On a theorem of Wilder

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1. **Wilder's theorem.** In this section, we state a theorem due to M. Zisman and the author. Proofs can be found in [**1**, Exposé 9].

1.1. Let X be a locally compact space of finite cohomological dimension [1], A a unitary ring which is a flat algebra over some commutative ring and let $F^\cdot$ be a complex of sheaves of right A-modules on X which has only finitely many nonzero cohomology sheaves. Let $F^\cdot \to \Omega(F^\cdot)$ be a resolution (i.e. a morphism of complexes which induces isomorphisms on the cohomology) of $F^\cdot$ by injective sheaves of A-modules. For any paracompact open subset $U \subset X$, the nth cohomology module of the complex $\Gamma(U, \Omega(F^\cdot))$ (resp. $\Gamma_c(U, \Omega(F^\cdot))$) is denoted by $H^n(U, F^\cdot)$ (resp. $H_c^n(U, F^\cdot)$) and is called the nth hypercohomology module (resp. the nth hypercohomology module with compact supports) of U with values in $F^\cdot$.

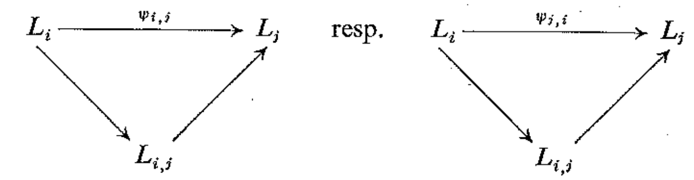
Since the complex $\Gamma(U, \Omega(F^\cdot))$ (resp. $\Gamma_c(U, \Omega(F^\cdot))$) has only finitely many nonzero cohomology modules, there exists a complex $L(U, F^\cdot)$ (resp. $L_c(U, F^\cdot)$) whose components are projective modules and a map of complexes $\phi_U: L(U, F^\cdot) \to \Gamma(U, \Omega(F^\cdot))$ (resp. $\phi_{c,U}: L_c(U, F^\cdot) \to \Gamma_c(U, \Omega(F^\cdot))$) such that

(1) the component $L^n(U, F^\cdot)$ (resp. $L_c^n(U, F^\cdot)$) is zero for n big enough; and

(2) the map ϕ_U (resp. $\phi_{c,U}$) induces isomorphisms on the cohomology. Moreover if $U \subset V$ are two paracompact open subsets of X, the restriction map $\rho_{V,U}: \Gamma(V, \Omega(F^\cdot)) \to \Gamma(U, \Omega(F^\cdot))$ (resp. the extension map $\varepsilon_{U,V}: \Gamma_c(U, \Omega(F^\cdot)) \to \Gamma_c(V, \Omega(F^\cdot))$) can be lifted to the projective resolutions. The lifting is unique up to homotopy, is called once again the restriction map (resp. the extension map) and is denoted once again by $\rho_{V,U}$ (resp. $\varepsilon_{U,V}$).

1.2 Definition. Let $(L_i,\ i \in I,\ i < j \mapsto \psi_{i,j}: L_i \mapsto L_j)$ be a directed (resp. inverse) system of complexes of right A-modules (we require only that the transition morphisms agree up to homotopy), and let $[m, n] \subset \mathbf{Z}$ be an interval. The system $(L_i, i \in I, \psi_{i,j})$ is said to be essentially of finite type of amplitude contained in $[m, n]$ if, for any $i \in I$, there exists a $j > i$ (resp. $j < i$), a complex $L_{i,j}$ whose components $L_{i,j}^k$ are projective modules of finite type, zero whenever $k \notin [m, n]$,

and a diagram commutative up to homotopy:



1.3 In the following theorem we use the notation of 1.1.

Theorem. *The following conditions are equivalent:*

(i) *There exists an interval* $[m, n] \subset \mathbf{Z}$ *such that for any compact* K *in* X, *the directed system indexed by the paracompact open neighborhood of* K:

$$(U \to L(U, F^\cdot),\ V \subset U \mapsto \rho_{U,V}: L(U, F^\cdot) \to L(V, F^\cdot))$$

is essentially of finite rank of amplitude contained in $[m, n]$.

(ii) *There exists an interval* $[m', n'] \subset \mathbf{Z}$ *such that for any compact* K *in* X, *the inverse system indexed by the paracompact open neighborhood of* K:

$$(U \to L(U, F^\cdot),\ V \subset U \to \rho_{U,V}: L(U, F^\cdot) \to L_c(U, F^\cdot))$$

is essentially of finite rank of amplitude contained in $[m, n]$.

(iii) *Same as* (i) *but consider only the compact subsets reduced to one point.*

(iv) *Same as* (ii) *but consider only the compact subsets reduced to one point.*

1.4 Definition. A complex of sheaves $F^\cdot$ on X is said to be perfect if it has the equivalent properties of Theorem 1.3.

It is clear that these properties depend only on $F^\cdot$ and not on the different resolutions.

1.5 Corollary. *Let* $f: X \to Y$ *be a proper map between two locally compact spaces of finite cohomological dimension, and* $F^\cdot$ *a perfect complex of injective sheaves*[1] *of* A*-modules on* X. *Then the complex* $f_*(F^\cdot)$ *is a perfect complex on* Y. *In particular, when* Y *is a point and* X *compact, the complex* $L(X, F^\cdot)$ *is homotopic to a bounded complex of projective* A*-modules of finite rank.*

1.6 Proposition. *Let* $0 \to F'^\cdot \to F^\cdot \to F''^\cdot \to 0$ *be an exact sequence of complexes of sheaves on* X. *If two of the complexes of the sequence are perfect, so is the third one.*

1.7 Proposition. *Assume that* A *is right noetherian and that* $F^\cdot$ *has a finite flat amplitude (i.e. that* $F^\cdot$ *has a resolution by a bounded complex whose components are sheaves with flat stalks). Then* $F^\cdot$ *is perfect if and only if it has one of the two following equivalent properties:*

(a) *For any* $x \in X$, *and for any open neighborhood* U *of* x, *there exists an open neighborhood* V *of* x, $V \subset U$, *such that for any* $n \in \mathbf{Z}$, *the restriction map* $H^n(U, F^\cdot) \to H^n(V, F^\cdot)$ *has a finitely generated image.*

[1] Or more generally of sheaves acyclic for the direct image functor.

(b) *For any $x \in X$ and for any open neighborhood U of x, there exists an open neighborhood V of x, $U \subset V$, such that for any $n \in \mathbf{Z}$, the extension map*

$$H_c^n(U, F^\cdot) \to H_c^n(V, F^\cdot)$$

has a finitely generated image.

Applying Theorem 1.3 and Proposition 1.7 to the case $A = \mathbf{Z}$ and $F =$ constant sheaf free of rank one, we get Wilder's theorem [**2**].

2. **The Wall invariant.**

2.1 Let $f: X \to Y$ be a continuous map between locally compact spaces and let G be a sheaf of A-modules on X (say on the right). For any open set U of Y denote by $\psi(U)$ the family of closed subsets S of the space $f^{-1}(U)$ such that the map f induces a proper map from S to U. We denote by $f_! G$ the sheaf

$$U \to \Gamma_{\psi(U)}(f^{-1}(U), G).$$

The sheaf $f_! G$ is called the direct image of G with proper supports.

2.2 Let X be a connected, locally simply connected, finite dimensional, compact space and $X^\sim \xrightarrow{p} X$ its universal covering. Choose a base point in $X^\sim$. Then for any complex of sheaves of right A-modules $F^\cdot$ on X, the complex $p_! p * F^\cdot$ is canonically equipped with a right $A[\Pi_1(X)]$-structure. Furthermore, when $F^\cdot$ is perfect one checks immediately by local inspection that $p_! p^* F^\cdot$ is a perfect $A[\Pi_1(X)]$-complex. Therefore, in that case, the complex $L(X, p_! p^* F^\cdot)$ (1.1) (the resolutions being taken in the category of right $A[\Pi_1(X)]$-modules) is homotopic to a bounded complex $C^\cdot(X, F^\cdot)$ of projective $A[\Pi_1(X)]$-modules of finite rank.

2.3 Definition. Let X be a connected, locally simply connected, finite dimensional, compact space and let $F^\cdot$ be a perfect complex of sheaves of right A-modules on X. The element $\sum_{i \in \mathbf{Z}} (-1)^i \mathrm{cl}(C^i(X, F^\cdot))$, in the Grothendieck group $K(A[\Pi_1(X)])$ of projective right $A[\Pi_1(X)]$-modules of finite type, is called the *Wall Invariant* of $F^\cdot$ and denoted by $W_A(X, F^\cdot)$.

It is easy to check that the element $W_A(X, F^\cdot)$ does not depend on the choice of the point in $X^\sim$ used to define the action of $\Pi_1(X)$ and does not depend on the choice of the different resolutions.

As an immediate consequence of the definitions we have the following proposition:

2.4 Proposition. (1) *Let $u: F^\cdot \to F'^\cdot$ be a map between two perfect complexes which induces an isomorphism on the sheaves of cohomology. Then $W_A(X, F^\cdot) = W_A(X, F'^\cdot)$.*

(2) *Let $0 \to F'^\cdot \to F^\cdot \to F''^\cdot \to 0$ be an exact sequence of perfect complexes. Then $W_A(X, F^\cdot) = W_A(X, F'^\cdot) + W_A(X, F''^\cdot)$.*

In particular if $F^\cdot[1]$ denotes the complex $F^\cdot$ shifted one degree to the left, we have $W_A(X, F^\cdot[1]) = -W_A(X, F^\cdot)$.

2.5 Let $f: Y \to X$ be a continuous map of connected, locally simply connected, finite dimensional compact spaces. The map f induces a homomorphism

$$\Pi_1(f): \Pi_1(Y) \to \Pi_1(X)$$

so that the ring $A[\Pi_1(X)]$ becomes an $A[\Pi_1(Y)]$-algebra. The tensor product—$\otimes_{A[\Pi_1(Y)]} A[\Pi_1(X)]$ defines a map denoted $f_*: K(A[\Pi_1(Y)]) \to K(A[\Pi_1(X)])$ on the corresponding Grothendieck groups. Let now $F^\cdot$ be a perfect complex of right A-modules on Y. Assume that the components of $F^\cdot$ are acyclic for the functor direct image by f. The direct image $f_* F^\cdot$ is then a perfect complex on X (1.5).

2.6 Proposition. *With the notation of* (2.5), *we have*

$$W(X, f_* F^\cdot) = f_* W(Y, F^\cdot).$$

Proof. Let $p: X^\sim \to X$ the universal covering of X and $X^\sim \times_X Y$ the fiber product. We have a cartesian square:

$$\begin{array}{ccc} X^\sim \times_Y X & \xrightarrow{f^\sim} & X^\sim \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$$

and the map $\mathrm{pr}_2: X^\sim \times_X Y \to Y$ is a principal covering with group $\Pi_1(X)$. Furthermore, the base change theorem for direct images by proper morphism yields a canonical isomorphism, compatible with the action of $\Pi_1(X)$; $f_* \mathrm{pr}_{2!} \mathrm{pr}_2^* F^\cdot \simeq p_! p^* f_* F^\cdot$.

Let $\mathrm{pr}_{2!} \mathrm{pr}_2^* F^\cdot \to I^\cdot$ be a resolution by injective sheaves of right $A[\Pi_1(X)]$-modules. The complex $f_* I^\cdot$ is an injective resolution of the complex $p_! p^* f_* F^\cdot$ and we have $\Gamma(X, f_* I^\cdot) \simeq \Gamma(Y, I^\cdot)$. By Corollary 1.5, there exists a bounded complex whose components are finitely generated projective $A[\Pi_1(X)]$-modules $P^\cdot$ and a resolution $P^\cdot \to \Gamma(Y, I^\cdot)$. We have therefore the equality $W_A(X, f_* F^\cdot) = \sum_{i \in \mathbf{Z}} (-1)^i \mathrm{cl}(P^i)$. Let $q: Y^\sim \to Y$ be the universal covering of Y. There exists a canonical isomorphism $q_! q^* F^\cdot \otimes_{A[\Pi_1(Y)]} A[\Pi_1(X)] \simeq \mathrm{pr}_{2!} \mathrm{pr}_2^* F^\cdot$. Hence we have a homotopy equivalence $C^\cdot(Y, F^\cdot) \otimes_{A[\Pi_1(Y)]} A[\Pi_1(X)] \to P^\cdot$ and the proposition follows.

2.7 Corollary. *Let X be a finite polyhedron (geometric realization of a finite semisimplicial complex), and let $F^\cdot$ be a complex of sheaves on X which induces on each cell a perfect complex. Then $F^\cdot$ is perfect. Let $\chi(F^\cdot) \in K(A)$ be the Euler-Poincaré characteristic of $F^\cdot$ in the Grothendieck group of finitely generated projective right A-modules. Then*

$$W_A(X, F^\cdot) = \chi(F^\cdot) \cdot \mathrm{cl}(A[(\Pi_1 X)]).$$

Proof. Follows, by induction on the number of cells, from Propositions 2.4 and 2.6.

2.8 Let X be a connected, locally simply connected, finite dimensional compact space on which the constant sheaf $\mathbf{Z}$ is perfect. Let $T_X^{\cdot}$ be the orientation complex introduced by Borel and Moore [3]. Then the complex $T_X^{\cdot}$ is perfect [1]. The cohomology groups $H_i(X, \mathbf{Z}) = H^{-i}(X, T_X^{\cdot})$ are the homology groups of X. The results quoted above (1.6) show that the homology groups of $X^{\sim}$ equipped with their natural action of $\Pi_1(X)$ are the homology modules of a finite $\mathbf{Z}[\Pi_1(X)]$-complex whose components are finitely generated projective $\mathbf{Z}[\Pi_1(X)]$-modules, this complex being well defined up to homotopy, so that our Wall invariants are natural translations in the theory of sheaves of the invariant introduced by Wall.

In particular, when the Borel-Moore homology of X is isomorphic to the singular homology, the element $W_{\mathbf{Z}}(X, T_X^{\cdot})$ is the invariant introduced by Wall [4].

3. **Properties of the Wall invariant.** In this section we want to show that some properties of the Wall invariant are immediate consequences of the definitions and of the Poincaré duality theorem (see [5] for analogous results in singular homology).

3.1 Let $F^{\cdot}$ be a complex of sheaves with finitely many nonzero cohomology sheaves on a locally compact, finite dimensional space X. Let $T_{X,A}^{\cdot}$ be the orientation complex of X with respect to the ring A [3]. (This is a complex of sheaves of A-bimodules injective on the right.) We denote by $D_{X,A}F^{\cdot}$ the complex $\mathcal{H}om_A^{\cdot}(F^{\cdot}, T_{X,A}^{\cdot})$ ($\mathcal{H}om^{\cdot}$ is the complex of sheaf homomorphisms). The complex $D_{X,A}F^{\cdot}$ is a complex of sheaves of right A^0-modules and is called the dual complex of $F^{\cdot}$. When $F^{\cdot}$ is perfect, $D_{X,A}F^{\cdot}$ is perfect [1, Exposé 9].

Let $F^{\cdot} \mapsto S(F^{\cdot})$ be a bounded c-soft resolution of $F^{\cdot}$ and $A \to I^{\cdot}(A)$ be a resolution by A-bimodules injective on the right. The complex of presheaves on X:

$$U \mapsto \mathrm{Hom}_A^{\cdot}(\Gamma_c(U, S(F^{\cdot}), I^{\cdot}(A)), \tag{3.1.1}$$

is a complex of flabby sheaves of right A^0-modules [3], and is denoted by $F^{\cdot *}$. It follows from the Poincaré duality theorem [1, Exposé 4] that the complexes $D_{X,A}F^{\cdot}$ and $F^{\cdot *}$ have isomorphic injective resolutions.

3.2 Let Π be a group and let P be a projective right $A[\Pi]$-module of finite type. Set

$$P^* = \mathrm{Hom}_{A[\Pi]}(P, A[\Pi]). \tag{3.2.1}$$

The module P^* is a right projective $A^0[\Pi]$-module of finite type, (Π acting on the right via its left action on $A[\Pi]$). The map $P \mapsto P^*$ induces an isomorphism $K(A[\Pi]) \to K(A^0[\Pi])$ on the corresponding Grothendieck groups.

3.3 Proposition. *Let X be a connected, locally simply connected, finite dimensional compact space. Then, for any perfect complex $F^{\cdot}$ on X, we have:*

$$W_A(X, F^{\cdot})^* = W_{A^0}(X, D_{X,A}F^{\cdot}).$$

Proof. We sketch the proof. Let $p: X^{\sim} \to X$ be the universal covering of X. By (2.4) we may assume that $F^{\cdot}$ is a bounded complex of c-soft sheaves. Let us consider the complex $D_{X,A[\Pi_1(X)]}p_!p^*F^{\cdot}$ viewed as an $A^0[\Pi_1(X)]$-complex (via the isomorphism $A^0[\Pi_1(X)] \xrightarrow{\sim} A[\Pi_1(X)]^0$). Using (3.1.1) and the definition of $p!$

(2.1), we see that the complexes $p_!D_{X^{\sim},A}\,p^*F^{\cdot}$ and $D_{X,A[\Pi_1(X)]}p_!p^*F^{\cdot}$ have isomorphic injective resolutions. Moreover, since p is a covering, we have a canonical isomorphism $D_{X^{\sim},A}p^*F^{\cdot} \simeq p^*D_{X,A}F^{\cdot}$. Therefore the complexes $\Gamma(X, p_!p^*D_{X,A}F^{\cdot})$ and $\Gamma(X, D_{X,A[\Pi_1(X)]}p_!p^*F^{\cdot})$ have isomorphic injective resolutions. Let

$$I^{\cdot}(A[\Pi_1(X)])$$

be a resolution of $A[\Pi_1(X)]$ by $A[\Pi_1(X)]$-bimodules injective on the right. It follows from the Poincaré duality theorem (3.1) that the complexes

$$\Gamma(X, D_{X,A[\Pi_1(X)]}p_!p^*F^{\cdot}) \quad \text{and} \quad \mathrm{Hom}_{A[\Pi_1(X)]}^{\cdot}(\Gamma(X, p_!p^*F^{\cdot}), I^{\cdot}(A[\Pi_1(X)]))$$

have isomorphic injective resolutions. There exist (2.2) a bounded complex of finitely generated projective $A[\Pi_1(X)]$-modules $C(X, F^{\cdot})$ and a resolution

$$C^{\cdot}(X, F^{\cdot}) \to \Gamma(X, p_!d^*F^{\cdot}).$$

We have therefore two maps of complexes

$$\mathrm{Hom}_{A[\Pi_1(X)]}^{\cdot}(C^{\cdot}(X, F^{\cdot}), A[\Pi_1(X)]) \to \mathrm{Hom}_{A[\Pi_1(X)]}^{\cdot}(C^{\cdot}(X, F^{\cdot}), I^{\cdot}(A[\Pi_1(X)]))$$

$$\mathrm{Hom}_{A[\Pi_1(X)]}^{\cdot}(\Gamma(X, p_!p^*F^{\cdot}), I^{\cdot}(A[\Pi, (X)])) \to \mathrm{Hom}_{A[\Pi_1(X)]}^{\cdot}(C^{\cdot}(X, F^{\cdot}), I^{\cdot}(A[\Pi_1(X)])),$$

which induce isomorphisms on the cohomology and the proposition follows.

3.4 Let Π be a group and A a commutative ring. We denote by $G(A[\Pi])$ the Grothendieck group of $A[\Pi]$-modules that are finitely generated and projective as A-modules. The tensor product over A of two such $A[\Pi]$-modules endowed with the diagonal action of Π is an $A[\Pi]$-module of the same kind. Hence, the tensor product defines a ring structure over G. Let M be an $A[\Pi]$-module that is finitely generated and projective as an A-module and P a finitely generated projective and projective $A[\Pi]$-module. The tensor product $A[\Pi] \otimes_A P$ with the diagonal action of Π is a finitely generated projective $A[\Pi]$-module. Hence the tensor product defines on $K(A[\Pi])$ a structure of module over $G(A[\Pi])$.

3.5 Let X be a connected and locally simply connected space and M a locally constant sheaf of finitely generated and projective A-modules. Then the stalk of M is an $A[\Pi_1(X)]$-module that is finitely generated and projective as an A-module. Its image in $G(A[\Pi_1(X)])$ is denoted by $\mathrm{cl}(M)$.

3.6 Proposition. *Let X be a connected locally simply connected finite dimensional compact space, $F^{\cdot}$ a perfect complex of sheaves of A-modules, M a locally constant sheaf of finitely generated and projective A-modules. Then*

$$W_A(X, M \otimes_A F^{\cdot}) = \mathrm{cl}(M) \cdot W_A(X, F^{\cdot}).$$

Proof. By (2.4) we may assume that $F^{\cdot}$ is a bounded complex of c-soft A-modules. Let $p: X^{\sim} \to X$ be the universal covering of X. The $A[\Pi_1(X)]$-complex $\Gamma_c(X^{\sim}, p^*F^{\cdot})$ has a resolution $C^{\cdot}(X, F^{\cdot})$ by a bounded complex of finitely generated and projective $A[\Pi_1(X)]$-modules. Since we have a canonical isomorphism $\Gamma_c(X^{\sim}, p^*M \otimes_A F^{\cdot}) \simeq \Gamma_c(X, p^*F^{\cdot}) \otimes_A M_x$, where M_x is any stalk of

M with its natural action of $\Pi_1(X)$, the complex $C^{\cdot}(X, F^{\cdot}) \otimes_A M_\infty$ is a resolution of the complex $\Gamma_c(X^{\sim}, p^*M \otimes_A F^{\cdot})$. q.e.d.

3.7 COROLLARY. *Let X be a compact connected n dimensional topological variety with boundary. Let ∂X_j, $1 \leq j \leq q$, be the different connected components of its boundary, $i_j : \partial X_j \to X$ the inclusion maps, and let Λ_X be the orientation $\mathbf{Z}[\Pi_1(X)]$-module. Then*

$$(-1)^n W_{\mathbf{Z}}(X, \mathbf{Z}) + (-1)^{n-1} \textstyle\sum_j i_{j*}(W_{\mathbf{Z}}(\partial X_j, \mathbf{Z})) = \mathrm{cl}(\Lambda_X) \cdot W_{\mathbf{Z}}(X, \mathbf{Z})^*.$$

PROOF. We have $W_{\mathbf{Z}}(X, \mathbf{Z})^* = W_{\mathbf{Z}}(X, T^{\cdot}_X)$ where $T^{\cdot}_X$ is the orientation complex of X (3.3). The complex $T^{\cdot}_X$ has only one zero cohomology sheaf 0_X in dimension $-n$. Hence $W_{\mathbf{Z}}(X, \mathbf{Z}) = (-1)^n{}_{W_{\mathbf{Z}}}(X, 0_X)$. The sheaf 0_X is locally constant free of rank one on $X - \partial X$ and its restriction to ∂X is zero. Let $j : X - \partial X \to X$ be the inclusion map. We have an exact sequence

$$0 \to 0_X \to j_* 0_X \to j_* 0_X / \partial X \to 0.$$

Hence (2.4) $W_{\mathbf{Z}}(X, 0_X) = W_{\mathbf{Z}}(X, j_* 0_X) + W_{\mathbf{Z}}(X, j_* 0_X / \partial X)$. The sheaf $j_* 0_X$ is locally free of rank one and is defined by the orientation module Λ_X. The formula follows from (3.6), (2.6) and trivial manipulations.

4. Fibration.

4.1 PROPOSITION. *Let X and Y be two connected, locally simply connected finite dimensional compact spaces, and let $F^{\cdot}$ and $G^{\cdot}$ be two perfect complexes of sheaves of A-modules on X and Y respectively (A is commutative). Assume that the stalks of the components of $F^{\cdot}$ are flat A-modules. Then*

$$W_A(X, F^{\cdot}) \cdot W_A(Y, G^{\cdot}) = W_A(X \times Y, F^{\cdot} \otimes_A G^{\cdot}),$$

where $F^{\cdot} \otimes_A G^{\cdot}$ denote the cartesian product of the two complexes of sheaves (tensor product of the two inverse images by the two projections of the product $X \times Y$).

PROOF. Immediate consequence of the Künneth formula [1 Exposé 3].

4.2 Let Π be a group and $G'(\mathbf{Z}[\Pi])$ the Grothendieck group of the $\mathbf{Z}[\Pi]$-module which are finitely generated as abelian groups. It is easy to check that the canonical map $G(\mathbf{Z}[\Pi]) \to G'(\mathbf{Z}[\Pi])$ is an isomorphism. Hence any $\mathbf{Z}[\Pi]$-module M, finitely generated as abelian group, yields an element in $G(\mathbf{Z}[\Pi])$ denoted by $\mathrm{cl}(M)$.

4.3 Let X be a connected and locally simply connected space and M a locally constant sheaf whose stalks are finitely generated abelian groups. The stalk at any point is a finitely generated abelian group on which $\Pi_1(X)$ acts hence yields an element in $G(\mathbf{Z}[\Pi_1(X)])$ denoted by $\mathrm{cl}(M)$. Let now $f : E \to X$ be a continuous map such that for any $q \in \mathbf{Z}$, $R^q f \mathbf{Z}$ is a locally constant sheaf whose stalks are of finite type, zero for q big enough.

We denote by $\mathrm{cl}(f)$ the element $\sum_q (-1)^q \mathrm{cl}(R^q f_* \mathbf{Z})$ in $G(\mathbf{Z}[\Pi_1(X)])$.

4.4 PROPOSITION. *Let $f : E \to X$ be a locally trivial fibration, where X is a connected, locally simply connected, finite dimensional compact space and where the fiber is a finite dimensional compact space on which the constant sheaf $\mathbf{Z}$ is perfect. Assume E is connected and locally simply connected. Then, for any q in $\mathbf{Z}$, the sheaf $R^q f\, \mathbf{Z}$ is a locally constant sheaf whose stalks are finitely generated abelian groups, zero for q big enough. Moreover, for any perfect complex $F^{\cdot}$ on X, $f^*F^{\cdot}$ is a perfect complex on E and*

$$f_* W_{\mathbf{Z}}(E, f^*F^{\cdot}) = W_{\mathbf{Z}}(X, F^{\cdot}) \cdot \mathrm{cl}(f).$$

PROOF. It is clear that the $R^q f_* Z$ are locally constant sheaves whose stalks are isomorphic to the cohomology of the fiber hence finitely generated abelian groups. It is also clear that $f^*F^{\cdot}$ is perfect whenever $F^{\cdot}$ is perfect. Let us prove the equality. By 2.4 we may assume that $F^{\cdot}$ is a bounded complex of c-soft sheaves whose stalks are torsion free. We have $f_* W_{\mathbf{Z}}(E, f^*F^{\cdot}) = W_{\mathbf{Z}}(X, f_* f^* F^{\cdot})$ (2.6). Let $\mathbf{Z} \to \Omega^{\cdot}(\mathbf{Z})$ be a c-soft resolution of the constant sheaf $\mathbf{Z}$ on E. The projection formula yields a resolution $f_* f^* F^{\cdot} \to F^{\cdot} \otimes_{\mathbf{Z}} f_* \Omega^{\cdot}(\mathbf{Z})$ [**1**, Exposé 3]. Therefore (2.4) we have $W_{\mathbf{Z}}(X, f_* f^* F^{\cdot}) = W_{\mathbf{Z}}(X, F^{\cdot} \otimes_{\mathbf{Z}} f_* \Omega^{\cdot}(\mathbf{Z}))$. The cohomology sheaves of $f_* \Omega^{\cdot}(\mathbf{Z})$ are the $R^q f_* \mathbf{Z}$ and the complexes $F^{\cdot} \otimes_{\mathbf{Z}} R^q f_* \mathbf{Z}$ are perfect. Hence, by 2.4, we have

$$W_{\mathbf{Z}}(X, F^{\cdot} \otimes_{\mathbf{Z}} f_* \Omega^{\cdot}(\mathbf{Z})) = \sum_q (-1)^q W_{\mathbf{Z}}(X, F^{\cdot} \otimes_{\mathbf{Z}} R^q f_* \mathbf{Z}).$$

Therefore, we are reduced to proving that, for any locally constant sheaf M whose stalks are finitely generated abelian groups, we have

$$W_{\mathbf{Z}}(X, F^{\cdot} \otimes_{\mathbf{Z}} M) = W_{\mathbf{Z}}(X, F^{\cdot}) \cdot \mathrm{cl}(M).$$

When M is locally free, this equality follows from (3.6), so that, using (2.4), we may assume that the stalks of M are torsion groups. But then M has a resolution of length two by locally free sheaves of finite rank and the equality follows from (3.6) and (2.4).

Analogous results for singular homology can be found in [**6**] and [**7**].

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