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BRANCHED COVERINGS OF MANIFOLDS WITH BOUNDARY AND LINK INVARIANTS. I

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Abstract. In this paper relationships are found between invariants of a knot $K \subseteq S^{2k-1}$ of codimension 2 and invariants of cyclic branched coverings of the ball D^{2k} branched over orientable compact submanifolds spanning K; analogous relationships are studied in a more general situation.

Bibliography: 10 titles.

The present paper deals with a new interpretation of some well-known invariants of knot theory, for example the signature and the Minkowski units. It turns out that they can be regarded as invariants of cyclic branched coverings of a ball branched over an orientable surface spanning a link. In particular, the signature of a onedimensional link is equal to the signature of a branched double covering manifold of a ball.

This interpretation allows us to define similar invariants in a more general situation. Moreover, on the basis of this we succeed in: 1) showing a relationship between the approaches of Rohlin [3] and Tristram [9] to the problem of realizing two-dimensional homology classes of four-dimensional manifolds; 2) generalizing the results of Rohlin [3] to the case of a surface with singularities; 3) generalizing the inequalities of Murasugi [8] and Tristram [9] to higher dimensions. The second part of the article will deal with these applications.

The terminology of differential topology will be used in the article. In particular, manifolds will mean smooth manifolds and submanifolds will mean submanifolds of smooth manifolds in the sense of differential topology.

In the first three sections mainly well-known material is presented in a form suited to the needs of the present paper. In §1 we describe a construction of canonical branched coverings that represents a modification of a construction in §2 of [3]; in §§2 and 3 we study invariants of an even-dimensional \mathbb{Z}_m -manifold with boundary related to its quadratic form. The main results are formulated and partially proved in §4; in §5 the proof of the main theorem is completed.

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§1. Canonical coverings

1.1. Cyclic branched coverings. A submanifold A of a manifold X will be called proper if $\partial X \cap A = \partial A$ and at points of its boundary A is transversal to ∂X .

Let Y be a manifold of dimension n + 2 and let A be a proper n-dimensional submanifold of the (n + 2)-dimensional manifold X. The mapping P: $Y \rightarrow X$ is called an *m*-fold cyclic branched covering of X with branching over A or, shorter, an *m*-fold cyclic branched covering of the pair (X, A) if it satisfies the following conditions: a) the mapping $p: Y \setminus P^{-1}(A) \rightarrow X \setminus A$ determined by P is a smooth *m*-fold cyclic covering; b) for every point $a \in A$ there exists embeddings $\tilde{\phi}_a: D^n \times D^2 \rightarrow Y$ and $\phi_a: D^n \times D^2 \rightarrow X$ such that $\phi(D^n \times D^2)$ is a neighborhood of a and

$$P \circ \widetilde{\varphi_a}(v, w) = \varphi_a(v, w^m),$$

where $v \in D^n$ and $w \in D^2$ (w is viewed as a complex number).

1.2. Definition. A pair consisting of an oriented connected compact (n + 2)dimensional manifold X with $H^{1}(X) = H^{1}(\partial X) = 0$ and an oriented compact proper *n*dimensional submanifold A realizing the zero of the group $H_{n}(X, \partial X)$ will be called a special pair.

1.3. Construction of canonical coverings. Let (X, A) be a special pair with dim A = n, and let λ : $H_1(X \setminus A) \rightarrow \mathbb{Z}$ be the linking coefficient with the fundamental class of A. For each natural number m we construct a covering $p_m: \mathbb{Z}_m \rightarrow X \setminus A$ corresponding to the kernel of the composition

$$\pi_1(X \setminus A) \xrightarrow{h} H_1(X \setminus A) \xrightarrow{\lambda} \mathbb{Z} \xrightarrow{\zeta_m} \mathbb{Z}_m,$$

where h is the Hurewicz homomorphism and ζ_m is the natural projection.

Let B be a (closed) tubular neighborhood of A in X; it has the structure of a smooth SO(2)-fibration over A with fiber D^2 . Insofar as the boundary C of a fiber of this fibration is singly linked with A, the composition of the inclusion homomorphism $\pi_1(C) \rightarrow \pi_1(X \setminus A)$ with the homomorphism $\zeta_m \lambda h$ is surjective. From this it follows that the fibers of the fibration $p_m^{-1}(B \setminus A) \rightarrow A$ obtained as the composition of the restriction $p_m^{-1}(B \setminus A) \rightarrow B \setminus A$ of the covering p_m : $\mathbb{Z}_m \rightarrow X \setminus A$ and the restriction $B \setminus A \rightarrow A$. of the fibration $B \rightarrow A$ are connected, and therefore the fibration $p_m^{-1}(B \setminus A) \rightarrow A$ is provided with the structure of a SO(2)-fibration with fiber $D^2 \setminus \{0\}$. Let p'_m : $B_m \rightarrow A$ be the associated fibration with fiber D^2 , and let $i: p_m^{-1}(B \setminus A) \rightarrow B_m$ be the natural embedding. We define the fiber embedding $i': p_m^{-1}(B \setminus A) \rightarrow B_m$ by the formula $i'(w) = |w|^{(1-m)/m} i(w)$, and we attach the manifolds \mathbb{Z}_m and B_m define the differential structure of the manifold obtained. We denote it by $N_m(X, A)$, and we define a mapping $P_m: N_m(X, A) \rightarrow X$ by the formula

$$P_m(x) = \begin{cases} p_m(x), & \text{if } x \in \mathbb{Z}_m, \\ p_m(x), & \text{if } x \notin \mathbb{Z}_m. \end{cases}$$

Obviously this mapping is an *m*-fold cyclic branched covering of the pair (X, A).

We orient $N_m(X, A)$ such that the projection P_m has degree + m. Automorphisms of the branched covering P_m : $N_m(X, A) \rightarrow X$ (i.e. diffeomorphisms $f: N_m(X, A) \rightarrow N_m(X, A)$ with $P_m f = P_m$) preserve the orientation; they form a group canonically isomorphic to the group

$$\pi_1(X \setminus A)/p_{m_*}\pi_1(Z_m) = \pi_1(X \setminus A)/\text{Ker}(\zeta_m \lambda h)$$

of automorphisms of the covering $p_m: \mathbb{Z}_m \to X \setminus A$, which in turn is canonically isomorphic to \mathbb{Z}_m . Let $T: N_m(X, A) \to N_m(X, A)$ be the automorphism of the covering $P_m: N_m(X, A) \to X$ corresponding to the standard generator of the group \mathbb{Z}_m .

The branched covering $P_m: N_m(X, A) \to X$ and the \mathbb{Z}_m -manifold(1)($N_m(X, A), T$) depend only on the initial data: the special pair (X, A) and the natural number m. We shall call $P_m: N_m(X, A) \to X$ the canonical m-fold covering, and the \mathbb{Z}_m -manifold $(N_m(X, A), T)$ will be called the canonical m-fold cover of the pair (X, A).

1.4. Cobordism of \mathbb{Z}_m -manifolds with boundary. We shall call the n-dimensional \mathbb{Z}_m -manifolds (X_1, T_1) and (X_2, T_2) cobordant if there exists a \mathbb{Z}_m -manifold (X, T) of dimension n + 1 whose boundary can be obtained by attaching (X_1, T_1) to $(-X_2, T_2)$ by means of some equivariant diffeomorphism $(\partial X_1, T_1 | \partial X_1) \rightarrow (\partial X_2, T_2 | \partial X_2)$.

For closed manifolds this definition is equivalent to the usual one ([2], Chapter V). Similar to the way it is done in the closed case, it can be shown that cobordism of \mathbb{Z}_m -manifolds with boundary is an equivalence. However, the classes of cobordant \mathbb{Z}_m -manifolds with boundary do not form a group with respect to disjoint summation.

1.5. If (X_0, A_0) and (X_1, A_1) are special pairs with $X_0 = X_1 = X$ and $\partial A_0 = \partial A_1$, then for each natural number m the canonical m-fold covers $(N_m(X_0, A_0), T)$ and $(N_m(X_1, A_1), T)$ are cobordant.

Proof. Let dim $A_0 = n$. The submanifold $\{0\} \times (-A_0) \cup I \times \partial A_1 \cup \{1\} \times A_1$ realizes the zero of the group $H_n(\partial(I \times X))$. Therefore there exists a proper submanifold A with boundary

$$\partial A = \{0\} \times (-A_{\mathfrak{o}}) \cup I \times \partial A_{\mathfrak{i}} \cup \{1\} \times A_{\mathfrak{i}}$$

realizing the zero of $H_{n+1}(I \times X, \partial(I \times X))$. Obviously $(I \times X, A)$ is a special pair.

We define the embedding $i: X \setminus A_0 \to I \times X \setminus A$ by the formula i(x) = (0, x). From the commutativity of the diagram

⁽¹⁾By a \mathbb{Z}_m -manifold we shall mean a pair consisting of an oriented compact manifold X and an orientation-preserving diffeomorphism $T: X \to X$ with $T^m = 1$.

$$\begin{array}{cccc} \pi_1 (X \ A_0) \xrightarrow{h_0} & H_1 (X \ A_0) \end{array} \xrightarrow{\lambda_0} \\ & & I_1 (X \ A_0) \xrightarrow{h_0} & H_1 (X \ A_0) \xrightarrow{\lambda_0} \\ & & I_1 (I \ X \ A) \xrightarrow{h} & H_1 (I \ X \ A) \end{array}$$

in which h_0 and h are Hurewicz homomorphisms, ζ_m is the natural projection, λ_0 is the linking coefficient with A_0 in X, and λ is the linking coefficient with A in $I \times X$, it follows that the covering $p_m: N_m(X, A_0) \setminus P_m^{-1}(A_0) \to X \setminus A_0$ corresponding to the kernel of $\zeta_m \lambda_0 h_0$ is equivalent to the restriction of the covering

$$p_m: N_m(I \times X, A) \setminus P_m^{-1}(A) \to I \times X \setminus A,$$

corresponding to the kernel of $\zeta_m \lambda h$; this equivalence extends naturally to the embedding

$$j_0: N_m(X, A_0) \rightarrow \partial N_m(I \times X, A).$$

Obviously j_0 reverses orientation and commutes with T. In an analogous way we construct an embedding

$$j_1: N_m(X, A_1) \rightarrow \partial N_m(I \times X, A)$$

with $\operatorname{Im} j_1 = \partial N_m(I \times X, A)$ Int $\operatorname{Im} j_0$, which agrees with the orientations and commutes with T. The embeddings j_0 and j_1 give a representation of the boundary of the \mathbb{Z}_m manifold $N_m(I \times X, A)$ in the form of the result of attaching $(-N_m(X, A_0), T)$ to $(N_m(X, A_1), T)$.

§2. Invariants of Z_m -manifolds

2.1. Forms with isometries. Let \mathcal{F} be a field of characteristic 0. The triple consisting of a finite-dimensional vector space \mathcal{O} over \mathcal{F} , a bilinear symmetric or skew-symmetric form q on \mathcal{O} , and an isometry $\tau: \mathcal{O} \to \mathcal{O}$ will be called a Z-form (over \mathcal{F}). A Z-form (\mathcal{O} , q, τ) with $\tau^m = 1$ will be called a Z_m-form.

The nondegenerate part of the Z-form $(\circlearrowright, q, \tau)$ will mean the Z-form obtained from $(\circlearrowright, q, \tau)$ after factoring by the radical of q (i.e. by the annihilator of the whole space \circlearrowright). The Z-form $(\circlearrowright, q, \tau)$ will be called *null-cobordant* if its nondegenerate part $(\circlearrowright, q, \tau)$ is such that \circlearrowright contains the $\widetilde{\tau}$ -invariant totally isotropic (i.e. self-annihilating) subspace of half the dimension. The Z-forms $(\circlearrowright, q_1, \tau_1)$ and $(\circlearrowright, q_2, \tau_2)$ will be called *cobordant* if they are both symmetric or both skew-symmetric and if the Z-form

$$(\circlearrowright, \oplus \circlearrowright_2, q_1 \oplus (-q_2), \tau_1 \oplus \tau_2)$$

is null-cobordant. It is not hard to verify that cobordism of Z-forms is an equivalence (cf. Levine [5, 6]).

The set of classes of cobordant symmetric Z-forms over \mathcal{F} will be denoted by $\Lambda_{+1}(\mathcal{F})$, and the set of classes of cobordant skew-symmetric Z-forms over \mathcal{F} will be denoted by $\Lambda_{-1}(\mathcal{F})$. As is easy to see, these sets are groups with respect to orthogonal summation.

2.2. The form of a $\mathbb{Z}_{-manifold}$. Let (X, T) be a $\mathbb{Z}_{-manifold}$ of dimension 2k. The diffeomorphism T induces a linear mapping.

$$T_{\star}: H_{k}(X; \mathbf{Q}) \rightarrow H_{k}(X; \mathbf{Q})$$

of period *m*, which preserves the intersection index

 $Q: H_{\mathbb{A}}(X; \mathbf{Q}) \otimes H_{\mathbb{A}}(X; \mathbf{Q}) \rightarrow \mathbf{Q}.$

Thus $(H_{\mu}(X; \mathbf{Q}), Q, T^*)$ is a Z-form over Q; we shall call it the form of the form of the \mathbb{Z}_{m} -manifold (X, T).

2.3. The forms of cobordant Z_-manifolds are cobordant.

This theorem follows from the following two lemmas in an obvious way:

1) The form of the \mathbb{Z}_{m} -manifold obtained by attaching the \mathbb{Z}_{m} -manifolds (X_{1}, T_{1}) and (X_2, T_2) by means of an equivariant diffeomorphism

 $(\partial X_1, T_1 | \partial X_1) \rightarrow (-\partial X_2, T_2 | \partial X_2),$

is cobordant to the orthogonal sum of the forms of (X_1, T_1) and (X_2, T_2) .

2. The form of a null-cobordant Z_-manifold is null-cobordant.

The proof of the first lemma is contained in the proof of the additivity of the Gsignature ([1], Proposition 7.1); the proof of the second is contained in the proof of the fact that the G-signature of a null-cobordant G-manifold is equal to zero.

§3. Invariants of Z-forms

3.1. Decomposition of a Z-form. Let \mathcal{F} be a field of characteristic 0, and let $\Phi =$ (\mathcal{O}, q, τ) be a Z-form over \mathcal{F} . If λ is an irreducible polynomial over \mathcal{F} , then by \mathcal{O}_{λ} we shall denote the λ -primary component of the space \mathcal{O} :

 $\mathcal{O}_{\lambda} = \operatorname{Ker} \lambda(\tau)^{N}$ for large N;

the restriction of Φ to \mathbb{O}_{λ} will be denoted by Φ_{λ} or by $(\mathbb{O}_{\lambda}, q_{\lambda}, r_{\lambda})$. If $\lambda(t) = t^{k} + a_{1}t^{k-1} + \cdots + a_{k}$ is a polynomial with coefficients in \mathcal{F} , then the polynomial

$$\bar{\lambda}(t) = \frac{1}{a_k} \left(a_k t^k + a_{k-1} t^{k-1} + \ldots + 1 \right)$$

will be called symmetric to the polynomial λ . As Milnor has shown ([7], Lemma 3.1), if the polynomial λ is not symmetric to the polynomial λ' , then the subspace \mathcal{O}_{λ} is orthogonal (with respect to q) to \mathcal{O}_{λ} . From this it follows that the Z-form $\Phi =$ (\mathcal{O} , q, r) can be decomposed into the orthogonal sum of Z-forms of two types: 1) Φ_{λ} , where λ is a symmetric (i.e. symmetric to itself) irreducible polynomial over \mathcal{F} , and 2) $(\mathcal{O}_{\lambda} + \mathcal{O}_{\overline{\lambda}}, q | \mathcal{O}_{\lambda} + \mathcal{O}_{\overline{\lambda}}, \eta | \mathcal{O}_{\lambda} + \mathcal{O}_{\overline{\lambda}})$, where λ is an asymmetric irreducible polynomial over \mathcal{F} , and that terms of the second type are null-cobordant. Furthermore, the Z-form Φ is null-cobordant if and only if all of its terms of the first type are null-cobordant; see Levine [6].

3.2. Extensions of a Z-form. If $\Phi = (\mathcal{O}, q, r)$ is a Z-form over \mathcal{F} and K is an extension of the field \mathcal{F} , then there is an obvious extension of Φ to a Z-form over K. We shall denote this Z-form by Φ^{K} or by $(\circlearrowright^{K}, q^{K}, \tau^{K})$.

3.3. Lemma (Levine [6], Proposition 17). A symmetric Z-form over Q is nullcobordant if and only if its extensions to Z-forms over every completion of Q are nullcobordant.

3.4. Lemma (Levine [6], Proposition 16). If $(\circlearrowright, q, \tau)$ is a nondegenerate symmetric Z-form over one of the completions of Q and if the characteristic polynomial of τ is equal to λ^e , where λ is an irreducible symmetric polynomial, then the Z-form $(\circlearrowright, q, \tau)$ is null-cobordant if and only if the exponent e is even.

3.5. Invariants of symmetric \mathbb{Z}_1 -forms. Let \mathbb{O} be a vector space of dimension n over \mathcal{F} , and let q be a bilinear symmetric form on \mathbb{O} . Let e_1, \dots, e_n be an orthogonal (with respect to q) basis of \mathbb{O} with $q(e_i \otimes e_i) = a_i$, where $a_i \in \mathcal{F}$, $a_i \neq 0$ for $1 \le i \le r$, and $a_i = 0$ for $r < i \le n$.

We shall denote by d(q) the element of the factor group $\dot{\mathcal{F}}/(\dot{\mathcal{F}})^2$ of the multiplicative group of \mathcal{F} by the subgroup of squares defined by the formula

$$d(q) = (-1)^{\left\lfloor \frac{r}{2} \right\rfloor} \prod_{i=1}^{r} a_i \pmod{(\bar{\mathcal{F}})^2}.$$

We shall denote by $\epsilon(q)$ the number τ reduced modulo 2. It is clear that d(q) and $\epsilon(q)$ are invariants of the class of \mathbb{Z}_1 -forms cobordant to the \mathbb{Z}_1 -form ($\mathbb{C}, q, 1$).

Let $\mathcal{F} = \mathbb{Q}_p$ be the field of *p*-adic numbers. We shall denote the product $\prod_1^r a_i$ by D; let $D = p^a d$, where *d* is a unit of the ring of *p*-adic integers. From the classification of quadratic forms over \mathbb{Q}_p (see, for example, [4]) it follows that the class of \mathbb{Z}_1 forms cobordant to the \mathbb{Z}_1 -form $(\mathcal{O}, q, 1)$ is completely determined by the invariants $\epsilon(q)$, d(q), and the Minkowski unit

$$C_{q} = \begin{cases} (p, D)^{\alpha} \prod_{1 \le i \le j \le r} (a_{i}, a_{j}), & \text{if } p \neq 2, \\ \\ \begin{pmatrix} r \\ (-1) \end{bmatrix} \begin{pmatrix} r \\ 4 \end{pmatrix} + \begin{pmatrix} 1 + \begin{bmatrix} r \\ 2 \end{bmatrix} \end{pmatrix} \begin{pmatrix} d+1 \\ 2 \end{pmatrix} + \frac{(d^{2}-1)\alpha}{2} + \frac{(d^{2}-1)\alpha}{8} + 1 \prod_{1 \le i \le j \le r} (a_{i}, a_{j}), & \text{if } p = 2 \end{cases}$$

(here (,) is the Hilbert symbol). When r is even the Minkowski unit can be replaced by the invariant introduced by Levine [6]:

$$\mu(q) = (-1, -1)^{\frac{r(r+6)}{8}} (-1, D)^{\frac{r}{2}} \prod_{1 \leq i \leq j \leq r} (a_i, a_j).$$

Indeed, it is not hard to verify that

$$C(q) = (d(q), (-1)^{p-1} p^{a}) \mu(q).$$
⁽¹⁾

3.6. Invariants of symmetric Z-forms. Let $\Phi = (\mathfrak{O}, q, \tau)$ be a symmetric Z-form over Q, and let (\mathfrak{O}, q, τ) be its nondegenerate part.

a) For each symmetric irreducible polynomial λ we denote by $\epsilon_{\lambda}(\Phi)$ the exponent reduced modulo 2 with which λ appears in the characteristic polynomial of \tilde{r} .

b) For each symmetric irreducible polynomial λ we denote the invariant $d(q_{\lambda})$ by $d_{\lambda}(\Phi)$.

c) For each symmetric polynomial λ irreducible over **R** we denote the signature $\sigma(q_{\lambda}^{\mathbf{R}})$ by $\sigma_{\lambda}(\Phi)$.

Note. The signature $\sigma_{\lambda}(\Phi)$ can be obtained by starting with the Hermitian form q^{HC} : $(\mathfrak{O} \otimes \mathbb{C}) \otimes (\mathfrak{O} \otimes \mathbb{C}) \to \mathbb{C}$ defined by the formula

$$q^{HC}\left((v_1 \otimes z_1) \otimes (v_2 \otimes z_2)\right) = \overline{z_1 z_2} q\left(v_1 \otimes v_2\right).$$
⁽²⁾

Namely, for $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ and $\zeta \neq \pm 1$ the signature $\sigma_{t^2 - 2t \operatorname{Re} \zeta + 1}(\Phi)$ is equal to the doubled signature of the restriction of q^{HC} to $(v \otimes \mathbb{C})_{t-\zeta}$.

d) For each symmetric polynomial λ irreducible over Q_p we denote the Minkowski unit $C(q_{\lambda}^{Q_p})$ by $C_{\lambda}^{p}(\Phi)$.

e) Since every symmetric polynomial of odd degree is divisible by t-1 or t+1, every irreducible symmetric polynomial λ with $\lambda(1) \neq 0 \neq \lambda(-1)$ has even degree. Therefore for such λ the λ -primary components of Z-forms are even dimensional. Consequently for each symmetric polynomial λ irreducible over Q_p with $\lambda(1) \neq 0 \neq \lambda(-1)$ Levine's invariant $\mu(q_{\lambda}^{Qp})$ is defined. This invariant will be denoted by $\mu_{\lambda}^{p}(\Phi)$.

3.7. The symmetric Z-forms Φ and Φ' are cobordant if and only if $d_{t-1}(\Phi) = d_{t-1}(\Phi')$, $d_{t+1}(\Phi) = d_{t+1}(\Phi')$, $\epsilon_{\lambda}(\Phi) = \epsilon_{\lambda}(\Phi')$, $\sigma_{\lambda}(\Phi) = \sigma_{\lambda}(\Phi')$ and $C_{\lambda}^{p}(\Phi) = C_{\lambda}^{p}(\Phi')$ for every λ and p for which these invariants are defined.

Proof. Let us represent our Z-forms in the form

$$\Phi = \Phi_{t-1} \oplus \Phi_{t+1} \oplus \overline{\Phi} \text{ and } \Phi' = \Phi_{t-1}^{'} \oplus \Phi_{t+1}^{'} \oplus \overline{\Phi}'.$$

The cobordism of the Z-forms Φ and Φ' is equivalent to the cobordism of corresponding terms in these decompositions (see §3.1). By virtue of the results of §§ 3.3-3.5, it is necessary and sufficient for the cobordism of the Z-forms Φ_{t-1} , Φ'_{t-1} and Φ_{t+1} , Φ'_{t+1} that $d_{\lambda}(\Phi) = d_{\lambda}(\Phi') = \epsilon_{\lambda}(\Phi) = \epsilon_{\lambda}(\Phi')$, $\sigma_{\lambda}(\Phi) = \sigma_{\lambda}(\Phi')$ and $C^{p}_{\lambda}(\Phi) = C^{p}_{\lambda}(\Phi')$ for $\lambda(t) = t - 1$ and $\lambda(t) = t + 1$ and for every prime p.

For every symmetric polynomial λ irreducible over Q_p with $\lambda(1) \neq 0 \neq \lambda(-1)$ we have

$$d(q_{\lambda}^{\mathbf{Q}_{p}}) = ((-1)^{\deg \lambda} \lambda(1) \lambda(-1))^{\epsilon_{\lambda}(\mathbf{\Phi})}$$

where deg λ is the degree of λ (see [6], §§7 and 21). Therefore formula (1) permits us to express the invariants μ_{λ}^{p} in terms of C_{λ}^{p} and ϵ_{λ} :

$$\mu_{\lambda}^{p}(\Phi) = \left((-1)^{\deg\lambda}\lambda(1)\lambda(-1), (-1)^{p-1}p^{\alpha}\right)^{\epsilon_{\lambda}(\Phi)}C_{\lambda}^{p}(\Phi), \tag{3}$$

where α denotes the exponent of p in the factorization of the number $(\lambda(1)\lambda(-1))^{\epsilon_{\lambda}}(\Phi)$. On the other hand, as Levine has shown ([6], Theorem 21), the invariants ϵ_{λ} , σ_{λ} and μ_{λ}^{p} completely determine the class of Z-form with trivial (t-1)-primary and (t+1)-primary components. Consequently the Z-forms Φ and Φ' are cobordant if and only if $\epsilon_{\lambda}(\Phi) = \epsilon_{\lambda}(\Phi')$, $\sigma_{\lambda}(\Phi) = \sigma_{\lambda}(\Phi')$ and $C_{\lambda}^{p}(\Phi) = C_{\lambda}^{p}(\Phi')$ for every prime p and for every λ with $\lambda(1) \neq 0 \neq \lambda(-1)$ for which these invariants are defined.

3.8. Lemma. Every skew-symmetric Z-form coinciding with its λ -primary component, where $\lambda(t) = t - 1$ or $\lambda(t) = t + 1$, is null-cobordant.

Proof. As Levine has shown ([6], Lemma 12), a Z-form coinciding with its λ primary component is cobordant to a Z-form (\emptyset , q, τ) such that λ is the minimal polynomial of τ . Therefore in the case $\lambda(t) = t \pm 1$ we can suppose that every subspace is τ -invariant. On the other hand, the nondegenerate part of the form q has (as a nondegenerate skew-symmetric form) a totally isotropic subspace of half the dimension.

3.9. The homomorphism M. We define the homomorphism

$$M: \Lambda_{-1}(\mathscr{F}) \to \Lambda_{+1}(\mathscr{F}),$$

taking the class of the symmetric Z-form $(\circlearrowright, \mu(q, \tau), \tau)$ to the class of the skewsymmetric Z-form $(\circlearrowright, q, \tau)$, where the form $\mu(q, \tau)$ is defined by the formula

$$\mu(q, \tau) (\upsilon_1 \otimes \upsilon_2) = q((\tau - \tau^{-1}) \upsilon_1 \otimes \upsilon_2)$$

(cf. Milnor [7]). It follows directly from the definition that this mapping is well defined and a homomorphism.

3.10. The homomorphism M is injective. In fact, let (\ddot{U}, q, τ) br a skew-symmetric Z-form such that the Z-form $(\ddot{U}, \mu(q, \tau), \tau)$ is null-cobordant. According to the discussion in §§3.1 and 3.8, it is sufficient to consider the case $\ddot{U} = \ddot{U}_{\lambda}$, where λ is a symmetric irreducible polynomial with $\lambda(1) \neq 0 \neq \lambda(-1)$. In this case $\tau - \tau^{-1}$ is an automorphism, and therefore q can be expressed by $\mu(q, \tau)$:

$$q\left(v_1\otimes v_2\right)=\mu\left(q,\tau\right)\left(\left(\tau-\tau^{-1}\right)^{-1}v_1\otimes v_2\right).$$

Consequently the radicals and the r-invariant totally isotropic subspaces of the forms q and $\mu(q, r)$ coincide, and hence the Z-form (\mathcal{O}, q, r) is null-cobordant.

3.11. The skew-symmetric Z-forms Φ and Φ' are cobordant if and only if $\epsilon_{\lambda}(M\Phi) = \epsilon_{\lambda}(M\Phi'), \sigma_{\lambda}(M\Phi) = \sigma_{\lambda}(M\Phi')$ and $C_{\lambda}^{p}(M\Phi) = C_{\lambda}^{p}(M\Phi')$ for every prime p and every λ with $\lambda(1) \neq 0 \neq \lambda(-1)$ for which these invariants are defined.

This theorem follows from the results of §§3.7, 3.8 and 3.10.

Note. By virtue of equation (3) Theorem 3.11 remains true if in its formulation we replace the condition $C_{\lambda}^{p}(M\Phi) = C_{\lambda}^{p}(M\Phi')$ by the condition $\mu_{\lambda}^{p}(M\Phi) = \mu_{\lambda}^{p}(M\Phi')$,

§4. Main results

4.1. The construction of invariants of a closed special pair. Let (M, L) be a special pair with dim M = 2k + 1 consisting of closed manifolds, and let A be an

oriented compact proper 2k-dimensional submanifold of the product $l \times M$ having boundary $\partial A = \{1\} \times L$. By virtue of Theorem 1.5 the class of \mathbb{Z}_m -manifolds cobordant to the canonical cover $(N_m(l \times M, A), T)$ does not depend on the submanifold A but only on the differential topological type of the pair (M, L), and therefore invariants of this class are invariants of the pair (M, L). In particular, as Theorem 2.3 shows, the class of \mathbb{Z}_m -forms cobordant to the form of the canonical cover $(N_m(l \times M, A), T)$ belongs to the set of invariants of the pair (M, L), and hence so do the invariants of this class described in §3.

4.2. Seifert pairing. Let M be a closed oriented (2k + 1)-dimensional manifold and N a compact oriented 2k-dimensional submanifold of M. We shall denote by $\hat{H}_k(N)$ the kernel of the inclusion homomorphism

$$H_{\mathbf{k}}(N; \mathbf{Q}) \longrightarrow H_{\mathbf{k}}(M; \mathbf{Q})$$

The orientations of N and M determine a normal vector field on N. Let $s: N \to M \setminus N$ be a small translation along this field. The pairing

$$\theta: \hat{H}_{k}(N) \otimes \hat{H}_{k}(N) \to \mathbf{Q}_{k}$$

defined by the formula

$$\theta(v_1 \otimes v_2) = \lambda(v_1 \otimes s_{\bullet}(v_2)), \qquad (4)$$

is called the Seifert pairing of the pair (M, N), where λ is the linking coefficient.

4.3. \mathbb{Z}_m -forms of a pairing. Let \emptyset be a finite-dimensional vector space over \mathbb{Q} . For each pairing $q: \emptyset \otimes \emptyset \to \mathbb{Q}$ and each natural number $m \ge 2$ we shall construct in this subsection two \mathbb{Z}_m -forms over \mathbb{Q} : a symmetric one $(\mathbb{O}^{m-1}, q_{+1}, \tau)$, and a skew-symmetric one $(\mathbb{O}^{m-1}, q_{-1}, \tau)$; we shall call them the \mathbb{Z}_m -forms of the pairing q.

Let ξ_1, \dots, ξ_{m-1} be the coordinate projections of the (m-1)th Cartesian power \mathbb{O}^{m-1} of the space \mathbb{O} , and let $\eta_1, \dots, \eta_{m-1}$ be the coordinate embeddings. We define the operator $r: \mathbb{O}^{m-1} \to \mathbb{O}^{m-1}$ by the formula

$$\tau(v) = \sum_{i=1}^{m-2} \eta_{i+1} \xi_i(v) - \sum_{i=1}^{m-1} \eta_i \xi_{m-1}(v).$$
(5)

Obviously $\tau^m = 1$. For $\epsilon = \pm 1$ we define the form $q_{\epsilon}: \mathbb{O}^{m-1} \otimes \mathbb{O}^{m-1} \to \mathbb{Q}$ by the formula

$$q_{\epsilon}(v_{1} \otimes v_{2}) = \sum_{i=1}^{m-1} (q(\xi_{i}(v_{1}) \otimes \xi_{i}(v_{2})) + \epsilon q(\xi_{i}(v_{2}) \otimes \xi_{i}(v_{1}))) - \sum_{i=1}^{m-2} (q(\xi_{i+1}(v_{1}) \otimes \xi_{i}(v_{2})) + \epsilon q(\xi_{i+1}(v_{2}) \otimes \xi_{i}(v_{1}))).$$
(6)

It is not hard to verify that q is invariant with respect to τ .

4.4. Main Theorem. Let (M, L) be a special pair with dim M = 2k + 1 consisting of closed manifolds; let N be a compact oriented 2k-dimensional submanifold of M spanning L (i.e. with $\partial N = L$), and let A be a compact oriented proper 2k-dimensional submanifold of the product $I \times M$ having boundary $\partial A = \{1\} \times L$. Then for each $m \ge 2$ the form of the canonical m-fold cover $(N_m(I \times M, A), T)$ is cobordant to the (symmetric when k is odd and skew-symmetric when k is even) \mathbb{Z}_m -form of the Seifert pairing of (M, N).

4.5. Corollary. Let L be a closed oriented (2k - 1)-dimensional submanifold of the sphere S^{2k+1} ; let N be a compact oriented 2k-dimensional submanifold of S^{2k+1} spanning L, and let A be a compact oriented proper 2k-dimensional submanifold of the ball D^{2k+2} having boundary $\partial A = L$. Then for each $m \ge 2$ the form of the canonical m-fold cover $(N_m(D^{2k+2}, A), T)$ is cobordant to the (symmetric when k is odd and skew-symmetric when k is even) Z_m -form of the Seifert pairing of (S^{2k+1}, N) .

Derivation of the corollary from the main theorem. Let D be a submanifold of D^{2k+2} diffeomorphic to D^{2k+2} and not intersecting $A \cup S^{2k+1}$. Since the restriction of the canonical covering $P_m: N_m(D^{2k+2}, A) \rightarrow D^{2k+2}$ to D^{2k+2} . Int D is diffeomorphic to the canonical covering $N_m(D^{2k+2}$. Int $D, A) \rightarrow D^{2k+2}$. Int D, we can obtain the \mathbb{Z}_m -manifold $(N_m(D^{2k+2}, A), T)$ by attaching the \mathbb{Z}_m -manifold $(P_m^{-1}(D), T | P_m^{-1}(D))$ to $(N_m(D^{2k+2} \setminus \operatorname{Int} D, A) T)$. But $P_m^{-1}(D)$ is diffeomorphic to the disjoint union of m copies of D. Consequently the forms of $(N_m(D^{2k+2}, A), T)$ and $(N_m(D^{2k+2} \setminus \operatorname{Int} D, A), T)$ are isomorphic. The pair $(D^{2k+2} \setminus \operatorname{Int} D, S^{2k+1})$ is diffeomorphic to

$$(I \times S^{2^{k+1}}, \{1\} \times S^{2^{k+1}})$$

and therefore by virtue of the main theorem the form of the canonical cover

$$(N_m(D^{2^{k+2}} \setminus \operatorname{Int} D, A), T)$$

is cobordant to the corresponding Z_m -form of the Seifert pairing of (S^{2k+1}, N) .

4.6. Classical knot theory invariants. Let (S^{2k+1}, K) be a knot of dimension 2k - 1 (i.e. a pair consisting of a sphere S^{2k+1} and an orientable submanifold K homeomorphic to S^{2k-1}). The Seifert pairings of pairs of form (S^{2k+1}, N) , where N is a compact oriented submanifold of S^{2k+1} spanning K, are also called Seifert pairings of the knot (S^{2k+1}, K) , and the quadratic forms obtained by symmetrization of these pairings are called quadratic forms of the knot (S^{2k+1}, K) . The signature and Minkowski units of a quadratic form of a knot are called the signature and Minkowski units of the knot; they are actually invariants of it, i.e. they do not depend on the choice of N (see, for example, [10]).

The symmetric \mathbb{Z}_2 -form of a pairing is obtained by the usual symmetrization of the pairing. Therefore if k is odd, then the quadratic form of the knot (S^{2k+1}, K) is cobordant (as a \mathbb{Z}_1 -form) to the quadratic form of a branched double covering manifold of D^{2k+2} with branching over an arbitrary compact oriented proper submanifold

spanning K, and hence the signature and Minkowski units of (S^{2k+1}, K) are equal respectively to the signature and Minkowski units of the quadratic form of this manifold.

4.7. Hermitian forms of a pairing. Let \emptyset be a finite-dimensional vector space over Q, and let $q: \emptyset \otimes \emptyset \to Q$ be a pairing. For $\zeta \in C$ with $|\zeta| = 1$ and $\zeta \neq 1$ we define the Hermitian form $q_{(\zeta)}: (\emptyset \otimes C) \otimes (\emptyset \otimes C) \to C$ by the formula

$$q_{(\zeta)}((v_1 \otimes z_1) \otimes (v_2 \otimes z_2)) = z_1 \overline{z_2} ((1 - \overline{\zeta}) q (v_1 \otimes v_2) + (1 - \zeta) q (v_2 \otimes v_1))$$
(7)

(cf. [5], §25, or [9]).

If ζ is a primitive mth root of 1, then the form $q_{(\zeta)}$ is isomorphic to the restriction to $(\mathbb{O}^{m-1} \otimes \mathbb{C})_{t-\zeta}$ of the Hermitian form $(q_{+1})^{HC}$ (see §3.6) constructed from the symmetric \mathbb{Z}_m -form of q.

Proof. Define the embedding $\nu: \mathbb{C} \otimes \mathbb{C} \to \mathbb{O}^{m-1} \otimes \mathbb{C}$ by the formula

$$v(\boldsymbol{v}\otimes\boldsymbol{z}) = \frac{1}{\sqrt{m}}\sum_{i=1}^{m-1}\eta_i(\boldsymbol{v})\otimes(\boldsymbol{\zeta}^{m-i}-1)\boldsymbol{z}.$$
(8)

We shall show that Im $\nu = (\mathbb{O}^{m-1} \otimes \mathbb{C})_{t-\zeta}$. In fact, let

$$v \in (\mathcal{O}^{m-1} \otimes \mathbf{C})_{t-\xi} = \operatorname{Ker}(\tau - \xi).$$

Then

$$(\tau - \zeta) v = \sum_{l=2}^{m-1} \eta_l (\xi_{l-1} - \xi_{m-1} - \zeta \xi_l) v - \eta_1 (\xi_{m-1} + \zeta \xi_l) v = 0.$$

Consequently $(\xi_{m-1} + \zeta \xi_1)v = 0$, and $(\xi_{i-1} - \xi_{m-1} - \zeta \xi_i)v = 0$ for $i = 2, \dots, m-1$. From these equations we obtain

$$\xi_i(v) = \frac{\zeta^{m-i} - 1}{\zeta - 1} \,\xi_{m-1}(v). \tag{9}$$

We set $w = \sqrt{m}(\zeta - 1)^{-1}/\xi_{m-1}(v)$. By virtue of (9) we have

$$v(\omega) = \frac{1}{\sqrt{m}} \sum_{i=1}^{m-1} (\zeta^{m-i} - 1) \eta_i \frac{\sqrt{m}}{\zeta - 1} \frac{\zeta - 1}{\zeta^{m-i} - 1} \xi_i(\upsilon) = \sum_{i=1}^{m-1} \eta_i \xi_i(\upsilon) = \upsilon.$$

Now we shall show that ν is an isometry. In fact, by virtue of (8), (2), (6) and (7),

$$(q_{+1})^{HC} (v (v_1 \otimes z_1) \otimes v (v_2 \otimes z_2))$$

$$= z_1 \overline{z}_2 \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} (\zeta^{m-i} - 1) \overline{(\zeta^{m-i} - 1)} \frac{1}{m} q_{+1} (\eta_i (v_1) \otimes \eta_j (v_2))$$

$$= \frac{z_1 \overline{z}_2}{m} \left\{ \sum_{i=1}^{m-1} (\zeta^{m-i} - 1) \overline{(\zeta^{m-i} - 1)} (q (v_1 \otimes v_2) + q (v_3 \otimes v_1)) - \sum_{i=1}^{m-2} (\zeta^{m-i-1} - 1) \overline{(\zeta^{m-i} - 1)} q (v_1 \otimes v_2) - \sum_{i=1}^{m-2} (\zeta^{m-i} - 1) \overline{(\zeta^{m-i-1} - 1)} q (v_3 \otimes v_1) \right\}$$

$$= \frac{z_1 \overline{z}_2}{m} \left\{ 2m (q (v_1 \otimes v_2) + q (v_2 \otimes v_1)) - m (1 + \overline{\zeta}) q (v_1 \otimes v_2) - m (1 + \zeta) q (v_2 \otimes v_1) \right\}$$

$$z_1 \overline{z}_2 ((1 - \overline{\zeta}) q (v_1 \otimes v_2) + (1 - \zeta) q (v_3 \otimes v_1)) = q_{(\zeta)} ((v_1 \otimes z_1) \otimes (v_2 \otimes z_2))$$

4.8. Levine-Tristram signature. The signatures of the Hermitian forms of a Seifert pairing (in the case of knots of arbitrary dimension and one-dimensional links) have been considered by Tristram [9] and Levine [5, 6]. (In Tristram's article [9] the signature of the form $q(\exp(p-1)\pi i/p)$ is denoted by $\sigma_p(q)$, and in Levine's article [5] if A is the matrix of q, then the signature of the form $q_{(\overline{\zeta})}$ is denoted by $\sigma_A(-\zeta)$.)

By virtue of the main theorem and the result of the preceding subsection the signatures of the Hermitian forms of a Seifert pairing corresponding to roots of unity can be obtained as invariants of cyclic branched covers. More precisely, if in the conditions of the main theorem k is odd and ζ is a primitive *m*th root of 1, then the signature of the Hermitian form of the Seifert pairing of (M, N) corresponding to ζ is equal to the signature of the restriction to $\operatorname{Ker}(T_* - \zeta)$ of the Hermitian form constructed from the form of $(N_m(I \times M, A), T)$.

§5. Proof of the main theorem

5.1. Choice of the submani/old A. As was proved in §4.1, the class of Z-forms cobordant to the form of the canonical cover $(N_m(I \times M, A), T)$ does not depend on A. Therefore it is sufficient to prove the main theorem for any particular A. We shall suppose that A has been obtained by rounding the corners of the "submanifold with corners" $[\frac{1}{2}, 1] \times L \cup [\frac{1}{2} \times N]$. More precisely, let $c: I \times L \to N$ be an embedding with c(1, x) = x for $x \in L$, and let $f: I \to I$ be a smooth function with $f(t) = \frac{1}{2}$ for $t \leq \frac{1}{2}$, f(1) = 1, and f'(t) > 0 for $t > \frac{1}{2}$. Define the function $h: N \to I$ by the formula

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$$h(x) = \begin{cases} f(t), & \text{if } x = c(t, y) \in c(I \times L) \\ \frac{1}{2}, & \text{if } x \in N \setminus c(I \times L). \end{cases}$$

Set $A = \{(t, x) \in I \times N | t \ge h(x)\}$. 5.2. Auxiliary objects. We shall denote by S the set

$$\{(t, x) \in I \times N | t \ge h(x)\}$$



Figure 1

(see Figure 1); it is clear that S is a submanifold of dimension 2k + 1 with corners on the boundary along $\{1\} \times L$. We introduce the abbreviations

$$U = I \times M \setminus S, \quad X = I \times M, \quad \widetilde{U} = P_m^{-1}(U).$$
$$\widetilde{X} = N_m(I \times M, A), \quad \widetilde{A} = P_m^{-1}(A).$$

Since the linking coefficient of the class $x \in H_1(X \setminus A)$ with the fundamental class [A] of A is equal to the intersection index of x with the class in $H_{2k+1}(X, A \cup \partial X)$ realizable by S, the linking coefficients of the classes in the image of the inclusion homomorphism $H_1(U) \rightarrow H_1(X \setminus A)$ with the class [A] are equal to zero. Therefore the restriction $U \rightarrow U$ of the canonical covering $P_m: X \rightarrow X$ is trivial, i.e. U consists of m components each of which is mapped diffeomorphically onto U by P_m . Let U_0 be one of the components of U; we set $U_j = T^j(U_0)$ for $j = 1, \dots, m-1$.

We shall denote by π the natural projection of $X = I \times M$ onto M. We fix an arbitrary Riemannian metric on X. Let C_1, \dots, C_g be compact, connected, oriented, (k + 1)-dimensional proper submanifolds of $[0, \frac{1}{2}] \times M$ satisfying the following conditions:

1) $\partial C_1, \dots, \partial C_p \in A$.

2) The submanifolds $\pi(\partial C_1), \dots, \pi(\partial C_g)$ realize some basis d_1, \dots, d_g of the space $\hat{H}_k(N) = \text{Ker}(H_k(N; \mathbf{Q}) \to H_k(M; \mathbf{Q})).$

3) C_1, \dots, C_g are orthogonal to the boundary of $[0, \frac{1}{2}] \times M$.

For $i = 1, \dots, g$ and $j = 0, \dots, m-2$ we denote by $C_{i,j}$ the "submanifold with corners"

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$$P_m^{-1}(C_i) \cap (U_{j+1} \cup \tilde{A} \cup U_j);$$

we orient $C_{i,j}$ so that the restriction $C_{i,j} \cap (U_{j+1} \cup \widetilde{A}) \to C_i$ of the projection P_m has degree + 1. We denote by $e_{i,j}$ the element of $H_{k+1}(\widetilde{X}; \mathbb{Q})$ whose representative is $C_{i,j}$; let \mathcal{E} be the subspace of $H_{k+1}(\widetilde{X}; \mathbb{Q})$ generated by the vectors $e_{i,j}$ $(i = 1, \dots, g; j = 0, \dots, m-2)$.

It is clear that $T_{*}^{j}e_{i,0} = e_{i,j}$ for $j \le m-2$ and $T_{*}^{m-1}e_{i,0} = -\sum_{j=0}^{m-2}e_{i,j}$. Thus \mathcal{E} is T_{*} -invariant.

5.3. Reduction to lemmas. The main theorem obviously follows from the following two lemmas.

Lemma 1. The nondegenerate part of the \mathbb{Z}_m -form $(H_{k+1}(\tilde{X}; Q), Q, T_*)$ and its restriction to \mathcal{E} are canonically isomorphic.

Lemma 2. The restriction of $(H_{k+1}(\widetilde{X}; \mathbb{Q}), \mathbb{Q}, T_*)$ to \mathfrak{E} is isomorphic to the \mathbb{Z}_m -form of the pairing $\theta: \hat{H}_k(N) \otimes \hat{H}_k(N) \to \mathbb{Q}$ (symmetric if k is odd and skew-symmetric if k is even).

5.4. Proof of Lemma 1. Let R be a regular neighborhood of N in M. We set

$$V = I \times R, \quad W = U \cup V, \quad \tilde{V} = P_m^{-1}(V), \quad \tilde{W} = P_m^{-1}(W),$$
$$\tilde{M} = P_m^{-1}(\{0\} \times M), \quad \tilde{N} = P_m^{-1}(\{0\} \times N),$$
$$W_j = \tilde{W} \cap U_j, \quad N_j = \tilde{N} \cap U_j.$$

The natural deformation retraction $V \rightarrow I \times N \rightarrow A$ induces a deformation retraction $\widetilde{V} \rightarrow \widetilde{A}$. It is clear that the composition of the inclusion $\{0\} \times N \rightarrow V$ and the retraction $V \rightarrow A$ is a diffeomorphism. This composition induces a mapping $\widetilde{N} \rightarrow \widetilde{A}$ whose restriction to N_j is also a diffeomorphism. Therefore the inclusion homomorphism ν_* : $H_*(W; \mathbb{Q}) \rightarrow H_*(V; \mathbb{Q})$ is surjective.

Consider the segment of the homology addition sequence of the triad $(\widetilde{X}; \widetilde{U}, \widetilde{V})$:

$$H_{k+1}(\widetilde{W}; \mathbf{Q}) \to H_{k+1}(\widetilde{V}; \mathbf{Q}) \oplus H_{k+1}(\widetilde{U}; \mathbf{Q}) \stackrel{\Psi}{\to}$$
$$\stackrel{\Psi}{\to} H_{k+1}(\widetilde{X}; \mathbf{Q}) \stackrel{\chi}{\to} H_k(\widetilde{W}; \mathbf{Q}) \stackrel{\Psi}{\to} H_k(\widetilde{V}; \mathbf{Q}) \oplus H_k(\widetilde{U}; \mathbf{Q}).$$

Since ν_* is surjective, it follows that Im ϕ coincides with the image of the inclusion homomorphism κ_* : $H_{k+1}(\widetilde{U}; \mathbb{Q}) \to H_{k+1}(\widetilde{X}; \mathbb{Q})$. The natural deformation retraction $U \to \{0\} \times M$ induces a deformation retraction $\widetilde{U} \to \widetilde{M}$, and therefore Im κ_* is contained in the image of the inclusion homomorphism $H_{k+1}(\widetilde{\partial X}; \mathbb{Q}) \to H_{k+1}(\widetilde{X}; \mathbb{Q})$ coinciding with the radical of \mathcal{Q} . Thus Im ϕ is contained in the radical of \mathcal{Q} , and hence the nondegenerate parts of $(H_{k+1}(\widetilde{X}; \mathbb{Q}), \mathcal{Q}, T_*)$ and its restriction to any T_* -invariant direct completion of Im ϕ are isomorphic.

We shall show that \mathcal{E} is a direct completion of Im $\phi = \text{Ker } \chi$. For this it is sufficient to prove that the classes $\chi e_{i,j}$ with $i = 1, \dots, g$ and $j = 0, \dots, m-2$ form a basis of the space Im $\chi = \text{Ker } \psi$.

Let R' be a tubular neighborhood of A in W, and let F_i be $C_i \cap \partial R'$ oriented like the boundary of $C_i \setminus \text{Int } R'$. Set $F_{i,j} = P_m^{-1}(F_i) \cap U_j$. It is clear that $F_{i,j+1} \cup (-F_{i,j})$ realizes the class $\chi e_{i,j}$. On the other hand, $F_{i,j}$ realizes the image of d_i under the composition of the natural isomorphism $H_k(N; \mathbb{Q}) \to H_k(N_j; \mathbb{Q})$ and the inclusion homomorphism $H_k(N_j; \mathbb{Q}) \to H_k(\widetilde{W}; \mathbb{Q})$. The natural deformation retractions $W \to \{0\} \times N$ and $U \to \{0\} \times M$ induce deformation retractions $\widetilde{W} \to \widetilde{N}$ and $\widetilde{U} \to \widetilde{M}$. Therefore in the diagram

$$\begin{array}{c} H_k(\widetilde{N}; \mathbf{Q}) \to H_k(\widetilde{M}; \mathbf{Q}) \\ \downarrow \\ H_k(\widetilde{W}; \mathbf{Q}) \xrightarrow{\omega_*} H_k(\widetilde{U}; \mathbf{Q}) \end{array}$$

formed by inclusion homomorphisms, the vertical arrows are isomorphisms. Thus the classes $d_{i,j} \in H_k(\widetilde{W}; \mathbb{Q})$ realizable by the submanifolds $F_{i,j}$ form a basis of Ker ω_* . Since the composition of the inclusion $N_j \to \widetilde{V}$ and the deformation retraction $\widetilde{V} \to \widetilde{A}$ is a diffeomorphism and N_j is a deformation retract of W_j , the inclusion homomorphism

$$(\mathbf{v}_{\bullet} | H_k(W_j; \mathbf{Q})) : H_k(W_j; \mathbf{Q}) \to H_k(V; \mathbf{Q})$$

is an isomorphism. Furthermore, it is clear that $\nu_* d_{i,j_1} = \nu_* d_{i,j_2}$ for any i, j_1 and j_2 . Consequently the classes $\chi e_{i,j} = d_{i,j+1} - d_{i,j}$ form a basis for the space

Ker $v_* \cap$ Ker $\omega_* =$ Ker ψ .

5.5. Proof of Lemma 2. As was shown in the previous subsection, the class $\chi e_{i,j}$ form a basis for Im χ . From this it follows that the classes $e_{i,j}$ are linearly independent. Furthermore,

$$T^{j}_{*}e_{i,0} = \begin{cases} e_{i,j}, & \text{if } j \leq m-2, \\ -\sum_{l=0}^{m-2} e_{i,l}, & \text{if } j = m-1. \end{cases}$$

Therefore it is sufficient for the proof of Lemma 2 to prove that

$$Q(e_{i_{1},j_{1}} \otimes e_{i_{2},j_{2}}) = \begin{cases} 0, & \text{if } |j_{1}-j_{2}| > 1, \\ -\theta(d_{i_{1}} \otimes d_{i_{2}}), & \text{if } j_{1}=j_{2}+1, \\ -e\theta(d_{i_{2}} \otimes d_{i_{1}}), & \text{if } j_{2}=j_{1}+1, \\ \theta(d_{i_{1}} \otimes d_{i_{2}}) + e\theta(d_{i_{2}} \otimes d_{i_{1}}), & \text{if } j_{1}=j_{2}. \end{cases}$$

First we construct smooth manifolds realizing $e_{i,j}$. Let β be the set of unit vectors of the normal bundle of $A \cap \{\frac{1}{2}\} \times M$ in X directed to the inside of $[\frac{1}{2}, 1] \times M$, and let γ be the unit vector field on $A \cap \{\frac{1}{2}\} \times M$ tangent to $\{\frac{1}{2}\} \times M$ and normal to A determined by the orientations of A and M. We denote by β_i the preimage of ∂C_i under the natural projection $\beta \to A \cap \{\frac{1}{2}\} \times M$. We extend the fields $\gamma \cap \beta_1, \dots, \gamma \cap$ β_g to normal fields $\gamma_1, \dots, \gamma_g$ on C_1, \dots, C_g such that the sets $\gamma_1 \cup \beta_1, \dots, \gamma_g \cup \beta_g$ are smooth submanifolds of the tangent bundle τX and such that $\gamma_1, \dots, \gamma_g$ are transversal to each other and to the zero section.

Since $\bigcup_{i=1}^{g} C_{i}$ is compact, the lengths of the vectors in $\bigcup_{i=1}^{g} \gamma_{i}$ do not exceed some number r. Let $\rho > 0$ be a real number such that there exist geodesic tubular neighborhoods of the submanifolds C_{1}, \dots, C_{g} and $A \cap \{\frac{1}{2}\} \times M$ in X of radius ρ . For $i = 1, \dots, g$ and $t \in [-1, 1]$ we denote by C_{i}^{t} the image of the submanifold $t\gamma_{i}$ of the tangent bundle rX under the mapping $x \to \exp(\rho x/r)$, and for $i = 1, \dots, g$ and $t \in (0, 1]$ we denote by D_{i}^{t} the image of $t\beta_{i}$ under the same mapping. The sets C_{i}^{t} and D_{i}^{t} are smooth submanifolds of X. We orient C_{i}^{t} such that the natural diffeomorphism $C_{i}^{t} \to C_{i}$ has degree +1 if $t \ge 0$ and degree -1 in the opposite case.

Set $C_{i,j}^t = P_m^{-1}(C^t) \cap U_j$; we denote by $D_{i,j}^t$ the component of $P_m^{-1}(D_i^t)$ intersecting $C_{i,j}^t$, and for t > 0 we set

$$E_{i,j}^t = C_{i,j+1}^t \bigcup D_{i,j}^t \bigcup C_{i,j}^{-t}.$$

Orient $E_{i,j}^t$ to correspond to the orientation of $C_{i,j+1}^t$. Obviously $E_{i,j}^t$ is *pl*-isotopic to the submanifold with corners $C_{i,j}^t$. Thus $E_{i,j}^t$ realizes $e_{i,j}^t$.

Now we shall concern ourselves with calculating $Q(e_{i_1,j_1} \otimes e_{i_2,j_2})$. For this we shall find the intersection index of $E_{i_1,j_1}^{\frac{1}{2}} \cdot E_{i_2,j_2}^1$. Obviously

$$D_{i_1,j_1}^{\frac{1}{2}} \cap D_{i_1,j_2}^{1} \subset P_m^{-1} (D_{i_1}^{\frac{1}{2}} \cap D_{i_2}^{1}) \subset P_m^{-1} \left(\exp \left(\frac{\rho}{2r} \beta \cap \frac{\rho}{r} \beta \right) \right) = \emptyset$$

Consequently

$$E_{i_1,j_1}^{\frac{1}{2}} \cdot E_{i_2,j_2}^{1} = (C_{i_1,j_1+1}^{\frac{1}{2}} \cup C_{i_1,j_2}^{\frac{1}{2}}) \cdot (C_{i_2,j_2+1}^{1} \cup C_{i_2,j_2}^{-1})$$

Moreover, $C_{i,j}^i \subseteq U_j$. Therefore, if $|j_1 - j_2| > 1$, then

$$E_{i_{1},j_{2}}^{\frac{1}{2}} \cdot E_{i_{1},j_{2}}^{1} = 0.$$

Furthermore, if $t_1, t_2 \in [-1, 1]$ and $t_1 \neq t_2$, then the intersection of $C_{i_1, j}^{t_1} \cdot C_{i_2, j}^{t_2} = C_{i_1}^{t_1} \cdot C_{i_2}^{t_2}$ is equal to linking coefficient of $\partial C_{i_1}^{t_2}$ and $\partial C_{i_2}^{t_2}$ in $\{\frac{1}{2}\} \times M$. We shall denote this linking coefficient by $\mathfrak{L}(\partial C_{i_1}^{t_1}, \partial C_{i_2}^{t_2})$.

Now let $j_1 = j_2 = j$. We have

$$E_{i_{1},j_{1}}^{\frac{1}{2}} \cdot E_{i_{s},j_{s}}^{1} = C_{i_{1}}^{\frac{1}{2}} \cdot C_{i_{s}}^{1} + C_{i_{1}}^{-\frac{1}{2}} \cdot C_{i_{s}}^{-1} = \mathcal{L}(\partial C_{i_{s}}^{\frac{1}{2}}, \partial C_{i_{s}}^{1}) + \mathcal{L}(\partial C_{i_{s}}^{-\frac{1}{2}}, \partial C_{i_{s}}^{-1}).$$

Obvious isotopies take $\partial C_{i_1}^{\vee} \bigcup \partial C_{i_2}^1$ to $\partial C_{i_1} \bigcup \partial C_{i_2}^{\vee}$ and $\partial C_{i_1}^{-\vee} \bigcup \partial C_{i_2}^{-1}$ to $(-\partial C_{i_1}^{\vee}) \bigcup (-\partial C_{i_2})$. Thus

$$E_{i_{\mathbf{s}},j}^{\frac{1}{2}} \cdot E_{i_{\mathbf{s}},j}^{l} = \mathcal{L}(\partial C_{i_{\mathbf{s}}}, \partial C_{i_{\mathbf{s}}}^{\frac{1}{2}}) + \mathcal{L}(-\partial C_{i_{\mathbf{s}}}^{\frac{1}{2}}, -\partial C_{i_{\mathbf{s}}}) = \mathcal{L}(\partial C_{i_{\mathbf{s}}}, \partial C_{i_{\mathbf{s}}}^{\frac{1}{2}}) + (-1)^{k+1} \mathcal{L}(\partial C_{i_{\mathbf{s}}}, \partial C_{i_{\mathbf{s}}}^{\frac{1}{2}}).$$

But by definition of the Seifert pairing we have

$$\mathscr{L}(\partial C_{l_1}, \partial C_{l_2}^{\frac{1}{2}}) = \theta(d_{l_1} \otimes d_{l_2})$$

and

$$\mathscr{L}(\partial C_{l_1}, \partial C_{l_1}^{\frac{1}{2}}) = \theta(d_{l_2} \otimes d_{l_1});$$

hence, if $j_1 = j_2$, then

Analogous arguments show that if $j_1 = j_2 + 1$, then

$$E_{i_{s},i_{s}}^{\frac{1}{2}}\cdot E_{i_{s},i_{s}}^{t}=-\theta(d_{i_{s}}\otimes d_{i_{s}}),$$

and if $j_2 = j_1 + 1$, then

$$E_{l_1,j_1}^{\overline{2}} \cdot E_{l_2,j_2}^{\mathbb{I}} = (-1)^k \theta (d_{l_2} \otimes d_{l_1}).$$

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