"Nil-groups and regularity" by Pierre Vogel (ca. 1990)

§ 1 Regular rings

Definition 1-1: Let A be a ring. A class $\mathcal{C}$ of A-modules is called exact if it satisfies the following conditions:

E1 - $\mathcal{C}$ is stable under direct limit

E2 - for every short exact sequence of A-modules:

$$0 \to M \to M' \to M'' \to 0$$

if two of these modules are in $\mathcal{C}$, so is the third.

Definition 1-2: Let A be a ring. Let $\mathcal{C}_0$ be the intersection of all exact classes of A-modules containing the module A itself. The modules of $\mathcal{C}_0$ will be called weakly regular.

A ring A is regular if every A-module is weakly regular.

Remarks: Because of the condition E1, an exact class is stable under direct summand. Therefore an exact class contains A if and only if it contains all projective modules. Thus the regularity condition for ring is Morita invariant and can be defined for every abelian category.

This notion of regularity semes (seems) to be in conflict with the classical notion of regularity used for noetherian or coherent rings. But that is not the case. Actually if a ring is coherent, it is regular in the classical sense if and only if it is regular in this sense (corollary 1-10).

Example: Since every flat module is a direct limit of projective modules, a flat module is weakly regular. By E2, every module of finite homological dimension is weakly regular. Therefore if a ring has finite homological dimension, it is regular. More precisely, if every finitely presented A-module has finite homological dimension, A is regular.

Counterexample: Let $\mathcal{C}$ be the class of $\mathbb{Z}/_4$-modules M such that the sequence:

$$\xrightarrow{2} M \xrightarrow{2} M \xrightarrow{2} M \xrightarrow{2}$$

is exact. The class $\mathcal{C}$ is exact and contains $\mathbb{Z}/_4$. Hence it contains all weakly regular modules. But $\mathbb{Z}/_2$ which is not in $\mathcal{C}$, is not weakly regular.

In the same way, if G is a group with torsion, we can see that $\mathbb{Z}[G]$ is not regular. To do that, consider a non trivial finite subgroup F of G and the class of all $\mathbb{Z}[G]$-modules M such that the Tate cohomology $\hat{H}^*(F,M)$ vanishes. This class contains all weakly regular modules but not $\mathbb{Z}$.

The class of weakly regular modules is the smallest class of A-modules containig A and satisfying conditions E1 and E2. In this situation, the condition E2 may be simplify a little:

Proposition 1-3: The class of weakly regular A-modules is the smallest class $\mathcal{C}$ of A-modules containing free modules and satisfying conditions E1 and:

E'2 - for every short exact sequence of A-modules:

$$0 \to M \to M' \to M'' \to 0$$

if M and M' are in $\mathcal{C}$, so is M''.

Proof: For every ordinal α, we can construct a class $\mathcal{C}_\alpha$ by induction in the following way:

$\mathcal{C}_0$ is the class of free modules.

if α is a limit ordinal, a module M lies in $\mathcal{C}_\alpha$ if and only if it is a direct limit of modules lying in $\bigcup_{\beta<\alpha} \mathcal{C}_\beta$.

if $\alpha = \beta+1$, a module M lies in $\mathcal{C}_\alpha$ if and only if it is the Cokernel of a monomorphism $f : M' \to M''$ where M' and M'' are in $\mathcal{C}_\beta$.

The only thing to do is to prove that the union $\mathcal{C}$ of classes $\mathcal{C}_\alpha$ is exactly the class of weakly regular modules.

Lemma 1-4: For every ordinal α the kernel of an epimorphism from a free module to a module in $\mathcal{C}_\alpha$ belongs to $\mathcal{C}$.

Proof: This lemma is obviously true if $\alpha = 0$. Suppose, by induction, that the lemma is true for every $\beta<\alpha$. Let $f: F \to M$ be an epimorphism from a free module F to a module M in $\mathcal{C}_\alpha$. If $\alpha = \beta + 1$, we have an exact sequence:

$$0 \to M' \to M'' \to M \to 0$$

and M' and M'' belong to $\mathcal{C}_\beta$. It is possible to complete this sequence to a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M'' & \longrightarrow & M & \longrightarrow & 0 \\ & & f'\uparrow & & f''\uparrow & & f\uparrow & & \\ 0 & \longrightarrow & F' & \longrightarrow & F'' & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

where the lines are exact, the vertical arrows are surjective and F' and F'' are free. By induction, Ker f' and Ker f'' are in $\mathcal{C}$. Hence the Kernel of f which is the Cokernel of Ker f' $\to$ Ker f'' belongs to $\mathcal{C}$ also.

If α is a limit ordinal, M is the direct limit of a system of modules M_i, $i \in I$ where I is a filtering small category and for every $i \in I$, M_i belongs to some $\mathcal{C}_\beta$, $\beta < \alpha$. Denote by M_* this system of modules. For every $i \in I$, let F_{i*} be the following system:

- for every $j \in I$, F_{ij} is the free A-module generated by the set of maps in I from i to j. For every map $j \to k$ in I, the induced mad from F_{ij} to F_{ik} is given by the composition.

Clearly, $\mathrm{Hom}(F_{i*}, M_*)$ is isomorphic to M_i and the limit of F_{i*} is isomorphic to A. Let J be the set of couples (i, u) where i is in I and u is a map from F_{i*} to M_*. Let F_* be the direct sum of F_{i*} for all couples (i, u) in J. We have an obvious map φ_* from F_* to M_*. For every $i \in I$, $\varphi_i: F_i \to M_i$ is surjective with kernel K_i in $\mathcal{C}$. Moreover φ_* induces an epimorphism φ from $F' = \varinjlim F_*$ to $M = \varinjlim M_*$ and the kernel of φ is the limit of the K_i's. By induction, K_i belongs to $\mathcal{C}$ for every i. Hence K belongs to $\mathcal{C}$.

On the other hand it is easy to see by induction that, for every γ, for every module N in $\mathcal{C}_\gamma$, $N \oplus F$ belongs to $\mathcal{C}$. Hence $\mathrm{Ker} f \oplus F'$, isomorphic to $K \oplus F$ by Shanuel's lemma, belongs to $\mathcal{C}$. Since $\mathcal{C}$ satisfies the property E1, it is stable by direct summand and Ker f lies in $\mathcal{C}$.

Lemma 1-5: $\mathcal{C}$ is stable under extension.

Proof: Let $0 \to M' \to M'' \to M \to 0$ be an exact sequence such that M' and M are in $\mathcal{C}$. Let f: $F \to M$ be an epimorphism from a free module F to M and N be the pull-back of F and M" over M. The module N is isomorphic to $M' \oplus F$. Thus it belongs to $\mathcal{C}$ and M", cokernel of a monomorphism from Ker f to N, lies in $\mathcal{C}$ too.

Lemma 1-6: $\mathcal{C}$ is stabie under kernel of epimorphism.

Proof: Let $0 \to M' \to M'' \to M \to 0$ be an exact sequence such that M" and M are in $\mathcal{C}$. Let f: $F \to M''$ be an epimorphism from a free module F to M" and K be the kernel of $F \to M$. We have an exact sequence:

$$0 \to K \to M' \oplus F \to M'' \to 0$$

By lemma 1-4, K lies in $\mathcal{C}$. By lemma 1-5, $M' \oplus F$ lies in $\mathcal{C}$ too. Since $\mathcal{C}$ satisfies E1, M' belongs to $\mathcal{C}$.

We have seen that $\mathcal{C}$ satisfies the condition E2. This class is exact and it is exactly the class of weakly regular modules.

Let $C = (C_n)$ be a A-chain complex, i. e. a graded differential projective A-module bounded from below. The complex C is *finite* if $\oplus_n C_n$ is finitely generated, *quasi-coherent* if each C_n is finitely generated.

The main result of this section is the following:

Theorem 1-7: Let C be a quasi-coherent chain complex and M be a module. Then, if M is weakly regular, every chain map from C to M factors through a finite chain complex.

Proof: In this theorem, M is consider as a graded differential module with trivial differential concentrated in degree 0.

Let $\mathcal{C}$ be the class of A-modules M such that, for every quasi-coherent chain complex C every chain map from C to M factors through a finite chain complex.

Let F be a free A-module and f be a chain map from a quasi-coherent chain complex C to F. This map is given by the map f_0 from C_0 to F. Hence f factors through a finitely generated free module F' contained in F. Since F' is a finite chain complex, F belongs to $\mathcal{C}$.

Let M be a direct limit of modules M_i in the class $\mathcal{C}$. Let f be a chain map from a quasi-coherent chain complex C to M. Since the map is defined by a map from the finitely presented module $\mathrm{Coker}(d\colon C_1 \to C_0)$ to M, f factors through some M_i and the chain map $C \to M_i$ factors through a finite chain complex. Therefore M belongs to $\mathcal{C}$.

Let $0 \to M' \to M'' \to M \to 0$ be an exact sequence of A-modules. Suppose that M' and M" are in $\mathcal{C}$. Let f be a chain map from a quasi-coherent chain complex C to M. Let C' be the mapping cone of the identity from the desuspention $\Sigma^{-1}C$ to itself. The complex C' is contractible and quasi-coherent and maps surjectively onto C. Since C' is contractible, there is no obstruction to lift the chain map $C' \to C \to M$ through M" and we get the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M'' & \longrightarrow & M & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Sigma^{-1}C & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Since M' is in $\mathcal{C}$ and $\Sigma^{-1}C$ is quasi-coherent, the chain map $\Sigma^{-1}C \to M'$ factors through a finite chain complex K'. Let C" be the push-out of C' and K' over $\Sigma^{-1}C$. The complex C" is quasi-coherent and M" lies in $\mathcal{C}$. Hence the chain map $C'' \to M''$ factors through a finite chain complex K". Let CK' be the cone of K'. Since CK' is acyclic the chain map $K' \to CK'$ extends to C". Let L be the direct sum $K'' \oplus CK'$. The constructions above give a factorisation of C'' → M'' through L and the map from K' to L is injective with projective cokernel K. The chain complexes K', L and K are finite and the chain map from C to M factors through K.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & M'' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K' & \longrightarrow & L & \longrightarrow & K & \longrightarrow & 0 \\
 & & \uparrow & \searrow \; C'' \nearrow & \uparrow & & \uparrow & & \\
 & & & \nwarrow & & & & & \\
0 & \longrightarrow & \Sigma^{-1}C & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & 0
\end{array}
$$

Then $\mathcal{C}$ contains all free modules and satisfies conditions E1 and E'2. By proposition 1-3, $\mathcal{C}$ contains all weakly regular modules and the theorem is proven.

Corollary 1-8: Let A be a regular ring. Let C be a quasi-coherent chain complex and C' be a chain complex with only finitely many non trivial homology groups. Then every chain map from C to C' factors up to homotopy through a finite chain complex.

Proof: The proof is by induction on the number of non zero homology group of C'. If C' has non homology, C' is contractible and every chain map from C to C' factors, up to homotopy, through a trivial chain complex. Let C' be a chain complex with n non zero homology groups and f be a chain map from C to C'. We can kill the last non trivial homology group of C' by adding algebraic cells and we get new chain complexes C'_0 and C'_1 and a short exact sequence:

$$0 \to C' \to C'_0 \to C'_1 \to 0$$

such that C'_1 has only one non trivial homology group and C'_0 only n-1.

By induction the composite map $C \to C' \to C'_0$ factors, up to homotopy, through a finite chain complex K_0. Let E be the cone of C. The difference of the maps $C \to C' \to C'_0$ and $C \to K_0 \to C'_0$ is homotopic to 0 and factors through E. Therefore the composite map $C \to C' \to C'_0$ factors through the chain complex $K'_0 = K_0 \oplus E$ and K'_0 is quasi-coherent and has the homotopy type of a finite chain complex. Moreover the map $C \to K'_0$ is injective with projective cokernel C_1.

The chain complex C_1 is quasi-coherent and we have a chain map g from C_1 to C'_1. But C'_1 has only one non trivial homology group M. Thus C'_1 is a projective resolution of M. For every quasi-coherent chain complex L, the homotopy classes of chain maps from L to C'_1 is isomorphic to the homotopy classes of chain maps from L to M. By theorem 1-7, the map g factors, up to homotopy, through a finite chain complex K_1. As above, we can construct a quasi-coherent chain complex K'_1 of the homotopy type of a finite chain complex and a factorization of g through K'_1. Let K' be the homotopy kernel of the chain map $K'_0 \to K'_1$ (i. e. the desuspension of its mapping cone). By construction the map f factors through K' and K' has the homotopy type of a finite chain complex K and f factors, up to homotopy, through K.

$$\begin{array}{ccccccccc}
0 & \to & C' & \to & C'_0 & \to & C'_1 & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & K' & \to & K'_0 & \to & K'_1 & & \\
 & & \uparrow & & \uparrow {=} & & \uparrow & & \\
0 & \to & C & \to & K'_0 & \to & C_1 & \to & 0
\end{array}$$

<u>Corollary</u> 1-9: Let A be a regular ring. Then a quasi-coherent chain complex is homotopy equivalent to a finite chain complex if and only if it has finitely many non trivial homology groups.

<u>Proof</u>: Let C be a quasi-coherent chain complex with finitely many non trivial homology groups. By corollary 1-8, the identity from C to C factors through a complex K of the homotopy type of a finite chain complex. Hence C is, up to homotopy, a direct summand of K and C has the homotopy type of a finite chain complex.

<u>Corollary</u> 1-10: Let A be a ring. Then A is regular coherent in the sense of Waldhausen [] if and only if it is regular and coherent.

<u>Proof</u>: A ring A is regular coherent in the sense of [] if every finitely presented A-module has a projective resolution:

$$0 \to C_n \to \dots \to C_1 \to C_0 \to M \to 0$$

where all C_i's are finitely generated projective.

The only if part is clear. Suppose now that A is regular and coherent.

Let M be a finitely presented A-module. Since A is coherent, M has a projective resolution C which is a quasi-coherent chain complex with only one non trivial homology group. By corollary 1-9, C has the homotopy type of a finite chain complex. Hence M has a finite projective resolution and A is regular coherent.

§2 Reduction of Nil objects

Throughout this section A is a ring and S is a A-bimodule flat from the left. We denote by $\mathcal{P}_A$ the class of finitely generated projective right A-modules, and by $\mathfrak{N}il(A, S)$ the additive category of pairs (P, α) where P is in $\mathcal{P}_A$ and α is a linear map from P to $P \otimes_A S$ which is nilpotent is the following sense:

for some integer n the map α^n from P to $P \otimes_A S^{\otimes n}$ is zero.

If we consider the class Q_A of right A-modules having a finite projective resolution, we can define in the same way a other category $\mathfrak{N}il'(A, S)$ containing $\mathfrak{N}il(A, S)$. If A is regular coherent, Q_A form an abelian category and by the resolution theorem [] the two categories $\mathfrak{N}il(A, S)$ and $\mathfrak{N}il'(A, S)$ have the same K-theory. In this case Waldhausen [] compute this K-theory. He needs for that two ingredients: the dévissage theorem [] and the fact that for every N in $\mathfrak{N}il'(A, S)$, there is a filtration in $\mathfrak{N}il'(A, S)$:

$$0 = N_0 \subset N_1 \subset \dots \subset N_p = N$$

where N_i/N_{i-1} is an object in $\mathfrak{N}il'(A, S)$ on the from $(P, 0)$.

There is no good hope to generalize this facts if A is not coherent. If we want to obtain some information when A is only regular, we first have to change the category $\mathfrak{N}il$.

Notations 2-1: A chain complex C is *positive* if its -1-skeleton is trivial. A chain map f between two chain complexes is a *cofibration* if it is injective and its cokernel is projective. $\mathcal{C}_A$ is the class of all chain complexes having the homotopy type of a finite chain complex. $\mathfrak{Nil}_*(A, S)$ is the category of pairs (C, α) where C is in $\mathcal{C}_A$ and α is a chain map from C to $C \otimes S$ (tensor product over A) which is nilpotent is the following sense:

for some integer n the map α^n from C to $C \otimes S^{\otimes n}$ is null-homotopic.

The objects of $\mathfrak{N}_*(A, S)$ are called *nilpotent complexes*. A nilpotent complex $N = (C, \alpha)$ is *elementary* if α is null-homotopic. A nilpotent complex N is *reducible* if there exist a filtration of N by nilpotent complexes:

$$0 = N_0 \subset N_1 \subset \ .... \ \subset N_p = N$$

such that, for all i, N_{i+1}/N_i is an elementary nilpotent complex. It's *stably reducible* if there exists a nilpotent complex N' and a morphism from a reducible nilpotent complex to $N \oplus N'$ inducing a surjective homology isomorphism.

The main result of this section is the following:

Theorem 2-2: Suppose A is regular. Then every nilpotent complex N in $\mathfrak{Nil}_*(A, S)$ is stably reducible.

The proof of this theorem is quite long and will be done in several lemmas.

Lemma 2-3: Let (M, α) be an object in $\mathfrak{Nil}(A, S)$. Then there exists a filtration of M by sub-modules:

$$0 = I_0 \subset I_1 \subset I_2 \subset \ ... \ \subset I_p = M$$

such that, for every i, $\alpha(I_{i+1})$ is included in $I_i \otimes S$.

Proof: Let I_i be the kernel of the map α^i from M to $M \otimes S^{\otimes i}$. Since α is nilpotent, these modules give a finite filtration of M. Since S is flat from the left, I_{i+1} is exactly $\alpha^{-1}(I_i \otimes S)$.

Notation: Let p be an integer. Let $\mathcal{F}_p$ be the category of triples (M, I_*, α) where (M, α) is an object in $\mathfrak{Nil}(A, S)$ and $I_* = (I_0, I_1, \dots, I_p)$ is a filtration of M by sub-modules:

$$0 = I_0 \subset I_1 \subset I_2 \subset \dots \subset I_p = M$$

such that, for every i, $\alpha(I_{i+1})$ is included in $I_i \otimes S$.

An object (M, I_*, α) in $\mathcal{F}_p$ is called *of finite type* if all modules I_i and M/I_i are finitely generated projective.

If $E = (M, I_*, \alpha)$ is an object in $\mathcal{F}_p$, the underlying module M will be denoted by $\underline{E}$.

Lemma 2-4: Let E be an object in $\mathcal{F}_p$. Then there exists an object $E' \in \mathcal{F}_p$ of finite type and a morphism from E' to E inducing an epimorphism from $\underline{E}'$ to $\underline{E}$.

Proof: Let $E = (M, I_*, \alpha)$ be an object in $\mathcal{F}_p$. It is possible, by decreasing induction, to construct finitely generated projective modules M_i, $i = p, \dots, 0$, maps β_i from M_{i+1} to $M_i \otimes S$ and maps f_i from M_i to I_i such that:

- f_p is an isomorphism and $M_0 = 0$.
- for every $i < p$, the following diagram commutes:

$$\begin{array}{ccc} M_i \otimes S & \xleftarrow{\beta_i} & M_{i+1} \\ \downarrow{\scriptstyle f_i \otimes 1} & & \downarrow{\scriptstyle f_{i+1}} \\ I_i \otimes S & \xleftarrow[\alpha_i]{} & I_{i+1} \end{array}$$

For every i, denote by J_i the module $M_0 \oplus \dots \oplus M_i$ and by $\beta = \oplus \beta_i$ the map from J_p to $J_p \otimes S$. We get an object $E' = (J_p, J_*, \beta)$ in $\mathcal{F}_p$ and a morphism from E' to E which is surjective from J_p to M.

Lemma 2-5: Let E be an object in $\mathcal{F}_p$. Then there exist a infinite sequence:

$$\dots \xrightarrow{d} E_n \xrightarrow{d} \dots \xrightarrow{d} E_1 \xrightarrow{d} E_0 \xrightarrow{d} E \to 0$$

such that:

i) E_0, E_1, ... are objects of finite type in $\mathcal{F}_p$.

ii) $d \circ d = 0$ (in the category $\mathcal{F}_p$)

iii) the induced sequence

$$\dots \xrightarrow{d} \underline{E}_n \xrightarrow{d} \dots \xrightarrow{d} \underline{E}_1 \xrightarrow{d} \underline{E}_0 \xrightarrow{d} \underline{E} \to 0$$

is exact.

Proof: By induction the kernel of the last constructed morphism d is an object in $\mathcal{F}_p$, and the next E_i can be defined by lemma 2-4.

Lemma 2-6: Let $E = (M, I_*, \alpha)$ be an object in $\mathcal{F}_p$. Then there exists a commutative diagram:

$$\begin{array}{ccccccc} 0 = C_0 & \subset & C_1 & \dots & \subset & C_p = C \\ \downarrow f_0 & & \downarrow f_1 & & & \downarrow f_p \\ 0 = I_0 & \subset & I_1 & \dots & \subset & I_p = M \end{array}$$

and a chain map β from C to $C \otimes S$, such that

- $C_0, \dots, C_p$ are positive quasi-coherent chain complexes and the inclusions are cofibrations.

- $f_0, \dots, f_p$ are chain maps and f_p is a homology equivalence

- $\beta(C_{i+1}) \subset C_i \otimes S$ and the following diagram commutes:

$$\begin{array}{ccc} C_i \otimes S & \xleftarrow{\beta} & C_{i+1} \\ \downarrow f_i \otimes 1 & & \downarrow f_{i+1} \\ I_i \otimes S & \xleftarrow{\alpha} & I_{i+1} \end{array}$$

Proof: Let $\dots \to E_n \to \dots \to E_1 \to E_0 \to E \to 0$ be a sequence in the category $\mathcal{F}_p$ given by the lemma 2-5. The object E_i is a triple (M_i, I_{i*}, α_i) and $\dots \to I_{in} \to \dots \to I_{i1} \to I_{i0} \to 0$ is a quasi-coherent chain complex C_i.

The map $I_{i0} \to I_i$ is a chain map f_i from C_i to I_i and α_0 is a chain map β from $C = C_p$ to $C_p \otimes S$.

Lemma 2-7: Let (M, α) be an object in $\mathfrak{Nil}(A, S)$, considered as a particular nilpotent complex N_0. Then there exist nilpotent complexes N and N' and morphisms $N \to N'$ and $N' \to N_0$ such that:

i) N' is reducible

ii) the composite map $N \to N_0$ induces a surjective isomorphism in homology

Proof: Because of lemmas 2-3 and 2-6, there exists a filtration I_* of M, a nilpotent chain complex (C, β) and a diagram:

$$\begin{array}{ccccccc} 0 = C_0 & \subset & C_1 & \dots & \subset & C_p = C \\ \downarrow f & & \downarrow f & & & \downarrow f \\ 0 = I_0 & \subset & I_1 & \dots & \subset & I_p = M \end{array}$$

satisfying conditions of lemma 2-6.

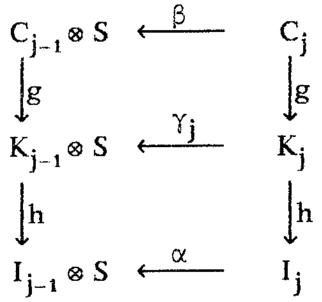
Suppose, by induction, that we have construct the following data:

- finite positive complexes $K_0 \subset \dots \subset K_i$
- chain maps γ_j, $j = 1, \dots, i$, from K_j to $K_{j-1} \otimes S$
- a commutative diagram:

$$\begin{array}{ccccccc} 0 = C_0 & \subset & C_1 & \dots & \subset & C_i \\ \downarrow g & & \downarrow g & & & \downarrow g \\ 0 = K_0 & \subset & K_1 & \dots & \subset & K_i \\ \downarrow h & & \downarrow h & & & \downarrow h \\ 0 = I_0 & \subset & I_1 & \dots & \subset & I_i \end{array}$$

where g and h are chain maps, such that $h \circ g = f$, and:

- for every $j = 1, \dots, i$, $K_{j-1} \subset K_j$ is a cofibration of finite chain complexes
- maps γ_j are compatible
- for every $j = 1, \dots i$, the following diagram commutes:



This construction is done if i = 0. To extend it, we'll procede as follows:

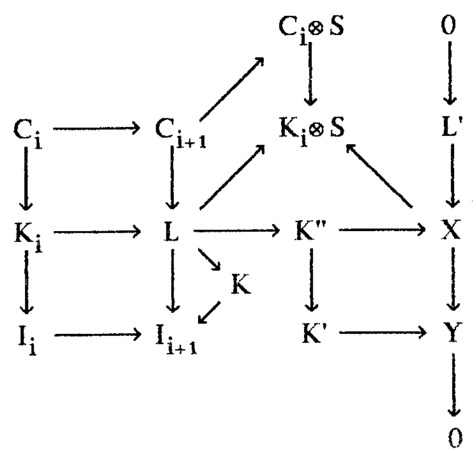
Let L be the push-out of K_i and C_{i+1} over C_i. There is a unique way to extend g from C_{i+1} to L, h from L to I_{i+1} and γ_i to $\gamma': L \to K_i \otimes S$. Moreover L is quasi-coherent and positive and $K_j \subset L$ is a cofibration.

Since A is regular, the chain map h: $L \to I_{i+1}$ factors through a finite chain complex K (theorem 1-7). By killing the -1-skeleton of K, we may as well suppose that K is positive.

Denote by L_n the modules of L and by d the differential. Since $K_i \otimes S$ is finite dimensional, γ' is trivial on L_i for $i > n = \dim K_i$, hence γ' factors through the following graded differential A-module X:

$$X = (\ldots \to 0 \to 0 \to dL_{n+1} \to L_n \to \ldots)$$

Let L' be the n-skeleton of X and Y be the quotient $X/_{L'}$. By theorem 1-7, the composite map $L \to X \to Y$ factors through a finite chain complex K'. Up to killing the n-skeleton of K', we may as well suppose that K' vanishes in dimension $\leqslant$ n. Therefore the chain map $L \to X$ factors through the pull-back K" of X and K' over Y. The complex K" is finite and the composite chain map $K_i \to K"$ is a cofibration.



Let C be the direct sum $K \oplus K''$. We have composite maps: $K_i \to K'_1 \to C$, $C \to K \to I_{i+1}$ and $C \to K'_1 \to K_i \otimes S$ and the following diagram:

$$\begin{array}{ccccc}
K_i & \longrightarrow & L & & \\
\downarrow & \searrow & \downarrow & \searrow & \\
 & & C & \longrightarrow & K_i \otimes S \\
\downarrow & & \downarrow & & \downarrow \\
I_i & \longrightarrow & I_{i+1} & \longrightarrow & I_i \otimes S
\end{array}$$

which is commutative except in the small square. Moreover C is a positive finite complex and $K_i \to C$ is a cofibration. Let u be the difference of the two maps from C to $I_i \otimes S$ given by this square and Z be the kernel of u. Let Z_0 be the 0-skeleton of Z. Z_0 is only a module and Z/Z_0 is a finite complex. Let E be an acyclic finite complex and λ be a surjective map from E to Z/Z_0. Since E is acyclic, λ factors through Z by a chain map μ from E to Z. Let Σ be the kernel of the composite map: $L \oplus E \to Z \to Z/Z_0$. The complex Σ is quasi-coherent and the chain map from Σ to Z_0 factors through a finite positive complex H. Hence the composite map: $L \to L \oplus E \to Z$ factors through the push-out H' of H and $L \oplus E$ over Σ, which is finite and positive.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_0 & \longrightarrow & Z & \longrightarrow & Z/Z_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & H & \longrightarrow & H' & \longrightarrow & Z/Z_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \Sigma & \longrightarrow & L \oplus E & \longrightarrow & Z/Z_0 & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & L & & & &
\end{array}$$

Let E' be a positive finite acyclic chain complex and v be a cofibration from K_i to E'. Since E' is acyclic v extends to a chain map v' from L to E'. Set $K_{i+1} = H' \oplus E'$. The direct sum of v' and the map $L \to H'$ is a chain map from L to K_{i+1} inducing a cofibration from K_i to K_{i+1}. The desired map g from C_{i+1} to K_{i+1} is the composite: $C_{i+1} \to L \to K_{i+1}$, the map h from K_{i+1} to I_{i+1} is the composite: $K_{i+1} \to H' \to Z \to C \to I_{i+1}$ and the map γ_{i+1} is the composite: $K_{i+1} \to H' \to Z \to C \to K_i \otimes S$

$$\begin{array}{ccccc}
C_i & \subset & C_{i+1} & \xrightarrow{\beta_{i+1}} & C_i \otimes S \\
\downarrow g & & \downarrow g & & \downarrow g \\
K_i & \subset & K_{i+1} & \xrightarrow{\gamma_{i+1}} & K_i \otimes S \\
\downarrow h & & \downarrow h & & \downarrow h \\
I_i & \subset & I_{i+1} & \xrightarrow{\alpha} & I_i \otimes S
\end{array}$$

The next step of the induction is now finish and, at the end of this construction, we get two nilpotent complexes $N = (C_p, \beta_p)$ and $N' = (K_p, \gamma_p)$ and morphisms from N to N' and from N' to N_0. The composite $N \to N' \to N_0$ induces a surjective isomorphism on homology and N' is obviously reducible.

Lemma 2-8: Let $N = (C, \alpha)$ be a nilpotent complex such that C is a chain complex of length 1. Then N is stably reducible.

Proof: If the length of C is 1, C is only a finitely generated projective A-module M, and (M, α) is an object in $\mathfrak{Nil}(A, S)$. By lemma 2-7, there exist two nilpotent chain complexes $N' = (C', \alpha')$ and $N'' = (C'', \alpha'')$ and morphisms $N' \to N'' \to N$ such that the composite morphism $N' \to N$ induces a surjective homology isomorphism and N" is reducible. Since $C' \to C$ is a surjective homotopy equivalence, its kernel E is contractible and C' is isomorphic to $C \oplus E$. The composite $C \to C' \to C''$ gives a section of $C'' \to C$ and C" is isomorphic to the direct sum of C and the kernel K of $C'' \to C$. Up to isomorphism, we may as well suppose that C' is equal to $C \oplus E$ and that C" is equal to $C \oplus K$. The chain map $C' \to C''$ is given by the matrix: $\begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$ and the chain map $C'' \to C$ is the first projection. Since the maps $C' \to C''$ and $C'' \to C$ respect the nilpotent maps, α' and α'' are given by matrices:

$$\alpha' : \begin{bmatrix} \alpha & 0 \\ x' & y \end{bmatrix} \qquad \alpha'' : \begin{bmatrix} \alpha & 0 \\ x'' & \beta \end{bmatrix}$$

and we have:

$$x'' = u x' \qquad \beta u = u y$$

Consider the complex $\Sigma = C \oplus K \oplus E$. We have a chain map γ from Σ to $\Sigma \otimes S$ given by the matrix:
$$\begin{bmatrix} \alpha & 0 & 0 \\ x'' & \beta & 0 \\ -x' & 0 & y \end{bmatrix}$$
and a chain map φ from Σ to $C \oplus K$ given by the matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u \end{bmatrix}$

We have an exact sequence of nilpotent complexes:
$$0 \to (E, y) \to (\Sigma, \gamma) \to (C'', \alpha'') \to 0$$
and (Σ, γ) is a reducible nilpotent complex. The map φ is a morphism from (Σ, γ) to $N \oplus (K, \beta)$, inducing a surjective homology equivalence from Σ to $C \oplus K$.

Lemma 2-9: An extension of two stably reducible nilpotent complexes is stably reducible.

Proof: Consider a short exact sequence in $\mathfrak{N}il_*(A, S)$:
$$0 \to N \to N' \to N'' \to 0$$
where N and N'' are stably reducible. There exist nilpotent complexes N_0, N''_0, N_1, N''_1 and surjective morphisms: $N_1 \to N \oplus N_0$ and $N''_1 \to N'' \oplus N''_0$ inducing isomorphisms on homology such that N_1 and N''_1 are reducible. Let L be the pull back of $N' \oplus N_0 \oplus N''_0$ and N''_1 over $N'' \oplus N''_0$. Set: $(C, \alpha) = N \oplus N_0$, $(C', \alpha') = L$, $(E, \beta) = \mathrm{Ker}(N_1 \to N \oplus N_0)$.

Since E is acyclic the underlying complex of N_1 is $C \oplus E$, and the nilpotent morphism from $C \oplus E$ to $(C \oplus E) \otimes S$ is given by the matrix $\begin{bmatrix} \alpha & 0 \\ x & y \end{bmatrix}$.

Since E is acyclic, there is no obstruction to extend the map $x: C \to E \otimes S$ to a map $x': C' \to E \otimes S$. Therefore the complex $C' \oplus E$ and the matrix $\begin{bmatrix} \alpha & 0 \\ x' & y \end{bmatrix}$ define a nilpotent complex L'. Since N_1 and N''_1 are reducible and the sequence
$$0 \to N_1 \to L' \to N''_1 \to 0$$
is exact, L' is reducible and the composite map $L' \to L \to N' \oplus N_0 \oplus N''_0$ induces a surjective homology equivalence.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N \oplus N_0 & \longrightarrow & N' \oplus N_0 \oplus N'_0 & \longrightarrow & N'' \oplus N''_0 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N \oplus N_0 & \longrightarrow & L & \longrightarrow & N''_1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & N_1 & \longrightarrow & L' & \longrightarrow & N''_1 & \longrightarrow & 0
\end{array}$$

Lemma 2-10: Let f be a morphism from a stably reducible nilpotent complex to a nilpotent complex N inducing a surjective homology isomorphism. Then N is stably reducible.

Proof: Obvious.

Proof of theorem 2-2: Let $N = (C, \alpha)$ be a nilpotent complex. Let n be the smallest length of complexes C' homotopically equivalent to C. If C is not acyclic, let p be the degree of the first non trivial homology group of C and q the smallest integer such that α^q is trivial on $H_p(C)$. The pair (n, q) will be denoted by $|N|$.

Let $N = (C, \alpha)$ be a nilpotent complex. If C is acyclic, N is obviously reducible. If not, set $|N| = (n, q)$. Let $H_p(C)$ be the first non trivial homology group of C. Since $H_p(C)$ is finitely generated, the image of α_*^{q-1} is a finitely generated sub-module of $H_p(C) \otimes S^{\otimes q-1}$contained in $(\mathrm{Ker}\,\alpha_*) \otimes S^{\otimes q-1}$, and there exists a finitely generated submodule I of $\mathrm{Ker}\,\alpha_*$ such that the image of α_*^{p-1} is contained in $I \otimes S^{\otimes q-1}$. Let P be a finitely generated projective A-module and f be an epimorphism from P onto I. The module P may be considered as a chain complex C' concentrated in dimension p. Let φ: $C' \to C$ be any chain map inducing f in homology, and ψ: $E \to C$ be a surjective chain map from an acyclic chain complex E onto C. Since E is acyclic, there is no obstruction to construct a chain map α' from $C' \oplus E$ to $(C' \oplus E) \otimes S$ such that: $\alpha \circ (\varphi \oplus \psi) = (\varphi \oplus \psi) \circ \alpha'$ and $\alpha'(C' \oplus E)$ is included in $E \otimes S$. Let E_0 be an acyclic chain map and u a cofibration from $C' \oplus E$ to E_0. Since E_0 is acyclic there exists a chain map β from E_0 to $E_0 \otimes S$ such that: $\beta \circ u = u \circ \alpha'$. The chain maps $\varphi \oplus \psi$ and u induce a morphism from the reducible nilpotent complex $(C' \oplus E, \alpha')$ to $N \oplus (E_0, \beta)$ and this map has a cokernel N'. If the underlying complex of N' is acyclic, n is equal to 1. If the underlying complex of N' is not acyclic, set $|N'| = (n', q')$. If n>1, n' is less than n or n' is equal to n and q' is equal to q-1. By induction on (n, q), N' is stably reducible. By lemma 2-9, $N \oplus (E_0, \beta)$ is stably reducible, and by lemma 2-10, N is stably reducible too.

Suppose now that n is equal to 1. The module $H_p(C)$ is projective and may be considered as a chain complex C' concentrated in dimension p. By lemma 2-8, the pair (C', α_*) is stably reducible. Let $\varphi: C' \to C$ be a chain map inducing the identity on homology and, as above, $\psi: E \to C$ be a chain map from an acyclic chain complex E onto C. Since E is acyclic, there is no obstruction to construct a chain map α' from $C' \oplus E$ to $(C' \oplus E) \otimes S$ such that: $(\varphi \oplus \psi) \circ \alpha' = \alpha \circ (\varphi \oplus \psi)$ and $\alpha'(E)$ is included in $E \otimes S$. The nilpotent complex $(C' \oplus E, \alpha')$ is an extension of two stably reducible complexes. By lemma 2-9, it is stably reducible, and, by lemma 2-10, N is stably reducible too.

§3 Algebraic K-theory and localization of complexes

In section 1 and 2, the chain complexes we have consider, were bounded from below. This complexes form a category which is good for many reasons except for one point: a direct sum of complexes like that is not necessary in this category. On the other hand the category of all graded differentiel projective modules is bad in one sense: acyclicity is not necessary equivalent to contractibility. For all these reasons we'll consider another category of graded differentiel modules:

Notation: Let A be a ring. We denote by $\overline{C}_A$ the category of all graded differential projective (right-) A-module C satisfying the following condition:

- for every graded differential projective acyclic A-module E, every chain map from C to E is null-homotopic.

From now on, we'll work every time with these categories. Thus, for simplicity, the objects of $\overline{\mathcal{C}}_A$ will be just called A-complexes.

A chain map between two A-complexes is a *cofibration* if it is injective with degreewise projective cokernel.

Remark: This category has direct sum and is exact in the following sense:

If $0 \to C \to C' \to C'' \to 0$ is a short exact sequence of graded differential projective A-modules, if two of C, C', C'' is in $\overline{\mathcal{C}}_A$, so is the third.

Moreover, in this category homology equivalence implies homotopy equivalence.

Definitions: Let I be a small category and J be a subcategory of I. A *diagram of rings* $\mathcal{D} = (A_*, S_*)$ over (I, J) is a covariant functor from I to the category $\mathcal{RB}$ of rings and bimodules, in the following sense: $\mathcal{D}$ associates to every $i \in I$ a ring $\mathcal{D}(i) = A_i$ and to every morphism u: $i \to j$ in I, an $A_j \times A_i$-bimodule S_u. Moreover for every u: $i \to j$ and every v: $j \to k$, a morphism $S_v \otimes S_u \to S_{v \circ u}$ is given and all these morphisms are compatible. If u= $i \to j$ is in J, we have: $A_i = A_j$ and S_u is the standard bimodule $A_i = A_j$.

Let $\mathcal{D}$ be a diagram of rings. A *$\mathcal{D}$-complex* (C_*, α_*) is a collection of complexes C_i and chain maps α_u such that:

- for every $i \in I$, C_i is a A_i-complex.
- for every morphism u: $i \to j$ in I, α_u is a chain map from C_i to $C_j \otimes S_u$
- for every morphim u= $i \to j$ in J, α_u is a cofibration from C_i to $C_j \otimes S_u = C_j$
- for every composable morphisms u and v in I, $\alpha_v \circ \alpha_u = \alpha_{v \circ u}$

If (C_*, α_*) and (C'_*, α'_*) are two $\mathcal{D}$-complexes, a *chain map* from (C_*, α_*) to (C'_*, α'_*) is a collection f_* of chain maps f_i: $C_i \to C'_i$, compatible with chain maps α_* and α'_*. A chain map f_* is a *homology equivalence* if, for every $i \in I$, f_i is a homology equivalence . The chain map f_* is a *cofibration* if f_i is a cofibration for every $i \in I$ and α_u induces a cofibration from Coker(f_i) to Coker(f_j) for every u : $i \to j$ in J. A $\mathcal{D}$-complex (C_*, α_*) is *acyclic* if C_i is acyclic for every $i \in I$. A sequence of

$\mathcal{D}$-complexes:

$$\dots \to (C_*, \alpha_*) \to (C'_*, \alpha'_*) \to (C''_*, \alpha''_*) \to \dots$$

is *exact* if the corresponding sequence: $\dots \to C_i \to C'_i \to C''_i \to \dots$

is exact for every $i \in I$.

The category of $\mathcal{D}$-complexes will be denoted by $\bar{C}_{\mathcal{D}}$.

Notice that all natural constructions on the category of chain complexes like: mapping-cone, mapping-cylinder, mapping-telescope, suspension , desuspension, ... may be generalized in the category $\bar{C}_{\mathcal{D}}$.

A class $\mathcal{A}$ of $\mathcal{D}$-complexes is called *exact* if it contains all acyclic $\mathcal{D}$-complexes and satisfies the following property:

- for every short exact sequence $0 \to K \to K' \to K'' \to 0$

if two of K, K', K" are in $\mathcal{A}$, so is the third.

Let $\mathcal{A}$ be an exact class of $\mathcal{D}$-complexes. A chain map f between two $\mathcal{D}$-complexes is an *$\mathcal{A}$-equivalence* if its mapping-cone is in $\mathcal{A}$. If $\mathcal{A}$ is the class of all acyclic $\mathcal{D}$-complexes, an $\mathcal{A}$-equivalence is nothing else but a homology equivalence.

Lemma 3-1: A class $\mathcal{A}$ of $\mathcal{D}$-complexes is exact if and only if it contains all acyclic $\mathcal{D}$-complexes and is stable under quotient by cofibration.

Proof: The only if part is clear. Let

$$0 \to X \to Y \to Z \to 0$$

be an exact sequence in $\bar{C}_{\mathcal{D}}$. Suppose that Z is in $\mathcal{A}$. Let f: E $\to$ Y be an epimorphism from an acyclic $\mathcal{D}$-complex E onto Y. We have the following exact sequence:

$$0 \to \Sigma Z \to X \oplus E \to Y \to 0$$

where ΣZ is the suspension of Z. The $\mathcal{D}$-complex ΣZ is the quotient of the mapping cylinder of the identity of Z, which is acyclic, by Z, and belongs to $\mathcal{A}$. If X is in $\mathcal{A}$, Y is in $\mathcal{A}$ and $\mathcal{A}$ is stable under extension. If Y is in $\mathcal{A}$, $X \oplus E$ is in $\mathcal{A}$, and X cokernel of the map $E \to X \oplus E$ is in $\mathcal{A}$ too. Therefore $\mathcal{A}$ is exact.

Consider two exact classes $\mathcal{A} \subset \mathcal{B}$ of $\mathcal{D}$-complexes. The cofibrations in $\mathcal{B}$ and the $\mathcal{A}$-equivalences define a structure of a category of cofibrations and weak equivalences in the sense of Waldhausen [] and the K-theory spectrum of this category of cofibrations and weak equivalences is defined. This spectrum will be denoted by $K(\mathcal{B}, \mathcal{A})$, or simply $K(\mathcal{B})$ if $\mathcal{A}$ is the class of all acyclic $\mathcal{D}$-complexes, Actually the category of $\mathcal{A}$-equivalences in $\mathcal{B}$ satisfies the saturation axiom and the extension axiom. Moreover the mapping-cylinder construction gives rise to a cylinder functor and the category of $\mathcal{A}$-equivalences satisfies the cylinder axiom.

In the definition of the spectrum $K(\mathcal{B}, \mathcal{A})$ there is a set-theoretical problem. The category $\mathcal{B}$ is not necessary small. We'll say that an exact class $\mathcal{A}$ of $\mathcal{D}$-complexes is *not too big* if there is a set of $\mathcal{D}$-complexes X_i in $\mathcal{A}$ such that for every $\mathcal{D}$-complex Y in $\mathcal{A}$ there is a homology equivalence from some X_i to Y.

Lemma 3-2: Let $\mathcal{A}$ be a not too big exact class of $\mathcal{D}$-complexes. Let $\{X_i\}$ be a set of objects in $\mathcal{A}$. Then there is a set $\mathcal{A}_0 \subset \mathcal{A}$, containing all X_i, and satisfying the following:

- for every $X \in \mathcal{A}$, there is a homology equivalence from an object $Y \in \mathcal{A}_0$ to X
- for every short exact sequence in $\mathcal{A}$:

$$0 \to X \to Y \to Z \to 0$$

if two of X, Y, Z are in $\mathcal{A}_0$, the third one is isomorphic to some object in $\mathcal{A}_0$.

- the cylinder functor is defined in $\mathcal{A}_0$.

Proof: Let $\mathcal{B}$ be a subset in $\mathcal{A}$ such that for every $X \in \mathcal{A}$ there is a homology equivalence from an object Y in $\mathcal{B}$ to X. For each cofibration (resp. epimorphism) $X \to Y$, $X \in \mathcal{B}$, $Y \in \mathcal{B}$, take a representative of its kernel (resp. cokernel). By adding these representatives to $\mathcal{B}$, we get a bigger subset $\mathcal{B}' \subset \mathcal{A}$. For every $X, Y \in \mathcal{A}$, the isomorphism class of extensions of X by Y is a set. By taking representatives of these extensions, we get another set $\mathcal{B}''$ containing $\mathcal{B}'$. If we add also all mapping-cylinder of maps between objects in $\mathcal{B}$ we get a third set $\mathcal{B}_1$.

If we apply this construction to $\mathcal{B}_1$, we get a bigger set $\mathcal{B}_2$, etc. ... Let $\mathcal{A}_0$ be the union of $\mathcal{B} \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots$. This set satisfies obviously the desired conditions.

Proposition 3-3: Let $\mathcal{A} \subset \mathcal{B}$ be two exact classes of $\mathcal{D}$-complexes. Suppose that $\mathcal{B}$ is not too big. Let $\mathcal{B}_0$ be a subset of $\mathcal{B}$ satisfying the conditions of 3-1. Then the K-theory spectrum $K(\mathcal{B}_0, \mathcal{A})$ of $(\mathcal{B}_0$, cofibrations, $\mathcal{A}$-equivalences) is well-defined (without set-theoretical problem). Moreover, if $\mathcal{B}'_0$ is another subset of $\mathcal{B}$ satisfying the conditions of 3-2 and containing $\mathcal{B}_0$, the map $K(\mathcal{B}_0, \mathcal{A}) \to K(\mathcal{B}'_0, \mathcal{A})$ is a homotopy equivalence.

Proof: The simplicial set $S.\mathcal{B}_0$ of filtered objects in $\mathcal{B}_0$ is well defined, and the notion of $\mathcal{A}$-equivalence extends to every map between two objects in $S_n\mathcal{B}_0$. Therefore $(S.\mathcal{B}_0$, $\mathcal{A}$-equivalences) is a simplicial category, and the space $K(\mathcal{B}_0, \mathcal{A})$ is well defined. To prove that the map $K(\mathcal{B}_0, \mathcal{A}) \to K(\mathcal{B}'_0, \mathcal{A})$ is a homotopy equivalence, we just have to apply the approximation theorem [] to the inclusion functor from $\mathcal{B}_0$ to $\mathcal{B}'_0$.

In this situation the homotopy type of $K(\mathcal{B}, \mathcal{A})$ is well defined, it doesn't depend on the choice of to subset $\mathcal{B}_0$.

Remark: Another possibility to solve the set-theoretical problem is the following: It is possible to take a universe $\mathcal{U}'$ containing the universe $\mathcal{U}$ where we are, and such that $\mathcal{B}$ is a set in $\mathcal{U}'$. Then $K(\mathcal{B}, \mathcal{A})$ is a spectrum in the universe $\mathcal{U}'$. If $\mathcal{B}$ is not too big, the above proposition said that $K(\mathcal{B}, \mathcal{A})$ has the homotopy type of a spectrum in the universe $\mathcal{U}$.

So we have two possibility to work with these spectra $K(-, -)$. We left the choice to the reader.

Example: Let $I = J$ be the trivial category with one object and one morphism. A diagram of rings (I, A_*, S_*) is just a ring A. The class $\mathcal{C}_A$ of all A-e-complexes having the homotopy type of a finite A-chain complex is an exact class.

Proposition 3-4: The spectrum $K(\mathcal{C}_A)$ is the connective spectrum associated to the K-theory spectrum K(A).

Proof: We have homology functors H_* from $\mathcal{C}_A$ to the category of A–modules. Let $\mathcal{E}$ be the full subcategory of all finitely generated projective A-modules. Following Waldhausen's terminology the category $\mathcal{C}^n$ of all $(H_*, \mathcal{E})$-spherical object of dimension n is the category of all complexes having the homotopy type of one finitely generated projective module concentrated in dimension n. In this situation, the suspension from $\mathcal{C}^n$ to $\mathcal{C}^{n+1}$ is an equivalence of categories. Thus Waldhausen's theorem relating the K-theory of a given category of cofibrations and weak equivalences and the category of spherical objects, implies that the inclusion functor from $\mathcal{C}^0$ to $\mathcal{C}_A$ induces a homotopy equivalence from $K(\mathcal{C}^0)$ to $K(\mathcal{C}_A)$.

Consider the following sub-categories of $\mathcal{C}^0$: $\mathcal{C}_+$ (resp. $\mathcal{C}_-$) is the class of acyclic complexes concentrated in non negative (resp. non positive) degree. Since all map in $\mathcal{C}_+$ or in $\mathcal{C}_-$ are weak equivalences; the identities in $K(\mathcal{C}_+)$ and in $K(\mathcal{C}_-)$ are homotopic to constant maps, and these spaces are contractible.

Let $C = (\dots \to C_{n+1} \to C_n \to \dots)$ be a complex in $\mathcal{C}^0$. There is a canonical filtration $C' \subset C'' \subset C$ such that C' is in $\mathcal{C}_+$, C''/C' is concentrated in dimension 0, C/C'' is in $\mathcal{C}_-$. In zero degree C' is the image of $C_1 \to C_0$, C'' is the kernel of $C_0 \to C_{-1}$. By the additivity theorem $K(\mathcal{C}^0)$ has the homotopy type of $K(\mathcal{C}_+) \times K(\mathcal{P}_A) \times K(\mathcal{C}_-)$ and the inclusion $\mathcal{P}_A \subset \mathcal{C}_A$ induces a homotopy equivalence in K-theory.

Since $K(\mathcal{P}_A)$ is the connective spectrum of the classical K-theory spectrum K(A), we get the result.

Proposition 3-5: Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$ be three exact classes of $\mathcal{D}$-complexes. Suppose that $\mathcal{C}$ is not too big. Then the inclusion maps induce a homotopy fibration of spectra:

$$K(\mathcal{B}, \mathcal{A}) \to K(\mathcal{C}, \mathcal{A}) \to K(\mathcal{C}, \mathcal{B})$$

Proof: This proposition is a direct consequence of the fibration theorem of [].

Proposition 3-6: Let $\mathcal{A} \subset \mathcal{B}$ be two exact classes of $\mathcal{D}$-complexes. Suppose that $\mathcal{B}$ is not too big. Suppose also that, for every X in $\mathcal{B}$, there is a Y in $\mathcal{B}$ such that $X \oplus Y$ is in $\mathcal{A}$. Then the spectrum $K(\mathcal{B}, \mathcal{A})$ is an Eilenberg-McLane spectrum $K(?, 0)$.

Proof: Consider the following relation on $\mathcal{B}$:

$$\forall\ X, Y \in \mathcal{B},\ X \equiv Y \Leftrightarrow X \oplus \Sigma Y \in \mathcal{A}$$

where ΣY is the suspension of Y defined by the cylinder functor.

Let X be an object in $\mathcal{B}$. By asumption, there exists $Y \in \mathcal{B}$ such that $X \oplus Y$ is in $\mathcal{A}$. Since $\mathcal{A}$ is exact, the suspension $\Sigma X \oplus \Sigma Y$ is in $\mathcal{A}$. But $\Sigma X \oplus \Sigma Y$ is the mapping-cone of the zero map from Y to ΣX. Then this map is an $\mathcal{A}$-equivalence and the map $1 \oplus 0$ from $X \oplus Y$ to $X \oplus \Sigma X$ is an $\mathcal{A}$-equivalence too. Let Z be the mapping-cone of this last map. We have an exact sequence:

$$0 \to X \oplus \Sigma X \to Z \to \Sigma X \oplus \Sigma Y \to 0$$

The objects Z and $\Sigma X \oplus \Sigma Y$ are in $\mathcal{A}$, so is $X \oplus \Sigma X$, and the relation $\equiv$ is reflexive. Suppose now that $X \equiv Y$ are two objects in $\mathcal{B}$. Then the zero map from Y to X is a $\mathcal{A}$-equivalence. Since the composite of zero maps from Y to X and from X to Y is an $\mathcal{A}$-equivalence, the zero map from X to Y is an $\mathcal{A}$-equivalence and: $Y \equiv X$. The same kind of argument show that $\equiv$ is transitive and thus an equivalence relation.

The direct sum operation on $\mathcal{B}$ induces an abelian group structure on $G = \mathcal{B}/_{\equiv}$. Let χ be the quotient map from $\mathcal{B}$ to G. The class $\mathcal{A}$ is exactly the class of objects X in $\mathcal{B}$ such that $\chi(X)$ vanishes.

We may consider G as a category of cofibrations and weak equivalences in the following way:

- $Obj(G) = G$

- for every u, v in G there is exactly one morphism from u to v and this morphism is a cofibration

- the weak equivalences are the identities.

The K-theory space of this category G is the space $\Omega|wS.G| = \Omega BG \simeq G$ and the associated spectrum is the Eilenberg-Mclane spectrum K(G, 0).

The approximation theorem of Waldhausen [] implies the result.

This approximation theorem is very important, but it will be usefull to reformulate it in the following words:

Approximation lemma 3-7: Let $\mathcal{D}$ and $\mathcal{D}'$ be two diagrams of rings, and $\mathcal{A} \subset \mathcal{B}$ be exact classes of $\mathcal{D}$-complexes . Let $\mathcal{B}'$ be a class of $\mathcal{D}'$-complexes containing an exact class $\mathcal{A}'$. Let F be a functor from $\mathcal{B}$ to $\mathcal{B}'$. Suppose F satisfies the following properties:

i) F is exact i. e. it sends exact sequences to exact sequences.

ii) for every X in $\mathcal{B}$, F(X) is in $\mathcal{A}'$ if and only if X is in $\mathcal{A}$

iii) if there is a $\mathcal{A}'$-equivalence from a $\mathcal{D}'$-complex X in $\mathcal{B}'$ to a $\mathcal{D}'$-complex Y, Y is in $\mathcal{B}'$

iv) F is surjective in the following sense: for every X in $\mathcal{B}$ and Y in $\mathcal{B}'$, and every map f from F(X) to Y, there exists X' in $\mathcal{B}$, a map u from X to X', an $\mathcal{A}'$-equivalence g from F(X') to Y such that f is the composite:

$$F(X) \xrightarrow{F(u)} F(X') \xrightarrow{g} Y$$

Then $\mathcal{B}'$ is an exact class and F induces a homotopy equivalence of spectra:

$$F_*: K(\mathcal{B}, \mathcal{A}) \xrightarrow{\sim} K(\mathcal{B}', \mathcal{A}')$$

Proof: Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of $\mathcal{D}'$-complexes such that X and Y are in $\mathcal{B}'$. By condition iii) applied to the map $F(0) \to X$, there exists a $\mathcal{D}$-complex X_0 and a $\mathcal{A}'$-equivalence from $F(X_0)$ to X. Apply again this condition to the map $F(X_0) \to X \to Y$. We construct a map u: $X_0 \to Y_0$ and a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & F(X_0) & \longrightarrow & F(Y_0) & & & & \end{array}$$

If M is the mapping-cone of u, we get a $\mathcal{A}'$-homology equivalence from F(M) to Z, and Z is in $\mathcal{B}'$. Then it is easy to see that $\mathcal{B}'$ is exact. Moreover all conditions for the approximation theorem [] are satisfied and F induces a homotopy equivalence of spectra from $K(\mathcal{B}, \mathcal{A})$ to $K(\mathcal{B}', \mathcal{A}')$.

Definitions: Let $\mathcal{A}$ be an exact class of $\mathcal{D}$-complexes. This class is *complete* if it is stable under direct sum. The smallest complete exact class $\bar{\mathcal{A}}$ containing $\mathcal{A}$ is called the *completion* of $\mathcal{A}$.

The class $\mathcal{A}$ has the *finiteness property* if, for every map f from an object X in $\mathcal{A}$ to a direct sum of objects Y_i in $\bar{\mathcal{A}}$, there exists a homology equivalence g from an object X' in $\mathcal{A}$ to X such that the composite $g \circ f$ factors through a finite sum of the Y_i's.

Let $\mathcal{A} \subset \mathcal{B}$ be two classes of $\mathcal{D}$-complexes. We say that $\mathcal{A}$ is *closed* in $\mathcal{B}$ if:

$$\forall\ X, Y \in \mathcal{B} : X \oplus Y \in \mathcal{A} \Rightarrow X \in \mathcal{A}$$

Definition: Let $\mathcal{A}$ be an exact class in $\bar{C}_{\mathcal{D}}$. A $\mathcal{D}$-complex X is called *$\mathcal{A}$-local* if every morphism from a $\mathcal{D}$-complex $Y \in \mathcal{A}$ to X factors through an acyclic $\mathcal{D}$-complex.

Proposition 3-8: Suppose $\mathcal{A}$ is not too big and satisfies the finiteness property. Let X be a $\mathcal{D}$-complex. Then there exists a $\bar{\mathcal{A}}$-equivalence from X to a $\mathcal{A}$-local $\mathcal{D}$-complex.

Proof: Let X be a $\mathcal{D}$-complex. We'll construct, by induction a sequence:

$$X = Z_0 \to Z_1 \to Z_2 \to \dots$$

where all maps $Z_i \to Z_{i+1}$ are cofibrations and $\bar{\mathcal{A}}$-equivalences, and where the limit Z of this sequence is $\mathcal{A}$-local. Since $\mathcal{A}$ is not too big, there exists a set of $\mathcal{D}$-complexes $\{X_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{A}$, such that, for every $\mathcal{D}$-complex Y in $\mathcal{A}$, there is a homology equivalence from some X_λ to Y. Suppose Z_j is constructed for $j \leq n$. Let T_n be the set of all (λ, u) where λ is in Λ and u is a map from X_λ to Z_n, and U_n be the direct sum of all X_λ, for all (λ, u) in T_n. We have an obvious map from U_n to Z_n. Let E_n be the cone of U_n, and Z_{n+1} be the push-out of E_n and Z_n over U_n.

$$\begin{array}{ccc} \bigoplus_{T_n} X_\lambda = U_n & \longrightarrow & E_n \\ \downarrow & & \downarrow \\ Z_n & \longrightarrow & Z_{n+1} \end{array}$$

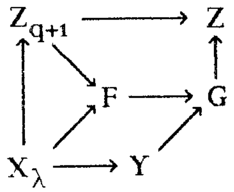
Since all the X_λ's are in $\mathcal{A}$, U_n is in $\bar{\mathcal{A}}$, and the map $Z_n \to Z_{n+1}$ is a $\bar{\mathcal{A}}$-equivalence. Let Z be the limit of this sequence, and $Y \to Z$ be any map from a $\mathcal{D}$-complex Y in $\mathcal{A}$ to Z. We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_0^\infty Z_n & \xrightarrow{\alpha} & \bigoplus_0^\infty Z_n & \longrightarrow & Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

where α is the difference of the identity and the stabilisation $Z_n \to Z_{n+1}$, and E is acyclic. Because of the finiteness property, there is a homology equivalence $K' \to K$ such that the left arrow sends K' to $\bigoplus_0^p Z_n$. Up to adding to K and E some acyclic $\mathcal{D}$-complex, we may as well suppose that $K' \to K$ is a cofibration. Since α sends $\bigoplus_0^p Z_n$ to $\bigoplus_0^{p+1} Z_n$ the middle vertical arrow sends $E/_{K'}$ to $\bigoplus_{p+2}^\infty Z_n$. Because of the finiteness property, there exists a homology equivalence from a $\mathcal{D}$-complex V to $E/_{K'}$ such that this arrow sends V to a finite sum. Let E' be the pull-back of E and V over $E/_{K'}$. By construction we have, for some integer q the following diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_0^{q-1} Z_n & \xrightarrow{\alpha} & \bigoplus_0^q Z_n & \longrightarrow & Z_q & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K' & \longrightarrow & E' & \longrightarrow & Y' & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccc} Z_q & \longrightarrow & Z \\ \uparrow & & \uparrow \\ Y' & \longrightarrow & Y \end{array}$$

and the map $Y' \to Y$ is a homology equivalence. Since Y' is in $\mathcal{A}$, there exists, for some $\lambda \in \Lambda$ a homology equivalence from X_λ to Y'. Let u be the map $X_\lambda \to Y' \to Z_q$. the pair (λ, u) is an element of the index set T_q and the map $X_\lambda \to Z_{q+1}$ factors, by construction through an acyclic $\mathcal{D}$-complex F. Therefore the map $Y \to Z$ factors through the push-out G of Y and F over X_λ, which is acyclic, and Z is $\mathcal{A}$-local.



On the other hand, there is an exact sequence:

$$0 \longrightarrow \overset{\infty}{\underset{0}{\oplus}} Z_n/_X \longrightarrow \overset{\infty}{\underset{0}{\oplus}} Z_n/_X \longrightarrow Z/_X \longrightarrow 0$$

Hence $Z/_X$ is in $\bar{\mathcal{A}}$ and the map $X \to Z$ is an $\bar{\mathcal{A}}$-equivalence.

Proposition 3-9: Every $\mathcal{A}$-local $\mathcal{D}$-complex is $\bar{\mathcal{A}}$-local.

Proof: Let C be the class of all $\mathcal{D}$-complexes Y such that, for every L in $\mathcal{LA}$, every map from Y to L factors through an acyclic $\mathcal{D}$-complex. This class C contains $\mathcal{A}$. Let Y_i be $\mathcal{D}$-complexes in C. Let f: $\oplus Y_i \to$ L be a map, where L is a $\mathcal{A}$-local. All maps $Y_i \to$ L factor through acyclic $\mathcal{D}$-complexes E_i and f factors through $\oplus E_i$. Thus C is stable under direct sum.

Let $0 \to X \to Y \to Z \to 0$

be an exact sequence of $\mathcal{D}$-complexes, where X and Y are in C, and f be a map from Z to a $\mathcal{A}$-local $\mathcal{D}$-complex L. Since Y is in C, the composite $Y \to Z \to L$ factors through an acyclic $\mathcal{D}$-complex E. And E may be chosen so that $E \to L$ is surjective with kernel L'. Since $\mathcal{LA}$ is exact, the map from X to L' factors through an acyclic $\mathcal{D}$-complex F. Let E' be the sum of E and an acyclic $\mathcal{D}$-complex containing F by a cofibration. We have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L' & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

where G is the cokernel of the cofibration $F \to E'$ and then is acyclic.

Hence the class C is exact and stable under direct sum. It contains $\bar{\mathcal{A}}$ and the lemma is proven.

Proposition 3-10: Let $\mathcal{A}$ be an exact class in $\bar{\mathcal{C}}_{\mathcal{D}}$. Then the class $\mathcal{L}\mathcal{A}$ of all $\mathcal{A}$-local $\mathcal{D}$-complexes is exact.

Proof: The class $\mathcal{L}\mathcal{A}$ contains all acyclic $\mathcal{D}$-complexes. Then the only thing to do is to prove that $\mathcal{L}\mathcal{A}$ is stable under cokernel of cofibration. Let

$$0 \to X \to Y \to Z \to 0$$

be an exact sequence in $\bar{\mathcal{C}}_{\mathcal{D}}$, where X and Y are $\mathcal{A}$-local. Let U be an object in $\mathcal{A}$ and f: U $\to$ Z be any map. Let E be an acyclic $\mathcal{D}$-complex going surjectively onto the pull-back of Y and U over Z. We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & V & \longrightarrow & E & \longrightarrow & U & \longrightarrow & 0 \end{array}$$

Where V is the kernel of E $\to$ U. Since E is acyclic, V is in $\mathcal{A}$ and the map V $\to$ X factors through an acyclic object F in $\bar{\mathcal{C}}_{\mathcal{D}}$. Since Y is in $\mathcal{A}$, the map from the push-out of F and E over V to Y, factors through an acyclic $\mathcal{D}$-complex G. Up to adding to G an acyclic $\mathcal{D}$-complex containing F by a cofibration, we may as well suppose that the map F $\to$ G is a cofibration with cokernel H. Therefore the map U $\to$ Z factors through the acyclic $\mathcal{D}$-complex H.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & V & \longrightarrow & E & \longrightarrow & U & \longrightarrow & 0 \end{array}$$

Theorem 3-11: Let $\mathcal{A} \subset \mathcal{B}$ be two exact classes of $\mathcal{D}$-complexes. Suppose that $\mathcal{B}$ is not too big and satisfies the finiteness property and that $\mathcal{A}$ is closed in $\mathcal{B}$.

Let $\mathcal{L}$ be the class of all $\mathcal{A}$-local complexes L such that there exists a $\mathcal{D}$-complex X in $\mathcal{B}$ and an $\bar{\mathcal{A}}$-equivalence from X to L . Then $\mathcal{L}$ is exact and there is a homotopy equivalence of spectra from K($\mathcal{B}$, $\mathcal{A}$) to K($\mathcal{L}$).

The rest of this section will be devoted to the proof of this theorem.

Lemma 3-12: Every map from a $\mathcal{D}$-complex X in $\mathcal{B}$ to a $\mathcal{D}$-complex Y in $\bar{\mathcal{A}}$ factors through a $\mathcal{D}$-complex in $\mathcal{A}$.

Proof: Let $\mathcal{C}$ be the class of all $\mathcal{D}$-complexes Y in $\bar{\mathcal{B}}$ such that, for every X in $\mathcal{B}$, every map from X to Y factors through a $\mathcal{D}$-complex in $\mathcal{A}$. Let Y_i be a family of $\mathcal{D}$-complexes in $\mathcal{C}$. Let X be a $\mathcal{D}$-complex in $\mathcal{B}$ and f be a map from X to $\oplus Y_i$. By the finiteness property, there exists a $\mathcal{D}$-complex X' and a homology equivalence g from X' to X such that $f \circ g$ factors through a finite sum Y' of the Y_i's. Since the class $\mathcal{C}$ is clearly stable under finite sum, the map from X' to Y' factors through a $\mathcal{D}$-complex Z in $\mathcal{A}$, and the map from X to $\oplus Y_i$ factors through the push-out U of X and Z over X', which is in $\mathcal{A}$.

$$\begin{array}{ccccc} X' & \longrightarrow & Z & \longrightarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & U & \longrightarrow & \oplus Y_i \end{array}$$

So the class $\mathcal{C}$ is stable under direct sum.

Let $0 \to X \to Y \to Z \to 0$

be an exact sequence of $\mathcal{D}$-complexes, where X and Y are in $\mathcal{C}$. Let U be a $\mathcal{D}$-complex in $\mathcal{B}$ and $f: U \to Z$ be a map. Let E be an acyclic $\mathcal{D}$-complex and g be a surjective map from E to the pull-back of Y and U over Z. Since X is in $\mathcal{C}$, the map from the kernel K of $E \to U$ to X factors through a $\mathcal{D}$-complex X_1 in $\mathcal{A}$. Since Y is in $\mathcal{C}$, the map from the push-out of X_1 and E over K factors (via a cofibration) through a $\mathcal{D}$-complex Y_1 in $\mathcal{A}$. Hence the map from U to Z factors through the cokernel Z_1 of $X_1 \to Y_1$ which is in $\mathcal{A}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & Y_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & U & \longrightarrow & 0 \end{array}$$

Therefore the class $\mathcal{C}$ is exact and contains $\bar{\mathcal{A}}$.

<u>Lemma</u> 3-13: Let $\mathcal{B}'$ be the class of all $\mathcal{D}$-complexes X such that there exists a complex U in $\mathcal{B}$ and a $\bar{\mathcal{A}}$-equivalence from U to X. Then $\mathcal{B}'$ is exact and the inclusion $\mathcal{B} \subset \mathcal{B}'$ induces a homotopy equivalence of spectra from $K(\mathcal{B}, \mathcal{A})$ to $K(\mathcal{B}', \bar{\mathcal{A}})$.

<u>Proof</u>: We just have to check the conditions of the approximation lemma 3-7 for the inclusion functor F: $\mathcal{B} \subset \mathcal{B}'$. If X is a $\mathcal{D}$-complex in $\mathcal{B}$ such that F(X) is in $\bar{\mathcal{A}}$, the identity $X \to X$ factors through a $\mathcal{D}$-complex in $\mathcal{A}$, and there exists a $\mathcal{D}$-complex X' in $\mathcal{B}$ such that $X \oplus X'$ is in $\mathcal{A}$. Since $\mathcal{A}$ is closed in $\mathcal{B}$ X is in $\mathcal{A}$.

Let X be a $\mathcal{D}$-complex in $\mathcal{B}$ and f be a map from X to a $\mathcal{D}$-complex Y in $\mathcal{B}'$. There exists a $\mathcal{D}$-complex Z in $\mathcal{B}$ and a $\bar{\mathcal{A}}$-equivalence $Z \to Y$. Let $Z \to E$ be a cofibration from Z to an acyclic $\mathcal{D}$-complex E and U be the cokernel of the cofibration $Z \to Y \oplus E$. The $\mathcal{D}$-complex U is in $\bar{\mathcal{A}}$ and the map $X \to Y \to U$ factors through a $\mathcal{D}$-complex V in $\mathcal{A}$. Let X' be the pull-back of V and $Y \oplus E$ over U. The $\mathcal{D}$-complex X' is in $\mathcal{B}$ and the map $X' \to Y$ is a $\mathcal{A}$-equivalence.

$$\begin{array}{ccccc} & & & & Z\\ & & & & \downarrow\\ X & \longrightarrow & X' & \longrightarrow & Y\oplus E\\ & \searrow & \downarrow & & \downarrow\\ & & V & \longrightarrow & U\end{array}$$

Then the approximation lemma holds and the lemma is proven.

<u>Lemma</u> 3-14: Let $\mathcal{E}$ be the category of exact sequences:

$$0 \to X \to Y \to Z \to 0$$

where X is in $\bar{\mathcal{A}}$, and Z is in $\mathcal{L}$. $\mathcal{E}$ is a category with cofibrations and weak equivalences, where the weak equivalences are the maps inducing a homology equivalence on the quotient term. Then the functor sending each such exact sequence to its middle term, induces a homotopy equivalence:

$K(\mathcal{E}) \simeq K(\mathcal{B}', \bar{\mathcal{A}})$

Proof: Let F be the functor sending each exact sequence in $\mathcal{E}$ to its middle term. Once again we want to apply the approximation theorem for F. Let S be an exact sequence in $\mathcal{E}$:

$$0 \to X \to Y \to Z \to 0$$

If Y is in $\bar{\mathcal{A}}$, Z is in $\mathcal{L}$ and in $\bar{\mathcal{A}}$. By lemma 3-9, Z is $\bar{\mathcal{A}}$-local and the identity of Z factors through an acyclic $\mathcal{D}$-complex. Hence Z is acyclic. We may apply that for the mapping-cone of any map f in $\mathcal{E}$, and we deduce that F(f) is in $\bar{\mathcal{A}}$ if and only if f is a weak equivalence in $\mathcal{E}$.

Let Y as above, and f: $Y \to Y_0$ be any map in $\mathcal{B}'$. Let Z' be the push-out of Z and Y_0 over Y. Since $\mathcal{B}$ is not too big and satisfies the finiteness property, so it is for $\mathcal{A}$, and by proposition 3-8, there exists an $\bar{\mathcal{A}}$-equivalence from Z' to a $\mathcal{A}$-local complex Z'. By adding to the map $Y_0 \to Z'$ a surjective map $E \to Z'$ where E is acyclic, we get a surjective map from $Y' = Y_0 \oplus E$ to Z', and a new sequence S' in $\mathcal{E}$:

$$0 \to X' \to Y' \to Z' \to 0$$

Moreover we have a map φ from S to S', and f is the map $F(\varphi)$ composed with a homology equivalence. Thus the approximation theorem holds and the lemma is proven.

Lemma 3-15: The class $\mathcal{L}$ is exact and the functor sending each exact sequence in $\mathcal{E}$ to the quotient term, induces a homotopy equivalence from $K(\mathcal{E})$ to $K(\mathcal{L})$.

Proof: The class $\mathcal{L}$ is the intersection $\mathcal{B}' \cap \mathcal{L}\mathcal{A}$ and then is exact. By the additivity theorem [], $K(\mathcal{E})$ is homotopy equivalent to the product $K(\bar{\mathcal{A}}, \bar{\mathcal{A}}) \times K(\mathcal{L})$, and then to $K(\mathcal{L})$.

§ 4 Applications to algebraic K-theory of rings and Nil-groups.

Let A, B, B' be three rings and $A \subset B$ and $A \subset B'$ be pure inclusions, i. e. there exists decompositions of B and B' as A-bimodules:

$$B = A \oplus S \qquad B' = A \oplus S'$$

where S and S' are flat from the left.

We can define the amalgamated free product R of B and B' over A. The standard example is $A = \mathbb{Z}[H]$, $B = \mathbb{Z}[G]$, $B' = \mathbb{Z}[G']$, where H is a subgroup of G and G'. In this case, R is the group ring $\mathbb{Z}[G \underset{H}{*} G']$.

If S and S' are free from the left, we have a fundamental result of Waldhausen:

<u>Theorem</u> 4-1: [] In this situation, the algebraic K-theory spectrum of R decomposes into two spectra: $K(R) \simeq K'(R) \times K''(R)$

The first piece fit in a homotopy cartesian square of spectra:

$$\begin{array}{ccc} K(A) & \longrightarrow & K(B) \\ \downarrow & & \downarrow \\ K(B') & \longrightarrow & K'(R) \end{array}$$

and the loop spectrum $\Omega K''(R)$ of the second piece has the homotopy type of a spectrum $\widetilde{K}\mathfrak{N}il(A; S, S')$ depending only on A, S, S'.

<u>Remark</u>: In this theorem all spectra are not connective in general. The perturbating Nil term represents the deffect of a Mayer-Vietoris exact sequence in algebraic K-theory. Under some conditions it vanishes [].

We'll give a sketch of proof of this theorem, using the machinery of the last section.

Let I be the following category:

$$\begin{array}{ccc} * & \xrightarrow{u} & * \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle v} \\ * & \xrightarrow[v']{} & * \end{array}$$

it has 4 objects, 4 identities, 5 other maps: u, u', v, v', $v \circ u = v' \circ u'$. Let J be the subcategory of I given by the identities.

We have the following diagram of rings over (I, J):

$$\mathcal{D} = \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ B' & \longrightarrow & R \end{array}$$

where the bimodule corresponding to every morphism in I is the ring corresponding to its target. If X is a $\mathcal{D}$-complex, the corresponding chain complexes over A, B, B', R will be denoted by X_A, X_B, $X_{B'}$, X_R.

Let $\mathcal{B}$ be the class od all $\mathcal{D}$-complexes

$$X = \begin{array}{ccc} X_A & \longrightarrow & X_B \\ \downarrow & & \downarrow \\ X_{B'} & \longrightarrow & X_R \end{array}$$

where X_A, X_B, $X_{B'}$, X_R have the homotopy type of finite chain complexes over A, B, B', R, and such that this diagram becomes homotopy cartesian, after tensoring by R, over A, B, B'. Let $\mathcal{A}$ be the class of all $\mathcal{D}$-complexes X in $\mathcal{B}$ such X_R is acyclic. Both classes $\mathcal{A}$ and $\mathcal{B}$ are exact. It is easy to see that for every $\mathcal{D}$-complex X in $\mathcal{B}$, there is a homology equivalence from a $\mathcal{D}$-complex X' to X, where X' involves only finite chain complexes over A, B, B', R. Therefore the class $\mathcal{B}$ is not too big and satisfies the finiteness property.

<u>Lemma</u> 4-2: The functor $X \mapsto (X_A, X_B, X_{B'})$ induces a homotopy equivalence of spectra:
$$K(\mathcal{B}) \to K(\mathcal{C}_A) \times K(\mathcal{C}_B) \times K(\mathcal{C}_{B'})$$

<u>Proof:</u> Let $\mathcal{E}$ be the category of such diagrams in $\mathcal{B}$:

$$X \to Y \to Z$$

where maps are cofibrations and the following holds:

X_A and $X_{B'}$ are acyclic

$(Y/_X)_A$ and $(Y/_X)_B$ are acyclic

$(Z/_Y)_B$ and $(Z/_Y)_{B'}$ are acyclic

$\mathcal{E}$ is a category with cofibrations and weak equivalences (homology equivalences). It is easy to see that the conditions above determine completely the homology type of X and Y in term of Z:

$$X_A \sim Y_A \sim 0 \qquad X_B \sim Y_B \sim Z_B$$

$$X_{B'} \sim 0 \qquad Y_{B'} \sim Z_{B'} \qquad X_R \sim Z_B \otimes R \qquad Y_R \sim Z_{B'} \otimes R$$

More precisely the functor from $\mathcal{E}$ to $\mathcal{B}$: $(X \to Y \to Z) \mapsto Z$ satisfies the conditions of the approximation theorem [] and induces a homotopy equivalence from $K(\mathcal{E})$ to $K(\mathcal{B})$. By the additivity theorem [], the functor $(X \to Y \to Z) \mapsto (Z/_Y, X, Y/_X)$ induces a homotopy equivalence from $K(\mathcal{E})$ to $K(\mathcal{B}_0) \times K(\mathcal{B}_1) \times K(\mathcal{B}_2)$, where $\mathcal{B}_0$ (resp. $\mathcal{B}_1$, $\mathcal{B}_2$) is the class of the $\mathcal{D}$-complexes U in $\mathcal{B}$ satisfying:

$$U_B \sim U_{B'} \sim 0 \quad (\text{resp. } U_A \sim U_{B'} \sim 0,\ U_A \sim U_B \sim 0)$$

Moreover the functor $X \mapsto X_A$ from $\mathcal{B}_0$ to $\mathcal{C}_A$ (resp. $X \mapsto X_B$ from $\mathcal{B}_1$ to $\mathcal{C}_B$, $X \mapsto X_{B'}$ from $\mathcal{B}_2$ to $\mathcal{C}_{B'}$) satisfies all conditions of the approximation lemma 3-7 and induces a homotopy equivalence from $K(\mathcal{B}_0)$ to $K(\mathcal{C}_A)$ (resp. from $K(\mathcal{B}_1)$ to $K(\mathcal{C}_B)$, from $K(\mathcal{B}_2)$ to $K(\mathcal{C}_{B'})$). On the other hand, for every $(X \to Y \to Z)$ in $\mathcal{E}$, we have natural homotopy equivalences:

$$Z_A \xrightarrow{\sim} (Z/_Y)_A \qquad X_B \xrightarrow{\sim} Z_B \qquad Z_{B'} \xleftarrow{\sim} Y_{B'} \xrightarrow{\sim} (Y/_X)_{B'}$$

Therefore the functor: $(X \to Y \to Z) \mapsto (Z_A, Z_B, Z_{B'})$ induces a homotopy equivalence from $K(\mathcal{E})$ to $K(\mathcal{C}_A) \times K(\mathcal{C}_B) \times K(\mathcal{C}_{B'})$ and the lemma follows.

<u>Lemma</u> 4-3: Let $\mathcal{C}$ be the class of all $\mathcal{A}$-local $\mathcal{D}$-complexes X such that there exists a $\bar{\mathcal{A}}$-equivalence from a $\mathcal{D}$-complex in $\mathcal{B}$ to X. Let $\mathcal{C}'_R$ be the class of all A-complexes in $\mathcal{C}_R$ with Euler characteristic in the image of $K_0(B) \oplus K_0(B') \to K_0(R)$.

Then the class $\mathcal{C}$ is exact and the functor $X \mapsto X_R$ induces a homotopy equivalence from $K(\mathcal{C})$ to $K(\mathcal{C}'_R)$.

Proof: Let X be a $\mathcal{A}$-local $\mathcal{D}$-complex. Let C (resp. C') be an acyclic B-e-complex (resp. B'-e-complex) and f (resp. f') be a surjective chain maps from C (resp. C') onto X_R. The $\mathcal{D}$-complex X has the homology type of the following $\mathcal{D}$-complex:

$$X' = \begin{array}{ccc} X_A & \longrightarrow & X_B \oplus C \\ \downarrow & & \downarrow \\ X_{B'} \oplus C' & \longrightarrow & X_R \end{array}$$

If we add to X_A an acyclic A-e-complex which is going onto the pull-back of X'_B and $X'_{B'}$ over X'_R, we get a new $\mathcal{D}$-complex X" such that the four maps in the diagram X" are surjective. Moreover we have a homology equivalence from X to X", and X" is $\mathcal{A}$-local. Let Y and Y' be the following $\mathcal{D}$-complexes in $\mathcal{A}$:

$$Y = \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array} \qquad Y' = \begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ B' & \longrightarrow & 0 \end{array}$$

Since X" is $\mathcal{A}$-local, every map from every suspension of Y or Y' to X" factors through an acyclic $\mathcal{D}$-complex, and that implies that the kernels of $X''_A \to X''_B$ and $X''_A \to X''_{B'}$ are acyclic. Therefore $X_A \to X_B$ and $X_A \to X_{B'}$ induce bijections in homology.

Let H be the homology of X_A and H' be the homology of X_R. Since the chain map from X_A to X_B and $X_{B'}$ are homology isomorphism, H has structures of B- and B'-module and these two structures agree over A. Thus H is a R-module, and we have a long exact sequence:

$$\ldots \to H \underset{A}{\otimes} R \xrightarrow{\alpha} H \underset{B}{\otimes} R \oplus H \underset{B'}{\otimes} R \to H' \to H \underset{A}{\otimes} R \to \ldots$$

The morphism α is the tensor product over R by the map: $R \underset{A}{\otimes} R \to R \underset{B}{\otimes} R \oplus R \underset{B'}{\otimes} R$.

The A-bimodule R has the following description:

$$R = A \oplus S \oplus S' \oplus S \otimes S' \oplus S' \otimes S \oplus S \otimes S' \otimes S \oplus S' \otimes S \otimes S' \oplus \ldots$$

Let R_+ (resp. R_-) be the sum of all bimodule begining with S (resp. S'). We have:

$$R = A \oplus R_+ \oplus R_- = B \otimes (A \oplus R_-) = B' \otimes (A \oplus R_+)$$

and the following sequence is exact:

$$0 \to R \underset{A}{\otimes} R \to R \underset{B}{\otimes} R \oplus R \underset{B'}{\otimes} R \to R \to 0$$

Since all this bimodules are flat from the left, α is injective and its cokernel is isomorphic to H. Hence: H = H' and all the maps of the diagram X induce bijections in homology.

Conversely, it is not difficult to check that every $\mathcal{D}$-complex X such that all maps of the diagram X induce bijections in homology, is $\mathcal{A}$-local.

Consider the functor F from $\mathcal{C}$ to $\mathcal{C}_R$: $X \mapsto X_R$. Let $\mathcal{C}'$ be the class of all R-e-complexes Y such that there exists a $\mathcal{D}$-complex X in $\mathcal{C}$ and a homology equivalence from X_R to Y. The conditions of the approximation lemma 3-7 are easy to check. Therefore the class $\mathcal{C}'$ is exact and F induces a homotopy equivalence from $K(\mathcal{C})$ to $K(\mathcal{C}')$. If Y is a $\mathcal{D}$-complex in $\mathcal{C}'$, there exists a $\mathcal{D}$-complex X in $\mathcal{B}$ such that Y_R is homotopy equivalent to Y. Thus $\mathcal{C}'$ is contained in the class $\mathcal{C}_R^{\cdot}$. On the other hand, if P and P' are finitely generated projective modules over B and B', consider as complexes concentrated in degree 0, the following complex is in $\mathcal{B}$:

$$\begin{array}{ccc} 0 & \longrightarrow & P \\ \downarrow & & \downarrow \\ P' & \longrightarrow & M = P \underset{B}{\otimes} R \oplus P' \underset{B'}{\otimes} R \end{array}$$

and M is in $\mathcal{C}'$. Therefore $\mathcal{C}'$ is the class $\mathcal{C}_R^{\cdot}$.

By the approximation lemma 3-7, the map $K(\mathcal{C}) \to K(\mathcal{C}_R^{\cdot})$ is a homotopy equivalence.

Lemma 4-4: There is a fibration of spectra:

$$K(\mathcal{A}) \to K(\mathcal{C}_A) \times K(\mathcal{C}_B) \times K(\mathcal{C}_{B'}) \to K(\mathcal{C}_R^{\cdot})$$

where the first map is given by:

$$X \mapsto (X_A, X_B, X_{B'})$$

and the second one by $-\alpha + \beta + \beta'$, where α, β, β' are the functors from $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_{B'}$ to $\mathcal{C}_R^{\cdot}$ given by $- \otimes R$.

Proof: We have a fibration of spectra:

$$K(\mathcal{A}) \to K(\mathcal{B}) \to K(\mathcal{C})$$

By the lemma 4-2, the composite $K(\mathcal{A}) \to K(\mathcal{B}) \xrightarrow{\sim} K(\mathcal{C}_A) \times K(\mathcal{C}_B) \times K(\mathcal{C}_{B'})$ is given by the functor $X \mapsto (X_A, X_B, X_{B'})$. We have a commutative diagram:

$$\begin{array}{ccccc} K(\mathcal{B}, \mathcal{A}) & \xrightarrow{\sim} & K(\mathcal{B}', \bar{\mathcal{A}}) & \xleftarrow{\sim} & K(\mathcal{L}) \\ & \searrow & \downarrow & \swarrow & \\ & & K(\mathcal{C}_R) & & \end{array}$$

and the map $K(\mathcal{B}) \to K(\mathcal{C}) \to K(\mathcal{C}_R)$ is homotopic to the map given by the functor $X \mapsto X_R$.

On the other hand, for every X in $\mathcal{B}$, we have a homotopy equivalence from the mapping cone of $X_A \otimes R \to X_B \otimes R \oplus X_{B'} \otimes R$ to X_R. Hence the map from $K(\mathcal{B})$ to $K(\mathcal{C}_R)$ given by $X \mapsto X_R$ is homotopic to $f_B + f_{B'} - f_A$, where f_A (resp. f_B, $f_{B'}$) is given by: $X \mapsto X_A$ (resp. X_B, $X_{B'}$). Therefore the map $K(\mathcal{B}) \to K(\mathcal{C})$ is homotopic to $\beta + \beta' - \alpha$.

In order to describe $\mathcal{A}$ in term of nil-objects, we have to work with another diagram of rings. Consider the following categories I' and J': I' has two objects: 0 and 1, two arrows: $0 \to 1$ and $1 \to 0$ and all possible composite of these:

$$0 \rightleftarrows 1$$

and J' has only the identities of I.

A diagram of rings $\mathcal{D}'$ over I' is given by two rings A_0 and A_1, one $A_0 \times A_1$-bimodule ${}_0S_1$ and one $A_1 \times A_0$-bimodule ${}_1S_0$. A $\mathcal{D}'$-complex is determined by a A_0-e-complex C_0, a A_1-e-complex C_1, a chain map α_0 from C_0 to $C_1 \otimes_{A_1} {}_1S_0$, a chain map α_1 from C_1 to $C_0 \otimes_{A_0} {}_0S_1$.

The exact class $\mathfrak{Nil}(\mathcal{C}_{A_0}, \mathcal{C}_{A_1}; {}_0S_1, {}_1S_0)$ is the class of all $\mathcal{D}'$-complexes $N = (C_0, C_1, \alpha_0, \alpha_1)$ such that C_0 and C_1 have the homotopy type of finite complexes and N is nilpotent in the following sense: $(\alpha_1 \circ \alpha_0)^n$ is null-homotopic for some n. If A_0 is equal to A_1 this class will be denoted only by $\mathfrak{Nil}(\mathcal{C}_{A_0}; {}_0S_1, {}_1S_0)$.

The forgetfull functor $u: (C_0, C_1, \alpha_0, \alpha_1) \mapsto (C_0, C_1)$ has a section:

$$(C_0, C_1) \mapsto (C_0, C_1, 0, 0)$$

Therefore u induces a split morphism on K-spectra and $K(\mathfrak{N}il(\mathcal{C}_{A_0}, \mathcal{C}_{A_1}; {}_0S_1, {}_1S_0))$ decomposes into three pieces:

$$K(\mathfrak{N}il(\mathcal{C}_{A_0}, \mathcal{C}_{A_1}; {}_0S_1, {}_1S_0)) \simeq K(\mathcal{C}_{A_0}) \times K(\mathcal{C}_{A_1}) \times \widetilde{K}\mathfrak{N}il(\mathcal{C}_{A_0}, \mathcal{C}_{A_1}; {}_0S_1, {}_1S_0)$$

Likewise:

$$K(\mathfrak{N}il(\mathcal{C}_{A}; {}_0S_1, {}_1S_0)) \simeq K(\mathcal{C}_{A}) \times K(\mathcal{C}_{A}) \times \widetilde{K}\mathfrak{N}il(\mathcal{C}_{A}; {}_0S_1, {}_1S_0)$$

<u>Lemma</u> 4-5: The functor F from $\mathfrak{N}il(\mathcal{C}_A ; S, S')$ to the class $\mathcal{A}$, given by:

$$(C_0, C_1, \alpha_0, \alpha_1) \mapsto \begin{array}{ccc} C_0 \oplus C_1 & \xrightarrow{1+\alpha_1} & C_0 \otimes B \\ \downarrow{\scriptstyle \alpha_0+1} & & \downarrow \\ C_1 \otimes B' & \longrightarrow & 0 \end{array}$$

induces a homotopy equivalence from $K(\mathfrak{N}il(\mathcal{C}_A; S, S'))$ to $K(\mathcal{A})$.

<u>Proof</u>: Let X be a $\mathcal{D}$-complex in $\mathcal{A}$. By definition of $\mathcal{A}$, the chain map $X_A \underset{A}{\otimes} R \to X_B \underset{B}{\otimes} R \oplus X_{B'} \underset{B'}{\otimes} R$ is a homotopy equivalence and may be written as follows:

$$\begin{array}{cccccccccccccccccccc} \dots & CSS'SS' & & CS'SS' & & CSS' & & CS' & & C & & CS & & CS'S & & CSS'S & & CS'SS'S & \dots \\ & \swarrow\ \searrow & & \swarrow\ \searrow & & \swarrow\ \searrow & & \swarrow\ \searrow & & \swarrow\ \searrow & & \swarrow\ \searrow & & \swarrow\ \searrow & & \swarrow\ \searrow & & \swarrow\ \searrow & \\ \dots & & KS'SS' & & K'SS' & & KS' & & K' & & K & & K'S & & KS'S & & K'SS'S & & \dots \end{array}$$

where $C = X_A$, $K = X_B$, $K' = X_{B'}$, and the tensor products are omitted. The map goes from the direct sum of the top terms to the sum of the bottom terms and is a homotopy equivalence. Let U (resp. U') be the sum of all terms appearing to the right hand side (resp. left hand side) of C and V (resp. V') be the sum of all terms in the bottom line appearing to the right hand side (resp. left hand side) of C. Then we get maps $U \to V$, $U' \to V'$, $C \to V$, $C \to V'$ inducing a homotopy equivalence from $U' \oplus C \oplus U$ to $V' \oplus V$. Let C_1 and C_0 be the mapping-cone of $U \to V$ and $U' \to V'$. We get a homotopy equivalence from C to $C_1 \oplus C_0$ and C_0 and C_1 have the homotopy type of finite complexes. We may use now the arguments of [](p 146-147) and we prove that the map $C \to X_B$ induces a homotopy equivalence from $C_0 \otimes B$ to X_B and a homotopy equivalence from C_1 to $X_{B'}$. Up to homotopy the maps $X_A \to X_B$ and X_A

$\to X_{B'}$ have the following form:

$$1\oplus\alpha_1 : C_0 \oplus C_1 \to C_0 \oplus C_0 \otimes S \qquad \text{and} \qquad \alpha_0 \oplus 1 : C_0 \oplus C_1 \to C_1 \oplus C_1 \otimes S'$$

for some maps α_0 and α_1. Furthermore the condition that $X_A \otimes R \to X_B \otimes R \oplus X_{B'} \otimes R$ induces an isomorphism in homology is exacly equivalent to the fact that $\alpha_1 \circ \alpha_0$ is nilpotent in homology and therefore to the fact that some power of $\alpha_1 \circ \alpha_0$ is null-homotopic.

So we have prove that every X in $\mathcal{A}$ has the homology type of some $\mathcal{D}$-complex F(N) for N in $\mathfrak{N}\mathrm{il}(\mathcal{C}_A;\ S,\ S')$. Since this construction is functorial enough the approximation lemma 3-7 applies and F induces a homotopy equivalence in K-theory.

4-6: <u>End of proof of the theorem</u>

We have now a fibration of spectra:

$$\tilde{K}\mathfrak{N}\mathrm{il}(\mathcal{C}_A;\ S,\ S') \times K(\mathcal{C}_A) \times K(\mathcal{C}_A) \xrightarrow{\varepsilon} K(\mathcal{C}_A) \times K(\mathcal{C}_B) \times K(\mathcal{C}_{B'}) \to K(\mathcal{C}_R^{\cdot})$$

Moreover, if φ and φ' are the maps from $K(\mathcal{C}_A)$ to $K(\mathcal{C}_B)$ and $K(\mathcal{C}_{B'})$ induced by the inclusions $A \subset B$ and $A \subset B'$, the map ε is given by the matrix:

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & \varphi & 0 \\ 0 & 0 & \varphi' \end{bmatrix}$$

Therefore the fibration above contains a subfibration:

$$\tilde{K}\mathfrak{N}\mathrm{il}(\mathcal{C}_A;\ S,\ S') \times K(\mathcal{C}_A) \xrightarrow{\varepsilon'} K(\mathcal{C}_B) \times K(\mathcal{C}_{B'}) \to K(\mathcal{C}_R^{\cdot})$$

where ε' is given by the matrix: $\begin{bmatrix} 0 & -\varphi \\ 0 & \varphi' \end{bmatrix}$

If we replace in this fibration, A, B, B', S, S', R by their suspensions $\Sigma^n A$, $\Sigma^n B$, $\Sigma^n B'$ etc ... we get other fibrations. But the family of spaces $\Omega^\infty K(\mathcal{C}_{\Sigma^n A})$ gives rise to the non connective Quillen's spectrum K(A) [ ]. Therefore the family of spaces $\Omega^\infty \tilde{K}\mathfrak{N}\mathrm{il}(\mathcal{C}_{\Sigma^n A};\ \Sigma^n S,\ \Sigma^n S')$ gives rise to a non connective spectrum $\tilde{K}\mathfrak{N}\mathrm{il}(A;\ S,\ S')$

and we have a fibration of spectra:

$$\tilde{K}\mathfrak{Nil}(A;\ S,\ S') \times K(A) \to K(B) \times K(B') \to K(R)$$

Moreover the inclusion of the fiber is null-homotopic on the $\mathfrak{Nil}$ part and the theorem follows.

§ 5 Properties of the $\mathfrak{Nil}$ functor

In [], Waldhausen defined other $\mathfrak{Nil}$ spectra. In particular he defines a spectrum $K\mathfrak{Nil}(A;\ S)$ where A is a ring and S is a A-bimodule. In our language this spectrum may be define in the following way:

Let I" be the category with one object 0, where the arrows are the powers of one map $0 \to 0$ and J" be the subcategory $(0,\ Id_0)$ of I". A diagram of rings $\mathcal{D}$" over (I", J") is a ring A and an A-bimodule S. A $\mathcal{D}$"-complex is an A-e-complex C endowed with a map α from C to $C \otimes S$. The class $\mathfrak{Nil}(\mathcal{C}_A;\ S)$ is the class of all $\mathcal{D}$"-complexes $(C,\ \alpha)$ where C has the homotopy type of a finite complex, and α is nilpotent (i. e. for some n, α^n is null-homotopic).

As before there is a split forgetfull functor $(C,\ \alpha) \mapsto C$ inducing a split fibration $K(\mathfrak{Nil}(\mathcal{C}_A;\ S)) \to K(\mathcal{C}_A)$ with fiber $\tilde{K}\mathfrak{Nil}(\mathcal{C}_A;\ S)$:

$$K(\mathfrak{Nil}(\mathcal{C}_A;\ S)) \xrightarrow{\sim} K(\mathcal{C}_A) \times \tilde{K}\mathfrak{Nil}(\mathcal{C}_A;\ S)$$

<u>Proposition</u> 5-1: Let A be a ring and S be a A-bimodule flat from the left. Then the collection of spaces $\Omega^{\infty}\ \tilde{K}\mathfrak{Nil}(\mathcal{C}_{\Sigma^n A};\ \Sigma^n S)$ gives rise to a non connective spectrum $\tilde{K}\mathfrak{Nil}(A;\ S)$.

<u>Theorem</u> 5-2: Let A_0 and A_1 be two rings, and ${}_0S_1$ (resp. ${}_1S_0$) be a $A_0 \times A_1$-bimodule (resp. $A_1 \times A_0$-bimodule) flat from the left. Then the functor F: $(C_0,\ C_1;\ \alpha_0,\ \alpha_1) \mapsto (C_0,\ \alpha_1 \circ \alpha_0)$ induces a homotopy equivalence of spectra:

$$\tilde{K}\mathfrak{Nil}(A_0,\ A_1;\ {}_0S_1,\ {}_1S_0) \to \tilde{K}\mathfrak{Nil}(A_0;\ {}_0S_1 \otimes_{A_1} {}_1S_0).$$

Proofs: Let $\mathcal{A}$ be the subclass of $\mathfrak{N}\mathrm{il}(A_0, A_1; {}_0S_1, {}_1S_0)$ defined by:

$$(C_0, C_1; \alpha_0, \alpha_1) \in \mathcal{A} \quad \Leftrightarrow \quad C_0 \text{ is acyclic}$$

and $\mathcal{L}$ be the class of all $\mathcal{A}$-local $\mathcal{D}'$-complexes. Let $L = (C_0, C_1; \alpha_0, \alpha_1)$ be a $\mathcal{D}'$-complex in $\mathcal{L}$. Up to homotopy, we may as well suppose that α_1 is surjective. The class $\mathcal{A}$ contains the particular $\mathcal{D}'$-complex $N_0 = (0, A_1; 0, 0)$, and every map from N_0 to L factors through an acyclic $\mathcal{D}'$-complex. Therefore the kernel of α_1 is acyclic and α_1 induces a bijection in homology. Conversely it is easy to see that a $\mathcal{D}'$-complex N $= (C_0, C_1; \alpha_0, \alpha_1)$ is $\mathcal{A}$-local if and only if α_1 induces a bijection in homology.

Let $\bar{\mathcal{A}}$ be the completion of $\mathcal{A}$, and $\mathcal{L}$ be the class of all $\mathcal{A}$-local $\mathcal{D}'$-complexes N such that there exists a $\bar{\mathcal{A}}$-equivalence from a $\mathcal{D}'$-complex in $\mathfrak{N}\mathrm{il}(\mathcal{C}_{A_0}, \mathcal{C}_{A_1}; {}_0S_1, {}_1S_0)$ to N. If N$= (C_0, C_1; \alpha_0, \alpha_1)$ is in $\mathcal{L}$, α_1 induces a bijection in homology and C_0 has the homotopy type of a finite complex. Conversely let $N = (C_0, C_1; \alpha_0, \alpha_1)$ be any $\mathcal{D}'$-complex such that α_1 induces a bijection in homology and C_0 is in $\mathcal{C}_{A_0}$. Since C_0 has the homotopy type of a finite complex, there exists a A_1-e-complex C'_1 in $\mathcal{C}_{A_1}$ and maps $\alpha'_0: C_0 \to C'_1 \otimes {}_1S_0$ and $f: C'_1 \to C_1$ such that α_0 is the composite $f \circ \alpha'_0$. Then we get a $\mathcal{D}'$-complex $N' = (C_0, C'_1; \alpha'_0, \alpha_1 \circ f)$ and a $\bar{\mathcal{A}}$-equivalence from N' to N. Therefore N is in $\mathcal{L}$ and $\mathcal{L}$ is exactly the class of $\mathcal{D}'$-complexes $(C_0, C_1; \alpha_0, \alpha_1)$ such that C_0 is finite up to homotopy and α_1 induces a bijection in homology.

By the approximation lemma 3-7, the functor $(C_0, C_1; \alpha_0, \alpha_1) \mapsto (C_0; \alpha_1 \circ \alpha_0)$ induces a homotopy equivalence from $K(\mathcal{L})$ to $K(\mathfrak{N}\mathrm{il}(\mathcal{C}_{A_0}; {}_0S_1 \otimes {}_1S_0)$ and the functor $(C_0, C_1; \alpha_0, \alpha_1) \mapsto C_1$ induces a homotopy equivalence from $K(\mathcal{A})$ to $K(\mathcal{C}_A)$. Then we get a fibration of spectra (theorem 3-11):

$$K(C_{A_1}) \to K\mathfrak{N}\mathrm{il}(\mathcal{C}_{A_0}, \mathcal{C}_{A_1}; {}_0S_1, {}_1S_0) \to K\mathfrak{N}\mathrm{il}(\mathcal{C}_{A_0}; {}_0S_1 \underset{A_1}{\otimes} {}_1S_0)$$

and a homotopy equivalence:

$$\tilde{K}\mathfrak{N}\mathrm{il}(\mathcal{C}_{A_0}, \mathcal{C}_{A_1}; {}_0S_1, {}_1S_0) \to \tilde{K}\mathfrak{N}\mathrm{il}(\mathcal{C}_{A_0}; {}_0S_1 \underset{A_1}{\otimes} {}_1S_0)$$

This homotopy equivalence holds for all suspensions of A_0, A_1, ${}_0S_1$, ${}_1S_0$. Therefore for every ring A and every A-bimodule S, flat from the left, the family of spaces $\Omega^\infty \tilde{K}\mathfrak{N}\mathrm{il}(\mathcal{C}_{\Sigma^n A}; \Sigma^n S)$ gives rise to a non connective spectrum $\tilde{K}\mathfrak{N}\mathrm{il}(A; S)$ and we have a homotopy equivalence of spectra:

$$\tilde{K}\mathfrak{Nil}(A_0,\ A_1;\ {}_0S_1,\ {}_1S_0) \to \tilde{K}\mathfrak{Nil}(A_0;\ {}_0S_1 \underset{A_1}{\otimes} {}_1S_0).$$

Corollary 5-3: Let A_0 and A_1 be two rings and ${}_0S_1$ be a $A_0 \times A_1$-bimodule and ${}_1S_0$ be a $A_1 \times A_0$-bimodule. Suppose that ${}_0S_1$ and ${}_1S_0$ are flat from the left. Then the two spectra $\tilde{K}\mathfrak{Nil}(A_0;\ {}_0S_1 \underset{A_1}{\otimes} {}_1S_0)$ and $\tilde{K}\mathfrak{Nil}(A_1;\ {}_1S_0 \underset{A_0}{\otimes} {}_0S_1)$ are homotopy equivalent.

Remark: In [], Waldhausen consider two other kind of $\mathfrak{Nil}$-spectra corresponding to the case of HNN-extension (or Laurent extension) and tensor algebra. These $\mathfrak{Nil}$-spectra may be defined in term of chain complexes instead of modules, and Waldhausen's theorems may be prove in this language in the same spirit as above. The proofs are shorter and these theorems remain true if all considered bimodules are only flat from the left and non necessary free. In particular if S is an A-bimodule, flat from the left, there is a homopoty equivalence from K(A[X]) to the product of K(A) and a delooping of $\tilde{K}\mathfrak{Nil}(A;\ S)$.

Because of theorem 5-2, the $\mathfrak{Nil}$ spectrum corresponding to two rings and two bimodules reduces to a spectrum $\tilde{K}\mathfrak{Nil}(A;\ S)$. In the Laurent case we have another kind of reduction: In this case, we have two rings A_0, A_1 (possibly equal) and four bimodules ${}_iS_j$ over $A_i \times A_j$, for $i,j = 0,1$. The category we have to consider is the class of triples $(C_0,\ C_1;\ {}_*\alpha_*)$, where for every i, j = 0, 1, C_i is a A_i-complex on the homotopy type of a finite complex, and ${}_i\alpha_j$ is a chain map from C_j to $C_i \underset{A_i}{\otimes} {}_iS_j$. Moreover maps ${}_*\alpha_*$ are nilpotent in the following sense: there exists n such that every composite of n ${}_i\alpha_j$'s is null-homotopic. But such a data is nothing else but a A-complex C in $\mathcal{C}_A$ endowed with a nilpotent map α from C to $C \otimes S$, where:

$$A = A_0 \times A_1 \qquad S = \bigoplus_{ij} {}_iS_j \qquad C = C_0 \otimes C_1 \qquad \alpha = \sum_{ij} {}_i\alpha_j$$

and for every i, j, k = 0, 1, the left A_i-action on ${}_jS_k$ is the given one if i = j and the trivial one otherwise, and the right A_i-action is the given one if i = k and the trivial one otherwise. Therefore the $\mathfrak{Nil}$-spectrum $\tilde{K}\mathfrak{Nil}(A_0,\ A_1;\ {}_0S_0,\ {}_1S_1,\ {}_0S_1,\ {}_1S_0)$ is homotopy equivalent to the spectrum $\widetilde{K\mathfrak{N}il}(A; S)$.

WARNING (A.Ranicki, 2009) The proof of Theorem 5-4 is incomplete. The statement is now known as the Vogel Conjecture. See the papers by Bihler in footnote (*) below for partial results on the Conjecture.

Theorem 5-4: Let A be a regular ring and S be a A-bimodule flat from the left. Then the K-theory spectrum $\widetilde{K\mathfrak{N}il}(A; S)$ is contractible.

Remark: This theorem was already proven by Waldhausen when A is regular coherent. Unfortulately the coherence condition is very strong and very difficult to check in general.

Proof: Consider the category Σ, where $Ob(\Sigma) = N$ and $Mor(\Sigma)$ is generated by $i_n: n \to n+1$ and $j_n: n+1 \to n$ $(n \geq 0)$ and the only relations are:

$$j_{n+1} \circ i_{n+1} = i_n \circ j_n \quad \text{for all } n \geq 0$$

Let Σ' be the subcategory of Σ generated by i_n, $n \geq 0$.

We have a particular diagram of rings: $\mathcal{D}_0 = (A_*, S_*)$ over (Σ, Σ') where:

$$A_n = A \qquad S_{i_n} = A \qquad S_{j_n} = S \qquad \text{for all } n \geq 0$$

A $\mathcal{D}_0$-complex is a triple $(C_*, \lambda_*, \alpha_*)$ where, for all $n \geq 0$:

- C_n is a A-e-complex
- $\lambda_n: C_n \to C_{n+1}$ is a cofibration
- $\alpha_n: C_{n+1} \to C_n \otimes S$ is a chain map
- $\alpha_{n+1} \circ \lambda_{n+1} = \lambda_n \circ \alpha_n$

Let $\mathcal{B}$ be the class of all $\mathcal{D}_0$-complexes $(C_*, \lambda_*, \alpha_*)$ where C_0 is acyclic, C_n is in $\mathcal{C}_A$ for all n, λ_n is a homotopy equivalence for n big enough. Let $\mathcal{A}$ be the class of all $\mathcal{D}_0$-complexes $(C_*, \lambda_*, \alpha_*)$ in $\mathcal{B}$ such that C_n is acyclic for n big enough. Every object in $\mathcal{B}$ has the homology type of a $\mathcal{D}_0$-complex $(C_*, \lambda_*, \alpha_*)$ where all C_n's are finite complexes and the system $C_0 \to C_1 \to \ldots$ is stationnary. Therefore the class $\mathcal{B}$ is not too big and satisfies the finiteness property.

For all n, denote by $\mathcal{B}_n$ the class of $\mathcal{D}_0$-complexes $(C_*, \lambda_*, \alpha_*)$ in $\mathcal{B}$ such that λ_i is a homotopy equivalence for $i \geq n$, and by $\mathcal{A}_n$ the class $\mathcal{B}_n \cap \mathcal{A}$.

(*) Frank Bihler, Vogel's notion of regularity for non-coherent rings, arXiv:math/0612569
Frank Bihler, Vanishing of the KNil groups: localization methods, arXiv:math/0702320

There is a functor T from $\mathcal{B}$ to $\mathcal{C}_A$, and a functor T$\mathfrak{N}$il from $\mathcal{B}$ to the class $\mathfrak{N}il(\mathcal{C}_A; S)$: for every $\mathcal{D}_0$-complexes $X = (C_*', \lambda_*, \alpha_*)$ in $\mathcal{B}$, T(X) is the limit of the system $C_0 \xrightarrow{\lambda_0} C_1 \xrightarrow{\lambda_1} \dots$. Since the maps α_* are compatible with maps λ_*, they induce a map $T(\alpha_*)$ from T(X) to $T(X)\otimes S$ and T$\mathfrak{N}$il(X) is the nilpotent complex $(T(X), T(\alpha_*))$.

A $\mathcal{D}_0$-complex $(C_*, \lambda_*, \alpha_*)$ is *reduced* if all maps α_n are surjective.

Lemma 5-5: For every X in $\mathcal{B}$ there exists a reduced $\mathcal{D}_0$-complex X' and a homology equivalence from X' to X.

Proof: Let $X = (C_*, \lambda_*, \alpha_*)$ be a $\mathcal{D}_0$-complex. Suppose we have construct complexes $C'_0, \dots, C'_n$ cofibrations $\mu_i: C'_i \to C'_{i+1}$, surjective homology equivalences $f_i: C'_i \to C_i$, surjective maps $\beta_i: C'_{i+1} \to C'_i \otimes S$ such that:

$$f_{i+1}\circ\mu_i = \lambda_i\circ f_i \qquad f_i\circ\beta_i = \alpha_i\circ f_{i+1} \qquad \text{for all } i<n$$

and: $\beta_{i+1}\circ\mu_{i+1} = \mu_i\circ\beta_i$ for all $i<n-1$

Let K be the pull-back of $C'_n\otimes S$ and C_{n+1} over $C_n\otimes S$ (defined by f_n and α_n). The map from the graded differential module K to C_{n+1} induces a surjective bijection in homology. The map from C'_n to K induced by $\mu_{n-1}\circ\beta_{n-1}$ and $\lambda_n\circ f_n$ factors through a complex C'_{n+1} in such a way that $C'_n \to C'_{n+1}$ is a cofibration μ_n and $C'_{n+1} \to K$ induces a bijection in homology. The map f_{n+1} is the composite: $C'_{n+1} \to K \to C_{n+1}$ and β_n is the composite: $C'_{n+1} \to K \to C'_n\otimes S$. The $\mathcal{D}_0$-complex X' is so constructed by induction.

Lemma 5-6: There is a fibration of spectra:

$$K(\mathcal{A}) \to K(\mathcal{B}) \to K(\mathfrak{N}il(\mathcal{C}_A; S))$$

where the map $K(\mathcal{B}) \to K(\mathfrak{N}il(\mathcal{C}_A; S))$ is induced by the functor T$\mathfrak{N}$il.

Proof: Let $\mathcal{L}$ be the class of $\mathcal{A}$-local $\mathcal{D}_0$-complexes X such that there exists a $\mathcal{D}_0$-complex X_0 in $\mathcal{B}$ and a $\overline{\mathcal{A}}$-equivalence from X_0 to X. We have a homotopy equivalence of spectra:

$$K(\mathcal{B}, \mathcal{A}) \simeq K(\mathcal{L})$$

For every X in $\mathcal{A}$, T(X) is acyclic. Therefore T(X) is acyclic for every X in $\bar{\mathcal{A}}$, and T(X) has the homotopy type of a finite complex for every X in $\mathcal{L}$.

Let $X = (C_*, \lambda_*, \alpha_*)$ be a reduced $\mathcal{D}_0$-complex in $\mathcal{L}$. Let n>0 be an integer and K be the complex A concentrated in some degree. Consider the following $\mathcal{D}_0$-complex $X_0 = (K_*, \mu_*, \beta_*)$:

$\forall\ i<n,\ K_i = 0 \qquad K_n = K \qquad \forall\ i>n,\ K_i = CK$ (the cone of K)

for every i, μ_i is the standard inclusion, and β_i is trivial.

Let ε the standard generator of K and $\{\varepsilon, \zeta\}$ the standard basis of CK. We have: $d\varepsilon = 0$ and $d\zeta = \varepsilon$. Therefore a map f from X_0 to X is characterized by two elements $u \in C_n$ and $v \in C_{n+1}$ satisfying the following conditions:

$$du = 0 \qquad \lambda_n(u) = dv \qquad \alpha_{n-1}(u) = 0 \qquad \alpha_n(v) = 0$$

Since X is $\mathcal{A}$-local, the map f factors through an acyclic $\mathcal{D}_0$-complex Y = (E_*, ν_*, γ_*). If M(-) denotes the mapping-cone functor, we have a commutatice diagram:

$$\begin{array}{ccccc} M(\beta_{n-1}) & \longrightarrow & M(\gamma_{n-1}) & \longrightarrow & M(\alpha_{n-1}) \\ \downarrow{\scriptstyle \mu_{n*}} & & \downarrow{\scriptstyle \nu_{n*}} & & \downarrow{\scriptstyle \lambda_{n*}} \\ M(\beta_n) & \longrightarrow & M(\gamma_n) & \longrightarrow & M(\alpha_n) \end{array}$$

Since S is flat from the left and complexes E_i are acyclic, the mapping-cone of ν_{n*} is acyclic. By construction, the pair (ε, ζ) define a cycle in the mapping-cone of μ_{n*}, and this element is null-homologous in the mapping-cone of λ_{n*}. Since α_i are surjective maps, the mapping-cone of λ_{n*} has the homology type of the suspension of the mapping-cone of $Ker(\alpha_{n-1}) \rightarrow Ker(\alpha_n)$. Therefore the pair (u, v) is null-homologous in this mapping-cone. But the only restriction on the pair (u, v) is the fact that it is a cycle in this mapping-cone. Hence the map $Ker(\alpha_{n-1}) \rightarrow Ker(\alpha_n)$ induces a bijection in homology.

Since S is flat from the left, the map $\mathrm{Ker}(\alpha^p_{n-1}) \to \mathrm{Ker}(\alpha^p_n)$ induces also a bijection in homology for every integer $n \geq p > 0$ and, for every $X = (C_*, \lambda_*, \alpha_*)$ in $\mathcal{C}$, and every $n \geq p > 0$, the following diagram is homology cartesian:

$$\begin{array}{ccc} C_p & \xrightarrow{\lambda^n} & C_{n+p} \\ \downarrow{\scriptstyle \alpha^p} & & \downarrow{\scriptstyle \alpha^p} \\ C_0 \otimes S^p & \xrightarrow{\lambda^n} & C_n \otimes S^p \end{array}$$

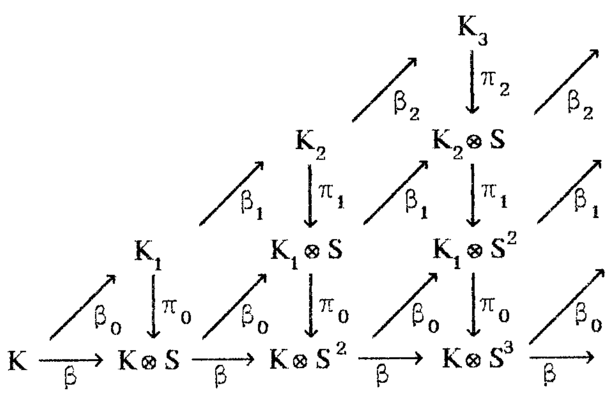
i. e. vertical maps induce bijection in homology between the mapping-cones of the horizontal maps. If we pass to the limit, we get a homology cartesian square:

$$(*) \qquad \begin{array}{ccc} C_p & \longrightarrow & T(X) \\ \downarrow{\scriptstyle \alpha^p} & & \downarrow{\scriptstyle \alpha^p} \\ C_0 \otimes S^p & \longrightarrow & T(X) \otimes S^p \end{array}$$

Consider the functor $T\mathfrak{N}il$ from $\mathcal{L}$ to $\mathfrak{N}il(\mathcal{C}_A; S)$. Let $\mathcal{C}$ be the class of all X in $\mathfrak{N}il(\mathcal{C}_A; S)$ such that there exists a $\mathcal{D}_0$-complex Y in $\mathcal{L}$ and a homology equivalence from $T\mathfrak{N}il(Y)$ to X.

In the approximation lemma 3-7 the only non trivial condition to check is the last one. Let $X = (C_*, \lambda_*, \alpha_*)$ be a $\mathcal{D}_0$-complex in $\mathcal{L}$, and f be a map from $T\mathfrak{N}il(X)$ to a nilpotent complex $Y = (K, \beta)$ in $\mathcal{C}$.

It is possible to construct A-e-complexes $K_0, K_1, K_2, \ldots$ and surjective maps $\pi_i: K_{i+1} \to K_i \otimes S$ inducing bijections in homology, such that K_0 is the complex K itself. There is no obstruction to lift β through K_1 by a map $\beta_0: K \to K_1$. By induction we can construct maps $\beta_n: K_n \to K_{n+1}$, compatible with the projections $K_{n+1} \to K_n \otimes S$.



Up to adding big acyclic complexes to complexes K_i, we may suppose that all maps β_n are cofibrations.

Let K'_p be the homotopy kernel (i. e. the desuspension of the mapping-cone) of the map $\beta_{p-1} \circ \beta_{p-2} \circ \dots \circ \beta_0$. We have obvious cofibrations λ'_p from K'_p to K'_{p+1} and β'_p from K'_{p+1} to $K'_p \otimes S$. It is clear that C'_0 is acyclic and that maps β' and λ' are compatible. So we get a $\mathcal{D}_0$-complex $DY = (K'_*, \lambda'_*, \beta'_*)$ and a homology equivalence from $T\mathfrak{N}il(DY)$ to Y. Moreover, for every p the following square is homology cartesian:

$$\begin{array}{ccc} K'_p & \longrightarrow & K \\ \big\downarrow {\scriptstyle \beta'^{p}} & & \big\downarrow {\scriptstyle \beta^{p}} \\ K'_0 \otimes S^p & \longrightarrow & K \otimes S^p \end{array}$$

A similar construction may be done for X. It is possible to construct A-e-complexes C_{ij}, surjective maps $\varepsilon_{ij}: C_{ij+1} \to C_{ij} \otimes S$ inducing bijections in homology, cofibrations $\lambda_{ij}: C_{ij} \to C_{i+1j}$, maps α_{ij} from C_{i+1j} to C_{ij+1}, maps f_{ij} from C_{ij} to K_j such that $C_{i0} = C_i$, $\lambda_{i0} = \lambda_i$, for every $i \geqslant 0$, and:

$$\alpha_{i+1j}\lambda_{i+1j} = \lambda_{ij+1}\alpha_{ij} \qquad \varepsilon_{i+1j}\lambda_{ij+1} = \lambda_{ij}\varepsilon_{ij} \qquad \varepsilon_{ij+1}\alpha_{ij+1} = \alpha_{ij}\varepsilon_{i+1j}$$

$$f_{i+1j}\lambda_{ij} = f_{ij} \qquad f_{ij+1}\alpha_{ij} = \beta_j f_{i+1j} \qquad f_{ij}\varepsilon_{ij} = \pi_j f_{ij+1}$$

for every $i\ j \geqslant 0$.

If all $C_{i'j'}$, all $\varepsilon_{i'j'-1}$, all $\lambda_{i'-1j'}$, all $\alpha_{i'-1j'-1}$, all $f_{i'j'}$ are defined, for all $i' < i$ and, if $i' = i$, for all $j' < j$, one constructs next data as follows:

Let E be the pull-back K_j and $C_{ij-1} \otimes S$ over $K_{j-1} \otimes S$. We can construct a complex C_{ij} C_{ij} and a surjective map $C_{ij} \to E$ inducing a bijection in homology. So we get composite maps $\varepsilon_{ij-1}: C_{ij} \to E \to C_{ij-1} \otimes S$ and $f_{ij}: C_{ij} \to E \to K_j$. The preceding data define maps from C_{i-1j} and C_{i+1j-1} to E, and there is no obstruction to lift these maps into maps $\lambda_{i-1j}: C_{i-1j} \to C_{ij}$ and $\alpha_{ij-1}: C_{i+1j-1} \to C_{ij}$. If C_{ij} is chosen to be big enough, λ_{i-1j} can be construted to be a cofibration.

Let C'_p be the homotopy kernel of the map $\alpha_{0p-1} \circ \alpha_{1p-2} \circ \dots \circ \alpha_{p-10}: C_{p0} \to C_{0p}$. The maps λ_{ij} and α_{ij} induce cofibrations λ'_p from C'_p to C'_{p+1} and maps α'_p from

C'_p to $C'_p \otimes S$ and we get a $\mathcal{D}_0$-complex $DX = (C'_*, \lambda'_*, \alpha'_*)$ and a canonical epimorphism from DX to X, and a map f_{**} from DX to DY. Moreover maps α'_* induce an epimorphism from $\mathrm{Ker}(C'_p \to C_p)$ to $\mathrm{Ker}(C'_{p-1} \to C_{p-1}) \otimes S$.

Let E_p be the kernel of the map $C'_p \to C_p$. The complex E_p is acyclic and we may identify C'_p with the sum $C_p \oplus E_p$. In this situation, the maps λ'_p and α'_p are represented by matrices:

$$\lambda'_p = \begin{bmatrix} \lambda_p & 0 \\ u_p & x_p \end{bmatrix} \qquad \alpha'_p = \begin{bmatrix} \alpha_p & 0 \\ v_p & y_p \end{bmatrix}$$

A section s_p of the map $C'_p \to C_p$ is the sum of the inclusion in the first factor and a map γ_p from C_p to E_p. Moreover theses maps commutes with the λ'_p's and the α'_p's if and only if:

$$\forall p \geqslant 0, \; u_p + x_p \gamma_p = \gamma_{p+1} \lambda_p \qquad \text{and} \qquad v_p + y_p \gamma_{p+1} = \gamma_p \alpha_p$$

Since the maps λ_p are cofibrations and the maps α_p are epimorphism with acyclic kernel, there is no obstruction to construct inductively the maps γ_p with the desired conditions. Therefore with have a cofibration s from X to DX and the following diagram commutes:

$$\begin{array}{ccc} & & T\mathfrak{N}il(DY) \\ & \nearrow^{T\mathfrak{N}il(f_{**}s)} & \downarrow \sim \\ T\mathfrak{N}il(X) & \longrightarrow & Y \end{array}$$

Since Y is in $\mathcal{C}$, there exists a $\mathcal{D}_0$-complex X_0 in $\mathcal{L}$ and a homology equivalence ε from $T\mathfrak{N}il(X_0)$ to Y. As above there exists a map u from X_0 to DY such that ε is the map $T\mathfrak{N}il(u)$ composed with the homology equivalence $T\mathfrak{N}il(DY) \to Y$. Since squares (*) are homology cartesian for both complexes X_0 and DY, the map u is a homology equivalence and DY is in $\mathcal{L}$.

The last condition of the approximation lemma 3-7 is now verified and this lemma holds. Therefore $\mathcal{C}$ is exact and $K(\mathcal{B}, \mathcal{A})$ has the homotopy type of the spectrum $K(\mathcal{C})$. More precisely we have a fibration of spectra:

$$K(\mathcal{A}) \to K(\mathcal{B}) \to K(\mathcal{C})$$

and the map on the right hand side is induced by the functor $T\mathfrak{N}il$.

Consider a $\mathcal{D}''$-complex $X = (C, \alpha)$ in the class $\mathfrak{N}il(\mathcal{C}_A; S)$. Since C has the homotopy type of a finite complex, there exists a homology equivalence from a $\mathcal{D}''$-complex $X_0 = (C_0, \alpha_0)$ to X, where C_0 is finite. Then X_0 is a nilpotent complex in the sense of §2 and, by theorem 2-2, there exists a nilpotent complex X'_0 and a homology equivalence from a reducible nilpotent complex Y to $X_0 \oplus X'_0$. But the reducible nilpotent complex Y is extension of nilpotent complexes $Y_i = (C_i, \alpha_i)$ where α_i is null-homotopic. Then all Y_i's are in the class $\mathcal{C}$, and Y is in $\mathcal{C}$ too. Therefore $X \oplus X'_0$ is in the class $\mathcal{C}$ and the lemma is a direct consequence of lemma 3-6.

<u>Lemma</u> 5-7: There is a fibration of spectra:

$$K(\mathcal{A}) \to K(\mathcal{B}) \to K(\mathcal{C}_A)$$

and the map on the right hand side is given by the functor T.

This lemma will be proven later.

Because of this lemma the forgetfull functor induces a split fibration

$$K(\mathfrak{N}il(\mathcal{C}_A; S) \to K(\mathcal{C}_A)$$

with K(G, 0) fiber. Therefore the spectrum $\tilde{K}\mathfrak{N}il(\mathcal{C}_A; S)$ is an Eilenberg-MacLane spectrum and the spectrum $\tilde{K}\mathfrak{N}il(A; S)$ has trivial homotopy groups in positive degree.

On the other hand for every ring R, there exist short exact sequences:

$$0 \to K_i(R) \to K_i(R[t]) \oplus K_i(R[t^{-1}]) \to K_i(R[t, t^{-1}]) \to K_{i-1}(R) \to 0$$

If we apply that for both rings A and A[S], we get corresponding exact sequences for Nil-terms. Therefore for every p there exists a surjection

$$\pi_i(\tilde{K}\mathfrak{N}il(A[\mathbb{Z}^p]; S[\mathbb{Z}^p])) \twoheadrightarrow \pi_{i-p}(\tilde{K}\mathfrak{N}il(A; S))$$

But if A is regular, $A[\mathbb{Z}^p]$ is regular for every p (see the apendix). Hence $\widetilde{K}\mathfrak{N}il(A; S)$ has trivial homotopy groups and the theorem is proven.

<u>Proof of lemma 5-7</u>: Since the spectrum $K(\mathcal{B}, \mathcal{A})$ is the limit of spectra $K(\mathcal{B}_n, \mathcal{A}_n)$, $n \to \infty$, it is enough to prove that T induces a homotopy equivalence from $K(\mathcal{B}_n, \mathcal{A}_n)$ to $K(\mathcal{C}_A)$, for every n>0. If n=1, $\mathcal{A}_1$ is the class of all acyclic $\mathcal{D}_0$-complexes. It is easy to chek the conditions of the approximation lemma 3-7 for the functor T from $\mathcal{B}_1$ to $\mathcal{C}_A$ and T induces a homotopy equivalence from $K(\mathcal{B}_1, \mathcal{A}_1)$ to $K(\mathcal{C}_A)$. Thus it is enough to prove that the inclusion functor induces a homotopy equivalence from $K(\mathcal{B}_n, \mathcal{A}_n)$ to $K(\mathcal{B}_{n+1}, \mathcal{A}_{n+1})$ for every $n > 0$. But that is equivalent to prove that the inclusion functor induces a homotopy equivalence from $K(\mathcal{A}_{n+1}, \mathcal{A}_n)$ to $K(\mathcal{B}_{n+1}, \mathcal{B}_n)$ for every $n > 0$ and that will be a consequence of:

<u>Lemma</u> 5-8: The functor:

$$(C_*, \lambda_*, \alpha_*) \mapsto C_{n+1}/C_n$$

induces a homotopy equivalence from $K(\mathcal{B}_{n+1}, \mathcal{B}_n)$ to $K(\mathcal{C}_A)$ and from $K(\mathcal{A}_{n+1}, \mathcal{A}_n)$ to $K(\mathcal{C}_A)$.

<u>Proof</u>: Let $\mathcal{E}_n$ be the class of all $\mathcal{D}_0$-complexes $(C_*, \lambda_*, \alpha_*)$ in $\mathcal{B}_n$ such that C_i is acyclic for all i<n. Let $\mathcal{L}\mathcal{B}_n$ be the class of all $\mathcal{B}_n$-local $\mathcal{D}_0$-complexes.

Let $X = (C_*, \lambda_*, \alpha_*)$ be a reduced $\mathcal{D}_0$-complex in $\mathcal{L}\mathcal{B}_n$. Let m<n. Consider the following $\mathcal{D}_0$-complex $X_0 = (C'_*, \lambda'_*, \alpha'_*)$:

$$\alpha'_* = 0 \qquad C'_i = 0 \text{ if } i<m \qquad C'_i = A \text{ and } \lambda'_i \text{ is the identity if } i \geq m$$

A morphism from X_0 to X is just a cycle in the kernel of α_m. Then it is easy to see that $\operatorname{Ker}\alpha_i$ are acyclic for every $i \leq n$. Therefore if X is no more reduced but only $\mathcal{B}_n$-local, α_i induce bijections in homology for $i \leq n$. Conversely a $\mathcal{D}_0$-complex $(C_*, \lambda_*, \alpha_*)$ is in $\mathcal{L}\mathcal{B}_n$ if and only if and only if α_i induce bijections in homology for $i \leq n$.

Therefore the class $\mathcal{L}$ of all $\mathcal{B}_n$-local $\mathcal{D}_0$-complexes X such that there exists a $\overline{\mathcal{B}}_n$-equivalence from a complex in $\mathcal{B}_{n+1}$ to X is the class $\mathcal{E}_{n+1}$, and we have a fibration of spectra:

$$K(\mathcal{B}_n) \to K(\mathcal{B}_{n+1}) \to K(\mathcal{E}_{n+1})$$

Since $\mathcal{E}_{n+1}$ is included in $\mathcal{B}_{n+1}$, this fibration split. Moreover, by the approximation lemma 3-7, the functor $(C_*, \lambda_*, \alpha_*) \mapsto C_{n+1}$ induces a homotopy equivalence:

$$K(\mathcal{E}_{n+1}) \to K(\mathcal{C}_A)$$

But there is a natural homotopy equivalence for every $(C_*, \lambda_*, \alpha_*)$ in $\mathcal{E}_{n+1}$: $C_{n+1} \to C_{n+1}/C_n$. Therefore the homotopy equivalence from $K(\mathcal{E}_{n+1})$ to $K(\mathcal{C}_A)$ may be defined by $(C_*, \lambda_*, \alpha_*) \mapsto C_{n+1}/C_n$, and the the first part of the lemma is proven.

On the other hand, for every $(C_*, \lambda_*, \alpha_*)$ in $\mathcal{A}_{n+1}$, we have a natural short exact sequence:

$$0 \to C_n \to C_{n+1} \to C_{n+1}/C_n \to 0$$

and C_{n+1} is acyclic. Hence the map above from $K(\mathcal{A}_{n+1}, \mathcal{A}_n)$ to $K(\mathcal{C}_A)$ is the opposite of the map given by the functor:

$$(C_*, \lambda_*, \alpha_*) \mapsto C_n$$