LOCALIZATION IN ALGEBRAIC L-THEORY

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Let $f : A \to B$ be a morphism of rings with involution. If B is a localization of A in the classical sense, Karoubi [1], Pardon [3], Ranicki [5] and Smith [6] have given exact sequences between the L-groups of A, the L-groups of B and relative groups which are defined in term of linking forms over torsion modules.

My purpose is to show that the localization exact sequence holds in a more general situation.

From f one can define a ring Λ endowed with a morphism $A \to \Lambda$ satisfying the following conditions :

i) for any matrix α with entries in A such that $\alpha \otimes B$ is invertible, $\alpha \otimes \Lambda$ is invertible too ;

ii) Λ is universal with respect to the property i).

We have a canonical homomorphism $\varepsilon : \Lambda \to B$. We will say that f is weakly locally epic if ε is epic and local if ε is an isomorphism. In this paper I will prove that the relative group $L_n(A \to \Lambda)$ depends only on the category $\mathcal{C}_B$ of finitely presented modules M with cohomology dimension 1 and satisfying $M \otimes B = \mathrm{Tor}_1(M,B) = 0$ if f is weakly locally epic and A doesn't contain any finitely generated submodule I which is B-perfect (i.e. $I \otimes B = 0$).

As a corollary we prove a Mayer-Vietoris exact sequence in L-theory for square of rings with involution :

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & D \end{array}$$

if $A \to B$ and $C \to D$ are local, A (resp. C) doesn't contain any finitely B-perfect (resp. D-perfect) submodule and the tensorization by C is an equivalence between the categories $\mathcal{C}_B$ dans $\mathcal{C}_D$.

For more simplicity I will consider the groups L_n^h only, but we have the same results with the groups L_n^{α}, α being any subgroup of $\tilde{K}_o$ or $\tilde{K}_1$ stable under involution ; we must just change a little the category $\mathcal{C}_B$.

§ 1 . QUADRATIC AND LINKING FORMS OVER COMPLEXES

Throughout this paper I will suppose that $f : A \to B$ is weakly locally epic and A doesn't contain any finitely generated B-perfect submodule.

Denote by $\mathcal{C}_B$ the class of "torsion modules" , i.e. the class of A-modules M having a resolution $0 \to C_1 \overset{d}{\to} C_o \to M \to 0$ by finitely generated free A-modules such that $d \otimes B$ is an isomorphism.

Let M be a torsion module and $0 \to C_1 \to C_o \to M \to 0$ be a resolution of M by finitely generated free A-modules. Since M is a finitely generated B-perfect module, $\mathrm{Hom}(M,A)$ is zero and we have an exact sequence :

$$0 \to \mathrm{Hom}(C_o,A) \to \mathrm{Hom}(C_1,A) \to \mathrm{Ext}^1(M,A) \to 0 \quad .$$

Then $\mathrm{Ext}^1(M,A)$ is a torsion module.

Denote by $\hat{M}$ the module $\mathrm{Ext}^1(M,A)$. The correspondance $M \mapsto \hat{M}$ is a contravariant functor from $\mathcal{C}_B$ to itself and $\hat{\hat{M}}$ is canonically isomorphic to M .

If M is a torsion module a bilinear form over M is a map $M \to \hat{M}$. The set $B(M)$ of bilinear forms over M is endowed with an involution in the following way : if $\varphi : M \to \hat{M}$ is a bilinear form, $t\varphi$ is the composite map : $M \simeq \hat{\hat{M}} \xrightarrow{\hat{\varphi}} \hat{M}$.

Let us consider now bilinearforms over complexes :

By definition a free (resp. torsion) complex will be a $\mathbb{Z}$-graded complex :

$$\longrightarrow C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \longrightarrow \cdots$$

where $\oplus_i C_i$ is a finitely generated free (resp. torsion) A-module. And a complex is a free or a torsion complex.

If C_* is a complex the dual complex $\hat{C}_*$ is the complex :

$$\longleftarrow \hat{C}_{n+1} \xleftarrow{(-1)^{n+1}\hat{d}} \hat{C}_n \xleftarrow{(-1)^n\hat{d}} \hat{C}_{n-1} \longleftarrow$$

where $\wedge$ is the fonctor $\mathrm{Hom}(\ ,A)$ if C_* is free and $\mathrm{Ext}^1(\ ,A)$ if C_* is torsion, and $\hat{C}_n$ is of degree $-n$.

A bilinear form over C_* is a linear map $C_* \to \hat{C}_*$. The set $B_*(C_*) = B^{-*}(C_*) = \mathrm{Hom}(C_*,\hat{C}_*)$ of bilinear forms over C_* is a graded differential $\mathbb{Z}$-module endowed with an involution t by :

$$\partial^\circ \varphi(u) = \partial^\circ\varphi + \partial^\circ u$$
$$d(\varphi(u)) = (d\varphi)u + (-1)^{\partial^\circ\varphi}\varphi(du)$$
$$(t\varphi)u = (-1)^{\partial^\circ u\ \partial^\circ\varphi(u)}\hat{\varphi}(u)$$

for any $u \in C_*$ and $\varphi \in B_*(C_*)$.

If $\varepsilon = \pm 1$, $B_*(C_*)^\varepsilon$ denote the complex $B_*(C_*)$ with the new involution $\varphi \mapsto \varepsilon t\varphi$, if C_* is free, and $\varphi \mapsto -\varepsilon t \varphi$ if C_* is torsion.

<u>Definition 1.1</u>

Let C_* be a free (resp. torsion) complex. A quadratic (resp. linking) n-form over C_* is an element of the group :

$$Q^n(C_*) = H_{-n}(\mathbb{Z}/2, B_*(C_*)^\varepsilon) \qquad \varepsilon = (-1)^n \quad .$$

If we take the standard resolution W_* of the $\mathbb{Z}[\mathbb{Z}/2]$-module $\mathbb{Z}$:

$$\mathbb{Z}[\mathbb{Z}/2] \xleftarrow{1-t} \mathbb{Z}[\mathbb{Z}/2] \xleftarrow{1+t} \mathbb{Z}[\mathbb{Z}/2] \longrightarrow \cdots$$

any quadratic (resp. linking) n-form over C_* is represented by :

$$e_o \otimes \varphi_o + e_1 \otimes \varphi_1 + \cdots \qquad \varphi_i \in B^{n+i}(C_*)$$

and we have :

$$\forall\ i \geqslant 0 \qquad d\,\varphi_i = ((-1)^{i+1} + (-1)^n t)\ \varphi_{i+1}$$
$$(\text{resp. } d\,\varphi_i = ((-1)^{i+1} - (-1)^n t)\ \varphi_{i+1}\ ) .$$

<u>Definition 1.2</u>

Let $\Sigma_* \to C_*$ be an epimorphism of free (resp. torsion) complexes. A quadratic (resp. linking) n-form over $\Sigma_* \to C_*$ is an element of the group :

$$Q^n(\Sigma_* \to C_*) = H_{-n}(\mathbb{Z}/2, B_*(\Sigma_*)^{-\varepsilon}/B_*(C_*)^{-\varepsilon}) \qquad \varepsilon = (-1)^n \quad .$$

Notations 1.3

Let C_* be a free (resp. torsion) complex and q be a quadratic (resp. linking) n-form over C_*. The image of q by the composite map :

$$Q^n(C_*) \xrightarrow{\text{transfert}} H_{-n}(1,B_*(C_*)^{\varepsilon}) \xrightarrow{\sim} H_{-n}(B_*(C_*))$$

give a chain map from C_* to $\hat{C}_*$ of degree $-n$. This chain map, well defined up to homotopy, will be denoted by $\tilde{q}$.

Let $\Sigma_* \to C_*$ be an epimorphisme of free (resp. torsion) complexes and q be a quadratic (resp. linking) n-form over $\Sigma_* \to C_*$. If K_* is the kernel of $\Sigma_* \to C_*$, we get a chain map from K_* to $\hat{\Sigma}_*$, well defined up to homotopy, as the image of q by the composite map :

$$Q^n(\Sigma_* \to C_*) \xrightarrow{\text{transfert}} H_{-n}(1,B_*(\Sigma_*)/B_*(C_*)) \longrightarrow H_{-n}(\mathrm{Hom}_*(K_*,\hat{\Sigma}_*)) \quad .$$

This chain map will be denoted by $\tilde{q}$.

Definition 1.4

Let C_* (resp. $\Sigma_* \to C_*$) be a complex (resp. an epimorphism of complexes) and q be a quadratic or linking n-form over C_* (resp. $\Sigma_* \to C_*$). The form q is said non singular if $\tilde{q}$ is a homology equivalence. If C_* is free (resp. Σ_* and C_* are free), q is said B-non singular if $\tilde{q}$ is a B-homology equivalence.

Definition 1.5

Let $\mathcal{C}$ be the word free (resp. free, resp. B-acyclic free, resp. torsion) and $\mathfrak{F}$ the words non singular quadratic (resp. B-non singular quadratic, resp. non singular quadratic, resp. non singular linking).

Let C_* be a $\mathcal{C}$ complex and q be a $\mathfrak{F}$ n-form over C_*. The object (C_*,q) is called cobordant to zero if there exists an exact sequence of $\mathcal{C}$ complexes $0 \to K_* \to \Sigma_* \to C_* \to 0$ such that q is the boundary of a $\mathfrak{F}$ n-1-form over $\Sigma_* \to C_*$.

Theorem 1.6 [7]

The group $L_n^h(A)$ (resp. $L_n^h(\Lambda)$) is isomorphic to the group of free complexes together with non singular (resp. B-non singular) quadratic n-forms modulo the following relation : (C_*,q) is cobordant to (C'_*,q') if $(C_*\oplus C'_*,q-q')$ is cobordant to zero.

Definition 1.7

The cobordism group of B-acyclic free (resp. torsion) complexes together with non singular quadratic (resp. linking) n-forms will be denoted by $L'_n(B,A)$ (resp. $L''_n(B,A)$).

Theorem 1.8

The group $L'_n(B,A)$ is isomorphic to $L^h_{n+1}(A \to \Lambda)$.

Proof

Since $L^h_n(\Lambda)$ is isomorphic to $\Gamma^h_n(A \to \Lambda)$ or $\Gamma^h_n(A \to B)$ [7], it suffices to prove that $L'_n(B,A)$ is isomorphic to the group $\Gamma^h_{n+1}\begin{pmatrix} A \to A \\ \downarrow \quad \downarrow \\ A \to B \end{pmatrix}$ and that is proved by Ranicki [5] and Smith [6] by using a dual point of view (a quadratic form over C_* in my sense is a quadratic form over C^* in the sense of Ranicki [4], [5]).

The main result of this paper is the following :

Theorem 1.9

The group $L''_n(B,A)$ is isomorphic to $L^h_{n+2}(A \to \Lambda)$.

§ 2 . RELATIONS BETWEEN FREE COMPLEXES AND TORSION COMPLEXES

The first informations about B-acyclic free complexes and torsion complexes are the following :

Lemma 2.1

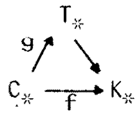
Let $\ldots \to 0 \to C_p \xrightarrow{d} C_{p+1} \to 0 \ldots$ be a B-acyclic free complex of length two. Then d is monic and Coker d is a torsion module.

Proof

Since $d \otimes B$ is bijective, the complex $\ldots \to 0 \to \hat{C}_p \xrightarrow{\hat{d}} \hat{C}_{p+1} \to 0 \to \ldots$ is B-acyclic and Coker $\hat{d}$ is B-perfect. But A doesn't contain any finitely generated B-perfect submodule. Then the map $\mathrm{Hom}(\hat{C}_{p+1},A) \xrightarrow{\hat{d}} \mathrm{Hom}(\hat{C}_p,A)$ is monic and d is monic and Coker d is a torsion module.

Lemma 2.2

Let C_* be a B-acyclic free complex and $f : C_* \to K_*$ a morphism from C_* to a torsion complex K_* . Then there is a commutative diagram :



such that T_* is a torsion complex and g is a homology equivalence.

Proof

Step 1

Suppose C_* is the complex $\ldots 0 \leftarrow C_p \leftarrow C_{p+1} \leftarrow 0 \leftarrow \ldots$. By setting :

$$T_i = 0 \qquad \text{for} \quad i \neq p, p+1$$

$$T_p = C_p \underset{C_{p+1}}{\oplus} K_{p+1} \quad \text{and} \quad T_{p+1} = K_{p+1}$$

we get a complex between C_* and K_* , and by lemma 2.1, T_* is a torsion complex, and $C_* \to T_*$ is a homology equivalence. Then the lemma is proved if C_* is of length two.

Step 2

Suppose we have an exact sequence of B-acyclic free complexes :

$$0 \to C_* \to C'_* \to C''_* \to 0$$

and suppose that C_* and C'_* satisfy the lemma. If $C''_* \to K_*$ is a morphism to a torsion complex, the composite map $C'_* \to K_*$ factorizes through a torsion complex T'_* and the map $C'_* \to T'_*$ is a homology equivalence. Up to homology equivalence we may suppose that $T'_* \overset{\alpha}{\to} K_*$ is epic. The map $C_* \to (\operatorname{Ker} \alpha)_*$ factorizes through a torsion complex T_* by a homology equivalence $C_* \to T_*$. Then it is not difficult to prove that the map $C''_* \to K_*$ factorizes by a homology equivalence through the maping cone of $T_* \to T'_*$:

$$\begin{array}{ccccccccc} 0 & \to & C_* & \to & C'_* & \to & C''_* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & T_* & \to & T'_* & \to & K_* & & \end{array}$$

The class of B-acyclic free complexes satisfying the lemma is stable under homotopy equivalence, suspension and quotient. Then it is stable under extension and contain all the complexes of length two. By [7] any B-acyclic complex is in this class and the lemma is proved.

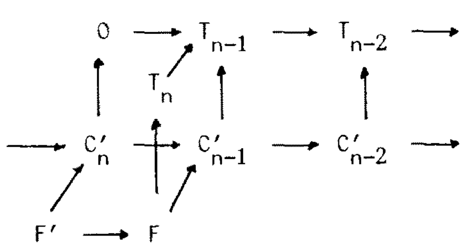
Conversely we have :

<u>Lemma 2.3</u>

Let T_* be a torsion complex. Then there exists a homology equivalence from a B-acyclic free complex to T_* .

<u>Proof</u>

This lemma will be proved by induction on the length of T_* . Let T_* be a n-dimensional torsion complex. By induction we have a homology equivalence f from a B-acyclic free complex C'_* to the complex $\to 0 \to T_{n-1} \to T_{n-2} \to \dots$. Take a finitely generated free resolution $0 \to F' \to F$ of T_n . Since the kernel of $C'_{n-1} \to C'_{n-2}$ maps onto the kernel of $T_{n-1} \to T_{n-2}$, the composite map $F \to T_n \to T_{n-1}$ factorizes through $\mathrm{Ker}(C'_{n-1} \to C'_{n-2})$. Moreover f is a homology equivalence and the composite map $F' \to F \to C'_{n-1}$ lift through C'_n . Then we get a morphism g from the complex $\to 0 \to F' \to F \to 0 \to \dots$ to C'_* and the induced map from the mapping cone of g to T_* is a homology equivalence :



<u>Lemma 2.4</u>

Let $\varepsilon = \pm 1$ and $0 \to K_* \overset{\alpha}{\to} C_* \to T_* \to 0$ be an exact sequence of complexes such that K_* is acyclic free, C_* is free and T_* is torsion. Then we have a canonical lóng exact sequence :

$$\dots \to H_{i+1}(\mathbb{Z}/2, B_*(T_*)^{-\varepsilon}) \to H_i(\mathbb{Z}/2, B_*(C_*)^{\varepsilon}) \to H_{i+2}(\mathbb{Z}/2, \hat{T}_* \otimes \hat{T}_*) \to H_i(\mathbb{Z}/2, B_*(T_*)^{-\varepsilon}) \to .$$

where $\hat{T}_* \otimes \hat{T}_*$ is endowed with the involution $a \otimes b \to -\varepsilon(-1)^{\partial^\circ a \partial^\circ b}\, b \otimes a$.

Proof

The above exact sequence induces the following :

$$0 \leftarrow \hat{T}_* \overset{\partial}{\leftarrow} \hat{K}_* \overset{\hat{\alpha}}{\leftarrow} \hat{C}_* \leftarrow 0$$

and we get a complex of complexes :

$$0 \leftarrow \hat{K}_* \otimes \hat{K}_* \overset{\mu}{\leftarrow} \hat{C}_* \otimes \hat{K}_* \oplus \hat{K}_* \otimes \hat{C}_* \overset{\lambda}{\leftarrow} \hat{C}_* \otimes \hat{C}_* \leftarrow 0$$

by setting : $\lambda(u \otimes v) = u \otimes \hat{\alpha}(v) + \hat{\alpha}(u) \otimes v$

$\mu(u \otimes b + a \otimes v) = \hat{\alpha}(u) \otimes b - a \otimes \hat{\alpha}(v)$

for any $a,b \in \hat{K}_*$ and $u,v \in \hat{C}_*$.

λ and μ are compatible with the differentials, and by setting :

$$t(u \otimes v) = \varepsilon(-1)^{\partial^\circ u \partial^\circ v}\, v \otimes u$$

$$t(u \otimes b + a \otimes v) = \varepsilon(-1)^{\partial^\circ a \partial^\circ v}\, v \otimes a + \varepsilon(-1)^{\partial^\circ u \partial^\circ b}\, b \otimes u$$

$$t(a \otimes b) = -\,\varepsilon(-1)^{\partial^\circ a \partial^\circ b}\, b \otimes a$$

for any $a,b \in \hat{K}_*$ and $u,v \in \hat{C}_*$, the morphisms λ and μ are equivariant for this involution.

Now we get three exact sequences of differential $\mathbb{Z}[\mathbb{Z}/2]$-modules :

$$0 \leftarrow \hat{T}_* \otimes \hat{T}_* \leftarrow \hat{K}_* \otimes \hat{K}_* \leftarrow \mathrm{Im}\,\mu \leftarrow 0$$

$$0 \leftarrow \mathrm{Im}\,\mu \leftarrow \hat{C}_* \otimes \hat{K}_* \oplus \hat{K}_* \otimes \hat{C}_* \leftarrow \mathrm{Ker}\,\mu \leftarrow 0$$

$$0 \leftarrow B_*(T_*)^{-\varepsilon} \leftarrow \mathrm{Ker}\,\mu \leftarrow B_*(C_*)^{\varepsilon} \leftarrow 0 \quad .$$

The involution on $\hat{T}_* \otimes \hat{T}_*$ is defined by :

$$t(a \otimes b) = -\,\varepsilon(-1)^{\partial^\circ a \partial^\circ b}\, b \otimes a \qquad \forall\, a,b \in \hat{T}_* \quad .$$

The third exact sequence comes from the isomorphism :

$$\hat{C}_* \otimes \hat{C}_* = \mathrm{Hom}(C_*,\hat{C}_*) = B_*(C_*)$$

$$\mathrm{Ker}\,\mu/\mathrm{Im}\,\mu \simeq \mathrm{Tor}_1(\hat{T}_*,\hat{T}_*) \simeq \mathrm{Hom}(T_*,\hat{T}_*) \simeq B_*(T_*) \quad .$$

On the other hand $\hat{K}_*$ is contractible and $\hat{K}_* \otimes \hat{K}_*$ and $\hat{C}_* \otimes \hat{K}_* \oplus \hat{K}_* \otimes \hat{C}_*$ are acyclic. That implies the isomorphisms :

$$H_{i+2}(\mathbb{Z}/2,\hat{T}_*\otimes\hat{T}_*) \tilde{\to} H_{i+1}(\mathbb{Z}/2,\mathrm{Im}\,\mu) \tilde{\to} H_i(\mathbb{Z}/2,\mathrm{Ker}\,\mu)$$

and by taking the homology exact sequence of the third above exact sequence, we prove the lemma.

Lemma 2.5

Let $C_* \xrightarrow{f} T_*$ be a homology equivalence from a free complex to a torsion complex. Then a linking n-1-form q over T_* induces a well defined quadratic n-form $f^*(q)$ on C_*. Furthermore q is non singular if and only if $f^*(q)$ is non singular.

Proof

Up to homotopy equivalence we may suppose that f is surjective with free kernel K_*. By 2.4 we get a map $Q^{n-1}(T_*) \xrightarrow{f^*} Q^n(C_*)$ and $f^*(q)$ is well defined.

Moreover f^* is induced by a boundary and the transfert commutes with the boundary. Then the cycle $\widetilde{f^*(q)}$ is the boundary of $\tilde{q}$ in the exact sequence $0 \leftarrow B_*(T_*)^{-\varepsilon} \leftarrow \text{Ker}\,\mu \leftarrow B_*(C_*)^{\varepsilon} \leftarrow 0$ (see the proof of 2.4).

More precisely $\widetilde{f^*(q)}$ is the boundary of the composite map $C_* \to T_* \overset{\tilde{g}}{\to} \hat{T}_*$ in the exact sequence $0 \to \hat{C}_* \to \hat{K}_* \to \hat{T}_* \to 0$. But $\hat{K}_*$ is acyclic. Hence $\tilde{q}$ is a homology equivalence if and only if $\widetilde{f^*(q)}$ is a homotopy equivalence.

§ 3 . THE ISOMORPHISM $L''_{n-1}(B,A) \tilde{\to} L'_n(B,A)$

Let T_* be a torsion complex and q a non singular linking n-1-form over T_*. By 2.3 there exists a homology equivalence f from a B-acyclic free complex C_* to T_*, and by 2.5 we get an element $F(T_*,q)$ in $L'_n(B,A)$ represented by $(C_*,f^*(q))$. If $f' : C'_* \to T_*$ is an other choice there is a homotopy equivalence $g : C'_* \to C_*$ and $f \circ g$ is homotopic to f'. Then $g^*(f^*(q))$ is equal to $f'^*(q)$ and $(C_*,f^*(q))$ is cobordant to $(C'_*,f'^*(q))$. Hence $F(T_*,q)$ depends only on (T_*,q).

Lemma 3.1

The correspondance F induces a morphism from $L''_{n-1}(B,A)$ to $L'_n(B,A)$.

Proof

Clearly F is additive. Then the only thing to do is to prove that $F(T_*,q)$ vanishes if (T_*,q) is cobordant to zero.

Suppose we have an exact sequence of torsion complexes :

$$0 \to R_* \to S_* \to T_* \to 0$$

and a non singular linking n-2-form u over $S_* \to T_*$ with boundary q .

By 2.3 there exist B-acyclic free complexes K_* , Σ_* , C_* and a commutative diagram :

$$\begin{array}{ccccccccc} 0 & \to & K_* & \to & \Sigma_* & \to & C_* & \to & 0 \\ \downarrow & & \downarrow & & \downarrow f & & \downarrow f & & \downarrow \\ 0 & \to & R_* & \to & S_* & \to & T_* & \to & 0 \end{array}$$

such that the lines are exact and the vertical maps are homology equivalences.

With a relative version of 2.5 we get a commutative diagram :

$$\begin{array}{ccc} Q^{n-2}(S_* \to T_*) & \xrightarrow{f^*} & Q^{n-1}(\Sigma_* \to C_*) \\ \partial \downarrow & & \downarrow \partial \\ Q^{n-1}(T_*) & \xrightarrow[f^*]{} & Q^n(C_*) \end{array} .$$

Then $f^*(u)$ is a quadratic n-1-form over $\Sigma_* \to C_*$ with boundary $f^*(q)$. Since u is non singular, $f^*(u)$ is non singular too and $(C_*,f^*(q))$ is cobordant to zero.

Lemma 3.2

Let $f : C_* \to T_*$ be a homology equivalence from a free complex to a torsion complex. Let $\hat{T}_* \otimes \hat{T}_*$ be the graded differential module endowed with the involution $t(a \otimes b) = -\varepsilon(-1)^{\partial^\circ a \partial^\circ b}\, b \otimes a$ $(\varepsilon = \pm 1)$ for any $a,b \in \hat{T}_*$. Then for any element $u \in H_*(\mathbb{Z}/2,\hat{T}_*\otimes\hat{T}_*)$ there exist a torsion complex T'_* and a homology equivalence $\alpha : T'_* \to T_*$ such that f lifts through α and $\alpha^*(u)$ vanishes.

Proof

Any element in $H_*(\mathbb{Z}/2,\hat{T}_*\otimes\hat{T}_*)$ is represented by $\sum_{i,p,q} e_i \otimes u_{pq}$, $u_{pq} \in \hat{T}_p \otimes \hat{T}_q$. Then it suffices to prove that for any $v \in \hat{T}_p \otimes \hat{T}_q$ there exists a surjective homology equivalence $\alpha : T'_* \to T_*$ such that f lifts through α and v goes to zero in $\hat{T}'_p \otimes \hat{T}_q$.

By the canonical isomorphism $\hat{T}_p \otimes \hat{T}_q \simeq \mathrm{Ext}^1(T_p, \hat{T}_q)$ an element $v \in \hat{T}_p \otimes \hat{T}_q$ gives an extension $0 \to \hat{T}_q \to T'_p \to T_p \to 0$ and v goes to zero in $\hat{T}'_p \otimes \hat{T}_q$.

By setting :

$$T'_i = \begin{cases} T_i & i \neq p, p+1 \\ T'_p \underset{T_p}{\times} T_{p+1} & i = p+1 \end{cases}$$

we get a torsion complex T'_* and a homology equivalence $\alpha : T'_* \to T_*$ such that v vanishes in $\hat{T}'_p \otimes \hat{T}_q$. Moreover f is a homology equivalence and C_* is free, then f lifts through α and the lemma is proved.

<u>Theorem 3.3</u>

The morphism $F : L''_{n-1}(B,A) \to L'_n(B,A)$ is an isomorphism.

<u>Proof</u>

<u>Surjectivity of F</u>

Let $w' \in L'_n(B,A)$ represented by a B-acyclic free complex C_* together with a non singular quadratic n-form q over C_*. By 2.2 there exists a homology equivalence f from C_* to a torsion complex T_*.

Consider the exact sequence (3.4) :

$$Q^{n-1}(T_*) \xrightarrow{f^*} Q^n(C_*) \xrightarrow{\partial} H_{-n+2}(\mathbb{Z}/2, \hat{T}_* \otimes \hat{T}_*) \quad .$$

By 3.2 there exist a torsion complex T'_* and a homology equivalence $\alpha : T'_* \to T_*$ such that f lifts by f' through T'_* and ∂q vanishes in $H_{-n+2}(\mathbb{Z}/2, \hat{T}'_* \otimes \hat{T}'_*)$. Then there exists $q' \in Q^{n-1}(T'_*)$ such that $q = f'^*(q')$. By 2.5 q' is non singular and (T'_*, q') gives an element in $L''_{n-1}(B,A)$ which is going to w' by F.

<u>Injectivity of F</u>

Let T_* be a torsion complex and q be a non singular linking n-1-form over T_* such that $F(T_*, q)$ vanishes. Take a homology equivalence f from a B-acyclic free complex C_* to T_*. Since $(C_*, f^*(q))$ is cobordant to zero there exists an exact sequence of B-acyclic free complexes :

$$0 \to K_* \to \Sigma_* \to C_* \to 0$$

together with a non singular quadratic n-1-form u over $\Sigma_* \to C_*$ with boundary $f^*(q)$.

By 2.2 we can construct a commutative diagram :

$$\begin{array}{ccccccccc} 0 & \to & K_* & \to & \Sigma_* & \to & C_* & \to & 0 \\ \downarrow & & \downarrow & & \downarrow f & & \downarrow f & & \downarrow \\ 0 & \to & R_* & \to & S_* & \to & T_* & \to & 0 \end{array}$$

such that the lines are exact and the vertical maps are homology equivalences.

Consider the commutative diagram :

$$\begin{array}{ccccccc} H_{-n+3}(\mathbb{Z}/2,\hat{S}_* \otimes \hat{S}_*) & & & & & & \\ \downarrow & & & & & & \\ Q^{n-1}(S_*) & \leftarrow & Q^{n-1}(T_*) & \overset{\partial}{\leftarrow} & Q^{n-2}(S_* \to T_*) & \leftarrow & H_{-n+2}(\mathbb{Z}/2,B_*(S_*)^{-\varepsilon}) \\ \downarrow & & \downarrow f^* & & \downarrow f^* & & \downarrow \\ Q^{n}(\Sigma_*) & \leftarrow & Q^{n}(C_*) & \overset{\partial}{\leftarrow} & Q^{n-1}(\Sigma_* \to C_*) & \leftarrow & H_{-n+1}(\mathbb{Z}/2,B_*(\Sigma_*)^{\varepsilon}) \\ & & & & & & \downarrow \\ & & & & & & H_{-n+3}(\mathbb{Z}/2,\hat{S}_* \otimes \hat{S}_*) \end{array}$$

All the lines and the columns of this diagram are exact.

Since $f^*(q)$ is the boundary of u , the image of q in $Q^{n-1}(S_*)$ comes from $H_{-n+3}(\mathbb{Z}/2,\hat{S}_* \otimes \hat{S}_*)$. By 3.2 we may as well suppose that q restricts to zero on S_* and is the boundary of an element $v' \in Q^{n-2}(S_* \to T_*)$. The obstruction to lift $f^*(v')-u$ in $Q^{n-2}(S_* \to T_*)$ is in $H_{-n+3}(\mathbb{Z}/2,\hat{S}_* \otimes \hat{S}_*)$. By 3.2 we may as well suppose that this obstruction vanishes and there exists an element $v \in Q^{n-2}(S_* \to T_*)$ such that $\partial v = q$ and $f^*v = u$.

Since u is non singular, v is non singular and (T_*,q) is cobordant to zero.

<u>Corollary 3.4</u>

The group $L''_{n-1}(B,A)$ is isomorphic to $L^h_{n+1}(A \to \Lambda)$.

<u>Corollary 3.5</u>

The group $L^h_{n+1}(A \to \Lambda)$ depends only on the category $\mathcal{C}_B$ of torsion modules.

Theorem 3.6

Let

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & D \end{array}$$

be a square of rings with involution such that $A \to B$ and $C \to D$ are local and A (resp. C) doesn't contain any finitely generated B-perfect (resp. D-perfect) submodule (that holds for example if $A \to B$ and $C \to D$ are monic). Suppose we have the following conditions :

i) for any torsion module $M \in \mathcal{C}_B$ the map $M \to M \otimes C$ is an isomorphism and $\mathrm{Tor}_1^A(M,C) = 0$

ii) any torsion module $N \in \mathcal{C}_D$, considered as A-module, is in $\mathcal{C}_B$.

Then we have an long exact sequence :

$$\ldots \overset{\partial}{\to} L_n^h(A) \to L_n^h(B) \oplus L_n^h(C) \to L_n^h(D) \overset{\partial}{\to} L_{n-1}^h(A) \to \ldots \quad .$$

Proof

The conditions i) and ii) imply that $\mathcal{C}_B$ and $\mathcal{C}_D$ are equivalent. Then the map $L_n^h(A \to B) \to L_n^h(C \to D)$ is an isomorphism and the Mayer-Vietoris exact sequence holds.

Remark 3.7

Actually it is possible to give an interpretation of the group $L_n^h(A \to \Lambda)$ in term of linking form over torsion modules as in [1], [3], [5], [6] for example. That will appear in a further paper.

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