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DELOOPING CLASSIFYING SPACES IN ALGEBRAIC K-THEORY

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FOR ANY associative ring with identity A let $BGL(A)^+$ denote the "classifying space" for algebraic K-theory given in [15] where a definition for the higher K-theory functors K_i, $i \geqslant 1$, is proposed by Quillen as $K_i(A) = \pi_i(BGL(A)^+)$. This K_1 and K_2 agree with the K_1 of Bass [2] and the K_2 of Milnor [14] and the theory has other very pleasant properties [15]. Now Anderson [1] and Segal [17] have shown how to associate a generalized cohomology theory $K^i(X; \mathscr{C})$ to any category $\mathscr{C}$ with a "commutative" and "associative" internal operation $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$. If one takes the category $\mathscr{P}$ of finitely generated projective modules over A with morphisms the isomorphisms and internal operation the direct sum, then $K_i(A) = K^{-i}(pt; \mathscr{P})$ for $i \geqslant 0$. See [18]. Our purpose here is to show that

$$K_0(A) \times BGL(A)^+ \cong \Omega(BGL(\mu A)^+)$$

where μA is the ring of "bounded operators modulo compact operators". This gives a more specific construction of the Ω-spectrum for algebraic K-theory. In §4 we show how to define relative K-groups $K_i(f)$ for a ring homomorphism $f\colon R \to S$ so that there is a long exact sequence

$$\ldots \to K_i(R) \to K_i(S) \to K_i(f) \to K_{i-1}(R) \to K_{i-1}(S) \to \ldots$$

This amounts to identifying, at least theoretically, the fiber of the map $BGL(R)^+ \to BGL(S)^+$.

The first six sections of this paper are concerned with algebraic K-theory; the last, which relies only on §1, §2, and (3.1), briefly speculates on Fredholm map germs in homotopy theory. We work with the B^+ construction of [15] which in our applications to algebraic K-theory is just the integral completion functor of [3].

Recall the definition of μA from [7]: Let ℓA denote the ring of *locally finite* matrices over A; that is, those infinite matrices (m_{ij}) with entries in A such that each row and each column has at most finitely many non-zero entries ($1 \leqslant i, j < \infty$). Let $mA \subset \ell A$ be the ideal of *finite* matrices; that is, those matrices with at most finitely many non-zero entries. Define $\mu A = \ell A/mA$. From an algebraic viewpoint the "cone" CA and "suspension" SA of [11] could be used in place of ℓA and μA in this paper. Recall that $CA \subset \ell A$ is the subring generated by the "permuting" matrices; that is, by those matrices of the form $P \cdot D$ where P is an infinite permutation matrix and D is an infinite diagonal matrix with entries coming from a finite subset of A. The suspension is defined as $SA = CA/mA$.

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In this paper Z will always denote the integers.

Many of the results in this paper have been obtained concurrently and independently by S. Gersten [8] who works with the cone and suspension.

I wish to thank F. T. Farrell for collaboration on the earlier work [7] which is the source of much in the present paper.

§1. GENERAL PROPERTIES OF B^+

First recall the definition of $K_i(R)$ given by Quillen in [15]: All spaces will have the homotopy type of a CW-complex and will have a base point. All maps and homotopies will preserve the base point. Now let X be a space and let $G \subset \pi_1 X$ be a perfect subgroup. There is a space X^+ and a map $i\colon X \to X^+$ such that:

(a) $\pi_1 X^+ \cong \pi_1(X)/\text{normal closure of } G$;

(b) For any $\pi_1(X^+)$-module A the map $i\colon X \to X^+$ induces an isomorphism $H_*(X; A) \to H_*(X^+; A)$.

One way to construct such a space X^+ is the following: Let Y be the covering space of X corresponding to the subgroup G. Attach 2-cells and 3-cells to Y to get a space Y^+ such that $\pi_1(Y^+) = 0$ and $Y \to Y^+$ induces an isomorphism $H_*(Y; Z) \to H_*(Y^+; Z)$. Then take X^+ to be the pushout of the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y^+ \arrow[d, dotted] \\ X \arrow[r, dotted] & X^+. \end{tikzcd} \tag{1.0}$$

Any map $i\colon X \to X^+$ satisfying (a) and (b) also satisfies the following universal property:

(c) Let $f\colon X \to Y$ be a map such that $f_\#(G) = 0$ where $f_\#$ is the induced map on π_1. Then there is a map $f^+\colon X^+ \to Y$, unique up to homotopy, which gives a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & X^+ \arrow[dl, dashed, "f^+"] \\ & Y & \end{tikzcd}$$

In particular if $f\colon X \to Y$ is a map such that $f_\#(G) \subset H$ where G and H are perfect subgroups of $\pi_1(X)$ and $\pi_1(Y)$ respectively and $i\colon X \to X^+$ and $j\colon Y \to Y^+$ satisfy (a) and (b) then there is a map $f^+\colon X^+ \to Y^+$, unique up to homotopy, that gives a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X^+ & \xrightarrow[f^+]{} & Y^+. \end{array}$$

Thus any two realizations of X^+ are homotopy equivalent in a natural way. Another useful property of the "+" construction is

(d) if Y and X are as in (1.0) and $G \subset \pi_1 X$ is normal then Y^+ is homotopy equivalent to the universal cover of X^+.

The following well-known lemma (cf. Lemma 6.2 of [5]) will be useful.

LEMMA 1.1. *Let $f: X \to Y$ be a map between weakly simple spaces (i.e. spaces whose fundamental groups act trivially on the homology of the universal cover). Suppose $f_*: H_*(X; Z) \to H_*(Y; Z)$ and $f_\#: \pi_1 X \to \pi_1 Y$ are isomorphisms. Then f is a homotopy equivalence.*
Note that any connected H-space is weakly simple because it is simple. For a very nice generalization of this Whitehead type theorem see [6].

In this paper a ring R will always be assumed to satisfy the condition

(*) for any finite set $r_1, \ldots, r_n \in R$ there is an idempotent $\rho \in R$ with $\rho \cdot r_i = r_i \cdot \rho = r_i$.
Any ring with identity satisfies (*). If R satisfies (*) so do mR, ℓR, and μR. In particular, if R has an identity, then mR satisfies (*) although it is not a ring with identity.

The general linear group $GL(n, R)$ can be defined as in [19] as follows: for any two $n \times n$-matrices P and Q over R let $P \circ Q = P + Q + P \cdot Q$. Then $GL(n, R)$ consists of those $n \times n$-matrices P for which there is a Q with $P \circ Q = Q \circ P = 0$. The group operation is $P \circ Q$. Let $E(n, R)$ be the subgroup generated by the $\delta_{ij}(r)$, which has r in the (i, j)th spot and zeroes elsewhere $(i \neq j)$. For $n \geqslant 3$, $E(n, R)$ is perfect since $[\delta_{ij}(\lambda), \delta_{jk}(\mu)] = \delta_{ik}(\lambda \cdot \mu)$. Let $GL(R) = \lim GL(n, R)$ and $E(R) = \lim E(n, R)$. Then as usual (i.e. when R has an identity) $E(R) = [GL(R), GL(R)]$. When R is a ring with unit the correspondence $P \to I + P$ defines an isomorphism between the $GL(n, R)$ as defined above and the usual group of $n \times n$ invertible matrices.

Now consider $GL(R)$ as a discrete topological group and form the classifying space $BGL(R)$ as in [18]. Form $BGL(R)^+$ using the perfect subgroup $E(R)$ and for $i \geqslant 1$ define as in [15]

$$K_i(R) = \pi_i(BGL(R)^+).$$

The (homotopy theoretic) universal cover is of $BGL(R)^+$ is $BE(R)^+$. Also, $BGL(R)^+$ is an H-space and is therefore simple. See (1.2) below.

In this paper the "+" construction will be applied more generally to the classifying space of a discrete topological group G which has an internal "direct sum"; that is, a homomorphism $\oplus: G \times G \to G$. Such a group will be called a *direct sum* group. $GL(R)$ is a direct sum group as follows: Partition the positive integers N into two disjoint infinite subsets $N = N_0 \cup N_1$ and choose bijections $\alpha: N \to N_0$ and $\beta: N \to N_1$. If $A = (a_{ij})$ and $B = (b_{ij})$ are in $GL(R)$ let

$$A \oplus B = (a_{\alpha(i), (j)}) \cdot (b_{\beta(i), \beta(j)}).$$

Now let G be any direct sum group such that $[G, G]$ is perfect and satisfies

(†) for $g_1, \ldots, g_n \in [G, G]$ and $g \in G$ there is an $h \in [G, G]$ with $gg_ig^{-1} = hh_ih^{-1}$ for $1 \leqslant i \leqslant n$.

Let e be the identity element of G. Suppose further that

(**) for any finite set $g_1, \ldots, g_n \in G$ there are elements c and d in G with $c(g_i \oplus e)c^{-1} = d(e \oplus g_i)d^{-1} = g_i$.

Form BG^+ with respect to the subgroup $[G, G]$, which we will do throughout the rest of the paper.

The group $GL(R)$ satisfies (†) and (**). For example, to see that (†) holds let $g_1, \ldots, g_n \in [GL(R), GL(R)]$ and let $g \in GL(R)$. Choose an integer k such that $g, g_1, \ldots, g_n \in GL(k, R)$. Then to get (†) we can choose

$$h = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \in E(2k, R).$$

Let G be a direct sum group satisfying (†) and (**) above and form BG^+ with respect to the subgroup $[G, G]$ which we have assumed to be perfect. Throughout the rest of the paper BG^+ will be constructed in this way.

PROPOSITION 1.2. *BG^+ is an H-space.*
For example, $BGL(R)^+$ is an H-space.

LEMMA 1.3. *Let f: $G \to G$ be an automorphism of the discrete group G such that for any set $g_1, \ldots, g_k \in G$ there is an element $h \in G$ such that $f(g_i) = hg_ih^{-1}$ for $1 \leqslant i \leqslant k$. Then $f_*: H_*(BG) \to H_*(BG)$ is the identity.*

Proof. Let $x \in H_n(BG)$ be represented by the chain $\Sigma n_i(g_1^i, \ldots, g_n^i)$. Then $f(x)$ is represented by

$$\sum_{i=1}^{m} n_i(f(g_1^i), \ldots, f(g_n^i)) = \sum n_i(h \cdot g_1^i \cdot h^{-1}, \ldots, h \cdot g_n^i \cdot h^{-1})$$

where the h is chosen with respect to the finite set g_j^i. Since conjugation induces the identity on $H_*(BG)$ we have $f(x) = x$.

Proof of 1.2. Let $E = [G, G]$. First note that BG^+ is a weakly simple space: By (1.3), $G \bmod[G, G]$ acts trivially on the homology of BE, a regular covering space of BG. The isomorphism $H_*(BE) \to H_*(BE^+)$ is compatible with the action of the covering translations and hence $G \bmod[G, G] = \pi_1 BG^+$ acts trivially on BE^+, the universal cover of BG^+. Similarly $B(G \times G)^+$ is a weakly simple space. Now let $\rho^+: B(G \times G)^+ \to BG^+ \times BG^+$ denote the map induced from the homeomorphism $\rho: B(G \times G) \to BG \times BG$ by the universal property (c). Since ρ induces an isomorphism on homology so does ρ^+ and this makes ρ^+ a homotopy equivalence in view of (1.1). Let δ be a homotopy inverse to ρ^+ and consider the map m induced by δ followed by $\oplus$:

$$m: BG^+ \times BG^+ \xrightarrow[\delta]{} B(G \times G)^+ \xrightarrow[\oplus]{} BG^+.$$

For m to be an H-space multiplication on BG^+ it must be homotopic to the identity (keeping the basepoint $p \in BG^+$ fixed) when restricted to the right and left factors of $BG^+ \times BG^+$. This may not be the case; however, if we set $r(x) = m(x, p)$ and $\ell(x) = m(p, x)$ for $x \in BG^+$, then the argument below shows that r and ℓ are at least homotopy equivalences. Choose homotopy inverses s and k for r and ℓ respectively. Then $m \circ (s \times r)$ is an H-space multiplication on BG^+. To see why, say, $r: BG^+ \to BG^+$ is a homotopy equivalence note that it is the map induced by the group homomorphism $\phi(g) = g \oplus e$. Property (**) and Lemma 1.3 show that $\phi(x) = x$ for any $x \in H_n(BG^+)$. Hence r is a homotopy equivalence. Similarly, ℓ is a homotopy equivalence.

Any homomorphism $\sigma: G_0 \to G_1$ of direct sum groups which preserves the direct sum operations induces an H-map $\sigma^+: BG_0{}^+ \to BG_1{}^+$. For any ring homomorphism $f: R \to S$ let $Bf^+: BGL(R)^+ \to BGL(S)^+$ denote the induced map.

LEMMA 1.4. *There is a natural homotopy equivalence*

$$BGL(R)^+ \xrightarrow[\simeq]{} BGL(mR)^+.$$

Proof. There is a natural isomorphism $GL(R) \cong GL(mR)$ because $mR \cong m(mR)$. Thus there is actually a homeomorphism $BGL(R)^+ \xrightarrow[\cong]{} BGL(mR)^+$.

§2. FLABBY GROUPS ARE ACYCLIC

One of the main applications of this section is to show in (2.5) that $H_n(BGL(\ell R)^+) = 0$ whenever $n > 0$ and hence that $BGL(\ell R)^+$ is contractible.

Let G be a discrete topological direct sum group satisfying (**). Following [11] we shall call G *flabby* provided there is a homomorphism $\tau: G \to G$ such that for any finite set $g_1, \ldots, g_n \in G$ there is a $c \in G$ such that

(***) $\quad c \cdot (g_i \oplus \tau(g_i)) \cdot c^{-1} = \tau(g_i).$

Examples below show that τ can be viewed as an infinite sum.

PROPOSITION 2.1. *If G is flabby, then $H_n(BG; Z) = 0$ for $n > 0$.*

COROLLARY 2.2. *If G is flabby, BG^+ is contractible.*

To prove (2.1) it suffices to show that the homology vanishes with coefficients in a field. Note the $\oplus: G \times G \to G$ induces a ring structure on $H_*(BG)$ where the multiplication, denoted by $\oplus$, is given by

$$H_*(BG) \otimes H_*(BG) \xrightarrow[\simeq]{} H_*(B(G \times G)) \xrightarrow[\oplus]{} H_*(BG).$$

The generator $1 \in H_0(BG)$ determined by the map of the standard 0-simplex to the base point is a unit for the multiplication. This is because for any $z \in H_n(BG)$ the correspondences $z \to z \oplus 1$ and $z \to 1 \oplus z$ are induced by the group homomorphisms $g \to g \oplus e$ and $g \to e \oplus g$. These latter induce the identity on homology by (1.3). Now let $\Delta_*: H_*(BG) \to H_*(BG) \otimes H_*(BG)$ be the algebraic diagonal map induced by the diagonal $\Delta: G \to G \times G$ and let $\tau_*: H_*(BG) \to H_*(BG)$ be the homomorphism induced by $\tau: G \to G$. Let $z \in H_n(BG)$. Then

$$\tau_*(z) = \oplus \circ (id \times \tau_*) \circ \Delta_*(z).$$

For let z be represented by the chain $\sum k_i(g_1^i, \ldots, g_n^i)$. The left hand side of the equation is represented by $\sum k_i(\tau(g_1^i), \ldots, \tau(g_o^i))$ and the right hand side is represented by $\sum k_i(g_1^i \oplus \tau(g_1^i), \ldots, g_n{}^i \oplus \tau(g_n^i))$. By (***) these chains are conjugate and therefore homologous.

Now to show that $H_n(BG) = 0$ when $n > 0$ proceed by induction and suppose that $H_i(BG) = 0$ for $0 < i < n$. Let $z \in H_n(BG)$. Then $\Delta_n(z) = z \otimes 1 + 1 \otimes z + \sum u_i \otimes v_i$ where $0 < \deg u_i < n$ and $0 < \deg v_i < n$. By the inductive assumption we know that $u_i = v_i = 0$. Hence

$$\begin{aligned}\tau_*(z) &= \oplus \circ (id \times \tau_*) \circ \Delta_*(z)\\ &= z \oplus 1 + 1 \oplus \tau_*(z)\\ &= z + \tau_*(z).\end{aligned}$$

This shows that $z = 0$ and completes the proof of (2.1).

A similar argument was used by J. Mather in [13] to show that the group of homeomorphisms of R^n with compact support (considered as a discrete group) has vanishing homology in positive dimensions.

Examples of direct sum groups and flabby groups

(2.3) For any ring R satisfying (*), $GL(R)$ is a direct sum group as in §1.

(2.4) Recall from [7] that an associative ring with identity R is a *sum-ring* provided there are elements $\alpha_0, \alpha_1, \beta_0, \beta_1 \in R$ such that

$$\alpha_0 \beta_0 = \alpha_1 \beta_1 = 1$$
$$\beta_0 \alpha_0 + \beta_1 \alpha_1 = 1.$$

Define the ring homomorphism $\oplus : R \times R \to R$ by

$$r \oplus s = \beta_0 r \alpha_0 + \beta_1 s \alpha_1$$

for $r, s \in R$.

Strictly speaking a sum-ring is a ring with a particular choice of α_i and β_i. Let R and R' be sum rings with respect to $\{\alpha_i, \beta_i\}$ and $\{\alpha_i', \beta_i'\}$ respectively. A morphism $f: R \to R'$ is an identity preserving ring homomorphism f such that $f(\alpha_i) = \alpha_i'$ and $f(\beta_i) = \beta_i'$. Suppose R is a sum-ring with respect to α_i and β_i and $f: R \to R'$ is an identity preserving ring homomorphism. Then R' is a sum-ring with respect to $\alpha_i' = f(\alpha_i)$ and $\beta_i' = f(\beta_i)$ and f becomes a morphism.

If R is a sum-ring, then $GL(R)$ is a direct sum group where

$$A \oplus B = (a_{ij} \oplus b_{ij})$$

when $A = (a_{ij})$ and $B = (b_{ij})$. We must see why the condition (**) is satisfied and in this case it sufficies to show that for A and $B \in GL(n, R)$ there is an invertible matrix $Q \in GL(3n, R)$ such that

$$Q^{-1}\cdot\begin{pmatrix} A\oplus B & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot Q=\begin{pmatrix} A & 0 & 0\\ 0 & B & 0\\ 0 & 0 & 1\end{pmatrix}\in GL(3n, R).$$

For $x = \alpha_i$ or β_i let $D_n(x)$ denote the $n \times n$ diagonal matrix with x along the diagonal. The right Q is then

$$Q=\begin{pmatrix} D_n(\beta_0) & D_n(\beta_1) & 0\\ 0 & 0 & D_n(\alpha_0)\\ 0 & 0 & D_n(\alpha_1)\end{pmatrix}$$

with

$$Q^{-1}=\begin{pmatrix} D_n(\alpha_0) & 0 & 0\\ D_n(\alpha_1) & 0 & 0\\ 0 & D_n(\beta_0) & D_n(\beta_1)\end{pmatrix}.$$

Any sum-ring R will be called an *infinite sum ring* provided there is an identity preserving ring homomorphism $\infty: R \to R$ such that $r \oplus r^\infty = r^\infty$ for any $r \in R$. A morphism $f: R \to R'$ will be required to satisfy the condition $f(r^\infty) = f(r)^\infty$. If R is an infinite sum ring, then $GL(R)$ is a flabby group where $\tau: GL(R) \to GL(R)$ is given by $\tau(A) = (a_{ij}^\infty)$ for $A = (a_{ij})$.

COROLLARY 2.5. *If Λ is an infinite sum ring, $BGL(\Lambda)^+$ is contractible.*

The ring ℓR is an infinite sum ring. To see this it will be convenient to identify ℓR with the ring $\ell_R(E)$ of locally finite R-linear transformations of the free right R-module E with countable basis $\{e_j^k\}$ where $1 \leqslant j, k < \infty$. Recall that $h: E \to E$ is locally finite provided that each e_i^k appears with a non-zero coefficient in $h(e_j^k)$ for at most finitely many e_j^k's. Now partition the basis $\{e_j^k\}$ into two disjoint infinite subsets $\{e_j^k\} = A_0 \cup A_1$. Let $\beta_i: \{e_j^k\} \to A_i$, $i = 0$ or 1, be any two bijections. Let $\beta_i \in \ell_R(E)$ denote the corresponding locally finite matrix. Define $\alpha_i \in \ell_R(E)$ for $i = 0$ or 1 by

$$\alpha_i(e_j^k)=\begin{cases}\beta_i^{-1}(e_j^k), \text{ if } e_j^k\in A_i\\ 0, \text{ otherwise.}\end{cases}$$

This gives a sum structure on $\ell_R(E)$ and hence on ℓR. There are many sum structures on $\ell_R(E)$ but the following is easy to use: choose β_0 to be any bijection of $\{e_j^k\}$, $1 \leqslant j < \infty$, onto $\{e_1^k\}$ and define β_1 by $\beta_1(e_j^k) = e_{j+1}^k$. Let α_0 and α_1 be as above. To make $\ell_R(E)$ into an infinite sum-ring recall that $E = \oplus E_j$ where E_j is the free submodule of E spanned by $\{e_j^k\}$, $1 \leqslant k < \infty$. Let $r \in \ell_R(E)$ and $e_j^k \in E$. Define

$$r^\infty(e_j^k) = \beta_1^{j-1}\beta_0 r\alpha_0\alpha_1^{j-1}(e_j^k).$$

Then r^∞ is just the infinite direct sum of copies of $\beta_0 r\alpha_0$ laid out on the E_j's. We have $r \oplus r^\infty = r^\infty$ because

$$r^\infty(e_j^k) = \beta_1^{j-1}\beta_0 r\alpha_0\alpha_1^{j-1}(e_j^k)$$

$$=\begin{cases}\beta_0 r\alpha_0(e_1^k) = r^\infty(e_1^k) \text{ for } j=1\\ \text{and } \beta_1 r^\infty\alpha_1(e_j^k) = \beta_1 r^\infty(e_{j-1}) = \beta_1(\beta_1^{j-1}\beta_0 r\alpha_0\alpha_1^{j-1}(e_{j-1}^k))\\ \qquad = \beta_1^j\beta_0 r\alpha_0\alpha_1^j(e_j^k) = r^\infty(e_j^k) \text{ for } j>1.\end{cases} \tag{2.7}$$

The ring μR is a sum-ring because it is the homomorphic image of ℓR; however, μR is not in general an infinite sum ring.

The two categories of sum-rings and of infinite sum rings are closed under the operations:

(i) fiber product of two morphisms;

(ii) forming the ring $R[M]$ where R is a ring and M is an associative monoid with identity; and

(iii) taking "ℓ" or "μ" of a ring.

For example (ii) implies that $K_0(\Lambda) = 0$ for any infinite sum ring Λ because

$$K_0(\Lambda) \subset K_1\ (\Lambda[t, t^{-1}]) = 0 \text{ (cf. §6)}.$$

§3. DELOOPING $BGL(R)^+$

Start with an exact sequence

$$1 \to H \to G \to K \to 1$$

in the category of direct sum groups satisfying (†) and (**) of §1.

LEMMA 3.1. *Assume* (i) *that* $[H, H]$, $[G, G]$, *and* K *are perfect and* (ii) *given* $g \in G$ *and* $h_1, \ldots, h_n \in H$ *there is an* $h \in H$ *such that* $g \cdot h_j \cdot g^{-1} = h \cdot h_j \cdot h^{-1}$ *for* $1 \leqslant j \leqslant n$. *Then*

$$BH^+ \to BG^+ \to BK^+$$

is a (homotopy theoretic) fibration, where the "+" is taken with respect to the commutator subgroup in each case.

Proof. Standard classifying space theory gives a homotopy theoretic fibration $BH \to BG \to BK$ and (ii) above implies that $\pi_1(BK) = K$ operates trivially on the $H_*(BH)$ by the following argument. Choose $k \in K$ and lift it to $g \in G$. Then k acts on $H_*(BH)$ by the automorphism induced by conjugation by g, which is the identity by (1.3). Consider the diagram

$$\begin{tikzcd} BH \arrow[r] \arrow[d, "\alpha"] & BG \arrow[r, "\pi"] \arrow[d, "\beta"] & BK \arrow[d, "\gamma"] \\ F \arrow[r] & BG^+ \arrow[r, "\pi^+"'] & BK^+ \end{tikzcd}$$

where the bottom sequence is π^+ made into a fibration. BG^+ and BK^+ are H-spaces by (1.2) and BK^+ is simply connected since K is perfect. Hence F is a connected H-space. Now the "+" construction says that β and γ are isomorphisms on homology. The Comparison Theorem for spectral sequences [12, p. 355] implies that α is an isomorphism on homology. Hence the induced map $BH^+ \to F$ is an isomorphism on homology between simple spaces, which makes it a homotopy equivalence by (1.1).

Let R be a ring with identity.

PROPOSITION 3.2. *There is a natural homotopy equivalence*

$$K_0(R) \times BGL(R)^+ \cong \Omega BGL(\mu R)^+.$$

Naturality means that for a ring homomorphism $f: R \to S$ there is a homotopy commutative diagram

$$\begin{array}{ccc} K_0(R) \times BGL(R)^+ & \cong & \Omega BGL(\mu R)^+ \\ \Big\downarrow {\scriptstyle f_* \times Bf^+} & & \Big\downarrow {\scriptstyle \Omega B\mu f^-} \\ K_0(S) \times BGL(S)^+ & \cong & \Omega BGL(\mu S)^+. \end{array}$$

Proof of 3.2. This is essentially an application of (3.1) to the exact sequence

$$1 \to GL(mR) \to E(\ell R) \to E(\mu R) \to 1$$

coming from the sequence $0 \to mR \to \ell R \to \mu R \to 0$. However, to get naturality it is necessary to be more precise than (3.1). Here is the idea of the proof. The exact sequence $0 \to mR \to \ell R \to \mu R \to 0$ of rings gives an exact sequence

$$1 \to GL(mR) \to GL(\ell R) \to GL(\mu R).$$

But $E(\ell R) = GL(\ell R)$ by (2.5) so there is an exact sequence

$$1 \to GL(mR) \to E(\ell R) \to E(\mu R) \to 1. \tag{3.3}$$

Now applying (3.1) we get a fibration

$$BGL(mR)^+ \to BE(\ell R)^+ \to BE(\mu R)^+$$

and hence a homotopy equivalence

$$BGL(mR)^+ \cong \Omega BE(\mu R)^+ \tag{3.4}$$

because $BE(\ell R)^+$ is contractible. Now $BE(\mu R)^+$ is the universal cover of $BGL(\mu R)^+$ so

$$BGL(mR)^+ \cong (\Omega BGL(\mu R)^+)_0,$$

where the subscript "0" denotes the component of loops contractible to a point. Finally, we know that $K_1(\mu R)$ is naturally isomorphic to $K_0(R)$ (cf. proof of (5.1) below, [7], or [11]) so that

$$K_0(R) \times BGL(R)^+ \cong \Omega BGL(\mu R)^+. \tag{3.5}$$

Now two things remain to be verified:

(a) The condition (ii) of (3.1) is satisfied.

(b) The equivalence can be constructed so it is natural.

First we show why (ii) holds in the special case when $n = 1$ and $g = e_{pq}(\lambda) = I + \delta_{pq}(\lambda)$. The extension to the general case is easy. Considered as an element of $E(mR)$, h_1 is of the form $I + (b_{ij})$ where $b_{ij} \in mR$ and at most finitely many of the b_{ij} are non-zero. Write $\lambda = \alpha + \beta$ where $\beta \cdot b_{ij} = b_{ij} \cdot \beta = 0$ and $\alpha \in mR$. Then

$$e_{pq}(\lambda) \cdot h_1 \cdot e_{pq}(-\lambda) = e_{pq}(\alpha) \cdot h_1 \cdot e_{pq}(-\alpha).$$

To construct (3.4) naturally consider a fixed realization of $BGL(mZ)^+$, $BE(\ell Z)^+$, and $BE(\mu Z)^+$ so that there is a strictly commutative diagram

$$\begin{array}{ccccc} BGL(mZ) & \xrightarrow[i]{} & BE(\ell Z) & \xrightarrow[\pi]{} & BE(\mu Z) \\ \downarrow & & \downarrow & & \downarrow \\ BGL(mZ)^+ & \xrightarrow[i^+]{} & BE(\ell Z)^+ & \xrightarrow[\pi^+]{} & BE(\mu Z)^+ \end{array}$$

where the vertical maps are inclusions. Let ε denote the trivial group. The image of $BGL(mZ)$ in $BE(\mu Z)$ is the contractible subcomplex $B\varepsilon$. Choose a homotopy, with support in $B\varepsilon$, of $\pi \circ i$ to the constant map and, using (b) of §1, extend this to a homotopy d_t of $\pi^+ \circ i^+$ to the constant map. Now let R be any ring with identity. The ring homomorphism $Z \to R$ induces group homomorphisms $GL(mZ) \to GL(mR)$, $E(\ell Z) \to E(\ell R)$, and $E(\mu Z) \to E(\mu R)$ together with maps of classifying spaces α: $BGL(mZ) \to BGL(mR)$, β: $BE(\ell Z) \to BE(\ell R)$, and γ: $BE(\mu Z) \to BE(\mu R)$. Note that the normal closure of the image of $E(mZ)$ in $GL(mR)$ is $E(mR)$. Similarly, the normal closures of the images of $E(\ell Z)$ and $E(\mu Z)$ in $E(\ell R)$ and $E(\mu R)$ are respectively just $E(\ell R)$ and $E(\mu R)$. Thus we have the following push-out diagram formulae:

$$\begin{aligned} BGL(mR)^+ &= BGL(mR) \cup_\alpha BGL(mZ)^+ \\ BE(\ell R)^+ &= BE(\ell R) \cup_\beta BE(\ell Z)^+ \\ BE(\mu R)^+ &= BE(\mu R) \cup_\gamma BE(\mu Z)^+. \end{aligned} \tag{3.6}$$

Let $d(R)$ denote the composite map

$$BGL(mR)^+ \xrightarrow[i^+]{} BE(\ell R)^+ \xrightarrow[\pi^+]{} BE(\mu R)^+.$$

Using the homotopy constructed above we get a homotopy $d(R)_t$ of $d(R)$ to the constant map which is natural; that is, for any ring homomorphism preserving the identity f: $R \to S$ we have

$$d(S)_t \circ Bf^+ = B\mu f^+ \circ d(R)_t. \tag{3.7}$$

Recall that any map π: $X \to Y$ is equivalent to a fibration as follows: let E_π = set of pairs (x, λ) where $x \in X$ and λ is a path in Y with $\pi(x) = \lambda(0)$. Then $E_\pi \to Y$ given by $(x, \lambda) \to \lambda(1)$ is a fibration with fiber F_π. For any map $\alpha: L \to X$ provided with a homotopy of $\pi \circ \alpha$: $L \to Y$ to the constant map to the base point there is a map $L \to F_\pi$ giving a commutative diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \\ F_\pi \arrow[r] & E_\pi \arrow[ur] & \end{tikzcd}$$

Applying this to π^+: $BE(\ell R)^+ \to BE(\mu R)^+$ produces a commutative diagram

$$\begin{tikzcd}
BGL(mR)^+ \arrow[r] \arrow[d] & BE(\ell R)^+ \arrow[d] \arrow[dr] & \\
F_{\pi+} \arrow[r] & E_{\pi+} \arrow[r] & BE(\mu R)^+ \\
\Omega BE(\mu R)^+ \arrow[r] \arrow[u] & PBE(\mu R)^+ \arrow[u] \arrow[ur] &
\end{tikzcd} \tag{3.8}$$

with the columns being homotopy equivalences by (3.1). For any ring homomorphism $R \to S$ preserving the identity there is a strictly commutative diagram

$$\begin{tikzcd}
BGL(mR)^+ \arrow[r] \arrow[d, "Bf^+"] & BE(\ell R)^+ \arrow[r] \arrow[d, "B\ell f^+"] & BE(\mu R)^+ \arrow[d, "B\mu f^+"] \\
BGL(mS)^+ \arrow[r] & BE(\ell S)^+ \arrow[r] & BE(\mu S)^+.
\end{tikzcd}$$

The formula (3.7) implies there is a map between the diagrams of type (3.8) for R and for S producing a homotopy commutative three dimension diagram. The left side of the diagram says that $f_* \times Bf^+$ is homotopic to $\Omega B\mu f^+$. This completes the proof of (3.2).

Essentially the above argument shows there is a natural equivalence of sequences

$$\begin{tikzcd}
BGL(mR)^+ \arrow[r] \arrow[d] & BGL(\ell R)^+ \arrow[r] \arrow[d] & BGL(\mu R)^+ \arrow[d] \\
{[\Omega BGL(\mu R)^+]_0} \arrow[r] & PBGL(\mu R)^+ \arrow[r] & BGL(\mu R)^+.
\end{tikzcd}$$

Now (3.2) allows us to define for any associative ring with identity R an Ω-spectrum $X(R) = \{X_i(R)\}$ such that for $0 \leqslant i < \infty$

$$X_{-i}(R) = \Omega^i[K_0(R) \times BGL(R)^+]$$

and

$$X_i(R) = K_0(\mu^i R) \times BGL(\mu^i R)^+.$$

Here $\mu^0 R = R$ and $\mu^i R = \mu(\mu^{i-1}R)$ for $i > 0$. For all $-\infty < i < \infty$ we set

$$K_i(R) = \pi_i(X(R)).$$

COROLLARY 3.9. *$K_i(\mu R) = K_{i-1}(R)$ for $-\infty < i < \infty$.*

§4. RELATIVE K-THEORY GROUPS

Let $f\colon R \to S$ be any identity preserving homomorphism. Define the ring γf as in [7] by the pull-back diagram

$$\begin{tikzcd}
\gamma f \arrow[r] \arrow[d] & \mu R \arrow[d, "\mu f"] \\
\ell S \arrow[r] & \mu S.
\end{tikzcd}$$

Define $\mu^i f\colon \mu^i R \to \mu^i S$ inductively as $\mu^0 f = f$ and $\mu^i f = \mu(\mu^{i-1} f)$ for $i > 0$.

Let $X(f) = \{X_i(f)\}$ be defined as

$$X_{-i}(f) = \Omega^i[K_0(\gamma f) \times BGL(\gamma f)^+]$$

and

$$X_i(f) = K_0(\gamma(\mu^i f)) \times BGL(\gamma(\mu^i f))^+$$

whenever $i \geqslant 0$.

In this section we show that $X(f)$ is a spectrum and, setting

$$K_i(f) = \pi_i(X(f))$$

for $i \in Z$, that there is a long exact sequence

$$\ldots \to K_i(R) \to K_i(S) \to K_i(f) \to K_{i-1}(R) \to K_{i-1}(S) \to \ldots \tag{4.1}$$

Consider the commutative diagram of rings

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & X \arrow[r] \arrow[d, "i"] & D \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & B \arrow[r, "\beta"'] \arrow[d, "="'] & C \arrow[r, "\gamma"'] \arrow[d] \arrow[dr, phantom, "1"] & D \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & B \arrow[r] & Y \arrow[r] & E \arrow[r] & 0
\end{tikzcd} \tag{4.2}$$

where

(a) the horizontal rows are exact;

(b) $BGL(X)^+$ and $BGL(Y)^+$ are contractible;

(c) the square 1 is a pull back;

(d) given $\beta_1, \ldots, \beta_n \in B \subset C$ and $x \in X$ there is a decomposition $x = \mu + \lambda$ such that $\mu \in A$ and $\beta_j \cdot {}_i(\lambda) = {}_i(\lambda) \cdot \beta_j = 0$.

For example, the following is a type (4.2) diagram:

$$\begin{tikzcd}
0 \arrow[r] & mR \arrow[r] \arrow[d, "\alpha"] & \ell R \arrow[r] \arrow[d] & \mu R \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & mS \arrow[r, "\beta"'] \arrow[d, "="] & \gamma f \arrow[r, "\gamma"'] \arrow[d] & \mu R \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & mS \arrow[r] & \ell S \arrow[r] & \mu S \arrow[r] & 0.
\end{tikzcd} \tag{4.3}$$

PROPOSITION 4.4. *Applying the functor $\pi_i(BGL(?)^+)$, for $i \geqslant 0$, to the diagram* (4.3) *gives an exact sequence*

$$\circ \xrightarrow[\alpha_*]{} \circ \xrightarrow[\beta_*]{} \circ \xrightarrow[\gamma_*]{} \circ \xrightarrow[\delta_*]{} \circ .$$

Proof of 4.4. For simplicity of notation let R^+ denote $BGL(R)^+$ for any ring R. Now for each ring in (4.3) we can construct the B^+ so that there is a commutative diagram

$$\begin{tikzcd} mR^+ \arrow[r] \arrow[d] & \ell R^+ \arrow[r] \arrow[d] & \mu R^+ \arrow[d] \\ mS^+ \arrow[r] \arrow[d] & \gamma f^+ \arrow[r] \arrow[d] & \mu R^+ \arrow[d] \\ mS^+ \arrow[r] & \ell S^+ \arrow[r] & \mu S^+ . \end{tikzcd} \tag{4.5}$$

This is done by considering the diagram (4.3) obtained from the identity map $id: Z \to Z$. We get

$$\begin{tikzcd} mZ \arrow[r] \arrow[d] & \ell Z \arrow[r] \arrow[d] & \mu Z \arrow[d] \\ mZ \arrow[r] \arrow[d] & \ell Z \arrow[r] \arrow[d] & \mu Z \arrow[d] \\ mZ \arrow[r] & \ell Z \arrow[r] & \mu Z . \end{tikzcd}$$

Applying B^+ gives a commutative diagram and to get (4.5) we set

$$BGL(mR)^+ = BGL(mR) \cup BGL(mZ)^+,$$

$$BGL(mS)^+ = BGL(mS) \cup BGL(mZ)^+,$$

etc., as in (3.6).

Now let $\pi: U \to V$ denote the map $\mu R^+ \to \mu S^+$ made into a fibration. There is a commutative diagram

$$\begin{tikzcd} \Omega U \arrow[r] \arrow[d] & PU \arrow[r] \arrow[d] & U \arrow[d, "="] \\ \Omega V \arrow[r] \arrow[d, "="] & W \arrow[r] \arrow[d] & U \arrow[d, "\pi"] \\ \Omega V \arrow[r] & PV \arrow[r, "p"'] & V \end{tikzcd}$$

where W is the pullback of ρ and π. Each horizontal row is a fibration. The argument which gives the homotopy exact sequence of a fibration shows that for $i \geqslant 0$ there is an exact sequence

$$\pi_i(\Omega U) \to \pi_i(\Omega V) \to \pi_i(W) \to \pi_i(U) \to \pi_i(V). \tag{4.7}$$

The method of §3 yields a map from (4.5) to (4.6):

$$\begin{tikzcd}
mR^+ \arrow[rr] \arrow[dr, "\delta_1"] \arrow[dd] & & \ell R^+ \arrow[rr] \arrow[dr] \arrow[dd] & & \mu R^+ \arrow[dr] \arrow[dd] & & \\
& \Omega U \arrow[rr] \arrow[dd] & & PU \arrow[rr] \arrow[dd] & & U \arrow[dd] & \\
mS^+ \arrow[rr] \arrow[dr, "\delta_2"] \arrow[dd] & & \gamma f^+ \arrow[rr] \arrow[dr, "\delta_3"] \arrow[dd] & & \mu R^+ \arrow[dr, "\delta_4"] \arrow[dd] & & = \\
& \Omega V \arrow[rr] \arrow[dd, equal] & & W \arrow[rr] \arrow[dd] & & U \arrow[dd] & \\
mS^+ \arrow[rr] \arrow[dr] & & \ell S^+ \arrow[rr] \arrow[dr] & & \mu S^+ \arrow[dr, "\delta_5"] & & \\
& \Omega V \arrow[rr] & & PV \arrow[rr] & & V &
\end{tikzcd}$$

Each square is homotopy commutative; δ_4 and δ_5 are homotopy equivalences by construction; δ_1 and δ_2 are isomorphisms on π_i for $i \geqslant 1$ by §3. We shall show that

$$\delta_3 \colon \gamma f^+ \to W_0 \text{ is a homotopy equivalence} \tag{4.8}$$

by showing that δ_3 induces an isomorphism on π_i for $i \geqslant 1$. This implies that the sequence in (4.4) is exact because it is isomorphic to the exact sequence (4.7) for $i \geqslant 1$. Note that (4.8) says $BGL(\lambda f)^+$ has the homotopy type of the base point component in the fiber of the map $BGL(\mu R)^+ \to BGL(\mu S)^+$.

Step 1. Isomorphism on π_1.

For any diagram (4.2) type diagram there is an exact sequence

$$K_1(A) \to K_1(B) \to K_1(C) \to K_1(D) \to K_1(E). \tag{4.9}$$

To see this apply GL to each ring in (4.2) to get the diagram

$$\begin{tikzcd}
1 \arrow[r] & GL(A) \arrow[r] \arrow[d, "\alpha"] & E(X) \arrow[r] \arrow[d, "i"] & GL(D) \arrow[d, "\cong"] \\
1 \arrow[r] & GL(B) \arrow[r, "\beta"'] \arrow[d, "\cong"'] & GL(C) \arrow[r, "\gamma"'] \arrow[d] & GL(D) \arrow[d, "\delta"] \\
1 \arrow[r] & GL(B) \arrow[r] & E(Y) \arrow[r, "j"'] & GL(E)
\end{tikzcd}$$

where the horizontal rows are exact except at the right hand end. Since $K_1(X) = K_1(Y) = 0$ we have replaced $GL(X)$ and $GL(Y)$ by $E(X)$ and $E(Y)$.

Exactness at $K_1(D)$. This follows easily because $GL(C)$ is the pull back of j and δ and $E(Y) \to E(E)$ is onto.

Exactness at $K_1(C)$. Let $[M] \in K_1(C)$ be represented by $M \in GL(C)$ with $[M] = 0$ in $K_1(D)$. There is an $N \in E(X)$ such that $\gamma(i(N)) = \gamma(M)$. Hence $[M] = [M \cdot i(N)^{-1}]$ pulls back to something in $K_1(B)$.

Exactness at $K_1(B)$. Let $[M] \in K_1(B)$ be represented by $M \in GL(B)$. Suppose $\beta(M) = \prod e_{i_k j_k}(y_k, d_k)$. Lift each d_k to some $x_k \in X$. Then $\prod e_{i_k j_k}(x_k)$ pulls back to some $N \in GL(A)$ because $\prod e_{i_k j_k}(d_k) = 1$. We shall show that it is possible to choose the x_k so that $\beta(M) \cdot \beta(\alpha(N))^{-1} = \prod e_{i_k j_k}(b_k, 0)$ for $b_k \in B$. Since β is a monomorphism this implies that $M = \alpha(N)$ mod $E(B)$.

Here is how to pick the x_k: note that for any choice of x_k

$$\beta(M) \cdot \beta(\alpha(N)^{-1}) = \prod_{k=1}^{n} e_{i_k j_k}(y_k, d_k) \cdot \prod_{k=n}^{1} e_{i_k j_k}(-z_k, -d_k)$$

where $i(x_k) = (z_k, d_k)$.

Now make an arbitrary choice of x_n which lifts d_n. Then

$$e_{i_n j_n}(y_n, d_n) \cdot e_{i_n j_n}(-z_n, -d_n) = e_{i_n j_n}(y_n - z_n, 0)$$

and $y_n - z_n \in B$. Use (d) to lift d_{n-1} to x_{n-1} so that $(y_n - z_n) \cdot z_{n-1} = 0$. Then

$$\begin{aligned} &e_{i_{n-1} j_{n-1}}(y_{n-1}, d_{n-1}) \cdot e_{i_n j_n}(y_n - z_n, 0) \cdot e_{i_{n-1} j_{n-1}}(-z_{n-1}, -d_{n-1}) \\ &= e_{i_{n-1} j_{n-1}}(y_{n-1} - z_{n-1}, 0) \cdot e_{i_n j_n}(y_n - z_n, 0). \end{aligned}$$

From this last step it is clear how to inductively choose the x_k as desired.

The sequences (4.7) and (4.9) plus the five-lemma imply that δ_3 induces an isomorphism on π_1.

Step 2. Isomorphism on π_i for $i \geqslant 2$.

Consider the diagram

$$\begin{tikzcd} 1 \arrow[r] & GL(mS) \arrow[r] \arrow[d] & E(\ell S) \tilde{\times} E(\mu R) \arrow[r] \arrow[d] & E(\mu R) \arrow[r] \arrow[d, "\delta"] & 1 \\ 1 \arrow[r] & GL(mS) \arrow[r] & E(\ell S) \arrow[r, "j"'] & E(\mu S) \arrow[r] & 1 \end{tikzcd}$$

where $E(\ell S) \tilde{\times} E(\mu R)$ is the pull back of j and δ. Note that $E(\gamma f) \subset E(\ell S) \tilde{\times} E(\mu R)$ is the commutator subgroup.

Now let $\pi' \colon U' \to Y'$ be the fibration obtained from $\pi \colon U \to V$ by taking the universal cover of U and of V. Applying B^+ "carefully" (e.g. as in §3) to (4.10) gives a homotopy commutative diagram

$$\begin{tikzcd}
BGL(mS)^+ \arrow[rr] \arrow[dd, "="'] \arrow[dr, "\Delta_1"] & & B[E(\ell S) \mathbin{\tilde{\times}} E(\mu R)]^+ \arrow[rr] \arrow[dd] \arrow[dr, "\Delta_2"] & & BE(\mu)R^+ \arrow[dd] \arrow[dr, "\Delta_3"] & \\
& \Omega V' \arrow[rr] \arrow[dd] & & Q \arrow[rr] \arrow[dd] & & U' \arrow[dd] \\
BGL(mS)^+ \arrow[rr] \arrow[dr] & & BE(\ell S)^+ \arrow[rr] \arrow[dr] & & BE(\mu S)^+ \arrow[dr] & \\
& \Omega V' \arrow[rr] & & PV' \arrow[rr] & & V'
\end{tikzcd}$$

where Δ_1 and Δ_3 are homotopy equivalences and Q is a covering space of W. By (3.1)

$$BGL(mS)^+ \to B[E(\ell S) \mathbin{\tilde{\times}} E(\mu R)]^+ \to BE(\mu R)^+$$

is a homotopy fibration; hence Δ_2 is a homotopy equivalence.

This gives a homotopy commutative diagram

$$\begin{tikzcd}
& B[E(\ell S) \mathbin{\tilde{\times}} E(\mu R)]^+ \arrow[r] \arrow[d] & Q \arrow[d] \\
BE(\gamma f)^+ \arrow[ur] \arrow[r] & BGL(\gamma f)^+ \arrow[r, "\delta_3"'] & W
\end{tikzcd}$$

where all the arrows (except possibly δ_3) are isomorphisms on π_i for $i \geqslant 2$. This completes the proof of (4.4).

PROPOSITION 4.11. *$X(f)$ is a spectrum.*

Proof. Consider the two diagrams of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mu(mR) \arrow[r] \arrow[d] & \mu(\ell R) \arrow[r] \arrow[d] & \mu^2 R \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mu(mS) \arrow[r] \arrow[d, "\simeq"'] & \mu(\gamma f) \arrow[r] \arrow[d] & \mu^2 R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mu(mS) \arrow[r] & \mu(\ell S) \arrow[r] & u^2 S \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & m(\mu R) \arrow[r] \arrow[d] & \ell(\mu R) \arrow[r] \arrow[d] & \mu^2 R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & m(\mu S) \arrow[r] \arrow[d] & \gamma(\mu f) \arrow[r] \arrow[d] & \mu^2 R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & m(\mu S) \arrow[r] & \ell(\mu S) \arrow[r] & \mu^2 S \arrow[r] & 0.
\end{tikzcd}$$

Let $M \to N$ be the map $BGL(\mu^2 R)^+ \to BGL(\mu^2 S)^+$ made into a fibration. Define H by the pull back diagram

$$\begin{tikzcd}
H \arrow[r] \arrow[d] & M \arrow[d] \\
PN \arrow[r] & N.
\end{tikzcd}$$

Then the argument proving (4.8) can be mimicked to show there are homotopy equivalences

$$BGL(\mu(\gamma f))^+ \xrightarrow[\cong]{} H_0 \xleftarrow[\simeq]{} BGL(\gamma)\mu f))^+. \tag{4.12}$$

Now for $i < 0$ it is clear that $X_i(f) \cong \Omega X_{i+1}(f)$. For $0 \leqslant i$ we have

$$\begin{aligned}
X_i(f) &= K_0(\gamma(\mu^i f)) \times BGL(\gamma(\mu^i f))^+ \\
&\cong \Omega[BGL(\mu\gamma(u^i f))^+] \\
&\cong \Omega[BGL(\gamma(\mu^{i+1} f))^+]. \\
&\cong \Omega[K_0(\gamma(u^{i+1} f)) \times BGL(\gamma(\mu^{i+1} f))^+] \\
&= \Omega X_{i+1}(f).
\end{aligned}$$

Now putting (3.2), (4.4), and (4.11) together gives the long exact sequence (4.1).

Whenever $f\colon R \to S$ is a surjection it has been known for some time that there is an exact sequence

$$K_2(R) \to K_2(S) \to K_1(f) \to K_1(R) \to K_1(S)$$

where $K_1(f)$ is the relative group defined by Bass. See [14]. In [7] it was shown that for any surjection $f\colon R \to S$ there is a natural isomorphism of sequences

$$\begin{tikzcd}
K_2(R) \arrow[r] \arrow[d] & K_2(S) \arrow[r] \arrow[d] & K_2(\gamma f) \arrow[r] \arrow[d, "\theta"] & K_1(R) \arrow[r] \arrow[d] & K_1(S) \arrow[d] \\
K_2(R) \arrow[r] & K_2(S) \arrow[r] & K_1(f) \arrow[r] & K_1(R) \arrow[r] & K_1(S).
\end{tikzcd}$$

The idea for defining θ is this: Let $z \in K_2(\gamma f)$ be represented by the word $\prod x_{i_\alpha j_\alpha}(a_\alpha, b_\alpha) \in St(\gamma f)$ where $a_\alpha \in \ell S$ and $b_\alpha \in \mu R$. Choose a lifting $b_\alpha' \in \ell R$ of b_α with $f(b_\alpha') = a_\alpha$. Then $M_z = \prod e_{i_\alpha j_\alpha}(b_\alpha) \in GL(\ell R)$ actually lies in the sub-group $GL(mR)$ and M_z goes to the identity in $GL(mS)$. This means that M_z determines an element of $K_1(f)$. The correspondence $z \to M_z$ defines θ.

§5. EQUIVALENCE OF ℓA AND μA WITH CA AND SA

The rings ℓA, μA, and γf arose from the smooth and piecewise linear topology of non-compact manifolds [7]. For algebraic purposes it is sometimes more convenient to use the rings CA and SA which came from more algebraic beginnings [11]. In this section it is shown that the two approaches give rise to the same K-theory. The isomorphism in (5.1) is a special case of the uniqueness theorem in [11] where it is also shown that $K_{-i}(A) = K_1(S^{i+1}A)$ is naturally isomorphic to the negative K-theory groups $L^iK_0(A)$ of [2].

For any ring homomorphism $f: A \to B$ preserving the identity, define Rf by the pull back diagram

$$\begin{tikzcd} Rf \arrow[r] \arrow[d] & SA \arrow[d] \\ CB \arrow[r] & SB. \end{tikzcd}$$

For $i > 0$ let $S^iA = S(S^{i-1}A)$ and $S^if = S(S^{i-1}f)$.

PROPOSITION 5.1. *For any $i \geqslant 0$ the natural maps $S^iA \to \mu^iA$ and $R(S^if) \to \gamma(\mu^if)$ induce an isomorphism of K-theory sequences* (cf. 4.9)

$$\begin{tikzcd} K_1(S^iA) \arrow[r] \arrow[d] & K_1(S^iB) \arrow[r] \arrow[d] & K_1(R(S^if)) \arrow[r] \arrow[d] & K_1(S^{i+1}A) \arrow[r] \arrow[d] & K_1(S^{i+1}B) \arrow[d] \\ K_1(\mu^iA) \arrow[r] & K_1(\mu^iB) \arrow[r] & K_1(\gamma(\mu^if)) \arrow[r] & K_1(\mu^{i+1}A) \arrow[r] & K_1(\mu^{i+1}B). \end{tikzcd}$$

Proof. It suffices to show that for all $i \geqslant 0$

$$K_1(S^iA) \to K_1(\mu^iA) \tag{5.2}$$

is an isomorphism. The five-lemma then shows the relative groups are isomorphic. Now for $i = 0$ (5.2) is certainly an isomorphism. Assume it is an isomorphism for $i < k$. Then using the excision theorem of [2] we get a diagram

$$\begin{tikzcd} & K_1(S^{k-1}(SA)) \arrow[d, "\alpha"] & & \\ K_1(\mu^{k-1}(CA)) \arrow[r] \arrow[d] & K_1(\mu^{k-1}(SA)) \arrow[r] \arrow[d, "\beta"] & K_0(\mu^{k-1}(mA)) \arrow[r] \arrow[d, "id"] & K_0(\mu^{k-1}(CA)) \arrow[d] \\ K_1(\mu^{k-1}(\ell A)) \arrow[r] & K_1(\mu^{k-1}(\mu A)) \arrow[r] & K_0(\mu^{k-1}(mA)) \arrow[r] & K_0(\mu^{k-1}(\ell A)) \end{tikzcd}$$

where α is an isomorphism by induction. The first and last groups in each of the horizontal rows vanish. Hence $\beta \circ \alpha$ is an isomorphism as required.

PROPOSITION 5.3. *The inclusion $S^iA \to \mu^iA$ induces a natural homotopy equivalence*

$$\theta: BGL(S^iA)^+ \xrightarrow[\cong]{} BGL(\mu^iA)^+.$$

Proof. By induction on i. The assertion for $i = 0$ is clear. Assume it is true for $i < k$ and consider the following map between fibrations

$$\begin{array}{ccccc} BGL(S^{k-1}A)^{+} & \longrightarrow & BE(C(S^{k-1}A))^{+} & \longrightarrow & BE(S^{k}A)^{+} \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} \\ BGL(\mu^{k-1}A)^{+} & \longrightarrow & BE(\ell(\mu^{k-1}A))^{+} & \longrightarrow & BE(\mu^{k}A)^{+}. \end{array}$$

By induction α is a homotopy equivalence. So is β because it is a map between contractible spaces. The comparison theorem implies γ is a homotopy equivalence. Hence θ is an isomorphism on π_i for $i \geqslant 2$; it is also an isomorphism on π_1 by (5.2). Hence it is a homotopy equivalence.

§6. $K_i(R[t, t^{-1}])$

It is interesting to calculate $K_i(A[t, t^{-1}])$ for $-\infty < i < \infty$. This has been done at least theoretically in [2] for $-\infty < i \leqslant 1$ where it is shown that

$$K_i(A[t, t^{-1}]) = K_i(A) \oplus K_{i-1}(A) \oplus \mathrm{Nil}^{+}_{i-1}(A) \oplus \mathrm{Nil}^{-}_{i-1}(A)$$

where $\mathrm{Nil}^{\pm}_{i-1}(A) = \operatorname{coker}\,(K_i(A) \to K_i(A[t^{\pm 1}]))$. One hopes a similar decomposition holds for all i. For $2 \leqslant i < \infty$ it is known that (cf. [8], or [20] and [10] for $i = 2$)

$$K_i(A[t, t^{-1}]) = K_i(A) \oplus K_{i-1}(A) \oplus (?).$$

Is $(?) = \mathrm{Nil}^{-}_{i-1}(A) \oplus \mathrm{Nil}^{+}_{i-1}(A)$? When $i \leqslant 0$ and A is regular $\mathrm{Nil}_i^{\pm}(A) = 0$ [2, Chap XII]. Is this true for $1 \leqslant i$? It has been shown in [9] that $\mathrm{Nil}_i^{\pm}$ (finite field) $= 0$ for $-\infty < i < \infty$.

The group $K_i(A)$ clearly sits in $K_i(A[t, t^{-1}]$ as a direct summand because A is a retract of $A[t, t^{-1}]$. The map $K_i(A[t, t^{-1}]) \to K_{i-1}(A)$ comes from the map of spectra $X(A[t, t^{-1}]) \to X(SA)$ induced by the ring homomorphism $\phi\colon A[t, t^{-1}] \to SA$ where

$$\phi(\textstyle\sum a_i t^i) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \\ & \vdots & & \ddots \end{pmatrix}.$$

Now let $\rho_i(A)\colon S^iA \to S^iA[t]$ denote the standard inclusion. Define a candidate for the "nil" spectrum N^+A as

$$N^{+}_{-i}A = \Omega^i[K_0(R\rho_0(A)) \times GBL(R\rho_0(A))^{+}]$$

and

$$N_i^{+}A = K_0(R\rho_i(A)) \times BGL(R\rho_i(A))^{+}$$

whenever $i \geqslant 0$. $N^{-}A$ is similarly defined using the inclusions $S^iA \to S^iA[t^{-1}]$ and is canonically isomorphic to $N^{+}A$. The results below are stated for $N^{+}A$ only; they are obviously true of $N^{-}A$ as well.

PROPOSITION 6.1. *$N^{+}A = \{N_i^{+}A\}_{i \in Z}$ is a spectrum.*

Proof. Note that $SA[t] = S(A[t])$ and hence

$$\rho_i(A) = \rho_{i-1}(SA) = S\rho_{i-1}(A).$$

For $i \geqslant 1$ it is clear that $N^+_{-1}A \equiv \Omega N^+_{-i+1}A$. For $i \geqslant 0$

$$\begin{aligned} N_i^+A &= K_0(R\rho_i(A)) \times BGL(R\rho^i(A))^+ \\ &= \Omega BGL(S(R\rho_i(A)))^+ \\ &= \Omega BGL(R(S\rho_i(A)))^+ \\ &= \Omega BGL(R\rho_i(SA))^+ = \Omega N_i{}^+(SA) \\ &= \Omega BGL(R\rho_{i+1}(A))^+ \\ &= \Omega N_{i+1}A. \end{aligned}$$

This shows N^+A is a spectrum.

COROLLARY 6.2. $\mathrm{Nil}^+_{i-1}(A) = \pi_i(N^+A)$ and $\mathrm{Nil}_i{}^+(SA) = \mathrm{Nil}^+_{i-1}(A)$.

D. Quillen has recently shown $\mathrm{Nil}_i^+(A) = 0$ for A left regular and $1 \leqslant i$.

§7. FREDHOLM PERMUTATIONS

Let Σ_∞ be the infinite symmetric group. It is a theorem of Barratt–Kahn–Priddy [4] and Quillen (see also [17]) that $Z \times B\Sigma_\infty^+ \cong \Omega^\infty S^\infty$ where the "+" is taken with respect to the infinite alternating group $A_\infty \subset \Sigma_\infty$. In this somewhat speculative section we suggest a way to construct $\Omega^{\infty-1}S^\infty = \lim_{n\to\infty} \Omega^{n-1}S^n$ using the classifying space of the group of Fredholm permutations. Presumably the program could also be extended to $\Omega^{\infty-p}S^\infty = \lim_{n\to\infty} \Omega^{n-p}S^n$ when $p > 1$.

Let S be any countable discrete set. By a *permutation* of S we mean a bijection $f: S \to S$ such that $f(s) = s$ for all but finitely many elements $s \in S$. By an *infinite permutation* we mean any bijection $f: S \to S$. Now let f and g be any two maps from S to itself. We say that f and g detetermine the same *germ* (at infinity), denoted by $f \sim g$, iff f and g agree except on a finite subset of S. A map $f: S \to S$ is called *Fredholm* iff there is a map $g: S \to S$ such that $f \circ g \sim id$ and $g \circ f \sim id$. Note that Fredholm maps are proper. Composition of germs of Fredholm maps is defined by taking the germ of the composition of representives.

Now let $E = \{e_i^k\}$ where $1 \leqslant i, k < \infty$. Identify Σ_∞ with the group of all permutations of E. Let P_n be the group of infinite permutations α of E such that $\alpha(e_i^k) = e_i^k$ whenever $i > n$. Let $P_\infty = \cup_n P_n$. Let F_n be the group of germs $\bar{\alpha}$ of Fredholm maps of E to itself with representatives $\alpha: E \to E$ satisfying $\alpha(e_i^k) = e_i^k$ whenever $i > n$. Let $F_\infty = \cup_n F_n$. There is an exact sequence

$$1 \longrightarrow \Sigma_\infty \xrightarrow[\sigma]{} P_\infty \xrightarrow[\rho]{} F_\infty. \tag{7.1}$$

There is also an exact sequence

$$P_\infty \xrightarrow[\rho]{} F_\infty \xrightarrow[Ind]{} Z \longrightarrow 1 \tag{7.2}$$

where the index homomorphism Ind: $F_\infty \to Z$ is defined as follows: let $[f] \in F_\infty$ be represented by $f: E \to E$. Write $E = U \cup V$ such that U is finite, $f: V \to E$ is injective, and $U \cap V = \emptyset$; then

$$\operatorname{Ind} f = \operatorname{card} U - \operatorname{card}(E - f(V)).$$

PROPOSITION 7.3. *BP_∞^+ is contractible and there is a homotopy theoretic fibration*

$$Z \times B\Sigma_\infty^+ \to BP_\infty^+ \to BF_\infty^+.$$

Thus

$$Z \times B\Sigma_\infty^+ \cong \Omega BF_\infty^+.$$

From (7.3) one might conjecture that $BF_\infty^+ \cong \Omega^{\infty-1}S^\infty$.

Added in proof. This conjecture has recently been verified by S. Priddy.

Remark. $P_\infty = [P_\infty, P_\infty]$ because BP_∞^+ is contractible. Hence exactness of (7.2) shows that $\rho(P_\infty) = [F_\infty, F_\infty]$.

Proof of (7.3). This is an easy corollary of (3.1) once we show that (7.1) is a sequence of direct sum groups and that P_∞ is flabby. Consider the following description of Σ_∞, P_∞, and F_∞. Let $S = \{e_i^{pq}\}$ where $1 \leqslant i, p, q < \infty$. Then Σ_∞ is just the permutations of S. P_∞ consists of those infinite permutations α of S such that there is an n for which $\alpha(e_i^{pq}) = e_i^{pq}$ whenever $i > n$. Similarly for F_∞. Now proceed as in §2: Let $\beta_0 \in \ell_Z(S)$ be given by any bijection of S onto $S_0 = \{e_i^{p1}\}$ where $1 \leqslant i, p < \infty$ such that for each fixed value i, β_0 takes $\{e_i^{pq}\}(1 \leqslant p, q < \infty)$ onto $\{e_i^{p1}\}(1 \leqslant p < \infty)$. Let α_0 be defined by (2.6). Let $\beta_1 \colon S \to S_1 = \{e_i^{pq}\}$ where $1 < q < \infty$ be given by $\beta_1(e_i^{pq}) = e_i^{p,q+1}$. Define α_1 as in (2.6). Then for A and B in Σ_∞, P_∞, or F_∞ define

$$A \oplus B = \beta_0 A \alpha_0 + \beta_1 B \alpha_1.$$

For $A \in P_\infty$ define $\tau(A) \in P_\infty$ by

$$\tau(A)(e_i^{pq}) = \beta_1^{q-1}\beta_0 A \alpha_0 \alpha_1^{q-1}(e_i^{pq}).$$

It is not hard to see that conditions (†) and (**) of §1 and (***) of §2 are satisfied. Now apply (3.1) to the sequence

$$1 \to \Sigma_\infty \to P_\infty \to [F_\infty, F_\infty] \to 1$$

to get a fibration

$$B\Sigma_\infty^+ \to BP_\infty^+ \to B[F_\infty, F_\infty]^+.$$

Since $\pi_1(BF_\infty^+) = F_\infty \bmod [F_\infty, F_\infty] = Z$, there is a fibration

$$Z \times B\Sigma_\infty^+ \to BP_\infty^+ \to BF_\infty^+$$

as asserted.

Perhaps methods of this section could be extended to give a new construction of $Q(S^iX)$ = base point component in $\lim \Omega^n S^{n+i}X$ for any space X.

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