SIGNATURE OF A BRANCHED COVERING

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This paper is written as differential-topological. Apparently, the formula proved is also true for piecewise-linear and topological branched coverings with locally flat branching manifolds, although the differential-topological proof given here does not admit automatic transfer to these categories.

The main result is formulated and proved in Sec. 2. In Sec. 1 we recount auxiliary material. In Paragraph 1.1 we reproduce definitions and some results relating to the theory of branched coverings. (In this theory there is no commonly accepted system of definitions, and hence some precision about terminology is necessary.) In Paragraph 1.2 we describe the familiar (cf., e.g., [1]) construction of self-intersections of a smooth submanifold which occurs in the following formulations. In Paragraph 1.3 we recount Hirzebruch's formula.

1. Preliminary Information

1.1. Smooth Branched Coverings. Let m and n be natural numbers greater than one. By model m-sheeted branched coverings of dimension n we mean the maps

$$(x, z) \mapsto (x, z^m)\colon \mathbf{R}^{n-2} \times \mathbf{C} \to \mathbf{R}^{n-2} \times \mathbf{C},$$

$$(x, z) \mapsto (x, z^m)\colon \mathbf{R}_+^{n-2} \times \mathbf{C} \to \mathbf{R}_+^{n-2} \times \mathbf{C}.$$

Let X and Y be (smooth) manifolds. A surjective map P: Y → X is called a (smooth) branched covering if each point of the manifold X has a neighborhood U such that its preimage $P^{-1}(U)$ can be represented as a union $\bigcup_\alpha V_\alpha$ of mutually disjoint open sets V_α, for each of which the map $V_\alpha \to U$ defined by the map P is either a diffeomorphism or is diffeomorphic with one of the model branched coverings. The manifold X is called the base of the branched covering P: Y → X, and Y is a branched covering of the manifold X. By the ramification index of the branched covering P: Y → X at the point $y \in Y$ is meant the absolute value of the local degree of the map P at the point y.

The set of points of the manifold Y at which the ramification index of the smooth branched covering P: Y → X is equal to the number m, greater than one, is a (smooth) proper submanifold of codimension 2 of the manifold Y, transverse to ∂Y. It is denoted by $B_{m,P}$. The manifolds $B_{2,P}, B_{3,P}, \ldots$ are mutually disjoint and their union $\bigcup_{m=2}^{\infty} B_{m,P}$ is called the branching manifold of the branched covering P: Y → X and is denoted by B_P.

If the base of the branched covering P: Y → X is oriented, then there exists a unique orientation of the branched covering such that with respect to these orientations the local degree of the map P is positive at any point of the manifold Y. Such orientations of the manifolds X and Y are said to be compatible.

A diffeomorphism $T\colon Y \to Y$ is called an automorphism of the branched covering $P\colon Y \to X$, if $P \circ T = P$. The automorphisms of a branched covering P form a group which is denoted by Aut(P). If the map of the space of orbits Y/Aut(P) onto X, induced by the map P, is a homeomorphism, then the branched covering P is called regular. A regular branched covering is called cyclic, if its group of automorphisms is a cyclic group.

Let B be a manifold, $\nu\colon N \to B$ and $\mu\colon M \to B$ be (smooth) bundles with fiber D^2. The map $\pi\colon N \to M$ is called an m-sheeted *branched morphism* of the bundle ν into the bundle μ, if each point of the manifold B has a neighborhood U such that there exist trivializations $\tilde{t}\colon U \times D^2 \to \nu^{-1}(U)$ and $t\colon U \times D^2 \to \mu^{-1}(U)$ of the restrictions of the bundles ν and μ to U, such that

$$t^{-1}\pi\tilde{t}\,(u, v) = (u, v^m)$$

for $(u, v) \in U \times D^2$ ($\subset U \times \mathbf{C}$). It is clear that π is an m-sheeted branched covering, whose branching manifold is the zero section of the bundle ν.

Leningrad State University. Translated from Matematicheskie Zametki, Vol. 36, No. 4, pp. 549-557, October, 1984. Original article submitted April 16, 1982.

 0001-4346/84/3634-0772$08.50

If m = 2, then this branched covering (as well as any two-sheeted branched covering in general) is cyclic.

1.1.A (cf. [2, Theorem B]). Let $\pi\colon N \to M$ be an m-sheeted branched morphism of the D^2-bundle $\nu\colon N \to B$ into the D^2-bundle $\mu\colon M \to B$. If m > 2, then the cyclicity of π as a branched covering is equivalent with the orientability of either of the bundles μ and ν.

If $p\colon \tilde{B} \to B$ is the two-sheeted covering defined by the class $w_1(\mu)$, and $\tilde{\nu}\colon \tilde{N} \to \tilde{B}$ and $\tilde{\mu}\colon \tilde{M} \to \tilde{B}$ are the bundles induced by the map p and the bundles ν and μ, then there exists an m-sheeted branched morphism $\tilde{\pi}\colon \tilde{N} \to \tilde{M}$ of the bundle $\tilde{\nu}$ into $\tilde{\mu}$, covering the morphism π, and which is a cyclic m-sheeted branched covering.

The following routine theorem shows that in a neighborhood of the branch manifold any branched covering can be constructed as a branched morphism.

1.1.B. Let $P\colon Y \to X$ be a branched covering. Then for each r > 1 there exist D^2-bundles $\nu_r\colon N_r \to B_{r,P}$ and $\mu_r\colon M_r \to B_{r,P}$, an imbedding $i_r\colon N_r \to Y$ and an immersion $j_r\colon M_r \to X$, extending the inclusion $B_{r,P} \subset Y$ and the immersion $P|_{B_{r,P}}\colon B_{r,P} \to X$, and an r-sheeted branched morphism $\pi_r\colon N_r \to M_r$, such that $\bigcup_{r=2}^{\infty} i_r(N_r)$ is a tubular neighborhood of the submanifold B_P and for each r the diagram

$$\begin{array}{ccc} N_r & \xrightarrow{i_r} & Y \\ {\scriptstyle \pi_r}\downarrow & & \downarrow{\scriptstyle P} \\ M_r & \xrightarrow{j_r} & X \end{array}$$

is commutative.

<u>1.2. Self-Intersections of Submanifolds.</u> Let L be a closed submanifold of the manifold M. We construct a sequence $L_0 \supset L_1 \supset L_2 \supset \dots$ of submanifolds of the manifold M. We set $L_0 = M$, $L_1 = L$, and if L_r is already constructed, then by making a small isotopy of the inclusion $L_r \subset M$, we get an imbedding $i_r\colon L_r \to M$, transverse to L, and we set $L_{r+1} = L \cap i_r(L_r)$.

If L and M are oriented, then all L_r also get the natural orientation (as transverse intersections of oriented submanifolds of an oriented manifold).

It turns out that if r is even, this orientation of the submanifold L_r is independent of the orientation of the submanifold L. Moreover, even if L is nonorientable, the submanifold L_r with even r is orientable and has a natural orientation which depends only on the orientation of the manifold M, and which is constructed as follows.

We shall denote the normal bundle of the submanifold A of the manifold X by $\nu_X A$. Obviously the bundle $\nu_M L_r$ splits into a direct sum:

$$\bigoplus_{k=1}^{r} (\nu_{L_{k-1}} L_k)|_{L_r}. \tag{1}$$

In view of the smallness of the isotopy which connects the inclusion $L_{k-1} \subset M$ with i_{k-1}, one can assume that it is fixed on L_k and hence induces an isomorphism

$$(\nu_{L_{k-2}} L_{k-1})|_{L_k} \to \nu_{L_{k-1}} L_k.$$

Thus, there are canonical isomorphisms between the summands of (1). Hence if the fiber of one summand is oriented, there arises an orientation of the fiber of each summand, which if r is even, is independent of the initial orientation. Thus, if r is even, the bundle $\nu_M L_r$ has a natural orientation. Together with the orientation of the manifold M this gives an orientation of the manifold L_r.

The manifolds L_r depend on the arbitrariness in the construction of the imbeddings i_r. However, for another choice of imbeddings, we get, as is easily seen, cobordant submanifolds, and in the cases when they have natural orientations, they are oriented cobordant to the previous submanifolds. In these cases the cobordism class of the submanifold L_r (which is an element of the group Ω_{n-2r}) is denoted by L^r.

<u>1.3. Hirzebruch's Formula.</u> The signature of an oriented closed 4k-dimensional manifold (i.e., the signature of its intersection index $H_{2k}(M;\mathbb{Q}) \times H_{2k}(M;\mathbb{Q}) \to \mathbb{Q}$) will be denoted by sign M. If L is a closed submanifold of codimension 2 of the oriented closed 4k-dimensional manifold M, and α is a function which can be represented by the power series $\sum_{r=0}^{\infty} \alpha_r t^r$, then by sign $\alpha(L)$ we denote the sum

$$\sum_{r=0}^{2k} \alpha_{2r} \operatorname{sign} L^{2r}.$$

1.3.A (Hirzebruch's Theorem [3, Sec. 5]). Let X, Y be closed 4k-dimensional manifolds, $P\colon Y \to X$ be a cyclic m-sheeted branched covering with $B_P = B_{m,P}$, and let X and Y be oriented compatibly. Then

$$\operatorname{sign} X = \operatorname{sign} \frac{(1+B_P)^m + (1-B_P)^m}{(1+B_P)^m - (1-B_P)^m} B_P = \frac{1}{m} \operatorname{sign} Y + \frac{m^2-1}{3m} \operatorname{sign} B_P^2 + \dots \tag{2}$$

Eq. (2) can be rewritten in the following form, which is more convenient for generalization:

$$\operatorname{sign} Y = m \operatorname{sign} X - \operatorname{sign} \frac{(1+mB_P)(1-B_P)^m - (1-mB_P)(1+B_P)^m}{(1+B_P)^m - (1-B_P)^m}. \tag{3}$$

We denote the function

$$x \mapsto \frac{(1+mx)(1-x)^m - (1-mx)(1+x)^m}{(1+x)^m - (1-x)^m}$$

by Π_m. Then (3) becomes

$$\operatorname{sign} Y = m \operatorname{sign} X - \operatorname{sign} \Pi_m (B_P). \tag{4}$$

2. Generalization of Hirzebruch's Formula

2.1. Basic Theorem. Let X, Y be closed 4k-dimensional manifolds, $P\colon Y \to X$ be an m-sheeted branched covering, and let X and Y be compatibly oriented. Then

$$\operatorname{sign} Y = m \operatorname{sign} X - \sum_{r=2}^{m} \operatorname{sign} \Pi_r (B_{r,P}). \tag{5}$$

Proof. Let $N_r, M_r, \nu_r\colon N_r \to B_{r,P}$, $\mu_r\colon M_r \to B_{r,P}$, $i_r\colon N_r \to Y$, $j_r\colon M_r \to X$ and $\pi_r\colon N_r \to M_r$ be the manifolds and mappings which exist by virtue of 1.1.B. We introduce a Riemannian metric in X. We set $N = \bigcup_{r=2}^{m} N_r$. In $Y \setminus \operatorname{int} N$ we take the Riemannian metric induced by P from X and we extend it to all of Y. In M_r we take the Riemannian metric induced by j_r from X.

We denote by $\mathcal{L}$ the differential form induced in the canonical way by the Riemannian metric, which realizes the Hirzebruch class $L_k(p_1, \dots, p_k)$. By Hirzebruch's theorem [4, Theorem 8.22],

$$\operatorname{sign} Y = \int_Y \mathcal{L} \tag{6}$$

and

$$\operatorname{sign} X = \int_X \mathcal{L}. \tag{7}$$

By the naturality of the form $\mathcal{L}$ with respect to local isometries,

$$\int_{Y \setminus N} \mathcal{L} = \int_{Y \setminus N} P^* (\mathcal{L}) = \int_X \varphi \mathcal{L}, \tag{8}$$

where $\varphi\colon X \to \mathbf{Z}$ is the function defined by

$$\varphi(x) = \operatorname{card} (P^{-1}(x) \setminus N).$$

For the same reason,

$$\int_{M_r} \mathcal{L} = \int_{M_r} j_r^* (\mathcal{L}) = \int_X \varphi_r \mathcal{L}, \tag{9}$$

where $\varphi_r\colon X \to \mathbf{Z}$ is the function defined by

$$\varphi_r(x) = \operatorname{card} (j_r^{-1}(x)).$$

Obviously,

$$\varphi(x) = m - \sum_{r=2}^{m} r \varphi_r(x)$$

for $x \in X$, and hence

$$\int_X \varphi \mathscr{L} = m\int_X \mathscr{L} - \sum_{r=2}^{m} r\int_X \varphi_r \mathscr{L}.$$

From this equation and from (7), (8), and (9), we get

$$\int_{Y\setminus N} \mathscr{L} = m\,\mathrm{sign}\,X - \sum_{r=2}^{m} r\int_{M_r} \mathscr{L}. \tag{10}$$

Representing the integral $\int_Y \mathscr{L}$ as the sum $\int_{Y\setminus N} \mathscr{L} + \sum_{r=2}^{m}\int_{N_r} \mathscr{L}$, and using (6) and (10), we get

$$\mathrm{sign}\,Y = m\,\mathrm{sign}\,X - \sum_{r=2}^{m}\left(r\int_{M_r} \mathscr{L} - \int_{N_r} \mathscr{L}\right). \tag{11}$$

By 1.1.A, the branched morphism $\pi_r\colon N_r \to M_r$ of the bundle $\nu_r\colon N_r \to B_{r,P}$ into the bundle $\mu_r\colon M_r \to B_{r,P}$ is two-sheeted covered by the branched morphism $\tilde{\pi}_r\colon \tilde{N}_r \to \tilde{M}_r$, which is the cyclic r-sheeted branched covering. In $\tilde{N}_r$ and $\tilde{M}_r$ we take the induced Riemannian metrics from N_r and M_r.

As follows from the theory of cobordisms of free actions constructed by Conner and Floyd [5], there exist a natural number c, compact oriented manifolds F_r and G_r, and an r-sheeted cyclic (unbranched) covering $Q_r\colon G_r \to F_r$, such that the restriction to $\partial G_r \to \partial F_r$ of the latter is diffeomorphic with the c-fold disconnected sum of the restriction to $(-\partial\tilde{N}_r) \to (-\partial\tilde{M}_r)$ of the branched covering $\tilde{\pi}_r\colon \tilde{N}_r \to M_r$. We extend the Riemannian metric from the boundary $\partial F_r = c(-\partial\tilde{M}_r)$ to the manifold F_r so that together with the existing Riemannian metric on $\tilde{M}_r$, it gives a Riemannian metric on the closed manifold $F_r \bigcup c\tilde{M}_r$. We let the Riemannian metric in F_r induce a Riemannian metric in G_r by means of Q_r.

By Hirzebruch's formula applied to the r-sheeted cyclic branched covering

$$Q_r \bigcup c\tilde{\pi}_r\colon G_r \bigcup_{\partial G_r} c\tilde{N}_r \to F_r \bigcup_{\partial F_r} c\tilde{M}_r,$$

$$\mathrm{sign}\,(G_r \bigcup_{\partial G_r} c\tilde{N}_r) = r\,\mathrm{sign}\,(F_r \bigcup_{\partial F_r} c\tilde{M}_r) - \mathrm{sign}\,c\Pi_r\,(B_{r,\tilde{\pi}_r}). \tag{12}$$

On the other hand, by Hirzebruch's theorem [4, Theorem 8.22],

$$\mathrm{sign}\,(G_r \bigcup_{\partial G_r} c\tilde{N}_r) = \int_{G_r} \mathscr{L} + c\int_{\tilde{N}_r} \mathscr{L} = \int_{G_r} \mathscr{L} + 2c\int_{N_r} \mathscr{L}, \tag{13}$$

$$\mathrm{sign}\,(F_r \bigcup_{\partial F_r} c\tilde{M}_r) = \int_{F_r} \mathscr{L} + c\int_{\tilde{M}_r} \mathscr{L} = \int_{F_r} \mathscr{L} + 2c\int_{M_r} \mathscr{L}. \tag{14}$$

Finally, by the naturality of the form $\mathscr{L}$ with respect to local isometries,

$$\int_{G_r} \mathscr{L} = r\int_{F_r} \mathscr{L}.$$

From this and from (12)-(14), we get

$$r\int_{F_r} \mathscr{L} + 2c\int_{N_r} \mathscr{L} = r\int_{F_r} \mathscr{L} + 2rc\int_{M_r} \mathscr{L} - c\,\mathrm{sign}\,\Pi_r\,(B_{r,\tilde{\pi}_r})$$

or

$$2\left(r\int_{M_r} \mathscr{L} - \int_{N_r} \mathscr{L}\right) = \mathrm{sign}\,\Pi_r\,(B_{r,\tilde{\pi}_r}). \tag{15}$$

The manifold $(B_{r,\tilde{\pi}_r})_s$ obviously covers $(B_{r,\pi_r})_s = (B_{r,P})_s$ two-sheetedly for any s. Consequently,

$$\mathrm{sign}\,\Pi_r\,(B_{r,\tilde{\pi}_r}) = 2\,\mathrm{sign}\,\Pi_r\,(B_{r,P}). \tag{16}$$

Eq. (5) is obtained by comparing (11), (15), and (16).

2.2. Four-Dimensional Case. Under the hypotheses of Theorem 2.1, let dim X = dim Y = 4. Then

$$\mathrm{sign}\,\Pi_r\,(B_{r,P}) = \frac{r^2-1}{3}\,e\,(B_{r,P}) = \frac{r^2-1}{3r}\,e\,(P\,|\,B_{r,P}),$$

where e denotes the normal Euler number of a submanifold and immersion.

Thus, by Theorem 2.1,

$$\operatorname{sign} Y = m \operatorname{sign} X - \sum_{r=2}^{m} \frac{r^2-1}{3} e(B_{r,P}) \tag{17}$$

$$= m \operatorname{sign} X - \sum_{r=2}^{m} \frac{r^2-1}{3r} e(P|_{B_{r,P}}). \tag{18}$$

2.3. Remark. Theorem 2.1 was formulated by me as a conjecture at the academic topology seminar of Leningrad University, after which A. Yu. Nenashev and N. Yu. Netsvetaev, who were participants in this seminar, proved special cases of it: A. Yu. Nenashev found a proof for the case $B_P = B_{2,P}$, and N. Yu. Netsvetaev for the case dim X = 4. Their proofs were modifications of the elementary proofs of the corresponding special cases of Hirzebruch's formula due to Jänich and Ossa [6] and Gordon [7]. I take this opportunity to thank N. Yu. Netsvetaev and A. Yu. Nenashev for helpful discussions.

LITERATURE CITED

1. M. F. Atiyah and I. M. Singer, "Index of elliptic operators. III," Usp. Mat. Nauk, 24, No. 1, 127-182 (1969).
2. I. V. Savel'ev, "Piecewise-linear branched coverings," Candidate's Dissertation, Moscow (1977).
3. F. Hirzebruch, "The signature of ramified coverings," in: Global Analysis, Papers in Honor of K. Kodaira, Princeton (1969), pp. 253-265.
4. F. Hirzebruch, Topological Methods in Algebraic Geometry, Springer-Verlag (1966).
5. P. E. Conner and E. E. Floyd, Smooth Periodic Mappings [Russian translation], Mir, Moscow (1969).
6. K. Jänich and E. Ossa, "On the signature of an involution," Topology, 8, No. 1 (1969).
7. C. McA. Gordon, "On the G-signature theorem in dimension four" (Preprint).

ANALYTIC EXTENSION OF LOCALLY DEFINED RIEMANNIAN MANIFOLDS

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We say that we are given a germ of a Riemannian real analytic manifold, if at the point $0 \in \mathbf{R}^n$ there are defined germs of real analytic functions g_{ij}, $1 \leqslant i \leqslant j \leqslant n$, such that the quadratic form $g_{ij}x^i x^j$ is positive-definite. On some open set $U \subset \mathbf{R}^n$, these germs define analytic functions, which define on U the structure of a Riemannian real analytic manifold. We call this manifold the carrier of the germ.

By a manifold we shall always mean a connected manifold without boundary.

Definition 1. The Riemannian real analytic manifold $\mathcal{N}$ is called an analytic extension of the Riemannian real analytic manifold $\mathcal{M}$, if there exists an analytic isometric embedding $i: \mathcal{M} \to \mathcal{N}$ whose image is an open subset of $\mathcal{N}$.

By the extension of a germ, we mean an extension of some carrier of it.

The Riemannian real analytic manifold $\mathcal{M}$ is called nonextendable if it does not admit nontrivial extension.

By a maximal extension of a germ, we mean its analytic extension to a nonextendable manifold.

Definition 2. A local isometry of the Riemannian real analytic manifold $\mathcal{M}$ into the Riemannian real analytic manifold $\mathcal{N}$ is an isometry $\varphi: W \simeq V$ between the open subsets $W \subset \mathcal{M}$ and $V \subset \mathcal{N}$.

Institute of the National Economy, Moscow. Translated from Matematicheskie Zametki, Vol. 36, No. 4, pp. 559-570, October, 1984. Original article submitted September 17, 1982.

0001-4346/84/3634-0776$08.50