Topology Vol. 15, pp. 189–210. Pergamon Press, 1976. Printed in Great Britain

FORMULAE FOR SURGERY OBSTRUCTIONS

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(*Received 7 July* 1975)

THE THEORY of surgery on manifolds which are compact but not necessarily simply-connected, as described in [14], is relatively ineffective due to lack of knowledge about the obstruction groups $L_i(\pi)$. In another paper [15] we have given extensive calculations of these groups for π finite: the details are in general so complicated that the situation is not much improved (except in some simple cases, such as when π has odd order). A second disadvantage is that computing the actual obstructions is even less effective: it is necessary first to perform surgery below the middle dimension to obtain equivalences, then impose a complicated structure on the homology kernels of the map.

A way round the second difficulty, proposed by the author in [14, §17G] and developed incompletely by Misčenko, has now been established by Ranicki (Geometric L-theory, I and II, to appear). It is now possible, without performing preliminary surgeries, to define structures at the chain complex level, sufficiently fine to detect surgery obstructions: these are usually done on the homology kernels, but with suitable conditions on the tangent bundle (much weaker than framing) can be defined intrinsically.

As to the calculation of surgery obstructions, it was observed by Sullivan that for maps between manifolds (as opposed to Poincaré complexes), these have the property of bordism invariance, and may indeed be considered as maps $\lambda : \Omega_*(K(\pi, 1) \times G/\mathrm{Top}) \to L_*(\pi)$, where Ω may be taken as topological bordism. The remarks above show that we may add—in a suitable sense—homology invariance. This is such a strong condition that we may hope to use it to obtain some computation of λ.

Although the above considerations motivated the paper, I proceed somewhat differently below, using numerous ideas and results from the recent paper [7] by Morgan and Sullivan (or equivalently, the corresponding paper [5] by Milgram): the key idea is, in fact, the introduction of so-called $\mathbf{Z}/n$-manifolds. Using a product formula (implicit rather than explicit) we will obtain an analogue to Sullivan's formula [8] for the Kervaire invariant of a map $g: M \to G/\mathrm{Top}$

$$c(M, g) = v^2(M) g^* k[M]$$

for a certain characteristic class k.

We present our arguments in an axiomatic framework—this will clarify just what geometry we need; also, some of the axioms are easier to verify for some groups π than for the general case; finally, this has the useful side-effect of simplifying our notation.

The first—and main—part of this paper will be devoted to this axiomatic development. We start with an introductory section, dealing with a simplified version of the problem. The next two sections expound respectively the notions of Bockstein functor and of $\mathbf{Z}/n$-manifold. The main part of the paper studies (for a fixed space X) a Bockstein functor $\lambda : \Omega_*(X \times G/\mathrm{Top}; \mathbf{Z}/n) \to F(\mathbf{Z}/n)$ with a product formula given by a commutative diagram

$$\begin{array}{ccc} \Omega_*(X; \mathbf{Z}/n) \otimes \Omega_*(G/\mathrm{Top}); \mathbf{Z}/n) & \longrightarrow & \Omega_*(X; \mathbf{Z}/n) \otimes P_*(\mathbf{Z}/n) \\ \downarrow & & \downarrow \\ \Omega_*(X \times G/\mathrm{Top}; \mathbf{Z}/n) & \xrightarrow{\lambda} & F(\mathbf{Z}/n). \end{array}$$

In §4 we study the case $n = 2^r$, and obtain a cohomological formula for λ, and in §5 we suppose n odd, and obtain a formula using K-theory. The formula holds for topological bordism, and the proof of this is much deeper than in the smooth case. In §6, we seek integral results: to obtain the neatest formulation here, it is necessary to impose finiteness conditions.

Next we verify that the axioms are in fact satisfied by surgery obstructions. This is accomplished by a slight modification of the techniques of [14, §9], in order to accommodate $\mathbf{Z}/n$-manifolds. This shows that λ is calculated by certain characteristic classes of π. These characteristic classes are natural. Applying this fact, we obtain a 'transfer formula' for surgery obstructions on closed manifolds which is extremely useful for applications. In the case when π is finite, it follows that the calculation can be reduced to the Sylow 2-subgroup.

In a final section it is shown how to utilise the fact that λ is far from being surjective, and we make further comments about possible relations with a more algebraic theory.

§1. W-MANIFOLDS

A *W-manifold* is a manifold whose orientation bundle is induced from the double covering bundle of S^1: equivalently, the cohomology Bockstein (with integer coefficients) $\beta w_1 = 0$. The map to S^1 is not unique up to homotopy; it is, however, unique up to bordism.

We write (following [12]) W_* for the bordism theory defined by W-manifolds. Using group structure on S^1, we see that this is a multiplicative theory. According to [12] for the smooth case, and [1] for the topological, there is a natural (if not canonical) isomorphism $W_*(X) \cong W_* \otimes H_*(X;\mathbf{Z}/2)$. Thus the product induces an isomorphism $W_*(X) \otimes_{W_*} W_*(Y) \cong W_*(X \times Y)$ for any X and Y.

Suppose now given a homomorphism $\lambda: W_*(X \times G/\mathrm{Top}) \to \mathbf{Z}/2$. We write this as $\lambda(N; f, g)$ where $f: N \to X$ and $g: N \to G/\mathrm{Top}$. Composing with the natural product yields a bilinear map $W_*(X) \times W_*(G/\mathrm{Top}) \to \mathbf{Z}/2$: we write $(N; f, g) = (M, f) \times (P; g)$. If we think in terms of surgery problems, it is natural to perform surgery on the P-part leaving M fixed.

Now $\pi_1(P)$ may vary with bordism, but since P is a W-manifold, we have a map $P \to S^1$ inducing $\pi_1(P) \to \mathbf{Z}$. Thus the universal surgery obstruction groups for W-manifolds are the groups $L_*(\mathbf{Z}^-)$ for nonorientable manifolds with fundamental group $\mathbf{Z}$.

We are thus led to

AXIOM 1. *$\lambda\{(M; f) \times (P; g)\}$ depends only on $(M; f)$ and the surgery obstruction in $L_*(\mathbf{Z}^-)$ for $(P; g)$.*

We will show that this axiom suffices for us to obtain a formula.

THEOREM 1. *Given a map λ satisfying Axiom 1, there are uniquely determined cohomology classes $\alpha_i \in H^{**}(X;\mathbf{Z}/2)$ ($i = 0, 2,$ and 3) such that*

$$\lambda(N; f, g) = \{v^2(N)(f^*\alpha_2 + v_1 f^*\alpha_3) + v\,\mathrm{Sq}^1 v(N)(f^*\mathrm{Sq}^1\alpha_0 + v_1 f^*\alpha_0)\}g^*k[N] + v^2(N)f^*\alpha_0 g^*l[N]. \tag{1.1}$$

Here k and l are the characteristic classes of Sullivan and Morgan [7], v is the total Wu class of N, $v_1 = w_1$, and $[N]$ is the fundamental homology class. It is simpler to combine all terms into a total formula than to have separate formulae for manifolds of different dimensions, particularly as λ need have no homogeneity.

Proof. The surgery obstruction groups $L_*(\mathbf{Z}^-) = P_*(\mathbf{Z}/2)$ are of order 2 in dimensions $n \equiv 0, 2, 3 \pmod 4$ and trivial for $n \equiv 1 \pmod 4$. We denote the isomorphisms $P_n(\mathbf{Z}/2) \to \mathbf{Z}/2$ by τ, c and $c\beta$ in the three cases. These are given by [5], [7]

$$\tau(P; g) = \{v^2(P)g^*l + v_1 v\,\mathrm{Sq}^1 v(P)g^*k\}[P], \quad c(P; g) = v^2(P)g^*k[P], \quad \text{and } c\beta(P; g) = v_1 v^2(P)g^*k[P] \text{ respectively.} \tag{1.2}$$

They satisfy the product formulae

$$\begin{aligned} \tau((P; g) \times Q) &= \tau(P; g)\chi(Q) + c(P; g)\delta\beta(Q) + c\beta(P; g)\delta(Q), \\ c((P; g) \times Q) &= c(P; g)\chi(Q) \text{ and} \\ c\beta((P; g) \times Q) &= c\beta(P; g)\chi(Q). \end{aligned} \tag{1.3}$$

Here, χ denotes the Euler characteristic, which satisfies $\chi(Q) = v^2(Q)[Q]$, and δ is the 'de Rham invariant', with $\delta(Q) = v\,\mathrm{Sq}^1 v(Q)[N]$. Throughout, β is to be interpreted as the 'Bockstein':

$\beta(Q)$ is any submanifold dual to $v_1(Q)$. Thus $\delta\beta(Q) = v_1 v \mathrm{Sq}^1 v(Q)[Q]$, and $\chi\beta(Q) = v_1 v^2(Q)[Q] = 0$: note that though $v(Q)|\beta Q - v(\beta Q)$ is not zero, it is divisible by v_1, so the difference vanishes when multiplied by v_1.

Since λ satisfies Axiom 1, it defines and is defined by a map $W_*(X) \otimes_{W_*} P_*(\mathbf{Z}/2) \to \mathbf{Z}/2$, and by our remarks on W_*, this tensor product is isomorphic to $H_*(X; \mathbf{Z}/2) \otimes P_*(\mathbf{Z}/2)$, or to a sum of 3 copies of $H_*(X; \mathbf{Z}/2)$. The map thus yields 3 cohomology classes as stated. It remains to make these isomorphisms explicit.

Now the structure of $P_*(\mathbf{Z}/2)$ as W_*-module is given by (1.3). Regard $P_*(\mathbf{Z}/2)$ as graded by $\mathbf{Z}/4$, and write ι_0, ι_2, ι_3 for the nonzero elements of the indicated degrees. Then ι_0 generates a W_*-submodule, which is trivial in the sense that each element of W_* acts through its Euler characteristic, and the quotient module is also trivial. We are thus led to seek a map θ: $W_*(X) \otimes P_*(\mathbf{Z}/2) \to H_*(X; \mathbf{Z}/2) \otimes P_*(\mathbf{Z}/2)$ of the form

$$\theta((M, f) \otimes \iota_0) = f_* v^2(M) \otimes \iota_0$$
$$\theta((M, f) \otimes \iota_2) = f_* v^2(M) \otimes \iota_2 + f_* a(M) \otimes \iota_0$$
$$\theta((M, f) \otimes \iota_3) = f_* v^2(M) \otimes \iota_3 + f_* b(M) \otimes \iota_0$$

for characteristic classes a and b yet to be determined. Here, and in the sequel, we identify the cohomology class $a(M)$ with its dual $a(M) \wedge [M]$. We must check whether, for any P and r, $\theta((M, f) \cdot P \otimes \iota_r) = \theta((M, f) \otimes P \cdot \iota_r)$. The coefficients of ι_r on both sides are equal to $f_* v^2(M) \cdot \chi(P)$. Equating coefficients of ι_0 gives

$$f_* a(M \times P) = f_* a(M)\chi(P) + f_* v^2(M)\delta\beta(P)$$
$$f_* b(M \times P) = f_* b(M)\chi(P) + f_* v^2(M)\delta(P).$$

In view of the formula for $\delta(P)$, this suggests taking $a(M) = v_1 v \mathrm{Sq}^1 v(M)$ and $b(M) = v \mathrm{Sq}^1 v(M)$. Now as $v \mathrm{Sq}^1 v(M \times P) = v \mathrm{Sq}^1 v(M) v^2(P) + v^2(M) v \mathrm{Sq}^1 v(P)$, the second formula holds. The first, unfortunately, does not: as well as the two desired terms on the right, we acquire two cross-terms. However, we also have $\delta\beta(P) = v_1 v \mathrm{Sq}^1 v[P] = \mathrm{Sq}^1(v \mathrm{Sq}^1 v)[P] = (\mathrm{Sq}^1 v)^2[P]$, and taking $a(M) = (\mathrm{Sq}^1 v)^2(M)$, the formula checks.

Now define α_r by the homomorphism $H_*(X; \mathbf{Z}/2) \otimes \iota_r \to \mathbf{Z}/2$ constructed from λ. This gives a formula for λ on product terms, viz.

$$\begin{aligned}\lambda((M, f) \times (P, g)) = {} & v^2(M) f^*\alpha_0[M]\tau(P, g) \\ & + (v^2(M) f^*\alpha_2 + (\mathrm{Sq}^1 v)^2(M) f^*\alpha_0)[M] c(P, g) \\ & + (v^2(M) f^*\alpha_3 + v \mathrm{Sq}^1 v(M) f^*\alpha_0)[M] c\beta(P, g),\end{aligned}$$

or, evaluated on $[M \times P]$, the class

$$\begin{aligned} & v^2(M) f^*\alpha_0 v^2(P) g^* l + v^2(M) f^*\alpha_0 v_1 v \mathrm{Sq}^1 v(P) v(P) g^* k \\ & \quad + v^2(M) f^*\alpha_2 v^2(P) g^* k + (\mathrm{Sq}^1 v)^2(M) f^*\alpha_0 v^2(P) g^* k \\ & \quad + v^2(M) f^*\alpha_3 v_1 v^2(P) g^* k + v \mathrm{Sq}^1 v(M) f^*\alpha_0 v_1 v^2(P) g^* k.\end{aligned}$$

We can rewrite this as

$$v^2(M \times P)(f^*\alpha_0 g^* l + f^*\alpha_2 g^* k) + v_1(P) v^2(M \times P) f^*\alpha_3 g^* k + \text{terms in } f^*\alpha_0 g^* k.$$

Now $(\mathrm{Sq}^1 v)^2(M) f^*\alpha_0[M] = v_1(v \mathrm{Sq}^1 v(M) f^*\alpha_0)[M] + v \mathrm{Sq}^1 v(M) f^* \mathrm{Sq}^1 \alpha_0[M]$, so we can rewrite the terms in $f^*\alpha_0 g^* k$ as

$$\begin{aligned} & v_1 v \mathrm{Sq}^1 v(M \times P) f^*\alpha_0 g^* k + v_1 v^2(M) f^*\alpha_0 v \mathrm{Sq}^1 v(P) g^* k \\ & \quad + v \mathrm{Sq}^1 v(M \times P) f^* \mathrm{Sq}^1 \alpha_0 g^* k + v^2(M) f^* \mathrm{Sq}^1 \alpha_0 v \mathrm{Sq}^1 v(P) g^* k.\end{aligned}$$

But $v_1 v^2(M) f^*\alpha_0[M] = \mathrm{Sq}^1(v^2(M) f^*\alpha_0)[M] = v^2(M) f^* \mathrm{Sq}^1 \alpha_0[M]$, so two terms here cancel out, and the others depend only on the characteristic classes of $M \times P$. The only term which does not is now $v_1(P) v^2(M \times P) f^*\alpha_3 g^* k$. It can now be verified that if we redefine θ on ι_2 terms so that

$$\begin{aligned}\theta((M, f) \otimes \iota_0) &= f_* v^2(M) \otimes \iota_0 \\ \theta((M, f) \otimes \iota_2) &= f_* v^2(M) \otimes \iota_2 + f_* v_1 v^2(M) \otimes \iota_3 + f_*(\mathrm{Sq}^1 v)^2(M) \otimes \iota_0 \\ \theta((M, f) \otimes \iota_3) &= f_* v^2(M) \otimes \iota_3 + f_* v \mathrm{Sq}^1 v(M) \otimes \iota_0\end{aligned} \tag{1.4}$$

then θ still factors through the tensor product over W_*, but we now have an extra term $v_1v^2(M)f^*\alpha_3c(P,g)$, corresponding to $v_1(M)v^2(M\times P)f^*\alpha_3g^*k$. To summarise, we now have

$$\lambda((M,f)\times(P,g)) = \{v^2(M\times P)(f^*\alpha_0g^*l + f^*\alpha_2g^*k) + v_1v^2(M\times P)f^*\alpha_3g^*k \\ + v_1v\,\mathrm{Sq}^1v(M\times P)f^*\alpha_0g^*k + v\,\mathrm{Sq}^1v(M\times P)f^*\mathrm{Sq}^1\alpha_0g^*k\}[M\times P].$$

Thus (1.1) holds on products, hence (by bordism invariance) in general.

We discuss the formula (1.1) briefly before proceeding to generalisations. First note that we can rewrite it as

$$\{v^2(N)(f^*\alpha_0g^*l + f^*\alpha_2g^*k) + v\,\mathrm{Sq}^1v(N)f^*(\mathrm{Sq}^1\alpha_0)g^*k\}[N] \\ + (v^2(\beta N)f^*\alpha_3 + v\,\mathrm{Sq}^1v(\beta N)f^*\alpha_0)g^*k[\beta N].$$

Next, observe that for λ satisfying the conditions of the Theorem, so does $\lambda\circ\beta$: what are the corresponding cohomology classes here? Well, we have

$$\lambda\beta(N;f,g) = \{v^2(\beta N)f^*\alpha_0g^*l + v^2(\beta N)f^*\alpha_2g^*k + v\,\mathrm{Sq}^1v(\beta N)f^*\mathrm{Sq}^1\alpha_0g^*k\}[\beta N]$$

and here

$$\begin{aligned} v^2(\beta N)f^*\alpha_0g^*l[\beta N] &= v_1v^2(N)f^*\alpha_0g^*l[N] \\ &= \mathrm{Sq}^1(v^2(N)f^*\alpha_0g^*l)[N] \\ &= v^2(N)f^*\mathrm{Sq}^1\alpha_0g^*l[N], \end{aligned}$$

since l is a class with $\mathbf{Z}_{(2)}$-coefficients. Hence

1.5. *If λ corresponds to $(\alpha_0, \alpha_2, \alpha_3)$, then $\lambda\beta$ corresponds to* $(\mathrm{Sq}^1\alpha_0, 0, \alpha_2)$.

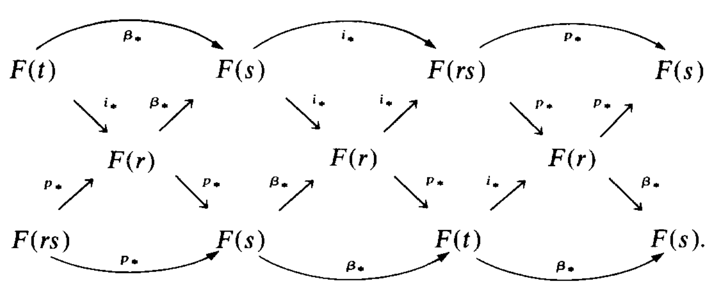
§2. BOCKSTEIN FUNCTORS

We now introduce the notion of Bockstein functor.

Let $\mathcal{A}$ be a class of abelian groups: we will not indicate precise axioms on $\mathcal{A}$, but for the next two sections will need only finite cyclic groups. We identify $\mathcal{A}$ with a full subcategory of the category $\mathcal{A}b$ of all abelian groups. A *Bockstein functor* on $\mathcal{A}$ consists in an additive functor $F:\mathcal{A}\to\mathcal{A}b$ together with, for each short exact sequence $S: 0\to A\to B\to C\to 0$ in $\mathcal{A}$, a connecting homomorphism $\beta_F(S): F(C)\to F(A)$ such that we have an exact triangle $F(A)\to F(B)\to F(C)\to F(A)$. This must be natural with respect to maps of short exact sequences S.

We now derive some simple consequences of the axioms which hold when $\mathcal{A}$ includes finite cyclic groups. Here we denote $F(\mathbf{Z}/r)$ by $F(r)$, the map $F(rs)\to F(s)$ induced by projection by p_*; the map $F(s)\to F(rs)$ induced by injection (multiplying by r) by i_*; and the Bockstein corresponding to the exact sequence $S_{r,s}: 0\to\mathbf{Z}/r\xrightarrow{i}\mathbf{Z}/rs\xrightarrow{p}\mathbf{Z}/s\to 0$ by β_*.

Now consider the short exact sequences $S_{s,t}$, $S_{t,s}$, $S_{r,s}$ and $S_{s,r}$ where $r = st$: the exact sequence fit into a braid diagram as follows



(2.1)

This diagram commutes: the triangles $p_*p_* = p_*$, $i_*i_* = i_*$ and the diamond $i_*p_* = p_*i_*$ since F is a functor, and the triangles $\beta_* = \beta_*i_*$, $p_*\beta_* = \beta_*$ and the diamond $i_*\beta_* = \beta_*p_*$ by naturality of β_* for the morphisms of exact sequences

$$\begin{array}{cccccccccc} & 0 \longrightarrow & \mathbf{Z}/s & \xrightarrow{i} & \mathbf{Z}/r & \xrightarrow{p} & \mathbf{Z}/t & \longrightarrow 0 \\ p_3 & & \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & \\ & 0 \longrightarrow & \mathbf{Z}/s & \xrightarrow{i} & \mathbf{Z}/rs & \xrightarrow{p} & \mathbf{Z}/r & \longrightarrow 0, \end{array} \qquad \begin{array}{cccccccccc} & 0 \longrightarrow & \mathbf{Z}/r & \xrightarrow{i} & \mathbf{Z}/rs & \xrightarrow{p} & \mathbf{Z}/s & \longrightarrow 0 \\ p_1 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \| & \\ & 0 \longrightarrow & \mathbf{Z}/t & \xrightarrow{i} & \mathbf{Z}/r & \xrightarrow{p} & \mathbf{Z}/s & \longrightarrow 0 \end{array}$$

and

$$p_2 \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}/s & \xrightarrow{i} & \mathbf{Z}/rs & \xrightarrow{p} & \mathbf{Z}/r & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & \| & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & \mathbf{Z}/r & \xrightarrow{i} & \mathbf{Z}/rs & \xrightarrow{p} & \mathbf{Z}/s & \longrightarrow & 0. \end{array}$$

We shall use these observations below; also the Mayer–Vietoris sequence[13] of the braid diagram which, since $p_*i_* = i_*p_*: F(r) \to F(r)$ coincides with multiplication by s, reduces to

$$F(r) \xrightarrow{(\beta_*, -p_*)} F(s) \oplus F(s) \xrightarrow{(i_*, \beta_*)} F(r) \xrightarrow{s} F(r) \cdots. \tag{2.2}$$

Note also that the composite of two Bocksteins always vanishes. This too follows by a standard argument. Given short exact sequences $S: 0 \to A \to B \to C \to 0$ and $T: 0 \to C \to D \to E \to 0$, choose a free abelian group F mapping into D. Let G be the kernel of $F \to E$, and H the pullback of $B \to C \leftarrow G$. Then we have maps of short exact sequences

$$\begin{array}{ccccccccc} Q: 0 & \to & A & \to & H & \to & G & \to & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \pi} & & \\ S: 0 & \to & A & \to & B & \to & C & \to & 0 \end{array} \qquad \begin{array}{ccccccccc} R: 0 & \to & G & \to & F & \to & E & \to & 0 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow & & \| & & \\ T: 0 & \to & C & \to & D & \to & E & \to & 0, \end{array}$$

so $\beta_S \circ \beta_T = \beta_S \circ \pi_* \circ \beta_R = \beta_Q \circ \beta_R$.

But G, a subgroup of F, is free so Q splits and $\beta_Q = 0$. (This follows since $F(H) \to F(G)$ is surjective, by exactness.)

This argument needs modification if $\mathscr{A}$ does not contain free groups. If, for example, we only have finite cyclic groups, and $|A| = a$, $|C| = c$, $|E| = e$, choose F or order ace. Then π is isomorphic to the map $B \to C$, so $\beta_S \circ \pi_* = 0$ by the exact sequence for S (we do not use Q, which is not in the category).

When we speak of a *natural transformation* of Bockstein functors, $\mu: F \to G$, we mean a transformation which is also natural with respect to Bockstein homomorphisms—i.e. for a short exact sequence $S: 0 \to A \to B \to C \to 0$,

$$\mu(A) \circ \beta_F(S) = \beta_G(S) \circ \mu(C): F(C) \to G(A).$$

For any space X, the (singular) homology or cohomology $H_*(X; A)$ provides an example of a Bockstein functor of A. The same also applies to generalised homology theories. We are particularly interested here in the case of bordism, and in the next section give a geometrical discussion of representatives following[7]; see also [2], [5].

We conclude this section by showing what is involved in verifying Axiom 2.

THEOREM 2. *To define a Bockstein functor on the category of cyclic groups* $\mathbf{Z}/m$ $(m \geq 0)$ *is equivalent to prescribing*

(i) *Abelian groups* A_m $(m \geq 0)$

(ii) *homomorphisms* $i_*: A_r \to A_{mr}$, $p_*: A_{mr} \to A_r$ *such that*

(iiia) *both* i_*p_* *and* p_*i_* *are multiplication by* m

(iiib) *the composite of two maps* p_* *is another*

(iiic) *the composite of two maps* i_* *is another, and* $i_* = 0$ *if* $m = 0$

(iv) *exact sequences*

$$A_0 \xrightarrow{r} A_0 \xrightarrow{p_*} A_r \xrightarrow{\beta_r} A_0 \xrightarrow{r} A_0$$

(v) *such that the following diagrams commute*:

$$\text{(a)} \quad \begin{array}{ccc} A_r & \xrightarrow{i} & A_{rs} \\ & \searrow{\scriptstyle \beta_r} & \downarrow{\scriptstyle \beta_{rs}} \\ & & A_0 \end{array} \qquad \text{(b)} \quad \begin{array}{ccc} A_{rs} & \xrightarrow{\beta_{rs}} & A_0 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle r} \\ A_s & \xrightarrow{\beta_s} & A_0. \end{array}$$

Proof. Certainly if F is a Bockstein functor, $A_m = F(\mathbf{Z}/m)$ has these properties: we must work to prove the converse. The proof is presented in outline only as the result is not really essential for our purposes: we use the properties above rather than any abstract results about Bockstein functors.

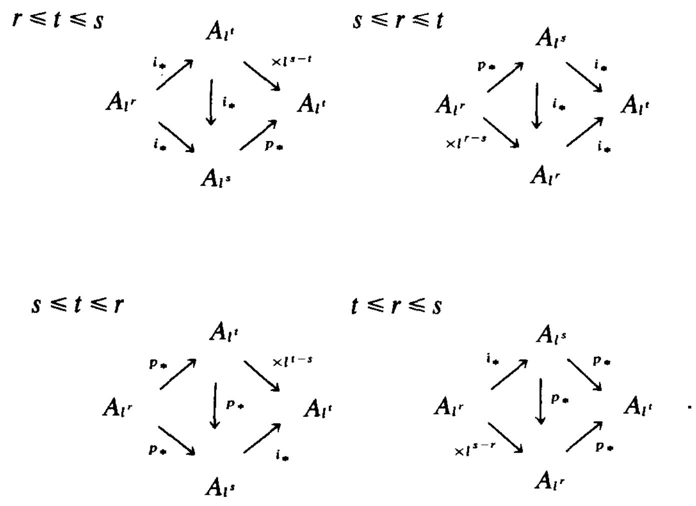
First (iv), with $r = 1$, shows that A_1 is trivial. Now putting $r = 1$ in (iiia) shows that A_m has exponent m. As any homomorphism $\mathbf{Z}/r \to \mathbf{Z}/s$ is a multiple of the composite $\mathbf{Z}/r \xrightarrow{p} \mathbf{Z}/(r, s) \xrightarrow{i} \mathbf{Z}/s$ we can define F on morphisms using p_* and i_*, but must work to establish the functor property for composites, $\mathbf{Z}/r \to \mathbf{Z}/s \to \mathbf{Z}/t$. All cases in which one of s, t is zero are trivial. If $r = 0$, the morphisms must be ap, bip for some integers a, b; then

$$\begin{aligned} F(bip) \circ F(ap) &= abi_* \circ p_* \circ p_* && \text{by definition} \\ &= abi_* \circ p_* && \text{by (iiib)} \\ &= F(bip \circ ap) && \text{by definition.} \end{aligned}$$

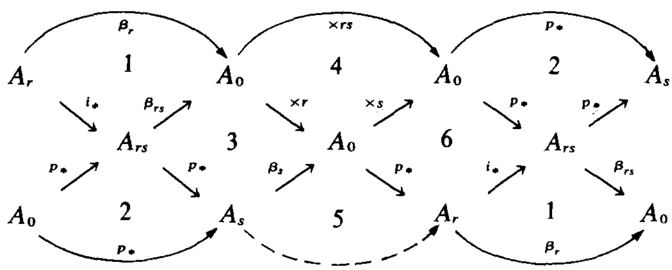
For other cases, suppose all groups l-primary and, since multiples cause no problem, that the morphisms to be composed are either i or p

$$\mathbf{Z}/l^r \to \mathbf{Z}/l^s \to \mathbf{Z}/l^t.$$

There are six cases according to relative sizes of r, s, t: $r \geq s \geq t$ is dealt with by (iiib), $r \leq s \leq t$ by (iiic), and the rest by commutative diagrams

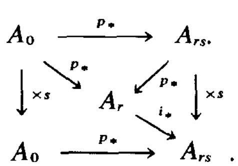


Next we consider the diagram



(2.3)

Here (1) commutes by (va), (2) by (iiib), (3) by (vb), (4) trivially and (6) because of the commutative diagram



Define the dotted arrow to make (5) commute. Now three of the sequences in (2.3) are exact, by (iv). It follows (c.f. [13]) that the fourth sequence is exact, provided we check that the composite $p_* i_*$ vanishes.

First suppose r, s coprime. Then the composite clearly vanishes, so we have an exact sequence $0 \to A_r \to A_{rs} \to A_s \to 0$ and A_{rs} is a direct sum of A_r and A_s: we can either choose injections i_* or projections p_* (but not both simultaneously; they need modifying by an integer). It follows now that it is sufficient to check the functor property on l-primary parts, which was done above. (Without this device, I think a further axiom would be necessary: commutativity of diagrams

$$\begin{array}{ccccc} & & A_{abc} & & \\ & {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle p_*} & \\ A_{ab} & & & & A_{ac} \\ & {\scriptstyle p_*}\searrow & & \nearrow{\scriptstyle i_*} & \\ & & A_a & & \end{array}.$$

So we have a functor. Now for arbitrary r, s above we have $p_* i_* = 0_* = 0$, and hence the sequence

$$\cdots A_r \xrightarrow{i_*} A_{rs} \xrightarrow{p_*} A_s \xrightarrow{p_*\beta_s} A_r \cdots$$

is exact.

We must now define β_* for arbitrary short exact sequences S, and check naturality for morphisms of S. If S involves infinite groups, we have an isomorphism

$$\begin{array}{lcccccccc} S: 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\times\epsilon t} & \mathbf{Z} & \xrightarrow{ap} & \mathbf{Z}/t & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle\times\epsilon} & & \| & & \uparrow{\scriptstyle\times a} & & \\ S_0: 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{t} & \mathbf{Z} & \xrightarrow{p} & \mathbf{Z}/t & \longrightarrow & 0 \end{array}$$

where $\epsilon = \pm 1$ and $(a, t) = 1$. We must thus define $\beta(S) = \epsilon a' \beta(S_0)$, where $aa' \equiv 1 \bmod t$. Otherwise, S is of the form $S: 0 \longrightarrow \mathbf{Z}/r \xrightarrow{ai} \mathbf{Z}/rs \xrightarrow{bp} \mathbf{Z}/s \longrightarrow 0$ with $(a, r) = 1$ and $(b, s) = 1$. Choosing a suitable representative of a, we may suppose further $(a, rs) = 1$. Then $(1, a, ab)$ give an isomorphism onto S of $S_1: 0 \to \mathbf{Z}/r \xrightarrow{i} \mathbf{Z}/rs \xrightarrow{p} \mathbf{Z}/s \to 0$ so we define $\beta(S) = c\beta(S_1)$ where $abc \equiv 1 \pmod s$. From now on, it suffices to consider the sequences S_0, S_1.

Next observe that morphisms of sequences S_0, S_1 are composites of a few 'elementary' morphisms. There are no nonzero morphisms $S_1 \to S_0$, and any morphism $S_0 \to S_1$ factors $S_0 \to S_0' \xrightarrow{q_1} S_1$ where q_1 is the canonical projection

$$q_1 \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\times s} & \mathbf{Z} & \xrightarrow{p} & \mathbf{Z}/s & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \| & & \\ 0 & \longrightarrow & \mathbf{Z}/r & \xrightarrow{i} & \mathbf{Z}/rs & \xrightarrow{p} & \mathbf{Z}/s & \longrightarrow & 0. \end{array}$$

For a morphism between two sequences S_0

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{r} & \mathbf{Z} & \xrightarrow{p} & \mathbf{Z}/r & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle bp} & & \\ 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{s} & \mathbf{Z} & \xrightarrow{p} & \mathbf{Z}/s & \longrightarrow & 0 \end{array}$$

$br = as$, so if $r = r'(r, s)$ and $s = s'(r, s)$ we can write $a = cr'$, $b = cs'$ for some c, and our morphism is a composite as follows:

$$\begin{array}{lccccccccc} & 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\times r} & \mathbf{Z} & \xrightarrow{p} & \mathbf{Z}/r & \longrightarrow & 0 \\ q_0 & & & \downarrow{\scriptstyle\times c} & & \downarrow{\scriptstyle\times c} & & \downarrow{\scriptstyle\times c} & & \\ & 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\times r} & \mathbf{Z} & \xrightarrow{p} & \mathbf{Z}/r & \longrightarrow & 0 \\ q_2 & & & \downarrow{\scriptstyle\times r'} & & \| & & \downarrow{\scriptstyle p} & & \\ & 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\times(r,t)} & \mathbf{Z} & \xrightarrow{p} & \mathbf{Z}/(r,t) & \longrightarrow & 0 \\ q_3 & & & \| & & \downarrow{\scriptstyle\times t'} & & \downarrow{\scriptstyle i} & & \\ & 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\times t} & \mathbf{Z} & \xrightarrow{p} & \mathbf{Z}/t & \longrightarrow & 0. \end{array}$$

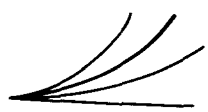
A similar but more complicated argument shows that any morphism between two sequences S_1 is a composite of one of four further types: multiplication through (p_0) by an integer, and the p_1, p_2, p_3 earlier in this section.

However, we do not need this. Given any morphism $f\colon (f_1, f_2, f_3)\colon S_1 \to S_1'$, consider $f \circ q_1\colon S_0 \to S_1'$. Suppose naturality of Bocksteins proved for $f \circ q_1$: then it follows for f. For since $\beta(S_1) = p_* \circ \beta(S_0)$ we then have $f_{1*} \circ \beta(S_1) = (f_1 \circ p)_* \circ \beta(S_0) = \beta(S_1') \circ f_*$ as required. By the above, $f \circ q_1$ can be factorised as a composite of morphisms q_i. But naturality for q_0 is trivial, for q_1 holds by definition, and for q_2 and q_3 is given by (vb) and (va) respectively.

This concludes the proof. We could discuss natural transformations similarly, but will be content with observing that it follows by the above analysis that if a transformation is natural on morphisms, then it commutes with all Bocksteins if it commutes with those for the sequences S_0.

§3. GEOMETRY OF $\mathbf{Z}/n$-MANIFOLDS

As it will be central to the arguments in this paper, we now recall the geometrical treatment [7] of bordism with $\mathbf{Z}/n$ coefficients. A $\mathbf{Z}/n$-manifold is defined by a pair $(\bar N, \beta N)$ of oriented manifolds, a partition $\partial \bar N = \cup \{\partial_i \bar N\colon 1 \le i \le n\}$ of the boundary of $\bar N$ into disjoint open submanifolds, and orientation-preserving homeomorphisms $h_i\colon \beta N \to \partial_i \bar N$. It is customary to identify βN with $\partial_i \bar N$ by h_i, obtaining a quotient space $N \supset \beta N$ of $\bar N$. We call βN the Bockstein of N. The identification is compatible with tangent bundles, so we obtain an oriented tangent bundle to $\bar N$. We visualise a neighbourhood of βN in N as a product with n mutually tangent rays.

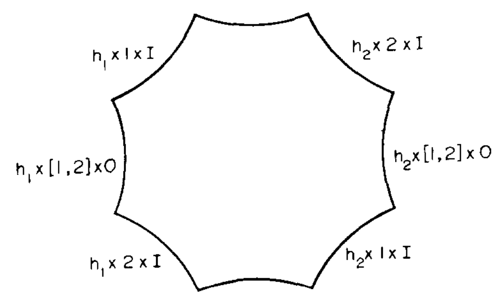


A $\mathbf{Z}/n$-manifold with boundary is defined in the obvious way: we have a pair $(\bar M, \beta M)$ of oriented manifolds, βM with boundary and $\bar M$ with a corner separating $\partial \bar M$ into submanifolds with closures $\partial_{\mathrm{Int}} \bar M$ and $\partial_{\mathrm{Ext}} \bar M$, say; and a partition $\partial_{\mathrm{Int}} \bar M = \bigcup \partial_i \bar M$ and homeomorphisms $h_i\colon \beta \bar M \to \partial_i \bar M$ as before. We can now extend the usual notions of cobordism (with extra structure as relevant).

It is important to observe that

3.1. The bordism class of $\bar N$ does not depend on the ordering of $\partial_r \bar N, \ldots, \partial_r \bar N$.

It suffices to show the class unaltered by interchanging ∂_1 and ∂_2. Choose collar neighbourhoods $\partial_i N \times [0, 2]$ of $\partial_i \bar N$ $(i = 1, 2)$ in $\bar N$, with $\partial_i \bar N \times 0 = \partial_i \bar N$. The desired cobordism is now constructed from $\bar N \times I$ by deleting $(\partial_1 \bar N \cup \partial_2 \bar N) \times]1, 2[+ I$ and glueing in its place, along $(\partial_1 \bar N \cup \partial_2 \bar N) \times \{1, 2\} \times I$, the product of βN with the octagon (c.f. [16]). The modified manifold, which may be



regarded as $\bar N \times I$ with a copy of $\partial_1 \bar N \times I \times I$ attached, gives a cobordism of $\bar N$ to $\bar N'$ say; and there is an obvious homeomorphism $\bar N \to \bar N'$ which interchanges ∂_1 and ∂_2.

We next consider the coefficient homomorphisms $i\colon \mathbf{Z}/r \to \mathbf{Z}/mr$, $p\colon \mathbf{Z}/mr \to \mathbf{Z}/r$. Geometrically, we represent i by taking m copies of the manifold identified along the Bockstein.

"Replication"

We represent p by splitting the mr sheets which meet along the Bockstein into m groups, each of r sheets—so the Bockstein is replaced by r copies of itself.

" Partial splitting "

Notice that by the observation (3.1) that the ordering of the sheets is unimportant, we see that the partition into m sets of r is likewise unimportant.

PROPOSITION 3.2. *$\mathbf{Z}/m$-bordism, as defined above, is a Bockstein functor.*

Proof. We consider here the simple bordism of a space X, though the argument extends without difficulty to more general situations, and one such extension will be needed later in the paper. Write $\Omega_*(X;\mathbf{Z}/m)$ for the bordism groups of X defined by $\mathbf{Z}/m$-manifolds.

It suffices to verify the conditions of Theorem 2. We have just defined i_* and p_*: properties (iiib) and (iiic) are clear. For (iiia), we can choose the partial splitting so that p_*i_* just yields m copies, so represents multiplication by m. But i_*p_* is also a partial splitting of $i_*\bar{N}$, where $i: \mathbf{Z}/mr \to \mathbf{Z}/m^2r$, so is cobordant to the other partial splitting $p_*i_*\bar{N} = m\bar{N}$.

We obtain the exact sequence (iv) by defining β_r to be represented by the Bockstein. Since $\bar{N}$ is nothing more nor less than a null-cobordism of r copies of βN, exactness follows by the standard cobordism argument. Finally, commutativity of (va) and (vb) follows at once from our geometrical definitions of i and p (and β).

To conclude our discussion of i and p, observe that the identification maps of degree 1 $mN \to i(N)$, $p(N) \to N$ are compatible with tangent bundles. Thus for any $f: N \to X$, class $x \in H^*(X)$ and characteristic class u of the tangent bundle,

$$f^*xu(N)[N] = f^*xu(pN)[pN] \tag{3.3}$$

and correspondingly for iN.

More troublesome is the study of products. Let N, P be $\mathbf{Z}/n$-manifolds. The link of βN in N is a discrete set $\mathbf{n}$ of n points. Thus the link of $\beta N \times \beta P$ in $N \times P$ is the join $\mathbf{n} * \mathbf{n}$. But this is a $\mathbf{Z}/n$ 1-manifold; since $\Omega_1(\mathbf{Z}/n)$ vanishes, it bounds a $\mathbf{Z}/n$ 2-manifold T_n, say. We resolve the singularity of $N \times P$ by deleting (the interior of) the neighbourhood $\beta N \times \text{cone}\,(\mathbf{n}) \times \beta P \times \text{cone}\,(\mathbf{n})$ of $\beta N \times \beta P$, and attaching in its place $\beta N \times \beta P \times T_n$. The result is denoted $N \otimes P$.

The product is well-defined up to bordism, for T_n is so since $\Omega_2(\mathbf{Z}/n) = 0$. It is also associative up to bordism since $(M \otimes N) \otimes P$ and $M \otimes (N \otimes P)$ agree except near $\beta M \times \beta N \times \beta P$, where each is a product with this as factor. Thus the obstruction to associativity is an element of $\Omega_3(\mathbf{Z}/n) = 0$. Similar arguments dispose of the expected commutative diagrams involving products with i_* and p_*: we shall not attempt to list all such (those we need could be done *ad hoc*).

The natural collapsing map $\rho: N \otimes P \to N \times P$ is not compatible with tangent bundles. If we denote by π the collapsing map $N \otimes P \to \beta N \times \beta P \times T_n/\partial T_n$ then [7, 1.5] $\tau_{N\otimes P}$ is stably equivalent to $\rho^*(\tau_N \times \tau_P) \oplus \pi^*\zeta$, where ζ is a vector bundle over $T_n/\partial T_n$. The bundle ζ need not be trivial. However, as all are oriented bundles, $v_1(\zeta) = 0$. It follows that $v^2(\zeta) = 1$ and $v\,\mathrm{Sq}^1 v(\zeta) = 0$. As these are the only characteristic classes which will enter our calculations, it will be possible to forget about ζ. For example, for any $x \in H^*(N \times P; \mathbf{Z}/n)$,

$$v^2(N)v^2(P)x[N \times P] = v^2(N \otimes P)\rho^*x[N \otimes P] \tag{3.4}$$

and similarly with $v\,\mathrm{Sq}^1 v$ or (c.f. [7, p. 485]) l replacing v^2.

Now compare $\beta(N \otimes P)$ with the *disjoint* union of $\beta N \times P$ and $N \times \beta P$ (which is a $\mathbf{Z}/n$-manifold). Again, these agree except near $\beta N \times \beta P$, and using the vanishing of $\Omega_1(\mathbf{Z}/n)$ we see that they are cobordant:

$$\beta(N \otimes P) \sim (\beta N \times P) \cup (N \times \beta P). \tag{3.5}$$

As a final topic in this section, we discuss the special case of $\mathbf{Z}/2$-manifolds. A $\mathbf{Z}/2$-manifold N

is already, as topological space, an unoriented manifold: the orientation cover consists of two sheets, cross-joined along βN. There is a natural collapsing map ϕ of N to cone (2)/boundary, which can be identified with a circle, S^1; the orientation cover of N is induced from the nontrivial double cover of S^1. Thus we have a W-manifold. Conversely, given a W-manifold $N \xrightarrow{\phi} S^1$ we can (save in the topological case, with dim $N = 4$ or 5, and even then up to bordism of N) make ϕ transverse to a point whose preimage βN then determines the orientation cover as above, so that $(N, \beta N)$ is a $\mathbf{Z}/2$-manifold, determined up to bordism.

We can thus identify W-manifolds with $\mathbf{Z}/2$-manifolds. Note that one can take T_2 to be a square, so that $M \otimes N = M \times N$ in this case. It is easy to see [7, 1.4] that the tangent bundles of N as manifold (τ_1) and as $\mathbf{Z}/2$-manifold (τ_2) are related by $\tau_1 \oplus \phi^*\epsilon \cong \tau_2 \oplus \phi^*\eta$ where ϵ, η are trivial and nontrivial line bundles over S^1. Hence for the characteristic classes, if $v_1(\eta) = w_1(\eta) = \alpha$, (so $\alpha^2 = 0$)

$$v(\tau_1) = v(\tau_2)(1 + \phi^*\alpha),\ \mathrm{Sq}^1 v(\tau_1) = \mathrm{Sq}^1 v(\tau_2)(1 + \phi^*\alpha),\ \text{so}$$
$$v_1(\tau_1) = \phi^*\alpha,\ v^2(\tau_1) = v^2(\tau_2)\ \text{and}\ v\,\mathrm{Sq}^1 v(\tau_1) = v\,\mathrm{Sq}^1 v(\tau_2). \tag{3.6}$$

Now the inclusion j: $\beta N \subset N$ has trivial normal bundle, so respects stable characteristic classes.

With these two remarks, we can rewrite (1.1). The terms $\{(v^2(N)f^*\alpha_2 + v\,\mathrm{Sq}^1 v(N)f^*\mathrm{Sq}^1\alpha_0)g^*k + v^2(N)f^*\alpha_0 g^*l\}[N]$ are, by the above, unaltered if we use τ_2 in place of τ_1. The others can be rewritten (as already noted) $v_1(N)(v^2(N)f^*\alpha_3 + v\,\mathrm{Sq}^1 v(N)f^*\alpha_0)[N] = (v^2(\beta N)f^*\alpha_3 + v\,\mathrm{Sq}^1 v(\beta N)f^*\alpha_0)[\beta N]$ and again reference to τ_1 has been omitted. We can now regard Theorem 1 as proved for $\mathbf{Z}/2$-manifolds.

§4. $\mathbf{Z}/2^r$-MANIFOLDS

For the next two sections, we shall investigate the consequences of the following two axioms.

AXIOM 2. $\lambda : \Omega_*(X \times G/\mathrm{Top}; \mathbf{Z}/n) \to F(\mathbf{Z}/n)$

is a natural transformation of Bockstein functors defined on cyclic groups.

Here, Ω_* denotes oriented smooth or topological bordism. In particular, we have composite maps

$$\lambda' : \Omega_*(X; \mathbf{Z}/n) \otimes \Omega_*(G/\mathrm{Top}; \mathbf{Z}/n) \to F(\mathbf{Z}/n).$$

Now surgery on $\mathbf{Z}/n$-manifolds without fundamental groups leads also to a ($\mathbf{Z}/4$-graded) Bockstein functor $P_*(\mathbf{Z}/n)$ (and $P_*(\mathbf{Z}/2)$ can be identified with $L_*(\mathbf{Z}^-)$). In analogy with Axiom 1, we impose

AXIOM 3. *λ' factors through a homomorphism*

$$\lambda'' : \Omega_*(X; \mathbf{Z}/n) \otimes P_*(\mathbf{Z}/n) \to F(\mathbf{Z}/n).$$

Here we restrict consideration to the 2-primary study of λ, leaving the odd torsion for §5.

We first apply the results of §1 to $\lambda(\mathbf{Z}/2)$. As F is additive, $F(\mathbf{Z}/2)$ has exponent 2 hence can be expressed as a sum of cyclic groups of order 2, with injections i_x and projections p_x. We can apply Theorem 1 to $p_x \circ \lambda(\mathbf{Z}/2)$: we obtain cohomology classes $\alpha_{r,x} \in H^{**}(X; \mathbf{Z}/2)$. Define $\alpha_r = \Sigma i_{x*}(\alpha_{r,x}) \in H^{**}(X; F(\mathbf{Z}/2))$. Note that the coefficients here really are the sum, not product, of copies of $\mathbf{Z}/2$: repeating the proof of Theorem 1 shows that α_r is defined by a homomorphism $H_*(X; \mathbf{Z}/2) \to F(\mathbf{Z}/2)$, and there is no problem here. Now the formula (1.1) yields $\lambda(\mathbf{Z}/2)$: note that $\mathrm{Sq}^1\alpha_0$ can be interpreted componentwise, or we can rewrite the terms so as not to involve it.

Next we apply (1.5). Here, β is the Bockstein for the short exact sequence $0 \to \mathbf{Z}/2 \to \mathbf{Z}/4 \to \mathbf{Z}/2 \to 0$. As λ is a Bockstein functor, $\lambda\beta(N; f, g) = \beta\lambda(N; f, g)$. Thus β_*: $F(\mathbf{Z}/2) \to F(\mathbf{Z}/2)$, as coefficient homomorphism, must satisfy

$$\beta_*\alpha_0 = \mathrm{Sq}^1\alpha_0,\ \beta_*\alpha_2 = 0,\ \text{and}\ \beta_*\alpha_3 = \alpha_2. \tag{4.1}$$

The formula (1.1) may now be written as

$$v^2(N)f^*\alpha_0 g^*l[N] + \{v^2(N)f^*(\beta_*\alpha_3) + v\,\mathrm{Sq}^1 v(N)f^*(\beta_*\alpha_0)\}g^*k[N]$$
$$+ \{v^2(\beta N)f^*\alpha_3 + v\,\mathrm{Sq}^1 v(\beta N)f^*\alpha_0\}g^*k[\beta N].$$

Now write

$$C(N;f,g)=(v^2(N)f^*\alpha_3+v\,\mathrm{Sq}^1v(N)f^*\alpha_0)g^*k[N]. \tag{4.2}$$

Then we conclude

$$\lambda(\mathbf{Z}/2)(N;f,g)=v^2(N)f^*\alpha_0 g^*l[N]+\beta_*C(N;f,g)+C(\beta N;f,g). \tag{4.3}$$

This form of the result is far more convenient for generalisation.

We are now ready to obtain general formulae, and an appropriate one is suggested by (4.3). The object of this Section is to prove the following.

THEOREM 3. *Let λ satisfy Axioms 2 and 3. Then there are unique classes $\eta_r\in H^{**}(X;F(\mathbf{Z}/2^r))$ such that for any $\mathbf{Z}/2^r$-manifold N:*

$$\lambda(\mathbf{Z}/2^r)(N;f,g)=l(N)f^*\eta_r g^*l[N]+\beta_*C(N;f,g)+i_*C(\beta N;f,g). \tag{4.4}$$

Moreover, $\eta_1=\alpha_0$ and $p_*\eta_r=\eta_s$ for $r>s$.

Here, in accord with our conventions, the precise groups involved for β_*, i_* and p_* are determined from the context: the two former have source $\mathbf{Z}/2$ and target $\mathbf{Z}/2^r$.

Proof. We shall prove the result by induction on r: we saw in (4.3) that it is true for $r=1$. We first consider λ': by Axiom 3, it suffices to consider λ''. Now $P_i(\mathbf{Z}/2^r)\cong\mathbf{Z}/2^r, 0, \mathbf{Z}/2, \mathbf{Z}/2$ for $i\equiv 0, 1, 2, 3 \pmod 4$; the isomorphisms may again be denoted τ, $-$, c and $c\beta$. There are thus three cases to examine. We take them in the reverse order.

The $c\beta$ case. We will use the fact that i_*: $P_3(\mathbf{Z}/2)\to P_3(\mathbf{Z}/2^r)$ is an isomorphism. Now the commutative diagram

$$\begin{array}{ccccc} \mathbf{Z}/2^r\otimes\mathbf{Z}/2 & & \xrightarrow{1\otimes i} & & \mathbf{Z}/2^r\otimes\mathbf{Z}/2^r \\ \downarrow{\scriptstyle p\otimes 1} & & & & \downarrow{\scriptstyle m} \\ \mathbf{Z}/2\otimes\mathbf{Z}/2 & \xrightarrow{m} & \mathbf{Z}/2 & \xrightarrow{i} & \mathbf{Z}/2^r \end{array}$$

(where m denotes isomorphisms induced by multiplication) yields a commutative diagram

$$\begin{array}{ccccc} \Omega_*(X;\mathbf{Z}/2^r)\otimes\Omega_*(Y;\mathbf{Z}/2) & & \xrightarrow{1\otimes i_*} & & \Omega_*(X;\mathbf{Z}/2^r)\otimes\Omega_*(Y;\mathbf{Z}/2^r) \\ \downarrow{\scriptstyle p_*\otimes 1} & & & & \downarrow \\ \Omega_*(X;\mathbf{Z}/2)\otimes\Omega_*(Y;\mathbf{Z}/2) & \longrightarrow & \Omega_*(X\times Y;\mathbf{Z}/2) & \xrightarrow{i_*} & \Omega_*(X\times Y;\mathbf{Z}/2^r) \end{array}$$

and surgery obstructions and λ map this diagram into

$$\begin{array}{ccccc} \Omega_*(X;\mathbf{Z}/2^r)\otimes P_*(\mathbf{Z}/2) & & \xrightarrow{1\otimes i_*} & & \Omega_*(X;\mathbf{Z}/2^r)\otimes P_*(\mathbf{Z}/2^r) \\ \downarrow{\scriptstyle p_*\otimes 1} & & & & \downarrow{\scriptstyle \lambda''(\mathbf{Z}/2^r)} \\ \Omega_*(X;\mathbf{Z}/2)\otimes P_*(\mathbf{Z}/2) & \xrightarrow{\lambda''(\mathbf{Z}/2)} & F(\mathbf{Z}/2) & \xrightarrow{i_*} & F(\mathbf{Z}/2^r) \end{array}$$

which thus also commutes.

Now if $(P;g)$ has $c\beta(P;g)=1$ (and $c(P;g)=\tau(P;g)=0$), we can write $(P;g)=i_*(P';g')$ where P', g' has the same properties. Thus

$$\lambda''(\mathbf{Z}/2^r)((M;f)\times(P;g))=i_*\{\lambda''(\mathbf{Z}/2)(p_*(M;f)\times(P';g'))\}.$$

If we write $p_*(M,f)=(M',f')$, then applying the result for $r=1$ shows that the expression in braces equals

$$(v^2(M')f'^*\alpha_3+v\,\mathrm{Sq}^1v(M')f'^*\alpha_0)[M']=C_0(M';f'),\ \text{say}. \tag{4.5}$$

But by (3.3), $C_0(M';f')=C_0(M;f)$. Thus when $c\beta(P;g)=1$ and $c(P;g)=\tau(P;g)=0$, we have $\lambda(\mathbf{Z}/2^r)((M;f)\times(P;g))=i_*C_0(M;f)$.

The c case. Here we suppose that $c(P;g)=1$ and that $c\beta(P;g)=\tau(P;g)=0$. As $(P;g)$ may be replaced by any other with the same properties, we choose a $\mathbf{Z}/2^r$-manifold $(P';g')$ with $c\beta(P';g')=1$ and $c(P',g')=\tau(P';g')=0$, and take $(P;g)=\beta(P';g')$.

Now the product $(M;f)\times(P';g')$ defines a class in $\Omega_*(X\times G/\mathrm{Top};\mathbf{Z}/2^r)$ which is represented by $(M\otimes P';f'',g'')$ say. By the $c\beta$ case, above, $\lambda(\mathbf{Z}/2^r)(M\otimes P';f'',g'')=i_*C_0(M,f)$. Since λ is a Bockstein functor, and by (3.5),

$$\begin{aligned}\beta_*i_*C_0(M;f)&=\beta_*\lambda(\mathbf{Z}/2^r)(M\otimes P';f'',g'')\\&=\lambda(\mathbf{Z}/2^r)\beta(M\otimes P';f'',g'')\\&=\lambda(\mathbf{Z}/2^r)((M;f)\times(P;g))+\lambda(\mathbf{Z}/2^r)((\beta M;f)\times(P';g'))\\&=\lambda(\mathbf{Z}/2^r)((M;f)\times(P;g))+i_*C_0(\beta M;f).\end{aligned}$$

Now since $\beta_*i_*=\beta_*$ by (2.1), we have

$$\lambda(\mathbf{Z}/2^r)((M;f)\times(P;g))=\beta_*C_0(M;f)+i_*C_0(\beta M;f).$$

The τ case. Taking λ'' with $P_0(\mathbf{Z}/2^r)$ yields a homomorphism $\lambda_r:\Omega_*(X;\mathbf{Z}/2^r)\to F(\mathbf{Z}/2^r)$, which is one of $\Omega_*(\mathbf{Z}/2^r)$-modules, where this ring operates on $F(\mathbf{Z}/2)^r)$ via the signature in $\mathbf{Z}/2^r$. Thus if $\Omega_*(X;\mathbf{Z}/2^r)\otimes_{\Omega_*(\mathbf{Z}/2^r)}\mathbf{Z}/2^r\cong H_*(X;\mathbf{Z}/2^r)$, the natural duality between homology and cohomology would yield the desired class. However, X may have torsion and we must be more circumspect.

According to Browder, Liulevicius and Peterson[1], when Ω_* is localised at 2, it gives a sum of homology functors. Equivalently, there exists a homotopy equivalence e of the spectrum $MSO\otimes\mathbf{Z}_{(2)}$ with $\Pi\{K(\mathbf{Z}_{(2)},4n)$: ω a partition of $n\}$. The inclusion of $K(\mathbf{Z}_{(2)},0)$, composed with e^{-1}, induces a right inverse j to the augmentation, or to the map $\Omega_*(\ ;\mathbf{Z}_{(2)})\to H_*(\ ;\mathbf{Z}_{(2)})$ defined by signatures. Then j induces a map of Bockstein functors. Further, since $\Omega_*(T;\mathbf{Z}/2^r)$ depends only on the chain complex of T, we can split this into elementary summands. For a summand $\mathbf{Z}$ or $\mathbf{Z}\xrightarrow{n}\mathbf{Z}$ with $2^r|n$ we obtain a free $\Omega_*(\mathbf{Z}/2^r)$-module with 1 or 2 basis elements, and the corresponding map

$$\Omega_*(\ ;\mathbf{Z}/2^r)\otimes_{\Omega_*(\mathbf{Z}/2^r)}\mathbf{Z}/2^r\to H_*(\ ;\mathbf{Z}/2^r)$$

is an isomorphism. The other summands yield elements of orders dividing 2^{r-1}.

Now $\lambda_r\circ j: H_*(X;\mathbf{Z}/2^r)\to F(\mathbf{Z}/2^r)$ determines by duality a class $\eta_r\in H^*(X;F(\mathbf{Z}/2^r))$. And since j is a natural transformation, the diagram

$$\begin{array}{ccc}H_*(X;\mathbf{Z}/2^r)&\xrightarrow{\lambda_r\circ j}&F(\mathbf{Z}/2^r)\\ \downarrow{\scriptstyle p_*}&&\downarrow{\scriptstyle p_*}\\ H_*(X;\mathbf{Z}/2^{r-1})&\xrightarrow{\lambda_{r-1}\circ j}&F(\mathbf{Z}/2^{r-1})\end{array}$$

commutes, implying that $p_*\eta_r=\eta_{r-1}$ whence $p_*\eta_r=\eta_s$ for all $s<r$. As j is right inverse to the natural map, we also deduce $\eta_1=\alpha_0$. Moreover, if the image of (M,f) in $\Omega_*(X;\mathbf{Z}/2^r)\otimes_{\Omega_*(\mathbf{Z}/2^r)}\mathbf{Z}/2^r$ is contained in that of $H_*(X;\mathbf{Z}/2^r)$ via j, we deduce that $\lambda_r(M;f)=l(M)f^*\eta_r[M]$.

But by the above analysis, the class of (M,f) is the sum of such a class and one of order 2^s $(s<r)$. By (2.2) a class of order 2^s is in $\operatorname{Im} i_*+\operatorname{Im}\beta_*$ where both i_*,β_* have source $\Omega_*(X;\mathbf{Z}/2^s)$. By our inductive hypothesis, the result holds on this group. We now compute $\lambda_ri_*(M',f')=i_*\lambda_s(M',f')=i_*(l(M')f'^*\eta_s[M'])$. By the analogue for i_* of (3.3), $2^{r-s}(l(M')f'^*\eta_r[M'])=l(i_*M')f^*\eta_r[i_*M']$, as desired. Finally observe that j, hence $\lambda\circ j$ is a Bockstein functor. The diagram

$$\begin{array}{ccc}H_*(X;\mathbf{Z}/2^s)&\xrightarrow{\lambda_sj}&F(\mathbf{Z}/2^s)\\ \downarrow{\scriptstyle\beta_*}&&\downarrow{\scriptstyle\beta_*}\\ H_*(X;\mathbf{Z}/2^r)&\xrightarrow{\lambda_rj}&F(\mathbf{Z}/2^r)\end{array}$$

thus commutes. Now

$$\lambda_s(M',f')=l(M')f'^*\eta_s[M']=\lambda_sjf'_*(l(M')\cap[M']).$$

Thus

$$\begin{aligned}\lambda_r\beta_*(M',f')&=\beta_*\lambda_s(M',f')=\beta_*\lambda_sjf'_*(l(M')\cap[M'])\\&=\lambda_rj\beta_*f'_*(l(M')\cap[M'])=\lambda_rjf_*(l(M)\cap[M])=l(M)f^*\eta_r[M],\end{aligned}$$

where

$$(M,f)=\beta(M',f').$$

Conclusion of the proof. What we have shown so far amounts to this, that

$$\lambda'((M;f)\times(P;g)) = i_*C_0(M;f)c\beta(P;g) + (\beta_*C_0(M;f) + i_*C_0(\beta M;f))c(P;g) + l(M)f^*\eta_r[M]\tau(P;g). \quad (4.6)$$

We must first deduce from this that (4.4) holds on decomposable elements, and then prove the general case.

Now

$$C((M;f)\times(P;g)) = C_0(M;f)c(P;g) + v^2(M)f^*\alpha_0[M]d(P;g), \quad (4.7)$$

where $d(P,g) = v\mathrm{Sq}^1v(P)g^*k[P]$ is not a surgery invariant. Thus taking $(N;f,g) = (M;f)\otimes(P,g)$, the right hand side of (4.4) yields, using (3.4) and (3.5),

$$\begin{aligned} &l(M)f^*\eta_r[M]l(P)g^*l[P] + \beta_*C_0(M;f)c(P;g)\\ &\quad + v^2(M)f^*(\beta_*\alpha_0)[M]d(P;g) + i_*C_0(\beta M;f)c(P,g)\\ &\quad + i_*C_0(M;f)c\beta(P;g) + v^2(\beta M)f^*(i_*\alpha_0)[\beta M]d(P;g)\\ &\quad + v^2(M)f^*(i_*\alpha_0)[M]d\beta(P;g). \end{aligned} \quad (4.8)$$

Here, the three terms involving C_0 are the same as the corresponding terms in (4.6). The two terms involving $d(P;g)$ cancel. For as $\alpha_0 = p_*\eta_r$, $i_*\alpha_0 = 2^{r-1}\eta_r$. Thus the two terms are $\beta_*\lambda_1p_*(M)$ and $2^{r-1}\lambda_r(\beta M)$. Now

$$\begin{aligned} \beta_*\lambda_1p_*(M) &= \beta_*p_*\lambda_r(M) && \text{as } \lambda \text{ a Bockstein transformation}\\ &= i_*\beta_*\lambda_r(M)\\ &= i_*p_*\beta_*\lambda_r(M) && \text{by properties (2.1) of Bockstein functors.} \end{aligned}$$

Here the indicated groups are $2^r \xrightarrow{\beta} 2^r \xrightarrow{p} 2 \xrightarrow{i} 2^r$, so $i_*p_* = 2^{r-1}$ and we have $2^{r-1}\beta_*\lambda_r(M) = 2^{r-1}\lambda_r(\beta M)$.

The final term of (4.8) equals $l(M)2^{r-1}f^*\eta_r[M]d\beta(P;g)$, so we can combine the first and last terms to give

$$l(M)f^*\eta_r[M]\{l(P)g^*l[P] + i_*d\beta(P,g)\} = l(M)f^*\eta_r[M]\tau(P,g).$$

Thus our formula (4.6) for λ' is indeed the same as that given by (4.4).

We can now complete the proof along the same lines as for the τ case above. For we see, just as there, that $\Omega_*(X\times G/\mathrm{Top};\mathbf{Z}/2^r)$ is the sum of the image of $\Omega_*(X;\mathbf{Z}/2^r)\otimes\Omega_*(G/\mathrm{Top};\mathbf{Z}/2^r)$ and a group of lower exponent, which is thus in the image of $i_* + \beta_*$ from groups with coefficient 2^s, $s < r$, where the formula holds by inductive hypothesis. Now λ is a Bockstein functor. Thus assuming (4.4) for $(N;f,g)$, we have

$$\begin{aligned} \lambda(\mathbf{Z}/2^r)i_*(N;f,g) &= i_*\lambda(\mathbf{Z}/2^s)(N;f,g)\\ &= i_*\{l(N)f^*\eta_sg^*l[N] + \beta_*C(N;f,g) + i_*C(\beta N;f,g)\}. \end{aligned}$$

Now $i_*\eta_s = 2^{r-s}\eta_r$, but $i_*(N)$ is defined by 2^{r-s}-fold replication. Also, $i_*\beta_* = 0$, even for $r = s+1$, by the natural exact sequence. As i_*N has the same Bockstein as N, however, we see that this agrees with (4.4) applied to $i_*(N;f,g)$. Similarly,

$$\begin{aligned} \lambda(\mathbf{Z}/2^r)\beta_*(N;f,g) &= \beta_*\lambda(\mathbf{Z}/2^s)(N;f,g)\\ &= \beta_*\{l(N)f^*\eta_sg^*l[N] + \beta_*C(N;f,g) + i_*C(\beta N;f,g)\}. \end{aligned}$$

Here the first term—arguing as in the τ case above—yields $l(\beta N)f^*\eta_rg^*l[\beta N]$; the second vanishes as the composite of two Bocksteins is always zero, and for the third we note by (2.1) $\beta_*i_* = \beta_*$. Thus again the result is as given by (4.4) for βN. This completes the proof of Theorem 3.

§5. $\mathbf{Z}/n$-MANIFOLDS, n ODD

The result in the odd case is analogous to that in the even case.

THEOREM 4. *Let λ satisfy Axioms 2 and 3. Then for n odd, there is a unique class $\theta_n \in KO^*(X;F(\mathbf{Z}/n))$ such that for any $\mathbf{Z}/n$-manifold N, $\lambda(\mathbf{Z}/n)(N;f,g) = f^*\theta_ng^*\Delta[N]_\Delta$, where*

$[N]_\Delta$ *denotes Sullivan's fundamental* $KO\otimes\mathbf{Z}[\frac{1}{2}]$*-homology class, and* $\Delta\in KO^*(G/\mathrm{Top})\otimes\mathbf{Z}[\frac{1}{2}]$ *is as defined in* [10]. *Moreover,* $p_*\theta_n=\theta_m$ *for* $m|n$.

First we prove the formula in the case when N is smooth.

Proof. We present this in outline only, as no new arguments are needed beyond those in the preceding section. First consider λ': by Axiom 3 this amounts to a homomorphism $\lambda_n:\Omega_*(X;\mathbf{Z}/n)\to F(n)$, since $P_i(\mathbf{Z}/n)\cong\mathbf{Z}/n$, for $i\equiv 0$ (mod 4), 0 otherwise. Thus we only have the analogue of the τ case to consider. Now in the case of smooth bordism, it follows from results of Milnor[6] and Conner and Floyd[4] (as was noted by Sullivan[10]) that if we regard $\mathbf{Z}/n$ as $\Omega_*(\mathbf{Z}/n)$-module via the signature, then $\Omega_*(X;\mathbf{Z}/n)\otimes_{\Omega_*(\mathbf{Z}/n)}\mathbf{Z}/n\cong KO_*(X;\mathbf{Z}/n)$, indeed; Sullivan proves the result for the odd localisation $\Omega_*(X)\otimes\mathbf{Z}[\frac{1}{2}]$. As λ_n is an $\Omega_*(\mathbf{Z}/n)$-module map, we have a homomorphism $KO_*(X;\mathbf{Z}/n)\to F(n)$, and hence by duality a class $\theta_n\in KO^*(X;F(\mathbf{Z}/n))$. The result $p_*\theta_n=\theta_m$ follows in the same way as the result for η. In view of the module structure used, we now see that $\lambda_n(M;f)=f^*\theta_n[M]_\Delta$. Hence $\lambda'((M;f)\times(P;g))=f^*\theta_n[M]_\Delta\tau(P;g)$ and we can compute τ using KO-theory as $\tau(P;g)=g^*\Delta[P]_\Delta$ by [10]. Thus if $N=M\otimes P$, $\lambda'(N;f,g)=f^*\theta_n g^*\Delta[N]_\Delta$.

Perhaps we should emphasise what is implicit in the above, that for X an infinite complex we consider $KO^*(X)$ as defined by maps $X\to BO$, not by vector bundles.

I will now show how Theorem 4 can be extended to the case of topological bordism. The same argument will also show that Sullivan's characterisation[10] of his $KO^*\otimes\mathbf{Z}[\frac{1}{2}]$-orientation of topological bundles, via signatures of transverse preimages for maps from manifolds to the total space, is valid for topological as well as for smooth manifolds.

The argument was developed in the course of conversations with Don Anderson, Peter May, Jim Milgram and Vic Snaith during a visit to the University of Chicago in August 1975: it is a pleasure to express my gratitude for this occasion.

The proof is divided into three sections: recall of the splitting of B Top, calculations in K-theory, and their application to the problem.

Most of the results on B Top are due to Sullivan[10], but we will follow more closely the account of May[19]. Throughout this section, we localise at a fixed odd prime p: this must be understood, so we omit it from our notation. Choose a number k generating the multiplicative group $(\mathbf{Z}/p^2\mathbf{Z})^\times$.

According to Adams, [17, lecture 4], there is a canonical splitting (π_1,π_2): $\mathrm{BSO}\to V_1\times V_2$, where $\pi_i(V_1)\neq 0\Leftarrow\Rightarrow i\equiv 0$ mod $(2p-2)$. Write (i_1,i_2) for its homotopy inverse.

Write BT for the classifying space for $KO\otimes\mathbf{Z}[\frac{1}{2}]$-oriented spherical fibrations. If U is the universal orientation of the Thom space, ψ^kU/U defines a class on BT classified by a map $s:BT\to BO$, say. Write j: $B\operatorname{Cok}J\to BT$ for the mapping fibre of s. Write also e_2: $BO\to BT$ for the fibre of the natural map f: $BT\to BG$. Finally, the Atiyah-Bott-Shapiro orientation for vector bundles induces a map e_1: $BO\to BT$. The product of e_1i_1: $V_1\to BT$, e_2i_2: $V_2\to BT$ and j: $B\operatorname{Cok}J\to BT$, composed with $(\pi_1,\pi_2)\times 1$, induces a homotopy equivalence $h:BO\times B\operatorname{Cok}J\to BT$.

Observe that since e_2 is the fibre of f, the spherical fibration induced by e_1i_2 is fibre homotopy trivial. The same holds also for e_1i_2, which factors through e_2[19, 4.11]—this amounts to using the Adams conjecture, and checking Adams' criterion for fibre homotopy triviality. Thus $h|BO$ induces a fibration fibre homotopy equivalent to the universal bundle, which is induced by e_1. Hence we have a homotopy equivalence of Thom spectra induced by h, $\hat{h}:\mathbf{M}O\wedge\mathbf{M}\operatorname{Cok}J\to\mathbf{M}T$.

Finally, we recall that the Sullivan orientation induces a homotopy equivalence B Top $\to BT$ which we use to identify the two, hence also $\mathbf{M}T$ and $\mathbf{M}$ Top. I repeat that the above only holds without qualification after localisation at the fixed odd prime p.

The key result for our argument is the following, due to Hodgkin and Snaith[18, Theorem 3.15]: $KO_*(B\operatorname{Cok}J)\otimes\mathbf{Z}[\frac{1}{2}]=0$. Localising further at the prime p, we noted above that the universal spherical fibration over $B\operatorname{Cok}J$ is $KO_*\otimes\mathbf{Z}[\frac{1}{2}]$-oriented. Hence by the Thom isomorphism for K-theory, $KO_*(\mathbf{M}\operatorname{Cok}J,\mathrm{S})\otimes\mathbf{Z}[\frac{1}{2}]=0$, where S denotes the sphere spectrum. In other words, the homotopy groups of the spectrum $(\mathbf{M}\operatorname{Cok}J/\mathrm{S})\wedge BO\otimes\mathbf{Z}[\frac{1}{2}]$ all vanish. Hence this spectrum is contractible. Hence for any X, its smash product with X is again contractible. Thus $KO_*(\mathbf{M}\operatorname{Cok}J\wedge X,\mathrm{S}\wedge X)\otimes\mathbf{Z}[\frac{1}{2}]=0$.

The proof of the smooth case of Theorem 4, and also Sullivan's construction of his orientation, depend on the natural equivalence $\Omega_*(X;\mathbf{Z}/n)\otimes_{\Omega_*(\mathbf{Z}/n)}\mathbf{Z}/n\cong KO_*(X;\mathbf{Z}/n)$ valid for

n odd—it is clearly sufficient to consider the case when n is a power of p. The extension to topological bordism is thus accomplished by

PROPOSITION 5.1. *For any X, and n odd, the inclusion of smooth in topological bordism induces an isomorphism*

$$\Omega_*(X;\mathbf{Z}/n)\otimes_{\Omega_*(\mathbf{Z}/n)}\mathbf{Z}_n \xrightarrow{\tau} \Omega_*^{\mathrm{Top}}(X;\mathbf{Z}_n)\otimes_{\Omega_*^{\mathrm{Top}}(\mathbf{Z}/n)}\mathbf{Z}/n.$$

Proof. Since Sullivan's orientation is defined also for topological manifolds, the natural transformation $\Omega_*(Y)\to KO_*(Y)\otimes\mathbf{Z}[\frac{1}{2}]$ which it defines factors through $\Omega_*^{\mathrm{Top}}(Y)$. It thus suffices to show that this transformation induces an isomorphism from both domain and range of τ onto $KO_*(X;\mathbf{Z}/n)$.

The result is known for the domain. As to the range we have, computing and using the results cited above,

$$\begin{aligned}\Omega_*^{\mathrm{Top}}(X;\mathbf{Z}/n) &= \pi_*(\mathbf{M}\,\mathrm{Top}\wedge X;\mathbf{Z}/n)\\ &= \pi_*(\mathbf{M}O\wedge\mathbf{M}\,\mathrm{Cok}\,J\wedge X;\mathbf{Z}/n)\\ &= \Omega_*(\mathbf{M}\,\mathrm{Cok}\,J\wedge X;\mathbf{Z}/n)\end{aligned}$$

and so

$$\begin{aligned}\Omega_*^{\mathrm{Top}}(X;\mathbf{Z}/n)\otimes_{\Omega_*(X;\mathbf{Z}/n)}\mathbf{Z}/n &= \Omega_*(\mathbf{M}\,\mathrm{Cok}\,J\wedge X;\mathbf{Z}/n)\otimes_{\Omega_*(\mathbf{Z}/n)}\mathbf{Z}/n\\ &= KO_*(\mathbf{M}\,\mathrm{Cok}\,J\wedge X;\mathbf{Z}/n) = KO_*(X;\mathbf{Z}/n),\end{aligned}$$

where Ω_*^{Top} is regarded as Ω_*-module using h, rather than e_1. It follows that the map above does induce an isomorphism $\Omega_*^{\mathrm{Top}}(X;\mathbf{Z}/n)\otimes_{\Omega_*^{\mathrm{Top}}(\mathbf{Z}/n)}\mathbf{Z}/n\to KO_*(X;\mathbf{Z}/n)$, and the proposition is proved The two applications mentioned are immediate corollaries.

§6. THE INTEGRAL CASE

We are now ready to attempt integer formulae: after all the preparatory work of the previous sections, this will not now be so difficult. We work with the same Axioms 2 and 3, reinterpreted now to include infinite cyclic groups in $\mathcal{A}$ (we can write $\mathbf{Z}=\mathbf{Z}/0$). It will still be necessary to work separately at even and odd primes, with the localisations $\mathbf{Z}_{(2)}$ and $\mathbf{Z}[\frac{1}{2}]$ respectively. However, since these are flat $\mathbf{Z}$-modules (localisation is exact) we can introduce these by tensor product without needing further geometrical axioms.

Our first concern is with formula for λ: the following, though not altogether satisfactory, is general and effective.

THEOREM 5. *Let λ satisfy Axioms 2 and 3. Then it determines homomorphisms*

$$\eta_0\colon H_*(X;\mathbf{Z}_{(2)})\to F(\mathbf{Z})\otimes\mathbf{Z}_{(2)},\quad \theta_0\colon KO_*(X)\otimes\mathbf{Z}[\tfrac{1}{2}]\to F(\mathbf{Z})\otimes\mathbf{Z}[\tfrac{1}{2}]$$

such that

(i) *we have commutative squares*

$$\begin{array}{ccc} H_*(X;\mathbf{Z}_{(2)}) & \xrightarrow{\eta_0} & F(\mathbf{Z})\otimes\mathbf{Z}_{(2)}\\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*}\\ H_*(X;\mathbf{Z}/2^r) & \xrightarrow{\eta'_r} & F(\mathbf{Z}/2^r)\end{array}\qquad \begin{array}{ccc} KO_*(X)\otimes\mathbf{Z}[\frac{1}{2}] & \xrightarrow{\theta_0} & F(\mathbf{Z})\otimes\mathbf{Z}[\frac{1}{2}]\\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*}\\ KO_*(X;\mathbf{Z}/n) & \xrightarrow{\theta'_n} & F(\mathbf{Z}/n)\end{array}$$

for $r\geq 1$ and n odd, where η'_r, θ'_n are given by Kronecker product with η_r, θ_n respectively,

(ii) *if $(N;f,g)$ determines a class in $\Omega_*(X\times G/\mathrm{Top})$, then the even and odd localisations of $\lambda(N;f,g)$ respectively are given by*

$$\eta_0(f_*\{l(N)g^*l\cap[N]\})+\beta_*C(N;f,g)\ \text{and}\ \theta_0(f_*\{g^*\Delta\cap[N]_\Delta\}). \tag{6.1}$$

Proof. It will suffice to describe the even case: the other is similar but simpler. Now λ induces as usual a map $\lambda'\colon\Omega_*(X)\otimes\Omega_*(G/\mathrm{Top})\to F(\mathbf{Z})$ and hence by Axiom 3 another map $\lambda''\colon\Omega_*(X)\otimes P_*\to F(\mathbf{Z})$. Now $P_0\cong\mathbf{Z}$: taking tensor product with the generator yields a map $\Omega_*(X)\to F(\mathbf{Z})$ and thus also $\lambda_0\colon\Omega_*(X;\mathbf{Z}_{(2)})\to F(\mathbf{Z})\otimes\mathbf{Z}_{(2)}$.

We may now simply define η_0 as $\lambda_0\circ j$, where j is as before. Assertion (i) of the Theorem follows since we have Bockstein functors (the same argument gave $p_*\eta_r=\eta_s$). The proof of assertion (ii) follows the same pattern as that of Theorem 3. First we show that

$$\lambda_0(M, f) = \eta_0(f_*\{l(M) \cap [M]\}). \tag{6.2}$$

For, as before, we have $\Omega_*(X; \mathbf{Z}_{(2)}) = P_*(X; \mathbf{Z}_{(2)}) + T_*(X; \mathbf{Z}_{(2)})$ where P_* is the Ω_* submodule generated by $j(H_*(X; \mathbf{Z}_{(2)})$, so the result holds here; and T_* is a torsion submodule, the sum of the images of the Bockstein maps $\beta_*: \Omega_*(X; \mathbf{Z}/2^r) \to \Omega_*(X; \mathbf{Z}_{(2)})$. But

$$\begin{aligned}\lambda_0\beta_*(M'; f') &= \beta_*\lambda_r(M'; f') \quad \text{as we have a Bockstein functor}\\ &= \beta_*\lambda_r j f'_*\{l(M') \cap [M']\}\\ &= \eta_0\beta_* f'_*\{l(M') \cap [M']\}\\ &= \eta_0 f_*\{l(M) \cap [M]\}\end{aligned}$$

for $(M; f) = \beta(M'; f')$. Thus (6.2) holds.

Next we obtain the formula for λ' on $\Omega_*(X) \otimes P_2$. We can choose $(P; g) = \beta(P'; g')$ with $c(P; g) = 1$ and $(P'; g')$ a $\mathbf{Z}/2$-manifold; then $(M; f) \times (P; g) = \beta\{M; f) \times (P'; g')\}$ so we obtain

$$\beta_*\lambda(\mathbf{Z}/2)((M; f) \times (P'; g')) = \beta_* C_0(M; f).$$

Thus for decomposable elements,

$$\lambda'((M; f) \times (P; g)) = \eta_0 f_*\{l(M) \cap [M]\}\tau(P; g) + \beta_* C_0(M; f)c(P; g).$$

We check that this agrees with (6.1). But this yields, using (4.7),

$$\eta_0 f_*\{l(M) \cap [M] \cdot g^* l[P]\} + \beta_* C_0(M; f)c(P; g) + v^2(M)f^*(\beta_*\alpha_0)[M]d(P; g).$$

Now $\tau(P; g) = g^* l[P]$, and $\beta_*\alpha_0 = 0$ since (i) shows that α_0 is in the image of $H^*(X; F(\mathbf{Z}) \otimes \mathbf{Z}_{(2)})$. Thus the formulae agree. Now, as above, $\Omega_*(X \times G/\mathrm{Top}; \mathbf{Z}_{(2)})$ is the sum of the images of $\Omega_*(X; \mathbf{Z}_{(2)}) \otimes \Omega_*(G/\mathrm{Top}; \mathbf{Z}_{(2)})$ and of the Bocksteins β_* on $\Omega_*(X \times G/\mathrm{Top}; \mathbf{Z}/2^r)$, so it remains only to check the formulae on these. But

$$\begin{aligned}\lambda\beta(N; f, g) &= \beta_*\lambda(\mathbf{Z}/2^r)(N; f, g)\\ &= \beta_*\{l(N)f^*\eta_r g^* l[N] + \beta_* C(N; f, g) + i_* C(\beta N; f, g)\}\end{aligned}$$

and, as before, the second term vanishes, the third gives $\beta_* C(\beta N; f, g)$ and the first is $\eta_0 f_*(l(\beta N) \cap [\beta N])$ as we see by following the image of $f_*\{l(N)g^* l \cap [N]\}$ round the commutative diagram

$$\begin{array}{ccc} H_*(X; \mathbf{Z}/2^r) & \xrightarrow{\eta'_r} & F(\mathbf{Z}/2^r)\\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*}\\ H_*(X; \mathbf{Z}_{(2)}) & \xrightarrow{\eta_0} & F(\mathbf{Z}) \otimes \mathbf{Z}_{(2)}. \end{array}$$

This completes the proof of Theorem 5.

The unsatisfactoriness of the above result lies in the fact that we have failed to obtain classes η_∞, θ_∞ in $H^*(X; F(\mathbf{Z}) \otimes \mathbf{Z}_{(2)})$ and $KO^*(X; F(\mathbf{Z}) \otimes \mathbf{Z}[\frac{1}{2}])$ inducing the η_r, θ_r $(r \geq 0)$. I do not see how to obtain such classes without imposing finiteness conditions. Let us begin with some further discussion of Bockstein functors.

Suppose now that F is a Bockstein functor on the class of all abelian groups. For any group A, choose a resolution $0 \to R \to G \to A \to 0$ where G, hence also R is a free abelian group. Now $F(G) = F(\mathbf{Z}) \otimes G$ holds by additivity for G finitely generated free abelian. If we suppose that F preserves direct limits (which we consider rather as a definition of F on groups not of finite type than as an axiom), it follows that $F(G) \cong F(\mathbf{Z}) \otimes G$ for any torsion free group G. In general, the exact triangle

$$\begin{array}{ccccccc} F(\mathbf{Z}) \otimes R & \longrightarrow & F(\mathbf{Z}) \otimes G & & & & \\ \| & & \| & & & & \\ F(R) & \longrightarrow & F(G) & \longrightarrow & F(A) & \longrightarrow & F(R) \end{array}$$

yields a short exact sequence $0 \to F(\mathbf{Z}) \otimes A \to F(A) \to \mathrm{Tor}\,(F(\mathbf{Z}), A) \to 0$, showing that F is effectively determined by $F(\mathbf{Z})$.

Now suppose $F(\mathbf{Z})$ of finite exponent N. Then the maps $\operatorname{Tor}(F\mathbf{Z}, \mathbf{Z}/Nr\mathbf{Z}) \to \operatorname{Tor}(F(\mathbf{Z}), \mathbf{Z}/r\mathbf{Z})$ induced by p_* vanish, so $\varprojlim F(\mathbf{Z}/r\mathbf{Z}) = \varprojlim F(\mathbf{Z}) \otimes \mathbf{Z}/r\mathbf{Z} = F(\mathbf{Z})$.

THEOREM 6. *Assume Axioms* 2 *and* 3, *and that* $F(\mathbf{Z})$ *has finite exponent. Then there are unique classes* $p_\infty \in H^{**}(X; F(\mathbf{Z}) \otimes \mathbf{Z}_{(2)})$, $\theta_\infty \in KO^*(X; F(\mathbf{Z}) \otimes \mathbf{Z}[\frac{1}{2}])$ *inducing the* p_r *and* θ_n. *They also induce* p_0 *and* θ_0.

This follows from the above discussion, noting that $H^{**}(X; F(A))$ is also a Bockstein functor having exponent N when $A = \mathbf{Z}$, and localising at 2 or away from 2 appropriately.

However, $F(\mathbf{Z})$ will not have finite exponent in most cases, so further discussion is needed. Next, we consider the case $F(\mathbf{Z}) = \mathbf{Z}$ and recall some ideas of Sullivan[7], [10], [11]. A transformation $H_*(X; \mathbf{Z}/n) \to \mathbf{Z}/n$ of Bockstein functors induces, taking direct limit under i_*, a map $H_*(X; \mathbf{Q}/\mathbf{Z}) \to \mathbf{Q}/\mathbf{Z}$. Now if each group $H_r(X; \mathbf{Z})$ is finitely generated, we can identify

$$\operatorname{Hom}(H_*(X; \mathbf{Q}/\mathbf{Z}), \mathbf{Q}/\mathbf{Z}) \cong H^{**}(X; \hat{\mathbf{Z}}),$$
$$\operatorname{Hom}(H_*(X; \mathbf{Q}), \mathbf{Q}) \cong H^{**}(X; \mathbf{Q}) \qquad \text{and}$$
$$\operatorname{Hom}(H_*(X; \mathbf{Q}), \mathbf{Q}/\mathbf{Z}) \cong H^{**}(X, \hat{\mathbf{Q}})$$

where $\hat{\mathbf{Z}} = \varprojlim \mathbf{Z}/n$ is the product over primes p of the p-adic integers $\hat{\mathbf{Z}}_p$, and $\hat{\mathbf{Q}} = \mathbf{Q} \otimes \hat{\mathbf{Z}}$. Now the exact coefficient sequence $0 \to \mathbf{Z} \to \mathbf{Q} \oplus \hat{\mathbf{Z}} \to \hat{\mathbf{Q}} \to 0$ induces a cohomology exact sequence, and the Bockstein vanishes (on account of the finiteness hypothesis). Thus $H^{**}(X; \mathbf{Z})$ is isomorphic to the group of commutative diagrams

$$\begin{array}{ccc} H_*(X; \mathbf{Q}) & \longrightarrow & \mathbf{Q} \\ \downarrow & & \downarrow \\ H_*(X; \mathbf{Q}/\mathbf{Z}) & \longrightarrow & \mathbf{Q}/\mathbf{Z}. \end{array}$$

PROPOSITION 7. *Let F be a Bockstein functor on the class of cyclic groups $\mathbf{Z}/n$ ($n \geq 0$), X a space with each $H_*(X; \mathbf{Z})$ finitely generated and $\mu: H_*(X; \mathbf{Z}/n) \to F(\mathbf{Z}/n)$ a Bockstein functor. Then if $F(\mathbf{Z})$ is the sum of a finitely generated group and a group of finite exponent, there is a (unique) class $\eta \in H^{**}(X; F(\mathbf{Z}))$ inducing μ via Kronecker product*

$$H_*(X; \mathbf{Z}/n) \otimes H^{**}(X; F(\mathbf{Z})) \to \mathbf{Z}/n \otimes F(\mathbf{Z}) \to F(\mathbf{Z}/n).$$

It may be observed that any such η induces a Bockstein functor, with no need for finiteness conditions.

Proof. Write $F(\mathbf{Z})$ as a direct sum of copies of $\mathbf{Z}$ and of a group of finite exponent. Then F decomposes correspondingly as a sum of functors (by arguments sketched above, using additivity), so it suffices to consider the cases separately

Now the result for $F(\mathbf{Z})$ of finite exponent was observed above, and the construction of η for $F(\mathbf{Z}) = \mathbf{Z}$ was described just before the statement of the Proposition: it suffices to note that the commutative diagrams

$$\begin{array}{ccc} H_*(X; n^{-1}\mathbf{Z}) & \longrightarrow & n^{-1}\mathbf{Z} \\ \downarrow & & \downarrow \\ H_*(X; n^{-1}\mathbf{Z}/\mathbf{Z}) & \longrightarrow & n^{-1}\mathbf{Z}/\mathbf{Z} \end{array}$$

given by the Bockstein functor yield the desired diagram on proceeding to direct limits.

I do not see how to weaken substantially the hypotheses of the Proposition, and accordingly introduce

AXIOM 4. *$F(\mathbf{Z})$ is the sum of a finitely generated group and a group of finite exponent.*

We have, however, obvious modifications of the Proposition in which we consider only certain primes (and then localise at them), or in which homology is replaced by K-theory. Applying these as appropriate to the Bockstein functors defined by the η_r and θ_n ($r \geq 0$, $n \geq 0$) we obtain our final result.

THEOREM 8. *If λ satisfies Axioms* 2, 3 *and* 4, *and if each $H_i(X; \mathbf{Z})$ is finitely generated, then there are classes η_∞, θ_∞ as in Theorem* 6.

We observe, in conclusion, that G/Top defines a cohomology theory $\mathscr{L}^*$ which is a mixture of KO^* at odd primes and H^{**} at even primes (with some coefficients $\mathbf{Z}/2$ thrown in). One can thus—at least informally—interpret α_3, η_∞ and θ_∞ as defining a single characteristic class $\chi \in \mathscr{L}^*(X; F(\mathbf{Z}))$ which completely determines λ. I leave the interested reader to pursue this: I see little prospect of such an interpretation simplifying any actual calculations. It may however suggest an alternative formulation: that the universal functor λ satisfying our axioms takes values in $F(A) = \mathscr{L}_*(X; A)$.

§7. SURGERY OBSTRUCTIONS SATISFY THE AXIOMS

In my book[14] I introduced, for any finitely presented group π and homomorphism $w: \pi \to \{\pm 1\}$, surgery obstruction groups $L_n(\pi, w)$; and (in §13B) I showed, following Sullivan, that surgery obstructions yield a map $\theta: \Omega_*(K(\pi, 1) \times G/\mathrm{Top}) \to L_*(\pi, w)$ which, moreover, vanishes on $\Omega_*(K(\pi, 1))$. I wish to apply the full results of the preceding chapters to this map. The crucial step is given by the following result.

THEOREM 9. (i) *Surgery on* $\mathbf{Z}/m$*-manifolds leads to a Bockstein functor* $L_*(\pi, w; \mathbf{Z}/n)$
(ii) θ *extends to a natural transformation of Bockstein functors.*

Thus our Axioms 1 and 2 hold for the geometrical problem.

Proof. The essential point of the proof consists in observing that the technique of [14, §9] is sufficiently flexible to cover the present situation, after one minor modification. Namely, a $(\mathbf{Z}/m)$-manifold can be regarded as an $(m+1)$-ad $(X; X_1, \ldots, X_m)$ such that $\partial X = \cup X_i$ and the X_i are disjoint, together with preferred (orientation-preserving) homeomorphisms between the X_i. Similarly, to any type $\mathbf{T}$ of manifold r − ad, we can define a "$\mathbf{Z}/m - \mathbf{T}$" to be an $(m+r)$ − ad where the first m faces are as above, the others have the extra structure imposed by $\mathbf{T}$. Now in [14, §9] I defined cobordism sets $L_n^1(K)$ of "objects", groups $L_n^2(K)$ of "restricted objects", and a natural map $L_n^1(K) \to L_n^2(K)$ bijective for $n \geq 4$. The identical reasoning, with the modification just indicated, (or rather, the proof of (9.5) loc. cit) yields a cobordism group $L_n^2(K; \mathbf{Z}/m)$ of $\mathbf{Z}/m$-objects, well behaved for $n \geq 5$.

Next I assert that the arguments of [14, 9.6] yield an exact sequence

$$\ldots L_n^2(K) \xrightarrow{m} L_n^2(K) \to L_n^2(K; \mathbf{Z}/m) \to L_{n-1}^2(K) \ldots$$

To the usual cutting and glueing arguments we need only add the remark that for X above, its boundary is a sum of m copies of X_1. Now, again as in [14, §9], we deduce from the Five Lemma that for $n \geq 6$, $L_n^2(K; \mathbf{Z}/m)$ depends only on $\pi_1(K)$, and we denote it by $L_n(\pi_1(K), w; \mathbf{Z}/m)$. That this is a Bockstein functor now follows exactly as in §3, using Theorem 2.

To extend θ to a transformation $\Omega_*(K(\pi, 1) \times G/\mathrm{Top}; \mathbf{Z}/n) \to L_*(\pi, w; \mathbf{Z}/n)$ we must again proceed geometrically. Given a $\mathbf{Z}/n$-manifold $(X; X_1, \ldots, X_n)$ and maps $f: X \to K(\pi, 1)$, $g: X \to G/\mathrm{Top}$ (such that all $f|X_i$ and all $g|X_i$ coincide), we apply the Browder–Thom transversality construction to $g|X_1$ to construct a normal map $Y_1 \to X_1$; we have n copies of this, and now apply the construction relative to ∂X to obtain a normal map $Y \to X$.

This determines a class in $L_*(\pi, w; \mathbf{Z}/n)$ as desired. The procedure is compatible with bordisms, so defines a map θ; also it is compatible with replication, with partial splitting and with taking Bocksteins, so θ is compatible with all the induced maps above, and is thus a natural transformation of Bockstein functors.

In order to apply to θ the results of the preceding sections, we must now verify the product formula (Axiom 3). It is no harder to prove a more general result, which will be very useful for calculations.

THEOREM 10. *There is a commutative diagram*

$$\begin{array}{ccc}
\Omega_*(K(\pi', 1); \mathbf{Z}/n) \otimes \Omega_*(K(\pi'', 1) \times G/\mathrm{Top}; \mathbf{Z}/n) & \xrightarrow{1 \otimes \theta} & \Omega_*(K(\pi', 1); \mathbf{Z}/n) \otimes L_*(\pi''; \mathbf{Z}/n) \\
\downarrow & & \downarrow \\
\Omega_*(K(\pi' \times \pi'', 1) \times G/\mathrm{Top}; \mathbf{Z}/n) & \xrightarrow{\theta} & L_*(\pi' \times \pi''; \mathbf{Z}/n).
\end{array}$$

Axiom 3 is the case of this in which π'' is trivial. The point of the theorem is, of course, the existence of the right hand arrow.

Proof. Suppose given $\mathbf{Z}/n$-manifolds N and P, and maps $f_1: N \to K(\pi', 1)$, $f_2: P \to K(\pi'', 1)$, $g: P \to G/\text{Top}$ compatible with the identifications on the boundary. Form the product $\rho: N \otimes P \to N \times P$. We must show that $\theta(N \otimes P, f_1 \times f_2, g)$ depends only on N, f_1 and $\theta(P, f_2, g)$.

Recall the definition of θ for closed manifolds P. Using transversality we construct from (P, g) a normal map $(\phi: Q \to P; \nu; F: \tau_Q \oplus \phi^* \nu \cong \epsilon)$ and we define $\theta(P, f, g)$ to be the equivalence class of the 'object' $(\phi: (Q, \emptyset) \to (P, \emptyset), \nu, F, f)$. We can obtain a corresponding object for $(N \times P, f_1 \times f_2, g)$ by taking products throughout with N: choose a normal bundle ν_0 over N and a framing $F_0: \tau_N \oplus \nu_0 \to \epsilon$ and take

$$(1_N \times \phi: (N \times Q, \emptyset) \to (N \times P, \emptyset), \nu_0 \oplus \nu, F_0 \oplus F, f_1 \times f_2).$$

To prove the theorem in the case $n = 0$, it now suffices to observe that for any object $(\phi: (R, Q) \to (Y, X), \nu, F, f)$ or cobordism of objects, we can multiply by N in the same way, so that multiplication by (N, f_1) (with ν_0, F_0 fixed) maps bordism classes to bordism classes and defines a map of groups.

But we can extend this argument to $\mathbf{Z}/n$-objects. If we fix the bordism T_n, the construction of $N \otimes P$ from P is completely canonical. So a $\mathbf{Z}/n$-object, or bordism of objects, can be multiplied by N in the same way as above. This proves the theorem.

§8. APPLICATIONS TO SURGERY THEORY

We have now shown that the surgery obstruction map θ verifies the hypothesis of Theorem 5. We conclude the existence (among others) of homomorphisms

$$\eta_0: H_*(\pi; \mathbf{Z}_{(2)}) \to L_*(\pi^+) \otimes \mathbf{Z}_{(2)}$$
$$\theta_0: KO_*(K(\pi, 1)) \otimes \mathbf{Z}[\tfrac{1}{2}] \to L_*(\pi^+) \otimes \mathbf{Z}[\tfrac{1}{2}]$$

and a class $\alpha_2 \in H^{**}(\pi; L_*(\pi^+; \mathbf{Z}/2))$ which determine surgery obstructions as in (5.1).

It does not quite follow that we can cover the nonorientable case as well, since there the restriction on $f: N \to K(\pi, 1)$ is that "f is orientable" (i.e. $f^*w = w_1(N)$) rather than that N is. We can, however, include this by observing that if $\tilde{K}(\pi)$ is the Thom space of the orientation line bundle over $K(\pi, 1)$, we have a natural isomorphism (of degree 1) between (reduced) f-orientable bordism of $K(\pi, 1)$ and oriented bordism of $\tilde{K}(\pi)$. We can thus apply Theorem 5 taking $X = \tilde{K}(\pi)$ and observing a shift in dimensions. In the case of α_2 this translates back to give a class as above; for η_0 we are led to homology of π, but with coefficients twisted by w. With these understandings, all holds in the nonorientable case also.

To obtain the stronger results of Theorem 8, we need finiteness conditions. We say that π (or more precisely, (π, w)) satisfies (F) if $L_*(\pi, w)$ is the sum of a finitely generated group and a group of finite exponent, and if each $H_i(\pi; \mathbf{Z})$ is finitely generated. Certainly finite groups satisfy (F) (see e.g. [15] for the L-theory). Next, if A, B and C satisfy (F) and we are given embeddings $A \leftarrow C \to B$, then the amalgamated free product $G = A *_C B$ satisfies (F). For there is a homology exact sequence

$$\ldots H(C) \to H_i(A) \oplus H_i(B) \to H_i(G) \to H_{i-1}(C) \ldots$$

and for the L-theory a similar sequence which "fails to be exact only by a group of exponent 2": for a precise statement, see Cappell[3]. A similar conclusion holds for *HNN* groups. Thus any group which can be built up from finite groups by these constructions satisfies (F). This is a satisfyingly wide (though not characterisable) class. It can be extended a little by sharper arguments, e.g. to include fundamental groups of closed surfaces. For groups satisfying (F), we have classes

$$p_\infty \in H^{**}(\pi; L_*(\pi) \otimes \mathbf{Z}_{(2)}), \qquad \theta_\infty \in KO^*(K(\pi, 1); L_*(\pi) \otimes \mathbf{Z}[\tfrac{1}{2}])$$

which, with α_3, determine all surgery obstructions.

We turn to the question of computing these classes. Although I will not discuss this numerically in this paper, calculations will surely depend heavily on natural properties of the classes, especially ones relating different groups. Theorem 10 is useful for this; even more so is straightforward naturality.

PROPOSITION 11. *All the characteristic classes above are natural for morphisms of groups.*

For example, let $\phi: \pi \to \pi'$. Then $H^{**}(\phi)(\alpha_3(\pi')) = (L_*(\phi))_* \alpha_3(\pi)$ (8.1)

and we have a commutative diagram

$$\begin{array}{ccc} H_*(\pi; \mathbf{Z}_{(2)}) & \xrightarrow{\eta_0(\pi)} & L_*(\pi) \otimes \mathbf{Z}_{(2)} \\ \downarrow{\scriptstyle H_*(\phi)} & & \downarrow{\scriptstyle L_*(\phi)} \\ H_*(\pi'; \mathbf{Z}_{(2)}) & \xrightarrow{\eta_0(\pi')} & L_*(\pi') \otimes \mathbf{Z}_{(2)}. \end{array} \tag{8.2}$$

The assertion follows immediately from commutativity of the diagram

$$\begin{array}{ccc} \Omega_*(K(\pi, 1) \times G/\mathrm{Top}) & \xrightarrow{\theta(\pi)} & L_*(\pi) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle L_*(\phi)} \\ \Omega_*(K(\pi', 1) \times G/\mathrm{Top}) & \xrightarrow{\theta(\pi')} & L_*(\pi') \end{array}$$

which follows e.g. from the "naturality for inclusion maps" in the Main Theorem of [14]; together with the corresponding diagrams with coefficients $\mathbf{Z}/n$.

Now consider the case of a finite group π, with Sylow 2-subgroup σ, and $\phi: \sigma \to \pi$ the inclusion. We know that apart from the signature (untwisted), the image of $\theta(\pi)$ is all 2-torsion. But $H^*(\phi)$ is injective on 2-torsion, so $\alpha_3(\pi)$ and $p_\infty(\pi)$ are determined by the values of α_3 and p_∞ for σ. We can formulate this conclusion explicitly as follows.

THEOREM 12. *Let $M \to V$ be a normal map between closed manifolds with finite fundamental group π. Then surgery on this map to obtain a [simple] homotopy equivalence is possible if and only if surgery is possible for the covering spaces $\tilde{M} \to \tilde{V}$ with fundamental group σ the Sylow 2—subgroup of π.*

More generally, we will show that if $i: \sigma \subset \pi$ denotes the inclusion of any subgroup, then

$$(L_*(i))_* \theta(\tilde{V}) = |\pi : \sigma| \theta(V). \tag{8.3}$$

This implies the theorem, since then the index of σ is odd. The proof proceeds by direct calculation: it suffices to illustrate the case of a single term, say $v^2(N) f^* \alpha_2 g^* k[N] = \theta'(N; f, g)$. Now i is classified by a covering map Bi, inducing $\tilde{V}$ as a pullback:

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{f}} & K(\sigma, 1) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle Bi} \\ V & \xrightarrow{f} & K(\pi, 1) \end{array}$$

and the normal map $\tilde{M} \to \tilde{V}$ is induced by $\tilde{g} = g \circ p$. Also, p is a submersion, hence covered by a map of tangent bundles, so $p^* v^2(V) = v^2(\tilde{V})$. Hence

$$\theta'(\tilde{V}; \tilde{f}, \tilde{g}) = v^2(\tilde{V}) \tilde{f}^* \alpha_2(\sigma) \tilde{g}^* k[\tilde{V}] = p^* v^2(V) \tilde{f}^* \alpha_2(\sigma) p^* g^* k[\tilde{V}].$$

Now applying $(L_* i)_*$, we replace the second term by $\tilde{f}^*(L_* i)_* \alpha_2(\sigma) = \tilde{f}^*(Bi)^* \alpha_2(\pi)$ by Proposition 11, (8.1). Hence

$$(L_* i)_* \theta'(\tilde{V}; \tilde{f}, \tilde{g}) = p^*(v^2(V) f^* \alpha_2(\pi) g^* k)[\tilde{V}] = v^2(V) f^* \alpha_2(\pi) g^* k[p_* \tilde{V}]$$

and since $p_*[\tilde{V}] = |\pi : \sigma| [V]$, (8.3) follows.

Versions of this have been conjectured for some years. Observe that it is *not* the case that $L_* \pi = L_* \sigma$. If, for example, σ has a normal complement ρ (e.g. if π is 2-hyperelementary), then $L_* \sigma$ is a summand of $L_* \pi$ and the theorem asserts that surgery obstructions for *closed* manifolds always lie in this summand. It is thus analogous to Atiyah's result that the equivariant signature of a closed manifold is always a multiple of the regular representation.

The theorem focusses attention on the $L_*(\sigma)$ with σ a finite 2-group. It is worth noting that according to [15, (5.2)] these are substantially easier to compute than L-groups of other finite groups.

§9. FURTHER COMMENTS

Although we will not enter here into any explicit calculations of the classes η_0, θ_0 and α_2 (or even less, of η_∞, θ_∞ and α_3), we now make a few observations which follow from the very existence of such classes.

First observe that θ_0: $KO_*(K(\pi, 1)) \to L_*(\pi, w) \otimes \mathbf{Z}[\frac{1}{2}]$ (which always exists) is none other than the homomorphism l_π defined in [14, §17H], by considering smooth bordism: indeed, the construction above is essentially the same as our former one.

Next, we note that no mention has been made of precisely which L-groups are in question. The above is valid not only for the L^s-groups of [14] but also for the L^h-groups of [9] and for Cappell's intermediate L-groups[3], in particular, those computed for finite groups in [15]. We may thus calculate with whichever are the most convenient.

Now observe that in general not every element of $L_*(\pi, w)$ can appear as a surgery obstruction. For example [14, §13B], if π is finite and w trivial, an element which can so appear has as signature a multiple of the regular representation of π. The calculation of the ξ_i will determine a subgroup $SL_*(\pi, w) \subset L_*(\pi, w)$ consisting of the elements which can so appear. Thus for a manifold M, the surgery exact sequence[14, §10] can be amended to read

$$L_{m+1}(\pi_1(M)) \to \mathcal{S}(M) \to \mathcal{T}(M) \to SL_m(\pi_1(M)).$$

This suggests that in general, the coset in $L_m(\pi)/SL_m(\pi)$ of the surgery obstruction does not depend on the choice of the normal invariant. I intend to publish the proof of this result subsequently: it depends on a discussion of composition of normal maps. It follows that for a Poincaré complex X with a normal invariant, the class in $L_m^h(\pi)/SL_m^h(\pi)$ is a homotopy invariant of X. Since $SL_m(\pi)$ is frequently a relatively 'small' subgroup of $L_m(\pi)$—for example, if π is finite we noted above that it maps injectively to $L_m(\sigma)$, σ a Sylow 2-subgroup—it follows that 'most' of the obstruction to having X a manifold can be defined (and perhaps computed) a priori, without reference to tangential structures.

In conclusion, I remark that the ideas of Ranicki which motivated (but were not used in) this paper have now been developed to yield the following:

(1) For any manifold or Poincaré complex M^m a fundamental hypercohomology class

$$[M] \in H^{-m}_{\mathbf{Z}[\mathbf{Z}/2]}(\mathbf{Z}, C_M{}^t \otimes_{\mathbf{Z}\pi} C_M),$$

where C_M is the chain complex of $\tilde{M}$. An algebraic notion of bordism of chain complexes with duality defined by such a class leads to a "symmetric L-group" $L^m(\mathbf{Z}\pi)$.

(2) A normal map $M \to X$ induces (using S-duality) a stable right inverse $\Sigma^k X \to \Sigma^k M$, which can be made equivariant, so that the induced chain map is "geometric". We can then interpret C_M as a direct sum $C_X \oplus K_M$, and there is a fundamental hyperhomology class in $H_m{}^{\mathbf{Z}[\mathbf{Z}/2]}(\mathbf{Z}, K_M{}^t \otimes_{\mathbf{Z}\pi} K_M)$. An algebraic notion of bordism of chain complexes with duality defined by such a class leads to the usual L-groups $L_m(\mathbf{Z}\pi)$, and the class represented by K_M is the surgery obstruction for the normal map.

(3) There are products such as, for example, $L^m(A) \times L_n(B) \to L_{m+n}(A \times B)$ and the surgery obstruction of a normal map given by the product of a Poincaré complex P and a normal map $M \to X$ is the product of the classes defined by these two.

For a fuller statement of these results, see Ranicki's recent preprint "An algebraic theory of surgery".

Comparing Ranicki's results with ours, we see that ours are less general but more explicit in that, for example, we show that the class in $L^m(\mathbf{Z}\pi)$ of a *manifold* only contributes to product formulae via characteristic invariants such as $f_*(v^2(M) \cap [M])$. To obtain a fuller understanding, however, it is clear that what is needed next is more calculations for non-trivial examples.

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Topology Vol. 16, pp. 495–496. Pergamon Press, 1977. Printed in Great Britain

FORMULAE FOR SURGERY OBSTRUCTIONS: CORRIGENDUM

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(Received 4th April 1977)

PERCEPTIVE comments by John Morgan on the main results of this paper led to the discovery of a rather stupid error in the proof. In consequence, the results originally stated are probably incorrect. However, without much modification one can obtain correct statements sufficiently strong for the applications I had in mind. Morgan informs me that he has obtained equivalent results. I shall give the full statements of the corrected results, but only specify which modifications are needed to the proofs.

No alterations are needed before Theorem 3. That result should read as follows:

THEOREM 3. *Let λ satisfy Axioms 2 and 3. Then there is a natural transformation η: $H_*(X; \mathbf{Z}/2^r) \to F(\mathbf{Z}/2^r)$ of Bockstein functors such that for any $\mathbf{Z}/2^r$-manifold N,*

$$\lambda(\mathbf{Z}/2^r)(N; f, g) = \eta f_*(l(N)g^*l \cap [N]) + \beta_* C(N; f, g) + i_* C(\beta N; f, g). \quad (4.4)$$

The initial comments, and sections on "the $c\beta$ case" and "the c case" need no alteration. The fundamental mistake occurs in the discussion of "the τ case", viz. the sentence "Now $\lambda_r j$: $H_*(X;\ \mathbf{Z}/2^r) \to F(\mathbf{Z}/2^r)$ determines by duality a class $\eta_r \in H^*(X;\ F(\mathbf{Z}/2^r))$". Since j and λ are transformations of Bockstein functors, we replace this by "Now define $\eta = \lambda \circ j$." The argument given before again shows that if the class of (M, f) in $\Omega_*(X; \mathbf{Z}/2^r) \otimes_{\Omega_*(\mathbf{Z}/2^r)} \mathbf{Z}/2^r$ lies in $j_* H_*(X; \mathbf{Z}/2^r)$, then

$$\lambda_r(M, f) = \eta f_*(l(M) \cap [M]).$$

The rest of the proof holds good with only trivial modifications.

THEOREM 4. *Let λ satisfy Axioms 2 and 3. Then there is a natural transformation θ: $KO_*(X; \mathbf{Z}/n) \to F(\mathbf{Z}/n)$ of Bockstein functors (for n odd) such that for any $\mathbf{Z}/n$-manifold N,*

$$\lambda(\mathbf{Z}/n)(N; f, g) = \theta f_*(g^*\Delta \cap [N]_\Delta).$$

Again, the proof is as before, but eliminating the reference to duality between KO^* and KO_*.

Now Theorem 5 is valid (essentially without alteration), but the remainder of §6 (including Theorem 6, Proposition 7 and Theorem 8) should be deleted.

The remainder of the paper needs only trivial changes (e.g. we can delete the discussion of finiteness conditions for groups π). In view of its proven importance for applications, we give the required changes for the case of Theorem 12.

By Proposition 11, if ϕ: $\pi \to \pi'$, the diagram

$$\begin{array}{ccc} H_*(\pi) & \xrightarrow{H_*(\phi)} & H_*(\pi') \\ \downarrow{\scriptstyle \eta(\pi)} & & \downarrow{\scriptstyle \eta(\pi')} \\ L_*(\pi) & \xrightarrow{L^*(\phi)} & L_*(\pi') \end{array}$$

of natural transformations is commutative.

THEOREM 12. *Let $M \to V$ be a normal map between closed manifolds with fundamental group π, i: $\sigma \subset \pi$ a subgroup of finite index, $\tilde{M} \to \tilde{V}$ the corresponding covering normal map. Write $\theta(V) \in L_*(\pi)$, $\theta(\tilde{V}) \in L_*(\sigma)$ for the surgery obstructions. Then $L_*(i)\theta(\tilde{V}) = |\pi : \sigma| \theta(V)$.*

Hence if π is finite and σ its Sylow 2-subgroup, $\theta(V) = 0$ if and only if $\theta(\tilde{V}) = 0$.

Proof. We calculate the surgery obstructions in terms of the classifying maps, lying in commutative diagrams

$$\begin{array}{ccc} \tilde V & \xrightarrow{\tilde f} & K(\sigma,1) \\ \downarrow p & & \downarrow Bi \\ V & \xrightarrow{f} & K(\pi,1) \end{array} \qquad\qquad \begin{array}{ccc} \tilde V & \overset{\tilde g}{\searrow} & \\ \downarrow p & & BG \\ V & \underset{g}{\nearrow} & \end{array}$$

The 2-localisation of $\theta(\tilde V)$ is given by

$$\eta(\sigma)\tilde f_*(l(\tilde V)\tilde g^*L \cap [\tilde V]) + \beta_* C(\tilde N; \tilde f, \tilde g).$$

The first term here is computed as

$$\eta(\sigma)\tilde f_*(p^*l(V)p^*g^*l \cap [\tilde V])$$

and its image by $L_*(i)$ is therefore

$\eta(\pi)H_*(i)\tilde f_*(p^*l(V)p^*g^*l \cup [\tilde V])$ by Proposition 11

$= \eta(\pi)f_*p_*(p^*l(V)p^*g^*(l) \cap [\tilde V])$ by the above diagram: $H_*(i) = (Bi)_*$

$= \eta(\pi)f_*(l(V)g^*(l) \cap p_*[\tilde V])$ naturality of cap product

$= \deg p.\ \eta(\pi)f_*(l(V)g^*(l) \cap [V])$.

The argument for the other term is similar (half was written out in the paper), and so is that for the odd localisation. Since $\deg p = |\pi : \sigma|$, this proves the first assertion. The second follows—as before, since $L_*(\pi)$ has no odd torsion.

Formulae for surgery obstructions. *Topology* **15** (1976), 189–210.

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