Proc. Camb. Phil. Soc. (1964), **60**, 807
Printed in Great Britain

Cobordism of combinatorial n-manifolds for $n \leqslant 8$

BY C. T. C. WALL
Trinity College, Cambridge

(*Received* 22 *May* 1963)

The object of this paper is two-fold: first to collect together the known facts about combinatorial cobordism in general, and then to calculate the groups for the first 8 dimensions. As in (29), we shall denote the unoriented and oriented cobordism groups in dimension n by $\mathfrak{N}_n$ and Ω_n, and will distinguish the combinatorial from the differential case by affixes c, d.

Using the results of Whitehead (30) on C^1 triangulations, we have natural maps

$$T_n : \mathfrak{N}_n^d \to \mathfrak{N}_n^c, \quad U_n : \Omega_n^d \to \Omega_n^c.$$

Now Stiefel numbers give a complete set of invariants of cobordism class in $\mathfrak{N}^d$ (23). These have been defined and proved invariant in the combinatorial case by Wu and Thom (22) (indeed, the proof is valid even for homology manifolds). Hence T_n is a monomorphism.

Similarly, Stiefel and Pontrjagin numbers together give a complete set of invariants of cobordism class in Ω^d (26). Pontrjagin numbers (with rational values) have also been defined in the combinatorial case by Thom (24), Rohlin and Svarc (17); so U_n is a monomorphism. Moreover, in dimension $n \not\equiv 0 \pmod 4$, if we choose a set of Stiefel numbers inducing an isomorphism of Ω_n^d, we obtain (effectively) a left inverse of U_n, and so Ω_n^d is a direct summand of Ω_n^c.

It has further been proved by Adams (1) that the relations between Stiefel numbers which hold for differential manifolds hold also for combinatorial (again, even homology) manifolds. This gives no direct information about combinatorial cobordism. The corresponding result for Pontrjagin numbers is false; this is essentially due to Milnor (8). Thus U_n is not onto and does not in general split if $n \equiv 0 \pmod 4$.

Finally, exact sequences which relate the combinatorial cobordism groups were obtained in the author's paper (29); they are as follows:

$$\Omega_n \xrightarrow{2} \Omega_n \xrightarrow{s} \mathfrak{W}_n \xrightarrow{\partial i} \Omega_{n-1} \xrightarrow{2} \Omega_{n-1}, \tag{1}$$

$$0 \to \mathfrak{W}_n \xrightarrow{i} \mathfrak{N}_n \xrightarrow{d} \mathfrak{N}_{n-2} \to 0, \tag{2}$$

$$\Omega_n \oplus \mathfrak{N}_{n-1} \xrightarrow{(2,0)} \Omega_n \xrightarrow{r} \mathfrak{N}_n \xrightarrow{(\partial,d)} \Omega_{n-1} \oplus \mathfrak{N}_{n-2} \xrightarrow{(2,0)} \Omega_{n-1}. \tag{3}$$

As far as the author is aware, no other facts are known about combinatorial cobordism which do not involve smoothing theory. We thus turn next to the groups Γ_{n-1} of differential structures on spheres (9).

LEMMA 1. *There is a natural homomorphism*

$$C_n : \Gamma_{n-1} \to \Omega_n^c / \Omega_n^d.$$

Proof. Let T^{n-1} be an $(n-1)$-sphere, with exotic differential structure, representing $x \in \Gamma_{n-1}$. Its tangent bundle is stably trivial, so all characteristic classes vanish. Hence it bounds a differential manifold V^n. Form a closed combinatorial manifold W^n by triangulating V^n and adjoining a cone on its boundary. We define $C_n(x)$ as the class of W^n.

This is well defined. For let V', leading to W', be another choice for V. The connected sum $W \# (-W')$ is defined by attaching V to $-V'$ by a diffeomorphism of the boundary, hence admits a differential structure. Thus its class in Ω_n^c/Ω_n^d is zero, and the classes of W, W' are equal.

Our construction is clearly compatible with connected sums, so we have a homomorphism. It is a more precise statement of a construction of Milnor (8).

We shall use the values of Γ_n in low dimensions. In fact, for $n \leqslant 6$, $\Gamma_n = 0$; $\Gamma_7 \cong \mathbf{Z}_{28}$. Proofs that $\Gamma_3 = 0$ may be found in (14), (18). A result of J. Cerf (31) states that $\Gamma_4 = 0$. Using Milnor's result (11) that a homotopy 5-sphere bounds a contractible manifold, it follows from Smale (19) that $\Gamma_5 = 0$. Smale also shows (20), (21) that for $n \geqslant 6$, Γ_n is isomorphic to Milnor's group Θ_n ((6); see also (11), (12)). But $\Theta_6 = 0$ ((6), (27)), and Θ_7 is cyclic of order 28 (11). This proves all the results; in particular, note that Γ_7 is the first non-vanishing group Γ_n.

We now invoke smoothing theory. A *smoothing* of a combinatorial manifold is a differential manifold which gives rise to it by C^1-triangulation. We need the following results.

PROPOSITION 1. *Let M be a triangulated combinatorial manifold, K a subcomplex. Let α be a smoothing of a neighbourhood of $K \cup M^i$. Then the obstruction to existence of a smoothing of a neighbourhood of $K \cup M^{i+1}$ agreeing with α on a neighbourhood of $K \cup M^{i-1}$ is an element of $H^{i+1}(M, K; \Gamma_i)$.*

We also need an interpretation of the obstruction in one case. Let M be closed, of dimension m, K contain all of M except one m-simplex, $i = m-1$. Then α induces a smoothing of a boundary of K (by the Cairns–Hirsch theorem (3), (4)), which defines an element of Γ_{m-1}.

PROPOSITION 2. *In this case, the obstruction may be identified with this element of Γ_{m-1}.*

It seems worth commenting on these results. The original obstruction theory of Thom (25) was not entirely rigorous. The work of Munkres (15), (16) does not give smoothings in the sense used here. Milnor's work with microbundles (13) is more relevant; however, his work does not give a relative theorem of the type above, and he did not know the coefficient groups—this difficulty, however, was settled by Mazur (7). The results above may be found in a paper of Hirsch (5). We observe that we possessed the arguments in this paper 2 years ago; however, this result of Hirsch, and Cerf's result $\Gamma_4 = 0$ were needed to justify them.

It is now trivial that if $m \leqslant 7$, M^m possesses a smoothing.

From results above, we deduce

LEMMA 2. *For $n \leqslant 7$, T_n and U_n are isomorphisms.*

We shall now prove

LEMMA 3. *C_8 is an isomorphism.*

It is easy to see that cobordism groups defined by connected manifolds coincide (in positive dimensions) with general ones; sum is defined by connected sum. We may, then, suppose all manifolds connected.

If W_8 is a closed combinatorial manifold, the unique obstruction to smoothing it is an element R_W of $H^8(W; \Gamma_7)$. If a simplex is removed, W becomes smoothable, giving rise to a differential manifold V. The boundary of V is a 7-sphere T with exotic differential structure; in fact, by Proposition 2, this is determined by the obstruction above. The construction of Lemma 1 leads from T to W; hence C_8 is onto.

Suppose x in the kernel of C_8. With the notation above, W is cobordant to a triangulation M of a differential manifold. Taking the connected sum of W with $-M$, we may suppose that W bounds a manifold N, say. It is clear from Proposition 1 that R_W is induced from the obstruction $R_N \in H^8(N; \Gamma_7)$ to smoothing N. But

$$i^*: H^8(N; \Gamma_7) \to H^8(W; \Gamma_7)$$

is zero, so $R_W = 0$ and $x = 0$. Hence C_8 is also (1-1).

We now use Lemma 3 to prove our main result.

THEOREM. *For $n \leqslant 7$, T_n and U_n are isomorphisms. We have $\Omega_8^c \cong \mathbf{Z}+\mathbf{Z}+\mathbf{Z}_4$, and $\mathfrak{N}_8^c$ has rank 6. Stiefel and Pontrjagin numbers do not determine oriented cobordism class nor Stiefel numbers unoriented cobordism class, in dimension* 8.

Proof. We first discuss Pontrjagin numbers in dimension 8. We shall work with the triple (p_1^2, p_2, σ), where σ is the signature; these are related in the differential case by $45\sigma = 7p_2 - p_1^2$ (due to Thom (23)) and by definition, this is valid also in the combinatorial case.

By a result of (23), $\Omega_8^d \cong \mathbf{Z}+\mathbf{Z}$, and a basis is given by the classes u_1, u_2 of $P_1(\mathrm{C}) \times P_1(\mathrm{C})$ and $P_2(\mathrm{C})$, which have Pontrjagin numbers (18, 9, 1) and (25, 10, 1). A more convenient basis consists of $v_1 = u_2 - u_1$, $v_2 = 9u_2 - 10u_1$, with Pontrjagin numbers (7, 1, 0) and (45, 0, -1), which puts the relation above in evidence.

Now σ remains integral for combinatorial manifolds. Also, Thom showed in (24) that the combinatorial class p_1 was always in the image of $H^4(M; \mathbf{Z})$. Hence p_1^2 is an integer. Thus the Pontrjagin numbers are linear combinations of the two sets for $\frac{1}{7}v_1$ and v_2: (1, $\frac{1}{7}$, 0) and (45, 0, -1). Alternatively, we could obtain this result using Lemma 3 and specific manifolds constructed in (8), (28) in the construction of Lemma 1.

Pontrjagin numbers thus define a homomorphism P of Ω_8^c onto the free Abelian group generated by $\frac{1}{7}v_1$ and v_2. Since the group is free, P splits as the projection of a direct sum. Now $P(\Omega_8^d)$ is a subgroup of index 7, so P induces a homomorphism

$$\lambda: \Omega_8^c/\Omega_8^d \to \mathbf{Z}_7$$

(see (8)), with Ker $P \cong$ Ker λ. By Lemma 3, $\Omega_8^c/\Omega_8^d \cong \Gamma_7 \cong \mathbf{Z}_{28}$, so Ker λ is cyclic of order 4. Hence $\Omega_8^c \cong \mathbf{Z}+\mathbf{Z}+\mathbf{Z}_4$ as stated. It follows that Stiefel and Pontrjagin numbers cannot determine cobordism class in it.

To deduce the unoriented groups we use the exact sequences (1) and (2). In particular, we have

$$\Omega_8 \xrightarrow{2} \Omega_8 \to \mathfrak{W}_8 \to \Omega_7 = 0.$$

Thus $\mathfrak{W}_8^c$ has rank 3 whereas $\mathfrak{W}_8^d$ has rank 2. The rank of $\mathfrak{N}_c^8$ now follows from Lemma 2 and sequence (2). The inadequacy of Stiefel numbers follows from the result of Adams (1) that Stiefel numbers take no new values. This completes the proof of the theorem.

In conclusion, it seems worth putting the following conjecture in print. The author is convinced of its truth, and feels that it will soon be proved—the main lacuna is a theory of t-regularity for combinatorial manifolds. (Apparently this gap has now been filled by Williamson.)

Added in proof. See *Notices American Math. Soc.* **11** (1964), p. 222.

CONJECTURE. *The cokernel of U_n is finite for all n.*

The corresponding result for T_n would follow as a corollary. It is hard to see further than this; the following is the simplest possibility.

GUESS. *The free part of Ω^c is a polynomial ring.*

Presumably, if this is so, a generator of Ω_{4n}^c is given (modulo decomposable elements) as a submultiple of a generator of Ω_{4n}^d, using the homomorphism C_{4n}. It seems that the relevant part of $\Gamma_{4n-1} \cong \Theta_{4n-1}$ lies in Milnor's subgroup $\Theta_{4n-1}(\partial\pi)$ (11). Perhaps it is the maximal subgroup of odd order: the order of this is known ((11), (2)) to be $(2^{2n-1}-1)$ times the numerator of (B_n/n).

REFERENCES

(1) ADAMS, J. F. On formulae of Thom and Wu. *Proc. London Math. Soc.* **11** (1961), 741–752.

(2) ADAMS, J. F. On the J-homomorphism. (To appear.)

(3) CAIRNS, S. S. The manifold smoothing problem. *Bull. American Math. Soc.* **67** (1961), 237–238.

(4) HIRSCH, M. W. On combinatorial submanifolds of differentiable manifolds. *Comment. Math. Helv.* **36** (1961), 103–111.

(5) HIRSCH, M. W. Obstruction theories for smoothing manifolds and maps. *Bull. American Math. Soc.* **69** (1963), 352–356.

(6) KERVAIRE, M. A. and MILNOR, J. W. Groups of homotopy spheres. I. *Ann. of Math.* **77** (1963), 504–537.

(7) MAZUR, B. *Séminaire de Topologie Combinatoire et Différentielle de l'I.H.E.S.*, 1962/1963.

(8) MILNOR, J. W. On manifolds homeomorphic to the 7-sphere. *Ann. of Math.* **64** (1956), 399–405.

(9) MILNOR, J. W. Sommes de variétés différentiables et structures différentiables de sphères. *Bull. Soc. Math. France*, **87** (1959), 439–444.

(10) MILNOR, J. W. Differentiable structures on spheres. *American J. Math.* **81** (1959), 962–972.

(11) MILNOR, J. W. *Differentiable manifolds which are homotopy spheres* (Princeton notes: Jan. 1959).

(12) MILNOR, J. W. *A procedure for killing homotopy groups of differentiable manifolds* (Proc. Symp. on Pure Math. III, American Math. Soc.: 1961).

(13) MILNOR, J. W. *Microbundles and differentiable structures* (Princeton notes: Sept. 1961).

(14) MUNKRES, J. Differentiable isotopies on the 2-sphere. *Michigan Math. J.* **7** (1960), 193–197.

(15) MUNKRES, J. *Obstructions to imposing differentiable structures* (Princeton notes: 1960; see also *Notices American Math. Soc.* **7** (1960), 204).

(16) MUNKRES, J. Obstructions to the smoothing of piecewise differentiable homeomorphisms. *Ann. of Math.* **72** (1960), 521–554.

(17) ROHLIN, V. A. and SVARC, A. S. The combinatorial invariance of Pontrjagin classes. *Dokl. Akad. Nauk SSSR*, **114** (1957), 490–493 (in Russian).

(18) SMALE, S. Diffeomorphisms of the 2-sphere. *Proc. American Math. Soc.* **10** (1959), 621–626.

(19) SMALE, S. Generalized Poincaré's Conjecture in dimensions greater than 4. *Ann. of Math.* **74** (1961), 391–406.
(20) SMALE, S. Differentiable and combinatorial structures on manifolds. *Ann. of Math.* **74** (1961), 498–502.
(21) SMALE, S. On the structure of manifolds. *American J. Math.* **84** (1962), 387–399.
(22) THOM, R. Espaces fibrés en sphères et carrés de Steenrod. *Ann. Sci. École Norm. Sup.* **69** (1952), 109–181.
(23) THOM, R. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.* **28** (1954), 17–86.
(24) THOM. R. *Les classes caractéristiques de Pontrjagin des variétés triangulées* (Symposium Internacional de Topologia Algebraica, Mexico: 1958).
(25) THOM, R. *Des variétés triangulées aux variétés différentiables* (Proc. Int. Congr. Math., 1958; Cambridge: 1960).
(26) WALL, C. T. C. Determination of the cobordism ring. *Ann. of Math.* **72** (1960), 292–311.
(27) WALL, C. T. C. Killing the middle homotopy groups of odd dimensional manifolds. *Trans. American Math. Soc.* **103** (1962), 421–433.
(28) WALL, C. T. C. Classification of $(n-1)$-connected $2n$-manifolds. *Ann. of Math.* **75** (1962), 163–189.
(29) WALL, C. T. C. Cobordism exact sequences for differential and combinatorial manifolds. *Ann. of Math.* **77** (1963), 1–15.
(30) WHITEHEAD, J. H. C. On C^1-complexes. *Ann. of Math.* **41** (1940), 809–824.
(31) CERF, J. La nullité de π_0 (Diff S^3). *Séminaire H. Cartan, nos.* 8, 9, 10, 20, 21, Paris, 1962/63.