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Visible L-theory

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Abstract. The visible symmetric L-groups enjoy roughly the same formal properties as the symmetric L-groups of Mishchenko and Ranicki. For a fixed fundamental group π, there is a long exact sequence involving the quadratic L-groups of π, the visible symmetric L-groups of π, and some homology of π. This makes visible symmetric L-theory computable for fundamental groups whose ordinary (quadratic) L-theory is computable.

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0. Introduction

The visible symmetric L-groups are a refinement of the symmetric L-groups of Mishchenko [12] and Ranicki [15]. They are much easier to compute, although they hardly differ from the symmetric L-groups in their formal properties. Cappell and Shaneson [3] use the visible symmetric L-groups in studying 4-dimensional s-cobordisms; see also Kwasik and Schultz [8]. The visible symmetric L-groups are related to the Ronnie Lee L-groups of Milgram [11].

The reader is assumed to be familiar with the main results and definitions of Ranicki [15]. Let R be a commutative ring (with unit), let π be a discrete group, and equip the group ring $R[\pi]$ with the w-twisted involution for some homomorphism $w\colon \pi \to \mathbb{Z}_2$. An n-dimensional symmetric algebraic Poincaré complex (SAPC) over $R[\pi]$ is a finite dimensional chain complex C of finitely generated free left $R[\pi]$-modules, together with a nondegenerate n-cycle

$$\phi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C^t \otimes_{R[\pi]} C)$$

where W is the standard resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}_2]$. Now fix a right free resolution P of R over $R[\pi]$. An n-dimensional visible symmetric algebraic Poincaré complex (VSAPC) is a chain complex C as above, together with an n-cycle

$$\phi \in P \otimes_{R[\pi]} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C)$$

whose image in

$$R \otimes_{R[\pi]} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C) \cong \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C^t \otimes_{R[\pi]} C)$$

under the chain map induced by the augmentation $P \to R$ is nondegenerate. (Here $R[\pi]$ acts on $C \otimes_R C$ by the w-twisted diagonal action, $g(x \otimes y) = (-1)^{w(g)} gx \otimes gy$ for $g \in \pi$.) The bordism group of n-dimensional SAPC's is denoted by $L^n(R[\pi])$, that of n-dimensional VSAPC's is written $VL^n(R[\pi])$. The augmentation $P \to R$ converts VSAPC's to SAPC's and so induces a homomorphism $VL^n(R[\pi] \to L^n(R[\pi])$. The following lemma should clarify the definition.

Lemma. *Let E be any chain complex of left free $R[\pi]$-modules. Then the chain map*

$$P \otimes_{R[\pi]} E \to R \otimes_{R[\pi]} E$$

induced by the augmentation $P \to R$ is a chain equivalence over R, provided E is bounded from below.

Proof. Induct over the skeletons of E. □

The hypothesis of the lemma is usually not satisfied if

$$E = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C),$$

with C as above; but it is satisfied if

$$E = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C \otimes_R C),$$

since C is finite dimensional. Therefore

$$\begin{aligned} W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C^t \otimes_{R[\pi]} C) &\cong R \otimes_{R[\pi]} (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C \otimes_R C)) \\ &\simeq P \otimes_{R[\pi]} (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C \otimes_R C)), \end{aligned}$$

which shows that the symmetrization map

$$1 + T\colon W \otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t \otimes_{R[\pi]} C \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C^t \otimes_{R[\pi]} C)$$

of Ranicki [15] has a factorization through $P \otimes_{R[\pi]} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C)$. In other words, the notion of a quadratic algebraic Poincaré complex (QAPC) is finer than the notion of a VSAPC, which in turn is finer than the notion of a SAPC; so we have homomorphisms

$$L_n(R[\pi]) \to VL^n(R[\pi]) \to L^n(R[\pi])$$

For another application of the lemma, let Y be a simplicial set whose geometric realization is a connected finite Poincaré space of formal dimension n; write $\tilde{Y}$ for the universal cover, and π for the group of covering translations ($\pi = \pi_1(Y)$). Let C be the cellular chain complex of $\tilde{Y}$ with R-coefficients. Let $R[\pi]$ act on C by

$g[y] = (-1)^{w(g)}[gy]$, where y denotes a nondegenerate simplex in $\tilde{Y}$ and $[y]$ is the corresponding generator C. Eilenberg–Zilber theory gives us a chain map

$$C \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C)$$

(see Ranicki [15]) which, by naturality, commutes with the $R[\pi]$-actions on source and target. Tensoring with P and using the lemma, we obtain

$$R \otimes_{R[\pi]} C \simeq P \otimes_{R[\pi]} C \to P \otimes_{R[\pi]} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C).$$

Choose an n-cycle in $R \otimes_{R[\pi]} C$ representing the fundamental class of Y; its image ϕ in $P \otimes_{R[\pi]} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C)$ is such that (C, ϕ) is an n-dimensional VSAPC. Its bordism class in $VL^n(R[\pi])$ is the visible symmetric signature of the Poincaré space Y. It refines Ranicki's symmetric signature in $L^n(R[\pi])$.

Working with an abstract group π again, one can define relative bordism groups $V\hat{L}^n(R[\pi])$ to fit into a long exact sequence

$$\to V\hat{L}^{n+1}(R[\pi]) \to L_n(R[\pi]) \to VL^n(R[\pi]) \to V\hat{L}^n(R[\pi]) \to \dots,$$

with $n \in \mathbb{Z}$. (Compare with the groups $\hat{L}^n$ in [15].) Decorations p, h or s can be attached to L_n and VL^n, but the relative groups $V\hat{L}^n$ are the same in all three cases.

0.1. Theorem. *There are isomorphisms*

$$V\hat{L}^n(R[\pi]) \cong \bigoplus_{j \in \mathbb{Z}} H_{n-j}(\pi; \hat{L}^j(R)),$$

where π acts on the group $\hat{L}^j(R) = V\hat{L}^j(R)$ by $gx = (-1)^{w(g)}x$ for $g \in \pi$ and $x \in \hat{L}^j(R)$. (If $R = \mathbb{Z}$, then $\hat{L}^j(R)$ is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2$, 0, $\mathbb{Z}_2$ if $j \equiv 0, 1, 2, 3 \bmod 4$). □

Note: Most of the chain complexes used in this paper are graded over the integers and finite dimensional, but not necessarily trivial in negative dimensions. For this reason $\hat{L}^j(R)$ is periodic in j with period 4; and it can therefore very well be nonzero for negative j. Consequently $V\hat{L}^n(R[\pi])$ can very well be infinitely generated, even if $R = \mathbb{Z}$ and π is finite. The following remark may be of use to the "working mathematician". If a class in $V\hat{L}^n(R[\pi])$ has a representative involving only chain complexes which are zero in dimensions < 0, then the class belongs to the subgroup

$$\bigoplus_{j \geq -1} H_{n-j}(\pi; \hat{L}^j(R)) \subset \bigoplus_{j \in \mathbb{Z}} H_{n-j}(\pi; \hat{L}^j(R)).$$

(An element of $V\hat{L}^n(R[\pi])$ is a bordism class of certain algebraic Poincaré pairs, and an algebraic Poincaré pair involves two chain complexes.)

The definition of the groups $VL^n(R[\pi])$ uses special properties of $R[\pi]$ which an arbitrary ring with involution may not have. The point is that $R[\pi]$ is a cocommutative Hopf algebra, projective as a module over R, graded over $\mathbb{Z}_2$, and the involution on $R[\pi]$ is the intrinsic Hopf algebra involution. (The Hopf algebra diagonal $R[\pi] \to R[\pi] \times R[\pi]$ comes from the diagonal map $\pi \to \pi \times \pi$; the grading over $\mathbb{Z}_2$ assigns grade $w(g)$ to $g \in \pi \subset R[\pi]$.)

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In the earlier version, Theorem 0.1 above was presented as a corollary to the main theorem of Weiss [23]. Later, Andrew Ranicki found a more elementary proof of 0.1 which fits very well into the general theory of assembly in algebraic surgery. This is the proof given here.

1. Constructing L-theory spectra

Let $\mathbb{A}$ be an additive category. Write $\mathbb{B}(\mathbb{A})$ for the category of chain complexes

$$C\colon \dots \longrightarrow C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \longrightarrow \dots,$$

graded over the integers, such that each C_r belongs to $\mathbb{A}$, the differentials d are morphisms in $\mathbb{A}$, and $C_r = 0$ for all but finitely many $r \in \mathbb{Z}$. We shall identify $\mathbb{A}$ with the full subcategory of $\mathbb{B}(\mathbb{A})$ consisting of the chain complexes concentrated in dimension 0.

A contravariant additive functor

$$T\colon \mathbb{A} \longrightarrow \mathbb{B}(\mathbb{A})$$

has a canonical extension

$$T\colon \mathbb{B}(\mathbb{A}) \longrightarrow \mathbb{B}(\mathbb{A})$$

defined as follows. Given an object C in $\mathbb{B}(\mathbb{A})$ let $T(C)$ be the chain complex with

$$T(C)_r = \bigoplus_{p+q=r} T(C_{-p})_q$$

and differentials

$$d_{T(C)} = \bigoplus_{p+q=r} (d_{T(C_{-p})} + (-1)^q T(d_C))\colon T(C)_r \longrightarrow T(C)_{r-1}.$$

(The direct sums in this formula had better be well-defined, and not just up to unique isomorphism, as they are by virtue of their universal property. Finite direct sums are well-defined in the additive category of abelian groups; they are also well-defined if we replace $\mathbb{A}$ by the larger but equivalent category of representable contravariant abelian group valued functors on $\mathbb{A}$: apply the direct sum operations to the values of the functors.)

1.1. Definition (Ranicki [17]). A *chain duality* on $\mathbb{A}$ is a contravariant additive functor $T: \mathbb{A} \to \mathbb{B}(\mathbb{A})$ together with a natural transformation e from $T^2: \mathbb{A} \to \mathbb{B}(\mathbb{A})$ to $1: \mathbb{A} \to \mathbb{B}(\mathbb{A})$ such that for each object M in $\mathbb{A}$

i) $e_{T(M)} \cdot T(e_M) = 1: T(M) \longrightarrow T^3(M) \longrightarrow T(M)$,

ii) $e_M: T^2(M) \longrightarrow M$ is a chain homotopy equivalence. □

Assume from now on that $\mathbb{A}$ is equipped with a chain duality (T, e). Given objects C, D in $\mathbb{B}(\mathbb{A})$ define

$$C \otimes_{\mathbb{A}} D = \mathrm{Hom}_{\mathbb{A}}(T(C), D),$$

a chain complex of abelian groups. Often we just write it $C \otimes D$. There is a canonical isomorphism

$$\tau: C \otimes D \longrightarrow D \otimes C$$

obtained by composing

$$T: \mathrm{Hom}_{\mathbb{A}}(T(C), D) \longrightarrow \mathrm{Hom}_{\mathbb{A}}(T(D), T^2(C))$$

with the chain map

$$\mathrm{Hom}_{\mathbb{A}}(T(D), T^2(C)) \longrightarrow \mathrm{Hom}_{\mathbb{A}}(T(D), C)$$

induced by $e_C: T^2(C) \to C$. Note that τ^2 is the identity.

1.2. Example. If $\mathbb{A}$ is the category of f.g. free left modules M over a ring with involution A let

$$T(M) = \mathrm{Hom}_A(M, A)$$

Use the involution on A to make $T(M)$ into a left module over A. There is a canonical identification $e: T^2(M) \cong M$. The pair (T, e) is a chain duality. □

In constructing L-theory spectra from $\mathbb{A}$, we shall find it useful to generate other additive categories with chain duality.

Let $\mathbb{X}$ be a finite category (finite in the sense that the total number of morphisms is finite, which implies that the class of objects is also finite). Let $\{m_x \mid x \in \mathrm{Ob}(\mathbb{X})\}$ be a family of objects in $\mathbb{A}$, indexed by the objects of $\mathbb{X}$. Define a covariant functor $F: \mathbb{X} \to \mathbb{A}$ by

$$F(y) = \bigoplus_{g: x \to y} m_x$$

where the direct sum ranges over all morphisms in $\mathbb{X}$ with target y.

1.3. Definition. Such a functor F, or any isomorphic functor, will be called *induced.* □

(This is a variation on a theme of Lück [9]. For example, if $\mathbb{X}$ is the category of free left modules over a ring R, then an induced functor F as above would be called a free $R[\mathbb{X}]$-module by Lück, and the finiteness condition could be dropped.)

Let X be a finite Δ-set (alias incomplete simplicial set, cf. Rourke and Sanderson [19]). This determines a category, also denoted by X, whose objects are the simplices of X. A morphism from an m-simplex x to an n-simplex y in X is a monotone map

$$f\colon \{0,1,\ldots,m\} \longrightarrow \{0,1,\ldots,n\}$$

such that $f^*(y) = x$.

1.4. Notation. We let $\mathbb{A}^*(X)$ be the category of induced (covariant) functors from X to $\mathbb{A}$, with arbitrary natural transformations as morphisms. We let $\mathbb{A}_*(X)$ be the category of induced functors from X^{op} to $\mathbb{A}$, with arbitrary natural transformations as morphisms.

1.5. Proposition (= 4.13 in Ranicki [17]). *The additive categories $\mathbb{A}^*(X)$ and $\mathbb{A}_*(X)$ inherit chain dualities from $\mathbb{A}$.*

Proof. We introduce some notation first. Let Δ be the category whose objects are the sets $[n] = \{0,1,\ldots,n\}$, for $n \geq 0$, and whose morphisms are the monotone injections. A contravariant functor C from Δ to $\mathbb{B}(\mathbb{A})$ determines a double chain complex which in bidegree (m,n) equals $C[m]_n$, with horizontal differentials given by

$$\sum_{i=0}^{m} (-1)^{i+m} C(\delta^i)\colon C[m]_n \longrightarrow C[m-1]_n$$

where $\delta^i\colon [m-1] \to [m]$ omits the element i, and with vertical differentials

$$d_{C[m]}\colon C[m]_n \longrightarrow C[m]_{n-1}$$

Certain finiteness conditions being understood, this can be worked into a single chain complex by the method of Cartan-Eilenberg [4], p. 60; call the result $\int_* C$.

A covariant functor C from Δ to $\mathbb{B}(\mathbb{A})$ determines a double chain complex which in bidegree $(-m,n)$ equals $C[m]_n$, with horizontal differentials

$$\sum_{i=0}^{m+1} (-1)^{i+m} C(\delta^i)\colon C[m]_n \longrightarrow C[m+1]_n$$

and vertical differentials

$$d_{C[m]}\colon C[m]_n \longrightarrow C[m]_{n-1}.$$

Finiteness conditions being understood, this can be worked into a single chain complex, written $\int^* C$.

More generally, a contravariant or covariant functor C from X to $\mathbb{B}(\mathbb{A})$ can be regarded as a contravariant or covariant functor from Δ to $\mathbb{B}(\mathbb{A})$: let

$$C[m] = \bigoplus_{|x|=m} C(x).$$

In this case it is convenient to write $\int_X C$ for $\int_* C$ and $\int^X C$ for $\int^* C$. Finiteness conditions are superfluous because X is finite. Note that $\mathbb{A}$ can be any additive category.

Given an object M in $\mathbb{A}^*(X)$ we let

$$T(M)(x) = \int_X (T\cdot M)_x \quad \text{for } x \in X,$$

where $(T\cdot M)_x$ is the contravariant functor

$$y \longrightarrow \bigoplus_{y \to x} T(M(y))$$

on X. (The direct sum runs over all the morphisms from y to x in X). Then $T(M)$ is a covariant functor, and belongs to $\mathbb{B}(\mathbb{A}^*(X))$. Given an object M in $\mathbb{A}_*(X)$ we let

$$T(M)(x) = \int^X (T\cdot M)^x \quad \text{for } x \in X,$$

where $(T\cdot M)^x$ is the covariant functor

$$y \longrightarrow \bigoplus_{x \to y} T(M(y))$$

on X. Then $T(M)$ is an object of $\mathbb{B}(\mathbb{A}_*(X))$.

For M and N in $\mathbb{A}^*(X)$ there are natural identifications of chain complexes

$$\operatorname{Hom}(T(M), N) \cong \int^X \operatorname{Hom}(T(M(-)), N(-)) \cong \int^X M(-) \otimes N(-)$$

where $M(-) \otimes N(-)$ denotes the functor $x \to M(x) \otimes N(x)$. Since the right-hand side is symmetric in M and N, so is the left-hand side; i.e. there is a natural identification

$$\tau\colon \operatorname{Hom}(T(M), N) \xrightarrow{\cong} \operatorname{Hom}(T(N), M).$$

For M and N in $\mathbb{A}_*(X)$ there are natural identifications

$$\operatorname{Hom}(T(M), N) \cong \int_X \operatorname{Hom}(T(M(-)), N(-)) \cong \int_X M(-) \otimes N(-)$$

giving again

$$\tau\colon \operatorname{Hom}(T(M), N) \xrightarrow{\cong} \operatorname{Hom}(T(N), M).$$

We extend these symmetry isomorphisms to natural isomorphisms

$$\tau\colon \operatorname{Hom}(T(C), D) \xrightarrow{\cong} \operatorname{Hom}(T(D), C)$$

with C and D both in $\mathbb{B}(\mathbb{A}^*(X))$ or both in $\mathbb{B}(\mathbb{A}_*(X))$. Taking $C = M$ and $D = T(M)$ gives

$$\tau\colon \operatorname{Hom}(T(M), T(M)) \xrightarrow{\cong} \operatorname{Hom}(T^2(M), M)$$

and we let

$$e_M = \tau(\mathrm{id}_{T(M)})\colon T^2(M) \longrightarrow M$$

General nonsense proves that e has property i) in 1.1 (this is another way of saying that the symmetry τ has order two). Property ii) in 1.1 is harder to establish. One can assume that M is induced by a collection $\{m_x | x \in X\}$ where only one of the m_x is nonzero, and then proceed by brute force. Use 1.6 iii) below. □

1.6. Remark. It is useful to know a little more about induced functors $F\colon \mathbb{X} \to \mathbb{A}$ as in 1.3.

i) Suppose that $v\colon F \to G$ is a natural transformation between induced functors from $\mathbb{X}$ to $\mathbb{A}$ such that

$$v_x\colon F(x) \longrightarrow G(x)$$

admits a section $s_x\colon G(x) \to F(x)$ for all objects x in $\mathbb{X}$ (so that $v_x s_x = \mathrm{id}$). Then v admits a section $s\colon G \to F$ (so that $vs = \mathrm{id}$). The proof is easy.

ii) Write $\mathbb{A}^{\mathbb{X}}$ for the category of induced (covariant) functors from $\mathbb{X}$ to $\mathbb{A}$, with natural transformations as morphisms. Let D be an object in $\mathbb{B}(\mathbb{A}^{\mathbb{X}})$ such that $D(x)$ is a contractible object (in $\mathbb{B}(\mathbb{A})$) for all objects x in $\mathbb{X}$. Then D is contractible. (Proof: Construct a contraction using i)).

iii) Let $f\colon D \to E$ be a morphism in $\mathbb{B}(\mathbb{A}^{\mathbb{X}})$ such that $f_x\colon D(x) \to E(x)$ is a homotopy equivalence for all objects x in $\mathbb{X}$. Then f is a homotopy equivalence. (This follows in the usual manner from ii)).

1.7. Remark. It can be shown that a map $f\colon X \to Y$ between Δ-sets induces functors

$$f^*\colon \mathbb{A}^*(Y) \longrightarrow \mathbb{A}^*(X),$$
$$f_*\colon \mathbb{A}_*(X) \longrightarrow \mathbb{A}_*(Y),$$

which are compatible with the respective chain dualities. See Ranicki [17], 4.20 for details. See also Ranicki and Weiss [18] for examples. We shall not use these induced functors f^*, f_* here, except for injections f in which case they are obvious: f^* is given by composition with f, and f_* is given by extending induced functors $X^{op} \to \mathbb{A}$ trivially to Y^{op}. □

If Y is an infinite Δ-set define $\mathbb{A}^*(Y)$ as the inverse limit of the $\mathbb{A}^*(X)$ where X ranges over the finite Δ-subsets of Y, and define $\mathbb{A}_*(Y)$ as the direct limit of the $\mathbb{A}_*(X)$.

We return to our original project, doing L-theory in $\mathbb{B}(\mathbb{A})$. At the most basic level, this looks as follows. A pairing of dimension n between objects C and D in $\mathbb{B}(\mathbb{A})$ is an n-cycle in $C \otimes D$. It is nondegenerate if the corresponding chain map $\Sigma^n T(C) \to D$ is a homotopy equivalence. (By the way, the sign conventions which I use in defining, say, Hom and $\otimes$ of chain complexes are those of Cartan–Eilenberg [4] and Dold [5]). An n-dimensional symmetric structure on C is a $\mathbb{Z}_2$-equivariant chain map

$$\phi\colon \Sigma^n W \longrightarrow C \otimes C,$$

where W is the standard free resolution of the trivial module $\mathbb{Z}$ over the ring $\mathbb{Z}[\mathbb{Z}_2]$. (See Ranicki [15]. Remember that $\mathbb{Z}_2$ acts on $C \otimes C$ via τ.) If ϕ sends the generator of $H_0(W) \cong \mathbb{Z}$ to a nondegenerate class in $H_n(C \otimes C) \cong H_0((\Sigma^{-n} C) \otimes C)$, then we speak of a nondegenerate symmetric structure. A pair (C, ϕ), where ϕ is an n-dimensional nondegenerate symmetric structure on C, is called an n-dimensional symmetric Poincaré object (in $\mathbb{B}(\mathbb{A})$). An n-dimensional quadratic structure on C is an n-cycle

$$\psi \in W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C).$$

Such a quadratic structure determines a symmetric structure

$$(1 + T)\psi\colon \Sigma^n W \longrightarrow C \otimes C$$

as follows: Think of ψ as a chain map of degree n (over the ring $\mathbb{Z}[\mathbb{Z}_2]$) from the dual chain complex W^{-*} of W to $C \otimes C$. Compose with the chain map from W to W^{-*} which sends the generator in $H_0(W) \cong \mathbb{Z}$ to the generator in $H_0(W^{-*}) \cong \mathbb{Z}$. This gives $(1 + T)\psi$. Call ψ nondegenerate if $(1 + T)\psi$ is nondegenerate. A pair (C, ψ), where C is in $\mathbb{B}(\mathbb{A})$ and ψ is an n-dimensional nondegenerate quadratic structure on C, is called an n-dimensional quadratic Poincaré object (in $\mathbb{B}(\mathbb{A})$).

We also want to talk about bordisms between quadratic or symmetric Poincaré objects in $\mathbb{B}(\mathbb{A})$. These bordisms are best considered as quadratic or symmetric Poincaré objects in $\mathbb{B}(\mathbb{A}^*(\Delta^1))$. To abbreviate, we write $\mathbb{A}^n$ for $\mathbb{A}^*(\Delta^n)$, and let d_i: $\mathbb{A}^n \to \mathbb{A}^{n-1}$ be the functor induced by the inclusion $\delta^i\colon \Delta^{n-1} \to \Delta^n$.

1.8. Definition. Two n-dimensional quadratic Poincaré objects (C, ψ) and (C', ψ') in $\mathbb{B}(\mathbb{A}) \cong \mathbb{B}(\mathbb{A}^0)$ are said to be *bordant* if there exists an n-dimensional quadratic Poincaré object (D, θ) in $\mathbb{B}(\mathbb{A}^1)$ such that $d_1(D, \theta) = (C, \psi)$ and $d_0(D, \theta) = (C', \psi')$. □

It is not hard to show that "bordant" is an equivalence relation. Direct sum makes the set of equivalence classes into an abelian group, where the inverse of $[(C, \psi)]$ is given by $[(C, -\psi)]$. We denote it by $L_n(\mathbb{B}(\mathbb{A}))$. The bordism group $L^n(\mathbb{B}(\mathbb{A}))$ of n-dimensional symmetric Poincaré objects in $\mathbb{B}(\mathbb{A})$ is defined similarly.

There is a slightly diferent way of describing the bordism relation. Let $\mathbb{A}^{1/2} \subset \mathbb{A}^1$ be the full subcategory consisting of those objects whose image under $d_0\colon \mathbb{A}^1 \to \mathbb{A}^0$ is zero. The chain duality on $\mathbb{A}^1$ restricts to one on $\mathbb{A}^{1/2}$. We can say that two n-dimensional quadratic Poincaré objects (C, ψ) and (C', ψ') in $\mathbb{B}(\mathbb{A})$ are bordant if there exists an n-dimensional quadratic Poincaré object (D, θ) in $\mathbb{B}(\mathbb{A}^{1/2})$ such that

$$d_1(D, \theta) = (C, \psi) \oplus (C', -\psi').$$

There is an analogous definition in the symmetric case.

1.9. Terminology. It is suggestive to use the expression "$(n+1)$-dimensional Poincaré pair in $\mathbb{B}(\mathbb{A})$" synonymously with "n-dimensional Poincaré object in $\mathbb{B}(\mathbb{A}^{1/2})$". Note the dimension shift. □

Following the method of Buoncristiano, Rourke and Sanderson [2], we now show that the bordism groups just defined can be regarded as the homotopy groups of a suitable spectrum. We write $*$ for the Δ-set with exactly one simplex in every dimension. A pointed Δ-set Y is a Δ-set Y together with a Δ-map from $*$ to Y. The suspension of a pointed Δ-set Y is the pointed Δ-set ΣY having one nontrivial $(k+1)$-simplex Σx for every nontrivial k-simplex x in Y, with

$$d_i \Sigma x = \begin{cases} \Sigma d_{i-1} x & \text{if } 0 < i \leq k+1 \\ \text{base point} & \text{otherwise.} \end{cases}$$

For $q \in \mathbb{Z}$, let $\mathbb{L}.(\mathbb{B}(\mathbb{A}), q)$ be the Δ-set whose k-simplices are the $(-q)$-dimensional symmetric Poincaré objects in $\mathbb{B}(\mathbb{A}^k)$.

1.10. Proposition. *These Δ-sets have the Kan extension property.*

Proof. (See Rourke and Sanderson [19] for the definition of the "Kan extension property".) Following [19], we write $\angle_{k,i}$ for the i-th horn of Δ^k, the union of all faces $d_j \Delta^k$ for $j \neq i$.) The following observation is crucial: If C is an object of $\mathbb{B}(\mathbb{A}^k)$, then the direct limit

$$\varinjlim_{s \subset \angle_{k,i}} C(s)$$

exists in $\mathbb{B}(\mathbb{A})$. This is because we assume that C_r is induced for all r (see 1.3). We will say that C is i-shallow if the chain maps

$$\varinjlim_{s \subset \angle_{k,i}} C(s) \longrightarrow C(\Delta^k)$$

and

$$C(d_j \Delta^k) \longrightarrow C(\Delta^k)$$

are homotopy equivalences. Given a Δ-map

$$g\colon \angle_{k,i} \longrightarrow \mathbb{L}.(\mathbb{B}(\mathbb{A}), q)$$

let us first search for an object C in $\mathbb{B}(\mathbb{A}^k)$ having the prescribed faces $d_j C$ for $j \neq i$. Now

$$\varinjlim_{s \subset \angle_{k,i}} C(s)$$

is prescribed, and it is not hard to define the missing values $C(\Delta^k)$ and $C(d_i \Delta^k)$ so that C belongs to $\mathbb{B}(\mathbb{A}^k)$ and is i-shallow. Now observe that the given quadratic structures on $d_j C$, for $j \neq i$, extend to an essentially unique quadratic structure on C, because

$$C(d_i \Delta^k) \longrightarrow C(\Delta^k)$$

is a homotopy equivalence. Then observe that this quadratic structure on C is nondegenerate. Therefore $\mathbb{L}.(\mathbb{B}(\mathbb{A}), q)$ has the Kan property. The proof for $\mathbb{L}^{\cdot}(\mathbb{B}(\mathbb{A}), q)$ is similar. □

If we regard the k-simplex Δ^k and the $(k+1)$-simplex Δ^{k+1} as Δ-sets for the moment, then there is a unique nontrivial Δ-map

$$p\colon \Delta^{k+1} \longrightarrow \Sigma(\Delta^k_+),$$

where Δ^k_+ is the disjoint union of Δ^k and a base 0-simplex. We use it to define a functor σ from $\mathbb{A}^k$ to $\mathbb{A}^{k+1}$: for F in $\mathbb{A}^k$, the object σF in $\mathbb{A}^{k+1}$ is given by

$$(\sigma F)(s) = \begin{cases} F(q) & \text{if } p(s) = \Sigma q \\ 0 & \text{if } p(s) \text{ is at the base point} \end{cases}$$

Note that a $(-q)$-dimensional quadratic structure on C in $\mathbb{B}(\mathbb{A}^k)$ is the same as a $(-q-1)$-dimensional quadratic structur on σC in $\mathbb{B}(\mathbb{A}^{k+1})$. Summarizing, σ gives rise to an injective Δ-map

$$\Sigma\mathbb{L}.(\mathbb{B}(\mathbb{A}), q) \longrightarrow \mathbb{L}.(\mathbb{B}(\mathbb{A}), q+1).$$

(Its image consists of all simplices in $\mathbb{L}.(\mathbb{B}(\mathbb{A}), q+1)$ having 0-th face and 0-th vertex at the base point.) Letting q vary we see that the $\mathbb{L}.(\mathbb{B}(\mathbb{A}), q)$, or the geometric realizations, form a spectrum $\mathbb{L}.(\mathbb{B}(\mathbb{A}))$. This is the quadratic L-theory spectrum of $\mathbb{B}(\mathbb{A})$. Similarly, we obtain $\mathbb{L}^{\cdot}(\mathbb{B}(\mathbb{A}))$, the symmetric L-theory spectrum of $\mathbb{B}(\mathbb{A})$.

It is an obvious consequence of 1.10 that $\pi_n(\mathbb{L}.(\mathbb{B}(\mathbb{A}), q))$ and $\pi_n(\mathbb{L}^{\cdot}(\mathbb{B}(\mathbb{A}, q))$ can be identified with the groups $L_{n-q}(\mathbb{B}(\mathbb{A}))$ and $L^{n-q}(\mathbb{B}(\mathbb{A}))$, respectively. It follows that

$$\pi_n(\mathbb{L}.(\mathbb{B}(\mathbb{A}))) \cong L_n(\mathbb{B}(\mathbb{A})),$$
$$\pi_n(\mathbb{L}^{\cdot}(\mathbb{B}(\mathbb{A}))) \cong L^n(\mathbb{B}(\mathbb{A})).$$

1.11. Notation. In later sections we shall use the abbreviations

$$Q_n(D) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (D \otimes_{\mathbb{A}} D))$$
$$Q^n(D) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, D \otimes_{\mathbb{A}} D))$$
$$\hat{Q}^n(D) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\hat{W}, D \otimes_{\mathbb{A}} D))$$

for objects D in $\mathbb{B}(\mathbb{A})$. (The symbol $\hat{W}$ denotes a complete resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}_2]$, as in Ranicki [15]).

2. The fibration theorem

By dropping the nondegeneracy condition in the definition of an n-dimensional symmetric or quadratic Poincaré object in $\mathbb{B}(\mathbb{A})$, one obtains the notion of an n-dimensional symmetric or quadratic object in $\mathbb{B}(\mathbb{A})$. A result of Ranicki [15], suitably reformulated in the language of additive categories, states that the classification up to homotopy equivalence of n-dimensional symmetric or quadratic Poincaré pairs in $\mathbb{B}(\mathbb{A})$ is the same as the classification up to homotopy equivalence of n-dimensional symmetric or quadratic objects in $\mathbb{B}(\mathbb{A})$. This is a useful ingredient in proving a fibration theorem which goes back to the generalization due to Vogel [21] of the L-theory localization exact sequence of Ranicki [16]. To simplify, let us concentrate on the symmetric case, the quadratic case being similar.

Two n-dimensional symmetric objects (C, ϕ) and (C', ϕ') in $\mathbb{B}(\mathbb{A})$ are considered homotopy equivalent if there exists a chain map $f\colon C \to C'$ which is a homotopy equivalence and satisfies

$$f_*[\phi] = [\phi'] \in H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C' \otimes_{\mathbb{A}} C'))$$

If one of the two objects is Poincaré, then so is the other.

For an object D in $\mathbb{B}(\mathbb{A}^{1/2})$, write

$$\begin{array}{lll} bC & \text{to mean} & C(d_1 \Delta^1), \\ wC & \text{to mean} & C(\Delta^1), \\ qC & \text{to mean} & C(\Delta^1)/C(d_1 \Delta^1). \end{array}$$

These are objects in $\mathbb{A}$; think of b as boundary, w as whole, q as quotient. (The quotient alias cokernel exists since D is induced in all dimensions.) For D and E in $\mathbb{B}(\mathbb{A}^{1/2})$ there is a natural chain map

$$\Sigma(D \otimes E) \longrightarrow qD \otimes qE$$

defined as follows: recall from the proof of 1.5 that

$$\begin{aligned} \Sigma(D \otimes E) &\cong \Sigma\left(\int^{\Delta^1} D(-) \otimes E(-)\right) \\ &\cong \text{mapping cone of } (bD \otimes bE \to wD \otimes wE) \end{aligned}$$

which projects to $qD \otimes qE$. Taking $D = E$, we see that an $(n-1)$-dimensional symmetric structure ϕ on D projects to an n-dimensional symmetric structure $q\phi$ on qD.

2.1. Proposition. *The rule*

$$(D, \phi) \longrightarrow (qC, q\phi)$$

yields a bijection between the set of homotopy equivalence classes of $(n-1)$-dimensional symmetric Poincaré objects in $\mathbb{B}(\mathbb{A}^{1/2})$ and the set of homotopy equivalence classes of n-dimensional symmetric objects in $\mathbb{B}(\mathbb{A})$.

Proof. See Ranicki [15] for the case where $\mathbb{A}$ is the additive category of example 1.2. See also section 4 of Weiss [23] if a more categorical but less explicit proof is required. These proofs carry over to the general case. □

2.2. Remark. The proposition implies that the boundary bD can be recovered from $(qC, q\phi)$, up to homotopy equivalence. Indeed $q\phi$ determines an n-dimensional pairing $q\phi_0$ of qC with itself, which is adjoint to a chain map

$$\Sigma^n(T(qC)) \longrightarrow qC$$

whose mapping cone is homotopy equivalent to the suspension of bD. See Ranicki [15], Weiss [23]. □

Let $\mathbb{E} \subset \mathbb{B}(\mathbb{A})$ be a full subcategory with the following properties:

(i) All contractible objects belong to $\mathbb{E}$.
(ii) If $C \to D \to E$ is a short exact sequence in $\mathbb{B}(\mathbb{A})$, and if two of the three objects C, D, E belong to $\mathbb{E}$, then so does the third. (The sequence is short exact if it is split short exact in every dimension.)
(iii) For any object E in $\mathbb{E}$ the dual TE belongs to $\mathbb{E}$.

Following Vogel [21], we call $\mathbb{E}$ an *exact symmetric subcategory* of $\mathbb{B}(\mathbb{A})$.

Assuming that $\mathbb{E}$ is exact symmetric, we can construct the quadratic L-theory spectrum $\mathbb{L}_\bullet(\mathbb{E})$ and the symmetric L-theory spectrum $\mathbb{L}^\bullet(\mathbb{E})$. In detail, this looks as follows. For any $k \geq 0$, let $\mathbb{E}^k \subset \mathbb{B}(\mathbb{A}^k)$ be the full subcategory consisting of those objects C such that $C(s)$ belongs to $\mathbb{E}$ for each face $s \subset \Delta^k$. Let $\mathbb{L}_\bullet(\mathbb{E}, q)$ be the Δ-set whose k-simplices are the $(-q)$-dimensional quadratic Poincaré objects in $\mathbb{E}^k$. Let $\mathbb{L}_\bullet(\mathbb{E})$ be the spectrum made up of the $\mathbb{L}_\bullet(\mathbb{E}, q)$ for $q \in \mathbb{Z}$. Define $\mathbb{L}^\bullet(\mathbb{E})$ similarly, replacing quadratic structures by symmetric structures.

We seek a non-relative description of the cofibres of the inclusion maps

$$\mathbb{L}_\bullet(\mathbb{E}) \longrightarrow \mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}))$$

and

$$\mathbb{L}^\bullet(\mathbb{E}) \longrightarrow \mathbb{L}^\bullet(\mathbb{B}(\mathbb{A})).$$

The idea is to modify the notion of nondegeneracy. We call a pairing of degree n between objects C and D in $\mathbb{B}(\mathbb{A}^k)$ "nondegenerate mod $\mathbb{E}^k$" if the mapping cone of the chain map

$$\Sigma^n(TC) \longrightarrow D$$

which it determines belongs to $\mathbb{E}^k$. An n-dimensional quadratic or symmetric structure on C will be called Poincaré (mod $\mathbb{E}^k$), or simply Poincaré (mod $\mathbb{E}$), if the underlying pairing is nondegenerate mod $\mathbb{E}^k$. Let $\mathbb{L}.(\mathbb{B}(\mathbb{A}), \mathbb{E}, q)$ be the Δ-set whose k-simplices are the $(-q)$-dimensional quadratic Poincaré (mod $\mathbb{E}$) objects in $\mathbb{B}(\mathbb{A}^k)$. Let $\mathbb{L}.(\mathbb{B}(\mathbb{A}), \mathbb{E})$ be the spectrum made up of the $\mathbb{L}.(\mathbb{B}(\mathbb{A}), \mathbb{E}, q)$ for $q \in \mathbb{Z}$, and call it the quadratic L-spectrum of $(\mathbb{B}(\mathbb{A}), \mathbb{E})$. Define similarly $\mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A}), \mathbb{E})$, replacing quadratic structures by symmetric ones. Note: if $\mathbb{E}$ consists of all the contractible objects in $\mathbb{B}(\mathbb{A})$, we recover $\mathbb{L}.(\mathbb{B}(\mathbb{A}))$ and $\mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A}))$.

2.3. Theorem. *The composite inclusion maps*

$$\mathbb{L}.(\mathbb{E}) \longrightarrow \mathbb{L}.(\mathbb{B}(\mathbb{A})) \longrightarrow \mathbb{L}.(\mathbb{B}(\mathbb{A}), \mathbb{E})$$

and

$$\mathbb{L}^{\bullet}(\mathbb{E}) \longrightarrow \mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A})) \longrightarrow \mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A}), \mathbb{E})$$

are nullhomotopic, by a preferred nullhomotopy. The resulting maps of spectra

$$\mathbb{L}.(\mathbb{B}(\mathbb{A}))/\mathbb{L}.(\mathbb{E}) \longrightarrow \mathbb{L}.(\mathbb{B}(\mathbb{A}), \mathbb{E})$$

and

$$\mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A}))/\mathbb{L}^{\bullet}(\mathbb{E}) \longrightarrow \mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A}), \mathbb{E})$$

are homotopy equivalences.

Proof. Again we concentrate on the symmetric case, the quadratic case being similar. Let $\mathbb{K} \subset \mathbb{B}(\mathbb{A}^{1/2})$ be the exact symmetric (full) subcategory consisting of all objects C such that $d_1 C$ belongs to $\mathbb{E} \subset \mathbb{B}(\mathbb{A})$ (and remember that $d_0 C = 0$ by the definition of $\mathbb{B}(\mathbb{A}^{1/2})$). If (C, ψ) is an n-dimensional symmetric Poincaré object in $\mathbb{K}$, then $(qC, q\psi)$ is an $(n+1)$-dimensional symmetric Poincaré (mod $\mathbb{E}$) object in $\mathbb{B}(\mathbb{A})$. (This remains correct with $\mathbb{K}$ and $\mathbb{B}(\mathbb{A})$ replaced by $\mathbb{K}^k$ and $\mathbb{B}(\mathbb{A}^k)$, respectively, where $k > 0$. Note that $\mathbb{B}((\mathbb{A}^{1/2})^k) \cong \mathbb{B}((\mathbb{A}^k)^{1/2})$.) In other words, the collapsing procedure of 2.1 gives a map

$$\Sigma\mathbb{L}^{\bullet}(\mathbb{K}) \longrightarrow \mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A}), \mathbb{E})\,.$$

Using 2.1, and using 2.2 to note that (C, ψ) in 2.1 has boundary $d_1 C$ in $\mathbb{E}$ if and only if $(qC, q\psi)$ is Poincaré mod $\mathbb{E}$, we see that this map is an isomorphism on homotopy groups. (In fact, 2.1 proves surjectivity, and 2.1 applied with $\mathbb{A}^1$ instead of $\mathbb{A}$ proves injectivity.) Furthermore, the inclusion

$$\mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A})) \longrightarrow \mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A}), \mathbb{E})$$

factors through the injection

$$\mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A})) \longrightarrow \Sigma\mathbb{L}^{\bullet}(\mathbb{K}).$$

(The injection comes from the fact that any $(n+1)$-dimensional symmetric Poincaré object in $\mathbb{B}(\mathbb{A})$ can be regarded as an n-dimensional symmetric Poincaré pair in $\mathbb{K}$ with zero boundary.) So it only remains to prove, firstly, that the composite map

$$\mathbb{L}^{\bullet}(\mathbb{E}) \longrightarrow \mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A})) \longrightarrow \Sigma\mathbb{L}^{\bullet}(\mathbb{K})$$

is nullhomotopic, and secondly, that the resulting map from the cofibre $\mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A}))/\mathbb{L}^{\bullet}(\mathbb{E})$ to $\Sigma\mathbb{L}^{\bullet}(\mathbb{K})$ is a homotopy equivalence. Now the map from $\mathbb{L}^{\bullet}(\mathbb{E})$ to $\Sigma\mathbb{L}^{\bullet}(\mathbb{K})$ factors through $\Sigma\mathbb{L}^{\bullet}(\mathbb{E}^{1/2})$, which is contractible for obvious reasons. This gives a preferred nullhomotopy. The proof is completed by comparing the homotopy groups of $\mathbb{L}^{\bullet}(\mathbb{B}(\mathbb{A}))/\mathbb{L}^{\bullet}(\mathbb{E})$ with those of $\Sigma\mathbb{L}^{\bullet}(\mathbb{K})$. □

3. Excision

By investigating the properties of the functor

$$X \longrightarrow \mathbb{L}_{\bullet}(\mathbb{B}(\mathbb{A}_*(X)))$$

from finite Δ-sets and injective Δ-maps to spectra, we shall be able to determine the homotopy type of $\mathbb{L}_{\bullet}(\mathbb{B}(\mathbb{A}_*(X)))$ for all X. We concentrate on the quadratic case (for a change), the symmetric case being similar. Write X^k for the k-skeleton of X, and $X[k]$ for the set of k-simplices.

3.1. Lemma. *The cofibre of the inclusion map*

$$\mathbb{L}_{\bullet}(\mathbb{B}(\mathbb{A}_*(X^{k-1}))) \longrightarrow \mathbb{L}_{\bullet}(\mathbb{B}(\mathbb{A}_*(X^k)))$$

is homotopy equivalent to

$$\bigvee_{x\in X[k]} \Sigma^k\mathbb{L}_{\bullet}(\mathbb{B}(\mathbb{A}))$$

Proof. Evaluation on the k-simplices gives a functor

$$\mathbb{A}_*(X^k) \longrightarrow \prod_{x\in X[k]} \mathbb{A}; \quad M \longrightarrow (M(x))_{x\in X[k]}.$$

This is compatible with the chain dualities, up to a dimension shift, and therefore induces a map

$$\mathbb{L}_{\bullet}(\mathbb{B}(\mathbb{A}_*(X^k)), q) \longrightarrow \mathbb{L}_{\bullet}(\mathbb{B}(\prod_{x\in X[k]} \mathbb{A}), q-k)$$

for all q, and even a map

$$\mathrm{ev}_k\colon \mathbb{L}_{\bullet}(\mathbb{B}(\mathbb{A}_*(X^k)), \mathbb{E}, q) \longrightarrow \mathbb{L}_{\bullet}(\mathbb{B}(\prod_{x\in X[k]} \mathbb{A}), q-k)$$

for all q, where $\mathbb{E}$ consists of all the objects C in $\mathbb{B}(\mathbb{A}_*(X^k))$ such that $C(x)$ is contractible for all $x \in X[k]$. Note that the inclusion of $\mathbb{B}(\mathbb{A}_*(X^{k-1}))$ in $\mathbb{E}$ induces a homotopy equivalence of quadratic L-theory spectra: check on homotopy groups, using 1.10. The source and target of ev_k are Kan Δ-sets, and a painful inspection which we omit reveals that ev_k is a Kan fibration. (The proof of 1.10 can serve as a model). The fibre of ev_k over $*$ is $\mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X^{k-1})), \mathbb{B}(\mathbb{A}_*(X^{k-1})), q)$, which is contractible by 2.3. (Here, the base point $*$ is to be interpreted as a Δ-subset having exactly one simplex in each dimension). It follows from Corollary 7.6 of Rourke and Sanderson [19] that ev_k is a componentwise homotopy equivalence for every q. Letting q vary, we see that ev_k gives a homotopy equaivalence of spectra

$$\mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X^k)), \mathbb{E}) \longrightarrow \Sigma^k \mathbb{L}.(\mathbb{B}(\prod_{x \in X[k]} \mathbb{A}))$$

Now apply 2.3 and note that the inclusions of the factors $\mathbb{A}$ into the product

$$\prod_{x \in X[k]} \mathbb{A}$$

induce a homotopy equivalence

$$\bigvee_{x \in X[k]} \mathbb{L}.(\mathbb{B}(\mathbb{A})) \longrightarrow \mathbb{L}.(\mathbb{B}(\prod_{x \in X[k]} \mathbb{A}))$$

(Check on homotopy groups, using 1.10). □

3.2. Corollary. *The functor*

$$X \longrightarrow \mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X)))$$

is homotopy invariant. That is, if $f\colon X \to Y$ is an injective Δ-map between finite Δ-sets which is a homotopy equivalence (after realization), then

$$f_*\colon \mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X))) \longrightarrow \mathbb{L}.(\mathbb{B}(\mathbb{A}_*(Y)))$$

is a homotopy equivalence of spectra.

Proof. The natural filtration of $\mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X)))$ leads to a spectral sequence converging to

$$\pi_*(\mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X)))).$$

According to 3.1, its E^1-term is

$$E^1_{p,q} \cong \mathrm{cl}(X)_p \otimes L_q(\mathbb{B}(\mathbb{A}))$$

(where cl denotes cellular chain complexes, and $L_q(\mathbb{B}(\mathbb{A})) = \pi_q(\mathbb{L}.(\mathbb{B}(\mathbb{A})))$. It is not hard to see that the differential in $E^1_{*,q}$ agrees with the differential in the chain complex

$$\mathrm{cl}(X) \otimes L_q(\mathbb{B}(\mathbb{A})),$$

so that

$$E^2_{p,q} \cong H_p(X; L_q(\mathbb{B}(\mathbb{A}))).$$

But if the E^2-term is already homotopy invariant, then so is the E^∞-term. □

3.3. Corollary. *The functor*

$$X \longrightarrow \mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X)))$$

is excisive. That is, if X' and X'' are finite Δ-subsets of the finite Δ-set X, with $X' \cup X'' = X$, then the square

$$\begin{array}{ccc} \mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X' \cap X''))) & \longrightarrow & \mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X'))) \\ \downarrow & & \downarrow \\ \mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X''))) & \longrightarrow & \mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X))) \end{array}$$

is a homotopy pushout square.

Proof. For a finite Δ-set Y, let $GT_k(Y) = \mathbb{L}.(\mathbb{B}(\mathbb{A}_*(Y^k)))$. The functor

$$Y \longrightarrow G_k(Y)/G_{k-1}(Y)$$

sends the square of Δ-sets under consideration to a homotopy pushout square, for all $k \geq 0$, by 3.1. By induction on k, the same is then true for the functor G_k itself. For $k = \dim(X)$, this is what we need. □

3.4. Remark. A functor G from finite Δ-sets and injective Δ-maps to spectra having the homotopy invariance and excision properties in 3.2 and 3.3 is a homology theory. That is,

$$G(X) \simeq |X|_+ \wedge G(\text{point}),$$

where $|X|_+$ is the geometric realization of X, with an extra base point.

Proof. The first step is to reduce to the case where $|X|$ is a simplicial complex (which means that simplices in X are determined by their vertex sets). If that is not the case, choose a diagram of finite Δ-maps

$$X \rightarrow Y \leftarrow Z$$

where both arrows are injections and homotopy equivalences, and $|Z|$ is a simplicial complex. Then

$$G(X) \simeq G(Y) \simeq G(Z)$$

since G is homotopy invariant. Assuming now that $|X|$ is a simplicial complex, we define three covariant functors from X to spectra, given by

$$F_1^X(y) = G(\Delta^{|y|}), \quad F_2^X(y) = G(\text{cone on } \Delta^{|y|}), \quad F_3^X(y) = G(\text{point})$$

for simplices $y \in X$. The inclusion of $\Delta^{|y|}$ into its cone and the inclusion of the apex into the cone induce homotopy equivalences

$$F_1^X(y) \to F_2^X(y) \leftarrow F_3^X(y);$$

therefore

$$\operatorname{hocolim} F_1^X \simeq \operatorname{hocolim} F_2^X \simeq \operatorname{hocolim} F_3^X \simeq |X'|_+ \wedge G(\text{point})$$

where X' is the barycentric subdivision of X. (See Bousfield and Kan [1] for homotopy direct limits, especially p. 327.) The maps $F_1^X(y) \to G(X)$ induced by the characteristic maps $c_y\colon \Delta^{|y|} \to X$ determine a map

$$v_X\colon \operatorname{hocolim} F_1^X \to G(X)$$

We regard this as a natural transformation between homotopy invariant and excisive functors in the variable X; the left-hand side is homotopy invariant and excisive because it is homotopy equivalent to $|X|_+ \wedge G(\text{point})$. Further,$v_X$ is a homotopy equivalence if X is a point. Arguments going back to Eilenberg-Steenrod [6] show that v_X is a homotopy equivalence for all finite X. □

3.5. Remark. In the same way one can prove that

$$\mathbb{L}^\bullet(\mathbb{B}(\mathbb{A}_*(X))) \simeq |X|_+ \wedge \mathbb{L}^\bullet(\mathbb{B}(\mathbb{A})),$$
$$\hat{\mathbb{L}}^\bullet(\mathbb{B}(\mathbb{A}_*(X))) \simeq |X|_+ \wedge \hat{\mathbb{L}}^\bullet(\mathbb{B}(\mathbb{A})),$$

where $\hat{\mathbb{L}}^\bullet$ is the cofibre of the symmetrization map from $\mathbb{L}_\bullet$ to $\mathbb{L}^\bullet$. These homotopy equivalences are also valid for infinite X, by a direct limit argument. □

4. Assembly and related topics

Fix a discrete group π, and a commutative ring R. From now on we write $\mathbb{A}$ for the additive category of f.g. free R-modules, and $\mathbb{A}[\pi]$ for the additive category of f.g. free left $R[\pi]$-modules (where $R[\pi]$ is the group ring). We equip R with the identity involution, and $R[\pi]$ with the usual involution

$$\sum_{g \in \pi} r_g \cdot g \longrightarrow \sum_{g \in \pi} r_g \cdot g^{-1}.$$

For C in $\mathbb{B}(\mathbb{A}[\pi])$ we let

$$VQ_n(C) = H_n(P \otimes_{\mathbb{Z}[\pi]} (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_R C))),$$
$$VQ^n(C) = H_n(P \otimes_{\mathbb{Z}[\pi]} (\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]} (W, C \otimes_R C))),$$
$$V\hat{Q}^n(C) = H_n(P \otimes_{\mathbb{Z}[\pi]} (\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]} (\hat{W}, C \otimes_R C)))$$

where P is the free resolution from the introduction. Remember that $VQ_n \cong Q_n$.

Let now $\tilde{X}$ be a Δ-set on which π acts freely, with quotient Δ-set $X = \tilde{X}/\pi$. Write $p: \tilde{X} \to X$ for the projection. If M is an object in $\mathbb{A}_*(X)$ define an object $\alpha(M)$ in $\mathbb{A}[\pi]$ by the formula

$$\alpha(M) = \operatorname{colim} M \cdot p.$$

Here we are talking about the direct limit of the composite functor

$$\tilde{X}^{op} \xrightarrow{p} X^{op} \xrightarrow{M} \mathbb{A} \subset \{\text{category of all } R\text{-modules}\},$$

in the sense of MacLane [10]. The action of π on $\tilde{X}$ leaves $M \cdot p$ invariant and so induces an action of π on $\alpha(M)$. This makes $\alpha(M)$ into a f.g. free left $R[\pi]$-module. (To verify this, assume that M is given by

$$M(y) = \bigoplus_{f: x \to y} m_x$$

where f denotes morphisms in X^{op} and $\{m_x | x \in X\}$ is a family of objects in $\mathbb{A}$. Then

$$\alpha(F) = \bigoplus_{y \in \tilde{X}} m_{p(y)}$$

by inspection, and this is free over $R[\pi]$.)

4.1. Definition. The functor $\alpha: \mathbb{A}_*(X) \to \mathbb{A}[\pi]$ is called the *assembly functor*. □

Clearly the next thing we need is a natural chain map

$$\omega: D \otimes_{\mathbb{A}_*(X)} E \longrightarrow \alpha(D) \otimes_{\mathbb{A}\pi} \alpha(E),$$

defined for objects D and E in $\mathbb{B}(\mathbb{A}_*(X))$, which maps nondegenerate homology classes to nondegenerate homology classes and respects the symmetries τ. The following construction gives that and more.

4.2. Construction. There is a chain map

$$\hat{\omega}: D \otimes_{\mathbb{A}_*(X)} E \longrightarrow \operatorname{cl}(\tilde{X})^t \otimes_{\mathbb{Z}[\pi]} (\alpha(D) \otimes_R \alpha(E)),$$

defined whenever D and E are objects in $\mathbb{A}_*(X)$, and natural in D and E. □

Explanation: Write

$$D \otimes_{\mathbb{A}_*(X)} E \cong \int_X D(-) \otimes E(-) \cong \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} \int_{\tilde{X}} D \cdot p(-) \otimes_R E \cdot p(-)$$

and map this to

$$\begin{aligned} \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} \int_{\tilde{X}} \alpha(D) \otimes \alpha(E) &\cong \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} (\operatorname{cl}(\tilde{X}) \otimes_{\mathbb{Z}} \alpha(D) \otimes_R \alpha(E)) \\ &\cong \operatorname{cl}(\tilde{X})^t \otimes_{\mathbb{Z}[\pi]} (\alpha(D) \otimes_R \alpha(E)) \end{aligned}$$

using the canonical transformations from $D \cdot p(-)$ and $E \cdot p(-)$ to the constant functors on $\tilde{X}$ with value $\alpha(D)$ and $\alpha(E)$, respectively.

Now the natural chain map

$$\omega: D \otimes_{\mathbb{A}_*(X)} E \longrightarrow \alpha(D) \otimes_{\mathbb{A}\pi} \alpha(E)$$

promised earlier can be obtained from $\hat{\omega}$ by applying the chain map

$$\text{augmentation}: \mathrm{cl}(\tilde{X}) \longrightarrow \mathbb{Z}.$$

In more detail, let

$$\text{augmentation} \otimes \mathrm{id} \otimes \mathrm{id}$$

follow upon $\hat{\omega}$. This gives a chain map from $D \otimes_{\mathbb{A}_*(X)} E$ to

$$\mathbb{Z} \otimes_{\mathbb{Z}[\pi]} (\alpha(D) \otimes_R \alpha(E)) \cong \alpha(D) \otimes_{\mathbb{A}[\pi]} \alpha(E).$$

Claims regarding nondegenerate homology classes can be verified by using suitable test objects. This is left to the reader. Together, α and ω give rise to maps of spectra

$$\begin{aligned} &\mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}_*(X))) \longrightarrow \mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}[\pi])) = \mathbb{L}_\bullet(R[\pi]) \\ &\mathbb{L}^\bullet(\mathbb{B}(\mathbb{A}_*(X))) \longrightarrow \mathbb{L}^\bullet(\mathbb{B}(\mathbb{A}[\pi])) = \mathbb{L}^\bullet(R[\pi]) \end{aligned}$$

which must be seen in the light of section 3. These are the assembly maps of Ranicki [17], the algebraic versions of the geometric assembly maps of Quinn [13], [14].

Let now $\mathbb{E}$ be the full subcategory of $\mathbb{B}(\mathbb{A}_*(X))$ consisting of those objects D such that $\alpha(D)$ is contractible. It is clear from the foregoing discussion that $\mathbb{E}$ is an exact symmetric subcategory, and that the assembly maps above factor in the following way:

$$\begin{aligned} &\mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}_*(X))) \longrightarrow \mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}_*(X)), \mathbb{E}) \longrightarrow \mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}[\pi])), \\ &\mathbb{L}^\bullet(\mathbb{B}(\mathbb{A}_*(X))) \longrightarrow \mathbb{L}^\bullet(\mathbb{B}(\mathbb{A}_*(X)), \mathbb{E}) \longrightarrow \mathbb{L}^\bullet(\mathbb{B}(\mathbb{A}[\pi])). \end{aligned}$$

(Just observe that a pairing in $\mathbb{B}(\mathbb{A}_*(X))$ which is nondegenerate mod $\mathbb{E}$ will still give rise to a nondegenerate pairing in $\mathbb{B}(\mathbb{A}[\pi])$ under assembly.) The following three propositions constitute Ranicki's argument proving Theorem 0.1.

4.3. Proposition. *If $\tilde{X}$ is contractible, then*

$$\pi_n(\mathbb{L}^\bullet(\mathbb{B}(\mathbb{A}_*(X)), \mathbb{E})) \cong VL^n(R[\pi]) \quad \text{for } n \in \mathbb{Z}. \quad \square$$

4.4. Proposition. *If $\tilde{X}$ is contractible, then the map*

$$\mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}_*(X)), \mathbb{E}) \longrightarrow \mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}[\pi]))$$

given by assembly is a homotopy equivalence. $\square$

4.5. Proposition. *The symmetrization map*

$$\mathbb{L}_\bullet(\mathbb{E}) \longrightarrow \mathbb{L}^\bullet(\mathbb{E})$$

is a homotopy equivalence (without any conditions on X). □

The proof of 0.1, modulo 4.3, 4.4 and 4.5, is as follows. Inspect the commutative diagram of spectra

$$\begin{array}{ccccc} \mathbb{L}_\bullet(\mathbb{E}) & \longrightarrow & \mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}_*(X))) & \longrightarrow & \mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}_*(X)), \mathbb{E}) \\ e\Big\downarrow & & f\Big\downarrow & & g\Big\downarrow \\ \mathbb{L}^\bullet(\mathbb{E}) & \longrightarrow & \mathbb{L}^\bullet(\mathbb{B}(\mathbb{A}_*(X))) & \longrightarrow & \mathbb{L}^\bullet(\mathbb{B}(\mathbb{A}_*(X)), \mathbb{E}) \end{array}$$

where the vertical arrows are symmetrization maps. By 3.1, the rows are "cofibrations up to homotopy". By 4.5, the symmetrization map e is a homotopy equivalence. It follows that

$$\pi_n(f) \cong \pi_n(g) \quad \text{for all } n \in \mathbb{Z}.$$

But by the results of section 3 we have

$$\pi_n(f) \cong \pi_n(|X|_+ \wedge \hat{\mathbb{L}}^\bullet(R))$$

where $\hat{\mathbb{L}}^\bullet(R)$ is the cofibre of the symmetrization map

$$\mathbb{L}_\bullet(R) \longrightarrow \mathbb{L}^\bullet(R),$$

with homotopy groups $\hat{L}^i(R)$. Up to homotopy equivalence the spectrum $\hat{\mathbb{L}}^\bullet(R)$ is a wedge of Eilenberg–MacLane spectra (Taylor and Williams [20]), so that

$$\pi_n(f) \cong \bigoplus_{i \in \mathbb{Z}} H_{n-i}(X; \hat{L}^i(R)).$$

If $\tilde{X}$ is contractible, the we may write

$$\pi_n(f) \cong \bigoplus_{i \in \mathbb{Z}} H_{n-i}(\pi; \hat{L}^i(R)),$$

and we also have

$$\pi_n(g) \cong V\hat{L}^n(R[\pi])$$

by 4.3 and 4.4. Now 0.1 follows.

The proofs of 4.3 and 4.4 require two lemmas and some notation.

4.6. Lemma. *Suppose that $\tilde{X}$ is contractible. Let B be an object in $\mathbb{B}(\mathbb{A}_*(X))$, and let E be an object in $\mathbb{B}(\mathbb{A}[\pi])$. Any morphism*

$$f\colon \alpha(B) \longrightarrow E$$

in $\mathbb{B}(\mathbb{A}[\pi])$ *can be written in the form* $e \cdot \alpha(g)$, *where* $g: B \to D$ *is a morphism in* $\mathbb{B}(\mathbb{A}_*(X))$ *and* $e: \alpha(D) \to E$ *is a homotopy equivalence in* $\mathbb{B}(\mathbb{A}[\pi])$. □

This is Corollary 0.2 in Ranicki and Weiss [18]. It looks different in the language of triangulated categories.

4.7. Lemma. *Let* C *be any contravariant functor from* X *to* $\mathbb{B}(\mathbb{A})$ *such that* $C(x) = 0$ *for all but finitely many simplices* x. *(We do not assume that* C *belongs to* $\mathbb{B}(\mathbb{A}_*(X))$, *see 1.4.) Then there exist an object* B *in* $\mathbb{B}(\mathbb{A}_*(X))$ *and a natural chain map* $\lambda: B \to C$ *such that* $\lambda_x: B(x) \to C(x)$ *is a homotopy equivalence for all* x. □

This is also proved in [18]. An explicit formula for B is as follows:

$$B(x) = \text{hocolim } C \cdot h_x$$

where

$$h_x: X^{op} \downarrow x \longrightarrow X^{op}$$

is the forgetful functor, and $X^{op} \downarrow x$ is the category whose objects are morphisms in X^{op} with target x. The definition of the homotopy direct limit (= hocolim) is summarized in [18], but it is of course due to Bousfield and Kan [1].

4.8. Notation. If D and E are arbitrary contravariant functors from X to the category of chain complexes of R-modules, we write $D \odot E$ to mean

$$\int_X D(-) \otimes_R E(-).$$

If F is a free f.g. left $R[\pi]$-module, let $\beta(F)$ be the contravariant functor from X to the category of free R-modules given by

$$\beta(F)(x) = \{\pi\text{-maps from } p^{-1}(x) \text{ to } F\}$$

where $p^{-1}(x)$ is the set of simplices of X lying over the simplex x in X. (Any element in $p^{-1}(x)$ determines an isomorphism $\beta(F)(x) \cong F$.) □

In a very informal sense, the functors α (= assembly) and β are adjoint. That is, if M is an object in $\mathbb{A}_*(X)$ and F is an object in $\mathbb{A}[\pi]$, then morphisms

$$\alpha(M) \longrightarrow F$$

in $\mathbb{A}[\pi]$ correspond to natural R-homomorphisms

$$M \longrightarrow \beta(F).$$

Proof of 4.3. First it has to be explained why an n-dimensional symmetric structure ϕ on an object D in $\mathbb{B}(\mathbb{A}_*(X))$ should give rise to an n-dimensional visible symmetric structure on $\alpha(D)$.

Composing ϕ with $\hat{\omega}$ of 4.2, we obtain an n-cycle

$$\hat{\omega} \cdot \phi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{cl}(\tilde{X})^t \otimes_{\mathbb{Z}\pi} (\alpha(D) \otimes_R \alpha(D)))$$

which, on inspection, turns out to belong to the chain subcomplex

$$\mathrm{cl}(\tilde{X})^t \otimes_{\mathbb{Z}\pi} (\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \alpha(D) \otimes_R \alpha(D))).$$

(Remember that D is zero off a finite Δ-subset of X. Therefore, informally speaking, so are ϕ and $\hat{\omega} \cdot \phi$, which proves the claim.) Now $\mathrm{cl}(X)^t$ is a right free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\pi]$, and therefore $\hat{\omega} \cdot \phi$ is a visible symmetric structure. If ϕ is non-degenerate, so is $\omega \cdot \phi$, and therefore so is $\hat{\omega} \cdot \phi$. This describes the homomorphisms in 4.3.

As for surjectivity, suppose that we are given an object E in $\mathbb{B}(\mathbb{A}[\pi])$ with an n-dimensional visible symmetric structure ψ. We can assume that ψ comes in the form of a chain map

$$\Sigma^n W \longrightarrow \mathrm{cl}(\tilde{X})^t \otimes_{\mathbb{Z}\pi} (E \otimes_R E) \cong \beta(E) \odot \beta(E),$$

cf. 4.8, but it should only involve finitely many simplices of X (just like $\hat{\omega} \cdot \phi$ above). Choose a graded $R[\pi]$-basis for E. This implies a graded R-basis for each $\beta(E)(x)$, where x can be any simplex in X. A subfunctor $C \subset \beta(E)$ will be called straight if each $C(x)$ has a graded R-basis contained in the given graded R-basis of $\beta(E)(x)$. It will be said to carry ψ if the chain map ψ factors through the inclusion

$$C \odot C \longrightarrow \beta(E) \odot \beta(E).$$

Among all straight subfunctors $C \subset \beta(E)$ which carry ψ there is a smallest one. It is such that $C(x)$ is f.g. free over R for all x, and $C(x) = 0$ for all but finitely many $x \in X$. To this C we now apply 4.7. We get

$$\lambda: B \longrightarrow C$$

with B in $\mathbb{B}(\mathbb{A}_*(X))$, and $\lambda_x: B(x) \to C(x)$ a homotopy equivalence for all x. The composition

$$B \xrightarrow{\lambda} C \longrightarrow \beta(E)$$

is adjoint to some

$$f: \alpha(B) \longrightarrow E$$

to which we can apply 4.6. We get $f = e \cdot \alpha(g)$, where $g: B \to D$ is a morphism in $\mathbb{B}(\mathbb{A}_*(X))$ and $e: \alpha(D) \to E$ is a homotopy equivalence in $\mathbb{B}(\mathbb{A}[\pi])$. Since C carries ψ, we can write

$$\psi: \Sigma^n W \longrightarrow C \odot C.$$

Since $B \odot B$ maps to $C \odot C$ by a homotopy equivalence, we can write ψ as a composition

$$\Sigma^n W \xrightarrow{\psi'} B \odot B \longrightarrow C \odot C,$$

subjecting ψ to a chain homotopy if necessary. The composite chain map

$$\psi'' = (g \odot g) \cdot \psi' : \Sigma^n W \longrightarrow D \odot D = D \otimes_{\mathbb{A}_*(X)} D$$

is an n-dimensional symmetric structure on D. It is not hard to verify that assembly transforms (D, ψ'') into something homotopy equivalent to (E, ψ). That is, the homotopy equivalence

$$e : \alpha(D) \longrightarrow E$$

sends the visible symmetric structure $\hat{\omega} \cdot \psi''$ to ψ. If ψ is nondegenerate, then $\hat{\omega} \cdot \psi''$ must be nondegenerate mod $\mathbb{E}$. This proves surjectivity in 4.3. Injectivity is proved by a relative version of the same argument, as is customary in such cases. This is left to the reader.

Proof of 4.4. This is practically contained in the proof of 4.3, as follows. Use the same arguments to prove that

$$\pi_n(\mathbb{L}.(\mathbb{B}(\mathbb{A}_*(X)), \mathbb{E})) \cong VL_n(R[\pi]),$$

where $VL_n(R[\pi])$ denotes the visible quadratic L-group. But

$$VL_n(R[\pi]) \cong L_n(R[\pi]),$$

a fact which we have verified in the introduction. So the map in 4.4 induces isomorphisms on homotopy groups. □

We come to the proof of 4.5. It will be sufficient to prove the following.

4.9. Lemma. *For any object D in $\mathbb{E} \subset \mathbb{B}(\mathbb{A}_*(X))$, the symmetrization homomorphisms*

$$Q_n(D) \longrightarrow Q^n(D)$$

are isomorphisms, for all $n \in \mathbb{Z}$.

Proof. The long exact sequence connecting Q_*, Q^* and $\hat{Q}^*$ shows that it is enough to prove that

$$\hat{Q}^*(D) = 0 \quad \text{if } D \text{ belongs to } \mathbb{E}.$$

In the proof of 4.3, we constructed (implicitly) homomorphisms

$$Q^n(D) \longrightarrow VQ^n(\alpha(D)); \quad [\phi] \longrightarrow [\hat{\omega} \cdot \phi]$$

for any D in $\mathbb{B}(\mathbb{A}_*(X))$ and $n \in \mathbb{Z}$; the same formula defines homomorphisms

$$z : \hat{Q}^n(D) \longrightarrow V\hat{Q}^n(\alpha(D)).$$

We shall show that z is an isomorphism for all D in $\mathbb{B}(\mathbb{A}_*(X))$ and $n \in \mathbb{Z}$. This does imply that $\hat{Q}^*(D) = 0$ if D belongs to $\mathbb{E}$, because then $\alpha(D)$ is contractible.

Note first that the functors

$$D \longrightarrow \hat{Q}^*(D), \quad D \longrightarrow V\hat{Q}^*(\alpha(D))$$

are homology theories on $\mathbb{B}(\mathbb{A}_*(X))$. That is, they are chain homotopy invariant, and a short exact sequence

$$(!) \qquad 0 \longrightarrow D \longrightarrow D' \longrightarrow D'' \longrightarrow 0$$

in $\mathbb{B}(\mathbb{A}_*(X))$ induces exact sequences

$$\hat{Q}^n(D) \longrightarrow \hat{Q}^n(D') \longrightarrow \hat{Q}^n(D''),$$
$$V\hat{Q}^n(\alpha(D)) \longrightarrow V\hat{Q}^n(\alpha(D')) \longrightarrow V\hat{Q}^n(\alpha(D'')).$$

(By definition, the sequence (!) is short exact if it is split exact in each dimension.) The proof can be adapted from Theorem 1.1 of Weiss [23]. Note that (!) can be expanded into a Puppe sequence

$$\ldots \longrightarrow \Sigma^{-1}(D'') \longrightarrow D \longrightarrow D' \longrightarrow D'' \longrightarrow \Sigma(D) \longrightarrow \ldots$$

in which any three term piece is short exact up to homotopy equivalence; therefore (!) gives rise to long exact sequences

$$\ldots \longrightarrow \hat{Q}^{n+1}(D'') \longrightarrow \hat{Q}^n(D) \longrightarrow \hat{Q}^n(D') \longrightarrow \hat{Q}^n(D'') \longrightarrow \hat{Q}^{n-1}(D) \longrightarrow \ldots$$
$$\ldots \longrightarrow V\hat{Q}^{n+1}(\alpha(D'') \longrightarrow V\hat{Q}^n(\alpha(D)) \longrightarrow V\hat{Q}^n(\alpha(D')) \longrightarrow V\hat{Q}^n(\alpha(D'')) \longrightarrow \ldots.$$

In checking that

$$z\colon \hat{Q}^*(D) \longrightarrow V\hat{Q}^*(\alpha(D))$$

is an isomorphism, we can therefore use the five lemma to induct over the skeletons of D. This leaves the case where D is concentrated in one dimension, without loss of generality dimension zero. Then D belongs to $\mathbb{A}_*(X) \subset \mathbb{B}(\mathbb{A}_*(X))$ and is a direct sum of objects Γ_y in $\mathbb{A}_*(X)$, where y can be any simplex in X and Γ_y is induced by the collection $\{m_x | x \in X\}$ with $m_x = 0$ for $x \neq y$ and $m_y = R$. We may therefore assume that $D = \Gamma_y$ for some y.

Now we calculate. If the characteristic map $c_y\colon \Delta^{|y|} \to X$ of y is injective, then

$$\Gamma_y \otimes_{\mathbb{A}_*(X)} \Gamma_y \cong \int_X \Gamma_y(-) \otimes \Gamma_y(-) \cong \mathrm{cl}(\Delta^{|y|}) \otimes R$$

which is equivariantly homotopy equivalent to R, so that

$$\hat{Q}^n(\Gamma_y) \cong \hat{H}^{-n}(\mathbb{Z}_2; R) \quad \text{for all } n.$$

If c_y is not injective, then we can nevertheless inject $\mathrm{cl}(\Delta^{|y|}) \otimes R$ equivariantly in $\Gamma_y \otimes_{\mathbb{A}_*(X)} \Gamma_y$ in such a way that $\mathbb{Z}_2$ acts freely on the quotient R-module chain complex. Applying

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\hat{W}, -)$$

turns this injection into a homotopy equivalence, and passing to homology we still obtain

$$\hat{Q}^n(\Gamma_y) \cong \hat{H}^{-n}(\mathbb{Z}_2; R) \quad \text{for all } n.$$

In calculating $V\hat{Q}^*(\alpha(\Gamma_y))$ we use a spectral sequence. Given any object C in $\mathbb{B}(\mathbb{A}[\pi])$, there is a spectral sequence with E^2-term

$$E^2_{p,q} = H_p(\pi; \hat{Q}^q(C^?))$$

(where $C^?$ is the underlying R-module chain complex) and converging to $V\hat{Q}^*(C)$. Namely, $V\hat{Q}^*(C)$ is the homology of the chain complex

$$P^t \otimes_{\mathbb{Z}[\pi]} (\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\hat{W}, C^? \otimes_R C^?))$$

which has a filtration coming from the skeleton filtration of the free resolution P. The filtration leads to a spectral sequence as usual. With

$$C = \alpha(\Gamma_y) \cong \mathbb{Z}[p^{-1}(y)] \otimes R \cong \bigoplus_{x \in p^{-1}(y)} R$$

we get

$$\hat{Q}^q(C^?) \cong \mathbb{Z}[p^{-1}(y)] \otimes \hat{H}^{-q}(\mathbb{Z}_2; R)$$

because $\hat{Q}^*$ preserves direct sums, and then

$$H_p(\pi; \hat{Q}^q(C^?)) \cong \begin{cases} 0 & \text{if } p \neq 0 \\ \hat{H}^{-q}(\mathbb{Z}_2; R) & \text{if } p = 0. \end{cases}$$

It follows that the spectral sequence collapses and

$$V\hat{Q}^n(\alpha(\Gamma_y)) \cong \hat{H}^{-n}(\mathbb{Z}_2; R) \cong \hat{Q}^n(\Gamma_y).$$

Inspection shows that this abstract isomorphism agrees with the map z (or its inverse), which is therefore an isomorphism. □

5. The twisted case

In proving theorem 0.1 in the general case (where $w: \pi \to \mathbb{Z}_2$ may be nontrivial) we shall again work with a Δ-set which is a $K(\pi, 1)$ and equip it with a double covering classified by $w: \pi \to \mathbb{Z}_2$.

For our purposes, the best way to codify double coverings is the following. We shall work in the category with objects (X, w), where X is a Δ-set and

$$w: X \longrightarrow \mathbb{Z}_2$$

is a covariant functor. (As usual, X is viewed as a category; $\mathbb{Z}_2$ is also viewed as a category, with one object and two morphisms.) A morphism from (X, w) to (X', w') consists of a Δ-map $f: X \to X'$ and a natural transformation from w to $w' \cdot f$. To an object (X, w) we associate a double covering of X with total Δ-set X^w: an n-simplex in

X^w is the same as a morphism $(\Delta^n, v) \to (X, w)$ where $v\colon \Delta^n \to \mathbb{Z}_2$ is the trivial functor. Every double covering of X arises from a suitable w, up to isomorphism. (Proof: Given a double covering of X, choose for any simplex $x \in X$ a simplex $\tilde{x}$ in the double cover which covers x; for a morphism $u\colon x \to y$ in X, let $w(u) = 0$ if u lifts to a morphism from $\tilde{x}$ to $\tilde{y}$, and $w(u) = 1$ otherwise.)

Given (X, w) and a covariant or contravariant functor

$$C\colon X \longrightarrow \mathbb{B}(\mathbb{A}),$$

we obtain a new covariant or contravariant functor

$$C \times w\colon X \longrightarrow \mathbb{B}(\mathbb{A})$$

by

$$\begin{cases} (C \times w)(x) = C(x) & \text{for } x \in X, \\ (C \times w)(f) = (-1)^{w(f)} \cdot C(f) & \text{for a morphism } f \text{ in } X. \end{cases}$$

Given (X, w) and a chain duality (T, e) on $\mathbb{A}$, we obtain a chain duality on $\mathbb{A}_*(X)$ as follows. For an object M in $\mathbb{A}_*(X)$ define $T(M)$ by

$$T(M)(x) = \int^X (T \cdot M)^x \times w.$$

(See the proof of 1.5). Then T is a contravariant functor from $\mathbb{A}_*(X)$ to $\mathbb{A}_*(X)$, and inspection reveals that

$$\mathrm{Hom}(T(C), D) \cong \int_X (C(-) \otimes D(-)) \times w$$

for C and D in $\mathbb{B}(\mathbb{A}_*(X))$. This implies a symmetry isomorphism

$$\tau\colon \mathrm{Hom}(T(C), D) \cong \mathrm{Hom}(T(D), C)$$

With $C = M$ and $D = T(M)$ we obtain

$$\tau\colon \mathrm{Hom}(T(M), T(M)) \cong \mathrm{Hom}(T^2(M), M)$$

and we let

$$e = \tau(\mathrm{id}_{T(M)}\colon T^2(M) \longrightarrow M.$$

This completes the construction of a chain duality on $\mathbb{A}_*(X)$; for trivial w we have already seen it in section 1.

Using this chain duality we form L-theory spectra

$$\mathbb{L}.(\mathbb{A}_*(X), w), \quad \mathbb{L}^{\cdot}(\mathbb{A}_*(X), w), \quad \hat{\mathbb{L}}^{\cdot}(\mathbb{A}_*(X), w)$$

where the w indirectly specifies the chain duality on $\mathbb{A}_*(X)$. These spectra behave naturally in (X, w), at least with respect to injective morphisms

$$(X, w) \longrightarrow (X', w').$$

As functors in (X, w) they have a homotopy invariance and an excision property. General nonsense as in 3.4, 3.5 proves that

$$\mathbb{L}_\bullet(\mathbb{A}_*(X), w) \simeq |X^w|_+ \wedge_{\mathbb{Z}_2} \mathbb{L}_\bullet(\mathbb{A})$$

where the generator of $\mathbb{Z}_2$ acts on $\mathbb{L}_\bullet(\mathbb{A})$ by changing the signs of all quadratic structures in sight. (More conceptually, identify $\mathbb{L}_\bullet(\mathbb{A})$ with $\mathbb{L}_\bullet(\mathbb{A}_*(\Delta^0), v)$, where $v: \Delta^0 \to \mathbb{Z}_2$ is the only possible functor, and note that the object (Δ^0, v) has automorphism group isomorphic to $\mathbb{Z}_2$ in our category with objects (X, w).) It is permitted to substitute $\mathbb{L}^\bullet$ or $\hat{\mathbb{L}}^\bullet$ for $\mathbb{L}_\bullet$ in this homotopy equivalence.

Assume now that $\mathbb{A}$ is the category of f.g. free R-modules, where R is a commutative ring. The methods of section 4 also show that

$$V\hat{L}^n(R[\pi]) \cong \pi_n(\hat{\mathbb{L}}^\bullet(\mathbb{A}_*(X), w))$$

provided X is a $K(\pi, 1)$ and the involution on $R[\pi]$ is the twisted involution corresponding to the homomorphism $\pi \to \mathbb{Z}_2$ which classifies the double covering $X^w \to X$. Since

$$\pi_n(\hat{\mathbb{L}}^\bullet(\mathbb{A}_*(X), w)) \cong \pi_n(|X^w|_+ \wedge_{\mathbb{Z}_2} \hat{\mathbb{L}}^\bullet(\mathbb{A}))$$

we can complete the proof of 0.1 in the general case by proving that the sign change involution on $\hat{\mathbb{L}}^\bullet(\mathbb{A}) = \hat{\mathbb{L}}^\bullet(R)$ respects the splitting of $\hat{\mathbb{L}}^\bullet(R)$ into Eilenberg–MacLane spectra. One way to prove this is to use products in L-theory. I shall be content with a sketch proof.

i) $\hat{\mathbb{L}}^\bullet(R)$ is a module spectrum over the ring spectrum $\mathbb{L}^\bullet(\mathbb{Z})$. (Here we are mostly interested in the action map

$$\mu: \mathbb{L}^\bullet(\mathbb{Z}) \wedge \hat{\mathbb{L}}^\bullet(R) \longrightarrow \hat{\mathbb{L}}^\bullet(R),$$

but not in its associativity properties. Still, to make sense of this, we have to think of the target of μ as a bispectrum made up of incomplete bisimplicial sets. This can be built using chain complexes modelled on standard bisimplices $\Delta^m \times \Delta^n$, rather than the standard simplices Δ^n. The details are omitted.)

ii) The action map

$$\mu: \mathbb{L}^\bullet(\mathbb{Z}) \wedge \hat{\mathbb{L}}^\bullet(R) \longrightarrow \hat{\mathbb{L}}^\bullet(R)$$

respects the $\mathbb{Z}_2$-actions provided $\mathbb{Z}_2$ acts on the right-hand side by sign change, and on the left-hand side via sign change on $\mathbb{L}^\bullet(\mathbb{Z})$, leaving the other smash factor $\hat{\mathbb{L}}^\bullet(R)$ alone.

iii) The spectrum $\mathbb{S}^0$, or a homotopy equivalent spectrum also denoted by $\mathbb{S}^0$, can be equipped with a $\mathbb{Z}_2$-action such that the unit map

$$\varepsilon: \mathbb{S}^0 \longrightarrow \mathbb{L}^\bullet(\mathbb{Z})$$

respects $\mathbb{Z}_2$-actions, with $\mathbb{Z}_2$ acting on $\mathbb{L}^\bullet(\mathbb{Z})$ by sign change. (Think of $\mathbb{S}^0$ as the bordism spectrum of smooth compact manifolds in $\mathbb{R}^\infty = \bigcup \mathbb{R}^n$ with framed normal bundle. The involution is given by the change of sign in the first vector in each frame. See Buoncristiano, Rourke and Sanderson [2].)

Now the composite map

$$\mathbb{S}^0 \wedge \hat{\mathbb{L}}^\bullet(R) \xrightarrow{\varepsilon \wedge \mathrm{id}} \mathbb{L}^\bullet(\mathbb{Z}) \wedge \hat{\mathbb{L}}^\bullet(R) \xrightarrow{\mu} \hat{\mathbb{L}}^\bullet(R)$$

is a homotopy equivalence, and respects $\mathbb{Z}_2$-actions. On the right-hand side, the action is by sign change, and on the left-hand side it only involves the smash factor $\mathbb{S}^0$. A splitting of $\mathbb{L}^\bullet(R)$ into Eilenberg–MacLane spectra implies a splitting of $\mathbb{S}^0 \wedge \hat{\mathbb{L}}^\bullet(R)$ into Eilenberg–MacLane spectra, and the involution on $\mathbb{S}^0 \wedge \hat{\mathbb{L}}^\bullet(R)$ will respect it as it only involves the factor $\mathbb{S}^0$.

6. Surgery obstructions

It is customary to say that a homotopy class $z \in \pi_k(M)$, where M^n is a compact manifold, can be killed by surgery if there exists a (co-)bordism $(V^{n+1}; M^n, N^n)$ such that the mapping cone of $z: S^k \to M$ is homotopy equivalent to V, relative to M. Similar terminology can be used in the context of Poincaré chain complexes. For example, let (C, ϕ) be an n-dimensional symmetric Poincaré object in $\mathbb{B}(\mathbb{A})$, where $\mathbb{A}$ is the category of f.g. free left modules over the ring with involution $R[\pi]$. We say that $z \in H_k(C)$ can be killed by surgery if there exists an n-dimensional symmetric Poincaré object (D, θ) in $\mathbb{B}(\mathbb{A}^*(\Delta^1))$ such that $d_1(D, \theta) = (C, \phi)$ and such that the mapping cone of

$$z: \Sigma^k R[\pi] \longrightarrow C$$

is homotopy equivalent to $D(\Delta^1)$, relative to C.

In the algebraic case there is a satisfactory obstruction theory. Let

$$z^* \in H^{n-k}(C; R[\pi])$$

be the Poincaré dual of $z \in H_k(C; R[\pi]) \cong H_k(C)$. Think of z^* as a homotopy class of chain maps from C to $\Sigma^{n-k} R[\pi]$. The image of $[\phi] \in Q^n(C)$ under z^* is a class in $Q^n(\Sigma^{n-k} R[\pi])$. This class in $Q^n(\Sigma^{n-k} R[\pi])$ is the obstruction to killing z by surgery. Consult Ranicki [15] for proofs. Had we worked with quadratic structures throughout, we would have found an obstruction in $Q_n(\Sigma^{n-k} R[\pi])$; with visible symmetric structures, an obstruction in $VQ^n(\Sigma^{n-k} R[\pi])$. It is therefore important to know $Q^n(\Sigma^p R[\pi])$, $Q_n(\Sigma^p R[\pi])$ and $VQ^n(\Sigma^p R[\pi])$ for all n and p. For the calculation of $Q^n(\Sigma^p R[\pi])$ and $Q_n(\Sigma^p R[\pi])$ see [15], or just calculate. In calculating $VQ^*(\Sigma^p R[\pi])$ we use the diagram with long exact columns

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
\hat{Q}^{n+1}(\Sigma^p R) & \xrightarrow{h} & V\hat{Q}^{n+1}(\Sigma^p R[\pi]) \\
\downarrow & & \downarrow \\
Q_n(\Sigma^p R) & \xrightarrow{f} & VQ_n(\Sigma^p R[\pi]) \\
\downarrow & & \downarrow \\
Q^n(\Sigma^p R) & \xrightarrow{g} & VQ^n(\Sigma^p R[\pi]) \\
\downarrow & & \downarrow \\
\hat{Q}^n(\Sigma^p R) & \xrightarrow{h} & V\hat{Q}^n(\Sigma^p R[\pi]) \\
\downarrow & & \downarrow
\end{array}
$$

where the horizontal arrows are induced by the inclusion of the trivial group 1 in π, and the resulting homomorphism of rings with involution

$$R \longrightarrow R[\pi].$$

6.1. Proposition

$$VQ^n(\Sigma^p R[\pi]) \cong Q^n(\Sigma^p R) \oplus (Q_n(\Sigma^p R[\pi])/Q_n(\Sigma^p R))$$

Proof. Recall that $VQ_n = Q_n$ for all n. From the proof of 4.9 we know that the arrows h in the diagram are all isomorphisms: both $\hat{Q}^n(\Sigma^p R) = V\hat{Q}^n(\Sigma^p R)$ and $V\hat{Q}^n(\Sigma^p R[\pi])$ were identified with $\hat{H}^{p-n}(\mathbb{Z}_2; R)$ at the very end of section 4. Furthermore

$$Q_n(\Sigma^p R) = H_{n-2p}(\mathbb{Z}_2; R^{(p)}),$$
$$VQ_n(\Sigma^p R[\pi]) \cong Q_n(\Sigma^p R[\pi]) = H_{n-2p}(\mathbb{Z}_2; R[\pi]^{(p)}),$$

where the superscript (p) indicates that $\mathbb{Z}_2$ acts by the usual involution multiplied by $(-1)^p$. It follows that the arrows f are all split injections.

7. An example

Using Theorem 0.1, we shall calculate

$$VL^*(\mathbb{Z}[\mathbb{Z}_2])$$

where $\mathbb{Z}[\mathbb{Z}_2]$ has the untwisted involution. Note that $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}_2])$ and $Wh(\mathbb{Z}_2)$ vanish, so decorations can be omitted. The calculation is particularly difficult in dimensions $0 \equiv (\text{mod } 4)$, and we start with this case.

7.1. Observation. The map

$$(i_*, f): VL^0(\mathbb{Z}[\mathbb{Z}_2]) \longrightarrow VL^0(\mathbb{R}[\mathbb{Z}_2]) \times V\hat{L}^0(\mathbb{Z}[\mathbb{Z}_2])$$

is injective, where i_* is induced by the inclusion of rings, and f is the forgetful map.

Proof. Use the exactness of

$$L_0(\mathbb{Z}[\mathbb{Z}_2]) \longrightarrow VL^0(\mathbb{Z}[\mathbb{Z}_2]) \xrightarrow{f} V\hat{L}^0(\mathbb{Z}[\mathbb{Z}_2])$$

and the injectivity of

$$L_0(\mathbb{Z}[\mathbb{Z}_2]) \longrightarrow L_0(\mathbb{R}[\mathbb{Z}_2]) \cong VL^0(\mathbb{R}[\mathbb{Z}_2]),$$

noting that $V\hat{L}^*(\mathbb{R}[\mathbb{Z}_2]) = 0$ by 0.1. □

To describe the image of (i_*, f) in 7.1, we use the isomorphism

$$VL^0(\mathbb{R}[\mathbb{Z}_2]) \longrightarrow L^0(\mathbb{R}) \times L^0(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$$

whose first component is induced by the augmentation homomorphism, and whose second component is the transfer (corresponding to the inclusion of $\mathbb{R}$ in $\mathbb{R}[\mathbb{Z}_2]$). We also use the isomorphism

$$V\hat{L}^0(\mathbb{Z}[\mathbb{Z}_2]) \cong \bigoplus_{j \geq 0} H_j(\mathbb{Z}_2; \hat{L}^{-j}(\mathbb{Z}))$$

of 0.1. The groups $\hat{L}^{-j}(\mathbb{Z})$ are described in 0.1.

7.2. Proposition. *The image of (i_*, f) in 7.1 consists of all the elements*

$$(s_1, s_2, a_0, a_1, a_2, \ldots) \in VL^0(\mathbb{R}[\mathbb{Z}_2]) \times V\hat{L}^0(\mathbb{Z}[\mathbb{Z}_2]) \cong \mathbb{Z} \times \mathbb{Z} \times \bigoplus_{j \geq 0} H_j(\mathbb{Z}_2; \hat{L}^{-j}(\mathbb{Z}))$$

satisfying the relations

$$s_2 \equiv 2s_1 (\text{mod } 8)$$
$$a_0 \equiv s_1 (\text{mod } 8)$$
$$a_1 = 0.$$

Proof. Given $x \in VL^0(\mathbb{Z}[\mathbb{Z}_2])$, we write $(s_1,(x), s_2(x), a_0(x), a_1(x), a_2(x), \ldots)$ for the image of x under (i_*, f). Applying the homomorphism $\mathbb{Z}[\mathbb{Z}_2] \to \mathbb{Z}$ we see that $a_0(x) \equiv s_1(x) (\text{mod } 8)$. Commutativity of

$$\begin{array}{ccc} VL^0(\mathbb{Z}[\mathbb{Z}_2]) & \xrightarrow{\text{forget}} & V\hat{L}^0(\mathbb{Z}[\mathbb{Z}_2]) \\ \Big\downarrow \text{transfer} & & \Big\downarrow \text{transfer} \\ VL^0(\mathbb{Z}) & \xrightarrow{\text{forget}} & V\hat{L}^0(\mathbb{Z}) \end{array}$$

shows that $2a_0(x) \equiv s_2(x) \pmod 8$, because the transfer on the right sends $(a_0, a_1, a_2, \ldots)$ to $2a_0$. This establishes the first two relations in 7.2. Looking at the image of $L_0(\mathbb{Z}[\mathbb{Z}_2])$ in $VL^0(\mathbb{Z}[\mathbb{Z}_2])$, we see that there are no further restrictions on $(s_1(x), s_2(x), a_0(x))$.

Therefore it is sufficient to prove that the boundary homomorphism

$$V\hat{L}^0(\mathbb{Z}[\mathbb{Z}_2]) \longrightarrow L_{-1}(\mathbb{Z}[\mathbb{Z}_2]) \cong \mathbb{Z}_2$$

is zero on the direct summand $H_j(\mathbb{Z}_2; \hat{L}^{-j}(\mathbb{Z}))$ $(j \geq 0)$ if and only if $j \neq 1$. (See Wall [22] for the computation of $L_*(\mathbb{Z}[\mathbb{Z}_2])$.) This is the difficult part. We can use the commutative diagramm

$$\begin{array}{ccc} \pi_0((B\mathbb{Z}_2)_+ \wedge \hat{\mathbb{L}}^\cdot(\mathbb{Z})) & \xrightarrow{\text{boundary}} & \pi_{-1}((B\mathbb{Z}_2)_+ \wedge \mathbb{L}.(\mathbb{Z})) \\ \Big\downarrow \text{assembly} & & \Big\downarrow \text{assembly} \\ V\hat{L}^0(\mathbb{Z}[\mathbb{Z}_2]) & \longrightarrow & L_{-1}(\mathbb{Z}[\mathbb{Z}_2]) \end{array}$$

where the upper horizontal arrow is induced by the boundary

$$\delta\colon \hat{\mathbb{L}}^\cdot(\mathbb{Z}) \longrightarrow S^1 \wedge \mathbb{L}.(\mathbb{Z})$$

This δ is a map between module spectra over the oriented bordism ring spectrum MSO, and localized at 2 over the ordinary homology spectrum $H\mathbb{Z}_2$. Therefore source and target of δ split into Eilenberg–MacLane spectra after localization at 2, and δ itself decomposes into cohomology operations of degree 0 and 1. So ist is sufficient to prove the following:

7.3. Proposition. *The assembly homomorphism*

$$\bigoplus_{j>0} H_{j-1}(\mathbb{Z}_2; L_{-j}(\mathbb{Z})) \longrightarrow L_{-1}(\mathbb{Z}[\mathbb{Z}_2]) \cong \mathbb{Z}_2$$

is zero on $H_{j-1}(\mathbb{Z}_2; L_{-j}(\mathbb{Z}))$ *if and only if* $j \neq 2$. □

Hambleton, Milgram, Taylor and Williams [7] denote the (partial) assembly homomorphism

$$H_p(\mathbb{Z}_2; L_q(\mathbb{Z})) \longrightarrow L_{p+q}(\mathbb{Z}[\mathbb{Z}_2])$$

by κ_p if $q \equiv 2 \pmod 4$, and by $\mathscr{I}_p$ if $q \equiv 0 \pmod 4$. They prove that κ_p is zero except possibly when $p = 0, 1, 2, 4, 8, \ldots$, and that κ_1 is nonzero. (This is Thm. 2.1, Thm. 1.16, Prop. 6.3 and remarks following Thm. A. in [7]). Since the relevant values of p in 7.3 are odd, this proves 7.3 and completes the proof of 7.2.

7.4. Observation. *The sequence*

$$0 \longrightarrow VL^3(\mathbb{Z}[\mathbb{Z}_2]) \xrightarrow{\text{forget}} V\hat{L}^3(\mathbb{Z}[\mathbb{Z}_2]) \xrightarrow{e} \hat{L}^3(\mathbb{Z}) \longrightarrow 0$$

is exact (where e is induced by the augmentation homomorphism). Therefore

$$VL^3(\mathbb{Z}[\mathbb{Z}_2]) \cong \bigoplus_{j>0} H_j(\mathbb{Z}_2; \hat{L}^{3-j}(\mathbb{Z})).$$

Proof. Use the long exact sequence just before 0.1, and the fact that the homomorphisms

$$\hat{L}^3(\mathbb{Z}) \xrightarrow{\text{boundary}} L_2(\mathbb{Z}) \xrightarrow{\text{induction}} L_2(\mathbb{Z}[\mathbb{Z}_2])$$

are isomorphisms.

7.5. Observation. The forgetful maps

$$VL^i(\mathbb{Z}[\mathbb{Z}_2]) \longrightarrow V\hat{L}^i(\mathbb{Z}[\mathbb{Z}_2])$$

are isomorphisms for $i = 1$ and $i = 2$.

Proof. Use the long exact sequence again, and the fact that

$$L_1(\mathbb{Z}[\mathbb{Z}_2]) = 0. \quad \square$$

From the first relation in 7.2, we obtain the following:

7.6. Observation. For an oriented Poincaré space X of formal dimension $4k$, with an oriented double covering $p: Y \to X$, we have

$$\text{signature}(Y) \equiv 2 \cdot \text{signature}(X) \pmod 8$$

As is well known, the signature defect $2 \cdot \text{signature}(X) - \text{signature}(Y) \in \mathbb{Z}$ is the signature of the Browder-Livesay form $(a, b) \to \langle a \cup Tb, [Y] \rangle$ on $H^{2k}(Y)$, which is even and so has signature $\equiv 0 \pmod 8$. $\square$

References

[1] Bousfield, A. K., Kan, D. M.: Homotopy limits, completions, and localizations. Springer Lecture Notes 304 (1972)
[2] Buoncristiano, S., Rourke, C., Sanderson, B.: A geometric approach to homology theory. London Math. Society Lecture Note Series Vol. 18, Cambridge (1976)
[3] Cappell, S., Shaneson, J.: On four-dimensional s-cobordisms, II. Comm. Math. Helv. **64** (1989), 338–346
[4] Cartan, H., Eilenberg, S.: Homological algebra. Princeton (1956)
[5] Dold, A.: Lectures on Algebraic Topology. Springer (1972)
[6] Eilenberg, S., Steenrod, N.: Foundations of Algebraic Topology. Princeton (1952)
[7] Hambleton, I., Milgram, J., Taylor, L., Williams, B.: Surgery with finite fundamental group. Proc. Lond. Math. Soc. (3) **56** (1988), 349–379
[8] Kwasik, S., Schultz, R.: On s-cobordisms of metacyclic prism manifolds II, preprint
[9] Lück, W.: Transformation Groups and Algebraic K-theory. Springer Lecture Notes in Mathematics 1408 (1989)
[10] MacLane, S.: Categories for the working mathematician. Grad. Texts in Mathematics Vol. 5, Springer (1971)
[11] Milgram, J.: Orientations for Poincaré duality spaces and applications. Proc. 1986 Arcata Conf. on Algebraic Topology. Springer Lecture Notes 1370 (1989), 293–324
[12] Mishchenko, A. S.: Homotopy invariants of nonsimply-connected manifolds III. Higher signatures. Izv. Akad. Nauk SSSR, ser. mat. **35** (1971), 1316–1355
[13] Quinn, F.: A geometric formulation of surgery. Topology of manifolds, Proc. 1969 Georgia Conf. Markham (1970), 500–511
[14] Quinn, F.: Assembly maps in bordism-type theories, preprint
[15] Ranicki, A. A.: The algebraic theory of surgery I, II. Proc. Lond. Math. Soc. **40** (1980), 87–283
[16] Ranicki, A. A.: Exact sequences in the algebraic theory of surgery. Math. Notes 26, Princeton (1981)
[17] Ranicki, A. A.: Algebraic L-theory assembly and topological manifolds, preprint
[18] Ranicki, A. A., Weiss, M.: Chain complexes and assembly. Math. Z. **204** (1990), 157–185
[19] Rourke, C., Sanderson, B.: Δ-sets I. Homotopy theory. Quart. J. Math. Oxford (2) **22** (1971), 321–338
[20] Taylor, L., Williams, B.: Surgery spaces: formulae and structure. Proc. 1978 Waterloo Algebraic Topology Conf.. Springer Lecture Notes 741 (1979), 170–195
[21] Vogel, P.: Une nouvelle famille de groupes en L-théorie algébrique. Proc. of 1982 Bielefeld Conf.. Springer Lecture Notes 1046 (1984), 385–421
[22] Wall, C.T.C.: Surgery on compact manifolds. Academic Press (1971)
[23] Weiss, M.: Surgery and the generalized Kervaire invariant I, II. Proc. Lond. Math. Soc. **51** (1985), 146–230

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