Similarly, we see that $f''(0) = m$. Thus $M = m$. Therefore, $f''(a)$ is a constant function in $[0, 2\pi]$. Let $f''(a) = \alpha$. Then f is given by the quadratic polynomial

$$f(a) = \alpha a^2/2 + \beta a + \gamma. \tag{11}$$

Our final step is to determine constants α, β, and γ explicitly. Substitute (11) in (6) and then equate the coefficients of the term a and the constant term to obtain

$$-\pi\alpha/2 = \beta/2, \quad \pi^2\alpha/2 + \beta\pi + 2\gamma = \gamma/2. \tag{12}$$

Notice that the coefficient of the term a^2 vanishes. By (1) and by a basic theorem on differentiation under the integral sign we obtain $f'(\pi/2) = \pi/2$, which, with (11), implies

$$\pi\alpha/2 + \beta = \pi/2. \tag{13}$$

It follows from (12) and (13) that $\alpha = -1$, $\beta = \pi$, and $\gamma = -\pi^2/3$. Thus, (11) yields (5). This completes the proof of Theorem.

The values of the Euler integrals (2), (3) and (4) now follow from (1) and (5) by setting a equal to 0, π, and $\pi/2$ respectively.

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A HOMOLOGY VERSION OF THE BORSUK-ULAM THEOREM

JAMES W. WALKER
Department of Mathematics, California Institute of Technology, Pasadena, CA 91125

An **involution** on a topological space X is a continuous map from X to X which is its own inverse. For example the antipodal map, which maps a point to the opposite end of the diameter on which it lies, is an involution on the n-sphere S^n.

Suppose X and Y are spaces equipped with involutions a and b, respectively. A map f from X to Y is **equivariant** if it respects the involutions, i.e., $b \circ f = f \circ a$.

One formulation of the Borsuk-Ulam theorem is that if m is greater than n, then there is no map from S^m to S^n which is equivariant with respect to the antipodal map. Many sources, for example [**1**, § 7.2], include proofs of the Borsuk-Ulam theorem, as well as applications such as the "ham sandwich theorem." We will use singular homology theory to prove a somewhat stronger theorem.

Our stronger theorem shows that the existence of any equivariant maps to S^n from any space X with an involution forces the existence of very special homology classes for X, so special that X could not be a sphere of dimension greater than n.

A few words about terminology: An **elementary 0-chain** is a singular 0-simplex with coefficient 1; loosely speaking, it's just a single point. We will use reduced homology, which essentially means that we consider the empty set to be a singular simplex of dimension -1, which is the boundary of every 0-simplex. It follows that $\tilde{H}_{-1}(X)$ vanishes unless X is empty, and $\tilde{H}_0(X)$ vanishes if X is path connected. Recall that each continuous map f induces a chain map $f_\#$, defined by composing f with singular simplices. In turn, such a chain map $f_\#$ induces a homology homomorphism f_*.

THEOREM. *Suppose X is a space with involution v, and $g\colon X \to S^n$ is an equivariant map. Then there exists an integer $j \leqslant n$, and a homology class β of $\tilde{H}_j(X; \mathbb{Z}/2)$ such that β is nonzero and $v_*(\beta) = \beta$. Furthermore, if no such β exists for j less than n, then β can be chosen such that $g_*(\beta)$ is the nonzero element of $\tilde{H}_n(S^n; \mathbb{Z}/2)$.*

In this framework our theorem applied to an equivariant map $g: S^m \to S^n$ with $m > n$ would require the existence of a special nonzero element of $\tilde{H}_j(S^m)$ for some $j \leqslant n < m$. But $\tilde{H}_j(S^m) = 0$ for all $j < m$ so our result generalizes the Borsuk-Ulam theorem.

Proof of theorem: The case $n = 0$ is straightforward, so assume that n is greater than 0. The proof will proceed by inductively constructing singular chains. (Similar methods were used on a higher level by P. A. Smith; see [**2**, chapter 13].) Bear in mind that signs can be ignored, since we are using coefficients in $\mathbb{Z}/2$.

It is convenient to define a "symmetrizer" chain map $\theta = \mathrm{id}_{\#} + v_{\#}$ on the singular chain complex of X, where "id" denotes the identity map. We use the same notation for the chain map $\mathrm{id}_{\#} + a_{\#}$ on S^n, where a is the antipodal map. These operators satisfy $\theta\theta = 0$ and $\theta g_{\#} = g_{\#}\theta$, as one can easily verify.

Assume that for all j less than n and for all β in $\tilde{H}_j(X; \mathbb{Z}/2)$, $v_*(\beta) = \beta$ implies $\beta = 0$. (Otherwise, we have a β which satisfies the first part of the theorem.) Hence if x_j is a j-cycle such that $\theta x_j = 0$, then x_j must be a boundary. Our goal is to produce a nontrivial element β of $\tilde{H}_n(X; \mathbb{Z}/2)$ such that $v_*(\beta) = \beta$ and $g_*(\beta) \neq 0$. Our strategy will be to make some observations about j-dimensional hemispheres h_j in S^n, construct chains c_j in X which behave much like the hemispheres, compare $g_{\#}c_j$ to h_j, and finally show that θc_n is a cycle which determines the desired homology class.

First we choose singular j-chains h_j in S^n, corresponding to hemispheres, such that

h_0 is an elementary 0-chain,

$\partial h_j = \theta h_{j-1}$ for $1 \leqslant j \leqslant n$, and

θh_n generates $\tilde{H}_n(S^n; \mathbb{Z}/2)$.

Next, we will construct singular j-chains c_j in X, for j ranging from 0 to n, such that

c_0 is an elementary 0-chain, and

$\partial c_j = \theta c_{j-1}$ for $1 \leqslant j \leqslant n$.

We have assumed that there is no nonzero β in $\tilde{H}_{-1}(X; \mathbb{Z}/2)$ such that $v_*(\beta) = \beta$, so X is nonempty. Pick a point in X, and let c_0 be the corresponding elementary 0-chain. Note that θc_0 is a cycle. Since $\theta\theta c_0 = 0$, there is a 1-chain c_1 such that $\partial c_1 = \theta c_0$.

Suppose that $\partial c_j = \theta c_{j-1}$ for some j less than n. We compute that $\partial\theta c_j = \theta\partial c_j = \theta\theta c_{j-1} = 0$, so θc_j is a cycle. Since $\theta\theta c_j = 0$, there exists a $(j+1)$-chain c_{j+1} such that $\partial c_{j+1} = \theta c_j$. This completes the inductive definition of $c_0, c_1, \dots, c_n$.

Now we will inductively construct j-chains e_j in S^n, for j ranging from 0 to n, such that

$$h_j - g_{\#}c_j - \theta e_j \text{ is a cycle.}$$

Note that $h_0 - g_{\#}c_0$ is a cycle, since h_0 and c_0 were chosen to be elementary 0-chains. Therefore we can let $e_0 = 0$.

Suppose that e_j is a j-chain, where j is less than n, such that $h_j - g_{\#}c_j - \theta e_j$ is a cycle. Since $\tilde{H}_j(S^n; \mathbb{Z}/2) = 0$, there is a $(j+1)$-chain e_{j+1} such that

$$\partial e_{j+1} = h_j - g_{\#}c_j - \theta e_j.$$

Apply θ and obtain

$$\partial\theta e_{j+1} = \theta h_j - g_{\#}\theta c_j.$$

Since $\theta h_j = \partial h_{j+1}$ and $\theta c_j = \partial c_{j+1}$, this becomes

$$\partial\theta e_{j+1} = \partial h_{j+1} - \partial g_{\#}c_{j+1}.$$

Therefore $h_{j+1} - g_{\#}c_{j+1} - \theta e_{j+1}$ is a cycle, as desired.

To complete the proof, we note that $h_n - g_{\#}c_n - \theta e_n$ is a cycle in S^n, which is therefore homologous to either zero or θh_n. In either case, when we apply θ, we find that $\theta h_n - g_{\#}\theta c_n$ is homologous to zero. That is, θh_n and $g_{\#}\theta c_n$ belong to the same homology class. Note that θc_n is a cycle, because $\partial\theta c_n = \theta\partial c_n = \theta\theta c_{n-1} = 0$. Therefore, if β is the homology class of θc_n, then $g_*(\beta)$ is the nonzero element of $\tilde{H}_n(S^n; \mathbb{Z}/2)$. It follows that β is nonzero. Finally, the fact that $\theta\theta c_n = 0$ means that $v_{\#}\theta c_n = \theta c_n$, so $v_*(\beta) = \beta$.

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ON THE NUMBER OF MULTIPLICATIVE PARTITIONS

JOHN F. HUGHES
Department of Mathematics, Bryn Mawr College, Bryn Mawr, PA 19010

J. O. SHALLIT
Department of Mathematics, University of California, Berkeley, CA 94720

I. A Number-Theoretic Function. In this note we show that if $f(n)$ is the number of essentially different factorizations of n, then

$$f(n) \leqslant 2n^{\sqrt{2}}.$$

In considering numbers that have exactly k divisors, one is led to examine this function $f(n)$, the number of ways to write n as the product of integers $\geqslant 2$, where we consider factorizations that differ only in the order of the factors to be the same. We call these representations of n **multiplicative partitions**. For example, $f(12) = 4$, since

$$12 = 6 \cdot 2 = 4 \cdot 3 = 3 \cdot 2 \cdot 2$$

are the four multiplicative partitions of 12. From these four representations, we can conclude that a number has exactly 12 divisors if and only if its prime factorization is one of the following:

$$p^{11}, p^5q, p^3q^2, p^2qr.$$

This follows from the expression for $\tau(n)$, the number of divisors of $n = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$.

$$\tau(n) = \prod_{j=1}^{k} (1 + a_j).$$

For example, see [1].

The behavior of $f(n)$ is quite erratic, and apparently has not been previously studied in this form. We observe that if q is prime, then $f(q^k) = p(k)$, the number of additive partitions of k. Also, if $q_1, q_2, \ldots, q_k$ are distinct primes, then $f(q_1q_2 \cdots q_k) = B(k)$, the kth Bell number. See [2].

More generally, $f(q_1^{a_1} \cdots q_k^{a_k})$ is the number of additive partitions of the "multi-partite number" $(a_1, a_2, \ldots, a_k)$, where addition is defined component-wise. See [3] for further details.

We will show that

$$f(n) \leqslant 2n^{\sqrt{2}}. \tag{1}$$

For a table of $f(n)$ for $1 \leqslant n \leqslant 100$, see the Appendix.

II. Proof of the Main Result. To prove (1) we first define an auxiliary function:

$g(m, n) =$ the number of multiplicative partitions of n with all elements $\leqslant m$.

Clearly $f(n) = g(n, n)$. We have the following