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Introduction

The aim of this thesis is the formulation and proof of a noncommutative analogue of a particular family index theorem, namely the one proven by Bunke and Koch in [BK]. Before precising the problem we sketch what is meant by "noncommutative analogue":

Family index theory describes Fredholm operators depending continuously on a parameter from some compact space. The index of a family of Fredholm operators $\{F_b\}_{b\in B}$ on a complex Hilbert space H is an element in $K^0(B)$. If $\{\mathrm{Ker}\, F_b\}_{b\in B}$ and $\{\mathrm{Coker}\, F_b\}_{b\in B}$ are vector bundles on B, then it is just the difference of the classes of these bundles.

One may reformulate this setting by replacing the space B by the C^*-algebra $C(B)$ of continuous functions on B and the Hilbert space H by the $C(B)$-module $C(B,H)$ of continuous H-valued functions on B. The family of operators yields a $C(B)$-module map on $C(B,H)$. The index, now a formal difference of projective $C(B)$-modules, is an element in the C^*-algebraic K-theory $K_0(C(B))$ which is naturally isomorphic to $K^0(B)$.

The theory of Fredholm operators on Hilbert C^*-modules introduced by Miščenko and Fomenko [MF] is a generalisation of this different view on family index theory to arbitrary C^*-algebras. Miščenko and Fomenko also elaborated the theory of Sobolev C^*-modules and elliptic operators over C^*-algebras on compact manifolds.

For the theory of operators on Hilbert C^*-modules we refer the reader to the books of Wegge-Olson [WO], in particular for the theory of Fredholm operators, and of Lance [La], in particular for the theory of unbounded operators.

In differential geometry a family index theorem usually calculates the Chern character of the index bundle in the de Rham cohomology of the base space that is assumed to be a manifold.

A straightforward noncommutative analogue of differential forms on a manifold was found by Karoubi [Kar] who defined a $\mathbb{Z}$-graded differential algebra $(\hat{\Omega}_*(\mathcal{A}), \mathrm{d})$ associated to a unital Fréchet algebra $\mathcal{A}$ and the de Rham homology of $\mathcal{A}$. He also introduced connections on projective right $\mathcal{A}$-modules and a Chern character associated to a connection. If $\mathcal{A}$ is a local Banach algebra [Bl], the Chern character yields a homomorphism

$$\mathrm{ch} : K_0(\mathcal{A}) \to H^{dR}_*(\mathcal{A}) \ .$$

Unfortunately the de Rham homology of a C^*-algebra does not behave well, in particular the de Rham homology of a commutative C^*-algebra is not the cohomology of the corresponding compact space.

In the general context one way out – unsatisfying because it depends on choices – is to introduce an additional structure on a C^*-algebra $\mathcal{A}$, namely a dense subalgebra $\mathcal{A}_\infty$ of $\mathcal{A}$ such that $K_0(\mathcal{A}_\infty)$ is canonically isomorphic to $K_0(\mathcal{A})$. Then the Chern character yields a map

$$\mathrm{ch} : K_0(\mathcal{A}) \to H_*^{dR}(\mathcal{A}_\infty) \ .$$

The choice of $\mathcal{A}_\infty$ depends on the situation, in the family situation one would choose $\mathcal{A}_\infty = C^\infty(B)$.

In the present thesis we define and investigate noncommutative η-forms associated to a Dirac operator on the unit interval. The algebra enters in the definition of boundary conditions that make the operator invertible. These η-forms occur in a noncommutative index theorem on a noncompact manifold with boundary which we prove. We give a preciser overview of our result:

Let $\{\mathcal{A}_i\}_{i\in\mathbb{N}_0}$ be a projective system of Banach algebras such that $\mathcal{A}_0 = \mathcal{A}$ is a C^*-algebra and such that there are dense embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}$ for all $i \in \mathbb{N}_0$ and let $\mathcal{A}_\infty$ be its projective limit. Some further technical conditions have to be imposed to the system in order to make the theory work ([Lo], §2.1).

Let

$$I_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in M_{2d}(\mathbb{C}) \ .$$

We call a projection $P \in M_{2d}(\mathcal{A})$ Lagrangian if $PI_0P = 0$ and $P(\mathcal{A}^{2d}) \oplus I_0P(\mathcal{A}^{2d}) = \mathcal{A}^{2d}$. To a pair (P_0, P_1) of Lagrangian projections onto transverse submodules with $P_0, P_1 \in M_{2d}(\mathcal{A}_\infty)$ we associate the operator D_I, that is defined to be the operator $I_0\frac{d}{dx}$ with

$$\mathrm{dom}\, D_I = \{f \in C^\infty([0,1], \mathcal{A}_\infty^{2d}) \mid P_0f(0) = f(0) \text{ and } P_1f(1) = f(1)\} \ .$$

Furthermore we define a superconnection A_I associated to D_I and an η-form $\eta(A_I)$ associated to A_I. It is an element in $\hat{\Omega}_*(\mathcal{A}_\infty)/\overline{[\hat{\Omega}_*(\mathcal{A}_\infty), \hat{\Omega}_*(\mathcal{A}_\infty)]_s}$, where $[\ ,\ ]_s$ denotes the supercommutator.

Let $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ with $\mathcal{P}_k \in M_{2d}(\mathcal{A}_\infty)$, $k = 0,1,2$, be a triple of Lagrangian projections on pairwise transverse submodules of $\mathcal{A}^{2d}$. Let D_{I_k}, $k = 0,1,2$ be the operator associated to $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{k+1 \bmod 3})$. As in the family case it turns out that there are superconnections A_{I_k}, $k = 0,1,2$, associated to D_{I_k}, $k = 0,1,2$ such that the sum $\eta(A_{I_0}) + \eta(A_{I_1}) + \eta(A_{I_2})$ is closed. Its homology class is the Chern character of the index of a Dirac operator D^+ on a two dimensional spin manifold M with cylindric ends isometric to $[0,\infty) \times [0,1]$. The projections $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$ enter in the definition of boundary conditions for D^+ making it a Fredholm operator between appropriate Hilbert $\mathcal{A}$-modules.

Furthermore it turns out that the index $\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) \in K_0(\mathcal{A})$ of D^+ can be calculated algebraically in terms of the projections $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$. It is a generalization of the Maslov index in the following sense: For $\mathcal{A} = \mathbb{C}$ the spaces

$L_k := \mathcal{P}_k(\mathbb{C}^{2d}),\ k = 0,1,2$, are Lagrangian subspaces of $\mathbb{C}^{2d}$ with respect to the skew-hermitian form induced by I_0 and the standard scalar product on $\mathbb{C}^{2d}$. The dimension of $\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ is the Maslov index associated to the triple (L_1, L_1, L_2) [Wa].

Our main result (th. 4.4.11) is:

Theorem. *It holds*

$$\operatorname{ch} \tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) = -2[\eta(A_{I_0}) + \eta(A_{I_1}) + \eta(A_{I_2})] \in H_*^{dR}(\mathcal{A}_\infty) \ .$$

It is the analogue of [BK], th. 1.1.

The proof follows the lines of the proof of Bunke and Koch based on heat kernel methods. The main problems in our context are to prove the existence of the heat semigroups associated to D_I and to $D = D^+ \oplus D^-$, where D^- is the adjoint of D^+, to show that they have smooth integral kernels and to find the heat kernel estimates needed in the proof.

We explain our methods for D_I:

On the Hilbert $\mathcal{A}$-module $L^2([0,1], \mathcal{A}^{2d})$ the semigroup $e^{-tD_I^2}$ can be easily defined by the functional calculus of regular operators [La]. The aim is to show that it is a family of integral operators with integral kernels in $C^\infty([0,1] \times [0,1], M_{2d}(\mathcal{A}_\infty))$ and to find estimates for the integral kernel, in particular for its behaviour at small times.

For that we have to show that the semigroup $e^{-tD_I^2}$ restricts to a semigroup on the Banach space $L^2([0,1], \mathcal{A}_i^{2n})$ for any $i \in \mathbb{N}$. We can easily construct the heat kernel if the boundary conditions are given by orthogonal submodules. It turns out that D_I with general boundary conditions is unitarily equivalent to a bounded perturbation of the operator with orthogonal boundary conditions.

Then using a perturbation lemma about holomorphic semigroups we can define the semigroup $e^{-tD_I^2}$ on $L^2([0,1], \mathcal{A}_i^{2n})$.

In order to get estimates for the norm of the semigroup we study the resolvent set of the generator.

We prove that $e^{-tD_I^2}$ is a family of integral operator by comparing it with an approximation by a family of integral operators using Duhamel's principle. It will follow that the integral kernel of $e^{-tD_I^2}$ is in $C^\infty([0,1] \times [0,1], M_{2n}(\mathcal{A}_\infty))$. The comparison also allows to obtain estimates for the heat kernel.

Before the organisation of this thesis is described, I want to thank all people who supported this work: in particular my supervisor Ulrich Bunke for discussions, Margit Rösler for the introduction into the theory of semigroups, and my friends and family for moral support.

Summary

The proof of the theorem is organised in the following way:

In the first chapter the manifold M is described and we explain in more detail the family index theorem of Bunke and Koch [BK].

Then the differential algebra $\hat{\Omega}_*(\mathcal{A}_\infty)$, the homology $H_*^{dR}(\mathcal{A}_\infty)$ and the Chern character are introduced and investigated. Furthermore Lagrangian projections and the Maslov index are defined and studied.

In the second chapter we introduce the operator $D = D^+ \oplus D^-$ on M with boundary conditions defined by a triple of Lagrangian projections $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$.

Furthermore the operator D_I is defined and its properties on the Hilbert $\mathcal{A}$-module $L^2([0,1], \mathcal{A}^{2n})$ are studied.

Then we show that D^+ is Fredholm when acting between appropriate Hilbert C^*-modules and that its index equals $\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$. We define a compact perturbation $D(\rho)$ of D with closed range.

The third chapter is devoted to heat semigroups and their integral kernel, in particular those associated to D_I and $D(\rho)$.

In chapter four we introduce superconnections in order to define the η-form. Now finally the statement of the index theorem is well-defined. The remainder of the chapter is devoted to its proof. We have to introduce another family of operators $e^{-A(\rho)_t^2}$ where $A(\rho)_t$ is a rescaled superconnection associated to $D(\rho)$. As before we show that it is a family of integral operators with smooth integral kernels and obtain estimates for the integral kernels for small t.

As usual the proof of the index theorem compares the limit of the supertrace of $e^{-A(\rho)_t^2}$ for $t \to \infty$, which is the Chern character of the index of D^+, with the limit for $t \to 0$.

In chapter five many tools are presented: We introduce the function spaces we deal with, as for example the L^2- spaces, and study operators on them. In particular we recall the properties of Fredholm operators and regular operators on Hilbert C^*-modules, study Hilbert-Schmidt operators and pseudodifferential operators and develop the tools from holomorphic semigroup theory we need.

Notation and conventions: If not specified the vector spaces and algebras are complex, manifolds are smooth.

We often deal with $\mathbb{Z}/2$-graded spaces. Then the suffix $[\ ,\ ]_s$ denotes the supercommutator, tr_s the supertrace. In a graded context the tensor products are graded. For an ungraded space V, we denote by V^+ resp. V^- the same space endowed with a graduation: all elements are positive resp. negative.

Tensor products denoted by $\otimes$ are completed. The way of completion is indicated by a suffix in all but two cases: In the case of Hilbert C^*-modules $\otimes$ means the

Hilbert C^*-module tensor product, and if one of the spaces is nuclear, then $\otimes$ means $\otimes_\pi$ or $\otimes_\varepsilon$. The algebraic tensor product is denoted by $\odot$.

By a differentiable function on an open subset of $[0,1]^n$ we understand a function that can be differentiably continued to an open subset of $\mathbb{R}^n$. This induces the notion of a differentiable function on a manifold with corners, in our case $M \times M$.

For a subset $S \subset X$ in some set X we denote by $1_S : X \to \{0,1\}$ its characteristic function. If X is a metric space, $y \in X$ and $S \subset X$ then $d(y,S) := \inf_{x \in S} d(x,y)$. For $S_1, S_2 \subset X$ we set $d(S_1,S_2) := \inf_{x \in S_1} d(x,S_2)$

If E resp. F is a vector bundle on a space X resp. Y and $p_i(x_1,x_2) = x_i,\ i = 1,2$ for $(x_1,x_2) \in X \times Y$, then $E \boxtimes F$ denotes the vector bundle $p_1^*E \otimes p_2^*F$ on $X \times Y$.

The value of the constant C which we use in estimates may vary during a series of estimates without an explicit remark.

Chapter 1

Preliminaries

1.1 The geometric situation

In this section we define the spin manifold M and a Clifford module E on it. The Dirac operator associated to it will be the main object of our study. Another aim of this section is to fix the notation.

For $k \in \mathbb{Z}/6$ let Z_k be a copy of $\mathbb{R} \times [0,1]$. With the metric and orientation induced by the euclidian metric of $\mathbb{R}^2$ it is an oriented Riemannian manifold with boundary. Let (x_1^k, x_2^k) be the euclidian coordinates of Z_k.

For $r \geq -\frac{1}{2}$, $b \leq \frac{1}{3}$ and $k \in \mathbb{Z}/6$ let

$$F_k(r,b) := \{(x_1^k, x_2^k) \in \big(]r,\infty[\times[0,1]\big) \cup \big(]-1,r] \times ([0,b[\cup]1-b,1])\big)\} \subset Z_k \ .$$

We define

$$F(r,b) := \Big(\bigcup_{k\in\mathbb{Z}/6} F_k(r,b)\Big)/\sim$$

with $(x_1^k, x_2^k) \sim (-\frac{3}{2} - x_1^{k-1}, 1 - x_2^{k-1})$ for $(x_1^k, x_2^k) \in (]-1,-\frac{1}{2}[\times[0,b[)$ and $k \in \mathbb{Z}/6$. Then $F(r,b)$ inherites the structure of an oriented Riemannian manifold from the sets $F_k(r,b)$.

The open set $F(-\frac{1}{2},\frac{1}{3})\backslash\overline{F(-\frac{1}{3},\frac{1}{4})}$ is diffeomorphic to the open ring $B_1(0)^\circ\backslash B_{1/2}(0) \subset \mathbb{R}^2$ via an oriented diffeomorphism ϕ. We define the manifold with boundary

$$M := F(-\tfrac{1}{2},\tfrac{1}{3}) \cup_\phi B_1(0)^\circ \ .$$

For $r > -\frac{1}{2}$, $b \leq \frac{1}{3}$ we identify $F(r,b)$ and $F_k(r,b)$ with the corresponding subsets in M. Note that with the coordinates above the sets $F_k(r,b)$ are coordinate patches of M.

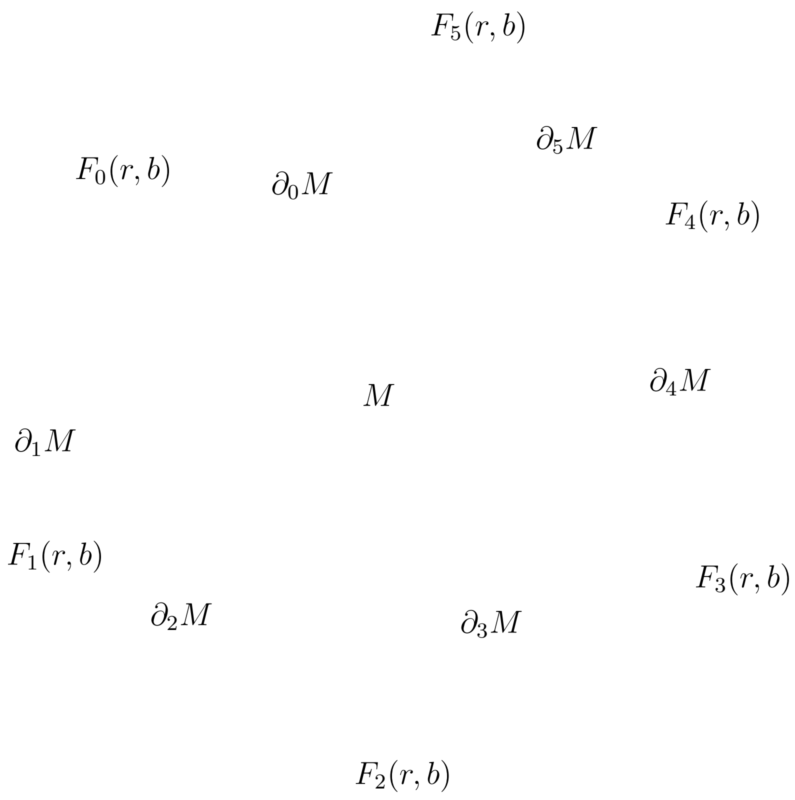
Make M into an oriented Riemannian manifold by extending the orientation and metric from $F := F(0,\frac{1}{4})$ to the whole of M. This allows us to identify TM and T^*M. Endow TM with the Levi-Civita connection.

The connected components of ∂M are labelled $\partial_k M$, $k \in \mathbb{Z}/6$, such that $\partial_k M \cup F_k(r,b) \subset \{x_2^k = 0\}$ and $\partial_{k+1} M \cup F_k(r,b) \subset \{x_2^k = 1\}$ for all $k \in \mathbb{Z}/6$.

For $r \geq 0$ and $b \leq \frac{1}{4}$ we define an open covering $\mathcal{U}(r,b) = \{\mathcal{U}_k\}_{k \in J}$ of M as follows: Let J be the disjoint union of $\mathbb{Z}/6$ with an one-element set $\{\clubsuit\}$. For $k \in \mathbb{Z}/6$ let $\mathcal{U}_k := F_k(r,b)$ and let $\mathcal{U}_\clubsuit := M \setminus \overline{F(r+1, b/2)}$.

For $r \geq 0$ let $M_r := M \setminus \overline{F(r,0)}$.

We can embed M diffeomorphically into $\mathbb{R}^2$, even with a diffeomorphism that is an isometry outside M_r for some $r > 0$. The image of the embedding is illustrated by the following picture:



Choose a spin structure on M and fix $d \in \mathbb{N}$. Let S be the spinor bundle endowed with a hermitian metric such that it is a selfadjoint Clifford module and with a Clifford connection. Let E be the graded vector bundle $((\mathbb{C}^+)^d \oplus (\mathbb{C}^-)^d) \otimes S$. The hermitian metric on S and the standard hermitian product on $\mathbb{C}^d$ induce a hermitian metric $< \cdot, \cdot >$ on E. Furthermore there are connections on E and its dual E^* induced by the Clifford connection on S.

Let $c : TM \to \mathrm{End}E$ be the Clifford multiplication. The Dirac operator associated to the Clifford module E is denoted by $\partial\!\!\!/_M$.

The oriented orthonormal frames $((-1)^k dx_1^k, (-1)^k dx_2^k)$ of $TM|_{F_k(0,\frac{1}{4})}$ patch together to an oriented orthonormal frame (e_1, e_2) of $TM|_F$.

The even endomorphism

$$I = c(e_1)c(e_2) : E^{\pm}|_F \to E^{\pm}|_F$$

defines a skew-hermitian form on $E^{\pm}|_F$, namely

$$E|_F \otimes E|_F \to \mathbb{C}, \ (x, y) \mapsto < x, Iy > \ .$$

The holonomy of $TM|_F$ is $4\pi\mathbb{Z}$ (measured with respect to any trivialisiation of TM on the whole of M), thus there are nonvanishing parallel sections of $S^+|_F$ and $S^-|_F$. We choose a parallel unit section s of $S^+|_F$ and fix once and for all the trivialisation of $E|_F$ defined for $x \in F$ by

$$E_x^+ = ((\mathbb{C}^+)^d \otimes S_x^+) \oplus ((\mathbb{C}^-)^d \otimes S_x^-) \to (\mathbb{C}^+)^d \oplus (\mathbb{C}^+)^d \ ,$$

$$(v \otimes s(x)) \oplus (w \otimes ic(e_1)s(x)) \mapsto (v, w) \ ,$$

and

$$E_x^- = ((\mathbb{C}^+)^d \otimes S_x^-) \oplus ((\mathbb{C}^-)^d \otimes S_x^+) \to (\mathbb{C}^-)^d \oplus (\mathbb{C}^-)^d \ ,$$
$$(v \otimes ic(e_1)s(x)) \oplus (w \otimes s(x)) \mapsto (v, w) \ .$$

With respect to the trivialisation the endomorphism I on E^+ corresponds to I_0 and $I|_{E^-}$ corresponds to $-I_0$ with

$$I_0 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in M_{2d}(\mathbb{C}) \ .$$

1.2 The commutative index problem

In this section we outline the index problem we want to transfer into a noncommutative context. For more details we refer to [BK].

Let $\mathbb{C}^{2d}$ be endowed with the standard hermitian product $< \ , \ >$ and the skew-hermitian form induced by I_0 and $< \ , \ >$.

Let B be a compact space and let (L_0, L_1, L_2) be a triple of pairwise transverse Lagrangian subbundles of the trivial bundle $B \times \mathbb{C}^{2d}$. For any $i = 0, 1, 2$ and $b \in B$ the Lagrangian subspace $L_i(b) \subset \mathbb{C}^{2d}$ defines a parallel Lagrangian subbundle of $E^+|_F$ via the trivialisation fixed in the previous section.

Let $D^+(b)$ be the Dirac operator associated to E with

$$\mathrm{dom}\, D^+(b) := \{s \in C_c^\infty(M, E^+) \mid s(x) \in L_i(b) \text{ for } x \in \partial_i M \cup \partial_{i+3} M, \ i = 0, 1, 2\} \ .$$

It turns out that for any $b \in B$ the kernel and cokernel of the closure of $D^+(b)$ are finite dimensional and that the K-theory class of the index bundle of the family $\{D^+(b)\}_{b \in B}$ equals the K-theory class of the Maslov index bundle of (L_0, L_1, L_2) whose definition is as follows:

The triple (L_0, L_1, L_2) induces a nondegenerate hermitian form h on L_0; namely let $v = v_1 + v_2,\ w = w_1 + w_2 \in L_0(b)$ with $v_1, w_1 \in L_1(b), v_2, w_2 \in L_2(b)$, then

$$h_b(v, w) :=< v_2, I_0 w_1 > \quad .$$

The Maslov index bundle is the element $[L_0^+] - [L_0^-] \in K^0(B)$ where L_0^+ and L_0^- are subbundles of L_0 with $L_0^+ \oplus L_0^- = L_0$ and such that h is positive on L_0^+ and negative on L_0^-.

If B is a manifold and the bundles are smooth, then by a generalization of the Atiyah-Patodi index theorem the Chern character of the index bundle can be expressed in terms of η-forms. We outline their definition:

Let ∂ be the differentiation operator on $C^\infty([0,1], \mathbb{C}^{2d})$.

For $i \neq j$ and any $b \in B$ the operator $D_I(b) := I_0 \partial$ with domain

$$\operatorname{dom} D_I(b) := \{s \in C^\infty([0,1], \mathbb{C}^{2d}) \mid s(0) \in L_i(b),\ s(1) \in L_j(b)\}$$

is essentially selfadjoint and its closure has a bounded inverse on $L^2([0,1], \mathbb{C}^{2d})$.

There is a family of rescaled superconnections A_t^I associated to σD_I on $C_1 \otimes \mathbb{C}^{2d}$ where C_1 is the Clifford algebra of the vector space $\mathbb{C}$: it is the graded complex unital algebra generated by the odd element σ with $\sigma^2 = 0$.

Let $\mathrm{Tr}_\sigma(a + \sigma b) := \mathrm{Tr}(a)$ (we do not want to specify here the space on which the trace Tr acts).

Then the η-form

$$\eta(L_i, L_j) := \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \mathrm{Tr}_\sigma D_I e^{-(A_t^I)^2} dt \in \Omega^*(B)$$

is well-defined.

The index theorem says

$$\mathrm{ch}(\operatorname{ind} D^+) = -2[\eta(L_0, L_1) + \eta(L_1, L_2) + \eta(L_2, L_0)] \in H^*_{dR}(B) \ .$$

Before we formulate a noncommutative version of this problem we present what will play the role of the algebra $\Omega^*(B)$, the Chern character, the de Rham cohomology $H^*_{dR}(B)$ and the Lagrangian subbundles. This is done in the next sections.

1.3 The algebra of differential forms

1.3.1 The universal graded differential algebra

Let $\mathcal{B}$ be an involutive locally m-convex Fréchet algebra with unit. In particular the group of invertible elements $\mathrm{Gl}(\mathcal{B})$ in $\mathcal{B}$ is open and the map $\mathrm{Gl}(\mathcal{B}) \to \mathrm{Gl}(\mathcal{B}),\ a \mapsto a^{-1}$ is continuous [Ma].

In this section we recall the definition of the topological universal graded differential algebra $\hat{\Omega}_*\mathcal{B}$ introduced by Karoubi [Kar] and collect its main properties. Most of the definitions of this section are taken from [Kar] or are an immediate generalisation from the algebraic case described in [CQ].

We write $\otimes_\pi$ for the completed projective tensor product.

Let

$$\hat{\Omega}_k\mathcal{B} := \mathcal{B} \otimes_\pi (\otimes_\pi^k(\mathcal{B}/\mathbb{C}))$$

and

$$\hat{\Omega}_*\mathcal{B} := \prod_{k=0}^{\infty} \hat{\Omega}_k\mathcal{B}\ .$$

Consider $\hat{\Omega}_*\mathcal{B}$ as a $\mathbb{Z}$-graded vector space with the following additional structures:

Product: There is a graded continuous product on $\hat{\Omega}_*\mathcal{B}$: It is defined for elementary tensors by

$$(b_0 \otimes b_1 \otimes \ldots \otimes b_k)(b_{k+1} \otimes b_{k+2} \otimes \ldots b_n) := \sum_{j=0}^{k} (-1)^{k-j} (b_0 \otimes b_1 \otimes \ldots \otimes b_j b_{j+1} \otimes b_{j+2} \otimes \ldots b_n)\ .$$

It is continuous, since the multiplication map $\mathcal{B} \otimes_\pi \mathcal{B} \to \mathcal{B}$ is continuous by the universal property of the projective tensor product. With the product $\hat{\Omega}_*(\mathcal{B})$ is a locally m-convex Fréchet algebra.

Differential: There is a continuous differential d of degree one on the graded algebra $\hat{\Omega}_*\mathcal{B}$ defined on elementary tensors by

$$\mathrm{d}(b_0 \otimes b_1 \otimes \ldots \otimes b_k) := 1 \otimes b_0 \otimes b_1 \otimes \ldots \otimes b_k\ .$$

It satisfies the graded Leipniz's rule: For $\alpha \in \hat{\Omega}_k\mathcal{B}$ and $\beta \in \hat{\Omega}_*\mathcal{B}$ it holds

$$\mathrm{d}(\alpha\beta) = (\mathrm{d}\,\alpha)\beta + (-1)^k \alpha(\mathrm{d}\,\beta)\ .$$

Furthermore it holds

$$b_0 \otimes b_1 \otimes \ldots \otimes b_k = b_0\,\mathrm{d}\,b_1\,\mathrm{d}\,b_2 \ldots \mathrm{d}\,b_k\ .$$

If $\mathcal{B}$ is a Banach algebra, then d is a map of norm one.

Involution: We extend the $*$-operation on $\mathcal{B}$ to a continuous involution on $\hat{\Omega}_*\mathcal{B}$ by setting

$$(b_0 \otimes b_1 \otimes \ldots \otimes b_k)^* := (1 \otimes b_k^* \otimes b_{k-1}^* \otimes \ldots \otimes b_1^*) b_0^*$$

or equivalently

$$(b_0 \,\mathrm{d}\, b_1 \,\mathrm{d}\, b_2 \ldots \mathrm{d}\, b_k)^* = (\mathrm{d}\, b_k^* \,\mathrm{d}\, b_{k-1}^* \ldots \mathrm{d}\, b_1^*) b_0^* \ .$$

For $\omega_1, \omega_2 \in \hat{\Omega}_*\mathcal{B}$ it holds

$$(\omega_1\omega_2)^* = \omega_2^*\omega_1^* \ ,$$

and for $\omega \in \hat{\Omega}_k\mathcal{B}$

$$(\mathrm{d}\,\omega)^* = (-1)^k \,\mathrm{d}(\omega^*) \ .$$

With these structures $\hat{\Omega}_*\mathcal{B}$ is an involutive graded differential Fréchet locally m-convex algebra. It behaves functorially in $\mathcal{B}$.

If $\omega \in \hat{\Omega}_*\mathcal{B}$, its part in degree n is denoted by ω^n .

Let

$$\hat{\Omega}_{\leq m}\mathcal{B} := \hat{\Omega}_*\mathcal{B} / \prod_{k=m+1}^{\infty} \hat{\Omega}_k\mathcal{B} \ .$$

We identify $\hat{\Omega}_{\leq m}\mathcal{B}$ as a graded vector space with the subspace of forms up to degree m in $\hat{\Omega}_*\mathcal{B}$.

Now we come to the definitions of the homology and the Chern character of $\mathcal{B}$:

The Fréchet space $\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}$ generated by the supercommutators in $\hat{\Omega}_*\mathcal{B}$ is preserved by d by Leipniz's rule. It follows that $(\hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}, \mathrm{d})$ is a complex.

Definition 1.3.1. *The de Rham homology of $\mathcal{B}$ is*

$$H_*^{dR}(\mathcal{B}) := H_*(\hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}, \mathrm{d}) \ .$$

On the right hand side we take the topological homology, i.e. we quotient out the closure of the range of d *in order to obtain a Hausdorff space.*

We extend d to the a map $\mathrm{d} : M_n(\hat{\Omega}_*\mathcal{B}) \to M_n(\hat{\Omega}_*\mathcal{B}),\ A \mapsto \mathrm{d}(A)$ by applying d component by component. Note the difference between $\mathrm{d}\, A = \mathrm{d} \circ A$ and $\mathrm{d}(A)$. Sometimes we write $(\mathrm{d}\, A)$ for $\mathrm{d}(A)$.

For $A \in M_n(\hat{\Omega}_k(\mathcal{B}))$ it holds

$$\mathrm{d}\, A = (\mathrm{d}\, A) + (-1)^k A \,\mathrm{d} \ .$$

Definition 1.3.2. *Let $P \in M_n(\mathcal{B})$ be a projection. Then*

$$\mathrm{ch}(P) := \sum_{k=0}^{\infty} (-1)^k \mathrm{tr} P(\mathrm{d}(P))^{2k} \in \hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}$$

is the Chern character form of P.

Proposition 1.3.3. *1. If $P : [0,1] \to M_n(\mathcal{B})$ is a differentiable path of projections, then*

$$\mathrm{ch}(P(1)) - \mathrm{ch}(P(0)) = \tfrac{1}{2} \sum_{k=1}^{\infty} (-1)^k \,\mathrm{d}\big(\mathrm{tr} P(1)(\mathrm{d}\, P(1))^{2k-1}) - \mathrm{tr} P(0)(\mathrm{d}\, P(0))^{2k-1})\big) \ .$$

In particular $\mathrm{ch}(P(1)) - \mathrm{ch}(P(0))$ *is exact.*

2. *The Chern character form is closed.*

3. *If $\mathcal{B}$ is a local Banach algebra [Bl], then the Chern character form induces a homomorphism*

$$\mathrm{ch} : K_0(\mathcal{B}) \to H_*^{dR}(\mathcal{B}), \ \ \mathrm{ch}([P] - [Q]) := \mathrm{ch}(P) - \mathrm{ch}(Q) \ .$$

Proof. First note that for a projection $P \in M_n(\mathcal{B})$ and for $v \in \mathcal{B}^n$ it holds

$$0 = \mathrm{d}((1-P)Pv) = (1-P)\,\mathrm{d}\, Pv - (\mathrm{d}\, P)Pv \ ,$$

hence $P(\mathrm{d}\, P)P = 0$ and $(1-P)\,\mathrm{d}\, P = (\mathrm{d}\, P)P$. From $0 = \mathrm{d}((1-P)P)$ it follows $(\mathrm{d}\, P)P = (1-P)(\mathrm{d}\, P)$, therefore

$$P(\mathrm{d}\, P)^2 = (\mathrm{d}\, P)^2 P \ .$$

1) Let P' be the derivative of P. It holds

$$0 = ((1-P)P)' = (1-P)P' - P'P \ ,$$

hence $PP'P = 0$ and $(1-P)P' = P'P$.
It holds

$$(\mathrm{tr} P(\mathrm{d}(P))^{2k})' = \mathrm{tr} P'(\mathrm{d}(P))^{2k} + \mathrm{tr} P((\mathrm{d}(P))^{2k})' \ .$$

The first term on the right hand side vanishes by

$$\begin{aligned} \mathrm{tr} P'(\mathrm{d}(P))^{2k} &= \mathrm{tr} P'P(\mathrm{d}(P))^{2k} + \mathrm{tr} PP'(\mathrm{d}(P))^{2k} \\ &= \mathrm{tr} P'P(\mathrm{d}(P))^{2k}P + \mathrm{tr} PP'(\mathrm{d}(P))^{2k}P \\ &= 2\ \mathrm{tr} PP'P(\mathrm{d}(P))^{2k} \\ &= 0 \ . \end{aligned}$$

The second term vanishes for $k = 0$ and for $k > 0$ it is exact by

$$\begin{aligned}
\mathrm{tr} P((\mathrm{d}(P))^{2k})' &= \sum_{i=0}^{2k-1} \mathrm{tr}(P(\mathrm{d}\, P)^i (\mathrm{d}\, P)' (\mathrm{d}\, P)^{2k-i-1}) \\
&= k \ \mathrm{tr}(\mathrm{d}\, P)^{2k-1} P(\mathrm{d}\, P)' + k \ \mathrm{tr}((\mathrm{d}\, P)^{2k-2} P(\mathrm{d}\, P)(\mathrm{d}\, P)' \\
&= k \ \mathrm{tr}(\mathrm{d}\, P)^{2k-1} (\mathrm{d}\, P)' \\
&= \tfrac{1}{2} \ (\mathrm{tr}(\mathrm{d}\, P)^{2k})' \\
&= \tfrac{1}{2} \ \mathrm{d}(\mathrm{tr} P(\mathrm{d}\, P)^{2k-1})'
\end{aligned}$$

2) It holds

$$\begin{aligned}
\mathrm{d}\, \mathrm{tr} P(\mathrm{d}\, P)^{2k} &= \mathrm{tr}(\mathrm{d}\, P)^{2k+1} \\
&= \mathrm{tr}(1-P)(\mathrm{d}\, P)^{2k+1}(1-P) + \mathrm{tr} P(\mathrm{d}\, P)^{2k+1} P \\
&= 0 \ .
\end{aligned}$$

3) follows from 1) and 2). □

1.3.2 Supercalculus

Let $\mathcal{B}$ be as before.

In this section all spaces and tensor products are $\mathbb{Z}/2$-graded. If no graduation is specified we assume the trivial graduation.

Let $V = V^+ \oplus V^-$ be a $\mathbb{Z}/2$-graded complex vector space with $\dim V^+ = m, \ \dim V^- = n$ and consider $\hat{\Omega}_*\mathcal{B}$ as a $\mathbb{Z}/2$-graded space wiht the even/odd-grading.

The space $V \otimes \hat{\Omega}_*\mathcal{B}$ is a free $\mathbb{Z}/2$-graded right $\hat{\Omega}_*\mathcal{B}$-module. It is furthermore a left supermodule of the superalgebra $\mathrm{End}(V) \otimes \hat{\Omega}_*\mathcal{B}$. (Note that our setting differs from the corresponding one defined in [BGV], since we consider right $\hat{\Omega}_*\mathcal{B}$-modules. This sometimes causes different signs.)

The supertrace $\mathrm{tr}_s : \mathrm{End}(V) \to \mathbb{C}$ extends to a supertrace

$$\mathrm{tr}_s : \mathrm{End}(V) \otimes \hat{\Omega}_*\mathcal{B} \to \hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}$$

$$\mathrm{tr}_s(T \otimes \omega) := \mathrm{tr}_s(T)\omega \ .$$

By dividing out the supercommutator we ensure that $\mathrm{tr}_s([T_1, T_2]_s) = 0$ for $T_1, T_2 \in \mathrm{End}(V) \otimes \hat{\Omega}_*\mathcal{B}$.

Note that the differential d acts on elements of $V \otimes \hat{\Omega}_*\mathcal{B}$ resp. $\mathrm{End}(V) \otimes \hat{\Omega}_*\mathcal{B}$ by

$$\mathrm{d}(A \otimes \omega) = (-1)^{\deg A} A \otimes \mathrm{d}\, \omega$$

for $A \in V^{\pm}$ resp. $A \in \mathrm{End}^{\pm}(V)$ and $\omega \in \hat{\Omega}_*\mathcal{B}$.

Since $\mathrm{tr}_s(A \otimes \omega) = 0$ if $A \in \mathrm{End}^-(V)$ it follows

$$\mathrm{tr}_s\, \mathrm{d}(T) = \mathrm{d}\, \mathrm{tr}_s T$$

for $T \in \mathrm{End}(V) \otimes \hat{\Omega}_* \mathcal{B}$.

Furthermore from Leipniz's rule it follows for $A \otimes \omega \in \mathrm{End}^{\pm}(V) \otimes \hat{\Omega}_k \mathcal{B},\ v \in V \otimes \hat{\Omega}_* \mathcal{B}$:

$$\mathrm{d}((A \otimes \omega)v) = \mathrm{d}(A \otimes \omega)v + (-1)^{\deg A + k}(A \otimes \omega)\, \mathrm{d}\, v\ .$$

Let now M be a Riemannian manifold, possibly with boundary. In §5.2.3 we introduce the notion of Hilbert-Schmidt operators. We use the notation of §5.2.2 and §5.2.3 in the following.

A trace class operator on $L^2(M, V \otimes \hat{\Omega}_{\le \mu} \mathcal{B})$ is an operator $T = \sum_{i=1}^n A_i B_i$ where A_i, B_i are Hilbert-Schmidt operators such that $(x \mapsto k_{A_i}(x, \cdot)) \in C(M, L^2(M, \mathrm{End} V \otimes \hat{\Omega}_{\le \mu} \mathcal{B}))$ and $(y \mapsto k_{A_i}(\cdot, y)) \in C(M, L^2(M, \mathrm{End} V \otimes \hat{\Omega}_{\le \mu} \mathcal{B}))$, and analogoulsy for $B_i,\ i = 1, \dots, n$.

In particular T is a Hilbert-Schmidt operator with continuous integral kernel k_T. We define

$$\mathrm{Tr}_s T := \int_M \mathrm{tr}_s k_T(x,x) \in \hat{\Omega}_{\le \mu} \mathcal{B} / \overline{[\hat{\Omega}_{\le \mu} \mathcal{B}, \hat{\Omega}_{\le \mu} \mathcal{B}]}\ .$$

It holds

$$|\mathrm{Tr}_s A_i B_i| \le \|A_i\|_{HS} \|B_i\|_{HS}$$

and

$$\mathrm{Tr}_s [A_i, B_i]_s = 0\ .$$

1.3.3 The algebra $\mathcal{A}_\infty$ and its properties

Let $(\mathcal{A}_j, \iota_{j+1,j} : \mathcal{A}_{j+1} \to \mathcal{A}_j)_{j \in \mathbb{N}_0}$ be a projective system of involutive Banach algebras with unit satisfying the following conditions ([Lo], §2.1):

- The algebra $\mathcal{A} := \mathcal{A}_0$ is a C^*-algebra.
- For any $j \in \mathbb{N}_0$ the map $\iota_{j+1,j} : \mathcal{A}_{j+1} \to \mathcal{A}_j$ is injective.
- For any $j \in \mathbb{N}_0$ the map $\iota_j : \mathcal{A}_\infty := \varprojlim_i \mathcal{A}_i \to \mathcal{A}_j$ has dense range.
- For any $j \in \mathbb{N}_0$ the algebra $\mathcal{A}_j$ is stable with respect to the holomorphic functional calculus in $\mathcal{A}$.

The projective limit $\mathcal{A}_\infty$ is an involutive locally m-convex Fréchet algebra with unit.

The motivating example is $\mathcal{A}_j = C^j(M)$ for a closed smooth manifold M.

Proposition 1.3.4. *The following properties hold for any $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$:*

1. *The map $\iota_j : \mathcal{A}_\infty \to \mathcal{A}_j$ is injective.*

2. *The algebras $M_n(\mathcal{A}_\infty)$ and $M_n(\mathcal{A}_j)$ are stable with respect to the holomorphic functional calculus in $M_n(\mathcal{A})$.*

3. *The map $\iota_{0*} : K_0(\mathcal{A}_\infty) \to K_0(\mathcal{A})$ is an isomorphism.*

Proof. 1) follows immediately.

2) follows from [Bo], prop. A.2.2.

3) follows from [Bo], th. A.2.1. $\square$

1.3.4 Properties of $\hat{\Omega}_*\mathcal{A}_\infty$

The projective system $(\mathcal{A}_j, \iota_{j+1,j})_{j\in\mathbb{N}_0}$ induces two projective systems of involutive graded differential Fréchet algebras.

One of them is given by the maps

$$\iota_{j+1,j*} : \hat{\Omega}_*\mathcal{A}_{j+1} \to \hat{\Omega}_*\mathcal{A}_j \ .$$

Furthermore $(\hat{\Omega}_{\le m}\mathcal{A}_j)_{m,j\in\mathbb{N}}$ is a projective system of involutive Banach graded differential algebras. The limits coincide:

The inclusion $\hat{\Omega}_{\le m}\mathcal{A}_j \to \hat{\Omega}_*\mathcal{A}_j$ is left inverse to the projection $\hat{\Omega}_*\mathcal{A}_j \to \hat{\Omega}_{\le m}\mathcal{A}_j$. The induced maps between the projective limits are inverse to each other. It follows

$$\varprojlim_j \hat{\Omega}_*\mathcal{A}_j \cong \varprojlim_{j,m} \hat{\Omega}_{\le m}\mathcal{A}_j \ .$$

Furthermore the inclusions $\iota_{j*} : \hat{\Omega}_*\mathcal{A}_\infty \to \hat{\Omega}_*\mathcal{A}_j$ induce a map

$$\iota_* : \hat{\Omega}_*\mathcal{A}_\infty \to \varprojlim_j \hat{\Omega}_*\mathcal{A}_j \ .$$

It is an isomorphism by the following proposition:

Proposition 1.3.5. *There are the following canonical isomorphisms of involutive Fréchet locally m-convex graded differential algebras:*

1. $\hat{\Omega}_*\mathcal{A}_\infty \cong \varprojlim_j \hat{\Omega}_*\mathcal{A}_j$,

2. $\hat{\Omega}_*\mathcal{A}_\infty \cong \varprojlim_{j,m} \hat{\Omega}_{\le m}\mathcal{A}_j$.

There is the following canonical isomorphism of graded Fréchet spaces:

3. $$\hat{\Omega}_*\mathcal{A}_\infty/\overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s} \cong \varprojlim_{j,m} \hat{\Omega}_{\le m}\mathcal{A}_j/\overline{[\hat{\Omega}_{\le m}\mathcal{A}_j, \hat{\Omega}_{\le m}\mathcal{A}_j]_s} \ .$$

Proof. 1): It is enough to prove that the right hand side and the left hand side are isomorphic as topological vector spaces. This follows from the fact that projective limits and projective tensor products commute ([Kö] 41.6).

2) follows from the remarks preceeding the proposition.

The following lemma implies 3). $\square$

The importance of this proposition for our purposes is the following: Since it is easier to deal with Banach spaces than with Fréchet spaces we will prove our index theorem in $\hat{\Omega}_{\le m}\mathcal{A}_j/\overline{[\hat{\Omega}_{\le m}\mathcal{A}_j, \hat{\Omega}_{\le m}\mathcal{A}_j]_s}$, making sure that the expressions in the index theorem behave well when taking the projective limit. By the proposition this will prove the index theorem in $\hat{\Omega}_*\mathcal{A}_\infty/\overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s}$.

Lemma 1.3.6. *Let $(A_j, \iota_{j+1,j})_{j\in\mathbb{N}}$ be a projective system of Banach spaces and let $(B_j, \iota_{j+1,j}|_{B_j})_{j\in\mathbb{N}}$ be a projective system such that B_j is a closed subspace of A_j for all $j \in \mathbb{N}$.*

Let $A_\infty := \varprojlim_j A_j$ and $B_\infty = \varprojlim_j B_j$ and assume further that the image of A_∞ is dense in A_j and the image of B_∞ is dense in B_j for any $j \in \mathbb{N}$.

Then it holds canonically

$$A_\infty/B_\infty \cong \varprojlim_j A_j/B_j \ .$$

Proof. For $j \in \mathbb{N}$ let $\iota_j : A_\infty \to A_j$ be the induced maps. We prove that the map

$$\iota_\infty : A_\infty/B_\infty \to \varprojlim_j A_j/B_j$$

is an isomorphism:

It is well-defined since $\iota_j(B_\infty) \subset B_j$ for all $j \in \mathbb{N}$.

It is injective: Let $a \in A_\infty/B_\infty$ with $\iota_\infty a = 0$, then for $\tilde{a} \in A_\infty$ with $[\tilde{a}] = a$ it holds $\iota_j\tilde{a} \in B_j$ for all $j \in \mathbb{N}$, so $\tilde{a} \in B_\infty$, thus $a = 0$.

It is surjective: For simplicity we assume that the norms of the maps $\iota_{j+1,j}$ are smaller or equal to one. This can be obtained by rescaling the norms inductively.

Let $a \in \varprojlim_j A_j/B_j$ and choose $a_j \in A_j$ such that $[a_j] \in A_j/B_j$ is the image of a with respect to the map

$$\varprojlim_j A_j/B_j \to A_j/B_j \ .$$

By modifying the sequence inductively we can assume that it holds :

$$(*) \qquad |\iota_{j+1,j}a_{j+1} - a_j| \le j^{-1} \ .$$

The modification is as follows: We define $a'_1 := a_1$. Assume a'_j is defined. It holds $\iota_{j+1,j}a_{j+1} - a'_j \in B_j$. Since the image of B_∞ is dense in B_j, we may choose $b_{j+1} \in B_\infty$ with

$$|\iota_j b_{j+1} - (\iota_{j+1,j}a_{j+1} - a'_j)| \le j^{-1}$$

and define $a'_{j+1} := a_{j+1} + \iota_{j+1} b_{j+1}$.

For $n \geq j$ let $\iota_{n,j} : A_n \to A_j$ be the injection. From $(*)$ it follows that for any $j \in \mathbb{N}$ the sequence $(\iota_{n,j}(a_n))_{n \geq j}$ is a Cauchy sequence in A_j. Its image in A_j/B_j is constant. Let $\tilde{a}_j \in A_j$ be its limit.

It holds $\iota_{j+1,j}\tilde{a}_{j+1} = \tilde{a}_j$ and $[\tilde{a}_j] = [a_j] \in A_j/B_j$. It follows that there is an element $\tilde{a} \in A_\infty$ with $\iota_j \tilde{a} = \tilde{a}_j$. It holds $\iota_\infty[\tilde{a}] = a$. □

1.4 Lagrangian projections

Let $\mathcal{A}$ be as before.

In this section we define and study the analogue of Lagrangian subbundles and of the Maslov index bundle.

Definition 1.4.1. *Let $n \in \mathbb{N}$.*

Two selfadjoint projections $P_1, P_2 \in M_n(\mathcal{A})$ are called transverse if

$$\operatorname{Ran} P_1 \oplus \operatorname{Ran} P_2 = \mathcal{A}^n \ .$$

We will often use the following criterium of transversality: Two selfadjoint projections P_1, P_2 are transverse if and only there exist $a, b \in \mathrm{Gl}(\mathcal{A})$ such that $aP_1 + bP_2 \in M_n(\mathcal{A})$ is invertible, and this is equivalent to the invertibility of $aP_1 + bP_2$ for any $a, b \in \mathrm{Gl}(\mathcal{A})$.

1.4.1 Definition and properties

Definition 1.4.2. *For $n \in \mathbb{N}$ let $\mathcal{A}^{2n}$ be the Hilbert $\mathcal{A}$-module endowed with the standard $\mathcal{A}$-valued scalar product and let*

$$I_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} : \mathcal{A}^n \oplus \mathcal{A}^n \to \mathcal{A}^n \oplus \mathcal{A}^n \ .$$

A Lagrangian projection on $\mathcal{A}^{2n}$ is a selfadjoint projection $P \in M_{2n}(\mathcal{A})$ with

$$PI_0 = I_0(1-P) \ .$$

Note that if $\mathcal{A} = C(B)$ for some compact space B, then the Lagrangian projections are in one-to-one correspondence with the subbundles of $B \times \mathbb{C}^{2n}$ that are Lagrangian with respect to the skew-hermitian form induced by I_0. The transversality of two projections is equivalent to the transversality of the corresponding subbundles.

Note furthermore that any Lagrangian projection P on $\mathcal{A}^{2n}$ is a Lagrangian projection on $M_n(\mathcal{A})^2$ as well, where we consider $M_n(\mathcal{A})^2$ as a Hilbert $M_n(\mathcal{A})$-module.

Denote

$$P_s := \tfrac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \ .$$

Lemma 1.4.3. *1. For any Lagrangian projection P of $\mathcal{A}^{2n}$ there is a unitary $p \in M_n(\mathcal{A})$ such that*

$$P = \tfrac{1}{2}\begin{pmatrix} 1 & p^* \\ p & 1 \end{pmatrix} .$$

2. For any Lagrangian projection P on $\mathcal{A}^{2n}$ the unitary

$$U = \begin{pmatrix} 1 & 0 \\ 0 & p^* \end{pmatrix} \in M_{2n}(\mathcal{A})$$

with p as in 1) fulfills $UI_0 = I_0U$ and $UPU^* = P_s$.

Proof. 1) Since P is selfadjoint, there are $a, b, c \in M_n(\mathcal{A})$ with $a = a^*, c = c^*$ such that $P = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$. From $PI_0 = I_0(1 - P)$ it follows

$$\begin{pmatrix} ia & -ib \\ ib^* & -ic \end{pmatrix} = \begin{pmatrix} i(1-a) & -ib \\ ib^* & -i(1-c) \end{pmatrix} ,$$

thus $a = c = \frac{1}{2}$. From $P^2 = P$ it follows furthermore that $2b$ is unitary.
2) is clear. □

Lemma 1.4.4. *Let $P \in M_{2n}(\mathcal{A}_\infty)$ be a Lagrangian projection of $\mathcal{A}^{2n}$ transverse to P_s. Let $\tilde{P} \in M_{2n}(\mathbb{C})$ be a complex Lagrangian projection. Then for any $0 < \varepsilon_1 < \varepsilon_2$ there is a smooth path of unitaries $U : [0, \varepsilon_2] \to M_{2n}(\mathcal{A}_\infty)$ such that*

1. *$U(0)PU(0)^* = \tilde{P}$,*
2. *U equals 1 on a neighbourhood of ε_2,*
3. *U is constant on $[0, \varepsilon_1]$,*
4. *U is diagonal with respect to the decomposition $\mathcal{A}^{2n} = \mathcal{A}^n \oplus \mathcal{A}^n$.*

Note that 4) implies $UI_0 = I_0U$.

Proof. It is enough to prove the assertion for $\tilde{P} = P_s$.
Let p be as in the previous lemma applied to P.
Since P and P_s are transverse, $P - P_s$ is invertible. It follows that $p - 1$ and $p^* - 1$ are invertible, so $\log p$ and $\log p^*$ are well-defined if we choose the complement of $[0, \infty)$ in $\mathbb{C}$ as a domain for the logarithm.
Let $\chi : [0, 1] \to [0, 1]$ be a smooth function with $\chi|_{[0,\varepsilon_1]} = 0$ and $\chi(t) = 1$ for $t \in [\frac{\varepsilon_2 - \varepsilon_1}{2}, \varepsilon_2]$. Then the smooth path of unitaries

$$\gamma : [0, \varepsilon_2] \to M_n(\mathcal{A}),\ \gamma(t) := \exp(2\pi i\chi(t) + (1 - \chi(t)) \log p^*)$$

connects p^* with 1. It holds $\gamma(t) \in M_n(\mathcal{A}_\infty)$ for all $t \in [0, \varepsilon_2]$ since $M_n(\mathcal{A}_\infty)$ is stable under holomorphic function calculus in $M_n(\mathcal{A})$ by prop. 1.3.4.
Let

$$U := \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} .$$

It satisfies the conditions. □

1.4.2 The Maslov index

Let (P_0, P_1, P_2) be a triple of pairwise transverse Lagrangian projections on $\mathcal{A}^{2n}$. Then we can write any $x \in \mathcal{A}^{2n}$ uniquely as $x = x_1 + x_2$ with $x_i \in \operatorname{Ran} P_i,\ i = 1, 2$, namely $x_1 = P_1(P_1 - P_2)^{-1}x$ and $x_2 = -P_2(P_1 - P_2)^{-1}x$.

The form

$$h : \operatorname{Ran} P_0 \times \operatorname{Ran} P_0 \to \mathcal{A},\ (x, y) \mapsto < x_2, I_0 y_1 >$$

is hermitian and its radical vanishes [Wa]. It can be represented by the matrix

$$A := -P_0(P_1 - P_2)^{-1}P_2 I_0 P_1 (P_1 - P_2)^{-1} P_0 \in M_{2n}(\mathcal{A}) \ .$$

Since A is the composition of projections and invertibles, the range of A is closed, thus the restriction of A to the range of P_0 is invertible. Hence the hermitian form h is non-singular and defines an element in $K_0(\mathcal{A})$ [Ros].

Definition 1.4.5. *The Maslov index $\tau(P_0, P_1, P_2)$ of a triple of pairwise transverse Lagrangian projections (P_0, P_1, P_2) is the class of the hermitian form h in $K_0(\mathcal{A})$.*

We can express the Maslov index in terms of A as follows:

$$\tau(P_0, P_1, P_2) = [1_{\{x>0\}}(A)] - [1_{\{x<0\}}(A)] \in K_0(\mathcal{A}) \ .$$

Note that an even permutation of the projections leaves the Maslov index unchanged whereas an odd permutation turns it into its negative.

Proposition 1.4.6. *For $i = 0, 1, 2$ let $P_i : [0, 1] \to M_{2n}(\mathcal{A})$ be continuous paths of Lagrangian projections such that $P_i(t) - P_j(t)$ is invertible for all $i \neq j$ and all $t \in [0, 1]$.*

Then the Maslov index $\tau(P_0(t), P_1(t), P_2(t))$ does not depend on t.

Proof. The selfadjoint element $A(t) \in M_{2n}(\mathcal{A})$ defined by $(P_0(t), P_1(t), P_2(t))$ as above depends continuously on t for all $t \in [0, 1]$. It follows that the projections $1_{\{x>0\}}(A(t))$ and $1_{\{x<0\}}(A(t))$ also depend continuously on t, thus their K-theory classes are constant. □

Let $\mathcal{A} = C(B)$ for some compact space B and let (P_0, P_1, P_2) be a triple of pairwise transverse Lagrangian projections in $M_{2n}(C(B))$. Let (L_0, L_1, L_2) be the corresponding triple of Lagrangian subbundles of $B \times \mathbb{C}^{2n}$. Then the Maslov index bundle $[L_0^+] - [L_0^-]$ defined in §1.2 corresponds to $\tau(P_0, P_1, P_2)$ under the canonical isomorphism $K^0(B) \cong K_0(C(B))$.

Now we study in more detail the Maslov index of a pair (P_0, P_1, P_2) with $P_0 = P_s$. The general case can be reduced to that case by lemma 1.4.3.

The Cayley transform yields a bijective map from the space of selfadjoint elements in $M_d(\mathcal{A})$ to space of projections transverse to P_s, namely

$$a \mapsto P(a) := \tfrac{1}{2} \begin{pmatrix} 1 & \frac{a-i}{a+i} \\ \frac{a+i}{a-i} & 1 \end{pmatrix} \ .$$

Lemma 1.4.7. *Let a_1, a_2 be selfadjoint elements of $M_n(\mathcal{A})$. Then $P(a_1)$ and $P(a_2)$ are transverse projections if and only if $a_1 - a_2$ is invertible.*

Proof. Let $U := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then it holds

$$UP(a_j)U^* = (a_j^2 + 1)^{-1} \begin{pmatrix} a_j^2 & -ia_j \\ ia_j & 1 \end{pmatrix} .$$

Now $P(a_1)$ and $P(a_2)$ are transverse if and only if

$$(a_1^2 + 1)UP(a_1)U^* - (a_2^2 + 1)UP(a_2)U^*$$

is invertible and this is the case if and only if $a_1 - a_2$ is invertible. □

Lemma 1.4.8. *Let (P_s, P_1, P_2) be a triple of pairwise transverse projections and let $a_1, a_2 \in M_n(\mathcal{A})$ be such that $P_i = P(a_i),\ i = 1, 2$. Then it holds:*

1. *There are continuous paths $P_1, P_2 : [0, 2] \to M_{2n}(\mathcal{A})$ such that $P_s, P_1(t), P_2(t)$ are pairwise transverse for all $t \in [0, 2]$ and such that $P_1(2) = P(2p^+ - 1)$ and $P_2(2) = P(1 - 2p^+)$ with $p^+ := 1_{\{x>0\}}(a_1 - a_2)$.*

2. *It holds*
$$\tau(P_s, P_1, P_2) = [1_{\{x>0\}}(a_1 - a_2)] - [1_{\{x<0\}}(a_1 - a_2)] .$$

Proof. 1) For $t \in [0, 1]$ define $a_1(t) := \frac{1}{2}(t(a_1 - a_2) + (1 - t)a_1)$ and $a_2(t) := \frac{1}{2}(t(a_2 - a_1) + (1 - t)a_2)$. Then $a_1(t) - a_2(t) = a_1 - a_2$ is invertible, thus the projections $P(a_1(t))$ and $P(a_2(t))$ are transverse. Furthermore $a_1(0) = a_1$ and $a_1(1) = a_1 - a_2$ whereas $a_2(0) = a_2$ and $a_2(1) = a_2 - a_1$.

Let $p^+ := 1_{\{x>0\}}(a_1 - a_2)$. For $t \in [1, 2]$ let $a_1(t)$ be a path of invertible selfadjoint elements with $a_1(1) = a_1 - a_2$ and $a_1(2) = p^+ - (1 - p^+)$ and let $a_2(t) = -a_1(t)$.

Then the paths $P(a_1(t))$ and $P(a_2(t))$ satisfy the conditions.

2) By 1) and prop. 1.4.6 it holds $\tau(P_s, P_1, P_2) = \tau(P_s, P(2p^+ - 1), P(1 - 2p^+))$.

Note that the Cayley transform of $2p^+ - 1$ is $i(2p^+ - 1)$. By computing the matrix A one sees that it holds $1_{\{x>0\}}(A) = p^+$ and $1_{\{x<0\}}(A) = (1 - p^+)$. □

Chapter 2

The Fredholm operator and its index

2.1 The operator D on M

2.1.1 Definition of D

Now we come back to the geometric situation described in §1.1.

We tensorize the bundle E with the C^*-algebra $\mathcal{A}$ in order to get an $\mathcal{A}$-vector bundle [MF]. Furthermore for $i, \mu \in \mathbb{N}_0$ we consider the bundle $E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ of right $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-modules. Keep in mind that E can be trivialized on M via a global orthonormal frame. Thus we do not need a theory of Banach space bundles in this context.

Parallel transport is defined on $E \otimes \mathcal{A}$ resp. $E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$.

The hermitian metrics on E extends to an $\mathcal{A}$-valued scalar product $< \cdot, \cdot >$ on $E \otimes \mathcal{A}$ and to an $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-valued non-degenerated product on $E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ (see §5.2.3 for this notion).

By the trivialisation of $E|_F$ fixed in §1.1 we identify $(E|_F \otimes \mathcal{A}, I)_x$ for $x \in F$ with $(\mathcal{A}^{4d}, I_0 \oplus (-I_0))$ as a Hilbert $\mathcal{A}$-module with a skew-hermitian structure, and $(E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)_x$ for $x \in F$ with $(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}$ as a right $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-module with an $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-valued non-degenerated product.

For a triple $R = (P_0, P_1, P_2)$ of Lagrangian projections of $\mathcal{A}^{2d}$ we define the following function spaces:

Let

$$\begin{aligned} C^0_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) \; &:= \; \{f \in C^0(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) \mid (P_i \oplus P_i)f(x) = f(x) \\ &\qquad \text{for } x \in \partial_i M \cup \partial_{i+3}M, \; i = 0, 1, 2\} \; . \end{aligned}$$

Define for $k \geq 1$ by induction

$$C^k_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) := \{f \in C^k(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) \mid f, \partial_M f \in C^{k-1}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)\}$$

and

$$C_R^\infty(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i) := \bigcap_{k\in\mathbb{N}} C_R^k(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i) \ .$$

Let these spaces be endowed with the subspace topologies.

Further suffixes, like c or $0 \dots$, have their usual meaning.

Furthermore we introduce Schwartz spaces on M:

Let $r \ge 0, \ 0 \le b \le \frac{1}{4}$.

Let $\{\phi_k\}_{k\in J}$ be a partition of unity subordinate to the covering $\mathcal{U}(r,b)$. For $k \in \mathbb{Z}/6$ the embedding $F_k(r,b) \hookrightarrow Z_k$ and the trivialisation of $E|_F$ induce a map

$$C^\infty(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i) \to C^\infty(Z_k, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}), \ f \mapsto \phi_k f \ .$$

First for $k \in \mathbb{Z}/6$ we define the Schwartz space

$$\mathcal{S}(Z_k, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}) := \mathcal{S}(\mathbb{R}) \otimes_\pi C^\infty([0,1], (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}) \ .$$

Let now $\mathcal{S}(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ as a vector space be the largest subspace of $C^\infty(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ such that for all $k \in \mathbb{Z}/6$ the maps

$$\mathcal{S}(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i) \to \mathcal{S}(Z_k, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}), \ f \mapsto \phi_k f$$

are well-defined, and let the topology on $\mathcal{S}(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ be the weakest topology such that these maps and the embedding into $C^\infty(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ are continuous. Then $\mathcal{S}(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ is a Fréchet space. It does not depend on the choice of r and b.

Let $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ be the space $\mathcal{S}(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i) \cap C_R^\infty(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ with the topology induced by $\mathcal{S}(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$.

Let $L^2(M, E \otimes \mathcal{A})$ be the completion of $C_c^\infty(M, E \otimes \mathcal{A})$ with respect to the norm induced by the $\mathcal{A}$-valued scalar product

$$< f, g >:= \int_M < f(x), g(x) > dx \ .$$

It is a Hilbert $\mathcal{A}$-module. (For fixing notation a short summary about Hilbert $\mathcal{A}$-modules is given in §5.1.1.) By lemma 5.1.20 any orthonormal basis of the Hilbert space $L^2(M, E)$ is an orthonormal basis of $L^2(M, E \otimes \mathcal{A})$ and hence $L^2(M, E \otimes \mathcal{A})$ is isomorphic to $l^2(\mathcal{A})$.

Now we are able to define the main object of our inquiry:

Let $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ be a triple of pairwise transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $\mathcal{P}_i \in M_{2d}(\mathcal{A}_\infty), \ i = 0, 1, 2$.

Let $D : \operatorname{dom} D \to L^2(M, E \otimes \mathcal{A})$ be the closure of the Dirac operator ∂_M with domain $C_{Rc}^\infty(M, E \otimes \mathcal{A})$ for $R = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ and let D^+ resp. D^- be the restriction of D to the sections of E^+ resp. E^-.

Note that D is a symmetric unbounded operator on $L^2(M, E \otimes \mathcal{A})$.

2.1.2 Comparison with D_s

Fix a triple of pairwise transverse Lagrangian projections $(\mathcal{P}_0^s := P_s, \mathcal{P}_1^s, \mathcal{P}_2^s)$ with $\mathcal{P}_1^s, \mathcal{P}_2^s \in M_{2d}(\mathbb{C})$. Define the operator D_s on $L^2(M, E \otimes \mathcal{A})$ analogously to the operator D for $R = (\mathcal{P}_0^s, \mathcal{P}_1^s, \mathcal{P}_2^s)$. By [BK] the operator D_s restricts to a selfadjoint operator on the Hilbert space $L^2(M, E)$. In this section we will see that the operator D is unitarily equivalent to a bounded perturbation of D_s.

Let $W \in C^\infty(M, \mathrm{End}^+ E \otimes \mathcal{A}_\infty)$ be such that

- $WW^* = 1$,
- $W(x)(\mathcal{P}_i \oplus \mathcal{P}_i)W(x)^* = (\mathcal{P}_i^s \oplus \mathcal{P}_i^s)$ for all $x \in \partial_i M \cup \partial_{i+3} M$ with $i = 0, 1, 2$,
- W is parallel on $M \setminus F$ and on a neighbourhood of ∂M,
- for all $k \in \mathbb{Z}/6$ the restriction of W on $F_k(0, \frac{1}{4})$ depends only on the coordinate x_2^k.
- W commutes with the Clifford multiplication.

The definition of W is motivated by the following result:

Proposition 2.1.1. *1. It holds $WDW^* = D_s + Wc(dW^*)$ with $c(dW^*)|_F := c(e_2)\partial_{e_2} W^*$ and $c(dW^*)|_{M \setminus F} := 0$. In particular $Wc(dW^*) \in C^\infty(M, \mathrm{End}^- E \otimes \mathcal{A}_\infty)$.*

2. The operator D is regular and selfadjoint.

Proof. 1) For $R = (\mathcal{P}_0^s, \mathcal{P}_1^s, \mathcal{P}_2^s)$ and $f \in C^\infty_{cR}(M, E \otimes \mathcal{A})$ it holds $(WDW^* f)|_{M\setminus F} = (D_s f)|_{M \setminus F}$ and

$$\begin{aligned}(WDW^* f)|_F &= (D_s f)|_F + W[c(e_2)\partial_{e_2}, W^*]_s (f|_F) \\ &= (D_s f)|_F + Wc(e_2)(\partial_{e_2} W^*)(f|_F) \ .\end{aligned}$$

2) The restriction of D_s to the Hilbert space $L^2(M, E)$ is selfadjoint. Hence $(1 + D_s^2)$ has a bounded inverse on $L^2(M, E)$. It follows that the range of $(1 + D_s^2)$ on $L^2(M, E \otimes \mathcal{A})$ is dense. Thus D_s is regular. By an analogous argument the operators $D_s \pm i$ have dense range. From lemma 5.1.17 it follows that D_s is selfadjoint.

By 1) the operator D is a bounded perturbation of D_s. It follows that D is selfadjoint. Prop. 5.1.13 implies that D is regular as well. $\square$

The existence of a section W fulfilling the properties above is proved in the following lemma and proposition:

Lemma 2.1.2. *There is a parallel unitary section $U \in C^\infty(M, \mathrm{End} E^+ \otimes \mathcal{A}_\infty)$ such that $U\mathcal{P}_0 U^* = \mathcal{P}_0^s = P_s$.*

Proof. By the isomorphism $E^+ \otimes \mathcal{A} \cong (\mathcal{A}^+)^d \otimes S^+ \oplus (\mathcal{A}^-)^d \otimes S^-$ any $A \in M_{2d}(\mathcal{A})$ of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a, b \in M_d(\mathcal{A})$ defines a parallel section of $\mathrm{End} E^+ \otimes \mathcal{A}$.

By lemma 1.4.3 there is a unitary $U \in M_{2d}(\mathcal{A}_\infty)$ of that form such that $U\mathcal{P}_0U^* = \mathcal{P}_0^s = P_s$. □

Proposition 2.1.3. *For any* $0 < b < \frac{1}{4}$ *there is a section* $W \in C^\infty(M, \mathrm{End}^+ E \otimes \mathcal{A}_\infty)$ *satisfying the properties above and such that* W *is parallel on* $M \backslash F(r, b)$ *for all* $r \geq 0$.

Proof. By the previous lemma we may assume $\mathcal{P}_0 = P_s$.

In the following we identify $\partial M \times [0, b]$ with $\{x \in M \mid d(x, \partial M) \leq b\}$.

Since for $i = 1, 2$ the projection $\mathcal{P}_i$ is transverse to P_s, by lemma 1.4.4 there are smooth paths $W_i : [0, b] \to M_{2d}(\mathcal{A}_\infty)$ of unitaries with $[W_i, I_0] = 0$ such that $W_i(0)\mathcal{P}_iW_i^*(0) = \mathcal{P}_i^s$ and such that W_i is equal to the identity in a neighbourhood of b and constant on $[0, \frac{b}{2}]$. They induce maps $\tilde{W}_i := \mathrm{id} \times W_i : (\partial_i M \cup \partial_{i+3} M) \times [0, b] \to M_{2d}(\mathcal{A}_\infty)$. The map

$$\cup_i(\tilde{W}_i \oplus \tilde{W}_i) : \partial M \times [0, b] \to M_{4d}(\mathcal{A}_\infty)$$

can be extended by the identity to a smooth section $W \in C^\infty(M, \mathrm{End}^+ E \otimes \mathcal{A}_\infty)$. By construction W has the right properties. □

2.2 The operator D_I on $[0, 1]$

2.2.1 Definition and comparison with D_{I_s}

Now we switch our attention to the unit interval.

Let ∂ be the differentiation operator.

For a pair $R = (P_0, P_1)$ of transverse Lagrangian projections with $P_j \in M_{2d}(\mathcal{A}_\infty)$, $j = 0, 1$, let

$$C^0_R([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}) := \{f \in C^0([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}) \mid P_j f(j) = f(j) \text{ for } j = 0, 1\} \ .$$

Furthermore for $k \in \mathbb{N}$ we define inductively the function spaces

$$\begin{aligned} & C^k_R([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}) \\ := \ & \{f \in C^k([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}) \mid f, I_0\partial f \in C^{k-1}_R([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})\} \ . \end{aligned}$$

Let

$$C^\infty_R([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}) := \bigcap_{k \in \mathbb{N}} C^k_R([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}) \ .$$

The topology on these spaces is the induced topology.

Given a pair (P_0, P_1) of transverse Lagrangian projections on $\mathcal{A}^{2d}$ we write D_I for the closure of the operator $I_0\partial$ as an unbounded operator on the Hilbert $\mathcal{A}$-module $L^2([0,1],\mathcal{A}^{2d})$ with domain $C^\infty_R([0,1],\mathcal{A}^{2d})$.

If $(P_0,P_1) = (P_s, 1-P_s)$ then we sometimes write D_{I_s} for D_I.

Proposition 2.2.1. *Let $P_0, P_1 \in M_{2d}(\mathcal{A}_\infty)$ be transverse Lagrangian projections of $\mathcal{A}^{2d}$. Then for any $0 < x_1 < x_2 < 1$ there is $U \in C^\infty([0,1], M_{2d}(\mathcal{A}_\infty))$ such that*

1. *$UU^* = 1$,*
2. *$UI_0 = I_0U$,*
3. *$U(0)P_0U(0)^* = P_s$,*
4. *$U(1)P_1U(1)^* = 1 - P_s$,*
5. *U is constant on $[0, x_1]$ and on $[x_2, 1]$.*

Proof. By lemma 1.4.3 there is a unitary $U_0 \in M_{2d}(\mathcal{A}_\infty)$ with $U_0I_0 = I_0U_0$ and $U_0P_0U_0^* = P_s$. Since $U_0P_1U_0^*$ is transverse to P_s, we can apply lemma 1.4.4 to $P = P_1$ and $\tilde{P} = 1 - P_s$ in order to get a smooth path of unitaries $U_1 : [0,1] \to M_{2d}(\mathcal{A}_\infty)$ such that $U(t) := U_1(t)U_0$ has the right properties. □

Proposition 2.2.2. *Let (P_0, P_1) a pair of transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_0, P_1 \in M_{2d}(\mathcal{A}_\infty)$ and let D_I be the associated operator.*

1. *Let U be as in the previous proposition with $U(0)P_0U(0)^* = P_s$ and $U(1)P_1U(1)^* = 1 - P_s$. Then it holds $UD_IU^* = D_{I_s} + UI_0(\partial U^*)$ and $UI_0(\partial U^*) \in C^\infty([0,1], M_{2d}(\mathcal{A}_\infty))$.*
2. *The operator D_I is regular and selfadjoint.*

Proof. 1) For $f \in C^\infty_R([0,1],\mathcal{A}^{2d}_\infty)$ with $R = (P_s, 1-P_s)$ it holds

$$\begin{aligned} UD_IU^*f &= UI_0\partial U^*f \\ &= D_{I_s}f + U[I_0\partial, U^*]f \\ &= D_{I_s}f + UI_0(\partial U^*)f \ . \end{aligned}$$

2) follows as in prop. 2.1.1. □

2.2.2 Generalized eigenspace decomposition

In the next two propositions we define a decomposition of $L^2([0,1],\mathcal{A}^{2d})$ in free finitely generated $\mathcal{A}$-modules that are preserved by D_I. For $d = 1$ these can be understood as analogues of eigenspaces of D_I.

Assume that the boundeary conditions of D_I are given by a pair (P_0, P_1) of transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_0 = P_s$.

By lemma 1.4.3 the general case can be reduced to that case.

Let $p \in M_d(\mathcal{A})$ be such that $P_1 = \frac{1}{2}\begin{pmatrix} 1 & p^* \\ p & 1 \end{pmatrix}$.

The transversality of P_0 and P_1 implies that $1-p$ is invertible. It follows that $\log p$ is defined for $\log : \mathbb{C} \setminus [0,\infty) \to \mathbb{C}$.

Proposition 2.2.3. *Assume $d = 1$.*

Then for $k \in \mathbb{Z}$ the right $\mathcal{A}$-module spanned by the function

$$f_k(x) = \tfrac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp[(-\tfrac{1}{2}\log p + \pi k i)x] + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp[(\tfrac{1}{2}\log p - \pi k i)x]\right)$$

is preserved by D_I and it holds $D_I f_k = \lambda_k f_k$ with $\lambda_k := -\frac{1}{2} i \log p - \pi k$.

The system $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of the Hilbert $\mathcal{A}$-module $L^2([0,1],\mathcal{A}^2)$.

Note that it holds $\lambda_k f_k = f_k \lambda_k$ and $\sigma(\lambda_k) \subset] -\pi k, -\pi(k-1)[$.

Proof. In the following we call a system $\{a_k\}_{k\in\mathbb{Z}} \subset \mathcal{A}$ square-summable if it is in $l^2(\mathcal{A})$.

It holds $< f_k, f_l >= \delta_{kl}$.

In order to prove that these functions form a basis we first show that there is an orthogonal projection onto the closure of their span. This implies that the span is orthogonally complemented. The second step will be to see that the complement is trivial. Then the claim follows from prop. 5.1.20.

The system

$$\{e_l^{\pm}(x) := v_{\pm} e^{2\pi i l x}\}_{l\in\mathbb{Z},\pm}$$

with $v_+ := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_- := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an orthonormal basis of $L^2([0,1],\mathcal{A}^2)$ by prop. 5.1.20.

It holds

$$\begin{aligned} < f_k, e_l^{\pm} > &= \tfrac{1}{\sqrt{2}} \int_0^1 \exp((\mp\tfrac{1}{2}\log p^* \mp \pi k i)x) \exp(2\pi i l x)\ dx \\ &= \tfrac{1}{\sqrt{2}} \left(\frac{(-1)^k \exp(\mp\frac{1}{2}\log p^*) - 1}{\mp\frac{1}{2}\log p^* + \pi i(2l \mp k)} \right) , \end{aligned}$$

hence for any $l \in \mathbb{Z}$ and for $\pm$ the system $\{< f_k, e_l^{\pm} >\}_{k\in\mathbb{Z}}$ is square-summable. Then

$$P e_l^{\pm} := \sum_{k\in\mathbb{Z}} f_k < f_k, e_l^{\pm} > ,$$

is well-defined.

The linear extension of P to the algebraic span of the functions $e_l^{\pm}$ has norm one. It follows that its closure is an orthogonal projection

$$P : L^2([0,1],\mathcal{A}^2) \to \overline{\operatorname{span}\{f_k \mid k \in \mathbb{Z}\}} \ .$$

It remains to show that the kernel of the projection P is trivial.

Let $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in L^2([0,1],\mathcal{A}^2)$ with $< f_k, g >= 0$ for all $k \in \mathbb{Z}$.

Hence for all $k \in \mathbb{Z}$ it holds

$$(*) \qquad \int_0^1 \exp((-\tfrac{1}{2}\log p^* - \pi k i)x)g_1(x) + \exp((\tfrac{1}{2}\log p^* + \pi k i)x)g_2(x)\ dx = 0 \ .$$

Since $\exp(-\frac{1}{2}(\log p^*)\ x)g_1(x)$ is in $L^2([0,1],\mathcal{A})$, there is a unique square-summable system $\{\lambda_l\}_{l\in\mathbb{Z}}$ in $\mathcal{A}$ such that

$$\sum_{l\in\mathbb{Z}} \lambda_l e^{2\pi i l x} = \exp(-\tfrac{1}{2}(\log p^*)\ x)g_1(x) \ .$$

Inserting this in $(*)$ and evaluating the integral for k even leads to

$$\lambda_{k/2} - \int_0^1 \exp((\tfrac{1}{2}\log p^* + \pi k i)x)g_2(x)dx = 0 \ .$$

It follows

$$\exp(\tfrac{1}{2}(\log p^*)\ x)g_2(x) = \sum_{l\in\mathbb{Z}}(-\lambda_l)e^{-2\pi i l x} \ .$$

Substituting again and evaluating $(*)$ for $k = 2\nu + 1$ with $\nu \in \mathbb{Z}$, we obtain

$$\begin{aligned} 0 &= \int_0^1 \sum_{l\in\mathbb{Z}} \lambda_l (e^{\pi i(2(l-\nu)-1)x} - e^{-\pi i(2(l-\nu)-1)x})dx \\ &= -\sum_{l\in\mathbb{Z}} \lambda_l \frac{4}{\pi i(2(l-\nu)-1)} \ . \end{aligned}$$

Note that for any $l \in \mathbb{Z}$ the function

$$a_l : \mathbb{Z} \to \mathbb{C},\ \nu \mapsto \frac{2}{\pi i(2(l-\nu)-1)}$$

is in $l^2(\mathbb{C})$. We claim that $\{a_l\}_{l\in\mathbb{Z}}$ is an orthonormal basis of $l^2(\mathbb{C})$. Then it follows that $\{a_l\}_{l\in\mathbb{Z}}$ is an orthonormal basis of $l^2(\mathcal{A})$ as well, thus $\lambda_l = 0$ for all $l \in \mathbb{Z}$ and hence $g_1 = g_2 = 0$.

The Fourier transform of

$$a_0 : \nu \mapsto \frac{-2}{\pi i(2\nu+1)}$$

is

$$h(x) = -ie^{-\pi ix}(1_{[0,1/2]}(x) - 1_{]1/2,1]}(x)) \in L^2(\mathbb{R}/\mathbb{Z}) \ .$$

From $a_l(\nu) = a_0(\nu - l)$ it follows that the Fourier transform of a_l is $h(x)e^{-2\pi ilx}$.
It holds $hh^* = 1$, hence $\{h(x)e^{-2\pi ilx}\}_{l\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R}/\mathbb{Z})$. This implies that $\{a_l\}_{l\in\mathbb{Z}}$ is an orthonormal basis of $l^2(\mathbb{C})$. $\square$

For general d there is a decomposition of $L^2([0,1]\mathcal{A}^{2d})$ in $\mathcal{A}$-modules of rank d:

Proposition 2.2.4. *For $k \in \mathbb{Z}$ let $U_k \subset L^2([0,1],\mathcal{A}^{2d})$ be the right $\mathcal{A}$-module spanned by the column vectors of*

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \exp[(-\frac{1}{2}\log p + \pi ki)x] \\ \exp[(\frac{1}{2}\log p - \pi ki)x] \end{pmatrix} \in C^\infty([0,1], M_{2d\times d}(\mathcal{A}_\infty)) \ .$$

Each U_k is a free right $\mathcal{A}$-module of rank d and it holds

$$L^2([0,1],\mathcal{A}^{2d}) = \hat{\bigoplus_{k\in\mathbb{Z}}} U_k \ .$$

The sum is orthogonal.
For $f \in U_k$ it holds $f \in \operatorname{dom} D_I$ and

$$D_I f = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{pmatrix} f \in U_k$$

with $\lambda_k = -\frac{1}{2}i\log p - \pi k$.

Proof. The projections $P_0, P_1 \in M_{2d}(\mathcal{A})$ are Lagrangian projections on $M_d(\mathcal{A})^2$ as we remarked after def 1.4.1.
Let $\tilde{D}_I$ be the closure of $I_0\partial$ on $L^2([0,1], M_d(\mathcal{A})^2)$ with domain $C^\infty_R([0,1], M_{2d}(\mathcal{A}))$ with $R = (P_0, P_1)$. Then it holds $\tilde{D}_I = \oplus^d D_I$ with respect to the decomposition

$$L^2([0,1], M_d(\mathcal{A})^2) = L^2([0,1],\mathcal{A}^{2d})^d$$

induced by the decomposition of a matrix into its column vectors.
By the previous proposition $L^2([0,1], M_d(\mathcal{A})^2)$ as a Hilbert $M_n(\mathcal{A})$-module has an orthonormal basis $\{f_k\}_{k\in\mathbb{Z}}$ such that

$$\tilde{D}_I f_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{pmatrix} f_k \ .$$

For $k \in \mathbb{Z}$ let P_k be the orthogonal projection onto the span of f_k in $L^2([0,1], M_d(\mathcal{A})^2)$. It is diagonal with respect to the decomposition $L^2([0,1], M_d(\mathcal{A})^2) = L^2([0,1],\mathcal{A}^{2d})^d$.
Hence it holds

$$L^2([0,1],\mathcal{A}^{2d}) = \hat{\bigoplus_{k\in\mathbb{Z}}} P_k L^2([0,1],\mathcal{A}^{2d}) \ .$$

The assertion follows now since $P_k L^2([0,1],\mathcal{A}^{2d}) = U_k$. The module U_k is free since $\exp[(-\frac{1}{2}\log p + \pi ki)x]$ is invertible in $M_d(\mathcal{A})$ for all $x \in [0,1]$. $\square$

Corollary 2.2.5. *Let* $\lambda \in \mathbb{C}$. *The operator* $D_I - \lambda$ *has a bounded inverse on* $L^2([0,1], \mathcal{A}^{2d})$ *if and only if* $\exp(-2i\lambda) \notin \sigma(p)$.

Proof. By the previous proposition $L^2([0,1], \mathcal{A}^{2d})$ decomposes into the orthogonal sum of submodules U_k preserved by $(D_I - \lambda)$, and $(D_I - \lambda)|_{U_k} = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{pmatrix} - \lambda$ is invertible if and only if $e^{-2i\lambda} \notin \sigma(p)$. Furthermore for λ with $e^{-2i\lambda} \notin \sigma(p)$ the inverse of $(D_I - \lambda)|_{U_k}$ is uniformly bounded in k. Hence the closure of $\oplus_k (D_I - \lambda)|_{U_k}$ has a bounded inverse by cor. 5.1.22. In particular $\oplus_k D_I|_{U_k}$ is selfadjoint. Since D_I is a selfadjoint extension of $\oplus_k D_I|_{U_k}$, it follows that D_I is the closure of $\oplus_k D_I|_{U_k}$. Hence $D_I - \lambda$ has a bounded inverse if $\exp(-2i\lambda) \notin \sigma(p)$. $\square$

2.3 The operator D_Z on the cylinder

Let $X = \mathbb{R}, \mathbb{R}/\mathbb{Z}$. Endow $X \times [0,1]$ with the euclidean metric and a spin structure and let (x_1, x_2) be the euclidian coordinates of $X \times [0,1]$. Let $S = S^+ \oplus S^-$ be the spinor bundle on $X \times [0,1]$ endowed with a hermitian metric such that it is selfadjoint as a Clifford module and with a Clifford connection. Then $((\mathcal{A}^+)^d \oplus (\mathcal{A}^-)^d) \otimes S$ is graded $\mathcal{A}$-vector bundle on $X \times [0,1]$ with an $\mathcal{A}$-valued scalar product induced by the hermitian metric on S and the standard $\mathcal{A}$-valued scalar product on $\mathcal{A}^d$. Let ∂_Z be the Dirac operator on the Clifford module $\big((\mathcal{A}^+)^{2d} \oplus (\mathcal{A}^-)^{2d}\big) \otimes S$.

We choose a parallel unit section s of S^+ and identify $((\mathcal{A}^+)^d \oplus (\mathcal{A}^-)^d) \otimes S$ with the trivial bundle $(X \times [0,1]) \times \big((\mathcal{A}^+)^{2d} \oplus (\mathcal{A}^-)^{2d}\big)$ by the isomorphisms

$$((\mathcal{A}^+)^d \otimes S_x^+) \oplus ((\mathcal{A}^-)^d \otimes S_x^-) \to (\mathcal{A}^+)^d \oplus (\mathcal{A}^+)^d \ ,$$

$$(v \otimes s(x)) \oplus (w \otimes ic(dx_1)s(x)) \mapsto (v, w) \ ,$$

and

$$((\mathcal{A}^+)^d \otimes S_x^-) \oplus ((\mathcal{A}^-)^d \otimes S_x^+) \to (\mathcal{A}^-)^d \oplus (\mathcal{A}^-)^d \ ,$$

$$(v \otimes ic(dx_1)s(x)) \oplus (w \otimes s(x)) \mapsto (v, w)$$

for $x \in X \times [0,1]$.

Let $I = c(dx_1)c(dx_2)$. It holds $I = I_0 \oplus (-I_0)$ and furthermore

$$\partial_Z = c(dx_1)(\partial_{x_1} - I\partial_{x_2}) \ .$$

We associate an unbounded operator D_Z on $L^2(X \times [0,1], \mathcal{A}^{4d})$ to a pair (P_0, P_1) of transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_i \in M_{2d}(\mathcal{A}_\infty)$, $i = 0,1$: Let D_Z be the closure of ∂_Z with domain

$$\{f \in C_c^\infty(X \times [0,1], (\mathcal{A}_\infty)^{4d}) \mid (P_i \oplus P_i)f(x,i) = f(x,i) \text{ for all } x \in X,\ i = 0,1\} \ .$$

As defined in §5.1.3 we write $H(D_Z)$ for the Hilbert $\mathcal{A}$-module whose underlying right $\mathcal{A}$-module is $\operatorname{dom} D_Z$ and whose $\mathcal{A}$-valued scalar product is $< f,g >_{D_Z} = < f,g > + < D_Z f, D_Z g >$.

Proposition 2.3.1. *1. The operator D_Z is selfadjoint on $L^2(X\times[0,1],\mathcal{A}^{4d})$ and has a bounded inverse.*

2. If $X=\mathbb{R}/\mathbb{Z}$, then the inclusion $i: H(D_Z)\to L^2(\mathbb{R}/\mathbb{Z}\times[0,1],\mathcal{A}^{4d})$ is compact.

Proof. The proof is as in the family case [BK]:

1) By lemma 1.4.3 we may assume $P_0=P_s$. Recall from prop. 2.2.4 that the operator D_I with boundary conditions (P_0,P_1) induces a decomposition

$$L^2([0,1],(\mathcal{A}^+)^{2d})=\hat{\bigoplus_{l\in\mathbb{Z}}}U_l$$

such that for $l\in\mathbb{Z}$ there is $\lambda_l\in M_d(\mathcal{A})$ with $D_If=\begin{pmatrix}\lambda_l & 0\\ 0 & \lambda_l\end{pmatrix}f$ for $f\in U_l$.

Let $U_{l,+}:=U_l$ and $U_{l,-}:=ic(dx_1)U_l$ in $L^2([0,1],\mathcal{A}^{4d})$. Then

$$\bigoplus_{l\in\mathbb{Z}}(U_{l,+}\oplus U_{l,-})L^2(X)$$

is dense in $L^2(X\times[0,1],\mathcal{A}^{4d})$.

First consider the case $X=\mathbb{R}$:

For $l\in\mathbb{Z}$ let $\partial_{l,\pm}$ be the closure of $(\partial_{x_1}\pm\lambda_l): U_{l,\pm}\mathcal{S}(\mathbb{R})\to U_{l,\pm}L^2(\mathbb{R})$ and let ∂_e be the closure of $c(dx_1)\big(\bigoplus_{l\in\mathbb{Z},\pm}\partial_{l,\pm}\big)$.

The operator D_Z is an extension of ∂_e.

We claim that ∂_e has a bounded inverse on $L^2(\mathbb{R}\times[0,1],\mathcal{A}^{4d})$.

The Fourier transform on $L^2(\mathbb{R})$ induces an automorphism on $U_{l,\pm}L^2(\mathbb{R})$. Conjugation by it transforms $\partial_{l,\pm}$ into multiplication by

$$ix_1\pm\begin{pmatrix}\lambda_l & 0\\ 0 & \lambda_l\end{pmatrix} .$$

Since $\sigma(\lambda_l)\subset\mathbb{R}^*$, the operator $\partial_{l,\pm}$ has a bounded inverse and the norm of the inverse tends to zero for $l\to\pm\infty$. It follows by cor. 5.1.22 that the closure of $\oplus_{l,\pm}\partial_{l,\pm}$ has a bounded inverse.

Hence the operator ∂_e has a bounded inverse. In particular it is selfadjoint. Since D_Z is a symmetric extension of ∂_e, it holds $D_Z=\partial_e$. For $X=\mathbb{R}$ the assertion follows.

Now we study $X=\mathbb{R}/\mathbb{Z}$. The spaces $U_{l,\pm}L^2(\mathbb{R}/\mathbb{Z})$ decompose further into the direct sum

$$U_{l,\pm}L^2(\mathbb{R}/\mathbb{Z})=\hat{\bigoplus_{k\in\mathbb{Z}}}V_{kl,\pm}$$

with $V_{kl,\pm}:=e^{2\pi ikx_1}U_{l,\pm}$. Note that $V_{kl,\pm}$ is isomorphic to $\mathcal{A}^d$ as a Hilbert $\mathcal{A}$-module.

Let $\partial_{kl,\pm} \in B(V_{kl,\pm})$ be defined by

$$\partial_{kl,\pm} f := (\partial_{x_1} \pm \lambda_l) f = \begin{pmatrix} 2\pi i k \pm \lambda_l & 0 \\ 0 & 2\pi i k \pm \lambda_l \end{pmatrix} f$$

and let ∂_e be the closure of $c(dx_1)\left(\bigoplus_{k,l\in\mathbb{Z},\pm} \partial_{kl,\pm}\right)$.

The operator D_Z is an extension of ∂_e.

Since $|(2\pi i k \pm \lambda_l)^{-1}|$ tends to zero for $k, l \to \pm\infty$, the closure of $\oplus_{kl,\pm}\partial_{kl,\pm}$ has a bounded – even compact – inverse by cor. 5.1.22 and hence ∂_e has a compact inverse as well.

Now 1) follows as above.

Furthermore it follows that for $X = \mathbb{R}/\mathbb{Z}$ the operator $D_Z^{-1} \in B(L^2(\mathbb{R}/\mathbb{Z} \times [0,1], \mathcal{A}^{4d}))$ is compact. This implies 2) since $i = D_Z^{-1} D_Z : H(D_Z) \to L^2(\mathbb{R}/\mathbb{Z} \times [0,1], \mathcal{A}^{4d})$. $\square$

2.4 The index of D^+

For an open precompact subset U of M we define $H^1_{R0}(U, E \otimes \mathcal{A})$ to be the closure of $C^\infty_{Rc}(U, E \otimes \mathcal{A})$ in $H(D)$ (see §5.1.3 for the definition of $H(D)$).

Note that $H(D)$ is isomorphic to $l^2(\mathcal{A})$ as a Hilbert $\mathcal{A}$-module, since $L^2(M, E \otimes \mathcal{A})$ is isomorphic to $l^2(\mathcal{A})$ and since $L^2(M, E \otimes \mathcal{A})$ and $H(D)$ are isomorphic by lemma 5.1.12. Recall that the boundary conditions of D are given by $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$.

Lemma 2.4.1. *For any $r \geq 0$ the inclusion $i : H^1_{R0}(M_r, E \otimes \mathcal{A}) \to L^2(M, E \otimes \mathcal{A})$ is compact.*

Proof. Let $\mathcal{V} := \{V_k\}_{k\in L}$ be an open covering of M_r such that the index set L is a finite subset of $\mathbb{N}$ with $1 \in L$ and satisfying the following conditions:

- $V_1 = M \setminus \overline{F(r, 1/6)}$;
- for $k > 1$ there is an isometry $V_k \cong]0, \frac{1}{2}[\times[0, \frac{1}{5}[$. In particular V_k is in the flat region and has one boundary component.

First we prove that the maps $i_k : H^1_{R0}(V_k, E \otimes \mathcal{A}) \to L^2(V_k, E \otimes \mathcal{A})$ are compact for all $k \in L$.

The compactness of i_1 follows from the Sobolev embedding theorem ([MF], lemma 3.3).

For $k \neq 1$ let $i \in \{0, 1, 2\}$ be such that $\partial V_k \subset (\partial_i M \cup \partial_{i+3} M)$ and set $P_k := \mathcal{P}_i$. Let D_k be the operator D_Z on the bundle $(\mathbb{R}/\mathbb{Z} \times [0,1]) \times \mathcal{A}^{4d}$ with boundary conditions given by $(P_k, 1 - P_k)$.

Since i_k factorizes through the map $H(D_k) \to L^2(\mathbb{R}/\mathbb{Z} \times [0,1], \mathcal{A}^{4d})$ and since this map is compact by prop. 2.3.1, the map i_k is compact as well.

Let $\{\phi_k\}_{k\in L}$ be a smooth partition of unity subordinate to the covering $\mathcal{V}$ such that for all $x \in \partial M$ and $k \in L$ it holds $\partial_{e_2}\phi_k(x) = 0$. Since multiplication with ϕ_k is a bounded map from $H^1_{R0}(M_r, E \otimes \mathcal{A})$ to $H^1_{R0}(V_k, E \otimes \mathcal{A})$ and since $i = \sum_{k\in L} i_k \phi_k$, the inclusion i is compact. □

Let $H(D)^+$ be the subspace of $H(D)$ containing all even elements.

Proposition 2.4.2. *The operator*

$$D^+ : H(D)^+ \to L^2(M, E^- \otimes \mathcal{A})$$

is a Fredholm operator.

Proof. We apply prop. 5.1.4: By constructing a parametrix for D^+ we show that D^+ is Fredholm in the sense of Miščenko/Fomenko. Then it follows $D^+ \in B(H(D)^+, L^2(M, E^- \otimes \mathcal{A}))$ from lemma 5.1.12, hence lemma 5.1.8 implies that D^+ is Fredholm.

The construction of the parametrix is analogous to the construction in the family case [BK]:

Choose a smooth partition of unity $\{\phi_k\}_{k\in J}$ subordinate to the covering $\mathcal{U}(0, \frac{1}{4})$ defined in §1.1 and a system of smooth functions $\{\gamma_k\}_{k\in J}$ on M such that for all $k \in J$ it holds

- $\operatorname{supp}\gamma_k \subset \mathcal{U}_k$ and $\gamma_k\phi_k = \phi_k$,
- $\partial_{e_2}\gamma_k(x) = 0 = \partial_{e_2}\phi_k(x) = 0$ for all $x \in \partial M$.

Define local parametrices Q_k of D on $\mathcal{U}_k$ as follows:

By §1.1 the set $\mathcal{U}_k$ is a subset of Z_k for $k \in \mathbb{Z}/6$. Let D_{Z_k} be the operator on $L^2(Z_k, \mathcal{A}^{4d})$ associated to $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{(k+1)\bmod 3})$ as defined in §2.3. By prop. 2.3.1 it is invertible, hence we can set

$$Q_k := D_{Z_k}^{-1} : L^2(Z_k, \mathcal{A}^{4d}) \to H(D_{Z_k}) \ .$$

Since the symbol of D is elliptic and since $\mathcal{U}_\clubsuit$ is precompact, there is a parametrix $Q_\clubsuit : L^2(\mathcal{U}_\clubsuit, E\otimes\mathcal{A}) \to H^1_{R0}(\mathcal{U}_\clubsuit, E\otimes\mathcal{A})$ such that $DQ_\clubsuit - 1$ resp. $Q_\clubsuit D - 1$ is compact on $L^2(\mathcal{U}_\clubsuit, E \otimes \mathcal{A})$ resp. $H^1_{R0}(\mathcal{U}_\clubsuit, E \otimes \mathcal{A})$ [MF]. Furthermore $Q_\clubsuit$ can be chosen to be an odd operator on $L^2(\mathcal{U}_\clubsuit, E \otimes \mathcal{A})$.

We claim that

$$Q := \sum_{k\in J} \gamma_k Q_k \phi_k : L^2(M, E \otimes \mathcal{A}) \to H(D)$$

is a parametrix of D. Since Q is odd, it then follows that $Q^- : L^2(M, E^- \otimes \mathcal{A}) \to H(D)^+$ is a parametrix of D^+.

In the following calculations the operators D_{Z_k} and the restriction of D to $\mathcal{U}_k$ are denoted by D as well. Let $\sim$ denote equality up to compact operators.

On $L^2(M, E \otimes \mathcal{A})$ it holds

$$\begin{aligned} DQ - 1 &= \sum_{k \in J} [D, \gamma_k] Q_k \phi_k + \sum_{k \in J} \gamma_k D Q_k \phi_k - 1 \\ &\sim \sum_{k \in J} c(d\gamma_k) Q_k \phi_k \ . \end{aligned}$$

Since $c(d\gamma_k)Q_k\phi_k$ is bounded from $L^2(M, E \otimes \mathcal{A})$ to $H^1_{R0}(M_r, E \otimes \mathcal{A})$ for any $k \in J$ and $r > 0$ big enough, $c(d\gamma_k)Q_k\phi_k$ is a compact operator on $L^2(M, E \otimes \mathcal{A})$ for any $k \in J$ by the previous lemma. Hence $DQ - 1$ is compact.

The proof that Q is a left parametrix of D is similar:

On $H(D)$ it holds

$$\begin{aligned} QD - 1 &= \sum_{k \in J} \gamma_k Q_k [D, \phi_k] + \sum_{k \in J} \gamma_k Q_k D \phi_k - 1 \\ &\sim \sum_{k \in J} \gamma_k Q_k c(d\phi_k) \ . \end{aligned}$$

Here $c(d\phi_k) : H(D) \to L^2(\mathcal{U}_k, E \otimes \mathcal{A})$ is a compact operator by the previous lemma since $\operatorname{supp} c(d\phi_k)$ is precompact for all $k \in J$. Since $\gamma_k Q_k : L^2(\mathcal{U}_k, E \otimes \mathcal{A}) \to H(D)$ is bounded, the operator $QD - 1 : H(D) \to H(D)$ is compact. $\square$

Lemma 2.4.3. *Let $P_0, P_1, P_2 : [0,1] \to M_{2d}(\mathcal{A})$ be continuous paths of Lagrangian projections that are pairwise transverse at any $t \in [0,1]$, and let D_t be the closure of ∂_M on $L^2(M, E \otimes \mathcal{A})$ with domain $C^\infty_{R(t)c}(M, E \otimes \mathcal{A})$ with $R(t) = (P_0(t), P_1(t), P_2(t))$.*

Let $\operatorname{ind} D_t^+$ be the index of the Fredholm operator $D_t^+ : H(D_t)^+ \to L^2(M, E \otimes \mathcal{A})$.

Then it holds

$$\operatorname{ind} D_0^+ = \operatorname{ind} D_1^+ \in K_0(\mathcal{A}) \ .$$

Proof. There is a continuous path of unitaries $[0,1] \to C^\infty(M, \mathrm{End} E^+ \otimes \mathcal{A}_\infty)$, $t \mapsto W_t$, such that $W_t D_t^+ W_t^* = D_s^+ + W_t c(dW_t^*)$. By prop. 5.1.13 it holds $D_s^+ + W_t c(dW_t^*) \in B(H(D_s)^+, L^2(M, E^- \otimes \mathcal{A}))$. Hence the family $D_s^+ + W_t c(dW_t^*) : H(D_s)^+ \to L^2(M, E^- \otimes \mathcal{A})$ is a continuous path of Fredholm operators. By prop. 5.1.9 it holds $\operatorname{ind} W_0 D_0^+ W_0^* = \operatorname{ind} W_1 D_1^+ W_1^*$. The assertion follows. $\square$

Proposition 2.4.4. *The index of $D^+ : H(D)^+ \to L^2(M, E^- \otimes \mathcal{A})$ is*

$$\operatorname{ind} D^+ = \tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) \in K_0(\mathcal{A}) \ .$$

Proof. The argument is analogous to the one in [BK].

By lemma 2.1.2 we may assume $\mathcal{P}_0 = P_s$.

Let $a_j := i\frac{p_j + 1}{p_j - 1} \in M_{2d}(\mathcal{A})$. Then in the notation of §1.4.2 it holds $\mathcal{P}_j = P(a_j)$, $j = 1, 2$.

Let $p^+ := 1_{\{x > 0\}}(a_1 - a_2)$.

From lemma 1.4.8 it follows $\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) = [p^+] - [1 - p^+] \in K_0(\mathcal{A})$.

Furthermore by lemma 1.4.8 there are continuous paths $P_1, P_2 : [0,2] \to M_{2d}(\mathcal{A})$ of Lagrangian projections with $P_j(0) = \mathcal{P}_j,\ j = 1, 2$, further with $P_1(2) = P(2p^+ - 1)$ and $P_2(2) = P(1 - 2p^+)$ and such that $P_s, P_1(t), P_2(t)$ are pairwise transverse for all $t \in [0,2]$.

For $t \in [0,2]$ let D_t be the Dirac operator on $L^2(M, E \otimes \mathcal{A})$ whose boundary conditions are given by the triple $(P_s, P_1(t), P_2(t))$. The previous lemma implies $\operatorname{ind} D_0^+ = \operatorname{ind} D_2^+$.

We show that the index of D_2^+ equals $[p^+] - [1 - p^+]$:

Let $Q_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{C})$, $Q_1 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and $Q_2 = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

It holds $P_1(2) = (Q_1 \otimes p^+) \oplus (Q_2 \otimes (1 - p^+))$ and $P_2(2) = (Q_2 \otimes p^+) \oplus (Q_1 \otimes (1 - p^+))$ with respect to the decomposition

$$E^+ \otimes \mathcal{A} = (S \otimes p^+ \mathcal{A}^n) \oplus (S \otimes (1 - p^+)\mathcal{A}^n) \ .$$

The Dirac operator ∂_M respects the decomposition. By [BK] the Dirac operator associated to the bundle $(\mathbb{C}^+ \oplus \mathbb{C}^-) \otimes S$ has index 1 if the boundary conditions are given by the triple (Q_0, Q_1, Q_2), and index -1 if the boundary conditions are given by (Q_0, Q_2, Q_1). It follows

$$\operatorname{ind} D_2^+ = [p^+] - [1 - p^+] \ .$$

$\square$

2.5 A perturbation with closed range

Choose an orthonormal basis $\{\psi_i\}_{i \in \mathbb{N}} \subset L^2(M, E^-)$ such that $\psi_i \in C_c^\infty(M, E^-)$ and $\operatorname{supp} \psi_i \in M \setminus \partial M$ for all $i \in \mathbb{N}$. By lemma 5.1.20 it is an orthonormal basis of $L^2(M, E^- \otimes \mathcal{A})$ as well.

Since D^+ is a Fredholm operator, there is a projective $\mathcal{A}$-module $P \subset L^2(M, E^- \otimes \mathcal{A})$ and a closed $\mathcal{A}$-module $Q \subset \operatorname{Ran} D^+$ such that $P \oplus Q = L^2(M, E^- \otimes \mathcal{A})$. By prop. 5.3.3 there is $\mathcal{N} \in \mathbb{N}$ such for $L_\mathcal{N} := \operatorname{span}\{\psi_i \mid i = 1, \ldots, \mathcal{N}\}$ it holds $L_\mathcal{N} + P = L^2(M, E^- \otimes \mathcal{A})$. In particular it follows

$$L_\mathcal{N} + \operatorname{Ran} D^+ = L^2(M, E^- \otimes \mathcal{A}) \ .$$

Let $M' := M \cup *$ be the disjoint union of M and one isolated point. Let E'^+ be the hermitian vector bundle $E^+ \cup (* \times \mathbb{C}^\mathcal{N})$ on M', where we endow $\mathbb{C}^\mathcal{N}$ with the standard hermitian product, and let E'^- be the hermitian bundle $E^- \cup (* \times \{0\})$. Let $E' = E'^+ \oplus E'^-$. Extend D by zero to a selfadjoint odd operator D on $L^2(M', E' \otimes \mathcal{A})$.

As D is regular, D' is regular as well.

Furthermore $D' : H(D') \to L^2(M', E' \otimes \mathcal{A})$ is a Fredholm operator and it holds

$$\operatorname{ind} D'^+ = \operatorname{ind} D^+ + [\mathcal{A}^{\mathcal{N}}] \ .$$

Let $e_k \in L^2(M', E'^+),\ k = 1, \ldots, \mathcal{N}$, with $e_k(x) = 0$ for $x \in M,\ k = 1, \ldots, \mathcal{N}$, and such that $\{e_k(*)\}_{k=1,\ldots,\mathcal{N}}$ is the standard basis of $\mathbb{C}^{\mathcal{N}}$.

Define the finite integral operator K on $L^2(M', E' \otimes \mathcal{A})$ by

$$Kf := \sum_{k=1}^{\mathcal{N}} e_k < \psi_k, f > + \psi_k < e_k, f > .$$

Since K is an odd selfadjoint compact operator, $D' + \rho K$ is selfadjoint and odd and for $\rho \in \mathbb{R}$ it holds

$$\operatorname{ind}(D'^+ + \rho K^+) = \operatorname{ind} D'^+ \ .$$

Furthermore $D' + \rho K$ is regular by prop 5.1.13.

By construction $D'^+ + \rho K^+$ is surjective for $\rho \neq 0$. Hence by prop. 5.1.13 and prop. 5.1.15 its kernel is a projective submodule of $H(D')^+$, and the kernel of $D'^- + \rho K^-$ is trivial.

Set $D(\rho) := D' + \rho K$.

Proposition 2.5.1. *For $\rho \in \mathbb{R}$ the operator $D'^+ + \rho K^+$ is a Fredholm operator. For $\rho \neq 0$ it holds*

$$\operatorname{ind} D^+ = \operatorname{ind}(D'^+ + \rho K^+) - [\mathcal{A}^{\mathcal{N}}] = [\operatorname{Ker} D(\rho)] - [\mathcal{A}^{\mathcal{N}}] \ .$$

From now on we write D, E, M for D', E', M' and we extend the operator D_s by zero to our new manifold M. Furthermore we redefine the open covering $\mathcal{U}(r, b)$ from §1.1 by adding the isolated point to the set $\mathcal{U}_{\clubsuit}$.

Chapter 3

Heat semigroups and kernels

3.1 Complex heat kernels

In this section we recall properties of the heat kernels associated to the Dirac operators on complex vector bundles.

3.1.1 The heat kernel of $e^{-tD_{I_s}^2}$

Since D_{I_s} is selfadjoint on $L^2([0,1],\mathbb{C}^{2d})$, the operator $-D_{I_s}^2$ generates a semigroup $e^{-tD_{I_s}^2}$ on $L^2([0,1],\mathbb{C}^{2d})$.

In this section we determine the corresponding family of integral kernels by using the method of images (see [Ta], 3.7) and study its properties.

The space $L^2([0,1],\mathbb{C}^{2d})$ decomposes into an orthogonal sum

$$L^2([0,1],P_s\mathbb{C}^{2d}) \oplus L^2([0,1],(1-P_s)\mathbb{C}^{2d})$$

and the semigroup $e^{-tD_{I_s}^2}$ is diagonal with respect to this decomposition.

We define a right inverse

$$\tilde{} : L^2([0,1],\mathbb{C}^{2d}) \to L^2(\mathbb{R}/4\mathbb{Z},\mathbb{C}^{2d})$$

of the map

$$L^2(\mathbb{R}/4\mathbb{Z},\mathbb{C}^{2d}) \to L^2([0,1],\mathbb{C}^{2d}),\ f \mapsto f|_{[0,1]}$$

by requiring that the image of $L^2([0,1],(1-P_s)\mathbb{C}^{2d})$ resp. $L^2([0,1],P_s\mathbb{C}^{2d})$ consists of functions that are odd resp. even with respect to $y=0$ and $y=2$ and even resp. odd with respect to $y=1$ and $y=3$.

Since for $R=(P_s,1-P_s)$ it holds

$$C_R^\infty([0,1],\mathbb{C}^{2d}) = C_l^\infty([0,1],P_s\mathbb{C}^{2d}) \oplus C_r^\infty([0,1],(1-P_s)\mathbb{C}^{2d})$$

with

$$
\begin{aligned}
&C_l^\infty([0,1],P_s\mathbb{C}^{2d})\\
&\quad := \quad \{f\in C^\infty([0,1],P_s\mathbb{C}^{2d}) \mid (i\partial)^{2k}f(1)=0,\ (i\partial)^{2k+1}f(0)=0\ \forall k\in\mathbb{N}_0\}
\end{aligned}
$$

and

$$
\begin{aligned}
&C_r^\infty([0,1],(1-P_s)\mathbb{C}^{2d})\\
&\quad := \quad \{f\in C^\infty([0,1],(1-P_s)\mathbb{C}^{2d}) \mid (i\partial)^{2k}f(0)=0,\ (i\partial)^{2k+1}f(1)=0\ \forall k\in\mathbb{N}_0\}\ ,
\end{aligned}
$$

the embedding $\tilde{}$ maps the space $C_R^\infty([0,1],\mathbb{C}^{2d})$ into $C^\infty(\mathbb{R}/4\mathbb{Z},\mathbb{C}^{2d})$.
It intertwines the operators $D_{I_s}^2$ on $C_R^\infty([0,1],\mathbb{C}^{2d})$ and $-\partial^2$ on $C^\infty(\mathbb{R}/4\mathbb{Z},\mathbb{C}^{2d})$.
It is well-known (see [Ta], 3.7) that the heat kernel for the holomorphic semigroup on $L^2(\mathbb{R}/4\mathbb{Z})$ generated by ∂^2 is

$$H(t,x,y)=(4\pi t)^{-\frac12}\sum_{k\in\mathbb{Z}}e^{-\frac{(x-y+4k)^2}{4t}}\ .$$

Hence for $f\in L^2([0,1],(1-P_s)\mathbb{C}^{2d})$ and $x\in[0,1]$ it holds

$$
\begin{aligned}
(e^{-tD_{I_s}^2}f)(x) &= (e^{t\partial^2}\tilde f)(x)\\
&= \int_0^1 H(t,x,y)f(y)dy+\int_1^2 H(t,x,y)f(2-y)dy\\
&\quad +\int_2^3 H(t,x,y)(-f(y-2))dy+\int_3^4 H(t,x,y)(-f(4-y))dy\ .
\end{aligned}
$$

Thus the action of $e^{-tD_{I_s}^2}$ on the space $L^2([0,1],(1-P_s)\mathbb{C}^{2d})$ is given by the scalar integral kernel

$$(x,y)\mapsto H(t,x,y)+H(t,x,2-y)-H(t,x,y+2)-H(t,x,4-y)\ .$$

Analogously we conclude that the action of $e^{-tD_{I_s}^2}$ restricted to $L^2([0,1],P_s\mathbb{C}^{2d})$ is given by the integral kernel

$$(x,y)\mapsto H(t,x,y)-H(t,x,2-y)-H(t,x,y+2)+H(t,x,4-y)\ .$$

This yields the integral kernel k_t of $e^{-tD_{I_s}^2}$.
In the following we write $C_R^\infty([0,1],M_{2d}(\mathbb{C}))$ for the subspace of those functions in $C^\infty([0,1],M_{2d}(\mathbb{C}))$ whose column vectors are in $C_R^\infty([0,1],\mathbb{C}^{2d})$.

Lemma 3.1.1. *The map*

$$(0,\infty)\to C^\infty([0,1],C_R^\infty([0,1],M_{2d}(\mathbb{C}))),\ t\mapsto \big(y\mapsto k_t(\cdot,y)\big)\ ,$$

is smooth.

For $\phi,\psi\in C^\infty([0,1])$ with $\operatorname{supp}\phi\cap\operatorname{supp}\psi=\emptyset$ the map $t\mapsto\big(y\mapsto \phi(\cdot)k_t(\cdot,y)\psi(y)\big)$ can be continued by zero to a smooth map from $[0,\infty)$ to $C^\infty([0,1],C_R^\infty([0,1],M_{2d}(\mathbb{C})))$.

Proof. This follows from the corresponding well-known properties of H. $\square$

Lemma 3.1.2. *Let $m, n \in \mathbb{N}_0$. Then there is $C > 0$ such that for all $x, y \in [0,1]$ and all $t > 0$ it holds*

$$|\partial_x^m \partial_y^n k_t(x,y)| \leq C(1 + t^{-\frac{m+n+1}{2}})(e^{-\frac{c(x,y)^2}{4t}}) \ .$$

Proof. This follows from the explicit formula of H above. When estimating the derivatives we take into account that for any $m \in \mathbb{N}$ the function $(x,y,t) \mapsto \frac{(x-y)^{2m}}{t^m} e^{-\frac{(x-y)^2}{4t}}$ can be continuously extended by zero to $t = 0$. $\square$

3.1.2 The heat kernel of $e^{-tD_s^2}$

The operator D_s that was defined in §2.1.2 acts on the Hilbert space $L^2(M,E)$ as a selfadjoint operator.

In this section we investigate the integral kernel of $e^{-tD_s^2}$ after having proved its existence.

Let $Z = [0,1] \times \mathbb{R}$.

Let D_Z be the operator defined in §2.3 with boundary conditions given by a pair (P_0, P_1) with $P_0, P_1 \in M_{2d}(\mathbb{C})$. Then D_Z is selfadjoint on $L^2(Z, \mathbb{C}^{4d})$. We study the semigroup $e^{-tD_Z^2}$ on $L^2(Z, \mathbb{C}^{4d})$ as well, since later we will compare $e^{-tD_s^2}$ on the cylindric ends with $e^{-tD_Z^2}$ for appropriate boundary conditions.

Since the proofs are standard, we only sketch them.

Note first that a solution $u : \mathbb{R} \to \operatorname{dom} D_s$ of the initial-value problem

$$\frac{d}{dt} u(t) = iD_s u(t), \ u(0) = f$$

with $f \in C^\infty_{Rc}(M,E)$ is unique, and also a solution of the corresponding problem for D_Z, this follows from an energy estimate.

Lemma 3.1.3. *For D_s on $L^2(M,E)$ it holds:*

If $f \in C^\infty_{Rc}(M,E)$, then for any $x \in \operatorname{supp}(e^{itD_s} f)$ it follows $d(x, \operatorname{supp} f) \leq |t|$.

An analogous result holds for D_Z on $L^2(Z, \mathbb{C}^{4d})$.

This property is called "finite propagation speed property".

Proof. We use a cutting-and-pasting argument.

For $j \in \mathbb{Z}/6$ let $V_j := \{x \in M \mid d(x, \partial_j M) < \frac{1}{4}\}$ and let $W := M \setminus \partial M$. These sets cover M.

There is an oriented isometric embedding $V_j \hookrightarrow \mathbb{R}^2$ mapping V_j to $\{x_2 \geq 0\}$ and ∂V_j to $\{x_2 = 0\}$. The operator ∂_M on V_j extends to a translation invariant differential operator $\partial_{\mathbb{R}^2} : C_c^\infty(\mathbb{R}^2, \mathbb{C}^{4d}) \to L^2(\mathbb{R}^2, \mathbb{C}^{4d})$. Let $D_{\mathbb{R}^2}$ be its closure as an unbounded operator on $L^2(\mathbb{R}^2, \mathbb{C}^{4d})$.

We apply the method of images in order to obtain an embedding

$$\tilde{} : C_{Rc}^\infty(V_j, E) \hookrightarrow C_c^\infty(\mathbb{R}^2, \mathbb{C}^{4d})$$

intertwining the operators D_s and $D_{\mathbb{R}^2}$:

Let

$$\begin{aligned} & C_l^\infty(V_j, \mathcal{P}^s_{j \bmod 3} E^+) \\ & := \{ f \in C_c^\infty(V_j, \mathcal{P}^s_{j \bmod 3} E^+) \mid (\partial_{e_2}^{2k+1} f)(x) = 0, \ \forall k \in \mathbb{N}_0, \ \forall x \in \partial_j M \} , \end{aligned}$$

and

$$\begin{aligned} & C_r^\infty(V_j, (1 - \mathcal{P}^s_{j \bmod 3}) E^+) \\ & := \{ f \in C_c^\infty(V_j, (1 - \mathcal{P}^s_{j \bmod 3}) E^+) \mid (\partial_{e_2}^{2k} f)(x) = 0, \ \forall k \in \mathbb{N}_0, \ \forall x \in \partial_j M \} . \end{aligned}$$

Then it holds

$$C_{Rc}^\infty(V_j, E^+) = C_l^\infty(V_j, \mathcal{P}^s_{j \bmod 3} E^+) \oplus C_r^\infty(V_j, (1 - \mathcal{P}^s_{j \bmod 3}) E^+) ,$$

and

$$C_{Rc}^\infty(V_j, E^-) = c(e_1) C_l^\infty(V_j, \mathcal{P}^s_{j \bmod 3} E^+) \oplus c(e_1) C_r^\infty(V_j, (1 - \mathcal{P}^s_{j \bmod 3}) E^+) .$$

For $f \in C_l^\infty(V_j, \mathcal{P}^s_{j \bmod 3} E^+)$ resp. $f \in C_r^\infty(V_j, (1 - \mathcal{P}^s_{j \bmod 3}) E^+)$ we define $\tilde{f}$ by first extending f by zero to the half plane $\{x_2 \geq 0\}$ and then reflecting such that $\tilde{f}$ is even resp. odd with respect to $\{x_2 = 0\}$.

For $f \in C_{Rc}^\infty(V_j, E^-)$ we set $\tilde{f} := ic(e_1)\tilde{}(ic(e_1) f)$.

For $D_{\mathbb{R}^2}$ the finite propagation speed property holds. Hence the assertion of the lemma holds for all $f \in C_{Rc}^\infty(V_j, E)$ with $\operatorname{supp} f \subset \{x \in M \mid d(x, \partial_j M) < \frac{3}{16}\}$ and for $|t| < \frac{1}{16}$.

For $f \in C_c^\infty(W, E)$ with $\operatorname{supp} f \subset \{x \in M \mid d(x, \partial M) > \frac{1}{16}\}$ and for $|t| < \frac{1}{16}$ the assertion holds by the standard theory of hyperbolic equations on open subsets of $\mathbb{R}^2$.

Since any $f \in C_c^\infty(M, E)$ can be written as $f = f_W + f_0 + \ldots f_5$ with $f_W \in C_c^\infty(W, E)$ and $f_j \in C_{Rc}^\infty(V_j, E)$, the assertion holds for any $f \in C_c^\infty(M, E)$ and for $|t| < \frac{1}{16}$, and by the group property of e^{itD_s} it follows for all $t \in \mathbb{R}$.

The proof for D_Z is analogous. $\square$

For $k \in \mathbb{N}_0$ let $H^k(\mathbb{C}, D_s)$ be the Hilbert space whose underlying vector space is $\operatorname{dom} D_s^k$ and whose scalar product is given by

$$< f, g >_{H^k} := < (1 + D_s^2)^{\frac{k}{2}} f, (1 + D_s^2)^{\frac{k}{2}} g > \quad .$$

Define $H^k(\mathbb{C}, D_Z)$ analogously.

Lemma 3.1.4. *Let $k \in \mathbb{N},\ k \geq 2$.*

1. *There is an embedding $H^k(\mathbb{C}, D_Z) \to C^{k-2}(Z, \mathbb{C}^{4d})$.*
2. *There is an embedding $H^k(\mathbb{C}, D_s) \to C^{k-2}(M, E)$.*

Proof. We sketch the proof of 2), the proof of 1) is analogous.

Let $D_{\mathbb{R}^2}$ be as in the previous proof.

For fixed $r > 0$ the constants in the Gårding inequality for the elliptic operator $(1 + D^2_{\mathbb{R}^2})^k$ on balls $B_r(x),\ x \in \mathbb{R}^2$, can be chosen independent of x.

Since for $j \in \mathbb{Z}/6$ the embedding $C^\infty_{Rc}(V_j, E) \hookrightarrow C^\infty_c(\mathbb{R}^2, \mathbb{C}^{4d})$ that was defined in the previous proof intertwines the Dirac operators D_s on $C^\infty_{Rc}(V_j, E)$ and $D_{\mathbb{R}^2}$ on $C^\infty_c(\mathbb{R}^2, \mathbb{C}^{4d})$, the Gårding inequality for the operator $(1+D_s^2)^k$ on balls $B_{1/8}(x) \subset M$ with $x \in \partial M$ holds with constants independent of x as well.

Since M is of bounded geometry we can also find global constants for the Gårding inequality for $(1 + D_s^2)^k$ on balls $B_r(x) \subset M$ not intersecting the boundary and with r small enough.

Then the assertion follows from the Sobolev embedding theorem on balls by standard arguments. $\square$

Corollary 3.1.5. *The operators $e^{-tD_Z^2}$ on $L^2(Z, \mathbb{C}^{4d})$ and $e^{-tD_s^2}$ on $L^2(M, E)$ are integral operators with smooth integral kernels.*

Proof. This follows from the previous two lemmata (see [Ro], lemma 5.6). $\square$

Lemma 3.1.6. *Let $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a function and assume that for any $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $r > \varepsilon$ it holds*

$$f(r, t) \leq Ce^{-\frac{(r-\varepsilon/2)^2}{(4+\delta)t}} \ .$$

Then for all $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $r > \varepsilon$ it holds

$$f(r, t) \leq Ce^{-\frac{r^2}{(4+\delta)t}} \ .$$

Proof. Choose $0 < a < 1$ with $\frac{1-a}{4+\delta/2} > \frac{1}{4+\delta}$ and let $m > \frac{2}{a}$.
Then there is $C > 0$ such that for all $r > \varepsilon$ it holds

$$f(r, t) \leq Ce^{-\frac{(r-\varepsilon/m)^2}{(4+\delta/2)t}} \ .$$

It follows

$$\begin{aligned} f(r,t) &\leq Ce^{-\frac{(1-a)r^2}{(4+\delta/2)t}} e^{\frac{r}{(4+\delta/2)t}(-ar+\frac{2\varepsilon}{m})} e^{-\frac{(\varepsilon/m)^2}{(4+\delta/2)t}} \\ &\leq Ce^{-\frac{r^2}{(4+\delta)t}} \ . \end{aligned}$$

In the last step we used the fact that $\frac{r}{(4+\delta/2)t}(-ar + \frac{2\varepsilon}{m}) < 0$ for $r > \varepsilon$. $\square$

Lemma 3.1.7. *Let N be closed spin manifold resp. let $N = M, Z$. If N is closed let E_N be a Clifford module on N and let D_N be the associated Dirac operator. If $N = M$ resp. $N = Z$, then let $E_N = E$ resp. $E_N = Z \times \mathbb{C}^{4d}$ and let $D_N = D_s$ resp. $D_N = D_Z$. Let k_t be the integral kernel of $e^{-tD_N^2}$.*

For any $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $t > 0,\ r > \varepsilon$ and $x \in N$ it holds

$$\int_{N \setminus B_r(x)} |k_t(x,y)|^2 dy \leq C e^{-\frac{r^2}{(4+\delta)t}} \ .$$

Analogous estimates hold for the partial derivatives in x and y with respect to unit vector fields on N.

Proof. Let $k \in 2\mathbb{N}$ with $k > \frac{\dim N}{2}$.

Let $S(x,\varepsilon) := \{u \in C_c^\infty(B_\varepsilon(x), E_N) \mid \|(1+D_N^2)^{-\frac{k}{2}} u\| \leq 1\}$.

Then by the Sobolev embedding theorem resp. by lemma 3.1.4 there is $C > 0$ such that for any $x \in N$ it holds

$$\int_{N \setminus B_r(x)} |k_t(x,y)|^2 dy \leq C \sup_{u \in S(x,\varepsilon/2)} \|e^{-tD_N^2} u\|^2_{N \setminus B_r(x)} \ .$$

As in the proof of [CGT], prop. 1.1, it follows:

$$\begin{aligned} \int_{N \setminus B_r(x)} |k_t(x,y)|^2 dy &\leq C t^{-1/2} \int_{r-\varepsilon/2}^{\infty} |(1 + (\frac{d}{ids})^2)^{k/2} e^{-s^2/4t}| ds \\ &= C \int_{\frac{r-\varepsilon/2}{\sqrt{t}}}^{\infty} |(1 + t^{-1}(\frac{d}{ids'})^2)^{k/2} e^{-s'^2/4}| ds' \ . \end{aligned}$$

Thus there is $l \in \mathbb{N}$ such that it holds

$$\begin{aligned} \int_{N \setminus B_r(x)} |k_t(x,y)|^2 dy &\leq C(1+t^{-l}) \int_{\frac{r-\varepsilon/2}{\sqrt{t}}}^{\infty} (1+s'^l) e^{-s'^2/4} ds' \\ &\leq C(1+t^{-l}) e^{-\frac{(r-\varepsilon/2)^2}{(4+\delta/2)t}} \\ &\leq C e^{-\frac{(r-\varepsilon/2)^2}{(4+\delta)t}} \ . \end{aligned}$$

Then the assertion follows by applying the previous lemma to $f(r,t) := \sup_{x \in N} \int_{N \setminus B_r(x)} |k_t(x,y)|^2 dy$. □

Lemma 3.1.8. *Let k_t be as in the previous lemma.*

For any $\varepsilon, \delta > 0$ there is $C < \infty$ such that for all $x, y \in N$ with $d(x,y) > \varepsilon$ and all $t > 0$ it holds

$$|k_t(x,y)| \leq Ce^{-\frac{d(x,y)^2}{(4+\delta)t}} .$$

Analogous estimates hold for the partial derivatives in x and y with respect to unit vector fields on N.

Proof. Let $S(y, \varepsilon)$ and $k \in 2\mathbb{N}$ be as in the proof of the previous lemma. By the Sobolev embedding theorem resp. lemma 3.1.4 there is $C > 0$ such that it holds for all $r > \varepsilon$ and all $x, y \in N$ with $d(x,y) \geq r$:

$$|k_t(x,y)| \leq C \sup_{u \in S(y,\varepsilon/4)} \|(1 + D_N^2)^{\frac{k}{2}} e^{-tD_N^2} u\|^2_{N \setminus B_{r-\varepsilon/4}(y)} .$$

Analogously to the proof of the previous lemma it follows

$$|k_t(x,y)| \leq Ce^{-\frac{(r-\varepsilon/2)^2}{(4+\delta)t}} .$$

Then the result follows from 3.1.6 with $f(r,t) := \sup_{x,y \in N:\ d(x,y)=r} |k_t(x,y)|$. $\square$

For the next lemma assume that $U \subset M$ is an open set for which one or both of the following properties hold:

1. U is precompact and $\overline{U} \cap \partial M = \emptyset$,
2. or there is $k \in \mathbb{Z}/6$ such that $U \subset F_k(0, \frac{1}{4})$.

In the first case there is a compact manifold N and a hermitian Clifford module E_N on N such that there is an isometric Clifford module isomorphism $E|_U \to E_N$, whose base map is an isometry. We identify $E|_U$ with its image in E_N. Then D_s coincides with D_N on U.

In the second case U is a subset of Z_k by §1.1. Let D_{Z_k} be the operator D_Z on $Z_k \times \mathbb{C}^{4d}$ with boundary conditions given by $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{(k+1) \bmod 3})$. Then D_s coincides with D_{Z_k} on U.

Lemma 3.1.9. *Let U be as in 1) resp. 2). Let k_t be the integral kernel of the heat semigroup of D_s on M and let k'_t be the integral kernel of the heat semigroup of D_N resp. D_{Z_k}.*

For any $T > 0$ and $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $0 < t < T$, $r > \varepsilon$ and $x, y \in U$ with $B_r(x) \subset U$ and $B_r(y) \subset U$ it holds

$$|k_t(x,y) - k'_t(x,y)| \leq e^{-\frac{r^2}{(4+\delta)t}} .$$

Analogous estimates hold for the partial derivatives with respect to unit vector fields on U.

Proof. The notation is as in the proof of lemma 3.1.7.

The estimate follows from

$$\begin{aligned}
|k_t(x,y) - k'_t(x,y)| &= \sup_{\phi\in S(x,\varepsilon)} \sup_{\psi\in S(y,\varepsilon/2)} |<\phi, e^{-tD_s^2}\psi> - <\phi, e^{-tD_N^2}\psi>| \\
&\leq Ct^{-\frac{1}{2}} \sup_{\phi\in S(x,\varepsilon)} \sup_{\psi\in S(y,\varepsilon/2)} |<\phi, \int_{\mathbb{R}} e^{-s^2/4t}(e^{isD_s} - e^{isD_N})\psi>| \\
&\leq Ct^{-\frac{1}{2}} \int\limits_{r-\varepsilon/2}^{\infty} |(1+(\frac{d}{ids})^2)^k e^{-s^2/4t}|
\end{aligned}$$

Here we used that it holds $(e^{isD_s} - e^{isD_N})\psi = 0$ for $|s| \leq r - \varepsilon/2$ by the finite propagation speed property (lemma 3.1.3) and the uniqueness of solutions of hyperbolic equations. □

3.2 The heat semigroup on compact manifolds

Let $\mathcal{B}$ be a Banach algebra with unit.

Let N be a closed spin manifold of dimension n. Let E_N be a hermitian Clifford module on N that is trivial as a vector bundle and let D_N be the associated Dirac operator.

Let $k_t \in C^\infty(N \times N, E_N \boxtimes E_N)$ be the induced heat kernel.

According to cor. 5.2.4 the heat kernel defines a family of bounded operators on $L^2(N, E_N \otimes \mathcal{B})$. The operators are smoothing, thus they restrict to a family of bounded operators on $C^m(N, E_N \otimes \mathcal{B})$ for any $m \in \mathbb{N}_0$. In order to show that the family extends to a holomorphic semigroup we have to study its behaviour for small t.

We define $-D_N^2$ as a closed operator on $L^2(N, E_N \otimes \mathcal{B})$ by requiring that $C^\infty(N, E_N \otimes \mathcal{B})$ is a core of $-D_N^2$.

For $t \to 0$ the heat kernel can be estimated as follows:

Lemma 3.2.1. *Let $\varepsilon > 0$ be smaller than the injectivity radius of N and let $\chi : [0,\infty) \to [0,1]$ be a smooth monotonously decreasing function such that $\chi(r) = 1$ for $r \leq \varepsilon/2$ and $\chi(r) = 0$ for $r \geq \varepsilon$.*

Let A be a differential operator of order m on $C^\infty(N, E_N)$.

Then there is $C > 0$ such that for all $x, y \in N$ and for all $t > 0$ it holds:

$$|A_x k_t(x,y)| \leq C + Ct^{-(n+m)/2} e^{-d(x,y)^2/4t} \chi(d(x,y)) \sum_{i=0}^{m} d(x,y)^i t^{\frac{i}{2}} \ .$$

Proof. This follows from [BGV], prop. 2.46, and its proof. □

Proposition 3.2.2. *Let A be a differential operator of order m on $C^\infty(N, E_N \otimes \mathcal{B})$. Then there is $C > 0$ such that the action of the integral kernel $A_x k_t(x,y)$ on $L^2(N, E_N \otimes \mathcal{B})$ is bounded by $C(1 + t^{-m/2})$ for all $t > 0$.*

Proof. Choose a finite open covering $\{U_\nu\}_{\nu \in I}$ of N of normal coordinate patches and assume that for any $x, y \in U_\nu$ the shortest geodesics connecting x and y is in U_ν.

Then there are $c_1, c_2 > 0$ such that for all $\nu \in I$ and all $x, y \in U_\nu$ it holds

$$c_1 |x - y|_\nu \leq d(x,y) \leq c_2 |x - y|_\nu \ ,$$

where $|\cdot|_\nu$ denotes the euclidian distance on U_ν defined by the coordinates.

Let $\{\phi_\nu\}_{\nu \in I}$ be a partition of unity subordinate to the covering $\{U_\nu\}_{\nu \in I}$.

Let $\varepsilon > 0$ be smaller than the injectivity radius of N and so small, that for any $\nu \in I$ it holds $\{x \in N \mid d(x, \operatorname{supp} \phi_\nu) \leq \varepsilon\} \subset U_\nu$. Then for χ as in the previous lemma it follows

$$\phi_\nu(x)\chi(d(x,y)) \leq \phi_\nu(x)\chi(c_1|x-y|_\nu) 1_{U_\nu}(y) \ .$$

By the previous lemma there is $C > 0$ such that for all $x, y \in N$ and $t > 0$ it holds:

$$|A_x k_t(x,y)| \leq C + C t^{-(n+m)/2} \sum_{\nu \in J} \phi_\nu(x) \Big(e^{-c_1^2 |x-y|_\nu^2/4t} \chi(c_1|x-y|_\nu) \sum_{i=0}^m c_2^i |x-y|_\nu^i t^{\frac{i}{2}} \Big) 1_{U_\nu}(y) \ .$$

The ν-th term of the outer sum is supported on $U_\nu \times U_\nu$ and is of the form $\phi_\nu(x) f(x-y) 1_{U_\nu}(y)$ in the coordinates of $U_\nu \times U_\nu$ with $f \in L^1(\mathbb{R}^n)$ and there is $C > 0$ such that

$$\|f\|_{L^1} \leq C t^{n/2}(1 + t^{m/2})$$

for all $t > 0$.

The assertion follows now from prop. 5.2.3 and cor. 5.2.5. □

Proposition 3.2.3.

1. *The family of integral kernels $k_t(x,y)$ defines a bounded strongly continuous semigroup on $L^2(N, E_N \otimes \mathcal{B})$ that may be extended to a bounded holomorphic semigroup. Its generator is $-D_N^2$.*
2. *The family of integral kernels $k_t(x,y)$ defines a bounded strongly continuous semigroup on $C^m(N, E_N \otimes \mathcal{B})$ for any $m \in \mathbb{N}_0$.*

Proof. 1) By the previous proposition the action of the integral kernel $k_t(x,y)$ on $L^2(N, E_N \otimes \mathcal{B})$ is uniformly bounded for $t > 0$. On $L^2(N, E_N) \odot \mathcal{B}$ it converges strongly to the identity. Thus $k_t(x,y)$ induces a bounded strongly continuous semigroup on $L^2(N, E_N \otimes \mathcal{B})$.

On $C^\infty(N, E_N \otimes \mathcal{B})$ the action of the generator coincides with the action of $-D_N^2$. Since $C^\infty(N, E_N \otimes \mathcal{B})$ is invariant under the semigroup and dense in $L^2(N, E_N \otimes \mathcal{B})$, it is a core for the generator. Hence the generator is $-D_N^2$.

By the previous proposition there is $C > 0$ such that on $L^2(N, E_N \otimes \mathcal{B})$ it holds for all $0 < t < 1$:

$$\|D_N^2 e^{-tD_N^2}\| < Ct^{-1} \ .$$

Since for $t > 0$ it holds $\operatorname{Ran} e^{-tD_N^2} \subset C^\infty(N, E_N \otimes \mathcal{B}) \subset \operatorname{dom} D_N^2$, it follows, by prop. 5.4.3, that $e^{-tD_N^2}$ extends to a holomorphic semigroup.

Since the integral kernel of $D_N^2 e^{-tD_N^2}$ is exponentially decaying in the supremum-snorm for $t \to \infty$ it follows that $D_N^2 e^{-tD_N^2}$ is exponentially decaying as an operator on $L^2(N, E_N \otimes \mathcal{B})$.

By prop. 5.4.3 this shows that the holomorphic extension is bounded.

2) follows from the fact that $k_t(x, y)$ defines a strongly continuous bounded semigroup on $C^m(N, E_N)$ by [BGV], th. 2.30, and that $C^m(N, E_N \otimes \mathcal{B}) \cong C^m(N, E_N) \otimes_\varepsilon \mathcal{B}$. □

3.3 The heat semigroup on $[0, 1]$

3.3.1 The semigroup

Let (P_0, P_1) be a pair of transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_0, P_1 \in M_{2d}(\mathcal{A}_\infty)$ and let D_I be the associated operator on $L^2([0, 1], \mathcal{A}^{2d})$ defined in §2.2.1.

We now define an action of D_I on $L^2([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$: Let D_I, as an unbounded operator on $L^2([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$, be the closure of $I_0\partial$ with domain $C_R^\infty([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

We determine the resolvent set of D_I:

Let $U \in M_{2d}(\mathcal{A}_\infty)$ be a unitary with $UI_0 = I_0U$ and $UP_0U^* = P_s$ and let $p \in M_d(\mathcal{A}_\infty)$ be such that

$$UP_1U^* = \tfrac{1}{2}\begin{pmatrix} 1 & p^* \\ p & 1 \end{pmatrix} .$$

The unitaries U and p exist by lemma 1.4.3.

Proposition 3.3.1. *Let $\lambda \in \mathbb{C}$ with $\exp(-2i\lambda) \notin \sigma(p)$. Then it holds:*

1. *The operator $D_I - \lambda$ has a bounded inverse on $L^2([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.*
2. *The inverse $(D_I - \lambda)^{-1}$ maps $C_R^l([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ isomorphically onto $C_R^{l+1}([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for any $l \in \mathbb{N}_0$.*
3. *The inverse $(D_I - \lambda)^{-1}$ maps $L^2([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ continuously to $C([0, 1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.*

Proof. The inverse of $D_I - \lambda$ is given by

$$((D_I - \lambda)^{-1} f)(x) = \int_0^x I_0 e^{I_0\lambda(x-y)} f(y)dy + \int_0^1 e^{I_0\lambda(x-y)} A(y) f(y)dy$$

with

$$A(y) = U^* \frac{i}{p - e^{-2i\lambda}} \begin{pmatrix} p & e^{-2i\lambda(1-y)} \\ pe^{-2i\lambda y} & e^{-2i\lambda} \end{pmatrix} U \ .$$

It is easily seen that this map fulfills 1),2) and 3). $\square$

We define D_I as an unbounded operator on $C^l_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}),\ l \in \mathbb{N}_0$, by setting $\operatorname{dom} D_I := C^{l+1}_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. By the previous proposition D_I is a closed operator on $C^l_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

As before D_I is also denoted by D_{I_s} if $R = (P_s, 1-P_s)$.

Now we show that $-D_I^2$ generates a holomorphic semigroup on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well as on $C^l_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. This will be done by first proving that $-D_{I_s}^2$ generates a holomorphic semigroup and then applying prop. 5.4.10.

Furthermore the knowledge of the resolvent set of $-D_I^2$ yields a norm estimate of the semigroup $e^{-tD_I^2}$ for large t.

Lemma 3.3.2. *Assume that $R = (P_s, 1-P_s)$.*

The operator $-D_{I_s}^2$ is the generator of a bounded holomorphic semigroup $e^{-tD_{I_s}^2}$ on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ and on $C^l_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for any $l \in \mathbb{N}_0$.

Proof. Let k_t be the integral kernel of $e^{-tD_{I_s}^2}$ (see §3.1.1) and let $S(t)$ be the induced integral operator.

By lemma 3.1.2 and prop. 5.2.3 the family $S(t)$ is uniformly bounded on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for $t > 0$ and the family $D_{I_s}^2 S(t)$ is bounded on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ by $C(1+t^{-1})e^{-\omega t}$ for some $C, \omega > 0$ and all $t > 0$.

Since $S(t)$ converges strongly to the identity on $L^2([0,1],\mathbb{C}^{2d}) \odot \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ for $t \to 0$, it is a strongly continuous semigroup.

By prop. 5.4.3 it extends to a bounded holomorphic semigroup.

The kernel H from §3.1.1 defines a bounded holomorphic semigroup on $C^k(\mathbb{R}/4\mathbb{Z},(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well by prop. 3.2.3. This implies that $S(t)$ restricts to a bounded holomorphic semigroup on $C^k_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. In particular it follows that $C^\infty_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ is a core of the generator of $S(t)$ on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. Thus the generator is $-D_{I_s}^2$. $\square$

Proposition 3.3.3. *The operator $-D_I^2$ generates a holomorphic semigroup $e^{-tD_I^2}$ on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well as on $C^k_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for all $k \in \mathbb{N}_0$, and there are $C, \omega > 0$ such that for all $t \geq 0$ it holds*

$$\|e^{-tD_I^2}\| \leq Ce^{-\omega t}$$

on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ and on $C^k_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

Proof. The following arguments hold on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well as on $C^k_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$:

The operator $D_I - U^* D_{I_s} U$ is bounded by prop. 2.2.2. Since $U^* D_{I_s} U$ has a bounded inverse by prop. 3.3.1 and $-U^* D_{I_s}^2 U$ generates a bounded holomorphic semigroup by the previous lemma, we can apply prop. 5.4.10: It follows that $-D_I^2$ generates a holomorphic semigroup.

By prop. 3.3.1 there is $\omega > 0$ such that the spectrum of D_I^2 on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well as on $C^l_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ is in $]\omega,\infty[$, hence by prop. 5.4.2 there is $C > 0$ with $\|e^{-tD_I^2}\| \leq C e^{-\omega t}$. $\square$

3.3.2 The integral kernel

Let D_I and let $R = (P_0, P_1)$ be as in the previous section.

By cutting and pasting we construct an approximation of the semigroup $e^{-tD_I^2}$:

Let D_{I_0} be defined analogously to D_I with boundary conditions given by $(P_0, 1 - P_0)$, and let D_{I_1} be defined analogously to D_I with boundary conditions given by $(1 - P_1, P_1)$.

From lemma 1.4.3 and §3.1.1 it follows that $e^{-tD_{I_k}^2},\ k = 0,1$ is an integral operator for $t > 0$. Let $e^k_t(x,y)$ be its integral kernel.

Let $\phi_0 : [0,1] \to [0,1]$ be a smooth function with $\operatorname{supp}\phi_0 \subset [0,\frac{2}{3}[$ and $\operatorname{supp}(1-\phi_0) \subset]\frac{1}{3},1]$ and let $\phi_1 := (1-\phi_0)$. Furthermore choose smooth functions $\gamma_0,\gamma_1 : [0,1] \to [0,1]$ with

- $\gamma_k|_{\operatorname{supp}\phi_k} = 1,\ k = 0,1$,
- $\operatorname{supp}\gamma_k' \cap \operatorname{supp}\phi_k = \emptyset,\ k = 0,1$,
- $\operatorname{supp}\gamma_0 \subset [0,\frac{5}{6}]$ and $\operatorname{supp}\gamma_1 \subset [\frac{1}{6},1]$.

Let

$$e_t(x,y) = \gamma_0(x) e^0_t(x,y)\phi_0(y) + \gamma_1(x) e^1_t(x,y)\phi_1(y)$$

and write E_t for the corresponding integral operator. Set $E_0 := 1$.

Then E_t is strongly continuous on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ and on $C^l_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ at any $t \geq 0$.

For $f \in C^\infty_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ the function $[0,\infty) \to L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}),\ t \mapsto E_t f$ is even differentiable. Hence by Duhamel's principle (prop. 5.4.5) it holds in $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for $f \in C^\infty_R([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$:

$$(*) \qquad e^{-tD_I^2} f - E_t f = -\int_0^t e^{-(t-s)D_I^2}\left(\frac{d}{ds} + D_I^2\right) E_t f \ ds \ .$$

In the next proposition we apply this equation: We prove that $e^{-tD_I^2}$ is an integral operator with smooth integral kernel for $t > 0$ by showing that the right hand side is an integral operator with smooth integral kernel.

Before we fix some notation: In the following the norm on $M_{2d}(\mathcal{A}_i)$ is denoted by $|\cdot|$.

Furthermore $C^k_R([0,1], M_{2d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i))$ with $k \in \mathbb{N}_0$ resp. $k = \infty$ means subspace of $C^k([0,1], M_{2d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i))$ containing those functions whose column vectors are elements of $C^k_R([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. Then any bounded operator on $C^k_R([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ acts as a bounded operator on $C^k_R([0,1], M_{2d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i))$ in an obvious way.

Proposition 3.3.4. *For $t > 0$ the operator $e^{-tD_I^2}$ is an integral operator. For its integral kernel k_t it holds:*

1. *The map*

$$(0,\infty) \to C^\infty([0,1], C^\infty_R([0,1], M_{2d}(\mathcal{A}_\infty))),\ t \mapsto \big(y \mapsto k_t(\cdot, y)\big)$$

   is well-defined and smooth.

2. *It holds $k_t(x,y) = k_t(y,x)^*$.*

3. *For any $m, n \in \mathbb{N}_0$ and any $\delta > 0$ there is $C > 0$ such that it holds*

$$|\partial_x^m \partial_y^n k_t(x,y) - \partial_x^m \partial_y^n e_t(x,y)| \leq Ct \sum_{k=0,1} e^{-\frac{d(y, \operatorname{supp}\gamma_k')^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y)$$

   for all $t > 0$ and all $x, y \in [0,1]$.

Proof. Let $f \in C^\infty_R([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. From $(*)$ it follows

$$e^{-tD_I^2} f - E_t f = \sum_{k=0,1} \int_0^t \int_{[0,1]} e^{-sD_I^2} (\gamma_k'\partial + \partial\gamma_k') e^k_{t-s}(\cdot, y)\phi_k(y) f(y)\ dyds\ .$$

By lemma 3.1.1 the map

$$t \mapsto \big(y \mapsto (\gamma_k'\partial + \partial\gamma_k') e^k_t(\cdot, y)\phi_k(y)\big)$$

can be extended by zero to a smooth map from $[0,\infty)$ to $C^\infty([0,1], C^\infty_R([0,1], M_{2d}(\mathcal{A}_i))$.

Since $e^{-sD_I^2}$ acts as a uniformly bounded operator on $C^\infty_R([0,1], M_{2d}(\mathcal{A}_i))$ by lemma 3.1.1 it follows that the operator on the right hand side is an integral operator with smooth integral kernel.

Hence also $e^{-tD_I^2}$ is an integral operator with smooth integral kernel. Its integral kernel satisfies 1) by the preceeding arguments.

The selfadjointness of $e^{-tD_I^2}$ implies 2).

Since $d(\operatorname{supp}\phi_k, \operatorname{supp}\gamma_k') > \varepsilon$ for some $\varepsilon > 0$, there is $C > 0$ by lemma 3.1.2 such that for all $x, y \in [0,1]$ and $t > 0$ it holds:

$$\begin{aligned}
&|\partial_x^m \partial_y^n \big(k_t(x,y) - e_t(x,y)\big)| \\
\leq\ & C \sum_{k=0,1} \int_0^t \|e^{-sD_I^2}(\gamma_k'\partial + \partial\gamma_k')\partial_y^n\big(e_{t-s}^k(\cdot,y)\phi_k(y)\big)\|_{C^m} ds\ 1_{\operatorname{supp}\phi_k}(y) \\
\leq\ & C \sum_{k=0,1} \int_0^t e^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{(4+\delta)(t-s)}} ds\ 1_{\operatorname{supp}\phi_k}(y) \\
\leq\ & C \sum_{k=0,1} t e^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{(4+\delta)t}}\ 1_{\operatorname{supp}\phi_k}(y)\ .
\end{aligned}$$

This shows statement 3). □

Corollary 3.3.5. *Let $k_t(x,y)$ be the integral kernel of $e^{-tD_I^2}$.*

For any $m, n \in \mathbb{N}_0$ and $\delta, \varepsilon > 0$ we find $C > 0$ such that for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and $t > 0$ it holds

$$|\partial_x^m \partial_y^n k_t(x,y)| \leq C e^{-\frac{d(x,y)^2}{(4+\delta)t}} + Ct \sum_{k=0,1} e^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y).$$

Proof. This follows from the previous proposition and lemma. 3.1.2. □

Corollary 3.3.6. *Let $k_t(x,y)$ be the integral kernel of $e^{-tD_I^2}$.*

Let ω be as in prop. 3.3.2. For any $m, n \in \mathbb{N}_0$ there is $C > 0$ such that for any $t > 0$ and any $x, y \in [0,1]$ it holds

$$|\partial_x^m \partial_y^n k_t(x,y)| \leq C(1 + t^{-\frac{m+n+1}{2}})e^{-\omega t}\ .$$

Proof. There is $C > 0$ such that for all $x, y \in [0,1]$ and all $0 < t < 1$ it holds

$$|\partial_x^m \partial_y^n k_t(x,y) - \partial_x^m \partial_y^n e_t(x,y)| \leq Ct \sum_{k=0,1} e^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{5t}}\ 1_{\operatorname{supp}\phi_k}(y)\ ,$$

hence by lemma 3.1.2

$$|\partial_x^m \partial_y^n k_t(x,y)| \leq C(1 + t^{-\frac{m+n+1}{2}})\ .$$

For all $t > 1$ and $y \in [0,1]$ it holds

$$k_t(\cdot,y) = e^{-(t-1)D_I^2} k_1(\cdot,y)\ .$$

The assertion follows now since $(y \mapsto k_1(\cdot,y)) \in C^n([0,1], C_R^m([0,1], M_{2d}(\mathcal{A}_i)))$ and since by prop. 3.3.3 implies the action of $e^{-(t-1)D_I^2}$ on $C_R^m([0,1], M_{2d}(\mathcal{A}_i))$ is bounded by $Ce^{-\omega t}$ for some $C, \omega > 0$ and any $t > 1$. □

The following facts will be needed when we define the η-form.

Lemma 3.3.7. *Let k_t be the integral kernel of $e^{-tD_{I_s}^2}$.*

Then for all $x, y \in [0,1]$ and $t > 0$ it holds

$$\operatorname{tr}(D_{I_s})_x k_t(x,y) = 0.$$

Proof. Let $S := 2P_s - 1 \in \mathrm{Gl}_{2d}(\mathbb{C})$. It holds $S^2 = 1$, $SI + IS = 0$, $SP_s = P_s$ and $S(1-P_s) = -(1-P_s)$.

This implies $SD_I e^{-tD_{I_s}^2} + D_I e^{-tD_{I_s}^2} S = 0$. Therefore it holds

$$S(D_{I_s})_x k_t(x,y) + (D_{I_s})_x k_t(x,y) S = 0 \ ,$$

hence

$$\operatorname{tr}(D_{I_s})_x k_t(x,y) = \operatorname{tr}(-S(D_{I_s})_x k_t(x,y)S) = -\operatorname{tr}(D_{I_s})_x k_t(x,y) \ .$$

It follows $\operatorname{tr}(D_{I_s})_x k_t(x,y) = 0$. □

Corollary 3.3.8. *Let now $(D_I k_t)$ be the integral kernel of $D_I e^{-tD_I^2}$. It holds uniformly on $[0,1]$:*

$$\lim_{t\to 0} \operatorname{tr}(D_I k_t)(x,x) = 0 \ .$$

Proof. By the previous lemma it holds $\operatorname{tr}(D_I)_x e_t(x,y) = 0$ for all $x, y \in [0,1]$. Then the assertion follows from the estimate in prop. 3.3.4. □

3.4 The heat semigroup on the cylinder

Let $Z = \mathbb{R} \times [0,1]$.

Let $R = (P_0, P_1)$ be a pair of pairwise transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_0, P_1 \in M_{2d}(\mathcal{A}_\infty)$ and let D_Z be the associated operator on $L^2(Z, \mathcal{A}^{4d})$ defined in §2.3.

In this section we define an action of D_Z as an unbounded operator on $L^2(Z, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$ and study its properties. If not specified the notation is as in §2.3.

First we define the following function spaces and operators:

Let

$$\begin{aligned} C^0_R(Z, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}) \ &:= \ \{f \in C^0(Z, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}) \mid \\ &\qquad (P_i \oplus P_i) f(x,i) = f(x,i) \text{ for } x \in \mathbb{R}, \ i = 0,1\} \ , \end{aligned}$$

and define for $k \in \mathbb{N}$ inductively

$$C^k_R(Z, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}) := \{f \in C^k(Z, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}) \mid f, \partial_Z f \in C^{k-1}_R(Z, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})\} \ .$$

Furthermore set

$$C_R^\infty(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}) := \bigcap_{k\in\mathbb{N}} C_R^k(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}) \ .$$

Further suffixes, like c or 0 ..., have their usual meaning.

These spaces are endowed with the subspace topologies.

For a Fréchet space V we define the Schwarz spaces

$$\mathcal{S}(Z,V) := \mathcal{S}(\mathbb{R}) \otimes C^\infty([0,1],V) \ .$$

Moreover let $\mathcal{S}_R(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$ be $\mathcal{S}(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}) \cap C_R^\infty(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$ as a vector space with the topology induced by $\mathcal{S}(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$.

Let D_Z as an unbounded operator on $L^2(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$ be the closure of ∂_Z with domain $\mathcal{S}_R(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$.

Note that at the moment it is not clear whether D_Z^2 is closed on $L^2(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$. We define Δ as the closure of $-\partial_{x_1}^2 - \partial_{x_2}^2$ with domain $\mathcal{S}_R(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$.

Let $\Delta_{\mathbb{R}}$ be the closure of $-\partial_{x_1}^2$ with domain $\mathcal{S}_R(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$.

Let $\tilde{D}_I$ be the closure of $I\partial_{x_2}$ as an unbounded operator on $L^2(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$ with domain $\mathcal{S}_R(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$.

Let D_I be the operator on $L^2([0,1],(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{2d})$ from §3.3.1 with boundary conditions defined by (P_0,P_1). Let k_t^I be the integral kernel of $e^{-tD_I^2}$. It exists by lemma 3.3.4.

By prop. 3.3.1 the operator D_I has a bounded inverse on $L^2([0,1],(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{2d})$. Since the space $L^2(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$ can be identified with $L^2(\mathbb{R},L^2([0,1],(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}))$ by lemma 5.2.2, it follows that $\tilde{D}_I$ has a bounded inverse on $L^2(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$. Hence $-\tilde{D}_I^2$ is closed.

By an analogous argument it follows that $-\tilde{D}_I^2$ generates a bounded holomorphic semigroup on $L^2(Z,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$ with integral kernel $k_t^I(x_2,y_2) \oplus k_t^I(x_2,y_2)$ for $t>0$, where k_t^I is the integral kernel of $e^{-tD_I^2}$. It exists by lemma 3.3.4.

It holds $\Delta = \Delta_{\mathbb{R}} + \tilde{D}_I^2$. Hence we have a natural candidate for the integral kernel of a semigroup generated by $-\Delta$, namely

$$k_t^Z(x,y) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}} (k_t^I(x_2,y_2) \oplus k_t^I(x_2,y_2)) \ .$$

In the next proposition the corresponding family of integral operator is studied.

In the following let $\omega > 0$ be such that it holds on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ for all $t \geq 0$:

$$\|e^{-t\tilde{D}_I^2}\| \leq Ce^{-\omega t} .$$

The existence of such an ω follows from prop. 3.3.3.

Proposition 3.4.1. *1. The integral kernel $k_t^Z(x,y)$ defines a holomorphic semigroup on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ whose generator is $-\Delta$.*

2. For any $m \in \mathbb{N}_0$ the kernel $k_t^Z(x,y)$ is the integral kernel of a holomorphic semigroup on $C_R^m(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. It is denoted by $e^{-t\Delta}$ as well.

3. Let A be a differential operator of order m on $C^\infty(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. Then it holds for the operator $Ae^{-t\Delta}$ on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as well as for $Ae^{-t\Delta} : C_R^n(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) \to C^n(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ with $n \in \mathbb{N}_0$:
There is $C > 0$ such that

$$\|Ae^{-t\Delta}\| \leq C(1 + t^{-m/2})e^{-\omega t}$$

for all $t > 0$.

Proof. 1) By lemma 5.2.3 the kernel $\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x_1-y_1)^2}{4t}}$ defines a uniformly bounded family of operators on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ for $t > 0$. For $t \to 0$ it converges strongly to the identity on $L^2(Z) \odot (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}$, thus it is a strongly continuous semigroup. The space $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ is invariant under the action of the semigroup and the generator's action on that space is given by $\partial_{x_2}^2$. Hence the generator is $-\Delta_{\mathbb{R}}$.

By checking the assumptions of prop. 5.4.3 we show that the semigroup $e^{-t\Delta_{\mathbb{R}}}$ extends to a holomorphic one:

The operator $(i\partial_{x_1})e^{-t\Delta_{\mathbb{R}}}$ equals the convolution with the function $g(x_1) := -c(dx_1)\frac{1}{\sqrt{4\pi t}}\left(\frac{x_1}{2t}\right)e^{-x_1^2/4t}$. Since there is $C > 0$ such that for $0 < t < 1$ it holds $\|g\|_{L^1} \leq Ct^{-1/2}$ it follows

$$\|(i\partial_{x_1})e^{-t\Delta_{\mathbb{R}}}\| \leq Ct^{-1/2} ,$$

on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$, hence it holds for $0 < t < 1$:

$$\|\Delta_{\mathbb{R}}e^{-t\Delta_{\mathbb{R}}}\| \leq \|(i\partial_{x_1})e^{-(t/2)\Delta_{\mathbb{R}}}\|^2 \leq Ct^{-1} .$$

It follows that the semigroup $e^{-t\Delta_{\mathbb{R}}}$ is holomorphic on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Note that this estimate also holds on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ showing that $e^{-t\Delta_{\mathbb{R}}}$ is a holomorphic semigroup on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as well.

Since the semigroups $e^{-t\Delta_{\mathbb{R}}}$ and $e^{-t\tilde{D}_I^2}$ commute with each other, their composition is a holomorphic semigroup. The space $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ is invariant under the action of the semigroup and the generator acts on it as $\partial_{x_1}^2 + \partial_{x_2}^2$. Thus the generator is $-\Delta$.

2) Using the fact that for $f \in C^n_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}), n \in \mathbb{N}$, it holds $(i\partial_{x_j})e^{-t\Delta}f = e^{-t\Delta}(i\partial_{x_j})f,\ j = 0, 1$, the assertion can be reduced to the case $n = 0$.

It follows from prop. 3.3.3 that the action of the integral kernel $k^I_t(x_2, y_2) \oplus k^I_t(x_2, y_2)$ on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ extends to a holomorphic semigroup, and in 1) we showed that the integral kernel $\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x_1-y_1)^2}{4t}}$ defines a holomorphic semigroup on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. Hence the kernel $k^Z_t(x, y)$ defines a semigroup on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ that extends to a holomorphic one.

3) We can restrict to the case that A has constant coefficients and furthermore to the case $n = 0$ by the argument in the proof of 2).

In the following the operator norms can be understood with respect to the action on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as well as with respect to the action on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

We decompose the differential operator A in a sum of operators $a_{hk}\tilde{D}^h_I(i\partial_{x_1})^k$ with $a_{hk} \in M_{4d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and $h + k \leq m$. It holds

$$\tilde{D}^h_I(i\partial_{x_1})^k e^{-t\Delta} = \tilde{D}^h_I e^{-t\tilde{D}^2_I}(i\partial_{x_1})^k e^{-t\Delta_{\mathbb{R}}} \ .$$

By cor. 5.4.9 there is $C > 0$ such that for $0 < t$ it holds:

$$\|D^h_I e^{-tD^2_I}\| \leq Ct^{-h/2}e^{-\omega t} \ .$$

It follows

$$\|\tilde{D}^h_I e^{-t\tilde{D}^2_I}\| \leq Ct^{-h/2}e^{-\omega t} \ .$$

By the estimate in the proof of 1) it holds for $0 < t < 1$:

$$\|(i\partial_{x_1})^k e^{-t\Delta_{\mathbb{R}}}\| \leq \|(i\partial_{x_1})e^{-(t/k)\Delta_{\mathbb{R}}}\|^k \leq Ct^{-k/2} \ .$$

Now the assertion follows from the fact that $e^{-t\Delta_{\mathbb{R}}}$ is uniformly bounded. $\square$

Corollary 3.4.2. *Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda^2 < \omega$.*

1. *The operator $D_Z - \lambda$ is invertible on $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.*
2. *The operator $D_Z - \lambda$ is invertible on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.*
3. *It holds $\Delta = D^2_Z$.*

Proof. 1) For any seminorm $\|\cdot\|$ of $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ there is $C > 0$ such that it holds $\|e^{-t\Delta_{\mathbb{R}}}f\| \leq C\|f\|$ and $\|e^{-t\tilde{D}^2_I}f\| \leq Ce^{-\omega t}$ for all $f \in \mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. Hence $e^{-t\Delta}$ restricts to a bounded operator on $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ and the integral

$$G(\lambda) := \int_0^\infty (D_Z + \lambda)e^{-t(\Delta-\lambda^2)}dt$$

converges as a bounded operator on $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. It inverts $D_Z - \lambda$ on $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

2) The operator $G(\lambda)$ extends to a bounded operator on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ since by the previous proposition there is $C>0$ such that for all $t>0$ it holds on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$:

$$\|(D_Z+\lambda)e^{-t(\Delta-\lambda^2)}\| \leq C(1+t^{-\frac{1}{2}})e^{-(\omega-\operatorname{Re}\lambda^2)t}\ .$$

From 1) it follows, that $G(\lambda)$ inverts $D_Z-\lambda$ on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as well.

3) From 1) it follows that $\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ is a core for the closure of D_Z^2, hence the closure of D_Z^2 equals Δ. From 2) it follows that the operator D_Z^2 is closed. $\Box$

Proposition 3.4.3. *Let* $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda<\omega$*.*

1. *The operator* $(D_Z^2-\lambda)^{-1}$ *maps* $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ *continuously to* $C(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$*.*
2. *Let* $n\in\mathbb{N}, n\geq 2$*. The operator* $(D_Z^2-\lambda)^{-n}$ *maps* $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ *continuously to* $C^{2n-3}(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$*.*

Proof. 1) For $\operatorname{Re}\lambda<\omega$ it holds on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$:

$$\begin{aligned}(D_Z^2-\lambda)^{-1} &= \int_0^\infty e^{-t(D_Z^2-\lambda)}dt\\ &= \int_0^\infty e^{\lambda t}e^{-t\Delta_{\mathbb{R}}^2}e^{-t\tilde{D}_I^2}dt\ .\end{aligned}$$

By prop. 3.3.1 the operator $D_I^{-1}: L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})\to C([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ is bounded.

Thus the family of operators

$$e^{-t\tilde{D}_I^2}=\tilde{D}_I^{-1}\tilde{D}_Ie^{-t\tilde{D}_I^2}: L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})\to L^2(\mathbb{R},C([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}))$$

is bounded by $C(1+t^{-\frac{1}{2}})e^{-\omega t}$ for all $t>0$.

Furthermore the family

$$e^{-t\Delta_{\mathbb{R}}}: L^2(\mathbb{R},C([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}))\to C(\mathbb{R},C([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}))$$

is bounded by $\sup_{x_1\in\mathbb{R}}\|\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x_1-y_1)^2}{4t}}\|_{L^2_{y_1}}$, hence by some $Ct^{-1/4}$. It follows that the integral converges in the bounded operators from $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ to $C(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

2) We show that for any $k\in\mathbb{N}_0,\ k\leq 2n-3$, the map

$$\tilde{D}_I^k(i\partial_{x_1})^{2n-3-k}(D_Z^2-\lambda)^{-n}: \mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})\to C(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$$

extends to a bounded operator $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})\to C(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$:

On $\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ it holds

$$(i\partial_{x_1})^2 = \Delta_{\mathbb{R}} = D_Z^2 - \tilde{D}_I^2$$

and

$$i\partial_{x_1} = -ic(dx_1)D_Z + i\tilde{D}_I \ .$$

Note that $c(dx_1)$ anticommutes with $\tilde{D}_I$. By these facts and by 1) it is enough to show that

$$(D_Z^2-\lambda)\tilde{D}_I^k D_Z^{2n-3-k}(D_Z^2-\lambda)^{-n} = \tilde{D}_I^k D_Z^{2n-3-k}(D_Z^2-\lambda)^{-n+1}$$

extends to a bounded operator on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.
Since D_Z and $\tilde{D}_I$ are invertible, prop. 5.4.7 implies that there are bounded involutions I_1, I_2 on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ satisfying $[I_1,\tilde{D}_I]=0$, $[I_2,D_Z]=0$, further $\tilde{D}_I = I_1|\tilde{D}_I|$ and $D_Z = I_2|D_Z|$. Thus we only need to consider the operator

$$|\tilde{D}_I|^k|D_Z|^{2n-3-k}(D_Z^2-\lambda)^{-n+1}$$

$$= \Big(|\tilde{D}_I|^{\frac{k}{n-1}}|D_Z|^{\frac{(2n-3-k)}{n-1}}(D_Z^2-\lambda)^{-1}\Big)^{n-1} \ .$$

On $\mathcal{S}_R(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ the term in brackets can be re-written as

$$\int_0^\infty |\tilde{D}_I|^{\frac{k}{n-1}} e^{-\frac{t}{2}\tilde{D}_I^2} e^{-\frac{t}{2}D_{\mathbb{R}}^2}|D_Z|^{\frac{(2n-3-k)}{n-1}} e^{-\frac{t}{2}D_Z^2} dt \ .$$

The integral converges as a bounded operator on $L^2(Z,(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ since by cor. 5.4.9 the integrand can be estimated for small t by

$$C\big(\frac{t}{2}\big)^{-(\frac{k}{2n-2})}\big(\frac{t}{2}\big)^{-(\frac{2n-3-k}{2n-2})} = C\big(\frac{t}{2}\big)^{-(\frac{2n-3}{2n-2})}$$

and since for large t it is exponentially decaying. □

In the following $|\cdot|$ denotes the norm on $M_{4d}(A_i)$.

Lemma 3.4.4. *For any $\varepsilon>0$ and $\alpha,\beta\in\mathbb{N}_0^2$ there are $c,C>0$ such that for all $x,y\in Z$ with $d(x,y)>\varepsilon$ and all $t>0$ it holds*

$$|\partial_x^\alpha\partial_y^\beta k_t^Z(x,y)| \leq Ce^{-\frac{d(x,y)^2}{ct}} \ .$$

Proof. For $m,n\in\mathbb{N}_0$ there are $C,c>0$ such that it holds

$$|\partial_{x_2}^m\partial_{y_2}^n k_t^I(x_2,y_2)| \leq Ce^{-\frac{(x_2-y_2)^2}{ct}}$$

for $|x_2-y_2|\geq\varepsilon/2$ and $t>0$. This follows from cor. 3.3.5 for $t<1$ and from cor. 3.3.6 for $t\geq 1$. For $|x_2-y_2|\leq\varepsilon/2$ the left hand side it bounded by $C(1+t^{-\frac{m+n+1}{2}})$ by cor. 3.3.6.

Hence there are $c_1, c_2, C > 0$ such that for $d(x,y) > \varepsilon$ and $t > 0$ it holds

$$\begin{aligned}
&|\partial_x^\alpha \partial_y^\beta \left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}} \left(k_t^I(x_2,y_2) \oplus k_t^I(x_2,y_2) \right) \right)| \\
\leq\ & C \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}} \Big(e^{-\frac{(x_2-y_2)^2}{c_1 t}} 1_{\mathbb{R}\setminus[0,\varepsilon/2]}(|x_2-y_2|) \\
& \qquad + (1 + t^{-\frac{|\alpha|+|\beta|+2}{2}}) 1_{[0,\varepsilon/2]}(|x_2-y_2|) \Big) \\
\leq\ & C e^{-\frac{d(x,y)^2}{c_2 t}} \ .
\end{aligned}$$

□

In the next lemma we write $\mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i))$ for the subspace of $\mathcal{S}(Z, M_{4d}(\mathcal{A}_i))$ containing those functions whose columns are in $\mathcal{S}_R(Z, \mathcal{A}_i^{4d})$. Then operators on $\mathcal{S}_R(Z, \mathcal{A}_i^{4d})$ act on $\mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i))$ as well – column by column. The space $C^\infty_{Rc}(Z, M_{4d}(\mathcal{A}_i))$ is analogously defined.

Lemma 3.4.5. *Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \omega$. Let $\xi_1, \xi_2 \in C^\infty(Z)$ be functions with disjoint support and assume that the support of ξ_2 is compact.*

Then for any $n \in \mathbb{N}$ the operator $\xi_1(D_Z^2 - \lambda)^{-n}\xi_2$ is an integral operator with integral kernel κ such that $(y \mapsto \kappa(\cdot,y)) \in C_c^\infty(Z, \mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i)))$ and $(x \mapsto \kappa(x,\cdot)^*) \in \mathcal{S}(Z, C^\infty_{Rc}(Z, M_{4d}(\mathcal{A}_i)))$.

In particular $\xi_1(D^2 - \lambda)^{-n}\xi_2$ maps $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ continuously to $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Proof. We prove the claim first for $n = 1$:
Let $f \in C^\infty_{Rc}(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.
Then it holds in $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$:

$$\begin{aligned}
\xi_1(D_Z^2 - \lambda)^{-1}\xi_2 f &= \int_0^\infty \xi_1 e^{-t(D_Z^2-\lambda)} \xi_2 f \ dt \\
&= \int_0^\infty \int_Z \xi_1 k_t^Z(\cdot,y) e^{\lambda t} \xi_2(y) f(y) \ dydt \ .
\end{aligned}$$

Let $\varepsilon > 0$ be such that $d(\operatorname{supp}\xi_1, \operatorname{supp}\xi_2) > \varepsilon$.
By the previous lemma it follows that there are $c, C > 0$ such that for all $x, y \in Z$ and all $t > 0$ it holds:

$$\begin{aligned}
|\xi_1(x) e^{\lambda t} k_t^Z(x,y) \xi_2(y)| \ &\leq\ C 1_{\{t>1\}}(t) |\xi_1(x)| e^{(\operatorname{Re}\lambda - \omega)t} e^{-\frac{(x_1-y_1)^2}{4t}} |\xi_2(y)| \\
&\quad + C 1_{\{t\leq 1\}}(t) |\xi_1(x)| e^{-\frac{d(x,y)^2}{ct}} |\xi_2(y)| \ .
\end{aligned}$$

Analogous estimates hold for the partial derivatives. Hence we can interchange the order of integration. It follows that $\xi_1(D_Z^2 - \lambda)^{-1}\xi_2$ is an integral operator with integral kernel

$$\kappa(x,y) := \int_0^\infty \xi_1(x) e^{-\lambda t} k_t^Z(x,y) \xi_2(y) dt \ .$$

The other statements of the lemma also follows from the estimates.

For $n > 1$ choose a smooth compactly supported function $\psi : Z \to [0,1]$ such that $\operatorname{supp}\psi \cap \operatorname{supp}\xi_1 = \emptyset$ and $\operatorname{supp}(1-\psi) \cap \operatorname{supp}\xi_2 = \emptyset$. Then

$$\begin{aligned}\xi_1(D_Z^2-\lambda)^{-n}\xi_2 &= \xi_1(D_Z^2-\lambda)^{-1}\psi(D_Z^2-\lambda)^{-n+1}\xi_2 \\ &\quad +\xi_1(D_Z^2-\lambda)^{-1}(1-\psi)(D_Z^2-\lambda)^{-n+1}\xi_2 \ .\end{aligned}$$

By induction the lemma can be applied to $\xi_1(D_Z^2-\lambda)^{-1}\psi$ and $(1-\psi)(D_Z^2-\lambda)^{-n+1}\xi_2$. The statement of the lemma follows for $\xi_1(D_Z^2-\lambda)^{-n}\xi_2$ from this and the fact that by cor. 3.4.2 the operator $(D_Z^2-\lambda)^{-m}$ acts continuously on $\mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i))$ for any $m \in \mathbb{N}$. $\square$

3.5 The heat semigroup on M

3.5.1 Definitions

Recall that we defined $D(\rho)^2$ as an unbounded operator on the Hilbert $\mathcal{A}$-module $L^2(M, E\otimes\mathcal{A})$ in §2.1.1 and §2.5. Now we define an action of it on $L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$:

Let $D(\rho)_i^2$ as an unbounded operator on $L^2(M, E\otimes\mathcal{A}_i)$ be the restriction of $D(\rho)^2$ to

$$\operatorname{dom} D(\rho)_i^2 := \{f \in \operatorname{dom} D(\rho)^2 \cap L^2(M, E\otimes\mathcal{A}_i) \mid D(\rho)^2 f \in L^2(M, E\otimes\mathcal{A}_i)\} \ .$$

It is closed.

Let $D(\rho)_{\mu,i}^2$ be the closure of the unbounded operator on $L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ whose domain is given by the right $\hat{\Omega}_{\le\mu}\mathcal{A}_i$-submodule of $L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ generated by $\operatorname{dom} D(\rho)_i^2$ and whose action is defined by $D(\rho)_{\mu,i}^2(f\omega) := (D(\rho)_i^2 f)\omega$ for $f \in \operatorname{dom} D(\rho)_i^2,\ \omega \in \hat{\Omega}_{\le\mu}\mathcal{A}_i$.

Note that the notation is misleading: It suggests that $D(\rho)_{\mu,i}^2$ is the square of some unbounded operator on $L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$, but this is not clear for the moment.

In the following we suppress the indices.

Define D_s^2 as a closed operator on $L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ in an analogous way.

We will often make use of cutting and pasting arguments. Here we fix the setting:

Let k_K be the integral kernel of K. Let $0 < b_0 \le \frac{1}{4}$ be small enough and $r_0 > 0$ large enough such that

$$\operatorname{supp} k_K \cap \big((F(r_0,b_0)\times M)\cup(M\times F(r_0,b_0))\big) = \emptyset \ ,$$

where $F(r_0,b_0)$ was defined in §1.1.

Let $\mathcal{U}(r_0,b_0)$ be the open covering that was defined in §1.1 and in §2.5.

Choose a smooth partition of unity $\{\phi_k\}_{k\in J}$ subordinate to $\mathcal{U}(r_0,b_0)$ and smooth functions $\{\gamma_k\}_{k\in J}$ on M such that for all $k \in J$ it holds

- $\operatorname{supp}\gamma_k \subset \mathcal{U}_k$,
- $\operatorname{supp}(1-\gamma_k) \cap \operatorname{supp}\phi_k = \emptyset$,
- the derivatives $\partial_{e_2}(\phi_k|_F)$ and $\partial_{e_2}(\gamma_k|_F)$ vanish in a neighbourhood of ∂M.

Let E_N be a hermitian Clifford module on a compact spin manifold N that is trivial as a vector bundle and assume that there is an isometric Clifford module isomorphism $E|_{\mathcal{U}_\clubsuit} \to E_N$, whose base map is an isometric embedding. Let D_N be the associated Dirac operator. We identify $\mathcal{U}_\clubsuit$ with its image in N and $E|_{\mathcal{U}_\clubsuit}$ with its image in E_N.

Since the support of k_K is in $\mathcal{U}_\clubsuit \times \mathcal{U}_\clubsuit$, the restriction of $D(\rho) = D + \rho K$ to $\mathcal{U}_\clubsuit$ extends to an operator $D_N + \rho K$ on the sections of E_N.

For $k \in \mathbb{Z}/6$ let D_{Z_k} be the operator D_Z on $L^2(Z, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$ defined in §3.4 with boundary conditions given by the pair $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{(k+1) \bmod 3})$.

3.5.2 The resolvents of $D(\rho)^2$

This section has three different aims:

Applying a method of Lott ([Lo], §6.1.) we investigate the resolvent set of $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$.

Furthermore we prove a kind of Sobolev embedding theorem – more precisely an analogue of lemma 3.1.4 for the operator $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$.

Third we obtain more information about the kernel of $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$: There is a projection on it, and this projection is a Hilbert-Schmidt operator with smooth integral kernel.

Let $\omega > 0$ be such that there is $C > 0$ with $\|e^{-tD^2_{Z_k}}\| \le Ce^{-\omega t}$ for all $t \ge 0$ and all $k \in \mathbb{Z}/6$.

Let $\nu \in \mathbb{N}$. For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \omega$ we define a parametrix of $(D^2 - \lambda)^\nu$:

For $k \in \mathbb{Z}/6$ let $Q_k(\lambda) = (D^2_{Z_k} - \lambda)^{-\nu}$. It is well-defined by cor. 3.4.2.

Let $Q_\clubsuit(\lambda)$ be a local parametrix of $(D^2 - \lambda)^\nu$ on $\mathcal{U}_\clubsuit$ defined by the symbol of $(D^2 - \lambda)^\nu$.

The operator

$$Q(\lambda) := \sum_{k \in J} \phi_k Q_k(\lambda)\gamma_k$$

acts as a bounded operator on the spaces $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ and $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ by §5.2.4 and by cor. 3.4.2.

Lemma 3.5.1. *For any $\rho \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \omega$ the closure of $Q(\lambda)(D(\rho)^2 - \lambda)^\nu - 1$ is an integral operator $\mathcal{K}$ with smooth integral kernel $\kappa \in L^2(M \times M, (E \boxtimes E^*) \otimes \mathcal{A}_i)$.*

It holds $(x \mapsto \kappa(x, \cdot)^*) \in \mathcal{S}(M, C^\infty_{cR}(M, E \otimes \mathcal{A}_i) \otimes E^*)$ and $(y \mapsto \kappa(\cdot, y)) \in C^\infty_c(M, \mathcal{S}_R(M, E \otimes \mathcal{A}_i) \otimes E^*)$.

In particular $\mathcal{K}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ continuously to $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Proof. The difference $(D(\rho)^2 - \lambda)^\nu - (D^2 - \lambda)^\nu$ is an integral operator with smooth integral kernel whose support is contained in $\operatorname{supp} k_K$.

Hence we need only investigate $Q(\lambda)(D^2 - \lambda)^\nu - 1$.

For any $k \in J$ choose a function $\xi_k \in C^\infty_c(M)$ with values in $[0, 1]$ and such that it holds $\operatorname{supp} \xi_k \subset \mathcal{U}_k$, $\xi_k|_{\operatorname{supp} d\gamma_k} = 1$ and $\operatorname{supp} \phi_k \cap \operatorname{supp} \xi_k = \emptyset$. Furthermore assume that $\partial_{e_2}(\xi_k|_F)$ vanishes in a neighbourhood of ∂M.

By induction on ν we show that it holds

$$[\gamma_k, (D^2 - \lambda)^\nu] = \xi_k[\gamma_k, (D^2 - \lambda)^\nu] \ :$$

For $\nu = 0$ the claim is trivial and for general ν it holds

$$\begin{aligned}
[\gamma_k, (D^2 - \lambda)^\nu] &= (D^2 - \lambda)^{\nu-1}\big(c(d\gamma_k)D + Dc(d\gamma_k)\big) \\
&\quad + [\gamma_k, (D^2 - \lambda)^{\nu-1}](D^2 - \lambda) \\
&= \xi_k(D^2 - \lambda)^{\nu-1}\big(c(d\gamma_k)D + Dc(d\gamma_k)\big) + [\gamma_k, (D^2 - \lambda)^{\nu-1}](D^2 - \lambda) \ .
\end{aligned}$$

In the following for simplicity the operators D_Z and D_N are denoted by D as well. Furthermore $\sim$ means equality up to integral operators with smooth compactly supported integral kernels.

Then on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ it holds

$$\begin{aligned}
Q(\lambda)(D^2 - \lambda)^\nu - 1 &= \sum_{k \in J} \phi_k Q_k(\lambda)[\gamma_k, (D^2 - \lambda)^\nu] + \sum_{k \in J} \phi_k Q_k(\lambda)(D^2 - \lambda)^\nu \gamma_k - 1 \\
&\sim -\sum_{k \in J} \phi_k Q_k(\lambda)\xi_k[\gamma_k, (D^2 - \lambda)^\nu]
\end{aligned}$$

For all $k \in J$ the operator $\phi_k Q_k(\lambda)\xi_k$ is an integral operator whose integral kernel has the properties stated in the lemma. This follows for $k \in \mathbb{Z}/6$ from lemma 3.4.5 and for $k = \clubsuit$ from the properties of pseudodifferential operators. This shows the assertion of the lemma. $\square$

Proposition 3.5.2. *Let $\rho \in \mathbb{R}$. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \omega$ such that $D(\rho)^2 - \lambda$ has a bounded inverse on the Hilbert $\mathcal{A}$-module $L^2(M, E \otimes \mathcal{A})$.*

Then $D(\rho)^2 - \lambda$ has a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

The inverse $(D(\rho)^2 - \lambda)^{-1}$ acts as a bounded operator on the space $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Note that in particular $D(\rho)^2 - \lambda$ has a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for all $\lambda < 0$.

Proof. The operator $Q(\lambda)$ is bounded on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and satisfies $Q(\lambda)(D(\rho)^2 - \lambda) = 1 - \mathcal{K}$ by the previous lemma. Here $\mathcal{K}$ is an integral operator with smooth integral kernel $\kappa \in L^2(M \times M, (E \boxtimes E^*) \otimes \mathcal{A}_i)$ and it holds $(x \mapsto \kappa(x, \cdot)) \in C(M, L^2(M, E \otimes \mathcal{A}_i) \otimes E^*)$ and $(y \mapsto \kappa(\cdot, y)) \in C(M, L^2(M, E \otimes \mathcal{A}_i) \otimes E^*)$.

We want to apply lemma 5.3.1. Since in general $1 - \mathcal{K}$ is not invertible on $L^2(M, E \otimes \mathcal{A})$, we modify the parametrix:

Choose an integral kernel $s \in C_c^\infty(M \times M, (E \boxtimes E^*) \otimes \mathcal{A}_i)$ vanishing near $(\partial M \times M) \cup (M \times \partial M)$ such that in $B(L^2(M, E \otimes \mathcal{A}))$ it holds

$$\|\mathcal{K} - S(D(\rho)^2 - \lambda)\| \leq \tfrac{1}{2}.$$

Since by assumption $(D(\rho)^2 - \lambda)$, hence also $(D(\rho)^2 - \overline{\lambda})$, has a bounded inverse on $L^2(M, E \otimes \mathcal{A})$, this is possible.

From the estimate it follows that

$$(Q(\lambda) + S)(D(\rho)^2 - \lambda) = 1 - \big(\mathcal{K} - S(D(\rho)^2 - \lambda)\big)$$

has a bounded inverse on $L^2(M, E \otimes \mathcal{A})$.

Lemma 5.3.1 implies that $1 - \mathcal{K} - S(D(\rho)^2 - \lambda)$ is invertible on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. Thus

$$\big(1 - (\mathcal{K} - S(D(\rho)^2 - \lambda))\big)^{-1}(Q(\lambda) + S)$$

is a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. On $L^2(M, E \otimes \mathcal{A})$ it inverts $D(\rho)^2 - \lambda$, hence it is the inverse of $D(\rho)^2 - \lambda$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ as well.

Since $Q(\lambda)$ acts continuously on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and $\mathcal{K}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ continuously to $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ by the previous lemma, the operator $(D(\rho)^2 - \lambda)^{-1}$ acts continously on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ by

$$\begin{aligned}(D(\rho)^2 - \lambda)^{-1} &= (1 - \mathcal{K})(D(\rho)^2 - \lambda)^{-1} + \mathcal{K}(D(\rho)^2 - \lambda)^{-1} \\ &= Q(\lambda) + \mathcal{K}(D(\rho)^2 - \lambda)^{-1}\ .\end{aligned}$$

□

Corollary 3.5.3. *The space $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is a core of $D(\rho)^2$.*

Proposition 3.5.4. *Let $\rho \in \mathbb{R}$ and let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \leq \omega$ be such that $(D(\rho)^2 - \lambda)$ has a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.*

Then for any $\nu \in \mathbb{N}$, $\nu \geq 2$, the operator $(D(\rho)^2 - \lambda)^{-\nu}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ continuously to $C_R^{2\nu-3}(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Proof. Let $Q(\lambda)(D(\rho)^2-\lambda)^\nu = 1-\mathcal{K}$ as before, thus

$$(D(\rho)^2-\lambda)^{-\nu} = Q(\lambda) + \mathcal{K}(D(\rho)^2-\lambda)^{-\nu} \ .$$

By prop. 3.4.3 and §5.2.4 the operator $Q(\lambda)$ maps $L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ continuously to $C_R^{2\nu-3}(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$. Furthermore $\mathcal{K}$ is smoothing. □

Corollary 3.5.5. *The kernel of $D(\rho)^2$ on $L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ is a subspace of $\mathcal{S}_R(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$.*

Proof. Let $\lambda\in\mathbb{C}$ be as in the previous proposition and assume furthermore $\lambda\neq 0$.

For $f\in \operatorname{Ker} D(\rho)^2$ it holds $(D(\rho)^2-\lambda)^{-\nu}f = (-\lambda)^{-\nu}f$ for any $\nu\in\mathbb{N}$.By the previous proposition it follows, that the elements of $\operatorname{Ker} D(\rho)^2$ are smooth.

For $k\in\mathbb{Z}/6$ and $f\in\operatorname{Ker} D(\rho)^2$ it holds $D^2_{Z_k}\phi_k f\in C^\infty_{cR}(Z_k, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$. From cor. 3.4.2 it follows

$$\phi_k f = D^{-2}_{Z_k}(D^2_{Z_k}\phi_k f)\in \mathcal{S}_R(Z_k, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d}) \ .$$

Hence $f\in\mathcal{S}_R(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$. □

Proposition 3.5.6. *Let $\rho\neq 0$.*

Let P be the projection onto the kernel of $D(\rho)$ on $L^2(M, E\otimes\mathcal{A})$.

Then P is a finite Hilbert-Schmidt operator whose integral kernel is of the form $\sum_{j=1}^m f_j(x)h_j(y)^*$ with $f_j, h_j\in\bigcap_i\big(\operatorname{Ker} D(\rho)\cap L^2(M, E\otimes\mathcal{A}_i)\big)\subset\mathcal{S}_R(M, E\otimes\mathcal{A}_\infty)$.

Furthermore it holds $\operatorname{Ker} D(\rho)^2 = PL^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ and $\operatorname{Ran} D(\rho)^2 = (1-P)L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$. Hence there is a decomposition

$$L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i) = \operatorname{Ker} D(\rho)^2\oplus\operatorname{Ran} D(\rho)^2$$

and

$$D(\rho)^2 = D(\rho)^2|_{\operatorname{Ker} D(\rho)^2}\oplus D(\rho)^2|_{\operatorname{Ran} D(\rho)^2} \ .$$

Proof. First consider the situation on $L^2(M, E\otimes\mathcal{A})$:

Since the range of $D(\rho)$ is closed, there is an orthogonal projection P onto the kernel of $D(\rho)$ by prop. 5.1.14. Furthermore $D(\rho)$ is selfadjoint on the Hilbert $\mathcal{A}$-module $L^2(M, E\otimes\mathcal{A})$, hence it holds $\operatorname{Ker} D(\rho) = \operatorname{Ker} D(\rho)^2$ by prop. 2.1.1. The range of $D(\rho)^2$ is closed, thus zero is an isolated point in the spectrum of $D(\rho)^2$ on $L^2(M, E\otimes\mathcal{A})$.

From prop. 3.5.2 it follows that zero is an isolated point in the spectrum of $D(\rho)^2$ on $L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ as well.

Hence, for r small enough, the projection

$$P = \frac{1}{2\pi i}\int_{|\lambda|=r}(D(\rho)^2-\lambda)^{-1}d\lambda$$

is well-defined and a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. By prop. 5.3.7 it is a Hilbert-Schmidt operator whose integral kernel is as asserted.

The remaining parts follow from the spectral theory for closed operators on Banach spaces ([Da], th. 2.14). □

Corollary 3.5.7. *Let* $\mathrm{Re}\,\lambda < \omega$.

1. *Let* $\rho \neq 0$ *and let* P *be the orthogonal projection onto the kernel of* $D(\rho)^2$.

   If $D(\rho)^2 + P - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \mathcal{A})$ *then* $D(\rho)^2 + P - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ *and the inverse acts as a bounded operator on the space* $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ *as well.*

2. *Let* P_0 *be the orthogonal projection onto* $\mathrm{Ker}\, D_s^2$. *If* $D_s^2 + P_0 - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \mathcal{A})$ *then* $D_s^2 + P_0 - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ *and the inverse acts as a bounded operator on the space* $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ *as well.*

In particular there is $c > 0$ *such that* $\{\mathrm{Re}\,\lambda < c\}$ *is in the resolvent set of* $D(\rho)^2 + P$ *resp.* $D_s^2 + P_0$.

Proof. From the previous proposition it follows that $P(1-\lambda)^{-1} + (1-P)(D(\rho)^2 - \lambda)^{-1}$ inverts $D(\rho)^2 + P - \lambda$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. Since P acts as a bounded operator on the space $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ by cor. 3.5.5 and $(D(\rho)^2 - \lambda)^{-1}$ is bounded on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ by prop. 3.5.2, the operator $(D(\rho)^2 + P - \lambda)^{-1}$ is bounded on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ as well.

2) follows analogously. □

3.5.3 An approximation of the semigroup

By cutting and pasting we construct an integral operator that for small t behaves similar to a semigroup generated by $-D(\rho)^2$:

Recall the definitions of §3.5.1.

Let $e(\rho)_t^{\clubsuit}(x, y)$ be the restriction of the integral kernel of $e^{-t(D_N + \rho K)^2}$ to $\mathcal{U}_{\clubsuit} \times \mathcal{U}_{\clubsuit}$ and for $k \in \mathbb{Z}/6$ let $e(\rho)_t^k(x, y)$ be the restriction of the integral kernel of $e^{-tD_{Z_k}^2}$ to $\mathcal{U}_k \times \mathcal{U}_k$ and extend these functions by zero to $M \times M$. Clearly for $k \in \mathbb{Z}/6$ it holds $e(\rho)_t^k(x, y) = e(0)_t^k(x, y)$.

Define the integral kernel

$$e(\rho)_t(x, y) := \sum_{k \in J} \gamma_k(x) e(\rho)_t^k(x, y) \phi_k(y)$$

Denote the corresponding family of integral operators on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ by $E(\rho)_t$ resp. E_t. By the results in §3.2 and §3.4 they are bounded and strongly continuous and have the following properties:

1. For $t \to 0$ the family $E(\rho)_t$ converges strongly to $E(\rho)_0 := 1$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

2. If $f \in C^\infty_{cR}(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, then the family $E(\rho)_t f \in L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ depends differentiably on t for $t \geq 0$.

3. For $t > 0$ it holds $\operatorname{Ran} E(\rho)_t \subset \mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

4. Let A be a differential operator on $C^\infty(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ of order m with bounded coefficients. By prop. 3.2.2 and prop. 3.4.1 the operator $AE(\rho)_t$ is bounded on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for $t > 0$. Moreover, for $T > 0$ there is $C > 0$ such that for all $T > t > 0$ it holds:
$$\|AE(\rho)_t\| \leq Ct^{-\frac{m}{2}} \ .$$

5. For any $m \in \mathbb{N}_0$ and $T > 0$ there is $C > 0$ such that it holds for all $y \in \mathcal{U}_\clubsuit$, $\rho \in [-1, 1]$ and $0 < t < T$ in $C^m_R(\mathcal{U}_\clubsuit, (E \otimes E_y) \otimes \mathcal{A}_i)$:
$$\|e(\rho)^\clubsuit_t(\cdot, y) - e(0)^\clubsuit_t(\cdot, y)\|_{C^m_R} \leq C|\rho| \ .$$
An analogous estimate holds for the partial derivatives in y with respect to unit vector fields on $\mathcal{U}_\clubsuit$.

The last statement follows by Volterra development (prop. 5.4.4): On N it holds
$$\begin{aligned} & e^{-tD_N(\rho)^2} - e^{-tD_N^2} \\ = \ & \rho \sum_{n=1}^\infty (-1)^n \rho^{n-1} t^n \int\int_{\Delta^n} e^{u_0 t D_N^2}\big([D_N, K]_s + \rho K^2\big) e^{u_1 t D_N^2} \dots e^{u_n t D_N^2} \, du_0 \dots du_n \ , \end{aligned}$$
and the sum is an integral operator whose integral kernel is uniformly bounded in $0 < t < T$ and $\rho \in [-1, 1]$.

3.5.4 The semigroup $e^{-tD_s^2}$

By cor. 3.1.5 the operator $e^{-tD_s^2}$ on the Hilbert space $L^2(M, E)$ is an integral operator with smooth integral kernel k_t for $t > 0$. In this section we show that k_t defines a strongly continuous semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and that this semigroup extends to a holomorphic one.

For D_s we define integral kernels $e^k_t(x, y)$ and
$$e_t(x, y) := \sum_{k \in J} \gamma_k(x) e^k_t(x, y) \phi_k(y)$$
analogously to $e(\rho)^k_t(x, y)$ and $e(\rho)_t(x, y)$ in the previous section.

The corresponding family of operators is denoted by E_t. We set $E_0 := 1$. For the properties of E_t see the previous section.

By Duhamel's principle it holds for $f \in C^\infty_{cR}(M,E)$:

$$e^{-tD_s^2}f - E_t f = \int_0^t e^{-sD_s^2}\left(\frac{d}{dt} + D_s^2\right) E_{t-s} f \; ds \; .$$

Proposition 3.5.8. *The heat kernel k_t defines a strongly continuous semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ with generator $-D_s^2$ that extends to a bounded holomorphic semigroup.*

Let A be a differential operator of order $m \in \mathbb{N}_0$ with bounded smooth coefficients. Then for any $t > 0$ the operator $Ae^{-tD_s^2}$ is bounded on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and for any $T > 0$ there is $C > 0$ such that for $0 < t < T$ it holds

$$\|Ae^{-tD_s^2}\| \leq Ct^{-\frac{m}{2}} \; .$$

Proof. First we show that for $T > 0$ there is $C > 0$ such that for $0 < t < T$ the difference $A_x k_t(x,y) - A_x e_t(x,y)$ is bounded by Ct in $L^2(M \times M, E \boxtimes E^*)$.

For $k \in J$ let $\chi_k \in C_c^\infty(M)$ be a function with values in $[0,1]$, with compact support in $\mathcal{U}_k$ and equal to one on a neighbourhood of $\operatorname{supp} d\gamma_k$.

From the equation above it follows

$$\begin{aligned}
&A_x k_t(x,y) - A_x e_t(x,y) \\
&= \; -\sum_{k\in J}\int_0^t \int_M A_x k_s(x,r)[\partial_M, c(d\gamma_k(r))]_s e^k_{t-s}(r,y)\phi_k(y) \; drds \; .
\end{aligned}$$

This can be re-written as

$$\begin{aligned}
&-\sum_{k\in J}\int_0^t \int_M (1-\chi_k(x))A_x k_s(x,r)[(\partial_M)_r, c(d\gamma_k(r))]_s e^k_{t-s}(r,y)\phi_k(y) \; drds \\
&-\sum_{k\in J}\int_0^t \int_M \chi_k(x)A_x(k_s(x,r) - e^k_s(x,r))[(\partial_M)_r, c(d\gamma_k(r))]_s e^k_{t-s}(r,y)\phi_k(y) \; drds \\
&-\sum_{k\in J}\int_0^t \int_M \chi_k(x)A_x e^k_s(x,r)[(\partial_M)_r, c(d\gamma_k(r))]_s e^k_{t-s}(r,y)\phi_k(y) \; drds \; .
\end{aligned}$$

Using lemma 3.1.8 and lemma 3.1.9 we estimate the norms of the three terms in $E_x \otimes E_y$:

Since $\operatorname{supp} d\gamma_k \cap \operatorname{supp}\phi_k = \emptyset$ and $\operatorname{supp} d\gamma_k \cap \operatorname{supp}(1-\chi_k) = \emptyset$, there is $C > 0$ such that for $x, y \in M$ and $t > 0$ the norm of the first term is bounded by

$$C\sum_{k\in J} t(1-\chi_k(x))e^{-\frac{d(x,\operatorname{supp} d\gamma_k)^2}{(4+\delta)t}} e^{-\frac{d(y,\operatorname{supp} d\gamma_k)^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y) \; ,$$

and such that for $x, y \in M$ and $t > 0$ the norms of the second and third term are bounded by

$$Ct\chi_k(x)e^{-\frac{d(y,\operatorname{supp} d\gamma_k)^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y) \; .$$

When estimating the third term we also used that the action of the integral kernel $e_s^k(x,r)\chi_k(r)$ is uniformly bounded for $k = \clubsuit$ on $C^n(\mathcal{U}_\clubsuit, E \otimes E_y)$ by prop. 3.2.2 and for $k \in \mathbb{Z}/6$ on $C^n_R(\mathcal{U}_k, E \otimes E_y)$ by prop. 3.4.1.

Analogous estimates holds for the derivatives in y with respect to unit vector fields on M.

Hence $A_x k_t(x,y) - A_x e_t(x,y)$ is bounded by Ct in $L^2(M \times M, E \boxtimes E^*)$ for $0 < t < T$ and some $C > 0$. By cor. 5.2.4 the corresponding family of operators on $L^2(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$ is bounded by Ct for $0 < t < T$, hence $A_x k_t(x,y)$ defines a family of bounded operators on $L^2(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$.

Write $S(t)$ for the integral operator induced by the integral kernel $k_t(x,y)$.

By property 4) in §3.5.3 there is $C > 0$ such that $\|AE_t\| \le Ct^{-\frac{m}{2}}$ on $L^2(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$ for $0 < t < T$ and some $C > 0$, hence

$$(*) \qquad \|AS(t)\| \le Ct^{-\frac{m}{2}} \ .$$

The fact that $S(t)$ extends to a bounded holomorphic semigroup on $L^2(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$ is an almost immediate consequence of $(*)$:

Since E_t converges strongly to the identity on $L^2(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$ for $t \to 0$, the operator $S(t)$ also does. Furthermore the kernels k_t obey the semigroup law, hence $S(t)$ is a strongly continuous semigroup on $L^2(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$.

Note that the range of $S(t) - E_t$ is a subset of $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$. Hence $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$ is invariant under the action of $S(t)$. It follows that $-D_s^2$ is the generator of $S(t)$.

From prop. 5.4.3 and the estimate $(*)$ applied to $A = D_s^2$ it follows that the semigroup $S(t) = e^{-tD_s^2}$ extends to a holomorphic semigroup on $L^2(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$.

It remains to show that the holomorphic semigroup is bounded:

Let P_0 be the projection onto the kernel of D_s^2.

By cor. 3.5.7 there is $c > 0$ such that $\{\operatorname{Re}\lambda \le c\}$ is in the resolvent set of $D_s^2 + P_0$ on $L^2(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$. Hence by prop. 5.4.2 the holomorphic semigroup $e^{-t(D_s^2+P_0)}$ is bounded. Thus

$$e^{-tD_s^2} = e^{-t(D_s^2+P_0)}(1 - P_0) + P_0$$

is bounded as well. □

Recall that in §3.5.1 we fixed the domain of D_s^2 as an unbounded operator on $L^2(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$, but not of D_s. This is done now:

Let D_s be the closure on $L^2(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$ of the Dirac operator ∂_M with domain $\mathcal{S}_R(M, E) \odot \hat{\Omega}_{\le \mu}\mathcal{A}_i$.

Corollary 3.5.9. *Let P_0 be the projection onto the kernel of D_s^2.*

Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda^2 < 0$.

Then the operators $D_s + P_0$ and $D_s - \lambda$ have a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\le \mu}\mathcal{A}_i)$.

Proof. By cor. 3.5.7 there is $c > 0$ such that $\{\mathrm{Re}\,\lambda \leq 2c\}$ is in the resolvent set of $D_s^2 + P_0$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

By the previous proposition and prop. 5.4.2 it follows that there is $C > 0$ such that for all $t > 0$ it holds on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$:

$$\|(D_s + P_0)e^{-t(D_s^2+P_0)}\| \leq C(1 + t^{-\frac{1}{2}})e^{-ct} \ .$$

Thus

$$G := \int_0^\infty (D_s + P_0)e^{-t(D_s^2+P_0)}dt$$

is a bounded operator on $L^2(M, E\otimes\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. On the Hilbert space $L^2(M, E)$ it holds $G = (D_s + P_0)^{-1}$. From cor. 3.5.7 it follows that G acts as a bounded operator on $\mathcal{S}_R(M, E)$. Hence G inverts $D_s + P_0$ on $\mathcal{S}_R(M, E) \odot \hat{\Omega}_{\leq\mu}\mathcal{A}_i$, thus that G inverts $D_s + P_0$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

The proof of the fact that

$$\int_0^\infty (D_s + \lambda)e^{-t(D_s^2-\lambda^2)}$$

inverts $(D_s - \lambda)$ for $\mathrm{Re}\,\lambda^2 < 0$ is analogous. □

From the corollary we conclude that D_s^2 as it was defined in the beginning of §3.5.1 is the square of D_s.

3.5.5 The semigroup $e^{-tD(\rho)^2}$

This section is devoted to the proof that $-D(\rho)^2$ generates a bounded holomorphic semigroup. We also study its smoothing properties.

But first we prove an analogon of cor. 3.5.9:

Recall that in §3.5.1 we only specified the domain of $D(\rho)^2$, not the one of $D(\rho)$. Let $D(\rho)$ be defined on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ as the closure of the operator $\partial_M + \rho K$ with domain $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Lemma 3.5.10. *There is $\varepsilon, R > 0$ such that for $\lambda \in \mathbb{C}$ with $-\lambda^2 \in \Sigma_{\pi/2+\varepsilon}$ and $|\lambda| > R$ the operator $(D(\rho) - \lambda)$ has a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.*

Proof. Let P_0 be the projection onto the kernel of D_s.

By prop. 3.5.8 and cor. 3.5.9 we can apply prop. 5.4.8 to $D_s + P_0$. This shows that there is $\delta > 0$ such that $D_s + P_0 - \lambda$ has a bounded inverse if $-\lambda^2 \in \Sigma_{\pi/2+\delta}$ and for $\varepsilon < \delta$ there is $C > 0$ such that for $\lambda \in \mathbb{C}$ with $-\lambda^2 \in \Sigma_{\pi/2+\varepsilon}$ it holds on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$:

$$\|(D_s + P_0 - \lambda)^{-1}\| \leq \frac{C}{|\lambda|} \ .$$

Since $D(\rho)-\lambda$ is a bounded perturbation of $W^*(D_s+P_0)W-\lambda$, there is $R>0$ such that the Neumann series expressing $(D(\rho)-\lambda)^{-1}$ in terms of $(W^*(D_s+P_0)W-\lambda)^{-1}$ is well-defined for $-\lambda^2 \in \Sigma_{\pi/2+\varepsilon}$ and $|\lambda|>R$. (You can find this argument in more detail in the proof of prop. 5.4.10.) $\square$

Corollary 3.5.11. *The operator $D(\rho)^2$ is the square of $D(\rho)$. In particular $D(\rho)$ commutes with all resolvents of $D(\rho)^2$.*

Proof. The assertions follow almost immediately from the fact that by the lemma and by prop. 3.5.2 there is $\lambda \in \mathbb{C}$ with

$$(D(\rho)^2-\lambda^2)^{-1} = (D(\rho)+\lambda)^{-1}(D(\rho)-\lambda)^{-1} .$$

$\square$

For $\rho \neq 0$ let P be the orthogonal projection onto the kernel of $D(\rho)^2$.

Proposition 3.5.12. 1. *Let $\rho \in \mathbb{R}$. The operator $-D(\rho)^2$ generates a holomorphic semigroup $e^{-tD(\rho)^2}$ on $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$. For any $t \geq 0$ the operator $e^{-tD(\rho)^2}$ depends analytically on ρ. For $\rho \neq 0$ the semigroup is bounded holomorphic.*

2. *For any $T>0$ there is $C>0$ such that for all $\rho \in [-1,1]$ and $0 \le t \le T$ it holds $\|e^{-tD(\rho)^2}\| \le C$.*

3. *For $\rho \neq 0$ there are $C, \omega > 0$ such that for all $t \geq 0$ it holds:*

$$\|(1-P)e^{-tD(\rho)^2}\| \le Ce^{-\omega t} .$$

Proof. 1) Let P_0 be the orthogonal projection onto the kernel of D_s^2. The family $e^{-t(D_s+P_0)^2} = e^{-tD_s^2}(1-P_0)+P_0$ is a holomorphic semigroup on $L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ by the previous section and D_s+P_0 has a bounded inverse on $L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ by cor. 3.5.9. Hence prop. 5.4.10 implies that for any $\rho \in \mathbb{R}$ the operator

$$WD(\rho)W^* = D_s + P_0 + (Wc(dW^*) + \rho WKW^* - P_0)$$

generates a holomorphic semigroup, thus the operator $-D(\rho)^2$ is the generator of a holomorphic semigroup as well.

It depends analytically on ρ by prop. 5.4.4.

Let now $\rho \neq 0$.

By cor. 3.5.7 there is $c>0$ such that $\{\mathrm{Re}\,\lambda \le 2c\}$ is in the resolvent set of $D(\rho)^2+P$ on $L^2(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$, hence by lemma 5.4.2 there is $C>0$ such that the holomorphic semigroup $e^{-t(D(\rho)^2+P)}$ is bounded by Ce^{-ct} for all $t\ge 0$, thus for $T>0$ there is $C>0$ such that for $T<t$ it holds

$$\|D(\rho)^2 e^{-tD(\rho)^2}\| = \|D(\rho)^2 e^{-t(D(\rho)^2+P)}\| \le Ce^{-ct} .$$

Prop. 5.4.3 implies that the semigroup $e^{-tD(\rho)^2}$ is bounded holomorphic.

The equality $(1-P)e^{-tD(\rho)^2} = (1-P)e^{-t(D(\rho)^2+P)}$ implies 3).

2) follows from prop. 5.4.4. $\square$

The following proposition shows that $e^{-tD(\rho)^2}$ is smoothing:

Proposition 3.5.13. *1. For any $n \in \mathbb{N}_0$, any $\rho \in \mathbb{R}$ and any $t > 0$ the operator $e^{-tD(\rho)^2}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ continuously to $C^n_R(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$.*

2. *For any $n \in 2\mathbb{N}$, $n \geq 4$, and any $\rho \neq 0$ the family $e^{-tD(\rho)^2} : C^n_{cR}(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i) \to C^{n-3}_R(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ is uniformly bounded.*

3. *For any $n \in 2\mathbb{N}$, $n \geq 4$, and any $T > 0$ the family $e^{-tD(\rho)^2} : C^n_{cR}(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i) \to C^{n-3}_R(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ is uniformly bounded in $0 \leq t < T$ and in $\rho \in [-1, 1]$.*

Proof. 1) follows from

$$e^{-tD(\rho)^2} = (D(\rho)^2 + 1)^{-n}(D(\rho)^2 + 1)^n e^{-tD(\rho)^2}$$

since $(D(\rho)^2+1)^n e^{-tD(\rho)^2}$ is bounded on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ for $t > 0$ by cor. 5.4.9 and since $(D(\rho)^2 + 1)^{-n}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ continuously to $C^{2n-3}_R(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ for $n \in \mathbb{N}$, $n \geq 2$ by prop. 3.5.4.

2) and 3) follow by prop. 3.5.4 from

$$e^{-tD(\rho)^2} = (D(\rho)^2 + 1)^{-n} e^{-tD(\rho)^2}(D(\rho)^2 + 1)^n \ .$$

□

3.5.6 The integral kernel

By comparison with $E(\rho)_t$ we prove that the operator $e^{-tD(\rho)^2}$ is an integral operator and study its integral kernel. In the following $|\cdot|$ denotes the fibrewise norm of $(E \boxtimes E^*) \otimes \mathcal{A}_i$.

Proposition 3.5.14. *For any $\rho \in \mathbb{R}$ and any $t > 0$ the operator $e^{-tD(\rho)^2}$ is an integral operator with smooth integral kernel. For its integral kernel $k(\rho)_t(x, y)$ it holds:*

1. *The map $(0, \infty) \to C^\infty(M \times M, (E \boxtimes E^*) \otimes \mathcal{A}_\infty)$, $t \mapsto k(\rho)_t$ is smooth.*

2. *For any $t > 0$ and $x, y \in M$ it holds $k(\rho)_t(x, y) = k(\rho)_t(y, x)^*$.*

3. *For any $T > 0$ there are $c, C > 0$ such that it holds*

$$|k(\rho)_t(x, y) - e(\rho)_t(x, y)| \leq Ct \left(\sum_{k \in J} e^{-\frac{d(y, \operatorname{supp} d\gamma_k)^2}{ct}} 1_{\operatorname{supp} \phi_k}(y) + |\rho| 1_{\mathcal{U}_\clubsuit}(y) \right)$$

for all $0 < t < T$, $\rho \in [-1, 1]$ and $x, y \in M$.

4. Let $\rho \neq 0$. There are $c, C > 0$ such that it holds

$$|k(\rho)_t(x,y) - e(\rho)_t(x,y)| \leq Cte^{-\frac{d(y,\mathcal{U}_\clubsuit)^2}{ct}}$$

for all $t > 0$ and all $x, y \in M$.

Statements analogous to 3) and 4) hold for the partial derivatives in x and y with respect to unit vector fields on M.

Proof. In order to prove 1) it is enough to consider the operator $e^{-tD(\rho)^2} - E(\rho)_t$. Let $f \in C^\infty_{cR}(M, E \otimes \mathcal{A}_i)$. Then by Duhamel's principle it holds

$$\begin{aligned} & e^{-tD(\rho)^2} f - E(\rho)_t f \\ = \; & -\sum_{k \in J} \int_0^t \int_M e^{-sD(\rho)^2} \big(\frac{d}{dt} + D(\rho)^2\big) \gamma_k e(\rho)^k_{t-s}(\cdot, y) \phi_k(y) f(y) \; dyds \end{aligned}$$

The map

$$\begin{aligned} & (0,\infty) \to C^\infty(M, C^\infty_{cR}(M, E \otimes \mathcal{A}_\infty) \otimes E) \;, \\ & \tau \mapsto \big(y \mapsto \big(\frac{d}{dt} + D(\rho)^2\big) e(\rho)^k_\tau(\cdot, y)\phi_k(y)\big) \end{aligned}$$

is smooth.

We can write

$$\begin{aligned} & \big(\frac{d}{dt} + D(\rho)^2\big)\gamma_k e(\rho)^k_\tau(\cdot,y)\phi_k(y) \\ = \; & -\sum_{k \in J} [\partial_M, c(d\gamma_k)]_s e(0)^k_\tau(\cdot,y)\phi_k(y) \\ & -\rho\big(\big[[\partial_M, K]_s + \rho K^2, \gamma_\clubsuit\big]_s e(\rho)^\clubsuit_\tau(\cdot,y)\phi_\clubsuit(y) \\ & -[\partial_M, c(d\gamma_\clubsuit)]_s \big(e(\rho)^\clubsuit_\tau(\cdot,y) - e(0)^\clubsuit_{t-s}(\cdot,y)\big)\phi_\clubsuit(y) \;. \end{aligned}$$

We estimate the three terms on the right hand side:

From the estimates in lemma 3.1.8 and 3.4.4 it follows that for any $m \in \mathbb{N}_0$ there are $C, c > 0$ such that it holds in $C^m_R(M, E \otimes \mathcal{A}_i) \otimes E^*_y$ for all $k \in J$, $0 < \tau < T$, $\rho \in [-1,1]$ and $y \in M$:

$$\|[\partial_M, c(d\gamma_k)]_s e(0)^k_\tau(\cdot,y)\phi_k(y)\|_{C^m_R} \leq Ce^{-\frac{d(y,\operatorname{supp} d\gamma_k)^2}{c\tau}} 1_{\operatorname{supp}\phi_k}(y) \;.$$

Since $\big[[\partial_M, K]_s + \rho K^2, \gamma_\clubsuit\big]_s$ is a finite Hilbert-Schmidt operator with smooth compactly supported integral kernel, for any $m \in \mathbb{N}_0$ there is $C > 0$ such that it holds for $0 < \tau < T$, $\rho \in [-1,1]$ and $y \in M$ in $C^m_R(M, E \otimes \mathcal{A}_i) \otimes E^*_y$:

$$\|\rho\big[[\partial_M, K]_s + \rho K^2, \gamma_\clubsuit\big]_s e(\rho)^\clubsuit_\tau(\cdot,y)\phi_\clubsuit(y)\;\|_{C^m_R} \leq C|\rho| 1_{\mathcal{U}_\clubsuit}(y) \;.$$

Furthermore by §3.5.3, property 5), there is $C > 0$ such that for $0 < \tau < T$, $\rho \in [-1,1]$ and $y \in M$ it holds in $C^m_R(M, E \otimes \mathcal{A}_i) \otimes E_y$:

$$\|[\partial_M, c(d\gamma_\clubsuit)]_s\big(e(\rho)^\clubsuit_\tau(\cdot,y) - e(0)^\clubsuit_\tau(\cdot,y)\big)\phi_\clubsuit(y)\|_{C^m_R} \le C|\rho| 1_{\mathcal{U}_\clubsuit}(y) \ .$$

Analogous estimates hold for the derivatives in y and also in τ since by the heat equation the derivatives with respect to τ can be expressed in terms of the derivatives with respect to x.

From these estimates and from prop. 3.5.13 it follows that we can interchange the order of integration, hence that $e^{-tD(\rho)^2} - E(\rho)_t$ is an integral operator and that for its integral kernel a statement analogous to 1) holds. This shows 1).

The property $k(\rho)_t(x,y) = k(\rho)_t(y,x)^*$ follows from the selfadjointness of $e^{-tD(\rho)^2}$.

Statement 2) and 3) follow from the estimates and from prop. 3.5.13. For the proof of 3) we also take the fact into account that fixed $\rho \neq 0$ the kernels $e(\rho)^\clubsuit_t$ and $e^\clubsuit_t$ and their derivatives are uniformly bounded in t with $t > T$. $\square$

Corollary 3.5.15. *For any $\nu \in \mathbb{N}_0$ and $T > 0$ there is $C > 0$ such that for $0 < t < T$ it holds on $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$:*

$$\|D(\rho)^\nu e^{-tD(\rho)^2}\| \le Ct^{-\frac{\nu}{2}} \ .$$

Proof. By Duhamel's principle it holds for $f \in C^\infty_{Rc}(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$:

$$\begin{aligned} &D(\rho)^\nu e^{-tD(\rho)^2} f - D(\rho)^\nu E(\rho)_t f \\ &= \ -\sum_{k\in J}\int_0^t \int_M e^{-sD(\rho)^2} D(\rho)^\nu [D(\rho)^2, \gamma_k]_s e(\rho)^k_{t-s}(\cdot,y)\phi_k(y) f(y) dy ds \ . \end{aligned}$$

There is $C > 0$ such that this term is bounded in $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ by some Ct for $0 < t < T$.

By prop. 3.2.2 and prop. 3.4.1 there is $C > 0$ such that on $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ it holds

$$\|D(\rho)^\nu E(\rho)_t\| \le Ct^{-\frac{\nu}{2}} \ .$$

The assertion follows. $\square$

Corollary 3.5.16. *For any $\rho \neq 0$ and $m \in \mathbb{N}$ the family of integral kernels $k(\rho)_t$ defines a strongly continuous semigroup on $C^m_R(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ bounded by $C(1+t)^{\frac{3}{2}}$ for some $C > 0$ and all $t > 0$.*

It is denoted by $e^{-tD(\rho)^2}$ as well.

Proof. By the estimates in the proposition the integral kernel $k(\rho)_t - e(\rho)_t$ defines an operator on $C^m_R(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ bounded by $Ct^{3/2}$ for any $t > 0$ and some $C > 0$.

For $k \in \mathbb{Z}/6$ the action of the integral kernel $e(\rho)^k_t$ on $C^m_R(\mathcal{U}_k, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ is strongly continuous and is uniformly bounded by prop. 3.4.1.

By cor. 3.2.3 and prop. 5.4.4 the family $e^{-t(D_N+\rho K)^2}$ is a strongly continuous semigroup on $C^m(N, E_N \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. It is bounded since its integral kernel is uniformly bounded for $t > 1$. Hence the action of $e(\rho)_t^{\clubsuit}$ on $C^m(\mathcal{U}_{\clubsuit}, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is strongly continuous and is uniformly bounded

Since furthermore the operator induced by $e(\rho)_t$ converges strongly to the identity for $t \to 0$, the operator induced by $k(\rho)_t$ also does. It satisfies the semigroup property, hence it is a strongly continuous semigroup. $\square$

Corollary 3.5.17. *Let $\rho \neq 0$ and $n \in \mathbb{N}$.*

For any $\varepsilon > 0$ we can find $C, c > 0$ such that for $x, y \in M$ with $d(x,y) > \varepsilon$ and $t > 0$ it holds

$$|k(\rho)_t(x,y)| \leq C(e^{-\frac{d(x,y)^2}{ct}} + te^{-\frac{d(y,\mathcal{U}_{\clubsuit})^2}{ct}}) \ .$$

An analogous statement holds for the derivatives in x and y with respect to unit vector fields on M.

In the next corollary we use the notion of a Hilbert-Schmidt operator and the Hilbert-Schmidt norm $\|\cdot\|_{HS}$ that are defined in §5.2.3.

Corollary 3.5.18. *Let $\rho \neq 0$.*

For any $r > 0$ and $\nu \in \mathbb{N}_0$ the operators $1_{M_r} D(\rho)^\nu e^{-tD(\rho)^2}$ and $D(\rho)^\nu e^{-tD(\rho)^2} 1_{M_r}$ are Hilbert-Schmidt operators and it holds for any $\nu \in \mathbb{N}_0$:

1. *For any $T > 0$ there is $C > 0$ such that for any $r > 0$ and $t > T$ it holds:*

$$\|1_{M_r} D(\rho)^\nu e^{-tD(\rho)^2}\|_{HS} \leq C(1+r)$$

   and

$$\|D(\rho)^\nu e^{-tD(\rho)^2} 1_{M_r}\|_{HS} \leq C(1+r) \ .$$

2. *For any $\varepsilon > 0$ there is $C > 0$ such that for any $r, t > 0$ it holds:*

$$\|1_{M_r} D(\rho)^\nu e^{-tD(\rho)^2}(1 - 1_{M_{r+\varepsilon}})\|_{HS} \leq C(1+r)t^{1/2}$$

   and

$$\|(1 - 1_{M_{r+\varepsilon}}) D(\rho)^\nu e^{-tD(\rho)^2} 1_{M_r}\|_{HS} \leq C(1+r)t^{1/2} \ .$$

Proof. 1) Since by prop. 3.5.12 the semigroup $e^{-tD(\rho)^2}$ is bounded, it follows from prop. 5.2.12 that there is $C > 0$ such that for all $r > 0$ and $t > T$ it holds:

$$\|1_{M_r} D(\rho)^\nu e^{-tD(\rho)^2}\|_{HS} \leq C\|1_{M_r} D(\rho)^\nu e^{-TD(\rho)^2}\|_{HS} \ .$$

By the previous corollary there are $C, c > 0$ such that for all $r > 0$ and $x, y \in M$ it holds

$$|1_{M_r}(x) D(\rho)_x^\nu k(\rho)_T(x,y)| \leq C 1_{M_r}(x)(e^{-cd(x,y)^2} + e^{-cd(y,M_r)^2}) \ .$$

This yields the asserted estimate. The second estimate in 1) is proven analogously.

2) follows from the previous corollary and 1). $\square$

Chapter 4

The superconnection and the index theorem

The notion of a superconnection on a free finitely generated $\mathbb{Z}/2$-graded module which we define now generalizes the notion of a connection on a free module [Kar].

In the family case ([BGV], ch. 9,10) superconnections usually act on infinite dimensional bundles. In analogy we will consider later superconnections acting on modules with infinitely many generators. The following definition should be merely seen as a motivation for the definitions of the superconnections to come.

Definition 4.0.19. *Let $\mathcal{B}$ be a locally m-convex Fréchet algebra and let $p, q \in \mathbb{N}_0$.*

Let $V := (\mathbb{C}^+)^p \oplus (\mathbb{C}^-)^q$. Consider $V \otimes \hat{\Omega}_*\mathcal{B}$ as a $\mathbb{Z}/2$-graded space.

A superconnection on $V \otimes \mathcal{B}$ is an odd linear map

$$A : V \otimes \hat{\Omega}_*\mathcal{B} \to V \otimes \hat{\Omega}_*\mathcal{B}$$

satisfying Leipniz's rule:

For $\alpha \in V^\pm \otimes \hat{\Omega}_k\mathcal{B}$ and $\beta \in \hat{\Omega}_*\mathcal{B}$ it holds

$$A(\alpha\beta) = A(\alpha)\beta + (-1)^{\deg \alpha}\alpha \, \mathrm{d}\, \beta$$

where $\deg \alpha$ is the degree of α with respect to the $\mathbb{Z}/2$-graduation of $V \otimes \hat{\Omega}_*\mathcal{B}$.

The map A^2 is called the curvature of A.

As for a connection [Kar] it follows that the curvature is a right $\hat{\Omega}_*\mathcal{B}$-module map.

4.1 The superconnection A_t^I associated to D_I

4.1.1 The family $e^{-(A_t^I)^2}$

Let C_1 be the $\mathbb{Z}/2$-graded unital algebra generated by an odd element σ with $\sigma^2 = 1$. As a vector space it is isomorphic to $\mathbb{C} \oplus \mathbb{C}$ via the map $C_1 \to \mathbb{C} \oplus \mathbb{C}, a + b\sigma \mapsto (a, b)$. We endow C_1 with the scalar product induced by the standard hermitian scalar product on $\mathbb{C} \oplus \mathbb{C}$.

We identify $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{2d})$ with $C_1 \otimes L^2([0,1], (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{2d})$.

Let (P_0, P_1) be a pair of transverse Lagrangian projections with $P_i \in M_{2d}(\mathcal{A}_\infty),\ i = 1, 2$ and let D_I be the associated unbounded operator on $L^2([0,1], (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{2d})$ from §3.3.1.

Following [BK] we define a superconnection A_I associated to the odd operator σD_I on the $\mathbb{Z}/2$-graded $\mathcal{A}_i$-module $L^2([0,1], C_1 \otimes \mathcal{A}_i^{2d})$:

Let $U \in C^\infty([0,1], M_{2d}(\mathcal{A}_\infty))$ be as in prop. 2.2.1 with $U(0)P_0U(0)^* = P_s$ and $U(1)P_1U(1)^* = 1 - P_s$. The differential $U^* \mathrm{d}\, U$ can be considered as a trivial superconnection on $L^2([0,1], C_1 \otimes \mathcal{A}_i^{2d})$.

Let

$$A_I := U^* \mathrm{d}\, U + \sigma D_I$$

and for $t \ge 0$ define

$$A_t^I := U^* \mathrm{d}\, U + \sqrt{t}\sigma D_I \ .$$

Since A_I is an odd map on $C_1 \otimes C_R^\infty([0,1], (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{2d})$ fulfilling Leipniz's rule, we call it a superconnection, and we call A_t^I the corresponding rescaled superconnection.

The curvature of A_I is

$$A_I^2 = U^* \mathrm{d}^2\, U + U^* \mathrm{d}\, U \sigma D_I + \sigma D_I U^* \mathrm{d}\, U + D_I^2 = D_I^2 + \sigma [D_I, U^* \mathrm{d}\, U] \ .$$

By

$$\begin{aligned} [D_I, U^* \mathrm{d}\, U] &= -U^*[\mathrm{d}, U D_I U^*]U \\ &= -U^*([\mathrm{d}, D_{I_s}] + [\mathrm{d}, U I_0(\partial U^*)])U \\ &= -U^* \mathrm{d}(U I_0(\partial U^*))U =: R \end{aligned}$$

it holds $A_I^2 = D_I^2 + \sigma R$ with $R \in C^\infty([0,1], M_{2d}(\hat{\Omega}_1\mathcal{A}_\infty))$ vanishing near the boundary and fulfilling $R^* = R$.

The curvature of the rescaled superconnection A_t^I is

$$(A_t^I)^2 = t D_I^2 + \sqrt{t}\sigma R \ .$$

We see that the curvature and the rescaled curvature are right $\hat{\Omega}_{\le\mu}\mathcal{A}_i$-module maps.

Since A_I^2 is a bounded perturbation of D_I^2, it defines a holomorphic semigroup $e^{-tA_I^2}$ on $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{2d})$.

In the following we restrict to the case $t \geq 0$:
By Volterra development it holds

$$\begin{aligned} e^{-tA_I^2} &= \sum_{n=0}^{\infty}(-1)^n t^n \int_{\Delta^n} e^{-u_0 tD_I^2}\sigma R e^{-u_1 tD_I^2}\sigma R \dots e^{-u_n tD_I^2}\, du_0 \dots du_n \\ &= \sum_{n=0}^{\infty}\sigma^n (-1)^{\frac{(n+1)n}{2}} t^n I_n(t) \end{aligned}$$

with

$$I_n(t) := \int_{\Delta^n} e^{-u_0 tD_I^2} R e^{-u_1 tD_I^2} R \dots e^{-u_n tD_I^2}\, du_0 \dots du_n \ .$$

The series is finite on $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.
It follows

$$e^{-(A_t^I)^2} = \sum_{n=0}^{\infty}\sigma^n (-1)^{\frac{(n+1)n}{2}} t^{n/2} I_n(t) \ .$$

The operators $I_n(t)$ obey the following recursive relation for $n \geq 1$:

$$\begin{aligned} I_n(t) &= \int_0^1 du_0\ e^{-u_0 tD_I^2} R \int_{(1-u_0)\Delta^{n-1}} e^{-u_1 tD_I^2} R \dots e^{-u_n tD_I^2}\, du_1 \dots du_n \\ &= \int_0^1 du_0\ (1-u_0)^{n-1} e^{-u_0 tD_I^2} R \int_{\Delta^{n-1}} e^{-(1-u_0)u_1 tD_I^2} R \dots e^{-(1-u_0)u_n tD_I^2}\, du_1 \dots du_n \\ &= \int_0^1 du_0\ (1-u_0)^{n-1} e^{-u_0 tD_I^2} R I_{n-1}((1-u_0)t) \ . \end{aligned}$$

Note that the operators $e^{-(A_t^I)^2}$ and $I_n(t)$ are selfadjoint on $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ in the sense of §5.2.3.

4.1.2 The integral kernel of $e^{-(A_t^I)^2}$

Since $e^{-tD_I^2}$ is a bounded semigroup on $C_R^m([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for any $m \in \mathbb{N}_0$ by prop. 3.3.3, the family $I_n(t) : C_R^m([0,1], \mathcal{A}_i^{2d}) \to C_R^m([0,1], (\hat{\Omega}_n\mathcal{A}_i)^{2d})$ is uniformly bounded in $t \geq 0$.
In the following we write $|\cdot|$ for the norm on $M_{2d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for any $\mu, i \in \mathbb{N}_0$.

Proposition 4.1.1. *For any $n \in \mathbb{N}_0$ and $t > 0$ the operator $I_n(t)$ is an integral operator. Let $p_t(x,y)^n$ be its integral kernel.*
Then it holds:

1. *The map*

   $$(0,\infty) \to C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_\infty))),\ t \mapsto \big(y \mapsto p_t(\cdot,y)^n\big) \ ,$$

   is smooth.

2. *It holds* $p_t(x,y)^n = (p_t(y,x)^n)^*$.
3. *For every* $l,m,n \in \mathbb{N}_0$ *there is* $C,\omega > 0$ *such that for* $t > 0$ *and* $x,y \in [0,1]$ *it holds*
$$|\partial_x^l \partial_y^m p_t(x,y)^n| \le C(1 + t^{-\frac{l+m+1}{2}})e^{-\omega t} \ .$$

Proof. We prove the claim by induction on n. In degree $n = 0$ the assertions hold by prop. 3.3.4 and cor. 3.3.6.

Let $k_t(x,y)$ be the integral kernel of $e^{-tD_I^2}$.

For $f \in C_R^\infty([0,1], (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{2d})$ it holds by induction and by the recursion formula above:

$$(I_n(t)f)(x) = \int_0^1 \int_{[0,1]} \int_{[0,1]} (1-s)^{n-1} k_{st}(x,r) R(r) p_{(1-s)t}(r,y)^{n-1} f(y) \ dydrds \ .$$

Since for fixed s the integrand is uniformly bounded in r and y we can interchange the integration over r and y.

For the proof of the existence of the integral kernel and of 1) it suffices to show that the map

$$(s,t) \mapsto \big(y \mapsto \int_{[0,1]} (1-s)^{n-1} k_{st}(\cdot,r) R(r) p_{(1-s)t}(r,y)^{n-1} dr\big)$$

is a smooth map from $[0,1] \times (0,\infty)$ to $C^\infty([0,1], C_R^\infty[0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i)))$:

For $s \ge \frac{1}{2}$ this follows from the fact that by induction the map

$$(s,t) \mapsto \big(y \mapsto Rp_{(1-s)t}(\cdot,y)^{n-1}\big)$$

is a smooth map from $[\frac{1}{2},1] \times (0,\infty)$ to $C^\infty\big([0,1], C_R^\infty([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i))\big)$. Furthermore the family $e^{-stD_I^2}$ is uniformly bounded on $C_R^l([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i))$ for any $l \in \mathbb{N}_0$ by prop. 3.3.3 and depends smoothly on s,t.

For $s \le \frac{1}{2}$ the map is smooth since

$$(s,t) \mapsto \big(x \mapsto Rk_{st}(x,\cdot)^*\big)$$

is a smooth map with values in $C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\hat{\Omega}_1\mathcal{A}_i)))$ and since the action of the family $I_{n-1}((1-s)t)$ on $C_R^m([0,1], M_{2d}(\hat{\Omega}_{\le\mu}\mathcal{A}_i)))$ depends smoothly on s,t and is uniformly bounded for any $m \in \mathbb{N}_0$.

Assertion 2) holds since $I_n(t)$ is selfadjoint in the sense of §5.2.3.

The preceeding arguments and the following facts imply the estimate in 3):

By induction there is $C > 0$ such that the norm of $\big(y \mapsto Rp_{(1-s)t}(\cdot,y)^{n-1}\big)$ in $C^m\big([0,1], C_R^l([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i))\big)$ is bounded by $C(1 + t^{-\frac{l+m+1}{2}})e^{-\omega t}$ for $0 \le s \le \frac{1}{2}$, $t > 0$. On the other hand by cor. 3.3.6 the norm of $\big(x \mapsto Rk_{st}(x,\cdot)^*\big)$ is bounded in $C^l([0,1], C_R^m([0,1], M_{2d}(\hat{\Omega}_1\mathcal{A}_i))$ by $C(1 + t^{-\frac{l+m+1}{2}})e^{-\omega t}$ for $s > \frac{1}{2}$, $t > 0$. $\square$

The proof of the previous proposition did not use the fact that $D_I^2+\sigma R$ is the curvature of a superconnection. Hence an analogous argument shows that the semigroup $e^{-t(D_{I_s}^2+\sigma R)}$ is an integral operator whose integral kernel can be written as

$$\sum_{n=0}^{\infty}(-1)^{\frac{(n+1)n}{2}}\sigma^n t^n p_t^s(x,y)^n$$

with $\big(y\mapsto p_t^s(\cdot,y)^n\big)\in C^\infty([0,1],C_R^\infty([0,1],M_{2d}(\hat\Omega_n\mathcal{A}_i))$ for $R=(P_s,1-P_s)$.

Lemma 4.1.2. *For any $l,m,n\in\mathbb{N}_0$ and $\varepsilon,\delta>0$ there is $C>0$ such that for all $x,y\in[0,1]$ with $d(x,y)>\varepsilon$ and all $t>0$ it holds*

$$|\partial_x^l\partial_y^m p_t^s(x,y)^n|\le Ce^{-\frac{d(x,y)^2}{(4+\delta)t}}.$$

Proof. We prove the assertion by induction on n.

In degree $n=0$ it holds by cor. 3.1.2.

Let $\eta=\varepsilon/4$.

Let $\chi:\mathbb{R}\to[0,1]$ be a smooth function with $\chi(x)=0$ for $x>\eta$ and $\chi(x)=1$ for $x\le\eta/2$.

Let k_t be the integral kernel of $e^{-tD_{I_s}^2}$.

For $l,m\in\mathbb{N}_0$ it holds

$$\partial_x^l\partial_y^m p_t^s(x,y)^n=\int_0^1\int_{[0,1]}(1-s)^{n-1}\partial_x^l k_{st}(x,r)R(r)\chi(d(x,r))\partial_y^m p_{(1-s)t}^s(r,y)^{n-1}drds$$

$$+\int_0^1\int_{[0,1]}(1-s)^{n-1}\partial_x^l k_{st}(x,r)R(r)\big(1-\chi(d(x,r))\big)\chi(d(r,y))\partial_y^m p_{(1-s)t}^s(r,y)^{n-1}drds$$

$$+\int_0^1\int_{[0,1]}(1-s)^{n-1}\partial_x^l k_{st}(x,r)R(r)\big(1-\chi(d(x,r))\big)\big(1-\chi(d(r,y))\big)\partial_y^m p_{(1-s)t}^s(r,y)^{n-1}drds\,.$$

We begin by estimating the first term on the right hand side: By induction there is $C>0$ such that for $x,y\in[0,1]$ with $d(x,y)>\varepsilon$ and $0<s<1$ and $t>0$ it holds in $C_R^l([0,1],M_{2d}(\hat\Omega_n\mathcal{A}_i))$:

$$\|R\chi(d(\cdot,x))\partial_y^m p_{(1-s)t}^s(\cdot,y)^{n-1}\|_{C_R^l}\le Ce^{-\frac{(d(x,y)-\eta)^2}{(4+\delta)t}}\,.$$

Furthermore the operator $e^{-stD_{I_s}^2}$ is uniformly bounded on $C_R^l([0,1],M_{2d}(\hat\Omega_n\mathcal{A}_i))$, hence the first term is bounded by $Ce^{-\frac{(d(x,y)-\eta)^2}{(4+\delta)t}}$.

An analogous bound exists for the second term: By cor. 3.1.2 and since the integral kernel $(y,r)\mapsto(\partial_y^m p_{(1-s)t}^s(r,y)^{n-1})^*$ induces a uniformly bounded family of operators from $C_R^m([0,1],M_{2d}(\hat\Omega_{\le\mu}\mathcal{A}_i))$ to $C_R([0,1],M_{2d}(\hat\Omega_{\le\mu}\mathcal{A}_i))$ there is $C>0$ such that for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and $0 < s < 1$ and $t > 0$ it holds in $C^m_R([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i))$:

$$\|\big(\partial^l_x k_{st}(x,\cdot)R\big)^*\big(1-\chi(d(\cdot,x))\big)\chi(d(\cdot,y))\|_{C^m_R} \leq Ce^{-\frac{(d(x,y)-\eta)^2}{(4+\delta)t}} .$$

Furthermore by cor. 3.1.2 it holds

$$|\partial^l_x k_{st}(x,r)R(r)\big(1-\chi(d(x,r))\big)| \leq Ce^{-\frac{d(x,r)^2}{(4+\delta)st}}$$

and by induction

$$|\big(1-\chi(d(x,r))\big)\partial^m_y p^s_{(1-s)t}(r,y)^{n-1}| \leq Ce^{-\frac{d(r,y)^2}{(4+\delta)(1-s)t}}$$

for all $x, y, r \in [0,1]$, all $0 < s < 1$ and $t > 0$, thus the third term is bounded by $Ce^{-\frac{d(x,y)^2}{(4+\delta)t}}$.

Hence there is $C > 0$ such that for all $x, y \in [0,1]$ and all $t > 0$ it holds

$$|\partial^l_x \partial^m_y p^s_t(x,y)^n| \leq Ce^{-\frac{(d(x,y)-\varepsilon/2)^2}{(4+\delta)t}} .$$

The assertion follows now from lemma 3.1.6.

□

We apply Duhamel's principle in order to obtain an analogous result for the kernel $p_t(x,y)^n$:

Recall the definitions of $\phi_k, \gamma_k,\ k = 0,1$, in §3.3.1.

Let $U_k \in M_{2d}(\mathcal{A}_\infty),\ k = 0,1$, be a unitary with $UI_0 = I_0U$ and $U_kP_kU_k^* = P_s$ (for its existence see lemma 1.4.3).

Let

$$w^k_t(x,y)^n := U_k^* p^s_t(x,y)^n U_k .$$

Let $W^k_n(t)$ be the integral operator with integral kernel $w^k_t(x,y)^n$.

We define the integral kernel

$$w_t(x,y)^n := \gamma_0(x)w^0_t(x,y)^n\phi_0(y) + \gamma_1(x)w^1_t(x,y)^n\phi_1(y)$$

and denote by $W_n(t)$ the corresponding integral operator. Let $W_0(0) := 1$ and $W_n(0) := 0$ for $n \geq 1$. Then $W_n(t),\ n \in \mathbb{N}_0$ is a strongly continuous family of operators on $L^2([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ and for $f \in C^\infty_R([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ the family $W_n(t)f \in L^2([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ depends even smoothly on t for all $t \in [0,\infty)$.

Furthermore for $t > 0$ the range of $W_n(t)$ is in $C^\infty_R([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

Hence Duhamel's principle yields for $f \in C_R^\infty([0,1], \mathcal{A}_i^{2d})$:

$$\begin{aligned}
&\big(e^{-tA_I^2} - \sum_{n=0}^\infty \sigma^n (-1)^{\frac{n(n+1)}{2}} t^n W_n(t)\big) f \\
&= \; -\int_0^t e^{-sA_I^2}\big(\frac{d}{dt} + A_I^2\big) \sum_{n=0}^\infty (t-s)^n W_n(t-s) f \; ds \\
&= \; \int_0^t e^{-sA_I^2} \sum_{k=0,1} [\gamma_k, D_I^2]_s \sum_{n=0}^\infty \sigma^n (-1)^{\frac{n(n+1)}{2}} (t-s)^n W_n^k(t-s)\phi_k f \; ds \\
&= \; \sum_{n=0}^\infty \int_0^t \sigma^n (-1)^{\frac{n(n+1)}{2}} \sum_{j=0}^n s^{n-j}(t-s)^j I_{n-j}(s) \sum_{k=0,1} (\gamma_k'\partial + \partial\gamma_k') W_j^k(t-s)\phi_k f \; ds \; .
\end{aligned}$$

It follows

$$\big(I_n(t) - W_n(t)\big) f = t^{-n} \int_0^t \sum_{j=0}^n s^{n-j}(t-s)^j I_{n-j}(s) \sum_{k=0,1} (\gamma_k'\partial + \partial\gamma_k') W_j^k(t-s)\phi_k f \; ds \; .$$

Proposition 4.1.3. *For any $l, m, n \in \mathbb{N}_0$ and any $\delta > 0$ there is $C > 0$ such that it holds*

$$|\partial_x^l \partial_y^m p_t(x,y)^n - \partial_x^l \partial_y^m w_t(x,y)^n| \le Ct \sum_{k=0,1} e^{-\frac{d(y, \operatorname{supp}\gamma_k')^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y)$$

for all $t > 0$ and all $x, y \in [0,1]$.

Proof. It holds in $C_R^l([0,1], M_{2d}(\hat{\Omega}_n \mathcal{A}_i))$:

$$\begin{aligned}
&\|\partial_y^m p_t(\cdot,y)^n - \partial_y^m w_t(\cdot,y)^n\|_{C_R^l} \\
&= \; \|t^{-n} \int_0^t \sum_{j=0}^n s^{n-j}(t-s)^j I_{n-j}(s) \sum_{k=0,1} (\gamma_k'\partial + \partial\gamma_k') \partial_y^m \big(w_{t-s}^k(\cdot,y)^j \phi_k(y)\big) ds \|_{C^l} \\
&\le \; \int_0^t \sum_{j=0}^n \|I_{n-j}(s) \sum_{k=0,1} (\gamma_k'\partial + \partial\gamma_k') \partial_y^m \big(w_{t-s}^k(\cdot,y)^j \phi_k(y)\big)\|_{C_R^l} ds \; .
\end{aligned}$$

From the previous lemma it follows that for any $j \in \mathbb{N}_0$ and $k = 0,1$ there is $C > 0$ such that

$$\|(\gamma_k'\partial + \partial\gamma_k') \partial_y^m \big(w_{t-s}^k(\cdot,y)^j \phi_k(y)\big)\|_{C_R^l} \le C e^{-\frac{d(y, \operatorname{supp}\gamma_k')^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y)$$

for all $y \in [0,1]$, all $t > 0$ and $0 < s < t$.

The assertion follows from the fact that $I_{n-j}(s)$ is a uniformly bounded family of operators on $C_R^l([0,1], M_{2d}(\hat{\Omega}_{\le n} \mathcal{A}_i))$.

□

Corollary 4.1.4. *For any $l, m, n \in \mathbb{N}_0$ and $\varepsilon > 0$ there are $c, C > 0$ such that it holds*

$$|\partial_x^l \partial_y^m p_t(x,y)^n| \leq C e^{-\frac{d(x,y)^2}{ct}}$$

for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and all $t > 0$.

Proof. Note that the assertion is equivalent to the assertion that for any $l, m, n \in \mathbb{N}_0$ and $\varepsilon > 0$ there are $c, C > 0$ such that it holds

$$|\partial_x^l \partial_y^m p_t(x,y)^n| \leq C e^{-\frac{1}{ct}}$$

for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and all $t > 0$. The estimate follows for $0 < t < 1$ from the previous proposition and lemma 4.1.2 since $\operatorname{supp} \gamma_k' \cap \operatorname{supp} \phi_k = \emptyset$. For $t > 1$ it holds by the estimate in prop. 4.1.1. □

4.1.3 The η-form

In the following the integral kernel of $D_I I_n(t)$ is denoted by $(D_I p_t)(x,y)^n$

Lemma 4.1.5. *For any $n \in \mathbb{N}_0$ the integral*

$$\int_0^\infty t^{\frac{n-1}{2}} \int_{[0,1]} \operatorname{tr}(D_I p_t)(x,x)^n dx dt$$

is well-defined in $\hat{\Omega}_* \mathcal{A}_\infty / \overline{[\hat{\Omega}_* \mathcal{A}_\infty, \hat{\Omega}_* \mathcal{A}_\infty]_s}$.

Proof. The integral converges in $\hat{\Omega}_{\leq \mu} \mathcal{A}_i / \overline{[\hat{\Omega}_{\leq \mu} \mathcal{A}_i, \hat{\Omega}_{\leq \mu} \mathcal{A}_i]_s}$ for all $n, \mu, i \in \mathbb{N}_0$: For $n = 0$ and $t \to 0$ the convergence follows from cor. 3.3.8; for $n > 0$ and $t \to 0$ and for $n \in \mathbb{N}_0$ and $t \to \infty$ the convergence follows from prop. 4.1.1. By prop. 1.3.5 the integral converges in $\hat{\Omega}_* \mathcal{A}_\infty / \overline{[\hat{\Omega}_* \mathcal{A}_\infty, \hat{\Omega}_* \mathcal{A}_\infty]_s}$ as well. □

Let $\operatorname{tr}_\sigma(a + \sigma b) := \operatorname{tr} a$ for $a, b \in M_{2d}(\hat{\Omega}_* \mathcal{A}_i)$ and let Tr_σ be the corresponding trace on integral operators.

Definition 4.1.6. *The η-form of the superconnection A_I associated to D_I is*

$$\eta(A_I) := \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}_\sigma D_I e^{-(A_t^I)^2} dt \ \in \hat{\Omega}_* \mathcal{A}_\infty / \overline{[\hat{\Omega}_* \mathcal{A}_\infty, \hat{\Omega}_* \mathcal{A}_\infty]_s} \ .$$

It holds

$$\eta(A_I) = \frac{1}{2\sqrt{\pi}} \sum_{n=0}^\infty (-1)^n \int_0^\infty t^{n-\frac{1}{2}} \int_{[0,1]} \operatorname{tr}(D_I p_t)(x,x)^{2n} \, dx dt \ .$$

Hence the η-form is well-defined by the previous lemma.

4.2 The superconnection associated to D_Z

Let (P_0, P_1) be a pair of transverse Lagrangian projections with $P_i \in M_{2d}(\mathcal{A}_\infty),\ i = 1,2$ and let D_Z be the asssociated operator on $L^2(Z, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$ defined in §3.4.

In this section we define a superconnection associated to D_Z.

Let $U \in C^\infty([0,1], M_{2d}(\mathcal{A}_\infty))$ be as in prop. 2.2.1 with $U(0)P_0U(0)^* = P_s$ and $U(1)P_1U(1)^* = 1 - P_s$. We consider U as a function on Z depending only on the variable x_2 and set

$$\tilde{W} := U \oplus U \in C^\infty(Z, M_{2d}(\mathcal{A}_\infty)) \ .$$

We call $A_Z := \tilde{W}^* \,\mathrm{d}\,\tilde{W} + D_Z$ a superconnection associated to D_Z and $A_t^Z := \tilde{W}^* \,\mathrm{d}\,\tilde{W} + \sqrt{t} D_Z$ the corresponding rescaled superconnection.

Its curvature is

$$\begin{aligned}
A_Z^2 &= D_Z^2 + [\tilde{W}^* \,\mathrm{d}\,\tilde{W}, D_Z]_s \\
&= D_Z^2 + \tilde{W}^*[\mathrm{d}, \tilde{W} c(dx_2)\partial_{x_2}\tilde{W}^*]_s \tilde{W} \\
&= D_Z^2 - c(dx_1)\tilde{W}^*[\mathrm{d}, \tilde{W} I \partial_{x_2}\tilde{W}^*]_s \tilde{W} \\
&= D_Z^2 - c(dx_1)\tilde{W}^* \,\mathrm{d}(\tilde{W} I \partial_{x_2}\tilde{W}^*)\tilde{W} \\
&= D_Z^2 + c(dx_1)(R \oplus (-R))
\end{aligned}$$

with $R = -U^* \,\mathrm{d}(U I_0(\partial U^*))U$.

Let $\tilde{R} = c(dx_1)(R \oplus (-R)) \in C^\infty(Z, M_{4d}(\hat{\Omega}_{\le\mu}\mathcal{A}_i))$. It holds $\tilde{R}^* = -\tilde{R}$.

The negative of the curvature generates a holomorphic semigroup $e^{-tA_Z^2}$ on $L^2(Z, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{4d})$.

Let D_I be the closed operator on $L^2([0,1], (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^{2d})$ associated to (P_0, P_1). Recall from §3.4 that for the operator $\tilde{D}_I$ it holds $e^{-tD_Z^2} = e^{-t\tilde{D}_I^2} e^{-t\Delta_{\mathbb{R}}}$.

By prop. 4.1.1 the operator

$$I_n(t) := \int_{\Delta^n} e^{-u_0 t D_I^2} R e^{-u_1 t D_I^2} R \dots e^{-u_n t D_I^2} \, du_0 \dots du_n$$

is an integral operator. Its integral kernel is denoted by $p_t^I(x,y)^n$.

Since $e^{-t\Delta_{\mathbb{R}}}$ commutes with $\tilde{R}$ and $c(dx_1)$ commutes with $e^{-tD_Z^2}$ and with $R \oplus (-R)$, we obtain from Volterra development:

$$\begin{aligned}
e^{-tA_Z^2} &= \sum_{n=0}^\infty (-1)^n t^n \int_{\Delta^n} e^{-u_0 t D_Z^2} \tilde{R} e^{-u_1 t D_Z^2} \tilde{R} \dots e^{-u_n t D_Z^2} \, du_0 \dots du_n \\
&= \sum_{n=0}^\infty (-1)^n t^n e^{-t\Delta_{\mathbb{R}}} \int_{\Delta^n} e^{-u_0 t \tilde{D}_I^2} \tilde{R} e^{-u_1 t \tilde{D}_I^2} \tilde{R} \dots e^{-u_n t \tilde{D}_I^2} \, du_0 \dots du_n \\
&= \sum_{n=0}^\infty (-1)^n t^n c(dx_1)^n e^{-t\Delta_{\mathbb{R}}} \big(I_n(t) \oplus (-1)^n I_n(t) \big) \ .
\end{aligned}$$

We define

$$p_t^Z(x,y)^n := c(dx_1)^n \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}} \left(p_t^I(x_2,y_2)^n \oplus (-1)^n p_t^I(x_2,y_2)^n\right) .$$

It follows that the integral kernel of $e^{-tA_Z^2}$ is

$$\sum_{n=0}^{\infty} (-1)^n t^n p_t^Z(x,y)^n$$

and the integral kernel of $e^{-(A_t^Z)^2}$ is

$$\sum_{n=0}^{\infty} (-1)^n t^{\frac{n}{2}} p_t^Z(x,y)^n .$$

Note that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^2$ it holds

$$\mathrm{tr}_s \partial_x^\alpha \partial_y^\beta p_t^Z(x,y)^n = 0 .$$

Furthermore it holds

$$(D_Z)_x p_t^Z(x,y)^n = c(dx_1)(\partial_{x_1} - I\partial_{x_2}) p_t^Z(x,y)^n .$$

Since $\mathrm{tr}_s c(dx_1)\partial_{x_1} p_t^Z(x,y)^n$ vanishes for $x_1 = y_1$ and $\mathrm{tr}_s c(dx_1) I \partial_{x_2} p_t^Z(x,y)^n$ vanishes for all $x, y \in Z$, it holds $\mathrm{tr}_s (D_Z)_x p_t^Z(x,y)^n = 0$ if $x = y$.

Furthermore the integral kernel $p_t^Z(x,y)^n$ satisfies the following Gaussian estimate:

Lemma 4.2.1. *Let $\alpha, \beta \in \mathbb{N}_0^2$. For any $\varepsilon > 0$ there are $c, C > 0$ such that for all $x, y \in Z$ with $d(x,y) > \varepsilon$ and all $t > 0$ it holds*

$$|\partial_x^\alpha \partial_y^\beta p_t^Z(x,y)^n| \leq C e^{-\frac{d(x,y)^2}{ct}} .$$

Proof. The assertion follows from prop. 4.1.1 and cor. 4.1.4: The proof is analogous to the proof of lemma 3.4.4. □

4.3 The superconnection $A(\rho)_t$ associated to $D(\rho)$

4.3.1 The family $e^{-A(\rho)_t^2}$

Recall that in §3.5.1 we fixed $r_0, b_0 > 0$ such that

$$\mathrm{supp}\, k_K \cap \left((F(r_0,b_0) \times M) \cup (M \times F(r_0,b_0))\right) = \emptyset .$$

. Let $W \in C^\infty(M, \mathrm{End}^+ E \otimes \mathcal{A}_\infty)$ be as in §2.1.2 and assume that W is parallel on $M \setminus F(r, b_0)$ for any $r \geq 0$. By prop. 2.1.3 this is possible. Then it holds $[W, K]_s = 0$.

We define a superconnection associated to $D(\rho)$ on the $\mathbb{Z}/2$-graded $\mathcal{A}_i$-module $L^2(M, E \otimes \mathcal{A}_i)$ by

$$A(\rho) := W^* \,\mathrm{d}\, W + D(\rho) \ .$$

The corresponding rescaled superconnection is

$$A(\rho)_t := W^* \,\mathrm{d}\, W + \sqrt{t} D(\rho) \ .$$

The curvature of $A(\rho)$ is

$$\begin{aligned} A(\rho)^2 &= W^* \,\mathrm{d}^2\, W + [D(\rho), W^* \,\mathrm{d}\, W]_s + D(\rho)^2 \\ &= D(\rho)^2 + W^*[\mathrm{d}, Wc(dW^*)]_s W \\ &=: D(\rho)^2 + \mathcal{R} \end{aligned}$$

by prop. 2.1.1 and the fact that $[W, K]_s = 0$.
It holds $\mathcal{R} \in C^\infty(M, \mathrm{End} E \otimes \hat{\Omega}_{\le 1}\mathcal{A}_i)$ with $\mathcal{R}|_{M\setminus F(r,b_0)} = 0$ for all $r \ge 0$ and

$$\begin{aligned} \mathcal{R}|_F &= W^*[\mathrm{d}, Wc(e_2)\partial_{e_2} W^*)]_s W \\ &= -c(e_1) W^* \,\mathrm{d}(WI\partial_{e_2} W^*) W \ . \end{aligned}$$

Furthermore for any $k \in \mathbb{Z}/6$ the restriction of $\mathcal{R}$ to $\mathcal{U}_k$ is of the form $c(e_1)(R \oplus (-R))$ with $R \in C^\infty(\mathcal{U}_k, M_{2d}(\hat{\Omega}_{\le 1}\mathcal{A}_i))$ independent of the variable x_2^k.
The rescaled curvature is

$$A(\rho)_t^2 = tD(\rho)^2 + \sqrt{t}\mathcal{R} \ .$$

Both, $A(\rho)^2$ and $A(\rho)_t^2$, are right $\hat{\Omega}_{\le\mu}\mathcal{A}_i$-module homomorphisms.
Since $A(\rho)^2$ is a bounded perturbation of $D(\rho)^2$, it generates a holomorphic semigroup $e^{-tA(\rho)^2}$ on $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$.
In the following we only consider $t \ge 0$.
The Volterra development of $e^{-tA(\rho)^2}$ is

$$\begin{aligned} e^{-tA(\rho)^2} &= \sum_{n=0}^{\infty} (-1)^n t^n \int_{\Delta^n} e^{-u_0 t D(\rho)^2} \mathcal{R} e^{-u_1 t D(\rho)^2} \mathcal{R} \dots e^{-u_n t D(\rho)^2} \, du_0 \dots du_n \\ &=: \sum_{n=0}^{\infty} (-1)^n t^n I_n(\rho, t) \ . \end{aligned}$$

It follows

$$e^{-A(\rho)_t^2} = \sum_{n=0}^{\infty} (-1)^n t^{n/2} I_n(\rho, t) \ .$$

For $n \in \mathbb{N}$ there is the following recursive relation:

$$I_n(\rho, t) = \int_0^1 du_0 \, (1-u_0)^{n-1} e^{-u_0 t D(\rho)^2} \mathcal{R} I_{n-1}(\rho, (1-u_0)t) \ .$$

For $\rho \ne 0$ the family $t \mapsto I_n(\rho, t)$ is uniformly bounded on $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$. By cor. 3.5.16 it acts as a strongly continuous family of operators on $C_R^m(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ and there are $C, l > 0$ such that the action is bounded by $C(1+t)^l$.
Note that it holds $I_n(\rho, t)^* = (-1)^n I_n(\rho, t)$.

4.3.2 The integral kernel of $e^{-A(\rho)_t^2}$

In this section we prove that $I_n(\rho,t)$ is an integral operator for $t>0$ by constructing an approximation of the family $I_n(\rho,t)$ by a family integral operators and comparing it with $I_n(\rho,t)$ by Duhamel's principle.

Let $\{\phi_k\}_{k\in J}$ and $\{\gamma_k\}_{k\in J}$ as defined in §3.5.1.

For $k\in\mathbb{Z}/6$ the function $W|_{\mathcal{U}_k}:\mathcal{U}_k\to M_{4d}(\mathcal{A}_\infty)$ does not depend on the coordinate x_1^k. We extend it to a section $\tilde{W}_k:Z_k\to M_{4d}(\mathcal{A}_\infty)$ which is independent of x_1^k as well. Then the superconnection $A_{Z_k}:=\tilde{W}_k^*\,\mathrm{d}\,\tilde{W}_k+D_{Z_k}$ coincides on $\mathcal{U}_k$ with the superconnection $A(\rho)$.

For $k\in\mathbb{Z}/6$ and $n\in\mathbb{N}_0$ let $w(\rho)_t^k(x,y)^n$ be the restriction of $p_t^{Z_k}(x,y)^n$ to $\mathcal{U}_k\times\mathcal{U}_k$.

Let furthermore $w(\rho)_t^{\clubsuit}(x,y)^0$ be the restriction of the integral kernel of $e^{-tD_N(\rho)^2}$ to $\mathcal{U}_\clubsuit\times\mathcal{U}_\clubsuit$ for $t>0$ and set $w(\rho)_t^{\clubsuit}(x,y)^n=0$ for $n>0$.

This is reasonable since $A(\rho)^2$ equals $D(\rho)^2$ on $\mathcal{U}_\clubsuit$.

We extend $w(\rho)_t^k(x,y)^n$ by zero to $M\times M$ and set

$$w(\rho)_t(x,y)^n:=\sum_{k\in J}\gamma_k(x)w(\rho)_t^k(x,y)^n\phi_k(y)\ .$$

Write $W_n(\rho,t)$ for the corresponding integral operator. It is a bounded operator from $L^2(M,E\otimes\mathcal{A}_i)$ to $L^2(M,E\otimes\hat{\Omega}_{\le n}\mathcal{A}_i)$ as well as from $C_R^m(M,E\otimes\mathcal{A}_i)$ to $C_R^m(M,E\otimes\hat{\Omega}_{\le n}\mathcal{A}_i)$.

Set $W_0(\rho,0)=1$ and $W_n(\rho,0)=0$ for $n>0$.

Then for $f\in L^2(M,E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ the family $W_n(\rho,t)f\in L^2(M,E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ depends continuously on t for all $t\in[0,\infty)$, and for $f\in C_{Rc}^\infty(M,E\otimes\mathcal{A}_i)$ even smoothly.

By Duhamel's principle it follows for $f\in C_{Rc}^\infty(M,E\otimes\mathcal{A}_i)$:

$$\begin{aligned}
&\big(e^{-tA(\rho)^2}-\sum_{n=0}^\infty(-1)^nt^nW_n(\rho,t)\big)f\\
&= -\int_0^t e^{-sA(\rho)^2}\Big(\frac{d}{dt}+A(\rho)^2\Big)\sum_{k\in J}\sum_{n=0}^\infty(-1)^n(t-s)^n\gamma_kW_n^k(\rho,t-s)\phi_kf\ ds\\
&= \int_0^t e^{-sA(\rho)^2}\sum_{k\in J}[\gamma_k,D(\rho)^2]_s\sum_{n=0}^\infty(-1)^n(t-s)^nW_n^k(\rho,t-s)\phi_kf\ ds\\
&= \sum_{n=0}^\infty(-1)^n\int_0^t\sum_{m=0}^n s^{n-m}(t-s)^mI_{n-m}(\rho,s)\sum_{k\in J}[\gamma_k,D(\rho)^2]_sW_m^k(\rho,t-s)\phi_kf\ ds
\end{aligned}$$

It follows

$$\big(I_n(\rho,t)-W_n(\rho,t)\big)f=t^{-n}\int_0^t\sum_{m=0}^n s^{n-m}(t-s)^mI_{n-m}(\rho,s)\sum_{k\in J}[\gamma_k,D(\rho)^2]_sW_m^k(\rho,t-s)\phi_kf\ ds\ .$$

In the following $|\cdot|$ denotes the fibrewise norm of $(E\boxtimes E^*)\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i$.

Proposition 4.3.1. *The operator $I_n(\rho,t)$ is an integral operator for $t>0$. For its integral kernel $p(\rho)_t(x,y)^n$ it holds:*

1. *The map $(0,\infty)\to C^\infty(M\times M,(E\boxtimes E^*)\otimes\hat\Omega_n\mathcal{A}_\infty), t\mapsto p(\rho)^n_t$ is smooth.*
2. *It holds $p(\rho)_t(x,y)^n=(-1)^n(p(\rho)_t(y,x)^n)^*$.*
3. *For any $T>0$ there are $C,c>0$ such that it holds*
$$|p(\rho)_t(x,y)^n-w(\rho)_t(x,y)^n|\le Ct\left(\sum_{k\in J}e^{-\frac{d(y,\operatorname{supp}d\gamma_k)^2}{ct}}1_{\operatorname{supp}\phi_k}(y)+\rho 1_{\mathcal{U}_\clubsuit}(y)\right)$$
for all $0<t<T$, $\rho\in[-1,1]$ and all $x,y\in M$.
4. *Let $\rho\ne 0$. Then there are $C,c>0$ and $j\in\mathbb{N}$ such that it holds*
$$|p(\rho)_t(x,y)^n-w(\rho)_t(x,y)^n|\le Ct(1+t)^je^{-\frac{d(y,\mathcal{U}_\clubsuit)^2}{ct}}$$
for all $t>0$ and all $x,y\in M$.
Statements analogous to 3), 4) hold for the partial derivatives of $p(\rho)_t(x,y)^n$ in x of y with respect to unit vector fields on M.

Proof. The proof is a generalization of the proof of prop. 3.5.14.
In order to show the existence of the integral kernel and 1) we need only investigate $I_n(\rho,t)-W_n(\rho,t)$.
It holds for $f\in C^\infty_{Rc}(M,E\otimes\mathcal{A}_i)$:

$$\begin{aligned}&\big(I_n(\rho,t)-W_n(\rho,t)\big)f\\ &=\ t^{-n}\int_0^t\int_M\sum_{m=0}^n s^{n-m}(t-s)^mI_{n-m}(\rho,s)\sum_{k\in J}[\gamma_k,A(\rho)^2]_sw(\rho)^k_{t-s}(\cdot,y)^m\phi_k(y)f(y)\,dyds\ .\end{aligned}$$

Note that for $k\in\mathbb{Z}/6$ it holds $[\gamma_k,A(\rho)^2]=[c(d\gamma_k),D]_s$.
For $\rho\in\mathbb{R}$ and $t>0$ the family $I_{n-m}(\rho,s):C^\nu_R(M,E\otimes\hat\Omega_{\le\mu}\mathcal{A}_i)\to C^\nu_R(M,E\otimes\hat\Omega_{\le\mu+n-m}\mathcal{A}_i)$ is uniformly bounded in $s<t$.
The function
$$\tau\mapsto\big(y\mapsto\sum_{k\in J}[\gamma_k,A(\rho)^2]_sw(\rho)^k_\tau(\cdot,y)^m\phi_k(y)\big)$$
is smooth from $(0,\infty)$ to $C^l(M,C^\nu_{Rc}(M,E\otimes\hat\Omega_{\le m+1}\mathcal{A}_i)\otimes E^*)$ for any $l,\nu\in\mathbb{N}_0$.
If $k\in\mathbb{Z}/6$, then by lemma 4.2.1 there are $c,C>0$ such that for all $y\in M$ and $0<\tau$ it holds
$$\|[\gamma_k,A(\rho)^2]_sw^k_\tau(\cdot,y)^m\phi_k(y)\|_{C^\nu}\le Ce^{-\frac{d(y,\operatorname{supp}d\gamma_k)^2}{c\tau}}1_{\operatorname{supp}\phi_k}(y)\ .$$

Furthermore it holds $w(\rho)^{\clubsuit}_{\tau}(x,y)^m = 0$ for $m > 0$. The kernel $w(\rho)^{\clubsuit}_{\tau}(x,y)^0$ is equal to the kernel $e(\rho)^{\clubsuit}_{\tau}(x,y)$ in the proof of prop. 3.5.14 and was estimated there. It follows that for $T,\delta > 0$ there is $C > 0$ such that for $y \in M$, $\rho \in [-1,1]$ and $0 < \tau < T$ it holds

$$\|[\gamma_{\clubsuit}, A(\rho)^2]_s w(\rho)^{\clubsuit}_{\tau}(\cdot,y)^m \phi_{\clubsuit}(y) f(y)\|_{C^\nu} \leq C\big(e^{-\frac{d(y,\operatorname{supp} d\gamma_{\clubsuit})^2}{(4+\delta)t}} + |\rho|\big) 1_{\operatorname{supp}\phi_{\clubsuit}}(y) \ .$$

Analogous estimates hold for the derivatives in y and also in τ by the heat equation.

The existence and 1) follow now by standard arguments.

Assertion 2) follows from $I_n(\rho,t)^* = (-1)^n I_n(\rho,t)$.

In order to prove 3) we combine the estimates above with the fact that for any $T > 0$ the operator $I_n(\rho,t) : C^l_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) \to C^l_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is uniformly bounded in $\rho \in [-1,1]$ and $0 \leq t < T$.

4) follows from the estimates by taking into account that for any $\rho \neq 0$ there is $C > 0$ and $j \in \mathbb{N}$ such that the norm of $I_n(\rho,t) : C^l_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) \to C^l_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is bounded by $C(1+t)^j$ for any $t > 0$. $\square$

Corollary 4.3.2. *Let $\rho \neq 0$. For any $\varepsilon > 0$ and $m,n \in \mathbb{N}_0$ there are $c, C > 0$ and $j \in \mathbb{N}$ such that for $t > 0$ and $x,y \in M$ with $d(x,y) > \varepsilon$ it holds*

$$|D(\rho)^m_x p(\rho)_t(x,y)^n| \leq C(1+t)^j\big(e^{-\frac{d(x,y)^2}{ct}} + e^{-\frac{d(y,\mathcal{U}_{\clubsuit})^2}{ct}}\big).$$

Proof. This follows from lemma 4.2.1. $\square$

Corollary 4.3.3. *Let $k \in \mathbb{Z}/6$.*

1. *For any $T > 0$ and $m,n \in \mathbb{N}_0$ there are $c, C > 0$ such that for all $x,y \in \mathcal{U}_k$, for $0 < t < T$ and $\rho \in [-1,1]$ it holds*

$$|D(\rho)^m_x p(\rho)_t(x,y)^n - (D_{Z_k})^m_x p^{Z_k}_t(x,y)^n| \leq C e^{-\frac{d(y,\mathcal{U}_{\clubsuit})}{ct}} \ .$$

2. *For any $\rho \neq 0$ and $n \in \mathbb{N}_0$ there is $c, C > 0$ and $j \in \mathbb{N}$ such that for all $x,y \in \mathcal{U}_k$ and $t > 0$ it holds*

$$|D(\rho)^m_x p(\rho)_t(x,y)^n - (D_{Z_k})^m_x p^{Z_k}_t(x,y)^n| \leq C(1+t)^j e^{-\frac{d(y,\mathcal{U}_{\clubsuit})}{ct}} \ .$$

4.4 The index theorem and its proof

4.4.1 The supertrace

Let tr_s be the supertrace on the fibres of $(E \otimes E^*) \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i = \mathrm{End}E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ and Tr_s the corresponding trace for trace class operators as defined in §1.3.2.

Let $\chi : \mathbb{R} \to [0,1]$ be a smooth function with $\chi(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1$. Let $\phi_r : M \to [0,1]$, $\phi_r(x) = \chi(d(M_r, x))$ for $r > 0$.

Definition 4.4.1. *Let K be a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ such that $\phi_r K \phi_r$ is a trace class operator for all $r \geq 0$. Then we define*

$$\mathrm{Tr}_s K := \lim_{r\to\infty} \mathrm{Tr}_s\big(\phi_r K \phi_r\big)$$

if the limit exists.

On trace class operators this definition coincides with the supertrace Tr_s defined in §1.3.2.

Proposition 4.4.2. *For $\rho \in \mathbb{R}$ and $t > 0$ the supertraces*

$$\mathrm{Tr}_s e^{-A(\rho)_t^2}$$

and

$$\mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2}$$

exist in the sense of the previous definition.

Proof. We show the assertion for $e^{-A(\rho)_t^2}$, the proof for $D(\rho)e^{-A(\rho)_t^2}$ is analogous.

The operator $\phi_r e^{-A(\rho)_t^2}\phi_r$ is trace class for $t > 0$ since

$$\phi_r e^{-A(\rho)_t^2}\phi_r = (\phi_r e^{-A(\rho)_t^2/2})(e^{-A(\rho)_t^2/2}\phi_r)$$

is an appropriate decomposition in Hilbert-Schmidt operators. Note that it follows that the operators $I_n(\rho, t)$ are also trace class for $t > 0$.

We show that $\mathrm{tr}_s p(\rho)_t(x,x)^n$ is in $L^1(M, \hat{\Omega}_{\leq\mu}\mathcal{A}_i/\overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]_s})$:

By cor. 4.3.3 there are $c, C > 0$ such that

$$|p(\rho)_t(x,x)^n - p_t^{Z_k}(x,x)^n| \leq Ce^{-cd(x,\mathcal{U}_\clubsuit)^2}$$

for all $x \in \mathcal{U}_k$.

Since $\mathrm{tr}_s p_t^{Z_k}(x,x)^n = 0$ by §4.2, it holds

$$|\mathrm{tr}_s p(\rho)_t(x,x)^n| \leq Ce^{-cd(x,\mathcal{U}_\clubsuit)^2} \ .$$

Now the assertion follows. $\square$

4.4.2 The limit of $\mathrm{Tr}_s e^{-A(\rho)_t^2}$ for $t \to \infty$

This section is devoted to the proof of

Theorem 4.4.3. *Let $\rho \neq 0$.*

Let P_0 be the projection onto the kernel of $D(\rho)$. Then for $T > 0$ there is $C > 0$ such that for all $t > T$ it holds

$$|\mathrm{Tr}_s e^{-A(\rho)_t^2} - \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \mathrm{Tr}_s (P_0 W \,\mathrm{d}\, W^* P_0)^{2n}| \leq C t^{-\frac{1}{2}}$$

and

$$|\mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2}| \leq C t^{-1}$$

in $\hat{\Omega}_{\leq\mu}\mathcal{A}_i / \overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]_s}$.

Note that it holds $(P_0 W \,\mathrm{d}\, W^* P_0)^{2n} = W(W^* P_0 W)(\mathrm{d}\, W^* P_0 W)^{2n} W^*$, hence by lemma 5.3.5 it follows that $(P_0 W \,\mathrm{d}\, W^* P_0)^{2n}$ is a trace class operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

The proof is subdivided in some lemmata. It relays on the tools developped in §5.2.3.

Throughout the section $\rho \neq 0$ is fixed.

In the following $|\cdot|$ denotes the norm of $\hat{\Omega}_{\leq\mu}\mathcal{A}_i / \overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]_s}$ resp. the fibrewise norm of $(E \boxtimes E^*) \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ (depending on the context), and $\|\cdot\|$ denotes the operator norm on $B(L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i))$.

Lemma 4.4.4. *Let $\nu = 0, 1$. For any $T > 0$ there are $\varepsilon, C > 0$ such that for all $t > T$ it holds:*

$$|\mathrm{Tr}_s D(\rho)^\nu e^{-A(\rho)_t^2} - \mathrm{Tr}_s \phi_t D(\rho)^\nu e^{-A(\rho)_t^2}| \leq C e^{-\varepsilon t}.$$

Proof. We prove the case $\nu = 0$, the case $\nu = 1$ can be proved analogously.

By cor. 4.3.3 for any $n \in \mathbb{N}_0$ there are $c, C, r > 0$ such that it holds

$$|p(\rho)_t(x,x)^n - p_t^{Z_k}(x,x)^n| \leq C(1+t)^j e^{-\frac{d(x,M_r)^2}{ct}}$$

for $t > 0$ and $x \in \mathcal{U}_k$ with $k \in \mathbb{Z}/6$.

Hence there are $c, C > 0$ such that for all $x \in M$ and $t > T$ it holds

$$|\mathrm{tr}_s p(\rho)_t(x,x)| \leq C e^{-\frac{d(x,M_r)^2}{ct}}$$

and by that there are $C, \varepsilon > 0$ such that for all $t > r$ it holds

$$\begin{aligned} |\mathrm{Tr}_s e^{-A(\rho)_t^2} - \mathrm{Tr}_s \phi_t e^{-A(\rho)_t^2}| &= |\mathrm{Tr}_s (1-\phi_t) e^{-A(\rho)_t^2}| \\ &\leq C \int_t^\infty e^{-\frac{(r'-r)^2}{ct}} dr' \\ &\leq C e^{-\varepsilon t} . \end{aligned}$$

$\square$

It holds

$$\mathrm{Tr}_s \phi_t D(\rho)^\nu e^{-A(\rho)_t^2} = \sum_{k=0}^{\infty} (-1)^k t^{k/2} \, \mathrm{Tr}_s \phi_t D(\rho)^\nu I_k(\rho, t)$$

and

$$\mathrm{Tr}_s \phi_t D(\rho)^\nu I_k(\rho, t) = \int_{\Delta^k} \mathrm{Tr}_s \big(\phi_t D(\rho)^\nu e^{-u_0 t D(\rho)^2} \mathcal{R} e^{-u_1 t D(\rho)^2} \mathcal{R} \dots e^{-u_k t D(\rho)^2} \phi_t\big) \, du_0 \dots du_k \, .$$

Note that on the right hand side the supertrace is applied to trace class operators.
Let $P_1 := 1 - P_0$. The decomposition $e^{-u_i t D(\rho)^2} = P_0 + P_1 e^{-u_i t D(\rho)^2}$ induces a decomposition of $\mathrm{Tr}_s \phi_t D(\rho)^\nu I_k(\rho, t)$ in a sum of 2^{k+1} terms. Let $P^\nu_{jk}(t)$ be the sum of all those terms containing j factors of the form $P_1 e^{-u_i t D(\rho)^2}$.
Hence

$$\mathrm{Tr}_s \phi_t D(\rho)^\nu e^{-A(\rho)_t^2} = \sum_{k=0}^{\infty} (-1)^k t^{\frac{k}{2}} \sum_{j=0}^{k+1} P^\nu_{jk}(t) \, .$$

From $P_0 \mathcal{R} P_0 = P_0 [W^* \mathrm{d}\, W, D(\rho)]_s P_0 = 0$ it follows $P^\nu_{jk}(t) = 0$ for $j < \frac{k}{2}$.
Note further that for k even it holds

$$\begin{aligned} P^\nu_{\frac{k}{2}k}(t) &= \int_{\Delta^k} \mathrm{Tr}_s \big(\phi_t D(\rho)^\nu P_0 \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_3 t D(\rho)^2} \mathcal{R} P_0 \dots \\ &\qquad \dots P_0 \mathcal{R} P_1 e^{-u_{k-1} t D(\rho)^2} \mathcal{R} P_0 \phi_t\big) \, du_0 \dots du_k \, . \end{aligned}$$

Since $D(\rho) P_0 = 0$, it follows $P^1_{\frac{k}{2}k}(t) = 0$.
In two lemmata we study the behaviour of the remaining cases for large t:

Lemma 4.4.5. *For $\nu = 0, 1$, for $k \in \mathbb{N}_0$ and $T > 0$ there are $C, \varepsilon > 0$ such that for all $t > T$ it holds*

$$|P^\nu_{(k+1)k}(t)| \le C e^{-\varepsilon t} \, .$$

Proof. It holds

$$\begin{aligned} P^\nu_{(k+1)k}(t) &= \int_{\Delta^k} \mathrm{Tr}_s \big(\phi_t D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} \dots \\ &\qquad \dots \mathcal{R} P_1 e^{-u_k t D(\rho)^2} \phi_t\big) \, du_0 \dots du_k \, . \end{aligned}$$

Note that for any $(u_0, \dots u_k) \in \Delta_k$ we can find $i \in \{0, 1, \dots, k\}$ such that $u_i \ge \frac{1}{k+1}$.
We begin by showing that for any $T > 0$ there are $C, \varepsilon > 0$ such that for $1 > u_i \ge \frac{1}{k+1}$, for $1 > u_0, \dots, u_{i-1} \ge 0$ and for $t > T$ the family

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} \dots \mathcal{R} P_1 e^{-\frac{u_i}{2} t D(\rho)^2}$$

is a family of Hilbert-Schmidt operators with Hilbert-Schmidt norm bounded by $C u_0^{-\frac{\nu}{2}} e^{-\varepsilon t}$. If not specified the estimates in the following hold for $1 > u_i > \frac{1}{k+1}$, for $1 > u_0, \dots, u_{i-1} \ge 0$ and for $t > T$.

It holds

$$
\begin{aligned}
&\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2}\mathcal{R}P_1 e^{-u_1 tD(\rho)^2}\mathcal{R}\dots\mathcal{R}P_1 e^{-\frac{u_i}{2}tD(\rho)^2}\\
&= \phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2}\mathcal{R}P_1 e^{-u_1 tD(\rho)^2}\mathcal{R}\dots\mathcal{R}P_1 e^{-\frac{u_i}{2}tD(\rho)^2}\phi_{(t+6)}\\
&\quad +\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2}\mathcal{R}\dots P_1 e^{-u_{i-1}tD(\rho)^2}\mathcal{R}\phi_{(t+6)}P_1 e^{-\frac{u_i}{2}tD(\rho)^2}(1-\phi_{t+6})\\
&\quad +\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2}\mathcal{R}\dots P_1 e^{-u_{i-1}tD(\rho)^2}\mathcal{R}(1-\phi_{(t+6)})P_1 e^{-\frac{u_i}{2}tD(\rho)^2}(1-\phi_{(t+6)})\ .
\end{aligned}
$$

Consider the first term on the right hand side: By cor. 3.5.18 there is $C>0$ such that the family $e^{-\frac{u_i}{4}tD(\rho)^2}\phi_{(t+6)}$ is family of Hilbert-Schmidt operators with norm bounded by Ct.

Furthermore by prop. 3.5.12 and cor. 3.5.15 there are $\varepsilon, C>0$ such that the operator norm of

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2}\mathcal{R}P_1 e^{-u_1 tD(\rho)^2}\mathcal{R}\dots\mathcal{R}P_1 e^{-\frac{u_i}{4}tD(\rho)^2}$$

is bounded by $Cu_0^{-\frac{\nu}{2}}e^{-\varepsilon t}$.

By prop. 5.2.12 it follows that the first term is a family of Hilbert-Schmidt operators with Hilbert-Schmidt norm bounded by $Cu_0^{-\frac{\nu}{2}}e^{-\varepsilon t}$ for some $C,\varepsilon>0$.

In the second term the factor

$$\phi_{(t+6)}P_1 e^{-\frac{u_i}{2}tD(\rho)^2}(1-\phi_{(t+6)}) = (\phi_{(t+6)}e^{-\frac{u_i}{4}tD(\rho)^2})P_1 e^{-\frac{u_i}{4}tD(\rho)^2}(1-\phi_{(t+6)})$$

is a Hilbert-Schmidt operator bounded by some $Ce^{-\varepsilon t}$ with $C,\varepsilon>0$. Hence the second term is bounded in the Hilbert-Schmidt norm by $Cu_0^{-\frac{\nu}{2}}e^{-\varepsilon t}$ for some $C,\varepsilon>0$.

The estimate of the third term requires more effort:

We prove by induction on $j\in\mathbb{N}$ that there is $C>0$ such that

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2}\mathcal{R}P_1 e^{-u_1 tD(\rho)^2}\mathcal{R}\dots P_1 e^{-u_j tD(\rho)^2}(1-\phi_{(t+6)})$$

is a Hilbert-Schmidt operator with Hilbert-Schmidt norm bounded by $Cu_0^{-\frac{\nu}{2}}(1+t)$ for $t>T$ and $0\le u_0,\dots,u_j\le 1$.

Then it follows that the third term is uniformly bounded by some $Ce^{-\varepsilon t}$ for some $C,\varepsilon>0$ since $P_1 e^{-\frac{u_i}{2}tD(\rho)^2}$ is exponentially decaying for $t\to\infty$ by prop. 3.5.12.

For $j=0$ the assertion follows from cor. 3.5.18 by

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2}(1-\phi_{(t+6)}) = \phi_t D(\rho)^\nu e^{-u_0 tD(\rho)^2}(1-\phi_{(t+6)}) - \phi_t D(\rho)^\nu P_0(1-\phi_{(t+6)})\ .$$

Now assume the assertion is true for $j-1$. It holds

$$
\begin{aligned}
&\phi_t P_1 D(\rho)^\nu e^{-u_0 tD(\rho)^2}\mathcal{R}P_1\dots\mathcal{R}P_1 e^{-u_j tD(\rho)^2}(1-\phi_{(t+6)})\\
&= \phi_t P_1 D(\rho)^\nu e^{-u_0 tD(\rho)^2}P_1\mathcal{R}P_1\dots P_1 e^{-u_{j-1}tD(\rho)^2}P_1\mathcal{R}\phi_{(t+3)}P_1 e^{-u_j tD(\rho)^2}(1-\phi_{(t+6)})\\
&\quad +\phi_t P_1 D(\rho)^\nu e^{-u_0 tD(\rho)^2}P_1\dots P_1 e^{-u_{j-1}tD(\rho)^2}P_1(1-\phi_{(t+3)})\mathcal{R}P_1 e^{-u_j tD(\rho)^2}(1-\phi_{(t+6)})\ .
\end{aligned}
$$

Both terms on the right hand side are bounded in the Hilbert-Schmidt norm by $Cu_0^{-\nu/2}(1+t)$ for some $C > 0$ and all $t > T$ and $0 \le u_0, \dots, u_j \le 1$: the first term since $\phi_{(t+3)} P_1 e^{-u_j tD(\rho)^2}(1-\phi_{(t+6)})$ is bounded by $C(1+t)$ by cor. 3.5.18, the second term by induction.

Now we have finished the claim the proof began with.

An analogous proof shows that for any $T > 0$ there are $C, \varepsilon > 0$ such that the family

$$P_1 e^{-\frac{u_i}{2} tD(\rho)^2} \mathcal{R} P_1 e^{-u_{i+1} tD(\rho)^2} \mathcal{R} \dots P_1 e^{-u_k tD(\rho)^2} \phi_t$$

is a family of Hilbert-Schmidt operators with Hilbert-Schmidt norm bounded by $Ce^{-\varepsilon t}$ for all $u_i > \frac{1}{k+1}$, $0 \le u_{i+1}, \dots, u_k \le 1$ and $t > T$.

It follows that

$$\int_{\Delta^k} \mathrm{Tr}_s\big(\phi_t P_1 D(\rho)^\nu e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \dots \mathcal{R} P_1 e^{-u_k tD(\rho)^2} \phi_t\big) du_0 \dots du_k$$

is bounded by $Ce^{-\varepsilon t}$ for some $C, \varepsilon > 0$ and all $t > T$.

This implies the assertion. □

Lemma 4.4.6. *Let $\nu = 0, 1$ and $j, k \in \mathbb{N}_0$ with $\frac{k}{2} \le j \le k$. For any $T > 0$ there is $C > 0$ such that it holds for $t > T$:*

$$|P^\nu_{jk}(t)| \le Ct^{-j} \ .$$

Furthermore for k odd, for any $n \in \mathbb{N}$ and $T > 0$ there is $C > 0$ such that for $t > T$ it holds:

$$|P^\nu_{\frac{k+1}{2}k}(t)| \le Ct^{-n} \ .$$

Proof. For $j \le k$ the operator $P^\nu_{jk}(t)$ is a sum of terms of the form

$$\int_{\Delta^k} \mathrm{Tr}_s(A(u_0, \dots u_i, t)P_0)(P_0 B(u_{i+1}, \dots u_k, t)) \, du_0 \dots du_k \ ,$$

where A and B are continuous families of bounded operators on $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$ for $u_0 \ne 0$.

Since P_0 is a Hilbert-Schmidt operator by prop. 5.3.7 it holds by prop. 5.2.12:

$$\Big|\int_{\Delta^k} \mathrm{Tr}_s\big(A(u_0, \dots u_i, t)P_0\big)\big(P_0 B(u_{i+1}, \dots u_k, t)\big) \, du_0 \dots du_k\Big|$$

$$\le \|P_0\|^2_{HS} \int_{\Delta^k} \|A(u_0, \dots u_i, t)\| \|B(u_{i+1}, \dots u_k, t)\| \, du_0 \dots du_k \ .$$

Let $\omega > 0$ be such that there is $C > 0$ with $\|P_1 e^{-tD(\rho)^2}\| \le Ce^{-\omega t}$ for all $t > 0$; then it follows:

$$\begin{aligned}
|P^\nu_{jk}(t)| &\le C \sum_{\lambda=0,1} \int_0^1 du_0\ (u_0 t)^{-\lambda/2} e^{-\omega u_0 t} \int_{(1-u_0)\Delta^{k-1}} \exp(-\sum_{i=1}^{j-1} \omega u_i t)\ du_1 \dots du_k \\
&= C \sum_{\lambda=0,1} \int_0^1 du_0\ (u_0 t)^{-\lambda/2} e^{-\omega u_0 t} \int_0^{(1-u_0)} e^{-\omega st}\ \mathrm{vol}(s\Delta^{j-2})\ \mathrm{vol}\big((1-u_0-s)\Delta^{k-j}\big)\ ds \\
&= C \sum_{\lambda=0,1} \int_0^1 du_0\ (u_0 t)^{-\lambda/2} e^{-\omega u_0 t} \int_0^{(1-u_0)} e^{-\omega st}\ \frac{s^{j-2}(1-u_0-s)^{k-j}}{(j-2)!(k-j)!} ds \\
&= Ct^{-j} \sum_{\lambda=0,1} \int_0^t dy\ y^{-\lambda/2} e^{-\omega y} \int_0^{(t-y)} e^{-\omega x}\ \frac{x^{j-2}(1-y/t-x/t)^{k-j}}{(j-2)!(k-j)!}\ dx \\
&\le Ct^{-j} \sum_{\lambda=0,1} \Big(\int_0^\infty y^{-\lambda/2} e^{-\omega y}\ dy\Big)\Big(\int_0^\infty e^{-\omega x}\ \frac{x^{j-2}}{(j-2)!(k-j)!}\ dx\Big) \\
&\le Ct^{-j}\ .
\end{aligned}$$

This shows the first statement.

For k odd it holds

$$\begin{aligned}
&P^\nu_{\frac{k+1}{2}k}(t) \\
&= \int_{\Delta^k} \mathrm{Tr}_s\big(\phi_t D(\rho)^\nu P_0 \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_3 t D(\rho)^2} \dots P_0 \mathcal{R} P_1 e^{-u_k t D(\rho)^2}\big)\ du_0 \dots du_k \\
&\quad + \int_{\Delta^k} \mathrm{Tr}_s\big(D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_2 t D(\rho)^2} \dots P_0 \mathcal{R} P_1 e^{-u_{k-1} t D(\rho)^2} \mathcal{R} P_0 \phi_t\big)\ du_0 \dots du_k\ .
\end{aligned}$$

From prop. 3.5.6 it follows that for any $n \in \mathbb{N}$ there is $C > 0$ such that

$$\|(1-\phi_t)P_0\|_{HS} + \|P_0(1-\phi_t)\|_{HS} \le Ct^{-n}\ .$$

The second estimate follow then from the cyclicity of the sypertrace since $P_0 P_1 = P_1 P_0 = 0$. □

From the results so far obtained it follows that the supertrace $\mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2}$ can be estimated by Ct^{-1} for $t > T$. This shows the second estimate in the theorem.

Furthermore it follows from the previous lemmata that for $t > T$ there is $C > 0$ such that it holds

$$|\mathrm{Tr}_s \phi_t e^{-A(\rho)_t^2} - \sum_{n=0}^\infty t^n\ P^0_{n(2n)}(t)| \le Ct^{-\frac{1}{2}}\ .$$

Define

$$\begin{aligned}
\mathcal{P}_n(t) &:= \int_{\Delta^{2n}} P_0 \mathcal{R} P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} P_0 \dots \\
&\qquad \dots P_0 \mathcal{R} P_1 e^{-u_{n-1} t D(\rho)^2} \mathcal{R} P_0\ du_0 \dots du_{2n}
\end{aligned}$$

Note that by $\mathcal{P}_n(t) = P_0\mathcal{P}_n(t)P_0$ this is a trace class operator.
It holds $P^0_{n(2n)}(t) = \mathrm{Tr}_s\phi_t\mathcal{P}_n(t)$.
From prop. 3.5.6 it follows that for any $j \in \mathbb{N}$ there is $C > 0$ such that

$$\|P_0 - \phi_t P_0\|_{HS} \le Ct^{-j} \ ,$$

hence

$$|\mathrm{Tr}_s\mathcal{P}_n(t) - P^0_{n(2n)}(t)| \le Ct^{-j} \ .$$

In the next lemma we show that for $T > 0$ there is $C > 0$ such that

$$\|t^n\mathcal{P}_n(t) - (-1)^n\frac{1}{n!}(P_0WdW^*P_0)^{2n}\| \le Ct^{-1}$$

in $B(L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i))$ for $t > T$. The first estimate of the theorem follows then since for any $j > 0$ there is $C > 0$ such that

$$\begin{aligned}
&|t^nP^0_{n(2n)}(t) - (-1)^n\frac{1}{n!}\mathrm{Tr}_s(P_0WdW^*P_0)^{2n}| \\
\le\ & Ct^{n-j} + |t^n\ \mathrm{Tr}_s[P_0^2\big(\mathcal{P}_n(t) - (-1)^n\frac{1}{n!}\mathrm{Tr}_s(P_0WdW^*P_0)^{2n}\big)]| \\
\le\ & Ct^{n-j} + \|P_0\|^2_{HS}\|\mathcal{P}_n(t) - (-1)^n\frac{1}{n!}(P_0WdW^*P_0)^{2n}\| \ .
\end{aligned}$$

Lemma 4.4.7. *Let $k, n \in \mathbb{N}_0$ with $n \le k$. For $t \to \infty$ the term*

$$t^n\int_{\Delta^k} P_0\mathcal{R}P_1e^{-u_0tD(\rho)^2}P_1\mathcal{R}P_0\mathcal{R}P_1e^{-u_1tD(\rho)^2}\mathcal{R}P_0\dots P_0\mathcal{R}P_1e^{-u_{n-1}tD(\rho)^2}\mathcal{R}P_0\,du_0\dots du_k$$

converges in $B(L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i))$ to

$$(-1)^n\frac{1}{(k-n)!}\ (P_0W\,\mathrm{d}\,W^*P_0)^{2n}$$

with $O(t^{-1})$.

Proof. For $i, j \in \{0, 1\}$ let ${}_id_j := P_iW\,\mathrm{d}\,W^*P_j$.
Note that by $\mathcal{R} = [W^*\,\mathrm{d}\,W, D(\rho)]_s$ it holds

$$\begin{aligned}
P_1\mathcal{R}P_0 &= P_1D(\rho)W^*\,\mathrm{d}\,WP_0 \\
P_0\mathcal{R}P_1 &= P_0W^*\,\mathrm{d}\,WD(\rho)P_1
\end{aligned}$$

and thus

$$P_0\mathcal{R}P_1e^{-tD(\rho)^2}\mathcal{R}P_0 = {}_0d_1D(\rho)^2e^{-tD(\rho)^2}{}_1d_0.$$

This term is uniformly bounded for $t \to 0$.
Hence the integral

$$\int_0^t {}_0d_1D(\rho)^2e^{-sD(\rho)^2}{}_1d_0\ ds$$

converges and equals ${}_0d_1(e^{-tD(\rho)^2}-1)\ {}_1d_0$.

For $n \in \mathbb{N}$ and $k \geq n$ it holds

$$t^n \int_{\Delta^k} P_0 \mathcal{R} P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} P_0 \dots P_0 \mathcal{R} P_1 e^{-u_{n-1} tD(\rho)^2} \mathcal{R} P_0 \; du_0 \dots du_k$$

$$= t^n \int_{\Delta^k} {}_0d_1 D(\rho)^2 e^{-u_0 tD(\rho)^2} {}_1d_0 {}_0d_1 D(\rho)^2 e^{-u_1 tD(\rho)^2} {}_1d_0 \dots {}_0d_1 D(\rho)^2 e^{-u_{n-1} tD(\rho)^2} {}_1d_0 \; du_0 \dots du_k \, .$$

For $n \in \mathbb{N}$ set

$$D_n := \{\sum_{i=0}^{n-1} u_i \leq 1;\ 0 \leq u_i \leq 1,\ i = 0, \dots, n-1\} \, .$$

By integration on $u_n, \dots, u_k$ the previous term equals

$$t^n \int_{D_n} {}_0d_1 D(\rho)^2 e^{-u_0 tD(\rho)^2} {}_1d_0 \ \dots {}_0d_1 D(\rho)^2 e^{-u_{n-1} tD(\rho)^2} {}_1d_0 \ \mathrm{vol}((1 - \sum_{i=0}^{n-1} u_i) \Delta^{k-n}) \, du_0 \dots du_{n-1} \, .$$

We prove by induction that this term converges to $\frac{1}{(k-n)!}({}_0d_1 {}_1d_0)^n$ with $O(t^{-1})$.

For $n = 1$ and $k = 1$ the term equals 1.

For $n = 1$ and $k \in \mathbb{N}$ the claim follows by induction on k: By partial integration it holds

$$\begin{aligned} &\int_0^1 {}_0d_1 D(\rho)^2 e^{-u_0 tD(\rho)^2} {}_1d_0 \, \mathrm{vol}((1-u_0)\Delta^{k-1}) \; du_0 \\ &= \ {}_0d_1 e^{-tD(\rho)^2} {}_1d_0 + \int_0^1 {}_0d_1 D(\rho)^2 e^{-u_0 tD(\rho)^2} {}_1d_0 \, \mathrm{vol}((1-u_0)\Delta^{k-2}) \; du_0 \end{aligned}$$

since

$$\partial_{u_0} \mathrm{vol}((1-u_0)\Delta^{k-1}) = - \mathrm{vol}((1-u_0)\Delta^{k-2}) \, .$$

Furthermore ${}_0d_1 e^{-tD(\rho)^2} {}_1d_0$ decays exponentially for $t \to \infty$.

Assume now that the claim holds for $n-1$ and any $k \geq n-1$.

Then by partial integration on u_{n-1} the term equals

$$t^n \int_{\Delta^{n-1}} {}_0d_1 D(\rho)^2 e^{-u_0 tD(\rho)^2} {}_1d_0 \ \dots {}_0d_1 \Big[-t^{-1} e^{-xtD(\rho)^2} \, \mathrm{vol}((1 - \sum_{i=0}^{n-2} u_i - x) \Delta^{k-n}) \Big]_0^{u_{n-1}} {}_1d_0 \;\; du_0 \dots du_{n-1}$$

$$+ \quad t^{n-1} \int_{D_n} {}_0d_1 D(\rho)^2 e^{-u_0 tD(\rho)^2} {}_1d_0 \ \dots {}_0d_1 e^{-u_{n-1} tD(\rho)^2} {}_1d_0 \Big(\partial_{u_{n-1}} \mathrm{vol}((1 - \sum_{i=0}^{n-1} u_i) \Delta^{k-n}) \Big) \, du_0 \dots du_{n-1}$$

Note that the first integral vanishes for $x = u_{n-1}$.

We obtain

$$t^{n-1}\int_{D_{n-1}} {}_0d_1D(\rho)^2e^{-u_0tD(\rho)^2}\dots{}_0d_1D(\rho)^2e^{-u_{n-2}tD(\rho)^2}{}_1d_{00}d_{11}d_0\,\mathrm{vol}((1-\sum_{i=0}^{n-2}u_i)\Delta^{k-n})\;du_0\dots du_{n-2}$$

$$-t^{n-1}\int_{D_{n-1}} {}_0d_1D(\rho)^2e^{-u_0tD(\rho)^2}{}_1d_0\dots{}_0d_1e^{-u_{n-1}tD(\rho)^2}{}_1d_0\,\mathrm{vol}((1-\sum_{i=0}^{n-1}u_i)\Delta^{k-n-1})\;du_0\dots du_{n-1}$$

There are $C,\omega > 0$ such that the last term is bounded by

$$Ct^{n-1}\int_0^1 e^{-s\omega t}\,\mathrm{vol}(s\Delta^{n-1})\,\mathrm{vol}((1-s)\Delta^{k-n-1})\;ds\;,$$

hence by an estimate analogous to the one in the proof of the previous lemma it follows that the last term vanishes with $O(t^{-1})$ for $t\to\infty$. Thus it will not contribute to the limit.

By induction the first term converges with $O(t^{-1})$ to

$$\frac{1}{(k-1-(n-1))!}({}_0d_{11}d_0)^n = \frac{1}{(k-n)!}({}_0d_{11}d_0)^n\;.$$

From $(W^*\,\mathrm{d}\,W)^2 = 0$ it follows ${}_0d_{11}d_0 = -{}_0d_0^2$. This shows the assertion of the lemma.

□

4.4.3 The limit of $\mathrm{Tr}_s e^{-A_t^2}$ for $t\to 0$

Recall the definition of $\mathcal{N}$ from §2.5.

Theorem 4.4.8. *It holds*

$$\lim_{t\to0}\mathrm{Tr}_s e^{-A_t^2} = \mathcal{N}$$

and

$$\lim_{t\to0}\mathrm{Tr}_s De^{-A_t^2} = 0\;.$$

Proof. From prop. 4.3.1 it follows

$$\lim_{t\to0}\mathrm{Tr}_s e^{-A_t^2} = \sum_{n=0}^{\infty}(-1)^n\lim_{t\to0}\;t^{\frac{n}{2}}\;\mathrm{Tr}_s W_n(0,t)\;.$$

It holds

$$\mathrm{Tr}_s W_n(0,t) = \sum_{k\in J}\int_{\mathcal{U}_k}\gamma_k(x)\;\mathrm{tr}_s w(0)_t^k(x,x)^n\phi_k(x)\;dx\;.$$

For $k \in \mathbb{Z}/6$ it holds $\mathrm{tr}_s w(\rho)_t^k(x,x)^n = 0$ for all $n \in \mathbb{N}_0$ and $t > 0$ by §4.2.
Furthermore $w(0)_t^\clubsuit(x,y)^n$ vanishes for $n > 0$.
Recall that $\mathcal{U}_\clubsuit$ contains the isolated point $*$. Since $w(0)_t^\clubsuit(x,y)^0$ is the integral kernel of $e^{-tD_N^2}$, it holds in $C(\mathcal{U}_\clubsuit)$ by the local index theorem ([BGV], th.4.2) and by $\mathrm{ch}(E/S) = \mathrm{ch}((\mathbb{C}^+)^d) - \mathrm{ch}((\mathbb{C}^-)^d) = 0$:

$$\lim_{t\to 0} \mathrm{tr}_s w(0)_t^\clubsuit(x,x)^0 = \mathcal{N} 1_*(x) \ .$$

Prop. 4.3.1 also implies

$$\lim_{t\to 0} \mathrm{Tr}_s D e^{-A_t^2} = \sum_{n=0}^{\infty} (-1)^n \lim_{t\to 0} \ t^{\frac{n}{2}} \ \mathrm{Tr}_s D W_n(0,t) \ .$$

Since $DW_0(0,t)$ is odd, its supertrace vanishes. It holds $\mathrm{Tr}_s DW_n(0,t) = 0$ by §4.2. $\square$

4.4.4 $\frac{d}{dt}\mathrm{Tr}_s e^{-A(\rho)_t^2}$ and $\frac{d}{d\rho}\mathrm{Tr}_s e^{-A(\rho)_t^2}$

Lemma 4.4.9. *1. It holds*

$$\frac{d}{dt}\mathrm{Tr}_s e^{-A(\rho)_t^2} = -\mathrm{Tr}_s \frac{dA(\rho)_t^2}{dt} e^{-A(\rho)_t^2} \ .$$

2. It holds

$$\frac{d}{d\rho}\mathrm{Tr}_s e^{-A(\rho)_t^2} = -\,\mathrm{d}\,\mathrm{Tr}_s \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2} \ .$$

Proof. 1) First we calculate $\frac{d}{dt}e^{-A(\rho)_t^2}$:
Consider the holomorphic semigroup $e^{-t'(D(\rho)^2 + z\mathcal{R})}$ depending on the parameter z. By the semigroup law it holds

$$\frac{d}{dt'} e^{-t'(D(\rho)^2+z\mathcal{R})} = -(D(\rho)^2 + z\mathcal{R}) e^{-t'(D(\rho)^2+z\mathcal{R})}$$

and by Duhamel's formula (prop. 5.4.4)

$$\frac{d}{dz} e^{-t'(D(\rho)^2+z\mathcal{R})} = -\int_0^{t'} e^{-(t'-s)(D(\rho)^2+z\mathcal{R})} \mathcal{R} e^{-s(D(\rho)^2+z\mathcal{R})} ds \ .$$

It follows

$$\begin{aligned}
\frac{d}{dt} e^{-A(\rho)_t^2} &= \frac{d}{dt} e^{-t(D(\rho)^2+t^{-1/2}\mathcal{R})} \\
&= \frac{d}{dt'} e^{-t'(D(\rho)^2+t^{-1/2}\mathcal{R})}|_{(t'=t)} - \tfrac{1}{2} t^{-3/2} \frac{d}{dz} e^{-t(D(\rho)^2+z\mathcal{R})}|_{(z=t^{-1/2})} \\
&= -(D(\rho)^2 + t^{-1/2}\mathcal{R}) e^{-t(D(\rho)^2+t^{-1/2}\mathcal{R})} \\
&\quad + \tfrac{1}{2} t^{-3/2} \int_0^t e^{-(t-s)(D(\rho)^2+t^{-1/2}\mathcal{R})} \mathcal{R} e^{-s(D(\rho)^2+t^{-1/2}\mathcal{R})} ds \\
&= -t^{-1} A(\rho)_t^2 e^{-A(\rho)_t^2} + \tfrac{1}{2} t^{-1/2} \int_0^1 e^{-(1-s)A(\rho)_t^2} \mathcal{R} e^{-sA(\rho)_t^2} ds \ .
\end{aligned}$$

Now we prove that for any $n \in \mathbb{N}_0$ and $t_1, t_2 > 0$ there are $r, c, C > 0$ such that

$$|\frac{d}{dt}\mathrm{tr}_s p(\rho)_t(x,x)^n| \le Ce^{-cd(x,M_r)^2}$$

for $t_1 < t < t_2$. Then it follows:

$$\frac{d}{dt}\mathrm{Tr}_s e^{-A(\rho)_t^2} = \mathrm{Tr}_s \frac{d}{dt} e^{-A(\rho)_t^2} \ .$$

By cor. 4.3.3 and the fact that the pointwise supertrace of the integral kernel $(A_t^{Z_k})^2 e^{-(A_t^{Z_k})^2}$ vanishes for $k \in \mathbb{Z}/6$ there are $r, c, C > 0$ such that the pointwise supertrace of the integral kernel of $A(\rho)_t^2 e^{-A(\rho)_t^2}$ can be estimated by $Ce^{-cd(x,M_r)^2}$ for $t_1 < t < t_2$.

The integral kernel of $\int_0^1 e^{-(1-s)A(\rho)_t^2}\mathcal{R}e^{-sA(\rho)_t^2}ds$ is the sum of the terms

$$(-1)^{m+n}t^{\frac{m+n}{2}}\int_0^1 (1-s)^{m/2}s^{n/2}\int_M p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y)p(\rho)_{st}(y,x)^n \ dyds$$

with $m, n \in \mathbb{N}_0$.

It holds

$$\begin{aligned}
&p(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)p(\rho)_{st}(y,x)^n \\
&\quad = \ p(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)(p(\rho)_{st}(y,x)^n - w(\rho)_{st}(y,x)^n) \\
&\qquad +(p(\rho)_{(1-s)t}(x,y)^m - w(\rho)_{(1-s)t}(x,y)^m)\mathcal{R}(y)w(\rho)_{st}(y,x)^n \\
&\qquad +w(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)w(\rho)_{st}(y,x)^n \ .
\end{aligned}$$

By prop. 4.3.1 and the fact that the operator $I_m(\rho,(1-s)t)$ is uniformly bounded on $C_R(M, E\otimes\hat{\Omega}_{\le\mu}\mathcal{A}_i)$ for $0 \le (1-s)t \le t_2$ there are $C, c, r > 0$ such that it holds for all $x \in M$ and $0 \le s \le 1$ and $t_1 \le t \le t_2$:

$$|\int_M p(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)(p(\rho)_{st}(y,x)^n - w(\rho)_{st}(y,x)^n)dy| \le Ce^{-cd(x,M_r)^2} \ .$$

An analogous estimate holds for the second term.

Since it holds $\mathcal{R}|_{\mathcal{U}_\clubsuit} = 0$ and since for $k \in \mathbb{Z}/6$ the section $\mathcal{R}|_{\mathcal{U}_k}$ is of the form $c(e_1)(R\oplus(-R))$, it follows by the results in §4.2:

$$\mathrm{tr}_s w(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)w(\rho)_{st}(y,x)^n = 0 \ .$$

Hence there are $r, c, C > 0$ such that for $x \in M_r$, $0 \le s \le 1$ and $t_1 \le t \le t_2$ it holds

$$(*) \qquad |\mathrm{tr}_s\int_M p(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)p(\rho)_{st}(y,x)^n \ dy| \le Ce^{-cd(x,M_r)^2} \ .$$

From the claim from the beginning of the proof follows.

Now we show

$$\mathrm{Tr}_s \int_0^1 e^{-(1-s)A(\rho)_t^2} \mathcal{R} e^{-sA(\rho)_t^2} ds = \mathrm{Tr}_s \mathcal{R} e^{-A(\rho)_t^2}$$

or equivalently

$$\int_M \int_0^1 (1-s)^{m/2} s^{n/2} \int_M \mathrm{tr}_s p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) p(\rho)_{st}(y,x)^n \; dy ds dx$$

$$= \int_0^1 (1-s)^{m/2} s^{n/2} \int_M \int_M \mathrm{tr}_s p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) p(\rho)_{st}(y,x)^n \; dx dy ds \; .$$

We can interchange the integration over s and x by the estimate $(*)$.

Fix s and t. Consider once more the decomposition of $p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) p(\rho)_{st}(y,x)^n$ from above:

From cor. 4.3.2 it follows that for $\varepsilon > 0$ there are $r, c, C > 0$ such that for all $x, y \in M$ it holds

$$\begin{aligned} & |p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) \big(p(\rho)_{st}(y,x)^n - w(\rho)_{st}(y,x)^n\big)| \\ \leq \;& C\big(1_{[0,\varepsilon]}(d(x,y)) + e^{-cd(x,y)^2} + e^{-cd(x,M_r)^2}\big) e^{-cd(y,M_r)^2} \; . \end{aligned}$$

The second term can be estimated in an analogous manner and the supertrace of the third term vanishes as we saw.

Hence we can interchange dx and dy.

Finally 1) follows from

$$\begin{aligned} \mathrm{Tr}_s \frac{d}{dt} e^{-A(\rho)_t^2} &= \mathrm{Tr}_s(-t^{-1} A(\rho)_t^2 + \tfrac{1}{2} t^{-1/2} \mathcal{R}) e^{-A(\rho)_t^2} \\ &= -\mathrm{Tr}_s \frac{dA(\rho)_t^2}{dt} e^{-A(\rho)_t^2} \end{aligned}$$

2) Since $A(\rho)_t^2 = tD^2 + t\rho[D,K] + t\rho^2 K^2 + \sqrt{t}\mathcal{R}$, it follows from Duhamel's formula (prop. 5.4.4) and the chain rule:

$$\frac{d}{d\rho} e^{-A(\rho)_t^2} = -\int_0^1 e^{-(1-s)A(\rho)_t^2} \frac{dA(\rho)_t^2}{d\rho} e^{-sA(\rho)_t^2} ds \; .$$

Note that the operator $\frac{dA(\rho)_t^2}{d\rho}$ is a finite integral operator whose integral kernel is compactly supported.

By arguments similar to those in the proof of 1) we can prove

$$\frac{d}{d\rho} \mathrm{Tr}_s e^{-A(\rho)_t^2} = \mathrm{Tr}_s \frac{d}{d\rho} e^{-A(\rho)_t^2} = -\mathrm{Tr}_s \frac{dA(\rho)_t^2}{d\rho} e^{-A(\rho)_t^2} \; .$$

Since $A(\rho)_t - \sqrt{t}D(\rho) = W^* \mathrm{d}\, W$ is nilpotent on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, the resolvent sets of $A(\rho)_t$ and $\sqrt{t}D(\rho)$ coincide. Hence by lemma 3.5.10 there is $\lambda \in \mathbb{C}$ such that

$A(\rho)_t \pm \lambda$ has a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\le\mu}\mathcal{A}_i)$. As in cor. 3.5.11 it follows that $A(\rho)_t$ commutes with $e^{-A(\rho)_t^2}$.

It follows

$$\begin{aligned} \frac{dA(\rho)_t^2}{d\rho} e^{-A(\rho)_t^2} &= [A(\rho)_t, \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s \\ &= [W^* \,\mathrm{d}\, W, \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s + \sqrt{t}[D(\rho), \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s \end{aligned}$$

The first term of the last line is an integral operator with integral kernel $W^* d(k) W$ if k is the integral kernel of $W \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2} W^*$. Hence the supertrace of the first term equals

$$\mathrm{d}\,\mathrm{Tr}_s \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2} \ .$$

Now consider the second term. Let P be the orthogonal projection onto the range of K. It is a Hilbert-Schmidt operator. Since $\frac{dA(\rho)_t}{d\rho} = \sqrt{t}K$ it holds

$$\mathrm{Tr}_s[D(\rho), \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s = \mathrm{Tr}_s[D(\rho)P, K e^{-A(\rho)_t^2}]_s \ .$$

Since $D(\rho)P$ and $Ke^{-A(\rho)_t^2}$ are Hilbert-Schmidt operators, the supertrace vanishes by def. 1.3.2. $\square$

Let D_{I_k} be the operator D_I from §3.3.1 with boundary conditions given by the pair $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{k+1 \bmod 3})$.

In §4.3.2 we defined $A_t^{Z_k} = \tilde{W}_k^* \,\mathrm{d}\, \tilde{W}_k + \sqrt{t} D_{Z_k}$ such that $A_t^{Z_k}$ coincides with $A(\rho)_t$ on $\mathcal{U}_k$. There is $U_k \in C^\infty([0,1], M_{2d}(\mathcal{A}_\infty))$ such that $\tilde{W}_k(x_1, x_2) = U_k(x_2) \oplus U_k(x_2)$. Let $A_t^{I_k} := U_k^* \,\mathrm{d}\, U_k + \sqrt{t}\sigma D_{I_k}$.

Lemma 4.4.10. *It holds*

$$\frac{d}{dt} \mathrm{Tr}_s e^{-A(\rho)_t^2} = -\frac{1}{\sqrt{4\pi t}} \sum_{k \in \mathbb{Z}/6} \mathrm{Tr}_\sigma D_{I_k} e^{-(A_t^{I_k})^2} - \frac{1}{2\sqrt{t}} \,\mathrm{d}\, \mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2} \ .$$

Proof. Since $\frac{dA(\rho)_t^2}{dt} = [\frac{dA(\rho)_t}{dt}, A(\rho)_t]_s$ it follows from the previous lemma

$$\frac{d}{dt} \mathrm{Tr}_s e^{-A(\rho)_t^2} = -\mathrm{Tr}_s [\frac{dA(\rho)_t}{dt}, A(\rho)_t]_s e^{-A(\rho)_t^2} \ .$$

Furthermore it holds

$$\begin{aligned} -[\frac{dA(\rho)_t}{dt}, A(\rho)_t]_s e^{-A(\rho)_t^2} &= -[A(\rho)_t, \frac{dA(\rho)_t}{dt} e^{-A(\rho)_t^2}]_s \\ &= -\frac{1}{2\sqrt{t}} [W^* \,\mathrm{d}\, W, D(\rho) e^{-A(\rho)_t^2}]_s - \frac{1}{2}[D(\rho), D(\rho) e^{-A(\rho)_t^2}]_s \ . \end{aligned}$$

The supertrace of the first supercommutator in the last line equals

$$-\frac{1}{2\sqrt{t}}\,\mathrm{d}\,\mathrm{Tr}_s D(\rho)e^{-A(\rho)_t^2} \ .$$

Now consider the second term:

Let $\chi : \mathbb{R} \to [0,1]$ be a smooth function with $\chi(x) = 1$ for $x \le 0$ and $\chi(x) = 0$ for $x \ge 1$. Let $\phi_r : M \to [0,1],\ \phi_r(x) = \chi(d(M_r, x))$ for $r > 0$. It is smooth as well.

It holds

$$\begin{aligned} &-\tfrac{1}{2}\mathrm{Tr}_s[D(\rho), D(\rho)e^{-A(\rho)_t^2}]_s \\ &\quad = \ -\tfrac{1}{2}\lim_{r\to\infty}\Big(\mathrm{Tr}_s\sqrt{\phi_r}D(\rho)^2 e^{-A(\rho)_t^2}\sqrt{\phi_r} + \mathrm{Tr}_s\sqrt{\phi_r}D(\rho)e^{-A(\rho)_t^2}D(\rho)\sqrt{\phi_r}\Big) \end{aligned}$$

Since for $\nu \in \mathbb{N}_0$ the operators $\sqrt{\phi_r}D(\rho)^\nu e^{-A(\rho)_t^2/2}$ and $e^{-A(\rho)_t^2/2}D(\rho)^\nu\sqrt{\phi_r}$ are odd Hilbert-Schmidt operators, it follows

$$\begin{aligned} &-\tfrac{1}{2}\mathrm{Tr}_s[D(\rho), D(\rho)e^{-A(\rho)_t^2}]_s \\ &= \ -\tfrac{1}{2}\lim_{r\to\infty}\Big(\mathrm{Tr}_s e^{-A(\rho)_t^2/2}\phi_r D(\rho)^2 e^{-A(\rho)_t^2/2} - \mathrm{Tr}_s e^{-A(\rho)_t^2/2}D(\rho)\phi_r D(\rho)e^{-A(\rho)_t^2/2}\Big) \\ &= \ \tfrac{1}{2}\lim_{r\to\infty}\mathrm{Tr}_s e^{-A(\rho)_t^2/2}c(d\phi_r)D(\rho)e^{-A(\rho)_t^2/2} \\ &= \ \tfrac{1}{2}\lim_{r\to\infty}\mathrm{Tr}_s c(d\phi_r)D(\rho)e^{-A(\rho)_t^2} \ . \end{aligned}$$

For $r > 0$ we define the function $\chi_r : Z \to \mathbb{R},\ \chi_r(x) := \chi(x_1 - r)$. Cor. 4.3.3 implies

$$\tfrac{1}{2}\lim_{r\to\infty}\mathrm{Tr}_s c(d\phi_r)D(\rho)e^{-A(\rho)_t^2} = \tfrac{1}{2}\lim_{r\to\infty}\sum_{k\in\mathbb{Z}/6}\mathrm{Tr}_s c(d\chi_r)D_{Z_k}e^{-(A_t^{Z_k})^2} \quad .$$

Recall from §4.2 that the integral kernel of $e^{-(A_t^{Z_k})^2}$ is $\sum\limits_{n=0}^{\infty}(-1)^n t^{\frac{n}{2}} p_t^{Z_k}(x,y)^n$ with

$$p_t^{Z_k}(x,y)^n = c(dx_1)^n \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}}\left(p_t^{I_k}(x_2,y_2)^n \oplus (-1)^n p_t^{I_k}(x_2,y_2)^n\right) \ .$$

Hence the integral kernel of $c(d\chi_r)D_{Z_k}e^{-(A_t^{Z_k})^2}$ is

$$\sum_{n=0}^{\infty}(-1)^n t^{\frac{n}{2}}\chi'(x_1 - r)(I\partial_{x_2} - \partial_{x_1})p_t^{Z_k}(x,y)^n \ .$$

An easy computation shows

$$\begin{aligned} &\mathrm{Tr}_s c(d\chi_r)D_{Z_k}e^{-(A_t^{Z_k})^2} \\ &= \ -\frac{1}{\sqrt{4\pi t}}\sum_{n=0}^{\infty} t^{\frac{n}{2}}\int_0^1 \mathrm{tr}_s c(dx_1)^n\big((D_{I_k}p_t^{I_k})(x_2,x_2)^n \oplus (-1)^{n+1}(D_{I_k}p_t^{I_k})(x_2,x_2)^n\big)dx_2 \end{aligned}$$

Comparison with def. 4.1.6 and the subsequent remark yields

$$\mathrm{tr}_s c(dx_1)^n \big((D_{I_k} p_t^{I_k})(x_2,x_2)^n \oplus (-1)^{n+1} (D_{I_k} p_t^{I_k})(x_2,x_2)^n \big) = 2i^n \mathrm{tr}_\sigma \sigma^n (D_{I_k} p_t^{I_k})(x_2,x_2)^n \ .$$

It follows:

$$\begin{aligned} \mathrm{Tr}_s c(d\chi_r) D_{Z_k} e^{-(A_t^{Z_k})^2} &= -\frac{2}{\sqrt{4\pi t}} \sum_{n=0}^{\infty} (-1)^n t^n \int_0^1 \mathrm{tr}(D_{I_k} p_t^{I_k})(x_2,x_2)^{2n} dx_2 \\ &= -\frac{1}{\sqrt{\pi t}} \mathrm{Tr}_\sigma D_{I_k} e^{-(A_t^{I_k})^2} \ , \end{aligned}$$

hence

$$-\tfrac{1}{2} \mathrm{Tr}_s [D(\rho), D(\rho) e^{-A(\rho)_t^2}]_s = - \sum_{k \in \mathbb{Z}/6} \frac{1}{\sqrt{4\pi t}} \mathrm{Tr}_\sigma D_{I_k} e^{-(A_t^{I_k})^2} \ .$$

□

4.4.5 The index theorem

Now we turn back to the notation used in prop. 2.5.1.

Let A_{I_k} be as in the previous section.

Theorem 4.4.11. *It holds*

$$\mathrm{ch}(\mathrm{ind}\, D^+) = -[\sum_{k \in \mathbb{Z}/6} \eta(A_{I_k})] \in H^{dR}_{ev}(\mathcal{A}_\infty) \ .$$

Here we understand $\mathrm{ind}\, D^+$ as a class in $K_0(\mathcal{A}_\infty)$ via the isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{A}_\infty)$ induced by the injection $\mathcal{A}_\infty \hookrightarrow \mathcal{A}$.

Proof. Let $\rho \neq 0$.

In $K_0(\mathcal{A})$ it holds by §2.5:

$$\mathrm{ind}\, D^+ = [\mathrm{Ker}\, D(\rho)^2] - [\mathcal{A}^{\mathcal{N}}] \ .$$

Let P_0 be the projection onto the kernel of $D(\rho)^2$.

From prop. 3.5.6 and prop. 5.3.7 it follows in $K_0(\mathcal{A}_\infty)$:

$$\mathrm{ind}\, D^+ = [\mathrm{Ran}_\infty P_0] - [\mathcal{A}_\infty^{\mathcal{N}}] = [\mathrm{Ran}_\infty W P_0 W^*] - [\mathcal{A}_\infty^{\mathcal{N}}] \ ,$$

hence in $H_{dR}(\mathcal{A}_\infty)$ it holds

$$\mathrm{ch}(\mathrm{ind}\, D^+) = \mathrm{ch}([\mathrm{Ran}_\infty W P_0 W^*]) - \mathcal{N}.$$

By prop. 5.3.7 it holds in $H_{dR}(\mathcal{A}_\infty)$:

$$\begin{aligned}\mathrm{ch}([\mathrm{Ran}_\infty WP_0W^*]) &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\mathrm{Tr}_s(WP_0W^*\,\mathrm{d}\,WP_0W^*)^{2n} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\mathrm{Tr}_s(P_0W^*\,\mathrm{d}\,WP_0)^{2n}\ .\end{aligned}$$

Let $\rho, T > 0$. By theorem 4.4.3 and theorem 4.4.8 it holds in $\overline{\hat\Omega_*(\mathcal{A}_\infty)/[\hat\Omega_*(\mathcal{A}_\infty),\hat\Omega_*(\mathcal{A}_\infty)]_s}$:

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\mathrm{Tr}_s(P_0W^*\,\mathrm{d}\,WP_0)^{2n} - \mathcal{N} &= \lim_{t\to\infty}\mathrm{Tr}_s e^{-A(\rho)_t^2} - \lim_{t\to 0}\mathrm{Tr}_s e^{-A_t^2} \\ &= \int_T^\infty \frac{d}{dt}\mathrm{Tr}_s e^{-A(\rho)_t^2}dt \\ &\quad + \int_0^\rho \frac{d}{d\rho'}\mathrm{Tr}_s e^{-A(\rho')_T^2}d\rho' \\ &\quad + \int_0^T \frac{d}{dt}\mathrm{Tr}_s e^{-A_t^2}dt\ .\end{aligned}$$

The integrals converge by the results of §4.4.4, the estimate of theorem 4.4.3 and theorem 4.4.8, and it holds in $\overline{\hat\Omega_*(\mathcal{A}_\infty)/[\hat\Omega_*(\mathcal{A}_\infty),\hat\Omega_*(\mathcal{A}_\infty)]_s}$:

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\mathrm{Tr}_s(P_0W^*\,\mathrm{d}\,WP_0)^{2n} - \mathcal{N} &= -\sum_{k\in\mathbb{Z}/6}\int_0^\infty \frac{1}{\sqrt{4\pi t}}\mathrm{Tr}_s D_{I_k}e^{-(A_t^{I_k})^2}dt \\ &\quad -\mathrm{d}\int_T^\infty \frac{1}{2\sqrt{t}}\mathrm{Tr}_s D(\rho)e^{-A(\rho)_t^2}dt \\ &\quad -\mathrm{d}\int_0^\rho \mathrm{Tr}_s\frac{dA(\rho')_t}{d\rho'}e^{-A(\rho')_t^2}d\rho' \\ &\quad -\mathrm{d}\int_0^T \frac{1}{2\sqrt{t}}\mathrm{Tr}_s D(0)e^{-A_t^2}dt\end{aligned}$$

The assertion follows since by def. 4.1.6 it holds

$$\eta(A_{I_k}) = \frac{1}{\sqrt{4\pi}}\int_0^\infty t^{-1/2}\mathrm{Tr}_\sigma D_{I_k}e^{-(A_t^{I_k})^2}dt\ .$$

$\square$

Chapter 5

Definitions and Technics

5.1 Hilbert C^*-modules

5.1.1 Bounded operators

Let $\mathcal{A}$ be a unital C^*-algebra with norm $|\cdot|$.

In order to fix notation we recall some facts about Hilbert $\mathcal{A}$-modules:

Definition 5.1.1. *A pre-Hilbert $\mathcal{A}$-module is a right $\mathcal{A}$-module H with an $\mathcal{A}$-valued scalar product $< \ , \ >: H \times H \to \mathcal{A}$, i.e. it holds:*

1. *$< \ , \ >$ is $\mathcal{A}$-linear in the second variable,*
2. *$< x, y > = < y, x >^*$,*
3. *$< x, x > \geq 0$ for all $x \in H$,*
4. *if $< x, x > = 0$, then $x = 0$.*

If H is complete with respect to the norm $\|v\| := | < v, v > |$, then H is called a Hilbert $\mathcal{A}$-module.

The right $\mathcal{A}$-module $\{(a_n)_{n\in\mathbb{N}} \mid \sum\limits_{n=1}^{\infty} a_n^* a_n \text{ converges}\}$ endowed with the $\mathcal{A}$-valued scalar product

$$< (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} > := \sum_{n=1}^{\infty} a_n^* b_n$$

is denoted by $l^2(\mathcal{A})$. It can also be defined with index set $\mathbb{Z}$.

Let M be a measure space and let $< \ , \ >$ be the standard $\mathcal{A}$-valued scalar product on $\mathcal{A}^n$. Then the Hilbert $\mathcal{A}$-module $L^2(M, \mathcal{A}^n)$ is defined in the following way: By

$$< f, g >_{L^2} = \int_M < f(x) g(x) > dx$$

an $\mathcal{A}$-valued scalar product is defined on the quotient of the space of simple functions on M with values in $\mathcal{A}^n$ by those simple functions that vanish on the complement of some set of measure zero. Hence the quotient is a pre-Hilbert $\mathcal{A}$-module. The corresponding Hilbert $\mathcal{A}$-module is $L^2(M, \mathcal{A}^n)$.

Let H be a Hilbert $\mathcal{A}$-module.

A submodule $U \subset H$ is called complemented if $U^\perp = \{x \in H \mid \, < x, u >= 0 \, \forall u \in U\}$ satisfies $U \oplus U^\perp = H$.

For example any projective submodule in H is complemented.

Let H_1, H_2 be Hilbert $\mathcal{A}$-modules. The elements of

$$\begin{aligned} B(H_1, H_2) &= \{T : H_1 \to H_2 \mid \exists T^* : H_2 \to H_1 \text{ with} \\ & \quad < Tv, w >_{H_2} = < v, T^* w >_{H_1} \ \ \forall v \in H_1, \ w \in H_2\} \end{aligned}$$

are called bounded operators from H_1 to H_2. They form a Banach space with respect to the operator norm. With the composition as a product $B(H_1) := B(H_1, H_1)$ is a C^*-algebra. Note that the existence of an adjoint must be required.

A continuous $\mathcal{A}$-module map $K : H_1 \to H_2$ is called compact, if it can be approximated in the operator norm topology by finite rank operators.

If K is compact, then K is adjointable, thus $K \in B(H_1, H_2)$.

A projection onto a projective submodule of H is compact.

If the range of $T \in B(H_1, H_2)$ is complemented, we call its complement the cokernel $\operatorname{Coker} T$. Clearly a necessary condition is that the range of T is closed. The following proposition shows that it is sufficient:

Proposition 5.1.2. *Suppose that $T \in B(H_1, H_2)$ has closed range.*

Then it holds

1. $\operatorname{Ker} T$ *is a complemented submodule of* H_1*,*
2. $\operatorname{Ran} T$ *is a complemented submodule of* H_2*,*
3. $T^* : H_2 \to H_1$ *also has closed range.*

Proof. [La], th. 3.2 □

5.1.2 Fredholm operators

Let H_1, H_2 be Hilbert $\mathcal{A}$-modules which are isomorphic to $l^2(\mathcal{A})$.

There are two different notions of Fredholm operators from H_1 to H_2. In the theory of Fredholm operators developped by Miščenko and Fomenko [MF] it is not required that the operators are adjointable.

Definition 5.1.3. *A Fredholm operator in the sense of Miščenko/Fomenko is an* $\mathcal{A}$*-linear continuous map* $F : H_1 \to H_2$*, not necessarily with adjoint, such that there are decompositions* $H_1 = M_1 \oplus N_1$ *and* $H_2 = M_2 \oplus N_2$ *with the following properties:*

1. N_1, N_2 *are projective* $\mathcal{A}$*-modules and* M_1, M_2 *are closed* $\mathcal{A}$*-modules.*
2. *The operator* F *is diagonal:* $F = F_M \oplus F_N$ *with* $F_M : M_1 \to M_2$ *and* $F_N : N_1 \to N_2$.
3. *The component* $F_M : M_1 \to M_2$ *is an isomorphism.*

The index of F *is defined as*

$$\operatorname{ind} F := [N_1] - [N_2] \in K_0(\mathcal{A}) \ .$$

Proposition 5.1.4. *An* $\mathcal{A}$*-linear continuous map* $F : H_1 \to H_2$ *is Fredholm in the sense of Miščenko/Fomenko if and only if there exists an* $\mathcal{A}$*-linear continuous map* $G : H_2 \to H_1$ *such that* $FG - 1$ *and* $GF - 1$ *are compact.*

Proof. [MF]. □

From the proposition it follows that if F is Fredholm in the sense of Miščenko/Fomenko then for any compact operator K the operator $F + K$ is Fredholm in the sense of Miščenko/Fomenko. Furthermore it holds:

Proposition 5.1.5. *If* $F : H_1 \to H_2$ *is a Fredholm operator in the sense of Miščenko/Fomenko and* $K : H_1 \to H_2$ *is a compact operator, then it holds*

$$\operatorname{ind} F = \operatorname{ind}(F + K) \ .$$

Proof. [MF], lemma 2.3. □

In our context the following definition, discussed in [WO], is more appropriate:

Definition 5.1.6. *An operator* $F \in B(H_1, H_2)$ *is Fredholm if there is a compact operator* $K : H_1 \to H_2$ *such that* $\operatorname{Ran}(F + K)$ *is closed and such that* $\operatorname{Ker}(F + K)$ *and* $\operatorname{Coker}(F + K)$ *are projective* $\mathcal{A}$*-modules.*

The index of F *is*

$$\operatorname{ind} F := [\operatorname{Ker}(F + K)] - [\operatorname{Coker}(F + K)] \ .$$

Proposition 5.1.7. *An operator* $F \in B(H_1, H_2)$ *is Fredholm if and only if there is* $G \in B(H_2, H_1)$ *such that* $FG - 1$ *and* $GF - 1$ *are compact.*

Proof. see [WO], th. 17.1.6. □

Fortunately for adjointable operators both notions are equivalent:

Lemma 5.1.8. *An operator $F \in B(H_1, H_2)$ is Fredholm if and only if it is Fredholm in the sense of Miščenko/Fomenko.*

Proof. If F is Fredholm, let $K \in B(H_1, H_2)$ be such that $\mathrm{Ran}(F+K)$ is closed and $\mathrm{Ker}(F+K)$ and $\mathrm{Coker}(F+K)$ are projective $\mathcal{A}$-modules. By prop. 5.1.2 the kernel of $F+K$ is complemented. Define $N_1 := \mathrm{Ker}(F+K)$, let M_1 be its orthogonal complement, let $N_2 := \mathrm{Coker}(F+K)$ and $M_2 := \mathrm{Ran}(F+K)$. These decompositions show that $F+K$ is Fredholm in the sense of Miščenko/Fomenko.

Assume now that F is Fredholm in the sense of Miščenko/Fomenko and let $H_1 = M_1 \oplus N_1$, $H_2 = M_2 \oplus N_2$ be the corresponding decompositions. Let P be the orthogonal projection onto N_1 and let P_{N_1} be the projection onto N_1 along M_1. Since N_1 is projective, P is compact, hence $P_{N_1} = PP_{N_1}$ is compact and thus FP_{N_1} is compact as well. Then $F - FP_{N_1}$ is an adjointable operator with closed range M_2, with kernel N_1 and cokernel N_2. It follows that F is Fredholm. □

Another important property of Fredholm operators is the following:

Proposition 5.1.9. *If $F : [0,1] \to B(H_1, H_2)$ is a continuous path of Fredholm operators, then the map $[0,1] \to K_0(\mathcal{A})$, $t \mapsto \mathrm{ind}\, F(t)$ is constant.*

Proof. see [WO], prop. 17.3.4. □

5.1.3 Regular operators

In this section we study unbounded operators on Hilbert $\mathcal{A}$-modules.

Let H be a Hilbert $\mathcal{A}$-module with $\mathcal{A}$-valued scalar product $<\ ,\ >$. Let $D : \mathrm{dom}\, D \to H$ be a densily defined operator on H.

Lemma 5.1.10. *If the adjoint D^* of D is densily defined, then D is closable.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{dom}\, D$ such that (f_n, Df_n) converges to $(0, f)$ in $H \oplus H$ for $n \to \infty$. Then for any $g \in \mathrm{dom}\, D^*$ we have

$$< f, g >= \lim_{n\to\infty} < Df_n, g >= \lim_{n\to\infty} < f_n, Dg >= 0.$$

Since $\mathrm{dom}\, D^*$ is dense in H, it follows $f = 0$. □

If D is closed, then

$$< f, g >_D := < f, g > + < Df, Dg >$$

is an $\mathcal{A}$-valued scalar product on $\mathrm{dom}\, D$ with respect to which $\mathrm{dom}\, D$ is a Hilbert $\mathcal{A}$-module. It is denoted by $H(D)$.

Lemma 5.1.11. *Assume that D is closed.*

1. *Suppose that D has a densily defined adjoint D^*. Then* $\operatorname{Ker} D^* = (\operatorname{Ran} D)^\perp$.

2. $\operatorname{Ker} D$ *is complemented in $H(D)$ if and only if* $\operatorname{Ker} D$ *is complemented in H.*

Proof. 1) Since for $f \in \operatorname{Ker} D^*$ and $h \in \operatorname{dom} D$ it holds

$$< f, Dh >=< D^* f, h >= 0 \ ,$$

the $\mathcal{A}$-module $\operatorname{Ker} D^*$ is a submodule of $(\operatorname{Ran} D)^\perp$.

On the other hand the linear functional

$$\operatorname{dom} D \to \mathcal{A}, \ f \mapsto < Df, g >$$

vanishes for $g \in (\operatorname{Ran} D)^\perp$. Thus $g \in \operatorname{dom} D^*$ and $D^* g = 0$.

2) Note first that for $g \in \operatorname{Ker} D$ and $f \in \operatorname{dom} D$ the conditions $< f, g >_D = 0$ and $< f, g >= 0$ are equivalent.

If $\operatorname{Ker} D$ is complemented in $H(D)$, then $\operatorname{Ker} D$ is complemented in H since it holds

$$\begin{aligned} H = \overline{H(D)} &= \overline{(\operatorname{Ker} D)^{\perp_{H(D)}} \oplus \operatorname{Ker} D} \\ &= (\operatorname{Ker} D)^\perp \oplus \operatorname{Ker} D \ . \end{aligned}$$

On the other hand if $\operatorname{Ker} D$ is complemented in H, we can decompose $g \in H(D)$ in a sum $g = g_1 + g_2$ with $g_1 \in \operatorname{Ker} D$ and $g_2 \in (\operatorname{Ker} D)^\perp$. Since $\operatorname{Ker} D \subset H(D)$, it holds $g_2 = g - g_1 \in H(D)$, hence $g_2 \in (\operatorname{Ker} D)^{\perp_{H(D)}}$. □

Recall that D is called regular if it is closed with densily defined adjoint D^* and if $1 + D^* D$ has dense range, or equivalently if it is closed with densily defined adjoint and if its graph is complemented in $H \times H$.

If D is regular, then $1 + D^* D$ has a bounded inverse.

In the following we denote the adjoint of an operator $A \in B(H(D), H)$ by $A^T \in B(H, H(D))$ in order to distinguish it from the adjoint A^* of A as an unbounded operator on H.

Lemma 5.1.12. *Assume that D is closed.*

1. *The operator D is regular if and only if the inclusion $\iota : H(D) \to H$ is in $B(H(D), H)$ and $(1 + D^* D)$ is selfadjoint.*

   Then $\iota^T = (1 + D^* D)^{-1} \in B(H, H(D))$ and $(1 + D^* D)^{-\frac{1}{2}} : H \to H(D)$ is an isometry.

2. *Assume that D is regular and selfadjoint. Then $D \in B(H(D), H)$ and it holds $D^T = D(1 + D^2)^{-1}$.*

Proof. If D is regular, it holds for $v \in H(D)$ and $w \in H$:

$$\begin{aligned} < \iota v, w > &= < v, w > \\ &= < v, (1 + D^*D)(1 + D^*D)^{-1} w > \\ &= < v, (1 + D^*D)^{-1} w > + < Dv, D(1 + D^*D)^{-1} w > \\ &= < v, (1 + D^*D)^{-1} w >_D \ . \end{aligned}$$

This shows $\iota^T = (1 + D^*D)^{-1}$.

Now the converse direction:

Let $v \in H$. Then for any $w \in \mathrm{dom}(1 + D^*D)$ it holds:

$$< v, w > = < \iota^T v, w >_D = < \iota^T v, (1 + D^*D) w > \ .$$

Since $(1+D^*D)$ is selfadjoint, it follows $\iota^T v \in \mathrm{dom}(1+D^*D)$ and $(1+D^*D)\iota^T v = v$. This shows that $(1 + D^*D)$ is surjective and that ι^T is a right inverse of $(1 + D^*D)$. Since $(1 + D^*D)$ is bounded below, it is injective as well. It follows that $(1 + D^*D)$ is invertible and ι^T is its inverse.

The remaining parts are immediate.

□

Proposition 5.1.13. *Let D_0 be a regular selfadjoint operator and assume $D = D_0 + V$ with $V \in B(H)$.*

1. *Then D is closed.*
2. *The identity map induces a continuous isomorphism from $H(D_0)$ to $H(D)$.*
3. *It holds: $D \in B(H(D_0), H)$.*
4. *Suppose D is selfadjoint. Then D is regular.*

Proof. 1) Because of $\mathrm{dom}\, D^{**} = \mathrm{dom}\, D_0^{**} = \mathrm{dom}\, D_0 = \mathrm{dom}\, D$ it holds $D = D^{**}$. Thus D is closed.

Assertion 2) follows from the fact that there is $C > 0$ such that for all $f \in H(D_0)$ it holds:

$$\begin{aligned} \|f\|_D^2 &\leq \|f\|^2 + \|D_0 f\|^2 + \| < Vf, D_0 f > \| + \| < D_0 f, Vf > \| + \|Vf\|^2 \\ &\leq C(\|f\|^2 + \|D_0 f\|^2) + 2\|Vf\| \|D_0 f\| \\ &\leq C\|f\|_{D_0}^2 \ . \end{aligned}$$

We applied Cauchy-Schwarz inequality.

3) By 2) the operator $D : H(D_0) \to H$ is continuous. By the previous lemma the adjoint of $D = D_0 + V\iota : H(D_0) \to H$ is

$$D^T = D_0(1 + D_0^2)^{-1} + (1 + D_0^2)^{-1} V^* : H \to H(D_0) \ .$$

4) By 3) the operator $D + i : H(D_0) \to H$ is an adjointable bounded operator. By [La], lemma 9.7, the range of $D + i$ is closed. Thus it is complemented by prop. 5.1.2. From lemma 5.1.11 it follows that the cokernel of $D + i$ agrees with the kernel of $D - i$. By [La], lemma 9.7, the operator $D - i$ is injective. It follows $\mathrm{Coker}(D + i) = \{0\}$. By [La], lemma 9.8, this shows that D is regular. $\square$

Proposition 5.1.14. *Let D be a regular and selfadjoint operator on H with closed range. Then it holds:*

1. *The cokernel of D exists and $\mathrm{Ker}\, D = \mathrm{Coker}\, D$. In particular $\mathrm{Ker}\, D$ is complemented.*
2. *The $\mathcal{A}$-module $\mathrm{dom}\, D \cap \mathrm{Ran}\, D$ is dense in $\mathrm{Ran}\, D$ and it holds $\mathrm{dom}\, D = \mathrm{Ker}\, D \oplus (\mathrm{dom}\, D \cap \mathrm{Ran}\, D)$, thus $D = 0 \oplus D|_{\mathrm{Ran}\, D}$ and $D|_{\mathrm{Ran}\, D}$ is invertible.*

Proof. 1) By lemma 5.1.12 it holds $D \in B(H(D), H)$. Since the range of D is closed, it is complemented by prop 5.1.2. Its complement is $\mathrm{Coker}\, D$. Since D is selfadjoint, it holds $\mathrm{Ker}\, D = \mathrm{Coker}\, D$ by lemma 5.1.11.

2) Let $P : H \to \mathrm{Ran}\, D$ be the orthogonal projection. From $(1 - P)(\mathrm{dom}\, D) \subset \mathrm{Ker}\, D \subset \mathrm{dom}\, D$ we conclude $P(\mathrm{dom}\, D) \subset \mathrm{dom}\, D$. The assertion follows because $P(\mathrm{dom}\, D)$ is dense in $P(H) = \mathrm{Ran}\, D$. $\square$

We will need the following $\mathbb{Z}/2$-version of the previous proposition:

If $H = H^+ \oplus H^-$ is $\mathbb{Z}/2$-graded, then we call a closed operator D on H even resp. odd if $\mathrm{dom}\, D$ decomposes in $(\mathrm{dom}\, D)^+ \oplus (\mathrm{dom}\, D)^-$ and if the action of D is even resp. odd.

Proposition 5.1.15. *Let H be a $\mathbb{Z}/2$-graded Hilbert $\mathcal{A}$-module and let D be an odd regular selfadjoint operator on H.*

Suppose $D^+ : (\mathrm{dom}\, D)^+ \to H^-$ is surjective.

Then the range of D is closed. It holds $\mathrm{Ker}\, D^+ = \mathrm{Ker}\, D = \mathrm{Coker}\, D = \mathrm{Coker}\, D^-$ and this module is complemented.

Proof. Since D^+ is surjective, D^- is injective and so $\mathrm{Ker}\, D^+ = \mathrm{Ker}\, D$.

Let P_+ be the orthogonal projection onto H^+. Since D is odd it holds $DP_+ = P_+D$.

By lemma 5.1.12 the operator $DP_+ : H(D) \to H$ is adjointable with adjoint $P_+D(D^2 + 1)^{-1}$. It follows that $D^-(D^2 + 1)^{-1} : H^- \to H(D)^+$ is the adjoint of $D^+ : H(D)^+ \to H^-$ since $P^+D|_{H(D)^\pm} = D^\pm$.

Since D^+ is surjective, $\mathrm{Ker}\, D^+$ is complemented in $H(D)^+$ and the adjoint $D^-(D^2 + 1)^{-1} : H^- \to H(D)^+$ has a closed range. Since it holds $D^-(D^2 + 1)^{-1} = (D^2 + 1)^{-\frac{1}{2}} D^-(D^2 + 1)^{-\frac{1}{2}}$ and since $(D^2 + 1)^{-1/2} : H^\pm \to H(D)^\pm$ is an isomorphism by lemma 5.1.12, we conclude that $\mathrm{Ran}\, D^-$ must be closed, too. $\square$

Proposition 5.1.16. *Let D be a regular selfadjoint operator on H.*

1. *For all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the operator $D - \lambda$ is invertible.*
2. *Assume that the range of D is closed and let P be the projection onto the kernel of D. Then there is $c > 0$ such that the spectrum of $(D + P)$ is contained in $\mathbb{R} \setminus] - c, c[$ and the spectrum of D is contained in $(\mathbb{R} \setminus] - c, c[) \cup \{0\}$.*

Proof. This follows from the functional calculus for regular operators ([La], th. 10.9) and from the decomposition in prop. 5.1.14. □

The following criteria for selfadjointness and regularity will be useful:

Lemma 5.1.17. *Let D be a symmetric regular operator such that the ranges of $D+i$ and of $D-i$ are dense in H. Then D is selfadjoint.*

Proof. By [La], lemma 9.7 the operators $D + i$ and of $D - i$ have closed range. It follows that they have a bounded inverse on H. Then they are adjoint to each other, thus D is selfadjoint. □

Lemma 5.1.18. *Assume that D is symmetric and has an inverse $D^{-1} \in B(H)$. Then D is regular.*

Proof. From $D^{-1} \in B(H)$ it follows that the graph of D^{-1} is complemented, hence the graph of D is complemented as well. Since D is symmetric, the adjoint is densily defined. Hence D is regular. □

5.1.4 Decompositions of Hilbert C^*-modules

Let H be a Hilbert $\mathcal{A}$-module with $\mathcal{A}$-valued scalar product $< \ , \ >$. Let $J = \{1, \dots, m\} \subset \mathbb{N}$ resp. $J = \mathbb{N}$. If $J = \mathbb{N}$, then set $m = \infty$.

Definition 5.1.19. *A system $\{f_k\}_{k \in J} \subset H$ is called orthonormal if for all $k, l \in J$ it holds*

$$< f_k, f_l >= \delta_{kl} \ .$$

It is called an orthonormal basis of H if for all $f \in H$ there is $(a_n)_{n \in J} \subset \mathcal{A}$ such that $f = \sum\limits_{n=1}^{m} f_n a_n$.

It holds $a_n =< f_n, f >$, thus the coefficients are uniquely defined by the system.

Lemma 5.1.20. *Let $\{f_k\}_{k \in J}$ be an orthonormal system in H whose span is dense in H. Then it is an orthonormal basis of H and H is isomorphic as a Hilbert $\mathcal{A}$-module to $\mathcal{A}^m$ if $m < \infty$ and to $l^2(\mathcal{A})$ else.*

Proof. Let P_n be the orthogonal projection onto the span of the first n vectors of the system $\{f_k\}_{k \in J}$. On the algebraic span of $\{f_k\}_{k \in J}$ the projection P_n converges strongly to the identity for $n \to \infty$. Since $\|P_n\| = 1$ for all $n \in \mathbb{N}$, it follows that P_n converges strongly to the identity on H.

The isomorphism is given by $f \mapsto (< f_n, f >)_{n \in J}$. □

Lemma 5.1.21. *Let $\{U_i\}_{i\in\mathbb{N}}$ be a family of pairwise orthogonal closed subspaces of H such that $\oplus_{i\in\mathbb{N}}U_i$ is dense in H. Let $\{T_i\}_{i\in\mathbb{N}}$ be a family of operators with $T_i \in B(U_i)$ and assume that there is $c \in \mathbb{R}$ such that for all $i \in \mathbb{N}$ it holds $\|T_i\| \le c$. Then the closure T of the operator $\oplus_{i\in\mathbb{N}}T_i$ is in $B(H)$ and it holds $\|T\| \le c$.*

Proof. The spectral radius of an operator $A \in B(H)$ is denoted by $r(A)$.

Write $T(n)$ for the restriction of T on $\oplus_{i=1}^n U_i$. Its norm is uniformly bounded in n by

$$\|T(n)\|^2 = r(T(n)^*T(n)) = \max_{1\le i\le n} r(T_i^*T_i) = \max_{1\le i\le n} \|T_i^*T_i\| \le c^2 ,$$

For $v \in \operatorname{dom} T$ there is $n \in \mathbb{N}$ such that $v \in \oplus_{i=1}^n U_i$. Then $Tv = T(n)v$ and thus

$$\|Tv\| = \|T(n)v\| \le c\|v\| .$$

It follows that $\oplus_{i\in\mathbb{N}}T_i$ is bounded. Since it is densily defined, its closure is a bounded operator on H. The adjoint is given by the closure of $\oplus_{i\in\mathbb{N}}T_i^*$. □

Corollary 5.1.22. *Let $\{U_i\}_{i\in\mathbb{N}}$ be a family of pairwise orthogonal closed subspaces of H such that $\oplus_{i\in\mathbb{N}}U_i$ is dense in H. Let $\{T_i\}_{i\in\mathbb{N}}$ be a family of operators such that $T_i^{-1} \in B(U_i)$ and assume that there is $c \in \mathbb{R}$ such that for all $i \in \mathbb{N}$ it holds $\|T_i^{-1}\| \le c$. Then the closure T of the map $\oplus_{i\in\mathbb{N}}T_i$ is invertible with inverse in $B(H)$.*

Proof. The operator $\oplus_{i\in\mathbb{N}}T_i^{-1}$ is inverse to $\oplus_{i\in\mathbb{N}}T_i$. It fulfills the conditions of the previous lemma and hence its closure is a bounded operator on H. It is the inverse of the closure of T. □

5.2 Banach space valued functions

5.2.1 Function spaces and tensor products

Let V be a Fréchet space.

A space of V-valued functions can sometimes be described as an ε- or π-tensor product of V with a complex function space. This property garanties that any bounded operator on the complex function space extends to a bounded operator on the corresponding space of V-valued functions.

In the following we list some important examples. The proofs can be found in [Tr] or they are an easy generalisation. If not specified the functions are assumed complex valued.

- Let M be a compact topological space.

  Then the inclusion $C(M) \odot V \to C(M,V)$ extends to an isomorphism

  $$C(M) \otimes_\varepsilon V \cong C(M,V) .$$

For two compact spaces M, N it holds

$$C(M \times N) \cong C(M) \otimes_\varepsilon C(N) .$$

- Let $U \subset \mathbb{R}^n$ be open and precompact. For all $m \in \mathbb{N}_0$ there is a canonical isomorphism
$$C_0^m(U) \otimes_\varepsilon V \cong C_0^m(U, V) .$$

- Let M be a closed smooth manifold. For all $m \in \mathbb{N}_0$ there is a canonical isomorphism
$$C^m(M) \otimes_\varepsilon V \cong C^m(M, V) .$$

- Let $U \subset \mathbb{R}^n$ be open and precompact. Then $C_0^\infty(U)$ is nuclear, in particular there are canonical isomorphisms
$$C_0^\infty(U) \otimes_\pi V \cong C_0^\infty(U) \otimes_\varepsilon V \cong C_0^\infty(U, V) .$$

- Let M be a closed smooth manifold. Then $C^\infty(M)$ is nuclear, in particular there are canonical isomorphisms
$$C^\infty(M) \otimes_\pi V \cong C^\infty(M) \otimes_\varepsilon V \cong C^\infty(M, V) .$$

  For closed smooth manifolds M, N it holds
$$C^\infty(M \times N) \cong C^\infty(M) \otimes C^\infty(N) .$$

- The space of Schwartz functions $\mathcal{S}(\mathbb{R})$ is nuclear, in particular it holds
$$\mathcal{S}(\mathbb{R}) \otimes_\pi V \cong \mathcal{S}(\mathbb{R}) \otimes_\varepsilon V \cong \mathcal{S}(\mathbb{R}, V) .$$

5.2.2 L^2-spaces and integral operators

Let E be Banach space with norm $|\cdot|$. Let End$E$ be the Banach algebra of bounded operators on E. We denote the operator norm on End$E$ by $|\cdot|$ as well.

Definition 5.2.1. *Let M be a measure space and $p \in \mathbb{N}$.*

The Banach space $L^p(M, E)$ is the completion of the quotient of the space of simple E-valued functions on M by the subspace of functions vanishing outside a set of measure zero with respect to the norm

$$\|f\| := \left(\int_M |f(x)|^p dx \right)^{\frac{1}{p}} .$$

In order to avoid confusion we make the following convention: If $E = \mathcal{A}^n$ for a C^*-algebra $\mathcal{A}$, then $L^2(M, E)$ denotes the Hilbert $\mathcal{A}$-module defined in §5.1.1 and not the space just defined. In general they do not coincide.

Lemma 5.2.2. *Let M_1, M_2 be σ-finite measure spaces. Then the map*

$$L^2(M_1 \times M_2, E) \cong L^2(M_1, L^2(M_2, E)),\ \ f \mapsto (x \mapsto f(x, \cdot))$$

is an isometry.

Proof. The lemma follows from Fubini. $\square$

Proposition 5.2.3. *Let M be a measure space.*

Let $k : M \times M \to \mathrm{End}E$ be such that the integral kernel $|k(x,y)|$ defines a bounded operator $|K|$ on $L^2(M)$. Then k defines a bounded operator on $L^2(M,E)$ with norm less than or equal as $\||K|\|$.

Proof. For a simple function $f : M \to E$ it holds $\| \int_M k(x,y)f(y)dy\|_{L^2} \leq \| \int_M |k(x,y)||f(y)|dy\|_{L^2} \leq \||K|\|\ \|f\|_{L^2}$. $\square$

Corollary 5.2.4. *Let M be a measure space.*

There is a continuous map

$$L^2(M \times M, \mathrm{End}E) \to B(L^2(M,E), L^2(M,E))$$

$$k \mapsto \Big(f \mapsto Kf := \int_M k(\ .\ ,y)f(y)dy\Big)$$

with $\|K\| \leq \|k\|$.

Corollary 5.2.5. *The convolution induces a continuous map*

$$L^1(\mathbb{R}^n, \mathrm{End}E) \to B(L^2(\mathbb{R}^n, E)),\ \ f \mapsto (g \mapsto f * g))\ .$$

Proof. The convolution with $f \in L^1(\mathbb{R}^n, \mathrm{End}E)$ is an integral operator with integral kernel $f(x-y)$. For $f \in L^1(\mathbb{R}^n, \mathrm{End}E)$ it holds $|f| \in L^1(\mathbb{R}^n)$, hence the convolution with $|f|$ is bounded on $L^2(\mathbb{R}^n)$. Thus the assertion follows from the previous proposition. $\square$

Lemma 5.2.6. *For any $f \in L^p(\mathbb{R}^n, E)$ the translation*

$$\tau f : \mathbb{R}^n \to L^p(\mathbb{R}^n, E),\ \ y \mapsto \tau_y f$$

with

$$\tau_y f(x) := f(x-y)$$

is continuous map.

Proof. The proof is analogous to the case $E = \mathbb{C}$, see [Co], ch. VII, prop. 9.2. $\square$

5.2.3 Hilbert-Schmidt operators

Let $\mathcal{B}$ be an involutive Banach algebra with unit, and let M be a σ-finite measure space. In this section operators are assumed to be right $\mathcal{B}$-module maps. In particular we identify $\mathrm{End}(\mathcal{B}^n)$ with $M_n(\mathcal{B})$.

Lemma 5.2.7. *Let $k \in L^2(M \times M, M_n(\mathcal{B}))$ and let K be the corresponding integral operator. Then k is uniquely defined by K.*

Proof. It is enough to show that k vanishes in $L^2(M \times M, M_n(\mathcal{B}))$ if $K = 0$.

Let $\mathcal{B}'$ be the topological dual of $\mathcal{B}$. Applying $\lambda \in \mathcal{B}'$ componentwise yields maps $\lambda : M_n(\mathcal{B}) \to M_n(\mathbb{C})$ and $\lambda : \mathcal{B}^n \to \mathbb{C}^n$. Both maps separe points.

It holds for $f \in L^2(M, \mathbb{C}^n)$ almost everywhere:

$$\lambda(\int_M k(x,y)f(y)dy) = \int_M \lambda(k(x,y))f(y)dy = 0 \ .$$

It follows $\lambda(k(x,y)) = 0$ in $L^2(M \times M, M_n(\mathbb{C}))$. Since $\mathcal{B}'$ is separable, the set containing all $(x,y) \in M \times M$ such that there is $\lambda \in \mathcal{B}'$ with $\lambda(k(x,y)) \neq 0$ has measure zero. Outside this set k vanishes. $\square$

Definition 5.2.8. *A Hilbert-Schmidt operator on $L^2(M, \mathcal{B}^n)$ is an operator with an integral kernel in $L^2(M \times M, M_n(\mathcal{B}))$. Let A be a Hilbert-Schmidt operator on $L^2(M, \mathcal{B}^n)$. Its integral kernel in $L^2(M \times M, M_n(\mathcal{B}))$ is denoted by k_A. We define*

$$\|A\|_{HS} := \|k_A\| \ ,$$

where the norm on the right hand side is taken in $L^2(M \times M, M_n(\mathcal{B}))$.

The normed space of Hilbert-Schmidt operators on $L^2(M, \mathcal{B}^n)$ is denoted by $HS(L^2(M, \mathcal{B}^n))$.

Note that $HS(L^2(M, \mathcal{B}^n))$ is a Banach algebra and that the inclusion $HS(L^2(M, \mathcal{B}^n)) \to B(L^2(M, \mathcal{B}^n))$ is bounded. Prop. 5.2.12 below shows that $HS(L^2(M, \mathcal{B}^n))$ is a left $B(L^2(M, \mathcal{B}^n))$-module.

Definition 5.2.9. *Let E be a Banach right $\mathcal{B}$-module with norm $|\cdot|$.*

A $\mathcal{B}$-valued non-degenerated product on E is a sesquilinear map $< \ , \ >: E \times E \to \mathcal{B}$ such that the following properties hold:

1. *$< v, wb >=< v, w > b$ and $< vb, w >= b^* < v, w >$ for all $v, w \in E$, $b \in \mathcal{B}$,*
2. *if $< v, w >= 0$ for all $w \in E$, then $v = 0$,*
3. *there is $C > 0$ such that $| < v, w > | \leq C|v|_E \ |w|_E$ for all $v, w \in E$.*

Definition 5.2.10. *Let $< \ , \ >$ be a $\mathcal{B}$-valued non-degenerated product on E. A bounded operator $T : E \to E$ is said to be adjointable if there is a map $T^* : E \to E$ satisfying*

$$< v, Tw > = < T^* v, w >$$

for all $v, w \in E$.

Lemma 5.2.11. *Let $T : E \to E$ be adjointable. Then it holds:*

1. *The adjoint T^* is unique.*
2. *T^* is a right $\mathcal{B}$-module map.*
3. *T^* is bounded.*
4. *$T^{**} = T$.*
5. *$T^* S^* = (ST)^*$.*

Proof. For the graph $\Gamma(T^*)$ of an adjoint T^* of T it holds

$$\Gamma(T^*) \subset G := \{(x, y) \in E \times E \mid \ < y, w > + < -x, Tw > = 0 \ \forall w \in E\} \ .$$

Let $v \in E$. Then there is a unique $v_1 \in E$ with $(v, v_1) \in G$ since from

$$< v_1, w > = < v, Tw > = < v_2, w > \ \ \forall w \in E$$

it follows $< v_1 - v_2, w > = 0$ for all $w \in E$ and therefore $v_1 = v_2$. This shows $\Gamma(T^*) = G$.

2) If $(x, y), (v, w) \in \Gamma(T^*)$ and $b \in \mathcal{B}$, then $(xb + v, yb + w) \in \Gamma(T^*)$ by the proof of 1).

3) Since $\Gamma(T^*)$ is closed, the operator T^* is bounded.

4) From 1) it follows $\Gamma(T^{**}) = \Gamma(T)$.

5) It holds $< (ST)^* v, w > = < v, STw > = < S^* v, Tw > = < T^* S^* v, w >$. □

On $\mathcal{B}^n$ there is a standard $\mathcal{B}$-valued non-degenerated product, namely

$$< v, w > := \sum_{i=1}^{n} v_i^* w_i \ .$$

All elements of $M_n(\mathcal{B})$ are adjointable and taking the adjoint is a bounded linear map.

Furthermore there is a standard $\mathcal{B}$-valued non-degenerated product on $L^2(M, \mathcal{B}^n)$ defined by

$$< f, g >_{L^2} := \int_M < f(x), g(x) > dx \ .$$

We check condition 2):

We use that any $\lambda \in \mathcal{B}'$ induces a map $\lambda : \mathcal{B}^n \to \mathbb{C}^n$ by componentwise application and that this map separes points in $\mathcal{B}^n$.

If $f \in L^2(M, \mathcal{B}^n)$ with $< f, g >_{L^2} = 0$ for all $g \in L^2(M, \mathcal{B}^n)$, then in particular it holds for all $g \in L^2(M, \mathbb{C}^n)$ and $\lambda \in \mathcal{B}'$:

$$\int_M \lambda(f(x)^*) g(x) dx = 0 \ ,$$

hence $\lambda(f(x))$ equals zero outside a set of measure zero. Since $\mathcal{B}'$ is separable, it follows $f = 0$ in $L^2(M, \mathcal{B}^n)$.

All operators in $HS(L^2(M, \mathcal{B}^n))$ are adjointable: If $A \in HS(L^2(M, \mathcal{B}^n))$, then A^* is an integral operator whose integral kernel is $k_{A^*}(x, y) = k_A(y, x)^*$, in particular it is in $L^2(M \times M, M_n(\mathcal{B}))$. Taking the adjoint is a bounded map on $HS(L^2(M, \mathcal{B}^n))$.

Proposition 5.2.12. *1. Let $A \in B(L^2(M, \mathcal{B}^n))$, $K \in HS(L^2(M, \mathcal{B}^n))$. Then it holds $AK \in HS(L^2(M, \mathcal{B}^n))$.*

Furthermore there is $C > 0$ such that

$$\|AK\|_{HS} \leq C \|A\| \|K\|_{HS}$$

for all $A \in B(L^2(M, \mathcal{B}^n))$, $K \in HS(L^2(M, \mathcal{B}^n))$.

2. It holds $KA \in HS(L^2(M, \mathcal{B}^n))$ for any adjointable bounded operator A on $L^2(M, \mathcal{B}^n)$ and any $K \in HS(L^2(M, \mathcal{B}^n))$. Furthermore there is $C > 0$ with

$$\|KA\|_{HS} \leq C \|A^*\| \|K\|_{HS}$$

for any such A and K.

Proof. 1) There is an isomorphism

$$L^2(M \times M, M_n(\mathcal{B})) \cong L^2(M \times M, \mathcal{B}^n)^n$$

that is equivariant with respect to the left $M_n(\mathcal{B})$-action on both spaces. Furthermore the map

$$L^2(M \times M, \mathcal{B}^n) \to L^2(M, L^2(M, \mathcal{B}^n)), \ k \mapsto (y \mapsto k(\cdot, y))$$

is an isomorphism by lemma 5.2.2. The operator A induces a bounded map on $L^2(M, L^2(M, \mathcal{B}^n))$, namely $k \mapsto (y \mapsto Ak(\cdot, y))$, clearly its norm is less than or equal as the norm of A on $L^2(M, \mathcal{B}^n)$.

2) The map is a composition of the following maps on $HS(L^2(M, \mathcal{B}^n))$:

$$K \overset{*}{\mapsto} K^* \overset{A^*}{\mapsto} A^* K^* = (KA)^* \overset{*}{\mapsto} KA \ .$$

By 1) and the fact that taking the adjoint is bounded on $HS(L^2(M, \mathcal{B}^n))$ these maps are bounded. $\square$

Note that the integral kernel of $(TA)^*$ is $A^*_x k_{T^*}(x,y)$. By the suffix we indicate on which variable A^* acts.

Definition 5.2.13. *1. Let*

$$< \ , \ >: \mathcal{B}^n \times \mathcal{B}^n \to \mathcal{B}$$

be as above. If $e \in \mathcal{B}^n$ then define

$$e^* : \mathcal{B}^n \to \mathcal{B}, \ v \mapsto < e, v > \ .$$

2. An integral operator A on $L^2(M,\mathcal{B}^n)$ is called finite if there is $k \in \mathbb{N}$ and there are functions $f_j, h_j \in L^2(M,\mathcal{B}^n), \ j = 1 \dots k$, such that

$$k_A(x,y) = \sum_{j=1}^{k} f_j(x) h_j(y)^* \ .$$

5.2.4 Pseudodifferential operators

Let E be a Banach space.

Let U be an open precompact subset of $\mathbb{R}^n$. Recall the notion of a symbol of order m on U:

Definition 5.2.14. *A function $a \in C^\infty(U \times \mathbb{R}^n, M_l(\mathbb{C}))$ is called a symbol of order $m \in \mathbb{R}$, if it is compactly supported in the first variable and if for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ the expressions*

$$\sup_{x \in U, \ \xi \in \mathbb{R}^n} (1+|\xi|)^{-m+|\beta|} |\partial_{x^\alpha} \partial_{\xi^\beta} a(x,\xi)|$$

are finite.

These are seminorms on the space $S^m(U, M_l(\mathbb{C}))$ of symbols of order m on U that turn $S^m(U, M_l(\mathbb{C}))$ into a Fréchet space.

In order to simplify formula involving Fourier transform we rescale the Lebesgue measure on $\mathbb{R}^n$ by setting $d'x := (2\pi)^{\frac{n}{2}} dx$.

We consider $L^2(U, E^l)$ as a subspace of $L^2(\mathbb{R}^n, E^l)$.

Note that the Fourier transform is bounded from $L^1(\mathbb{R}^n, E)$ to $C_0(\mathbb{R}^n, E)$.

A symbol $a \in S^m(U, M_l(\mathbb{C}))$ defines a continuous operator

$$\mathrm{Op}(a) : C_c^\infty(U, E^l) \to C_c^\infty(U, E^l), \ (\mathrm{Op}(a)f)(x) = \int_{\mathbb{R}^n} e^{ix\xi} a(x,\xi) \hat{f}(\xi) \ d'\xi \ .$$

An operator defined by a symbol of order m is called a pseudodifferential operator of order m.

Lemma 5.2.15. *1. For $m < -\frac{n}{2}$ and $a \in S^m(U, M_l(\mathbb{C}))$ the operator $\operatorname{Op}(a)$ extends to a bounded operator on $L^2(U, E^l)$ and the map*

$$\operatorname{Op} : S^m(U, M_l(\mathbb{C})) \to B(L^2(U, E^l))$$

is continuous.

2. Let $m < -\frac{n}{2}$ and $\nu, k \in \mathbb{N}_0$ with $k < -\frac{n}{2} - m$. Then for $a \in S^m(U, M_l(\mathbb{C}))$ the operators

$$\operatorname{Op}(a) : C_0^\nu(U, E^l) \to C_0^{\nu+k}(U, E^l)$$

and

$$\operatorname{Op}(a) : L^2(U, E^l) \to C_0^k(U, E^l)$$

are continuous.

Proof. 1) Let $m < -\frac{n}{2}$.
The Fouriertransform induces a bounded map

$$S^m(U, M_l(\mathbb{C})) \to C_c^\infty(U, L^2(\mathbb{R}^n, M_l(\mathbb{C}))),\ a \mapsto \big(x \mapsto \hat{a}(x, \cdot)\big) \ .$$

For $a \in S^m(U, M_l(\mathbb{C}))$ and $f \in L^2(U, E^l)$ we define

$$(\operatorname{Op}(a)f)(x) := \int_{\mathbb{R}^n} \hat{a}(x, z) f(-x - z) d'z \ .$$

The map

$$\mathbb{R}^n \to L^2(\mathbb{R}^n, E^l),\ x \mapsto (z \mapsto f(-x - z))$$

is continuous by lemma 5.2.6, hence the function $\operatorname{Op}(a)f$ is in $C_0(U, E^l)$ and satisfies

$$\| \operatorname{Op}(a)f\|_{C_0} \le \sup_{x \in U} \|\hat{a}(x, \cdot)\|_{L^2} \|f\|_{L^2} \ .$$

Since U is precompact, the inclusion $C_0(U, E^l) \hookrightarrow L^2(U, E^l)$ is well-defined and continuous. Hence $\operatorname{Op}(a)$ is a bounded operator on $L^2(U, E^l)$. Furthermore the estimate shows that the map

$$S^m(U, M_l(\mathbb{C})) \to B(L^2(U, E^l)),\ a \mapsto \operatorname{Op}(a)$$

is continuous.
2) First let $m < -\frac{n}{2}$ and $k = 0$.
If $f \in C_0^\nu(U, E^l)$ then $x \mapsto (z \mapsto f(-x - z))$ is in $C_0^\nu(\mathbb{R}^n, L^2(\mathbb{R}^n, E^l))$.
It follows as above that $\operatorname{Op}(a)f$ is in $C_0^\nu(U, E^l)$ and that it holds

$$\| \operatorname{Op}(a)f\|_{C^\nu} \le C \sup_{|\alpha| \le \nu} \sup_{x \in U} \|\partial_x^\alpha \hat{a}(x, \cdot)\|_{L^2} \|f\|_{C^\nu} \ .$$

We go on by induction on k: Assume the assertion holds for $k-1$ and all $a \in S^m(U, M_l(\mathbb{C}))$ with $m < -\frac{n}{2} - k + 1$.

We prove the assertion for k and $a \in S^m(U, M_l(\mathbb{C}))$ with $m < -\frac{n}{2} - k$:

If $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$, it holds for $f \in C_0^\infty(U, E^l)$:

$$\begin{aligned}
&\partial_x^\alpha \int_{\mathbb{R}^n} \hat{a}(x,z) f(-x-z) d'z \\
&= \int_{\mathbb{R}^n} \partial_x^\alpha \hat{a}(x,z) f(-x-z) d'z + \int_{\mathbb{R}^n} \hat{a}(x,z) \partial_x^\alpha f(-x-z) d'z \\
&= \int_{\mathbb{R}^n} \partial_x^\alpha \hat{a}(x,z) f(-x-z) d'z + \int_{\mathbb{R}^n} \hat{a}(x,z) \partial_z^\alpha f(-x-z) d'z \\
&= \int_{\mathbb{R}^n} \partial_x^\alpha \hat{a}(x,z) f(-x-z) d'z - \int_{\mathbb{R}^n} \partial_z^\alpha \hat{a}(x,z) f(-x-z) d'z \ .
\end{aligned}$$

The last step is justified since the symbols $\xi^\alpha a(x,\xi)$ and $\partial_x^\alpha a(x,\xi)$ are in $S^{m+1}(U, M_l(\mathbb{C}))$, thus by $m+1 < -\frac{n}{2}$ the functions

$$x \mapsto (z \mapsto \partial_z^\alpha \hat{a}(x,z))$$

and

$$x \mapsto (z \mapsto \partial_x^\alpha \hat{a}(x,z))$$

are both in $C_c^\infty(U, L^2(\mathbb{R}^n, M_l(\mathbb{C})))$.

It follows that the map

$$f \mapsto \partial^\alpha(\mathrm{Op}(a)f)$$

is a pseudodifferential operator of degree $m + |\alpha| = m+1$. By induction it is a bounded operator from $C^\nu(U, E^l)$ to $C^{\nu+k-1}(U, E^l)$. Since this holds for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$, it follows that $\mathrm{Op}(a)$ is continuous from $C^\nu(U, E^l)$ to $C^{\nu+k}(U, E^l)$.

From an analogous induction argument it follows that $\mathrm{Op}(a)$ is continuous from $L^2(U, E^l)$ to $C_0^k(U, E^l)$. For $k=0$ this was proven in 1). $\square$

5.3 Projective systems and function spaces

The projective systems $(\mathcal{A}_i)_{i \in \mathbb{N}_0}$ and $(\hat{\Omega}_{\le\mu}\mathcal{A}_i)_{i,\mu \in \mathbb{N}_0}$ from §1.3.3 and §1.3.4 induce projective systems of spaces $\big(L^2(M, \mathcal{A}_i^l)\big)_{i \in \mathbb{N}_0}$ and $\big(L^2(M, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)\big)_{i,\mu \in \mathbb{N}_0}$.

Recall our convention fixed in §5.2.2: The space $L^2(M, \mathcal{A}^l)$ is the Hilbert $\mathcal{A}$-module defined in §5.1.1. For $\mu \in \mathbb{N}_0$ and $i \in \mathbb{N}$ the space $L^2(M, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)$ was defined in §5.2.2.

In the following we investigate the behaviour some particular classes of operators under the projective limit.

5.3.1 Integral operators

Hilbert-Schmidt operators on $L^2(M, \mathcal{A}_i^l)$ have the property that they extend to bounded operators on $L^2(M, (\hat{\Omega}_{\le\mu}\mathcal{A}_j)^l)$ for all $j \in \mathbb{N}_0$ with $j \le i$ and all $\mu \in \mathbb{N}_0$. We investigate how the spectrum depends on j, μ.

A result in this direction was proved by Lott ([Lo], §6.1.) for the case that M is a closed manifold. We extend Lott's method in order to obtain an analogous result for certain non-compact manifolds M with boundary (in particular for the manifold defined in §1.1).

Let $[0,1]^n$ be endowed with a measure of the form hdx where h is a positive continuous function on $[0,1]^n$ and dx is the Lebesgue measure.

In the proof we use that there exists a Schauder basis of $C([0,1]^n)$ which is orthonormal in $L^2([0,1]^n)$ (here and in the following $L^2([0,1]^n$ is defined with respect to hdx). For $h = 1$ a Franklin system [Se] yields such a basis $\{f_n\}_{n\in\mathbb{N}}$, then for general h the system $\{h^{-\frac{1}{2}} f_n\}_{n\in\mathbb{N}}$ is one.

Proposition 5.3.1. *1. Let hdx be a measure on $[0,1]^n$ as above.*

Let $k \in C([0,1]^n \times [0,1]^n, M_l(\mathcal{A}_i))$ and let K be the corresponding integral operator.

Assume that $1-K$ is invertible in $B(L^2([0,1]^n, \mathcal{A}^l))$. Then the operator $1-K : L^2([0,1]^n, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l) \to L^2([0,1]^n, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)$ is invertible.

2. Let M be a Riemannian manifold of dimension n, possibly with boundary. Suppose there is an exaustion $\{K_m\}_{m\in\mathbb{N}}$ of M such that K_m is diffeomorphic to $[0,1]^n$ for any $m \in \mathbb{N}$. Let $k \in L^2(M \times M, M_l(\mathcal{A}_i)) \cap C(M \times M, M_l(\mathcal{A}_i))$ and assume furthermore that $x \mapsto k(x,\cdot)$ and $y \mapsto k(\cdot,y)$ are in $C(M, L^2(M, M_l(\mathcal{A}_i)))$.

Then it holds for the corresponding integral operator K:

If $1-K$ is invertible in $B(L^2(M, \mathcal{A}^l))$, then $1-K$ is invertible in $B(L^2(M, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l))$.

Proof. 1) Choose a basis of $C([0,1]^n)$ which is orthonormal with respect to hdx and let P_N denote the projection onto the first N basis vectors. It is an integral operator with integral kernel in $L^2([0,1]^n \times [0,1]^n)$, thus it acts continuously on $L^2([0,1]^n, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)$.

We decompose $L^2([0,1]^n, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)$ into the direct sum

$$P_N L^2([0,1]^n, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l) \oplus (1-P_N) L^2([0,1]^n, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)$$

and write

$$1-K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the decomposition.

If we find N such that d is invertible on $(1-P_N)L^2([0,1]^n,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)$ and prove that then $a-bd^{-1}c$ is invertible on $P_N L^2([0,1]^n,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)$, we can conclude that $(1-K)$ is invertible by the equality

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a-bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix} .$$

First we show that

$$d = (1-P_N)(1-K)(1-P_N)$$

is invertible for N big enough. By prop. 5.2.12 the operator $(1-P_N)K(1-P_N)$ is a Hilbert-Schmidt operator. Furthermore its integral kernel is continuous.

For $N\to\infty$ the projections P_N converge strongly to the identity on $C([0,1]^n)$. By

$$C([0,1]^n\times[0,1]^n, M_l(\mathcal{A}_i)) = C([0,1]^n)\otimes_\varepsilon C([0,1]^n)\otimes_\varepsilon M_l(\mathcal{A}_i) ,$$

the projection P_N acting on the first factor of $C([0,1]^n\times[0,1]^n, M_l(\mathcal{A}_i))$ converges strongly to the identity as well. It follows that there is N such that the norm of the integral kernel of $(1-P_N)K(1-P_N)$ is smaller than $\frac{1}{2}$ in $C([0,1]^n\times[0,1]^n, M_l(\mathcal{A}_i))$.

For that N the series

$$(1-P_N)+\sum_{\nu=1}^{\infty}((1-P_N)K(1-P_N))^\nu$$

converges as a bounded operator on $(1-P_N)L^2([0,1]^n,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)$ and inverts d. Hence $a-bd^{-1}c$ is well-defined.

Via the basis we identify $a-bd^{-1}c$ with an element of $M_{Nl}(\mathcal{A}_i)$. Since $1-K$ is invertible on $L^2(M,\mathcal{A}^l)$, the matrix $a-bd^{-1}c$ is invertible in $M_{Nl}(\mathcal{A})$. By lemma 1.3.4 it follows that $a-bd^{-1}c$ is invertible in $M_{Nl}(\mathcal{A}_i)$ as well.

2): Let $m\in\mathbb{N}$ be such that

$$\|(1-1_{K_m}(x))k(x,y)(1-1_{K_m}(y))\|_{L^2(M\times M,M_l(\mathcal{A}_i))} \le \frac{1}{2} .$$

Write

$$(1-K) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the decomposition

$$L^2(M,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l) = 1_{K_m}L^2(M,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)\oplus(1-1_{K_m})L^2(M,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l) .$$

By the choice of K_m the entry $d = 1-(1-1_{K_m})K(1-1_{K_m})$ is invertible on $(1-1_{K_m})L^2(M,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)$.

We prove that $a-bd^{-1}c$ is invertible on $L^2(K_m,(\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)$ and then the assertion follows as in the proof of 1).

On $L^2(K_m, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ it holds

$$a - bd^{-1}c = 1_{K_m} - (1_{K_m}K1_{K_m} + bd^{-1}c)$$

and $1_{K_m}K1_{K_m} + bd^{-1}c$ is an integral operator on $L^2(K_m, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ with continuous integral kernel: The integral kernel of b is $1_{K_m}(x)k(x,y)(1 - 1_{K_m}(y))$ and $x \mapsto 1_{K_m}(x)k(x,\cdot)(1 - 1_{K_m})$ is in $C(K_m, L^2(M, M_l(\mathcal{A}_i)))$. For the integral kernel $(1 - 1_{K_m}(x))k(x,y)1_{K_m}(y)$ of c it holds $y \mapsto (1 - 1_{K_m})k(\cdot,y)1_{K_m}(y) \in C(K_m, L^2(M, M_l(\mathcal{A}_i)))$. It follows that $bd^{-1}c$ is an integral operator with continuous kernel on $K_m \times K_m$. Clearly the integral kernel of $1_{K_m}K1_{K_m}$ is continuous as well.

Since $a - bd^{-1}c$ is invertible on $L^2(K_m, \mathcal{A}^l)$ and since the measure on K_m pulled back by an orientation preserving diffeomorphism $[0,1]^n \to K_m$ is of the form hdx, we conclude by 1) that $a - bd^{-1}c$ is invertible on $L^2(K_m, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ as well. □

Corollary 5.3.2. *Let k be an integral kernel as in part 2) of the proposition and let K be the corresponding integral operator. Then for $\lambda \in \mathbb{C}^*$ the operator $K - \lambda$ is invertible on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ if and only if $K - \lambda$ is invertible on $L^2(M, \mathcal{A}^l)$.*

Proof. For $\lambda \in \mathbb{C} \setminus \{0\}$ the integral kernel k/λ fulfills the conditions of the lemma. Thus if $\lambda - K = \lambda(1 - K/\lambda)$ is invertible on $L^2(M, \mathcal{A}^l)$ then $\lambda - K$ is invertible on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ as well. □

5.3.2 Projections and the Chern character

Proposition 5.3.3. *Let $\{e_i\}_{i\in\mathbb{N}}$ be the standard basis of $l^2(\mathcal{A})$. Let M be a closed and N a projective submodule of $l^2(\mathcal{A})$ such that $l^2(\mathcal{A}) = M \oplus N$. Let P be the projection onto N along M and let P_n be the orthogonal projection onto $L_n :=$* $\mathrm{span}\{e_i \mid i = 1, \dots, n\}$.

For all $n \in \mathbb{N}$ with

$$\|P(1 - P_n)\| \leq \frac{1}{2}$$

it holds:

(i) The $\mathcal{A}$-module $N' := P_n(N)$ is projective and the maps

$$P_n : N \to N' \text{ and } P : N' \to N$$

are isomorphisms.

(ii) $l^2(\mathcal{A}) = M \oplus N'$.

Note that there exists $n \in \mathbb{N}$ with $\|P(1 - P_n)\| \leq \frac{1}{2}$ since P is a compact operator.

Proof. From $\|P(1-P_n)\| \leq \frac{1}{2}$ it follows

$$\|1_N - (PP_n)|_N\| \leq \frac{1}{2} \ .$$

By that $(PP_n)|_N : N \to N$ is invertible and thus for any finite set G of generators of N the set $PP_n(G)$ generates N as well.

The module $N' := P_n(N)$ is projective. Furthermore, since $(PP_n)|_N$ is an isomorphism, the maps

$$P_n : N \to N' \text{ and } P : N' \to N$$

are isomorphisms as well.

It remains to show $N' \oplus M = l^2(\mathcal{A})$.

The intersection $N' \cap M$ is trivial: For $x \in N' \cap M$ it holds $Px = 0$, but as $P : N' \to N$ is an isomorphism, x must be zero.

Let now $x \in l^2(\mathcal{A})$. Since $PP_n : N \to N$ is invertible, there is $y \in N$ such that $Px = PP_n y$.

Then it holds

$$\begin{aligned} x &= (1-P)x + Px \\ &= (1-P)x + PP_n y \\ &= (1-P)x + P_n y - (1-P)P_n y \\ &= (1-P)(x - P_n y) + P_n y \ . \end{aligned}$$

Since $(1-P)(x-P_n y) \in M$ and $P_n y \in N'$ it follows $x \in M \oplus N'$. □

Definition 5.3.4. *Let V be a $\mathbb{Z}/2$-graded finite dimensional vector space. Let K be an integral operator on $L^2(M, V \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ with integral kernel $k : M \times M \to \mathrm{End}(V) \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$. Then $\mathrm{d}(K)$ is the integral operator on $L^2(M, V \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ with integral kernel $\mathrm{d}(k(x,y))$. (The action of d on $\mathrm{End}(V) \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ was described in §1.3.2.)*

Note that if K is of degree n with respect to the $\mathbb{Z}/2$-grading on $L^2(M, V \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, then it holds

$$\mathrm{d}(Kf) = \mathrm{d}(K)f + (-1)^n K \,\mathrm{d}\, f \ .$$

Lemma 5.3.5. *Let M be a σ-finite measure space.*

Let P be a Hilbert-Schmidt operator on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ with integral kernel in $L^2(M \times M, M_l(\mathcal{A}_i))$ and assume that $P^2 = P$. Then it holds

$$P \,\mathrm{d}\, P \,\mathrm{d}\, P = P(\mathrm{d}(P))^2 \ .$$

Proof. As for matrices (see the beginning of the proof of prop. 1.3.3) it holds $(\mathrm{d}\, P) = P(\mathrm{d}\, P) + (\mathrm{d}\, P)P$ and $P(\mathrm{d}\, P)P = 0$.

It follows

$$\begin{aligned} P \,\mathrm{d}\, P \,\mathrm{d}\, P &= P(\mathrm{d}\, P)\,\mathrm{d}\, P = P(\mathrm{d}\, P)^2 P \\ &= P(\mathrm{d}\, P)(\mathrm{d}\, P) - P(\mathrm{d}\, P)P(\mathrm{d}\, P) \\ &= P(\mathrm{d}(P))^2. \end{aligned}$$

□

We introduced the notion of trace class operators in §1.3.2. For the proof of the next proposition we have to define a more general trace:

If A, B are Hilbert-Schmidt operators on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ where M is a σ-finite measure space, then we define

$$\mathrm{Tr}(A,B) := \int_M \mathrm{tr} k_A(x,y) k_B(y,x)\; dydx \in \hat{\Omega}_{\leq\mu}\mathcal{A}_i/\overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]}\ .$$

It holds

$$|\mathrm{Tr}(A,B)| \leq \|A\|_{HS}\|B\|_{HS}$$

and

$$\mathrm{Tr}(A,B) = \mathrm{Tr}(B,A)\ .$$

If A, B are as in §1.3.2 such that in particular AB is trace class, then it holds $\mathrm{Tr}(A,B) = \mathrm{Tr} AB$.

Lemma 5.3.6. *Let $P : [0,1] \to HS(L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l))$ be a differentiable path of Hilbert-Schmidt operators with integral kernels in $L^2(M \times M, M_l(\mathcal{A}_i))$, and assume $P(t)^2 = P(t)$ for any $t \in [0,1]$.*

Then for any $k \in \mathbb{N}_0$ it holds in $\hat{\Omega}_{\leq\mu}\mathcal{A}_i/\overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]}$:

$$\begin{aligned} &\mathrm{Tr}\big(P(1), (\mathrm{d}\, P(1))^{2k}\big) - \mathrm{Tr}\big(P(0), (\mathrm{d}\, P(0))^{2k}\big) \\ &= \tfrac{1}{2}\,\mathrm{d}\big[\mathrm{Tr}\big(P(1), (\mathrm{d}\, P(1))^{2k-1}\big) - \mathrm{Tr}\big(P(0), (\mathrm{d}\, P(0))^{2k-1}\big)\big]\ . \end{aligned}$$

Proof. As for matrices (see prop. 1.3.3). □

If S, T are Hilbert-Schmidt operators on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ that restricts to Hilbert-Schmidt operators $S_{\nu,j}, T_{\nu,j}$ on $L^2(M, (\hat{\Omega}_{\leq\nu}\mathcal{A}_j)^l)$ for all $j, \nu \in \mathbb{N}$ with $j \geq i$ and $\nu \geq \mu$ then we define the trace of (S,T) in $\hat{\Omega}_*(\mathcal{A}_\infty)/\overline{[\hat{\Omega}_*(\mathcal{A}_\infty), \hat{\Omega}_*(\mathcal{A}_\infty)]_s}$ by $\varprojlim_{\nu,j} \mathrm{Tr}(S_{\nu,j}, T_{\nu,j})$.

It is denoted by $\mathrm{Tr}(S,T)$. This notation has still another meaning: namely the trace of (S,T) in $\hat{\Omega}_{\leq\mu}\mathcal{A}_i/\overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]_s}$. The context will decide what is meant. An analogous convention is made for the trace of trace class operators.

Proposition 5.3.7. *Let M be a Riemannian manifold of dimension d, possibly with boundary. Suppose that there is an exaustion $\{K_m\}_{m\in\mathbb{N}}$ of M such that K_m is diffeomorphic to $[0,1]^d$ for all $m \in \mathbb{N}$.*

Let $P \in B(L^2(M,\mathcal{A}^l))$ be a projection onto a projective submodule of $L^2(M,\mathcal{A}^l)$. Assume further that for any $i \in \mathbb{N}$ it restricts to a bounded projection on $L^2(M,\mathcal{A}_i^l)$ and that $P(L^2(M,\mathcal{A}_i^l)) \subset C(M,\mathcal{A}_i^l)$. Let

$$\operatorname{Ran}_\infty P := \bigcap_{i\in\mathbb{N}} P(L^2(M,\mathcal{A}_i^l)) \ .$$

Then the following assertions hold:

(i) The projection P is a Hilbert-Schmidt operator of the form $\sum_{j=1}^m f_j(x)h_j(x)^*$ with $f_j, h_j \in \operatorname{Ran}_\infty P$.

(ii) The intersection $\operatorname{Ran}_\infty P$ is a projective $\mathcal{A}_\infty$-module. The classes $[\operatorname{Ran} P] \in K_0(\mathcal{A})$ and $[\operatorname{Ran}_\infty P] \in K_0(\mathcal{A}_\infty)$ correspond to each other under the canonical isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{A}_\infty)$.

(iii) It holds in $H_*^{dR}(\mathcal{A}_\infty)$:

$$\operatorname{ch}([\operatorname{Ran}_\infty P]) = \sum_{n=0}^\infty (-1)^n \frac{1}{n!} \operatorname{Tr}(P \,\mathrm{d}\, P)^{2n} \ .$$

Proof. (i): Let $\{e_n\}_{n\in\mathbb{N}} \subset C_0^\infty(M,\mathbb{C}^l)$ be an orthonormal basis of $L^2(M,\mathbb{C}^l)$.

Let P_n be the projection onto the span of the first n basis vectors. It is a Hilbert-Schmidt operator with integral kernel in $C_0^\infty(M\times M, M_l(\mathbb{C}))$.

In particular $P_n \in B(L^2(M,\mathcal{A}_i^l))$ for any $i \in \mathbb{N}$.

First we consider the situation on $L^2(M,\mathcal{A}^l)$:

Since P is compact, there is $n \in \mathbb{N}$ such that on $L^2(M,\mathcal{A}^l)$ it holds

$$\|P(P_n - 1)\| \leq \frac{1}{2} \ .$$

Hence by prop. 5.3.3 the map $PP_nP : \operatorname{Ran} P \to \operatorname{Ran} P$ is an isomorphism.

It follows $\operatorname{Ker} PP_nP = (\operatorname{Ran} P)^\perp = \operatorname{Ker} P$ and therefore

$$P = 1 - P_{\operatorname{Ker} P} = 1 - P_{\operatorname{Ker} PP_nP} \ .$$

Here $P_{\operatorname{Ker} P}$ resp. $P_{\operatorname{Ker} PP_nP}$ denotes the orthogonal projection onto $\operatorname{Ker} P$ resp. $\operatorname{Ker} PP_nP$.

Furthermore we can find $r > 0$ such that $B_r(0) \setminus \{0\}$ is in the resolvent set of PP_nP.

Then it holds

$$\begin{aligned} P &= 1 - P_{\mathrm{Ker}\,PP_nP} \\ &= 1 - \frac{1}{2\pi i}\int_{|\lambda|=r}(\lambda - PP_nP)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_{|\lambda|=r}\left(\lambda^{-1} - (\lambda - PP_nP)^{-1}\right)d\lambda \\ &= -\frac{1}{2\pi i}PP_nP\int_{|\lambda|=r}\lambda^{-1}(\lambda - PP_nP)^{-1}d\lambda \end{aligned}$$

Since P_n is a Hilbert-Schmidt operator on $L^2(M, \mathcal{A}^l)$, the operator P is one as well. In order to show that the equation holds on $L^2(M, \mathcal{A}_i^l)$ as well for all $i \in \mathbb{N}$, we prove that

$$R := \int_{|\lambda|=r}\lambda^{-1}(\lambda - PP_nP)^{-1}d\lambda$$

is well-defined in $B(L^2(M, \mathcal{A}_i^l))$ or equivalently that $(\lambda - PP_nP)$ is invertible in $B(L^2(M, \mathcal{A}_i^l))$ for $|\lambda| = r$.

The integral kernel of PP_nP is $k(x,y) := \sum_{j=1}^{n} Pe_j(x)(Pe_j(y))^*$. It is in $L^2(M \times M, M_l(\mathcal{A}_i)) \cap C(M \times M, M_l(\mathcal{A}_i))$ for all $i \in \mathbb{N}_0$ and the maps $x \mapsto k(x,\cdot)$ and $y \mapsto k(\cdot,y)$ are in $C(M, L^2(M, M_l(\mathcal{A}_i)))$. From cor. 5.3.2 we conclude that the spectrum of PP_nP on $B(L^2(M, \mathcal{A}_i^l))$ is independent of i, thus R is a bounded operator on $L^2(M, \mathcal{A}_i^l)$ for all $i \in \mathbb{N}_0$.

This and the equation $PR = PRP$ show that P is an integral operator with integral kernel

$$k_P(x,y) = -\frac{1}{2\pi i}\sum_{j=1}^{n} Pe_j(x)(PR^*Pe_j(y))^* \; .$$

The integral kernel is in $L^2(M \times M, M_l(\mathcal{A}_i))$ for all $i \in \mathbb{N}_0$ and is of the form we asserted.

(ii): Since $\|(P_n - 1)P\| \le \frac{1}{2}$ in $B(L^2(M, \mathcal{A}^l))$, the operator $(1 + (P_n - 1)P)$ is invertible in $B(L^2(M, \mathcal{A}^l))$. The integral kernel κ of $(P_n - 1)P$ is continuous and the maps $x \mapsto \kappa(x,\cdot)$ and $y \mapsto \kappa(\cdot,y)$ are in $C(M, L^2(M, M_l(\mathcal{A}_i)))$. Hence by prop. 5.3.1 the operator $(1 + (P_n - 1)P)$ is invertible in $B(L^2(M, \mathcal{A}_i^l))$ for any $i \in \mathbb{N}_0$. From $P_nP = (1 + (P_n - 1)P)P$ it follows

$$P_nP(L^2(M, \mathcal{A}_i^l)) \cong \mathrm{Ran}\,P(L^2(M, \mathcal{A}_i^l)) \; .$$

Furthermore $Q := (1 + (P_n - 1)P)P(1 + (P_n - 1)P)^{-1} \in B(L^2(M, \mathcal{A}_i^l))$ is a projection onto $P_nP(L^2(M, \mathcal{A}_i^l))$. Hence $Q(L_n(\mathcal{A}_i)) = P_nP(L^2(M, \mathcal{A}_i^l))$. Identify $L_n(\mathcal{A}_i)$ with $\mathcal{A}_i^n$ via the basis. Then Q restricted to $L_n(\mathcal{A}_i)$ is given by a projection $Q' \in M_n(\mathcal{A}_\infty)$. From $\mathrm{Ran}_\infty P \cong Q'(\mathcal{A}_\infty^n)$ it follows $[\mathrm{Ran}\,P] = [Q']$ in $K_0(\mathcal{A})$ and $[\mathrm{Ran}_\infty P] = [Q']$ in $K_0(\mathcal{A}_\infty)$. This shows the assertion.

(iii): Let P_n, Q and Q' be as in the proof of (ii). In the following computations we use the fact that $P_n Q = Q$, hence Q is a finite Hilbert-Schmidt operator. In $H_{dR}(\mathcal{A}_\infty)$ it holds

$$\begin{aligned}
\mathrm{ch}([\mathrm{Ran}_\infty P]) &= \mathrm{ch}(Q') \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{1}{k!}\mathrm{tr}(Q' \,\mathrm{d}\, Q')^{2k} \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{1}{k!}\mathrm{Tr}((P_n Q P_n)\,\mathrm{d}(P_n Q P_n))^{2k} \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{1}{k!}\mathrm{Tr}\big((P_n Q),(\mathrm{d}\, P_n Q)^{2k}\big) \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{1}{k!}\mathrm{Tr}(Q,(\mathrm{d}\, Q)^{2k}) \ .
\end{aligned}$$

Since

$$H : [0,1] \to B(L^2(M, (\hat{\Omega}_{\le\mu}\mathcal{A}_i)^l)) \ ,$$
$$H(t) = (1 + t(P_n - 1)P)P(1 + t(P_n - 1)P)^{-1}$$

is a differentiable path of finite projections with $H(0) = P$ and $H(1) = Q$ for any $i, \mu \in \mathbb{N}$, the difference $\mathrm{Tr}\big(Q, (\mathrm{d}\, Q)^{2k}\big) - \mathrm{Tr}(P\,\mathrm{d}\, P)^{2k}$ is exact in $\hat{\Omega}_*\mathcal{A}_\infty/\overline{[\hat{\Omega}_*\mathcal{A}_\infty), \hat{\Omega}_*\mathcal{A}_\infty]_s}$ by the previous lemma.

This shows the assertion. $\square$

5.4 Holomorphic semigroups

5.4.1 Generalities

Let X be a Banach space.

Recall that an operator Z on X generates a holomorphic semigroup e^{tZ} if and only if there is $\omega \ge 0$ such that $Z - \omega$ is δ-sectorial for some $\delta \in]0, \pi/2]$, i.e.

$$\Sigma_{\delta+\pi/2} = \{\lambda \in \mathbb{C}^* \mid |\arg\lambda| < \pi/2 + \delta\}$$

is a subset of $\rho(Z - \omega)$ and for any ε with $0 < \varepsilon < \delta$ there is $C > 0$ such that for all $\lambda \in \Sigma_{\pi/2+\varepsilon}$ it holds

$$\|(Z - \omega - \lambda)^{-1}\| \le \frac{C}{|\lambda|} \ .$$

The semigroup is bounded holomorphic if we can choose $\omega = 0$. Then in particular e^{tZ} is uniformly bounded for $t \ge 0$.

Lemma 5.4.1. *Let Z be an operator on X and let $\omega \in \mathbb{R}$ and $\delta > 0$ be such that*

1. $\Sigma_{\delta+\pi/2} \cup \{0\} \subset \rho(Z+\omega)$,

2. *for any* $\alpha < \delta$ *there are* $R, C > 0$ *such that*

$$\|(Z-\lambda)^{-1}\| \leq \frac{C}{|\lambda|}$$

for $\lambda \in \Sigma_{\alpha+\pi/2} \setminus B_R(0)$.

Then $Z + \omega$ *is* δ*-sectorial.*

Proof. Let $\varepsilon < \delta$ and choose $\alpha \in]\varepsilon, \delta[$. We have to show that there is $C > 0$ such that

$$\|(Z+\omega-\lambda)^{-1}\| \leq \frac{C}{|\lambda|} \ .$$

Let $r > \max(R, |\omega|)$ be such that $\overline{\Sigma_{\varepsilon+\pi/2}} \subset \big((\overline{\Sigma_{\alpha+\pi/2}} \cup B_r(0)) + \omega\big)$. (If $\omega \leq 0$, this is fulfilled for all $r > \max(R, |\omega|)$.)
The compact set $K := \overline{\Sigma_{\varepsilon+\pi/2}} \cap (B_r(0) + \omega)$ is a subset of $\rho(Z+\omega)$. The resolvent $(Z+\omega-\lambda)^{-1}$ is uniformly bounded on K, in particular there is $C > 0$ such that for $\lambda \in K \setminus \{0\}$ it holds

$$\|(Z+\omega-\lambda)^{-1}\| \leq \frac{C}{|\lambda|} \ .$$

If $\lambda \in \Sigma_{\varepsilon+\pi/2} \setminus K$, then $\lambda - \omega \in \Sigma_{\alpha+\pi/2} \setminus B_r(0)$ thus by assumption there is $C > 0$ such that for all $\lambda \in \Sigma_{\varepsilon+\pi/2} \setminus K$ it holds

$$\|(Z+\omega-\lambda)^{-1}\| \leq \frac{C}{|\lambda-\omega|} \ .$$

It follows that there is $C > 0$ such that for all $\lambda \in \Sigma_{\varepsilon+\pi/2}$ it holds

$$\begin{aligned}
\|(Z+\omega-\lambda)^{-1}\| &\leq \left(\frac{C}{|\lambda|}\right)\left(\frac{|\lambda-\omega|+|\omega|}{|\lambda-\omega|}\right) \\
&\leq \left(\frac{C}{|\lambda|}\right)\left(1+\frac{|\omega|}{|\lambda-\omega|}\right) \\
&\leq \left(\frac{C}{|\lambda|}\right)\left(1+\frac{|\omega|}{(r-|\omega|)}\right) \\
&\leq \frac{C}{|\lambda|} \ .
\end{aligned}$$

This proves the assertion. □

The lemma yields the following connection between the spectrum of a δ-sectorial operator Z and the behaviour of the holomorphic semigroup e^{tZ} for $t \to \infty$.

Proposition 5.4.2. *If Z is a δ-sectorial operator and there is $\omega > 0$ such that*

$$\{\operatorname{Re}\lambda \geq 0\} \subset \rho(Z + \omega) \ ,$$

then for any $\omega' < \omega$ there is $C > 0$ such that for all $t \geq 0$ it holds

$$\|e^{tZ}\| \leq Ce^{-\omega' t} \ .$$

Proof. For all $\omega' < \omega$ there is $0 < \delta' \leq \delta$ such that $\Sigma_{\delta'+\pi/2} \cup \{0\} \subset \rho(Z + \omega')$ and thus by the previous lemma $Z + \omega'$ is δ'-sectorial. $\square$

Proposition 5.4.3. *If e^{tZ} is a strongly continuous semigroup with generator Z such that* $\operatorname{Ran} e^{tZ} \subset \operatorname{dom} Z$ *for all $t \in (0, \infty)$ and if there is $T > 0$ and $C > 0$ such that for $t \leq T$ it holds*

$$\|Ze^{tZ}\| \leq Ct^{-1}$$

then e^{tZ} extends to a holomorphic semigroup.

If the estimate holds for all $t > 0$, then the extension is bounded holomorphic.

Proof. The assertion follows immediately from [Da], th. 2.39. $\square$

Part 1) of the next proposition is known under the name Volterra development and the formula in part 2) is called Duhamel's formula:

Proposition 5.4.4. *Let Z be the generator of a strongly continuous semigroup and let $M, \omega \in \mathbb{R}$ be such that $\|e^{tZ}\| \leq Me^{\omega t}$ for all $t \geq 0$.*

1. *Let $R \in B(X)$. Then $Z+R$ is the generator of a strongly continuous semigroup and for all $t \geq 0$ it holds*

   $$e^{t(Z+R)} = \sum_{n=0}^{\infty} (-1)^n t^n \int_{\Delta^n} e^{u_0 tZ} R e^{u_1 tZ} R \dots e^{u_n tZ} \, du_0 \dots du_n$$

   with $\Delta^n = \{u_0 + \cdots + u_n = 1; u_i \geq 0, \ i = 0, \dots, n\}$. Furthermore

   $$\|e^{t(Z+R)}\| \leq Me^{(\omega + M\|R\|)t} \ .$$

2. *Let $R_1, \dots, R_n \in B(X)$. For $t \geq 0$ the map*

   $$\mathbb{C}^n \to B(X), \ (z_1, \dots, z_n) \mapsto e^{t(Z + z_1 R_1 + \cdots + z_n R_n)}$$

   is analytical and for $i = 0, \dots n$ it holds

   $$\frac{d}{dz_i} e^{t(Z+z_1R_1+\cdots+z_nR_n)} = \int_0^t e^{(t-s)(Z+z_1R_1+\cdots+z_nR_n)} R_i e^{s(Z+z_1R_1+\cdots+z_nR_n)} \, ds \ .$$

Proof. 1) follows from [Da], th. 3.1 and the proof of it.

2) The analyticity follows from 1).

For the formula it is enough to consider $n = 1$. Let $R := R_1$.

For $z_0 \in \mathbb{C}$ it holds by 1):

$$e^{(Z+zR)t} = \sum_{n=0}^{\infty}(z-z_0)^n(-1)^n t^n \int_{\Delta^n} e^{u_0 t(Z+z_0R)} R e^{u_1 t(Z+z_0R)} R \dots e^{u_n t(Z+z_0R)}\, du_0 \dots du_n$$

and the series converges absolutely for all $z \in \mathbb{C}$. This implies

$$\frac{d}{dz} e^{(Z+zR)t}|_{z_0} = -t \int_0^1 e^{u_0 t(Z+z_0R)} R e^{(1-u_0)t(Z+z_0R)} du_0 \ .$$

The assertion follows now by the change of variables $s := (1-u_0)t$. □

The following proposition is known in the literature as Duhamel's principle ([Ta], §9):

Proposition 5.4.5. *Let Z be the generator of a strongly continuous semigroup on X. Let $u \in C^1([0,\infty), X)$ such that $\frac{d}{dt}u(t) - Zu(t) \in \operatorname{dom} Z$ for all $t \in [0,\infty)$. Then it holds*

$$e^{tZ}u(0) - u(t) = -\int_0^t e^{sZ}\big(\frac{d}{dt} - Z\big)u(t-s)ds \ .$$

5.4.2 Square roots of generators and perturbations

Assume that D is a densily defined closed operator on a Banach space X with bounded inverse and such that $-D^2$ is δ-sectorial.

There are well-defined fractional powers $(D^2)^\alpha$ for $\alpha \in \mathbb{R}$ ([RR], 11.4.2). These are densily defined closed operators that coincide for $\alpha \in \mathbb{Z}$ with the usual powers and satisfy

$$(D^2)^{\alpha+\beta} f = (D^2)^\alpha (D^2)^\beta f$$

for all $\alpha, \beta \in \mathbb{R}$ and $f \in \operatorname{dom}(D^2)^\gamma$ with $\gamma = \max\{\alpha, \beta, \alpha+\beta\}$. For $\alpha \le 0$ the operator $(D^2)^\alpha$ is bounded and depends in a strongly continuous way of α.

It follows in particular that for $\alpha \ge 0$ the operator $(D^2)^{-\alpha}$ is a bounded inverse of $(D^2)^\alpha$.

Define

$$|D| := (D^2)^{\frac{1}{2}}.$$

By [Ka], th. 2, the operator $-|D|$ is $\delta + (\pi/2 - \delta)/2$-sectorial and can be expressed in terms of the resolvents of D^2.

Note that for any $n \in \mathbb{N}$ the domain of $|D|^n$ is a core of $|D|$ and $\operatorname{dom} D^n$ is a core of D.

Lemma 5.4.6. *1. Let P be a projection on X, i.e. we demand P^2 to be densily defined and $P^2|_M = P|_M$ for some set $M \subset \operatorname{dom} P^2$ dense in X.*

Then it holds: If P is closed, then P is bounded.

2. Let I be a involution on X, i.e. assume that I^2 is densily defined and that there is a set $M \subset \operatorname{dom} I^2$ dense in X with $I^2|_M = 1|_M$.

Then it holds: If I is closed, then I is bounded.

Proof. 1) For $f \in M$ it holds $f = (1-P)f + Pf$ and $(1-P)Pf = P(1-P)f = 0$. We conclude $M \subset \operatorname{Ker} P + \operatorname{Ker}(1-P) \subset \operatorname{dom} P$. If P is closed, then $\operatorname{Ker} P + \operatorname{Ker}(1-P)$ is closed, hence $\operatorname{Ker} P + \operatorname{Ker}(1-P) = X$, thus $\operatorname{dom} P = X$.

2) The operator $P := \frac{1}{2}(1-I)$ is a closed projection on X in the sense of 1). Thus P is bounded. It follows that I is bounded as well. □

Proposition 5.4.7. *1. The operator $|D|^{-1}D : \operatorname{dom} D \to X$ extends to a bounded involution I on X.*

2. It holds $\operatorname{dom} D = \operatorname{dom} |D|$ and $I(\operatorname{dom} D) \subset \operatorname{dom} D$.

3. It holds $|D| = ID = DI$ and $D = I|D| = |D|I$.

Proof. 1) The operator D^{-1} commutes with the resolvents of D^2. It follows $|D|^{-1}D^{-1} = D^{-1}|D|^{-1}$. Hence because of $\operatorname{dom} D = D^{-1}X$ it holds $|D|^{-1}(\operatorname{dom} D) \subset \operatorname{dom} D$, so $\operatorname{dom} D^2 \subset \operatorname{dom}(|D|^{-1}D)^2$.

For $f \in \operatorname{dom} D$ it holds $|D|^{-1}Df = D|D|^{-1}f$. For $f \in \operatorname{dom} D^2$ it follows $(|D|^{-1}D)^2 f = f$.

We prove that $|D|^{-1}D$ is closable, then the assertion follows from the previous lemma:

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{dom} D$ converging to zero. Then $|D|^{-1}f_n \in \operatorname{dom} D$ converges to zero for $n \to \infty$. But then if $|D|^{-1}Df_n = D|D|^{-1}f_n$ converges, the limit is zero since D is closed and injective. Thus $(|D|^{-1}D)$ is closable.

2) The composition $IDI : \operatorname{dom}|D| \to X$ is well defined and closed. It coincides with D on $\operatorname{dom} D^2$, hence it is a closed extension of D. It follows $\operatorname{dom} D \subset \operatorname{dom}|D|$. The inclusion $\operatorname{dom}|D| \subset \operatorname{dom} D$ is shown analogously. □

Let $P := \frac{1}{2}(1+I)$ with I as in the previous proposition. It is a bounded projection on X. By the proposition it holds $P \operatorname{dom} D \subset \operatorname{dom} D$, and P commutes with D and $|D|$.

From $ID = |D|$ and $I|D| = D$ it follows $PD = -P|D|$. Thus with respect to the decomposition $X = PX \oplus (1-P)X$ it holds

$$D = \begin{pmatrix} PDP & 0 \\ 0 & -(1-P)D(1-P) \end{pmatrix}$$

and

$$|D| = \begin{pmatrix} PDP & 0 \\ 0 & (1-P)D(1-P) \end{pmatrix} .$$

Taking into account that $-|D|$ is $\delta + (\pi/2 - \delta)/2$-sectorial it follows for the resolvent set of D:

Proposition 5.4.8. *For $\lambda \in \mathbb{C}$ it holds*

$$\{\lambda, -\lambda\} \subset \rho(D) \Leftrightarrow \{\lambda, -\lambda\} \subset \rho(|D|) .$$

Thus if $\lambda \in \mathbb{C}$ with $-\lambda^2 \in \Sigma_{\pi/2+\delta}$, then $\lambda \in \rho(D)$.

Furthermore for every $\delta' < \delta$ there is $C > 0$ such that for all λ with $-\lambda^2 \in \Sigma_{\pi/2+\delta'}$ it holds:

$$\|(D - \lambda)^{-1}\| \leq \frac{C}{|\lambda|} .$$

Corollary 5.4.9. *Let $\omega \geq 0$ be such that there is $C > 0$ with $\|e^{-tD^2}\| \leq Ce^{-\omega t}$ for all $t \geq 0$.*

For any $\alpha \in \mathbb{R}$ and $\omega' < \omega$ there is $C > 0$ such that for all $t > 0$ it holds

$$\||D|^\alpha e^{-tD^2}\| \leq Ct^{-\alpha/2} e^{-\omega' t}$$

For any $n \in \mathbb{N}$ and $\omega' < \omega$ there is $C > 0$ such that for all $t > 0$ it holds

$$\|D^n e^{-tD^2}\| \leq Ct^{-n/2} e^{-\omega' t}.$$

Proof. The first assertion is [RR], lemma 11.36, and the second one follows from the first one by $D = I|D|$ and $DI = ID$. □

Proposition 5.4.10. *Let A be a bounded operator. Then for every $\delta' < \delta$ there is $\omega > 0$ such that $-(D + A)^2 + \omega$ is δ'-sectorial.*

Proof. Let $\delta' < \delta$. We show that there is $\omega > 0$ such that $-(D + A)^2 + \omega$ and δ' satisfy the assumptions of lemma 5.4.1:

By prop. 5.4.8 there is $M > 0$ such that for all λ with $-\lambda^2 \in \Sigma_{\pi/2+\delta'}$ it holds:

$$\|(D - \lambda)^{-1}\| \leq \frac{M}{|\lambda|}.$$

Hence the Neumann series

$$(D + A - \lambda)^{-1} = (D - \lambda)^{-1} \sum_{n=0}^{\infty} (A(D - \lambda)^{-1})^n$$

converges for $|\lambda| > M\|A\|$ and $-\lambda^2 \in \Sigma_{\pi/2+\delta'}$.

If $|\lambda| > 2M\|A\|$ and $-\lambda^2 \in \Sigma_{\pi/2+\delta'}$, the norm is bounded by

$$
\begin{aligned}
\|(D+A-\lambda)^{-1}\| &= \|\sum_{n=0}^{\infty}(D-\lambda)^{-1}\left(A(D-\lambda)^{-1}\right)^n\| \\
&\leq \sum_{n=0}^{\infty}\|A\|^n\|(D-\lambda)^{-1}\|^{n+1} \\
&\leq \sum_{n=0}^{\infty}\frac{M^{n+1}\|A\|^n}{|\lambda|^{n+1}} \\
&= \frac{M}{|\lambda|}(1-\frac{M\|A\|}{|\lambda|})^{-1} \\
&\leq \frac{2M}{|\lambda|}
\end{aligned}
$$

Let $\mu \in \{|\mu| > 4M^2\|A\|^2\} \cap \Sigma_{\delta'+\pi/2}$. Then for $\lambda \in \mathbb{C}$ with $-\lambda^2 = \mu$ it holds $\lambda \in \rho(D+A)$, hence the resolvent

$$(-(D+A)^2-\mu)^{-1} = -(D+A-\lambda)^{-1}(D+A+\lambda)^{-1}$$

exists and is bounded by

$$\|(-(D+A)^2-\mu)^{-1}\| \leq \frac{4M^2\|A\|^2}{|\mu|} \ .$$

Furthermore there is $\omega > 4M^2\|A\|^2$ such that

$$\Sigma_{\delta'+\pi/2} \subset \left(\{|\mu| > 4M^2\|A\|^2\} \cap \Sigma_{\delta'+\pi/2} - \omega\right)$$

and thus

$$\Sigma_{\delta'+\pi/2} \subset \rho\left(-(D+A)^2+\omega\right) \ .$$

The assertion follows now from lemma 5.4.1. $\square$

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