ON SIMPLY-CONNECTED 4-MANIFOLDS

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This paper concerns (but does not succeed in performing) the diffeomorphism classification of closed, oriented, differential, simply-connected 4-manifolds. It arises out of the observation (due to Pontrjagin and Milnor [2]) that if two such manifolds M_1 and M_2 have isomorphic quadratic forms of intersection numbers on $H_2(M_i)$, then there is a map $f: M_1 \to M_2$ which is a homotopy equivalence and induces the tangent bundle of M_1 from that of M_2 (see below). The same result is also known to follow from h-cobordism of M_1 to M_2 [4]. This suggests

THEOREM 2. *Two simply-connected closed 4-manifolds with isomorphic quadratic forms are h-cobordant.*

This is our main result. We then use techniques of Smale [6]; although the "h-cobordism theorem", that h-cobordant manifolds are diffeomorphic, cannot yet be proved in dimension 4, we obtain

THEOREM 3. *If M_1, M_2 are h-cobordant simply-connected 4-manifolds, then for some k, $M_1 \# k(S^2 \times S^2) \cong M_2 \# k(S^2 \times S^2)$.*

Here k denotes k copies, and $\#$ connected sum. We obtain a number of corollaries; for example, the Grothendieck group of oriented simply-connected 4-manifolds is the free abelian group on P and Q, the complex projective plane with two orientations.

The proof that if the quadratic forms of M_1 and M_2 are isomorphic, they have the same homotopy type is due to Milnor—essentially, we have CW-complexes with one cell in dimensions 0 and 4, and several in dimension 2. The homotopy type is determined by the homotopy class of the attaching map of S^3 to a bouquet of 2-spheres, and this likewise is determined by the integers which are the coefficients of the quadratic form. Now the induced tangent bundle over a 2-cycle x is determined by an integer (in fact $x.x$) modulo 2; and given an SO_4-bundle over the 2-skeleton, it is determined over the whole of M by an element of $\pi_4(SO_4) \cong \mathbf{Z} \oplus \mathbf{Z}$; essentially by the Euler class and the Pontrjagin class. But the former is determined by the rank of $H_2(M)$; the latter is three times the signature.

1. *Manifolds with zero signature.*

By a result of Thom [7], an oriented 4-manifold with zero Pontrjagin class (or signature) bounds. We need a slight refinement of this.

Received 13 March, 1963.

[JOURNAL LONDON MATH. SOC., 39 (1964), 141–149]

LEMMA 1. *Suppose $\sigma(M^4) = 0$ and $W_2(M^4) = 0$. Then M^4 bounds an orientable manifold V^5 with $W_2(V^5) = 0$.*

Proof. According to Thom (*loc. cit.*) the spinor cobordism group in dimension 4, Ω^4 (Spin), is given by the stable groups $\pi_{N+4}\left(M(\mathrm{Spin}_N)\right)$. By a result of Adams [**1**], such groups are the end of a spectral sequence which starts with

$$\mathrm{Ext}^{**}_{A_2}\left(\mathbf{Z}_2, H^*\left(M(\mathrm{Spin}_N); \mathbf{Z}_2\right)\right).$$

For, by Milnor [**3**], these groups have no odd torsion. A straightforward calculation of the Ext groups in low dimensions now shows that Ω^4 (Spin) maps monomorphically into Ω^4, which is equivalent to the stated result.

THEOREM 1. *Let M^4 be simply-connected, and $\sigma(M^4) = 0$. Then M^4 bounds a manifold W^5 of the homotopy type of a bouquet of 2-spheres.*

Proof. We know that M^4 bounds an orientable manifold V^5, which we can take as a spinor manifold if M^4 is. Then (Milnor [**5**], Theorem 3) we can perform a series of spherical modifications to make V^5 simply-connected, and not destroy the condition: $W_2(M^4) = 0$ implies $W_2(V^5) = 0$. Then the homology exact sequence of the pair (V, M) has the form

$$0 \to H_3(V) \to H_3(V, M) \to H_2(M) \to H_2(V) \to H_2(V, M) \to 0,$$

and our task is to kill the group $H_2(V, M)$. First suppose it infinite; we choose an element x of infinite order, and an element y of $H_2(V)$ mapping onto x. If the value of W_2 on y is nonzero, by the condition above $W_2(M^4) \neq 0$, and we choose z in $H_2(M)$ with the value of W_2 on z nonzero, and add the image of z to y. Hence we can suppose that W_2 is zero on y.

We now represent y by an imbedding of S^2 in V, and since $W_2(y) = 0$, the image has trivial normal bundle. Make a spherical modification of V by first deleting a tubular neighbourhood $S^2 \times D^3$ to obtain a manifold X, and replacing it by $D^3 \times S^2$ to obtain a new manifold V', still with boundary M. Using Lemma 5 of [**5**], we see that we can still suppose $W_2(V') = 0$ in the case when $W_2(M) = 0$. To calculate the homology of V' modulo M, we consider the sequence

$$(S) \quad H_3(V, M) \xrightarrow{\alpha} H_3(V, X) \xrightarrow{\beta} H_2(X, M) \to H_2(V, M) \to 0$$

where, of course, the second term is infinite cyclic. Now since α represents intersection with y, and y has infinite order, the cokernel of α is finite, and so $H_2(X, M)$ is a finite extension of $H_2(V, M)$. Now consider the sequence (S') with V' in place of V; then the image of β maps to the element x in $H_2(V, M)$, so has infinite order. Thus the rank of $H_2(V', M)$ is one less than that of $H_2(X, M)$, *i.e.* of $H_2(V, M)$; by induction we may reduce this rank to zero.

We may now suppose $H_2(V, M)$ finite and nonzero. We shall also assume $H_2(V)$ infinite. Indeed, since the homomorphisms

$$H_3(V, M) \to H_2(M) \quad \text{and} \quad H_2(M) \to H_2(V)$$

correspond by Lefschetz duality and the universal coefficient theorem, they have the same rank; since they form an exact sequence, this rank is half that of $H_2(M)$. We temporarily impose the restriction that the rank of $H_2(M)$ be at least 4, and so that of $H_2(V)$ at least 2.

There exists an element y in $H_2(V)$ satisfying (i) $W_2(y) = 0$, (ii) the image of y in $H_2(V, M)$ is nonzero and (iii) y is a primitive element of $H_2(V)$. By (iii) we mean that the image of y in $H_2{}^{\omega}(V)$ (the torsion-free homology group) is indivisible and so generates a direct summand. To prove the existence of y, first suppose $W_2(M) = 0$. Then $W_2(V) = 0$, and condition (i) is void. If each primitive y in $H_2(V)$ has zero image in $H_2(V, M)$, the map of $H_2(V)$ to $H_2(V, M)$ is zero since $H_2(V)$, being infinite, is generated by primitive elements. But this contradicts the fact that $H_2(V)$ maps onto $H_2(V, M)$, and the assumption that the latter is nonzero.

If now $W_2(V) \neq 0$, the subgroup of $H_2(V)$ generated by primitive elements on which W_2 vanishes has index 2. For now the rank of $H_2(V)$ is at least 2, and we can choose a basis $\{x_i\}$ for the torsion free part with $W_2(x_i) = 0$ for $i > 1$ (and also $W_2(x_1) = 0$, unless W_2 vanishes on the torsion subgroup), which makes the remark obvious. But now if every y in this subgroup has zero image in $H_2(V, M)$, this subgroup is exactly the kernel, hence the image of $H_2(M)$. This contradicts the assumption that W_2 is not identically zero on this image. So the existence of y satisfying (i)–(iii) is established.

Perform a spherical modification (as above) starting with y. In the sequence (S), since y is indivisible, α is onto, and it follows that

$$H_2(X, M) \cong H_2(V, M).$$

Now the sequence (S') shows that in $H_2(V', M)$ we have killed the image of y, and so decreased the order of the group. Hence by induction we may reduce the group to zero. We then obtain a simply-connected manifold W whose only nonzero homology group is $H_2(W)$, which is free $\left[\text{being isomorphic to } H^3(W, M) \cong \operatorname{Hom}\left(H_3(W, M), \mathbf{Z}\right)\right]$. Thus W has the homotopy type of a bouquet of 2-spheres.

In the case when the rank of $H_2(M)$ is zero, M is a homotopy 4-sphere, and a proof of the theorem may be found in [9]. The proof when the rank is 2 we defer till the next theorem.

LEMMA 2. *Let $\partial W^5 = M^4$, where W and M are 1-connected. and W has the homotopy type of a bouquet of k 2-spheres. Then* (i) *W admits a handlebody $H \in \mathscr{H}(5, k, 2)$ as deformation retract.* (ii) *The closure C of $W - H$ gives an h-cobordism of M to ∂H.*

Proof. For the definition of handlebodies see [**6**]. To construct H we first imbed a disc D^5 in the interior of W, and then imbed discs D_i^2 with interiors avoiding D^5 and boundaries lying on it, such that their homology classes represent generators of $H_2(W)$. We take the D_i^2 disjoint (in these dimensions, the imbedding is easy). A smooth neighbourhood of $D^5 \cup U_{i=1}^k D_i^2$ now gives the required handlebody H. For W, H are simply-connected, and by construction, the inclusion of H in W is a homology equivalence.

For (ii) observe that since W is simply-connected, and the codimension of a disc D_i^2 is 3, C is also simply-connected. Now

$$H_i(C, \partial H) \cong H_i(W, H) = 0,$$

so ∂H is a deformation retract of C, and

$$H_i(C, M) \cong H^{5-i}(C, \partial H) = 0,$$

so M also is. This completes the proof of the lemma.

2. *The main theorem.*

We are now ready to prove

THEOREM 2. *Two simply-connected closed 4-manifolds with isomorphic quadratic forms are h-cobordant.*

Proof. Let the manifolds be M_1 and M_2. Form the connected sum $N = M_1 \# (-M_2)$. Since the signatures of M_1 and M_2 are equal, that of N is zero. First assume the rank of $H_2(M_1)$ not equal to 1, so that the part of Theorem 1 already proved does apply to N; by Theorem 1 and Lemma 2, N is h-cobordant to a handlebody boundary ∂V.

Now we know a good deal about handlebody boundaries. Indeed, a handlebody V in $\mathscr{H}(5, k, 2)$ is determined by k and whether $W_2(V)$ is zero or not (see *e.g.* [**10**]); and all such handlebodies are sums of ones in $\mathscr{H}(5, 1, 2)$, which are 3-disc bundles over S^2, and there are only two of these. Hence a handlebody boundary is a connected sum of 2-sphere bundles over S^2. We now refer to [**11**]. There are two such bundles, the product $S = S^2 \times S^2$ and a bundle T. The result we need from [**11**] is the Corollary to Theorem 2, in the form:

Let M be a connected sum of copies of S and T. Then any automorph of $H_2(M)$ can be represented by a diffeomorphism.

The term automorph refers to the quadratic form defined on $H_2(M)$ by intersection numbers.

Write K for the subgroup of $H_2(\partial V) \cong H_2(N) \cong H_2(M_1) \oplus H_2(M_2)$ given by the set of pairs (x, y) with $\alpha x = y$, where α is the given isomorphism of the quadratic forms on $H_2(M_1)$, $H_2(M_2)$. Clearly, K is a free abelian group, in fact a direct summand of $H_2(\partial V)$, with rank equal to that of

$H_2(M_1)$, and so half that of $H_2(\partial V)$. Moreover, the subgroup K is isotropic for if (x, y), $(x', y') \in K$, their intersection number is

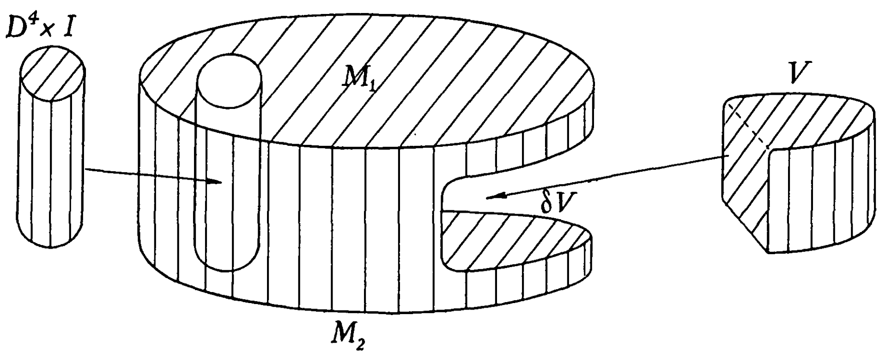
$$xx' - yy' = xx' - \alpha(x)\alpha(x') = 0,$$

[recall (a) that we reversed the orientation of M_2, (b) that α is an isomorphism]. This suggests that we can find a handlebody W, with $\partial W = \partial V$, and K the kernel of the inclusion $H_2(\partial V) \to H_2(W)$.

In fact let L be the kernel of $H_2(\partial V) \to H_2(V)$; then L satisfies the same conditions as K. We assert that for some automorph T of $H = H_2(\partial V)$, $T(L) = K$. This is quite a simple result, and we briefly sketch the proof. The stated conditions easily imply that L (or K) is its own annihilator in H. So L is the kernel of the map induced by the quadratic form

$$H \cong \mathrm{Hom}(H, \mathbf{Z}) \to \mathrm{Hom}(L, \mathbf{Z})$$

and if L' is a complement to L, we may identify L' with the dual of L. We choose dual bases $\{e_i\}$ in L, $\{e_i'\}$ in L', then $e_i e_j = 0$, $e_i e_j' = \delta_{ij}$. Then we must try varying the choice, by adding to each e_i' a linear combination of the e_j; when the quadratic form is even, we can make $e_i' e_j' = 0$, also; when it is odd, with a little more trouble, we obtain $e_i' e_j' = 0$ except $e_1' e_1' = 1$. Since the same considerations are valid for K, we may now use the chosen bases to define an isomorphism which carries L to K.



By the result quoted above, there is a diffeomorphism of ∂V which induces this isomorphism. So without loss of generality we can assume $K = L$, the kernel of $H_2(\partial V) \to H_2(V)$. We now take the h-cobordism of $M_1 \# (-M_2)$ to ∂V, and "fill in" ∂V by attaching V. Likewise, we can regard the connected sum as defined as the union

$$(M_1 - \mathring{D}^4) \cup (S^3 \times I) \cup (M_2 - \mathring{D}_4)$$

and fill in $S^3 \times I$ by attaching $D^4 \times I$. (See figure.) We have now constructed a manifold R whose boundary components are M_1 and M_2; we claim that R is an h-cobordism. First, it is clear from construction that R is simply-connected.

Now we calculate what has happened to the second homology group. We have already observed that

$$H_2(M_1) \oplus H_2(M_2) \cong H_2(N) \cong H_2(\partial V),$$

and this is clearly unaffected by filling in $D^4 \times I$. If we attach V, the kernel of $H_2(\partial V) \to H_2(V)$ is $L = K$, the set of pairs (x, y), $x \in H_2(M_1)$, $y \in H_2(M_2)$, with $\alpha x = y$. Since K is disjoint from $H_2(M_1)$, $H_2(M_2)$, we see that the induced maps $H_2(M_1) \to H_2(R)$ and $H_2(M_2) \to H_2(R)$ are isomorphisms, and indeed that the composite induces the isomorphism $-\alpha$ of $H_2(M_1)$ on $H_2(M_2)$.

Hence $H_k(R, M_i) = 0$ for $k \leqslant 2$ and $i = 1, 2$, and so also

$$0 = H^k(R, M_i) = H_{5-k}(R, M_{3-i}),$$

so that all the relative homology groups vanish, and R is indeed an h-cobordism.

We now return to the unsettled case of Theorem 1. If the rank of $H_2(M)$ is equal to 2, and $\sigma(M) = 0$, then the quadratic form of M has one of two types and these are the quadratic forms of S and T. By the result above M is h-cobordant to S or T, and then filling in the 2-sphere bundle $S(T)$ by the 3-disc bundle, we obtain the required manifold with boundary M. Having shown this, observe that the proof of Theorem 2 is now valid in the case (previously exceptional) when the rank of $H_2(M_1)$ is unity. Thus both theorems are established in full generality.

3. *h-cobordism.*

As we have already observed, Smale's proof of his h-cobordism theorem [6] breaks down for dimension 4. However, if we examine it, we find that a non-trivial result is nevertheless obtained.

THEOREM 3. *If M_1, M_2 are h-cobordant simply-connected 4-manifolds, then for some k, $M_1 \# k(S^2 \times S^2) \cong M_2 \# k(S^2 \times S^2)$.*

Proof. Let R be the 5-manifold providing the h-cobordism. We take a nice function on R (with minimum, $-\frac{1}{2}$, on M_1, and maximum, $5\frac{1}{2}$, on M_2); this yields a handle decomposition. Since the manifolds are connected, we can dispense with 0-handles (viz., those which correspond to critical points of index 0). Now (5.1) of [6] is valid in our case, and the argument given by Smale on pp. 404-5 also applies, and enables us to dispense with 1-handles. Dually, we can get rid of 5- and 4-handles. So there remain only handles of dimensions 2 and 3 (we remind the reader that the reason why Smale's arguments fail in our case is a lack of information on isotopies of imbeddings of 2-spheres in simply-connected 4-manifolds—a lack which no arguments in [11] or this paper have availed to circumvent).

Now denote by N the manifold at level $2\frac{1}{2}$. To pass from M_1 to N we have crossed a number, say k, of 2-handles. Hence N is derived from M_1 by k spherical modifications of type (1, 2)—in each we delete the interior of an imbedded $S^1 \times D^3$, and replace by a copy of $D^2 \times S^2$. But M_1 is simply-connected, and hence the circle $S^1 \times 0$ homotopic (and so diffeotopic) to zero. It follows, as in [11] ,that making a spherical modification is equivalent to taking the connected sum with a 2-sphere bundle over S^2, S or T. Hence N is obtained from M_1 by taking the connected sum with k copies of S or T. Similarly, it is so obtained from M_2, and by homology theory, we see that the same integer k is obtained.

To conclude our argument it remains only to observe that when the quadratic form of M_1 (hence also M_2) is odd, we have, by Theorem 1, Corollary 1 of [11], $M_1 \# T \cong M_1 \# S$, (and similarly for M_2), so that in this case all the summands T may be replaced by S. If, on the other hand, the quadratic forms are even, *i.e.* $W_2(M_1) = 0$, then since R admits M_1 as deformation retract, it follows that $W_2(R) = 0$; and since N has trivial normal bundle in R that $W_2(N) = 0$, so the quadratic form on N is also even. Thus in this case no summands T can occur.

We remark that our result is a pure existence theorem; we have obtained, even in principle, no bound whatever on the integer k. The natural conjecture to make is, of course, that the h-cobordism theorem still holds, and we can take $k = 0$. If this is so, then two simply-connected 4-manifolds with the same quadratic form are diffeomorphic.

4. *Deductions.*

We take the following oriented simply-connected 4-manifolds as basic: the complex projective plane with the usual orientation, P, or the other orientation, Q; $S = S^2 \times S^2$, and the other bundle T. By [11] Lemma 1, $T \cong P \# Q$.

(4.1) *If M_1, M_2 have isomorphic quadratic forms, then for some k, $M_1 \# kS \cong M_2 \# kS$.*

This follows at once from Theorems 2 and 3.

(4.2) *If M_1, M_2 have quadratic forms of the same genus, then for some k, $M_1 \# kS \cong M_2 \# kS$.*

For (see *e.g.* [2] or [8]), two indefinite forms of the same genus are isomorphic, and we can apply (4.1) to $M_1 \# S$ and $M_2 \# S$. Indeed, the converse of (4.2) is also true—this is essentially a statement about quadratic forms (same references).

(4.3) *There exists N with $M_1 \# N \cong M_2 \# N$ if and only if the $H_2(M_i)$ have the same rank and signature.*

The existence of N clearly implies the second condition; conversely, if that is satisfied, then $M_1 \# T$ and $M_2 \# T$ have quadratic forms of the same genus (even isomorphic) and we apply (4.2) or (4.1).

(4.4) *The Grothendieck group of simply-connected 4-manifolds is free abelian of rank 2, the classes of P and Q may be taken as generators.*

Taking rank r and signature σ clearly defines a homomorphism of the Grothendieck group into $2\mathbf{Z}$. By (4.3) this is a monomorphism. Now $r \equiv \sigma \pmod 2$ but P and Q have (r, σ) equal to $(1, 1)$ and $(1, -1)$ so the image is defined by $r \equiv \sigma \pmod 2$, and is generated by the images of P and Q.

A corresponding argument is valid if the quadratic form is restricted to be even (*i.e.* $W_2 = 0$). Let K be such a manifold with signature 16 (see [**2**]).

(4.5) *The Grothendieck group of simply-connected spinor 4-manifolds is free abelian of rank 2; the classes of S and K may be taken as generators.*

Recall that even quadratic forms have the same genus if and only if they have the same rank and signature. Then using (4.2) instead of (4.3) we see that (r, σ) again defines a monomorphism. But now r is even and, by a result of Rohlin, σ is divisible by 16. Since S has $(r, \sigma) = (2, 0)$, the result follows.

These results are of course much weaker than (4.1), and *à fortiori,* than Theorem 2; nevertheless, they seem worth stating. We also have a sort of imbedding theorem.

(4.6) *For any M there exist N, k with $M \# N \cong kT$. If $W_2(M) = 0$, we can also have $M \# N \cong kS$.*

We merely apply (4.1) to $M \# (-M) \# T$ [or to $M \# (-M)$] and use results quoted above.

Finally we use (4.1) to settle a problem raised in [**11**].

(4.7) *For any integer b we can find n such that each primitive characteristic element of $H_2(nT)$ of square $16i$ is represented by an imbedded 2-sphere with simply-connected complement, provided $-b \leqslant i \leqslant b$.*

Using the methods of Lemma 6 of [**11**], we see that it is sufficient to express nT as $A \# kP \# lQ$, where the quadratic form of A is even, and $k - l = 16i$. For i negative (for i positive we can deduce the same by reversing orientation) say $i = -j$, we take $A = jK$, where K is the manifold of (4.5), and apply (4.1) to $jK \# 16jQ$ to deduce that for some m, we have $mT \cong jK \# (16j + k)Q \# kP$. A similar decomposition is now valid for $n > m$, so we choose n large enough to accommodate each of the cases mentioned.

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