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Non-Additivity of the Signature

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It was recently observed by Novikov that if two compact oriented $4k$-manifolds are glued by a diffeomorphism (reversing orientation) of their boundaries, then the signature of their union is the sum of their signatures. The proof is given in [1, 7.1]; the result has been exploited by Jänich [2] to characterise the signature of closed manifolds.

In constructing such manifolds, it is often desirable to consider a more general case of glueing: viz. along a common submanifold, which may itself have boundary Z^{4k-2}, of the boundaries of the original manifolds. Additivity still holds if Z is empty, or more generally if $H^{2k-1}(Z;\mathbb{R})$ vanishes, by the same argument as before. However, it does not hold in general. The simplest counterexample is the Hopf bundle (with fibre D^2) over S^2, with signature ± 1 depending on the choices of sign: this is the union of the induced bundles over the upper and lower hemispheres of S^2, each of which (being contractible) has signature zero.

In this paper we investigate the deviation from additivity in the general case, and derive a formula for it. For this, we have first to construct a new invariant.

First, we recall a standard result from lattice theory, in the form we will use. Let V be an abelian group; A, B, C subgroups (all will be real vector spaces of finite dimension in practice). Consider the additive relation between A and B defined by

$$a R b \quad \text{if} \quad \exists c \in C \qquad \text{with } a+b+c=0.$$

Its domain is the set of $a \in A$ expressible as $-(b+c)$, $b \in B$, $c \in C$; i.e. it is $A \cap (B+C)$. Also $a R 0$ precisely when $A \in A \cap C$. Since corresponding results hold with A and B interchanged, the relation induces an isomorphism between

$$\frac{A \cap (B+C)}{A \cap C} \quad \text{and} \quad \frac{B \cap (C+A)}{B \cap C}.$$

Moreover, in this isomorphism, the images of $A \cap B$ correspond, and we can factor these out. The result is symmetrical in A, B and C: the correspondence $a+b+c=0$ induces isomorphisms between

$$W = \frac{A \cap (B+C)}{(A \cap B)+(A \cap C)}, \qquad \frac{B \cap (C+A)}{(B \cap C)+(B \cap A)}, \qquad \text{and} \qquad \frac{C \cap (A+B)}{(C \cap A)+(C \cap B)}.$$

Next, let Φ: $V \times V \to G$ be a bilinear map with $\Phi(A \times A) = \Phi(B \times B) = \Phi(C \times C) = 0$. Then if

$$a+b+c=a'+b'+c'=0,$$

we have

$$\begin{aligned} 0 &= \Phi(0,a') = \Phi(a+b+c,a') \\ &= \Phi(a,a') + \Phi(b,a') + \Phi(c,a') \\ &= \Phi(b,a') + \Phi(c,a'). \end{aligned}$$

Interchanging the roles of a and a', or a, b and c, this leads to the conclusions

$$\Phi(b,a') = -\Phi(c,a') = \Phi(c,b') = -\Phi(a,b') = \Phi(a,c') = -\Phi(b,c'),$$

so that these all vanish if any of a, b, c, a', b', c' do.

We define Ψ': $A \cap (B+C) \times A \cap (B+C) \to G$ by

$$\Psi'(a,a') = \Phi(a,b'),$$

where $a'+b'+c'=0$. If also $a'+b''+c''=0$, then $\Phi(a,b') - \Phi(a,b'') = \Phi(a,b'-b'') = \Phi(c,a'-a') = 0$; so Ψ' is well-defined. If a or a' is in $A \cap B$ or $A \cap C$, then Ψ' vanishes by a remark above; by linearity, $\Psi' = 0$ if a or a' is in $(A \cap B) + (A \cap C)$. Hence Ψ' induces a bilinear map of the quotient

$$\Psi\colon W \times W \to G.$$

We can identify W with any of three quotients by permuting the roles of A, B, C: the equations above show that Ψ is unaltered by even permutations of A, B, and C in its definition, and is replaced by $-\Psi$ under odd permutations.

Now let Φ be skew-symmetric. Then

$$\begin{aligned} \Psi(a,a') - \Psi(a',a) &= \Phi(a,b') - \Phi(a',b) \\ &= \Phi(a,b') + \Phi(b,a') \\ &= \Phi(a+b,a'+b') - \Phi(a,a') - \Phi(b,b') \\ &= \Phi(c,c') = 0. \end{aligned}$$

Thus Ψ is symmetric.

The case in which we are interested is that in which V is a finite dimensional real vector space; A, B, C supspaces and Φ: $V \times V \to \mathbb{R}$ skew-symmetric with $\Phi(A \times A) = \Phi(B \times B) = \Phi(C \times C) = 0$. We have constructed a symmetric bilinear map

$$\Psi\colon W \times W \to \mathbb{R}.$$

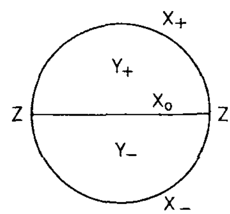
Its signature will be denoted by $\sigma(V; A, B, C)$. We have seen that if A, B, and C are permuted, this number is multiplied by the sign of the permutation. In fact in the case to be used, Φ is nonsingular and

$$\dim A = \dim B = \dim C = \tfrac{1}{2} \dim V$$

so that A, B and C are maximal isotropic subspaces; here it is easy to see that Ψ is also nonsingular. For if $\perp$ denotes orthogonal complements with respect to Φ, $A^{\perp} = A$, $B^{\perp} = B$, $C^{\perp} = C$ so $(B+C)^{\perp} = B \cap C$ and $\{A \cap (B+C)\}^{\perp} = A + (B \cap C)$. Thus if $a \in A \cap (B+C)$ has image in the radical of Ψ, the corresponding $b \in B \cap (C+A)$ lies also in $A + (B \cap C)$ and hence in $(B \cap A) + (B \cap C)$, so the image of a in W is zero.

Now suppose given manifolds, disjoint except as specified, with

$$\partial X_- = \partial X_0 = \partial X_+ = Z,$$
$$\partial Y_- = X_- \cup X_0, \qquad \partial Y_+ = X_0 \cup X_+;$$



write $Y = Y_- \cup Y_+$ and $X = X_- \cup X_0 \cup X_+$ (see figure). Let $\dim Y = 4k$, and suppose Y oriented, inducing orientations of Y_- and Y_+. Orient the rest so that

$$\partial_* [Y_-] = [X_0] - [X_-], \qquad \partial_* [Y_+] = [X_+] - [X_0],$$
$$\partial_* [X_-] = \partial_* [X_0] = \partial_* [X_+] = [Z].$$

Write $V = H_{2k-1}(Z; \mathbb{R})$; let A, B, C be the kernels of the maps induced by inclusions of Z in X_-, X_0 and X_+ respectively (equivalently, the images of $H_{2k}(X_-, Z; \mathbb{R})$ etc.). Let Φ denote intersection numbers in Z. It is easily seen that V, A, B, C and Φ have the properties mentioned in the algebraic discussion above.

Theorem.

$$\sigma(Y) = \sigma(Y_-) + \sigma(Y_+) - \sigma(V; A, B, C).$$

Proof. In the exact homology sequence

$$H_{2k}(\partial Y) \to H_{2k}(Y) \to H_{2k}(Y, \partial Y) \to H_{2k-1}(\partial Y),$$

the two vector spaces in the middle are dual, and if we choose dual bases, then the matrix of intersection numbers on $H_{2k}(Y)$ coincides with that of the middle map. Hence the radical of the quadratic form is the image of $H_{2k}(\partial Y)$, and the signature of the form is that of the associated nonsingular form on $G_{2k}(Y)=\operatorname{Im}(H_{2k}(Y)\to H_{2k}(Y,\partial Y))$.

The same remarks apply to Y_- and Y_+; we choose splittings

$$H_{2k}(Y_\varepsilon)=\operatorname{Im} H_{2k}(\partial Y_\varepsilon)\oplus G_{2k}(Y_\varepsilon) \qquad (\varepsilon=\pm)$$

of the exact sequences in these cases.

Lemma. *The subspace of $H_{2k}(Y,\partial Y)$ orthogonal to $H_{2k}(Y_-)\oplus H_{2k}(Y_+)$ is the image of $H_{2k}(X_0, Z)$.*

Proof. Since the dual to $H_{2k}(Y_-)$ is $H_{2k}(Y_-, \partial Y_-)$, we can interpret intersection numbers of $H_{2k}(Y,\partial Y)$ with $H_{2k}(Y_-)$ by considering the map

$$H_{2k}(Y,\partial Y)\to H_{2k}(Y, Y_+\cup\partial Y)\cong H_{2k}(Y_-, \partial Y_-);$$

similarly for Y_+. The subspace orthogonal to $H_{2k}(Y_-)$ is the kernel of this map; the desired subspace is the intersection of these, i.e. the kernel of

$$H_{2k}(Y,\partial Y)\to H_{2k}(Y,X)\cong H_{2k}(Y_-, \partial Y_-)\oplus H_{2k}(Y_+,\partial Y_+).$$

By exactness and excision, this is the image of

$$H_{2k}(X,\partial Y)\cong H_{2k}(X_0, Z).$$

Now the maps $G_{2k}(Y_\varepsilon)\to G_{2k}(Y)$ $(\varepsilon=\pm)$ induced by inclusion evidently preserve intersection numbers; since $G_{2k}(Y_\varepsilon)$ is nonsingular, and the two images are orthogonal to each other, both maps are injective and their images form a direct sum. Let K denote the orthogonal complement of $G_{2k}(Y_-)\oplus G_{2k}(Y_+)$ in $G_{2k}(Y)$. Since signature is additive for direct sums, $\sigma(Y)-\sigma(Y_-)-\sigma(Y_+)$ is the signature of the quadratic form on K.

The image S of $H_{2k}(\partial Y_-)\oplus H_{2k}(\partial Y_+)$ in $G_{2k}(Y)$ is evidently contained in K. Its orthogonal complement $S^\perp$ in K coincides with that of $\operatorname{Im}(H_{2k}(Y_-)\oplus H_{2k}(Y_+))$ in $G_{2k}(Y)$; by the lemma, this is given by

$$S^\perp=G_{2k}(Y)\cap\operatorname{Im}(H_{2k}(X_0, Z)\to H_{2k}(Y,\partial Y)).$$

Since the inclusion $\partial Y_\varepsilon\to(Y,\partial Y)$ factorises through $(\partial Y_\varepsilon, X_\varepsilon)$, and $H_{2k}(\partial Y_\varepsilon, X_\varepsilon)\cong H_{2k}(X_0, Z)$, we have $S\subset S^\perp$.

Now let L be a complement of S in $S^\perp$. As $(S^\perp)^\perp=S$, the radical of the form's restriction to $S^\perp$ is S, so the form is nonsingular on L. Hence it is also nonsingular on the orthogonal complement $L^\perp$ of L in K, and K is the orthogonal direct sum of L and $L^\perp$. But S coincides with its own orthogonal complement in $L^\perp$, so $L^\perp$ has signature zero by a standard argument. Thus K and L have the same signature.

It remains to evaluate the signature $\sigma(L)$. We recall that $L$was defined to be an additive complement of

$$S=\operatorname{Im}\left(H_{2k}(\partial Y_-)\oplus H_{2k}(\partial Y_+)\to H_{2k}(Y,\partial Y)\right)$$

in

$$\begin{aligned}S^\perp&=\operatorname{Im}\left(H_{2k}(Y)\to H_{2k}(Y,\partial Y)\right)\cap\operatorname{Im}\left(H_{2k}(X_0,Z)\to H_{2k}(Y,\partial Y)\right)\\&=\operatorname{Im}\left(H_{2k}(X)\to H_{2k}(Y,\partial Y)\right).\end{aligned}$$

Choosing an element x_1 of $S^\perp$, we can lift (non-uniquely) to $x_2\in H_{2k}(X_0,Z)$, and then form $x_3=\partial_* x_2\in H_{2k-1}(Z)$. We claim that this map induces an isomorphism of L onto

$$W=\frac{B\cap(C+A)}{B\cap C+B\cap A},$$

where A, B and C were defined just before the statement of the theorem.

First, if $x_1\in S$ we say that x_2 belongs to

$$\operatorname{Im}\left(H_{2k}(\partial Y_-)\oplus H_{2k}(\partial Y_+)\to H_{2k}(X,\partial Y)=H_{2k}(X_0,Z)\right),$$

and indeed (but this is clear) that any element of this group appears as such an x_2. For $x_1\in S$ evidently has such a lift, and the indeterminacy is the image of $H_{2k+1}(Y,X)\cong H_{2k+1}(Y_-,\partial Y_-)\oplus H_{2k+1}(Y_+,\partial Y_+)$ by the boundary homomorphism.

Next, the image of

$$H_{2k}(\partial Y_-)\to H_{2k}(\partial Y_-,X_-)\cong H_{2k}(X_0,Z)\xrightarrow{\partial_*}H_{2k-1}(Z)$$

is precisely $A\cap B$. This follows from the Mayer-Vietoris sequence

$$H_{2k}(\partial Y_-)\to H_{2k-1}(Z)\to H_{2k-1}(X_-)\oplus H_{2k-1}(X_0).$$

Hence the image of the subspace of the preceding paragraph is $(B\cap C)+(B\cap A)$.

Finally, the total range of values of x_2 is the set of elements of $H_{2k}(X_0,Z)$ whose image in $H_{2k}(Y,\partial Y)$ is also in the image of $H_{2k}(Y)$, or equivalently, maps to 0 in $H_{2k-1}(\partial Y)$. Thus the range of x_3 is

$$\operatorname{Im}\left(H_{2k}(X_0,Z)\to H_{2k-1}(Z)\right)\cap\operatorname{Ker}\left(H_{2k-1}(Z)\to H_{2k-1}(\partial Y)\right).$$

But the first member here is B; the second is

$$\operatorname{Im}\left(H_{2k}(\partial Y,Z)=H_{2k}(X_-,Z)\oplus H_{2k}(X_+,Z)\right),$$

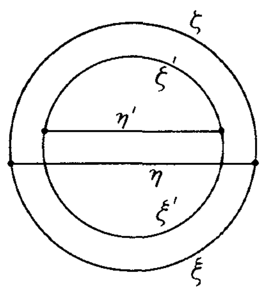
i.e. it is $C+A$. Combining the results of the last three paragraphs, we have established our isomorphism of L on W.

It remains only to identify the quadratic form. Let $b \in B \cap (A+C)$; represent by a cycle η of (X_0, Z), and write $\partial\xi + \partial\eta + \partial\zeta = 0$ with $\xi \in Z_{2k}(X_+, Z)$. Then an element of $S^\perp \subset H_{2k}(Y, \partial Y)$ corresponding to b is represented by η, and also by $\xi + \eta + \zeta$ which lifts it to $H_{2k}(Y)$. Let b', ξ', η', ζ' be another such system of elements. Deform ζ' and η' slightly inside Y_+, and ξ' inside Y.

Then

$$(\xi+\eta+\zeta)\cap(\xi'+\eta'+\zeta') = \eta \cap \xi' \qquad (\text{in } Y)$$

and this is the same as the intersection number of η and $\partial\xi'$ in X_0, hence equals the intersection number of $\partial\eta$ and $\partial\xi'$ in Z. But this is just $-\Psi(b, b')$. Thus we get the negative of the preceding form, so $\sigma(L) = -\sigma(V; A, B, C)$. This completes the proof.



If the compact group G acts on our manifolds, we note that all the splittings in the proof can be chosen to be equivariant; the G-signature of $L^\perp$ vanishes as before, and the result extends to G-signatures as defined by Atiyah and Singer [1]. Moreover for G-signatures the argument also yields a useful result when $\dim Y = 4k+2$; here Φ is symmetric and Ψ is skew (the proof is as above).

References

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