0, \infty] \to X - A$ such that:

(1) for each $t \in [0, \infty)$, $Cl_X R((X - A) \times [0, t]) \subset X - A$;
(2) each point $x \in X - A$ has a neighborhood U and a number $t_x \in [0, \infty)$ so that $R_t|_U = \text{identity}|_U$ for all $t \geq t_x$ (here $R_t : X - A \to X - A$ is defined by $R_t(x) = R(x, t)$);
(3) R extends continuously to a map $(X - A) \times [0, \infty] \cup (A \times \{\infty\}) \to X$ by setting $R(a, \infty) = a$, for all $a \in A$.

DEFINITION 2.10 (Tame). A closed set of X is tame if it is both forward and reverse tame.

DEFINITION 2.11. *A stratified space* is a finitely filtered, locally compact Hausdorff space $(X, \{X_i\}_0^n)$ such that:

(1) each stratum X_k is a k-dimensional topological manifold (possibly with boundary);
(2) for each $i < j$, X_i is tame in $X_i \cup X_j$;
(3) $\text{Holink}(X_i \cup X_j, X_i) \xrightarrow{p} X_i$ is a fibration, and the inclusion $\text{Holink}(\partial X_i \cup \partial X_j, \partial X_i) \to \text{Holink}(X_i \cup X_j, X_i)|_{\partial X_i}$ is a fiber homotopy equivalence over ∂X_i.

DEFINITION 2.12. Let X be an n-dimensional stratified space without boundary. By a *completion* of X we mean a compact stratified space $\bar{X}$ such that $X = \bar{X} - \partial\bar{X}$, and $\partial\bar{X}$ has a collar neighborhood in $\bar{X}$.

3. A continuously controlled h-cobordism theorem

The main goal of this section is to prove a version of the s-cobordism theorem in the context of relative manifolds. The arguments below will follow those of [13, 15, 18].

DEFINITION 3.1. A *relative manifold* is a compact Hausdorff pair (X, A), for which $X - A$ is a paracompact manifold of dimension $\dim(X - A) \geqslant 5$, possibly with boundary. We define the boundary of a relative manifold to be the relative manifold $\partial(X, A) = (\partial(X - A) \cup A, A)$. Here the topology on $\partial(X - A) \cup A$ is induced from X.

A *cobordism of relative manifolds* between (X, A) and (Y, A) is a relative manifold (W, A) such that $\partial W - A$ is the union of the two open sets $X - A$ and $Y - A$.

An *h-cobordism of relative manifolds* is a cobordism (W, A) between (X, A) and (Y, A) for which there are strict maps $r_i^t : (W, A) \to (W, A)$, $0 \leqslant t \leqslant 1$, $i \in \{1, 2\}$ deforming (W, A) into (X, A) and (Y, A) respectively, by a deformation which fixes X and Y respectively. We will denote the set of equivalence classes of h-cobordisms on (X, A) by $\text{hCob}(X; A)$. The equivalence relation is generated by relative homeomorphisms (a relative homeomorphism between (X, A) and (Y, A) is required to be the identity homeomorphism over A).

THEOREM 3.2. *Let* (W, A) *be an h-cobordism from* (X, A) *to* (Y, A)*, of* $\dim(W - A) \geqslant 5$. *Then the triple* (W, X, A) *specifies an element* $\tau(W, X, A) \in \text{Wh}(X, A)$ *such* *that the map* $\psi : \mathrm{hCob}(X; A) \to \mathrm{Wh}(X, A)$ *given by* $\psi((W, X, A)) = \tau(W, X, A)$ *is a bijection.*

PROOF. Let (W, A) be an h-cobordism between (X, A) and (Y, A) with strict maps $r_i^t : (W, A) \to (W, A)$, $0 \leqslant t \leqslant 1$, $i \in \{1, 2\}$ deforming (W, A) into (X, A) and (Y, A) respectively. By the results of Kirby and Siebenmann [11] one can choose a handle decomposition of the pair $(W - A, X - A)$; refining it, if necessary, one can assume without loss of generality that the resulting chain complex of geometric modules, say $\hat{C}(W, A)$, is a continuously controlled chain complex. Let $C(W, A) = r^1_{1*}\hat{C}(W, A)$. $C(W, A)$ is a contractible continuously controlled chain complex in $\mathcal{GM}(X, A)$, the contraction $H : C_i(W, A) \to C_{i+1}(W, A)$ being given by r^1_{1*}. So one defines $\psi((W, X, A)) = [C(W, A)] \in \mathrm{Wh}(X, A)$. The fact that ψ is a well-defined map follows from the homeomorphism invariance of the Whitehead torsion. It is essentially due to Chapman (see [3, 4]). We will explain it in more detail in 3.5.

It is not difficult to notice that the torsion of an h-cobordism satisfies 'Milnor' duality [14, p. 394]. That is to say, for any h-cobordism of relative manifolds (W, A) from (X, A) to (Y, A),

$$\tau(W, Y, A) = (-1)^{n-1}\bar{\tau}(W, X, A).$$

Here n is the dimension of $W - A$.

The main part of the proof is the injectivity of ψ. We will establish it first.

Suppose that $\tau(W, X, A) = 0$. One has to find a relative homeomorphism $\Phi : (X \times I, A \times I) \to (W, A)$, so that Φ is stationary along A. This amounts to showing that one can eliminate all the handles in the above handle decomposition. Recall that a handle decomposition consists of a collar along $X - A$ together with collections of j-handles $\{e_i^j, i \in \mathbb{Z}_+\}$, $0 \leqslant j \leqslant \dim(W - A)$. We will discuss separately the process of eliminating the 0- and 1-handles:

3.1. 0-Handles Let $\mathcal{H}^0 = \{e_i^0, i \in \mathbb{Z}_+\}$ be the collection of 0-handles. Each 0-handle of (W, X) can be connected to the collar on $X - A$ by a finite sequence of alternating 0- and 1-handles. Choose for every 0-handle e_k^0 a sequence of 0- and 1-handles, say $K(e_k^0)$ connecting e_k^0 to the collar. A collar means a neighborhood of X in W homeomorphic to the space $(X - A) \times I \cup A$; the topology on this space is the quotient topology making the map $X \times I \to X \times I/\{(a, t) \equiv (a, 0)\}$ continuous. We will further denote the space $(X - A) \times I \cup A$ with the above described topology by $X \times_\lambda I$. One can make the choice in such a way that the number of handles in each $K(e_k^0)$ is minimal; it is obvious that $K(e_k^0)$ is a compact set, for every $k \in \mathbb{Z}_+$, and the collection $\mathcal{K} = \{K(e_k^0) \mid e^0 \in \mathcal{H}^0\}$ is a continuously controlled family. By deleting some of the 0-handles, if necessary, we can assume that no element of $\mathcal{K}$ is contained in any other. The goal is to write $\mathcal{K}$ as a disjoint union of sets of the form $K(e_k^0)$, which can subsequently be absorbed in the collar. The argument is similar to that in [15, 18]. If $K(e_k^0)$ and $K(e_l^0)$ share a common 1-handle we subdivide, replacing it by two parallel 1-handles and a new 2-handle. This process eventually guarantees that no two elements of $\mathcal{K}$ share a common 1-handle. If $K(e_k^0)$ and $K(e_l^0)$ share a common 0-handle, we subdivide, replacing it by two 0-handles connected by a new 1-handle. So one can assume that $K(e_k^0) \cap K(e_l^0) = \emptyset$. It is obvious that $\mathcal{K}$ is still a continuously controlled family, and that every 0-handle lies in one of the sets $K(e_k^0)$. One can then absorb the disjoint sets $\{K(e_k^0)\}$ into the collar along $X - A$, getting a handle decomposition with no 0-handles. By duality one can remove the n-handles as well; here $n = \dim(W - A)$.

3.2. 1-handles By the previous paragraph we may assume that all the 1-handles are trivially attached. We will follow Milnor's argument to eliminate the 1-handles [13, Chapter 8]. Let e_α^1 be a 1-handle in the decomposition of (W, X), and let U_α be any open neighborhood of e_α^1. One can construct an embedding of S^1 in U_α whose image S intersects the right-hand sphere $S^{n-2}(e_\alpha^1)$ transversely at one point. It is easy to see that one can introduce a pair of canceling 2- and 3-handles in U_α, so that the boundary of the 2-handle is just S. We then cancel the 1-handle with the newly attached 2-handle. The process is local, the newly introduced 2- and 3-handles also form a continuously controlled family. In conclusion, one can eliminate all the 1-handles in the handle decomposition of (W, X). Likewise, by duality one can assume that we have also eliminated all the $(n-1)$-handles. Because $n \geqslant 5$, this will not create new 1-handles.

3.3. *k*-handles Suppose we have eliminated all the handles of dimension $0, 1, \ldots, k-1$, where $k \geqslant 2$ and $2k+1 \leqslant n$. We will show that one can trade the k-handles for $(k+2)$-handles. The argument follows closely those of [6, 10, 13, 15, 18]. The homotopy $H_k : C_k(W, A) \to C_{k+1}(W, A)$ is a left inverse for ∂_{k+1}. We introduce again the cancelation of pairs of $(k+1)$- and $(k+2)$-handles in a neighborhood of every k-handle, and use the change of basis theorem (see [13] or [10]) in order to modify the boundary map ∂_{k+1}; the geometric moves are prescribed by the map H_k, which is continuously controlled, so after using the Whitney trick one can cancel the k-handles with the newly attached $(k+1)$-handles and still get a cc-handle decomposition. In fact the algebraic moves are the same as for 1-handles, the only difference being the absence of the Whitney trick for 1-handles. Note that the Whitney isotopies are obviously continuously controlled. By induction one can then assume that the handle decomposition is concentrated in two adjacent dimensions, say k and $k+1$. So H_k and ∂_{k+1} are inverses.

We will use the vanishing of $\tau(W, X, A)$ at this point. It implies that after stabilization (that is, after introducing mutually canceling k and $(k+1)$-handles), the matrix of ∂_{k+1} is a finite product of deformations (a deformation is a product of elementary matrices and geometric isomorphisms). Algebraically, one can change the base of $C(W, X)$, so that ∂_{k+1} becomes the identity. It is obvious that the algebraic moves can be realized geometrically; moreover, the Whitney trick and the cancelation theorem allows us to cancel each k-handle with the corresponding $(k+1)$-handle. In conclusion, one can absorb all the handles in the collar along X. The h-cobordism then becomes a product cobordism, so ψ is injective.

The proof that the map $\psi : \mathrm{hCob}(X; A) \to \mathrm{Wh}(X, A)$ is surjective is quite easy. Each element of $\mathrm{Wh}(X, A)$ can be represented by a contractible chain complex of geometric modules $\mathcal{C}$, so that $\mathcal{C}_i = 0$, for all $i \neq 2, 3$. For each basis element of $\mathcal{C}_2$, we trivially attach a 2-handle to $X \times_\lambda I$; hence the geometric module formed by the 2-handles is just $\mathcal{C}_2$. We will then attach the 3-handles according to the instructions given by the continuously controlled boundary map $\partial : \mathcal{C}_3 \to \mathcal{C}_2$. This gives us an h-cobordism (W, A) from (X, A) to a relative manifold (Y, A) such that the torsion $\tau((W, X, A))$ is just $[\mathcal{C}]$.

LEMMA 3.3. *In the above notation, if X is a stratified space with singular set A and (W, A) is an h-cobordism of relative manifolds from (X, A) to (Y, A), then both W and Y are stratified spaces, so that A is the singular set of Y, and W contains X and Y as pure subsets.*

PROOF. The verification of the definition is simple: the h-cobordism deformation provides both the holink and the tameness condition. □

3.4. Additivity This is standard. Let (W, A) be an h-cobordism from (X, A) to (Y, A). Let (W_1, A) be an h-cobordism from (Y, A) to (Y_1, A). The retraction $\rho : Y \to X$ induces a map $\rho_* : \mathrm{Wh}(Y, A) \to \mathrm{Wh}(X, A)$. Then $(W_1 \cup_Y W, A)$ is an h-cobordism from (X, A) to (Y_1, A). Moreover,

$$\tau(W_1 \cup_Y W, A) = \tau(W, A) + \rho_* \tau(W_1, A).$$

The proof of the theorem is now complete. □

An important consequence of the theorem is the following result.

COROLLARY 3.4 (Invertibility of relative h-cobordisms). *Given any h-cobordism of relative manifolds (W, A) from (X, A) to (Y, A) there is an h-cobordism (W', A) such that $\tau(W' \cup_Y W, A) = 0$.*

The h-cobordism theorem for relative manifolds implies a similar theorem for stratified spaces. More precisely, for a stratified space X, define $\mathrm{hCob}(X, \sigma X)$ to be the set of h-cobordisms on X which admit a product structure along σX.

THEOREM 3.5. *Let W be a stratified h-cobordism between the stratified spaces X and Y. Assume that $\dim(W - A) \geqslant 6$. Suppose that W has a product structure along σW, that is to say, $\sigma W = \sigma X \times [0, 1]$. Then the pair (W, X) specifies an element $\tau(W, X) \in \mathrm{Wh}(X, \sigma X)$ such that the map $\psi : \mathrm{hCob}(X, \sigma X) \to \mathrm{Wh}(X, \sigma X)$ given by $\psi((W, X)) = \tau(W, X)$ is a bijection.*

PROOF. This is an easy consequence of Theorem 3.2 and Quinn's collaring theorem (see [21, p. 492]). Here are the details. W is a stratified space and ∂W contains Y as an open and closed subset. Moreover, σY has a collar in σW, by assumption. According to the lemma [21, p. 492] this collar can be extended to a collar of Y in

W, say $c : Y \times [0, 1] \to W$, so that $c(Y \times 0) = Y$. Now $W - c(Y \times [0, 1))$ is an h-cobordism of relative manifolds. The theorem follows now from the corresponding one for relative manifolds. □

3.5. The topological invariance of the Whitehead torsion We need to prove that, if $f : (X, A) \to (Y, A)$ is a homeomorphism of relative manifolds (that is, $f^{-1}(A) = A$ and $f|_A = 1_A$), then $\tau(f) = 0$ as an element in $\mathrm{Wh}(Y, A)$. This statement will follow from the following version of the α-approximation theorem, which is essentially due to Chapman [4].

THEOREM 3.6 [4, Theorem 1]. *Let Y be a locally compact polyhedron and β an open cover of Y. Then there exists another open cover of Y, α, so that if X is a locally compact polyhedron and $f : X \to Y$ a proper map which is an α-equivalence, then f is a β-simple homotopy equivalence.*

REMARKS 3.7.

(1) A map $f : X \to Y$ is a β-simple homotopy equivalence if there exist a locally compact polyhedron Z and cell like piecewise linear proper maps
$$X \xleftarrow{g_1} Z \xrightarrow{g_2} Y,$$
so that fg_1 is β homotopic to g_2.

(2) Chapman's proof is an inductive argument using the compact version of the α-approximation theorem. In fact, he does not keep track of the size of the simple homotopy, but if one replaces [4, Theorem 8.1] by the ϵ-δ approximation theorem (see [5] or [24]) the same argument applies to give the desired result. For completeness, here is the statement of the ϵ-δ approximation theorem phrased in terms of 'relaxing control' in the Whitehead groups [5, 24].

THEOREM 3.8. *For any compact polyhedron K and any $\epsilon > 0$ there is a $\delta > 0$ such that the stabilization map*
$$\mathrm{Wh}(K, 1_K, \delta) \longrightarrow \mathrm{Wh}(K, 1_K, \epsilon)$$
is the zero map.

The topological invariance can also be proved using a controlled approach similar to that of Kirby and Siebenmann [11, p. 117]. In this approach a simple homotopy equivalence will be a composition of expansions and inverses of expansions; it is necessary only to keep track of their sizes.

4. Finiteness obstructions

Let us now briefly recall the definitions of the controlled $\tilde{K}_0$-groups, and of the controlled finiteness obstruction. The groups $\tilde{K}_0(B, p)$ were introduced by Quinn in [19]. We follow the more algebraic approach of [22, 24].

4.1. The definition of $\tilde{K}_0(B, p)$ Let $p : E \to B$ be a continuous map from a topological space to a locally compact metric space, and let $\delta : B \to (0, \infty)$ be a continuous map. We say that a subset $S \subset E$ is a δ-*subset* if $p(S)$ lies in the $\delta(p(x))$ ball around $p(x)$, for each $x \in S$. A morphism $f = (f_t^s)_{(s,t)\in S\times T}$ in $GM^{\mathrm{lf}}(E)$, or $\mathrm{GH}^{\mathrm{lf}}(E)$, is a δ-*morphism* if $\mathrm{supp}(f_t^s)$ is a δ-subset for each (s, t). We write $\mathrm{GH}^{\mathrm{lf}}(E; \delta)$ for the δ-morphisms in $\mathrm{GH}^{\mathrm{lf}}(E)$.

$GM^{\mathrm{lf}}(B, p)$ ($\mathrm{GH}^{\mathrm{lf}}(B, p)$) is defined as the subcategory of $GM^{\mathrm{lf}}(E)$ ($\mathrm{GH}^{\mathrm{lf}}(E)$) consisting of all objects $M = (S, j)$, such that $p(M) = (S, pj)$ is an object of $GM^{\mathrm{lf}}(B)$, and all morphisms $f = (f_t^s)_{(s,t)\in S\times T}$ such that $p(f) = (pf_t^s)_{(s,t)\in S\times T}$ is a morphism of $GM^{\mathrm{lf}}(B)$ ($\mathrm{GH}^{\mathrm{lf}}(B)$). If, in addition, f is a δ-morphism we say that f is in $GM(B, p, \delta)$, ($\mathrm{GH}(B, p, \delta)$).

The δ-morphisms in $GM^{\mathrm{lf}}(B, p)(N, M)$ form a subgroup, which we denote by $GM^{\mathrm{lf}}(B, p; \delta)(N, M)$. The composition fg of a δ-morphism and a δ'-morphism is a $(\delta + \delta')$-morphism. We say that $f, g \in GM^{\mathrm{lf}}(B, p)(M, N)$ are δ-homotopic ($f \sim_\delta g$) if there is an $h \in \mathrm{GH}^{\mathrm{lf}}(B, p, \delta)(M, N)$ for which $f - g = (\partial_1 - \partial_0)(h)$. Following [24], we say that a δ-projective module is a pair (M, e), where $M \in |GM^{\mathrm{lf}}(B, p)|$, $e \in GM^{\mathrm{lf}}(B, p)(M, M)$, and $e^2 \sim_\delta e$. A δ-isomorphism between two of these, say $(M, e), (M', e')$, is a pair of δ-morphisms $f : M \to M'$, $g : M' \to M$, such that $fg \sim_\delta e'$, $gf \sim_\delta e$, $e'f \sim_\delta f \sim_\delta fe$, $eg \sim_\delta g \sim_\delta ge'$. A stable δ-isomorphism from (M, e) to (M', e') is a δ-isomorphism from $(M \oplus X, e \oplus 1_X)$ to $(M' \oplus X', e \oplus 1'_X)$ for some $X, X' \in |GM^{\mathrm{lf}}(B, p)|$. The paper by Yanicki and Yamasaki [24, proof of Theorem 1.4] shows that if two δ-projective modules, $(M, e), (M', e')$, are equivalent under the equivalence relation generated by stable 4δ-equivalence, then (M, e) and (M', e'), are stably 6δ-isomorphic. The set of all equivalence classes of δ-projective modules on (B, p) under the equivalence relation generated by stable 4δ-equivalence is an abelian group, written $\tilde{K}_0^{\mathrm{lf}}(B, p; \delta)$.

Now, by similarly mimicking the steps in [24], one defines $\tilde{K}_0^{\mathrm{lf}}(B, p; n, \delta)$ as the set of equivalence classes of δ-projective chain complexes of geometric modules, where the chain complexes, C, satisfy $C_i = 0$, unless $0 \le i \le n$. The equivalence relation is generated by stable 4δ-chain equivalence. Just as in [24, Proposition 1.7], there is an Euler characteristic map

$$\chi : \tilde{K}_0^{\mathrm{lf}}(B, p; n, \delta) \to \tilde{K}_0^{\mathrm{lf}}(B, p; 9\delta),$$

so that the composite map

$$\tilde{K}_0^{\mathrm{lf}}(B, p; \delta) \xrightarrow{i} \tilde{K}_0^{\mathrm{lf}}(B, p; n, \delta) \xrightarrow{\chi} \tilde{K}_0^{\mathrm{lf}}(B, p; 9\delta)$$

is induced by the inclusion $GM(B, p; \delta) \hookrightarrow GM(B, p; 9\delta)$. Here i is the map sending a module M to the n-dimensional complex, concentrated in degree 0, which M determines. Note that i is an epimorphism (see [24, Section 1.6]).

Definition 4.1. We define $\tilde{K}_0^{\mathrm{lf}}(B, p)$ as $\varprojlim_\delta \tilde{K}_0^{\mathrm{lf}}(B, p; \delta)$, as δ ranges over the continuous functions $\delta : B \to (0, \infty)$. It is clear that, if B is compact, ϵ and δ can be chosen to be positive real numbers. The corresponding groups will then be denoted by $\tilde{K}_0(B, p)$.

Let B be a compact metric space. The negative controlled K groups of (B, p) are defined in a standard way:

$$\tilde{K}_{-i}(B, p) = \tilde{K}_0^{\mathrm{lf}}(B \times \mathbb{R}^i, p \times 1_{\mathbb{R}^i}).$$

(Compare [16] and [24].)

In the last part of this section we will review the definition of Quinn's controlled finiteness obstruction.

4.2. The projective obstruction of a finitely dominated complex Let $p : E \to B$ and $\delta : B \to (0, \infty)$ be as before. Let C be a δ-chain complex over $GM^{\mathrm{lf}}(E)$ such that $(C, 1)$ is δ-homotopy equivalent to an n-dimensional projective δ-chain complex, (D, e), over $\widehat{GM}^{\mathrm{lf}}(B, p)$. The class

$$q_\delta(C) = \chi([D, e]) \in K_0^{\mathrm{lf}}(B, p; 9\delta)$$

depends only on C and is called the *controlled finiteness obstruction of C over B*.

Suppose C is an n-dimensional δ-chain complex over $GM^{\mathrm{lf}}(E)$ which is δ-dominated by a δ-chain complex over $GM^{\mathrm{lf}}(B, p)$. In [24, Section 3.1] it is shown that $(C, 1)$ is ϵ-chain equivalent to an n-dimensional ϵ-projective ϵ-chain complex, (D, e), over $\widehat{GM}^{\mathrm{lf}}(B, p)$. Here $\epsilon = (2n + 4)\delta$.

LEMMA 4.3. *Let $0 \to C' \xrightarrow{i'} C \xrightarrow{i''} C'' \to 0$ be an exact sequence of δ-chain complexes over $GM^{\mathrm{lf}}(E)$. Assume that $(C, 1), (C', 1), (C'', 1)$ are δ-homotopy equivalent to n-dimensional projective δ-complexes over the idempotent completion $\widehat{GM}^{\mathrm{lf}}(B, p)$. Then*

$$q_\epsilon(C) = q_\epsilon(C'') + q_\epsilon(C') \quad (\epsilon = 5\delta).$$

PROOF. The mapping cylinder and mapping cone of a δ-chain map between δ-complexes are 3δ-chain complexes. Using the hypotheses and standard mapping cylinder arguments (see Proposition 5.7 for more details) leads to a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & (D', e') \arrow[r, "\iota'"] \arrow[d, "\simeq"'] & (D, e) \arrow[r, "\iota''"] \arrow[d, "\simeq"'] & (D'', e'') \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & (C', 1) \arrow[r, "i'"] & (C, 1) \arrow[r, "i''"] & (C'', 1) \arrow[r] & 0
\end{tikzcd}$$

where the top row is an exact sequence of $(n + 1)$-dimensional ϵ-projective complexes over $\widehat{GM}^{\mathrm{lf}}(B, p)$, and the vertical maps are ϵ-homotopy equivalences over $\widehat{GM}^{\mathrm{lf}}(E)$. Here $\epsilon = 5\delta$. The construction of the top row goes as follows. First choose D' and D arbitrarily; the map ι' can be defined so that the diagram involving ι' and i' is commutative. By passing to the mapping cylinder of the map i' one can assure that D is the mapping cylinder of i'. Then construct D'' as the mapping cone of i'. See also the proof of Theorem 6.1 for more algebraic details.

By [24], $\chi[D, e] = \chi[D', e'] + \chi[D'', e'']$, from which the result follows. $\square$

4.3. The controlled end obstruction A *controlled end obstruction*, $q_0(M, f)$, for any tame end (M, f), is defined by Quinn [19, p. 410]. For the reader's convenience, we present the details of this definition for compact B, which is the case we use most.

DEFINITION 4.4. Let $f : M \to B$ be a map from a manifold M with compact boundary to a compact metric space B (an 'end' in the sense of [18]). Assume that $\dim(M) \geq 6$, and assume that (M, f) is tame [18, Section 1.1]. Quinn defines an obstruction $q_0(M, f)$ as follows.

Let $M = M(0) \supset M(1) \supset M(2) \supset \cdots$ be an infinite sequence of submanifolds with compact boundaries and relatively compact complements in M. Assume that $\bigcap_{k=1}^{\infty} M(k) = \emptyset$. Using the tameness condition one can choose this sequence so that $M(k)$ deforms, relative to $\partial M(k)$, into the set $N(k) = M(k) - \mathrm{Int}\,(M(k+1))$, and the diameter of the deformation (when measured in B), is less than $\delta(k)$, where $\delta(1) > \delta(2) > \cdots$, and $\lim_{k\to\infty} \delta(k) = 0$. Set $f(k) = f|_{M(k)} : M(k) \to B$.

Choose a handle decomposition of $(N(k), \partial M(k))$ and an infinite handle decomposition of $(M(k), \partial M(k))$ of diameter much less than $\delta(k)$ (when measured in B). Let $C(k) = C(M(k), \partial M(k))$. The deformation above shows that $C(k)$ is $\delta(k)$-finitely dominated by the finite complex $C(N(k), \partial M(k))$, so $q_{\epsilon(k)}(C(k))$ is defined, where $\epsilon(k) = C\delta(k)$, and C is a constant depending only on n. Moreover, the sequence

$$0 \to C(N(k), \partial M(k)) \to C(k) \to C(k+1) \to 0$$

is exact. Therefore by Lemma 4.3, $q_{\epsilon(k)}(C(k)) = q_{\epsilon(k)}(C(k+1))$. It follows that the sequence

$$q_0(M, f) = \{q_{\epsilon(k)}(C(k))\}_{k=1}^{\infty},$$

is an element of the group $\lim_k \tilde{K}_0(B, f(k), 9\epsilon(k))$.

It is not hard to imagine that $\tilde{K}_0(B, p) \approx \lim_k \tilde{K}_0(B, f(k), 9\epsilon(k))$ (see also [9]). Here p is the holink projection. We will give a sketch of the proof of this fact in the next section.

The element $q_0(M, f)$ is called the controlled end obstruction of (M, f).

REMARK 4.5. In [18, Ch. 7] and [19] Quinn proves that $q_0(M, f) = 0$ if and only if M is the interior of a compact manifold $\overline{M}$ such that f extends to a map $\overline{f} : \overline{M} \to B$.

5. The boundary map

In this section we will give an explicit description of the boundary map

$$\Delta : \mathrm{Wh}(X, B) \to \tilde{K}_0(B, p).$$

This will enable us to establish formula $(*)$ of Theorem 6.1 which is an important tool in the proof of the main theorems. We also conclude, using the description of Δ, that the sequence

$$\mathrm{Wh}(X, B) \xrightarrow{\Delta} \tilde{K}_0(B, p) \xrightarrow{\alpha} \tilde{K}_0(\mathbb{Z}\pi_1(X - B)),$$

is exact at $\tilde{K}_0(B, p)$.

Recall that $\mathcal{GM}(X, B)$ is an additive category, having $\mathcal{GM}(X - B)$ as a full subcategory.

DEFINITION 5.1. A chain complex C over $\mathcal{GM}(X, B)$ is dominable by a complex over $\mathcal{GM}(X - B)$ if there is a chain map, chain homotopic to 1_C, say $f = 1_C - h\partial - \partial h : C \to C$, and a factorization of f as a composite: $C \xrightarrow{f'} D \xrightarrow{f''} C$ where D is a complex over $\mathcal{GM}(X - B)$. A domination of C will be denoted by (D, f', f'', h).

The definition of an ϵ-domination is straightforward (see [24]).

For each such (D, f', f'', h), Ranicki [23, pp. 119–121] constructs a chain complex (E, q) over the idempotent completion $\widehat{\mathcal{GM}(X - B)}$, together with a chain homotopy equivalence, $(E, q) \simeq (C, 1)$.

This construction depends continuously on (D, f', f'', h), in the following sense. Let an integer n and a real number $\epsilon > 0$ be preassigned. There is a number $\delta > 0$, depending on n and ϵ, such that if D, f', f'', h, all have diameter less than δ, and $\dim(D) \leq n$, then E, $q : E \to E$ and the homotopy equivalence $(C, 1) \to (E, q)$ all have diameter less than ϵ (see [24, Section 3.1]).

Let X be a topological space. Let B be a metric space which is closed and forward tame in X. According to [1], there is an exact sequence

$$\cdots \longrightarrow \mathrm{Wh}(X, B) \xrightarrow{\Delta} \tilde{K}_0(B, p) \xrightarrow{\alpha} \tilde{K}_0(\mathbb{Z}\pi_1(X - B)) \longrightarrow \cdots .$$

The purpose of this section is to describe the construction of the boundary map Δ. We will not prove that the description we give below coincides with the one in [1]; instead, we will show in Proposition 5.8 why our definition of the boundary map makes this sequence exact at $\tilde{K}_0(B, p)$. This is sufficient for our purposes.

Let $\rho : N \to B$ be a neighborhood retraction and let $\{\rho_t \mid N \to X, 0 \leq t \leq 1\}$ be a homotopy, relative to B, from the inclusion $N \to X$ to ρ, such that $\rho_t^{-1}(B) = B$, for all $t < 1$. The adjoint of this homotopy is denoted $\lambda : N - B \to \mathrm{Holink}(X, B)$.

DEFINITION 5.2. Let C be a chain complex over $\mathcal{GM}(X, B)$. Let U be any neighborhood of B in X. We will denote by $C|_U$ the largest subcomplex of C which is supported on U. That is to say, $(C|_U)_0 = C_0$, $(C|_U)_1$ is the largest subcomplex of C_1, so that the map $\partial : C_1 \to C_0$ induces a map supported on U, denoted by $\partial|_U : (C|_U)_1 \to (C|_U)_0$, etc.

DEFINITION 5.3. Let $\delta > 0$ be arbitrary. Let U be a neighborhood of B in X. By a restriction of C to U, say (C_U, ∂_U) we mean a subcomplex of C such that:

(1) C_U is supported on U, by which we mean that C_U is a subcomplex of $C|_U$;
(2) there is a neighborhood $V \subset U$ so that $(C_U)|_V = C|_V$.

Note that such a subcomplex C_U obviously exists, but it is not uniquely determined by C and U.

LEMMA 5.4. *Let $\delta > 0$ be arbitrary. Let C be a chain complex over $\mathcal{GM}(X, B)$ dominated by the complex D over $\mathcal{GM}(X - B)$. For each sufficiently small neighborhood U of B, any choice of the complex C_U is a δ-complex over $\mathcal{GM}(U, B)$, δ-dominable (over $\mathcal{GM}(U, B)$) by a δ-complex over $\mathcal{GM}(U - B)$. Measurements are made in B, using the map ρ.*

PROOF. Let (D, f', f'', h) be a domination of C by a complex over $\mathcal{GM}(X - B)$. For all U sufficiently small, the continuous control condition ensures that C_U is a δ-complex and $f|_{C_U}$, $f'|_{C_U}$, $h|_{C_U}$ are δ-maps.

Let $\pi_U : C \to C_U$, and $i_U : C_U \to C$ be the projection and inclusion maps, respectively. Note that π_U is not a chain map; it is only a map of graded groups. Now introduce the chain homotopy $h_U = \pi_U h i_U : C_U \to C_U$, a δ-map. By the continuous control condition, there is a smaller neighborhood V of B, so that $h_U|_{C_V} = h|_{C_V}$, $\partial_U|_{C_V} = \partial|_{C_V}$ and $f'|_{C_V} = 0$.

Set $f_U = 1_{C_U} - h_U\partial_U - \partial_U h_U : C_U \to C_U$. Then f_U is a 2δ-map which is chain homotopic to 1_{C_U} by definition. Also $f_U|_{C_V} = (1 - h\delta - \delta h)|_{C_V} \sim_{4\delta} f'' f'|_{C_V} = 0$, because $f'|_{C_V} = 0$.

It follows that there is a δ-chain map $\hat{f}_U : C_U \to C_U$ such that $f_U \sim_{4\delta} \hat{f}_U$ and $\hat{f}_U|_{C_V} = 0$. Write $\hat{f}_U = g'' g'$, where $g' : C_U \to C_U/C_V$ is the natural epimorphism, and $g'' : C_U/C_V \to C_U$. Note that $(C_U/C_V, g', g'', h_U)$ is a 4δ-domination of C_U by a δ-complex over $\mathcal{GM}(U - B)$. □

DEFINITION 5.5. Let X be a topological space. Let $B \subset X$ be a metric space which is closed and forward tame in X. A complex over $\mathcal{GM}(X, B)$, dominable by a complex over $\mathcal{GM}(X - B)$, will be called a peripherally contractible *complex* (see Definition 5.1).

A chain map $f : C \to C'$ is said to be a *peripheral homotopy equivalence* if its mapping cone, $C(f)$, is peripherally contractible.

Note that a contractible complex is obviously peripherally contractible.

DEFINITION 5.6 (Boundary map Δ). Let C be a peripherally contractible complex. By Lemma 5.4, for each $\delta > 0$ and each sufficiently small neighborhood U of B, the complex $(C_U, 1)$ over $\widehat{\mathcal{GM}(U, B)}$ is δ-chain homotopic to a δ-projective complex, denoted by (E_δ, q_δ), over $\widehat{\mathcal{GM}(U - B)}$. Measurements are made in B, using ρ. The element $\chi(\lambda_*[E_\delta, q_\delta]) \in \tilde{K}_0(B, p; \delta)$ depends only on C and δ, and the family $\{\chi(\lambda_*[E_\delta, q_\delta]) \mid \delta > 0\}$ constitutes an element denoted $\Delta C \in \tilde{K}_0(B, p)$, depending only on C.

Here $p : \mathrm{Holink}(X, B) \to B$ is the projection.

The next proposition shows that the map Δ defined above gives a homomorphism

$$\Delta : \mathrm{Wh}(X, B) \to \tilde{K}_0(B, p),$$

which we deliberately denote by the same letter Δ, for the sake of simplicity.

In the following proposition all complexes and chain maps will be over the category $\mathcal{G}(X, B)$.

PROPOSITION 5.7.

(A) *If $0 \to C' \to C \to C'' \to 0$ is a short exact sequence of peripherally contractible complexes, then $\Delta C = \Delta C' + \Delta C''$.*

(B) *Suppose that $f : C' \to C$ is any chain map between peripherally contractible complexes. Then $\Delta(\mathrm{Cone}(f)) = \Delta C - \Delta C'$.*

(C) *Suppose that $f \circ g$ is a composite of two peripheral homotopy equivalences between any complexes. Then $\Delta(\mathrm{Cone}(f \circ g)) = \Delta(\mathrm{Cone}(g)) + \Delta(\mathrm{Cone}(f))$.*

(D) *Suppose that C, C' are peripherally contractible complexes, which are stably Σ-equivalent, in the sense of* [24]. *Then $\Delta C = \Delta C'$.*

PROOF. (A) This is a consequence of the companion exact sequence of chain complexes over (B, p): $0 \to C'_U \to C_U \to C''_U \to 0$ (defined for all U sufficiently small).

(B) If C' and C are peripherally contractible, then $\mathrm{Cone}(f)$ is also. There is an exact sequence

$$0 \longrightarrow C \longrightarrow \mathrm{Cone}(f) \longrightarrow \Sigma C' \longrightarrow 0.$$

We obtain from (A) that $\Delta(\mathrm{Cone}(f)) = \Delta(\Sigma C') + \Delta(C)$. Therefore,

$$\Delta(\mathrm{Cone}(f)) = \Delta(C) - \Delta(C').$$

(C) Given $C' \xrightarrow{g} C \xrightarrow{f} C''$, we let $f' : C \to \mathrm{Cyl}(f)$ be inclusion, and we let $j : \mathrm{Cone}(g) \to \mathrm{Cone}(f \circ g)$ be the natural map. We then get two exact sequences of chain complexes:

$$0 \to \mathrm{Cone}(g) \to \mathrm{Cone}(f' \circ g) \to \mathrm{Cone}(f) \to 0,$$

$$0 \to \mathrm{Cone}(f' \circ g) \to \mathrm{Cyl}(j) \to c(\Sigma C') \to 0,$$

where $c(\Sigma C') = \mathrm{Cone}(1_{\Sigma C'})$ is the cone on the suspension of C'. The statement follows at once from (B).

(D) It is enough to show the result when C and C' are Σ-isomorphic in the sense of [24]. But each Σ-isomorphism f is a product of geometric isomorphisms and elementary automorphisms. By (B), if f is an automorphism, then $\Delta(\mathrm{Cone}(f)) = 0$. Moreover, it is obvious that if $f : C \to C'$ is a geometric isomorphism, then $\Delta C = \Delta C'$, so that $\Delta(\mathrm{Cone}(f)) = 0$, by (B). Therefore $\Delta(\mathrm{Cone}(f)) = 0$ for any Σ-isomorphism by (C). So by (B), the result follows. □

By proceeding in the same way as before, we can also obtain a homomorphism in the locally finite case:

$$\Delta : \mathrm{Wh}^{\mathrm{lf}}(X, B) \to \tilde{K}_0^{\mathrm{lf}}(B, p).$$

We use Proposition 5.7 in the category $\mathcal{GM}^{\mathrm{lf}}(X, B)$.

PROPOSITION 5.8. *Let X be a topological space, and B be a metric space which is closed and forward tame in X. Assume also that $X - B$ is a topological manifold. Then the sequence*

$$\mathrm{Wh}(X, B) \xrightarrow{\Delta} \tilde{K}_0(B, p) \xrightarrow{\mathfrak{a}} \tilde{K}_0(\mathbb{Z}\pi_1(X - B))$$

is exact at $\tilde{K}_0(B, p)$.

PROOF. Here the map $p : \mathrm{Holink}(X, B) \to B$ is the projection. Recall that the map $\mathfrak{a}$ is a relaxation of control, induced, in the language of [24], by

$$\tilde{K}_0(B, p, \delta) \to \tilde{K}_0(B, p, \infty).$$

The map $\mathfrak{a}$ exists for any control map p.

It is easy to see that $\mathfrak{a} \circ \Delta = 0$: let C be a cc-chain complex over $\mathcal{GM}(X, B)$. In order to get the image of $[C]$ under the map $\mathfrak{a} \circ \Delta$ one needs to restrict C to a small enough neighborhood of B, obtaining a finitely dominated chain complex, then relax the control. But this is the same as using any restriction, and obviously a contractible cc-chain complex is dominated by the 0 complex. We conclude that $\mathrm{Im}(\Delta) \subset \mathrm{Ker}(\mathfrak{a})$.

The inclusion $\mathrm{Ker}(\mathfrak{a}) \subset \mathrm{Im}(\Delta)$ is a little more intricate. The rest of the section will be dedicated to its proof.

Consider the following, which can be easily deduced from the tameness of B in X.

(I) Let $U(1) \supset U(2) \supset U(3) \supset \cdots$ be a sequence of compact X-neighborhoods of B, such that $\bigcap_{k=1}^{\infty} U(k) = B$. We further assume that $M(k) := U(k) - B$ is a manifold with bicollared boundary. We set $N(k) = M(k) - \mathrm{Int}\,(M(k+1))$.

(II) Let $\rho : U(1) \times I \to X$: be a strong deformation retraction of $U(1)$, down to B, within X. We assume that ρ is *nearly strict. That is to say,*

$$\rho^{-1}(B) = (B \times I) \cup (U(1) \times 1).$$

We also assume that

$$\rho(U(k) \times I) \subset U(k-1)$$

for all $k > 1$.

(III) *Set $\pi(k) := \rho|_{M(k)} \times 1 : M(k) \to B$.*

(IV) *Let also $\delta(1) > \delta(2) > \delta(3) > \cdots$ be a sequence converging to zero, such that, for each k, $\rho|_{M(k)} \times I : M(k) \times I \to M(k-1)$ is a $\delta(k)$-homotopy, when measured using $\pi(k-1) : M(k-1) \to B$.*

(V) *Let $R(k) : M(k) \to \mathrm{Holink}(U(k-1), B)_{\delta(k-1)}$ be the adjoint to $\rho|_{M(k)} \times I$.*

(VI) *Let* $S(k) : \mathrm{Holink}(U(k), B)_{\delta(k)} \to M(k)$ *be defined by* $S(k)(\sigma) = \sigma(1)$. *Also* $p(k) : \mathrm{Holink}(U(k), B)_{\delta(k)} \to B$*: is the holink projection. Note that*

$$\pi(k) = p(k)R(k) : M(k) \to B.$$

(VII) *Denote the inclusion by*

$$\iota(k) : \mathrm{Holink}(U(k), B)_{\delta(k)} \to \mathrm{Holink}(U(k-1), B)_{\delta(k-1)}.$$

Note that there is a controlled homotopy $R(k)S(k) \simeq_{\delta(k-1)} \iota(k)$, *sending a pair* $(t, \sigma) \in I \times \mathrm{Holink}(U(k), B)_{\delta(k)}$ *to a path* $\sigma_t \in \mathrm{Holink}(U(k-1), B)_{\delta(k-1)}$. *Here* σ_t *and the path* σ *agree on* $[t, 1]$, *while* $\sigma_t|_{[0,t]}$ *is a reparametrization of* $R(k)(\sigma(t))$. *Measurements are taken in* B *using* $p(k)$.

(VIII) *Set* $i(k) := S(k-1)R(k) : M(k) \to M(k-1)$. *This is the inclusion also.*

(IX) *Let* $f_t(k) : M(k) \to M(k)$, $0 \le t \le 1$, *be a deformation relative to* $\partial M(k)$ *of* $M(k)$ *into* $N(k)$. *We assume that the homotopy has diameter less than* $\delta(k)$ *when measured in* B *using* $\pi(k)$.

Any sequence $\{U(k), \delta(k)\}_{k=1}^{\infty}$ and deformation ρ satisfying the above properties will be called a *tameness structure* for (X, B). Note that one can take any subsequence from this sequence of neighborhoods, and again get a tameness structure.

With these preliminaries, here is the promised description of the group $\tilde{K}_0(B, p)$ as an inverse limit of controlled groups.

THEOREM 5.9. *In the above notation, there is an isomorphism*

$$\varprojlim_k \tilde{K}_0(B, \pi(k); \delta(k)) \xrightarrow{\approx} \tilde{K}_0(B, p).$$

PROOF. Note that each inclusion $\mathrm{Holink}(U(k), B)_{\delta(k)} \to \mathrm{Holink}(X, B)$ is a homotopy equivalence of diameter 0 when measured in B. Therefore these inclusions induce an isomorphism

$$\varprojlim_k \tilde{K}_0(B, \partial(k); \delta(k)) \to \tilde{K}_0(B, p).$$

But the equations

$$S(k-1)R(k) = i(k),\ R(k)S(k) \simeq_{\delta(k-1)} \iota(k)$$

prove that the map

$$\lim_k R(k) : \varprojlim_k \tilde{K}_0(B, \pi(k); \delta(k)) \to \varprojlim_k \tilde{K}_0(B, p(k); \delta(k))$$

is also an isomorphism. The result follows. $\square$

Using the isomorphism above, we now interpret every element in $\tilde{K}_0(B, p)$ in the isomorphic group $\varprojlim_k \tilde{K}_0(B, \pi(k); \delta(k))$.

LEMMA 5.10. *An element of the group $\varprojlim_k \tilde{K}_0(B, \pi(k); \delta(k))$ is represented by a sequence $(A(k), e(k))$, where $k = 1, 2, 3, \ldots$, for which there exist $B(k)$, $f(k)$, $g(k)$, where $k = 1, 2, 3, \ldots$, with the following properties:*

(A) *$A(k)$, $B(k)$ are geometric modules on $N(k)$, and $(A(k), e(k))$ is a $\delta(k)$-projective module over $(B, \pi(k))$, with supports in $N(k)$.*

(B) *$A(k) \oplus B(k) \xrightarrow{f(k)} A(k+1) \xrightarrow{g(k)} A(k) \oplus B(k)$ are $\delta(k)$-morphisms in $GM(\pi(k))$ with supports in $M(k) - M(k+2)$.*

(C) *$f(k)g(k) \sim_{\delta(k)} e(k+1)$, $g(k)f(k) \sim_{\delta(k)} (e(k) \oplus 1_{B(k)})$.*

(D) *$e(k+1)f(k) \sim_{\delta(k)} f(k) \sim_{\delta(k)} f(k)(e(k) \oplus 1_{B(k)})$;*
$g(k)e(k+1) \sim_{\delta(k)} g(k) \sim_{\delta(k)} (e(k) \oplus 1_{B(k)})g(k)$.

PROOF. The above properties say that $i(k+1)_*(A(k+1), e(k+1))$ is $\delta(k)$-isomorphic to $(A(k), e(k)) \oplus (B(k), 1)$. This means that the sequence $\{A(k), e(k)\}_{k=1}^{\infty}$ constitutes a class in $\varprojlim_k \tilde{K}_0(B, \pi(k); \delta(k))$. It is therefore a consequence of the above constructions that these modules and morphisms can be chosen as required, except for the conditions on their supports. But since, for any k, the supports of $A(k)$, $B(k)$, $e(k)$, $f(k)$, $g(k)$ consist of finitely many compact sets in M, and since $\bigcap_{k=1}^{\infty} U(k) = B$, we can always find $k' > k$ such that $M(k')$ is disjoint from these supports. In this way we pass to a subsequence of the tameness structure to obtain the desired result. □

It is now easy to see that an element of $\tilde{K}_0(B, p)$ can be thought of geometrically as a chain complex C with 2- and 3-handles, such that $C_2(k) = A(k)$ and $C_3(k) = A(k) \oplus B(k)$. The 2-handles are trivially attached and the 3-handles have the boundary maps prescribed by

$$\partial(k) = \partial'(k) \oplus \partial''(k) : C_3(k) \to C_2(k) \oplus C_2(k+1),$$

where $\partial'(k) : C_3(k) \to C_2(k)$ and $\partial''(k) : C_3(k) \to C_2(k+1)$ satisfy

$$\begin{aligned} \partial'(k) &= (1 - e(k), 0) : A(k) \oplus B(k) \to A(k), \\ \partial''(k) &= f(k) : A(k) \oplus B(k) \to A(k+1). \end{aligned}$$

The next result provides a cc-chain homotopy between the identity map of C and a map with compact support (also mentioned in [9]).

LEMMA 5.11. *In the above notation, there is a cc-chain homotopy $H : C \to C$ such that $(\partial H + H\partial) \sim_{cc} (1_C - \phi)$, where $\phi : C \to C$ is a cc-chain map in $\mathcal{GM}(X, B)$ satisfying*

$$\phi_3 = 0 : C_3 \to C_3;\ \phi_2|_{C_2(k)} = 0 \quad \textit{for } k \geq 2;\ \phi_2 = e(1) : A \to A.$$

PROOF. Choose $\delta(k)$-maps $H'(k): C_2(k) \to C_3(k)$,

$$H''(k+1): C_2(k+1) \to C_3(k),$$

supported on $M(k) - M(k+2)$ so that $H''(1) = 0$ and so that, for $k \geq 1$,

$$H'(k) = \begin{pmatrix} 1-e(k) \\ 0 \end{pmatrix} : A(k) \to A(k) \oplus B(k),$$
$$H''(k+1) = g(k) : A(k+1) \to A(k) \oplus B(k);$$

Define $H = H' \oplus H'' : C_2 \to C_3$, where $H' = \sum_{k=1}^{\infty} H'(k)$, $H'' = \sum_{k=1}^{\infty} H''(k)$.

Note that ∂ and H are morphisms in $\mathcal{G}(X, B)$ because $\delta(k) \to 0$ as $k \to \infty$.

We first prove that $1 - H\partial_3 \sim_{cc} 0 : C_3 \to C_3$,

$$H\partial_3 = H'\partial' + H'\partial'' + H''\partial' + H''\partial''.$$

From the above definitions we get:

(I) $H'\partial' = \begin{pmatrix} 1-e \\ 0 \end{pmatrix} (1-e \;\; 0) \sim_{cc} \begin{pmatrix} 1-e & 0 \\ 0 & 0 \end{pmatrix} : A \oplus B \to A \oplus B$;
(II) $H'\partial'' = \begin{pmatrix} 1-e \\ 0 \end{pmatrix} f \sim_{cc} 0$ because $ef \sim_{cc} f$;
(III) $H''\partial' = g\ (1-e,\ 0) \sim_{cc} 0$ because $ge \sim_{cc} g$;
(IV) $H''\partial'' = gf \sim_{cc} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} : A \oplus B \to A \oplus B$.

Therefore, $(1 - H\partial_3) \sim_{cc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1-e & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} = 0$, as required.

Next we prove that: $1 - \partial_3 H \sim_{cc} \phi_2 : C_2 \to C_2$. As before, we notice that:

(I) $\partial' H' = (1-e \;\; 0) \begin{pmatrix} 1-e \\ 0 \end{pmatrix} \sim_{cc} (1-e) : A \to A$;
(II) $\partial' H'' = (1-e,\ 0)\ g \sim_{cc} 0$ because $(e \oplus 1_B)g \sim_{cc} g$;
(III) $\partial'' H' = f \begin{pmatrix} 1-e \\ 0 \end{pmatrix} \sim_{cc} 0$ because $f(e \oplus 1_B) \sim_{cc} f$;
(IV) $(\partial'' H'') = fg \sim_{cc} \sum_{k=1}^{\infty} e(k+1)$, which is equal to $e - e[1]$.

Adding, we get

$$p_*^1(1 - \partial_3 H) \sim_{cc} 1 - (1-e) - (e - e[1]) = e[1],$$

as required. It is easy to see that $1 - H\partial - \partial H \sim_{cc} \phi$. □

Now $(A(1), e(1))$ is the image of $[C] \in \varprojlim_k \tilde{K}_0(B, \pi(k); \delta(k))$ under the forget-control map. If the class of $(A(1), e(1))$ represents 0 in $\tilde{K}_0(\mathbb{Z}\pi_1(X - B))$, it follows that $(A(1), e(1)) \oplus (D, 1) \approx (E, 1)$, where $(D, 1)$ and $(E, 1)$ are chain complexes over $\mathcal{GM}(X - B)$. This means that the chain complex $C \oplus D$ admits a cc-chain contraction induced by H. So we have found an element of $\mathrm{Wh}(X, A)$ whose image under the boundary map Δ is $[C]$. This shows that Ker $(a) \subset$ Im (Δ), as required. □

We now prove that the sequence

$$\mathrm{Wh}(B, p) \xrightarrow{\mathfrak{A}} \mathrm{Wh}(\mathbb{Z}\pi_1(X - B)) \xrightarrow{\mathfrak{I}} \mathrm{Wh}(X, B) \xrightarrow{\Delta} \tilde{K}_0(B, p)$$
$$\xrightarrow{\mathfrak{a}} \tilde{K}_0(\mathbb{Z}\pi_1(X - B)) \xrightarrow{\mathfrak{i}} \tilde{K}_0(X, B)$$

is exact at every point.

Notice that the maps $\mathfrak{A}$ and $\mathfrak{a}$ are just the forget-control maps, as explained in the previous paragraph. The maps $\mathfrak{I}$ and $\mathfrak{i}$ are just the inclusion maps.

It is easy to notice that the composition $\Delta \circ \mathfrak{I} = 0$, because the restriction of a finite chain complex to a small enough neighborhood of B can be clearly chosen to be empty.

Also, the composition $\mathfrak{i} \circ \mathfrak{a} = 0$. This follows from Lemma 5.11 above: for any element $[C] \in \tilde{K}_0(B, p)$ one gets a continuously controlled chain homotopy between $(C, 1)$ and $(\mathfrak{a}(C), e|_{\mathfrak{a}(C)})$. This says that $[C] = 0$ in the group $\tilde{K}_0(X, B)$.

The next two lemmas will complete the proof.

LEMMA 5.12. *Let (C, p) be a finite chain complex representing an element of the group $\tilde{K}_0(\mathbb{Z}\pi_1(X - B))$. Assume that there exist continuously controlled chain complexes $(E, 1)$ and $(E', 1)$ such that $(C, p) \oplus (E, 1) \simeq (E', 1)$. Then there exists an element $x \in \tilde{K}_0(B, p)$ such that $\mathfrak{a}(x) = [(C, p)]$.*

PROOF. Suppose we choose ϵ such that (C, p) is an ϵ-chain complex. What we need to prove is that there exists an $(\epsilon/2)$-chain complex (C_1, p_1) such that (C, p) and (C_1, p_1) are ϵ-chain equivalent. We choose these chain complexes according to Lemma 5.10. □

LEMMA 5.13. *Let C be a continuously controlled contractible chain complex (this implies that $[C]$ is an element of the group $\mathrm{Wh}(X, B)$) such that $\Delta(C) = 0$. Then there exists a finite contractible chain complex D, such that $D \oplus T \approx C \oplus T'$, where T and T' are trivial chain complexes and $\approx$ means a Σ-isomorphism in the sense of* [24].

PROOF. By the definition of Δ we conclude that there exists a neighborhood U of B in X such that the restriction of C to U, say C_U, is a finitely dominated ϵ-chain complex which is chain equivalent to a finite complex. □

6. The torsion of an *h*-cobordism and the mapping cylinder obstructions

Using the description of the boundary map above, we will prove a formula that relates $\Delta(\tau)$, where τ is the torsion of an h-cobordism, to the mapping cylinder obstructions. This was first asserted by Quinn in [21].

THEOREM 6.1. *Let (X, B) be a relative manifold (see Definition 3.1). Let W be an h-cobordism of relative manifolds from (X, B) to (Y, B). Let $\tau \in \mathrm{Wh}(X, B)$ be the torsion of (W, B). Then*

$$\Delta(\tau) = r_* q_0(W, B) - q_0(X, B). \qquad (*)$$

The map Δ is constructed in the previous section and $r_* : \tilde{K}_0(B, p_W) \to K_0(B, p_X)$ is the isomorphism induced by the retraction r from W to X.

PROOF. We have the following exact sequence of relative pairs:

$$(X, B) \xrightarrow{\text{inclusion}} (W, B) \longrightarrow (W, X).$$

It is easy to see that, by passing to the chain complex level, we still get an exact sequence:

$$0 \longrightarrow C(X, B) \longrightarrow C(W, B) \longrightarrow C(W, X) \longrightarrow 0.$$

Let $\delta > 0$ be any fixed positive number. Let U be a small enough neighborhood of B in X, so that all the chain complexes above become δ-complexes when restricted to U. By choosing appropriate restrictions (remember that the restriction of a chain complex is not unique) we get the following exact sequence:

$$0 \longrightarrow C_U(X, B) \longrightarrow C_U(W, B) \longrightarrow C_U(W, X) \longrightarrow 0.$$

By [24] again, there exist δ-chain complexes

$$(E_\delta(X, B), p_\delta(X, B)), \quad (E_\delta(W, B), p_\delta(W, B)), \quad (E_\delta(W, X), p_\delta(W, X)),$$

over $\mathcal{G}(\widehat{U - B})$ so that $(C_U, 1)$ is 4δ-chain homotopic to (E_δ, p_δ). We will show below that, by appropriate choices, one can make the following sequence exact:

$$\begin{aligned} 0 \to (E_\delta(X, B), p_\delta(X, B)) \to (E_\delta(W, B), p_\delta(W, B)) \\ \to (E_\delta(W, X), p_\delta(W, X)) \to 0. \end{aligned} \tag{i}$$

The algebraic details are supplied below. Note that this is all we have to prove, since the class of $(E_\delta(W, X), p_\delta(W, X))$ represents $(\Delta(\tau))_\delta$, by construction of the boundary map. Also, $(E_\delta(X, B), p_\delta(X, B))$ and $(E_\delta(W, B), p_\delta(W, B))$ represent $q_{0\delta}(W, \sigma W)$ and $q_{0\delta}(X, \sigma X)$ respectively as elements of the group $\tilde{K}_0(B, p, \delta)$, as described in Section 4.2. □

6.1. Completion of the proof Let $\mathcal{A}$ be an additive category with the 'semisimple' exact structure (see Definition 2.6). Denote by $\mathcal{A}_*$ the derived category of chain complexes over $\mathcal{A}$. Recall the standard definition for the mapping cone.

DEFINITION 6.2. Let $f : A \to B$ be a map in $\mathrm{Hom}_{\mathcal{A}_*}(A, B)$. The mapping cone $C(f)$ of f is by definition the chain complex $B \oplus \Sigma A$. Here ΣA denotes the suspension of A. The map $d_{C(f)}$ is given by

$$(d_{C(f)})_i = \begin{pmatrix} d_B & (-1)^{i-1} f \\ 0 & d_{\Sigma A} \end{pmatrix} : B_i \oplus (\Sigma A)_i \to B_{i-1} \oplus (\Sigma A)_{i-1}. \tag{ii}$$

By [24, Proposition 2.4], $C(f)$ is contractible if and only if f is a chain equivalence. Here is a sketch of the proof.

PROOF (SKETCH). Let

$$\Gamma = \begin{pmatrix} k & ? \\ (-1)^r g & h \end{pmatrix} : C(f) \longrightarrow C(f) \tag{iii}$$

be a chain contraction. This means that g is the chain homotopy inverse for f, the homotopies being provided by k and h.

Conversely, given g, h, k, one defines Γ by

$$\Gamma = \begin{pmatrix} k + (fg - kf)g & (-1)^r(fh - kf)h \\ (-1)^r g & h \end{pmatrix}. \tag{iv}$$

See [24] for more details. □

LEMMA 6.3. *Let*

$$0 \longrightarrow C' \xrightarrow{\iota} C \longrightarrow C'' \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}_*$. Then there is a chain equivalence $C(\iota) \simeq C''$.

The proof is standard.

LEMMA 6.4. *Let*

$$\begin{tikzcd} C' \arrow[r, "f"] & C \\ D' \arrow[u] \arrow[r, "g"] & D \arrow[u] \end{tikzcd} \tag{v}$$

be a commutative diagram in $\mathcal{A}_*$. Then the map $C(g, f)_i : C(g)_i \longrightarrow C(f)_i$ given by the matrix $\left(\begin{smallmatrix} g & 0 \\ 0 & \Sigma f \end{smallmatrix}\right)$ is a chain map and makes the diagram.

$$\begin{tikzcd} C' \arrow[r, "f"] & C \arrow[r] & C(f) \arrow[r] & \Sigma C' \\ D' \arrow[u] \arrow[r, "g"] & D \arrow[u] \arrow[r] & C(g) \arrow[u, "C(g,f)"'] \arrow[r] & \Sigma D' \arrow[u] \end{tikzcd}$$

commutative.

PROOF. This is just an easy check, using the definition of $C(f)$. □

REMARK 6.5. Let $\alpha : A \to B$ be a morphism in $\mathcal{A}_*$. It is easy to see that $\mathrm{Hom}_{\mathcal{A}_*}[C(\alpha), X]$ can be identified with

$$\{(f, h) \in \mathrm{Hom}_{\mathcal{A}_*}(B, X) \times \mathrm{Hom}_{\mathcal{A}_*}(\Sigma A, X) \mid f\alpha \overset{h}{\simeq} 0\}.$$

PROPOSITION 6.6. *Suppose that the following diagram is commutative (the notation is as above):*

$$\begin{tikzcd}
C(\alpha) \arrow[r, "C(\beta,\alpha)"'] & C(\beta) & \\
C' \arrow[u] \arrow[r, "f"'] & C \arrow[u] \arrow[r] & C(f) \\
D' \arrow[u, "\alpha"] \arrow[r, "g"'] & D \arrow[u, "\beta"] \arrow[r] & C(g) \arrow[u, "C(g,f)"]
\end{tikzcd}$$

Then $C(C(g,\ f)) \approx C(C(\beta,\alpha))$. Moreover, the following (completed) diagram is commutative:

$$\begin{tikzcd}
C(\alpha) \arrow[r, "C(\beta,\alpha)"'] & C(\beta) \arrow[r] & C(C(g,\ f)) \approx C(C(\beta,\alpha)) \\
C' \arrow[u] \arrow[r, "f"'] & C \arrow[u] \arrow[r] & C(f) \arrow[u] \\
D' \arrow[u, "\alpha"] \arrow[r, "g"'] & D \arrow[u, "\beta"] \arrow[r] & C(g) \arrow[u, "C(g,f)"]
\end{tikzcd}$$

PROOF. Since

$$C(g) = D \oplus \Sigma D';\ (d_{C(g)})_i = \begin{pmatrix} d_D & (-1)^{i-1} g \\ 0 & d_{\Sigma D'} \end{pmatrix}, \tag{vi}$$

$$C(f) = C \oplus \Sigma C';\ (d_{C(f)})_i = \begin{pmatrix} d_C & (-1)^{i-1} f \\ 0 & d_{\Sigma C'} \end{pmatrix}, \tag{vii}$$

$$C(g,\ f) = \begin{pmatrix} \beta & 0 \\ 0 & \Sigma\alpha \end{pmatrix}, \tag{viii}$$

it follows that

$$C(C(g,\ f)) = C(f) \oplus \Sigma C(g) = C \oplus \Sigma C' \oplus \Sigma D \oplus \Sigma^2 D', \tag{ix}$$

$$(d_{C(C(g,f))})_i = \begin{pmatrix} d_C & (-1)^i f & \beta & 0 \\ 0 & d_{\Sigma C'} & 0 & \Sigma\alpha \\ 0 & 0 & d_{\Sigma D} & (-1)^{i-1}\Sigma g \\ 0 & 0 & 0 & d_{\Sigma^2 D'} \end{pmatrix}. \tag{x}$$

Similarly,

$$C(\alpha) = C' \oplus \Sigma D';\ (d_{C(\alpha)})_i = \begin{pmatrix} d_{C'} & (-1)^{i-1}\alpha \\ 0 & d_{\Sigma D'} \end{pmatrix}, \tag{xi}$$

$$C(\beta) = C \oplus \Sigma D;\ (d_{C(\beta)})_i = \begin{pmatrix} d_C & (-1)^{i-1}\beta \\ 0 & d_{\Sigma D} \end{pmatrix}, \tag{xii}$$

$$C(\beta,\ \alpha) = \begin{pmatrix} f & 0 \\ 0 & \Sigma g \end{pmatrix}, \tag{xiii}$$

Therefore

$$C(C(\beta,\alpha)) = C(\beta) \oplus \Sigma C(\alpha) = C \oplus \Sigma D \oplus \Sigma C' \oplus \Sigma^2 D', \tag{xiv}$$

$$(d_{C(C(\beta,\alpha))})_i = \begin{pmatrix} d_C & (-1)^{i-1}\beta & f & 0 \\ 0 & d_{\Sigma D} & 0 & \Sigma g \\ 0 & 0 & d_{\Sigma C'} & (-1)^{i-1}\Sigma\alpha \\ 0 & 0 & 0 & d_{\Sigma^2 D'} \end{pmatrix}. \tag{xv}$$

Now let $\mathfrak{M} \in \mathcal{A}_*$, $\mathfrak{M} = C \oplus \Sigma C' \oplus \Sigma D \oplus \Sigma^2 D'$, and

$$(d_{\mathfrak{M}})_i = \begin{pmatrix} d_C & f & (-1)^{i-1}\beta & 0 \\ 0 & d_{\Sigma C'} & 0 & (-1)^{i-1}\Sigma\alpha \\ 0 & 0 & d_{\Sigma D} & \Sigma g \\ 0 & 0 & 0 & d_{\Sigma^2 D'} \end{pmatrix}. \tag{xvi}$$

It is obvious that $\mathfrak{M} \approx C(C(\beta,\alpha))$. The isomorphism is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Also, if $A \xrightarrow{\gamma} (B, d)$ is an isomorphism, then $A \xrightarrow{(-1)^i\gamma} (B, (-1)^i d)$ is an isomorphism also. It follows that $\mathfrak{M} \approx C(C(g, f))$. □

Now, if the maps α and β are chain equivalences, it follows that $C(\alpha)$ and $C(\beta)$ are chain contractible. By Proposition 6.6 we get that $C(g, f)$ is a chain equivalence. This proves equation (i) and hence Theorem 6.1.

COROLLARY 6.7 (Realization theorem for mapping cylinder obstructions). *Let X be a stratified space, so that the singular stratum, say B, has a mapping cylinder neighborhood in X.*

According to [1], *there is an exact sequence:*

$$\cdots \operatorname{Wh}(X, B) \xrightarrow{\Delta} \tilde{K}_0(B, p_X) \xrightarrow{\mathfrak{a}} \tilde{K}_0(\mathbb{Z}\pi_1(X - B)) \longrightarrow \cdots.$$

Let $x \in \operatorname{Ker}(\mathfrak{a})$. Then there exists a stratified space Y, such that the singular stratum of Y is homeomorphic to B and the mapping cylinder obstruction of B in Y is represented by x. By this we mean that there is an isomorphism (which will be inclusion-induced) $\iota : \tilde{K}_0(B, p_X) \to \tilde{K}_0(B, p_Y)$, so that $q_0(Y, B) = \iota(x)$.

PROOF. From the hypotheses, it follows that there is a $\tau \in \operatorname{Wh}(X, A)$ such that $\Delta(\tau) = x$. By Theorem 3.5 one can construct on X an h-cobordism of stratified spaces of torsion τ, say (Y, B). There is an obvious isomorphism induced by the inclusion of X in Y, $\iota : \tilde{K}_0(B, p_X) \to \tilde{K}_0(B, p_Y)$. By the formula $(*)$, the mapping cylinder obstruction of B in Y turns out to be $\iota(x)$, as required. □

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BOGDAN VAJIAC, Department of Mathematics, Indiana University Northwest, 3400 Broadway, Gary IN 46408, USA
and
Current address: Department of Mathematics, Saint Mary's College, Notre Dame, IN 46556, USA
e-mail: bvajiac@saintmarys.edu