# FOUNDATIONS OF ALGEBRAIC L-THEORY

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In my book [17] I introduced certain algebraic functors $L_n$ which were then used to express the obstruction to doing surgery. I did not give a full account of the algebra, which at that time I did not yet have in sufficiently good shape. This paper (intended as my definitive account) is designed to fill this gap. A more immediate reason for writing it was the need for adequate foundation material for the papers [18] , and indeed it has enabled me to make my calculations much more effective.

I have presented this work as a sequel to the short paper [16].
I am grateful to Andrew Ranicki for sending a preprint to [12] and for showing me the proof of Lemma 7 below. The relation of this paper to other foundation material on the subject is discussed in § 5 below.

## §1 Preliminary definitions

For any category $\mathcal{C}$ with product (in the sense of Bass [4, p. 344] we define $k\mathcal{C}$ to be the monoid of isomorphism classes of objects of $\mathcal{C}$, $K_0(\mathcal{C})$ its universal (Grothendieck) group. Similarly, $K_1(\mathcal{C})$ is the universal group for functions on the automorphisms of $\mathcal{C}$ to (additive) abelian groups which are additive for sums and composites. If $A_1$, $A_1 \oplus A_2$, $A_1 \oplus A_2 \oplus A_3$, ... defines a cofinal sequence in $k(\mathcal{C})$, we can define Aut $\mathcal{C}$ as the direct limit of

$$\mathrm{Aut}_{\mathcal{C}}\, A_1 \subset \mathrm{Aut}_{\mathcal{C}}\, (A_1 \oplus A_2) \subset \dots ,$$

and $K_1(\mathcal{C})$ is then its commutator quotient group.

For any ring R, we write $\mathcal{P}(R)$ for the category of finitely generated projective (right) R -modules. There is a standard meaning for $\oplus$ here. The groups $K_0\mathcal{P}(R)$, $K_1\mathcal{P}(R)$ are written simply as $K_0(R)$, $K_1(R)$. Any automorphism of such a module thus has a 'determinant' in $K_1(R)$; in particular, so does any nonsingular matrix over R.

Now let $(R, \alpha, u)$ be an antistructure in the sense of [18] - i.e. $\alpha$ is an antiautomorphism and $u \in R^{\times}$ a unit of $R$ such that

$$x^{\alpha^2} = u x u^{-1} \qquad \text{for all } x \in R ,$$
$$u^{\alpha} = u^{-1} .$$

For an $R$-module $M$, the space $Q_{(\alpha,u)}(M)$ of $(\alpha, u)$-quadratic forms on $M$ was defined in [16], as was the concept of nonsingular form. We will write $\theta$ for a quadratic form and $(b_\theta, q_\theta)$ for the corresponding [16 Theorem 1] element of $\mathrm{Quad}_{(\alpha,u)}(M)$. We define $\mathcal{Q}(R, \alpha, u)$ to be the category whose objects are pairs $(P, \theta)$, $P$ a finitely generated projective $R$-module, $\theta \in Q_{(\alpha,u)}(P)$ nonsingular; and whose morphisms $(P, \theta) \to (P', \theta')$ are the isomorphisms $P \to P'$ which carry $\theta$ to $\theta'$. [A possible variant is to regard representatives $\phi \in S_{\alpha}(M)$ of $\theta$ as defining different objects, but still have morphisms as above.] An object of this category is called a quadratic module. There is an obvious notion of (orthogonal) direct sum. Forgetting the quadratic structure defines a functor

$$F : \mathcal{Q}(R, \alpha, u) \to \mathcal{P}(R) .$$

We also have a hyperbolic functor

$$H = H_{\alpha} : \mathcal{P}(R) \to \mathcal{Q}(R, \alpha, u) .$$

This was defined on objects on [16, p.249] : $H(M) = M \oplus M^{\alpha}$ as module; $\theta$ is the equivalence class of the pairing

$$(m, f) \cdot (m', f') = f(m').$$

The definition on morphisms is obvious: $H(f) = f \oplus (f^{\alpha})^{-1}$ it is clear that this does define a functor. It will be important for us to recognise hyperbolic modules; as a preliminary, if $(N, \theta)$ is a quadratic module, we define a submodule $M \subset N$ to be a <u>subkernel</u> [17] (alias Lagrangian subspace [11] [12]) if the identity on $M$ extends to an isomorphism of $(N, \theta)$ on $H(M)$. Clearly a necessary condition for this is that $M$ be <u>isotropic</u>, i.e. that $q_\theta(M) = 0$ and $b_\theta(M \times M) = 0$. Note that $M^\alpha$ is also a subkernel of $M$. Indeed, the map

$$M \oplus M^\alpha = H(M) \to H(M^\alpha) = M^\alpha \oplus M^{\alpha\alpha}$$

given by $(x, f) \mapsto (f, \lambda_u\, \omega_{M,\alpha}(x))$ is an isometry. For the sesquilinear form in $H(M^\alpha)$ is

$$\lambda_u \omega_{M,\alpha}(x)\ (f') = u\ \omega_{M,\alpha}(x)(f')$$
$$= u(f'(x))^{\alpha} = f'(x)^\alpha u$$

which comes from the defining form $f(x')$ for $H(M)$ by applying $T_u$. In general, two subkernels $E, F$ of $N$ are <u>complementary</u> (alias Hamiltonian complements) if there is an isomorphism of $(N, \theta)$ on $H(E)$ which is the identity on $E$ and takes $F$ to $E^\alpha$. We can weaken this condition as follows.

<u>Lemma 1</u> <u>Let</u> $(N, \theta)$ <u>be a nonsingular quadratic module, and</u> $E, F$ <u>isotropic subspaces with</u> $E + F = N$. <u>Then</u> $E$ <u>and</u> $F$ <u>are complementary subkernels.</u>

<u>Proof</u> Since $E \cap F$ is orthogonal to $E$ and to $F$, it is orthogonal to $E + F = N$, hence is zero by nonsingularity (it lies in $\mathrm{Ker}\,(Ab_\theta) = \{0\}$). Hence, additivity, $N = E \oplus F$. The isomorphism

$$E \oplus F = N \xrightarrow{Ab_\theta} N^\alpha = E^\alpha \oplus F^\alpha$$

has zero components $E \to E^\alpha$, $F \to F^\alpha$, hence yields isomorphisms $E \to F^\alpha$, $F \to E^\alpha$. Identifying $F$ with $E^\alpha$ by the isomorphism yields $N = E \oplus E^\alpha = H(E)$, an additive isomorphism which (we readily verify) is also an isometry.

<u>Lemma 2</u> <u>Let</u> $(N, \theta)$ <u>be a nonsingular quadratic module,</u> $E \subset N$ <u>an isotropic projective submodule. Then</u> $E$ <u>is a subkernel if and only if the map</u> $N/E \overset{b'}{\to} E^{\alpha}$ <u>induced by</u> $Ab_{\theta} : N \to N^{\alpha}$ <u>is an isomorphism.</u>

<u>Proof</u> The condition is clearly necessary: suppose it satisfied. Then $N/E$ is projective, so the extension $N$ of $E$ by it splits, and we can find an additive complement, $M$ say, to $E$, and identify $E$ with the dual $M^{\alpha}$.

Then $N$ is additively isomorphic to $M \oplus M^{\alpha}$, and $\theta$ is given by a sesquilinear form

$$(m, f).(m', f') = \xi(m, m') + f(m') .$$

(This can be seen from our description [ 16 , p. 246 ] of $Q_{(\alpha,u)}$ of a direct sum.) We now see that if $\zeta \in S_{\alpha}(M)$, and we embed $M$ in $M \oplus M^{\alpha}$ by the graph of $\zeta$, the induced quadratic form comes from the sesquilinear form $\xi + \zeta$. Thus if we choose $\zeta = -\xi$, we obtain an isotropic subspace, complementary to $M^{\alpha}$.

This last argument also yields the

<u>Corollary 1</u> <u>The subkernels of</u> $H(M)$ <u>complementary to</u> $M^{\alpha}$ <u>are the graphs of the</u> $Ab_{\theta} : M \to M^{\alpha}$ <u>corresponding to the</u> $\theta \in Q_{(\alpha,-u)}(M)$.

For here, $\xi = 0$, and $\zeta$ determines $0 \in Q_{(\alpha,u)}(M)$ if and only if

$$\zeta = \mathrm{Im}(1 - T_u) = \mathrm{Im}(1 + T_{-u})$$

is the bilinearisation of an $(\alpha, -u)$ quadratic form.

<u>Corollary 2</u> <u>Any automorphism of</u> $H(M)$ <u>leaving</u> $M^{\alpha}$ <u>pointwise fixed is given by</u> $x \in M \mapsto (x, Ab_{\theta}(x))$ <u>for some</u> $\theta \in Q_{(\alpha,-u)}(M)$.

For if $x \mapsto (p, q)$ we find $p = x$ since $(p, q)$ and $x$ have the same inner products with each element of $M^{\alpha}$. The conclusion now follows from the preceding.

We recall [16 Theorem 3]; that if $\theta$ is nonsingular, $(N, \theta) \oplus (N, -\theta) \cong H(N)$. The simplest way to see this is now to use Lemma 2 to show that the diagonal $\Delta(N) \subset N \oplus N$ is a subkernel. The special case when $(N, \theta) = H(M)$ will be important below.

We will need to study based modules. A based module is a pair $(M, v)$ where $M$ is a free $R$-module and $v$ an equivalence class of free (ordered) bases of $M$, two bases being equivalent if the automorphism of $M$ taking one to the other has determinant $0 \in K_1(R)$. We can regard $v$ as a sort of volume element on $M$. Now define $\mathcal{B}(R)$ as the category whose objects are based modules $(M, v)$ and morphisms are based isomorphisms (i.e. preserving preferred classes of bases). There is an obvious definition of sum in $\mathcal{B}(R)$, but it is not commutative (permutation matrices can have determinant $-1$). Hence we restrict to the subcategory $\mathcal{B}_0(R)$ of modules of even rank.

More interesting is the category $\mathcal{B}\mathfrak{Q}(R, \alpha, u)$ of based quadratic modules, i.e. triples $(N, v, \theta)$ where $(N, v)$ is a based module and $(N, \theta)$ a quadratic module. A morphism here is an isomorphism class of modules respecting both structures. Again we have a direct sum, which behaves well on the subcategory $\mathcal{B}_0\mathfrak{Q}(R, \alpha, u)$ of modules of even rank. There is an obvious forgetful functor $F : \mathcal{B}\mathfrak{Q}(R, \alpha, u) \to \mathcal{B}(R)$, but before we can define a hyperbolic functor, we must discuss duality in $\mathcal{B}(R)$: this needs some care.

We recall from [16] that for $\alpha$ an antiautomorphism of R and M a right R-module, the dual module $M^\alpha$ is $\mathrm{Hom}_R(M, R)$ with module structure defined by

$$fr(m) = r^\alpha f(m);$$

that the natural map of $M$ to its double dual is $\omega_{M,\alpha} : M \to (M^{\alpha})^{\alpha^{-1}}$, where

$$\omega_{M,\alpha}(m)(f) = f(m)^{\alpha^{-1}} ;$$

and that if $(R, \alpha, u)$ is an antistructure, there is an isomorphism $\lambda_u : M^{\alpha^{-1}} \to M^{\alpha}$ given by

$$\lambda_u(f)(m) = u^{-1} f(m) .$$

If $e_1, \ldots, e_n$ is a free basis of $M$, the 'dual basis' $e_1^*, \ldots, e_n^*$ of $\mathrm{Hom}_R(M, R)$ is defined by

$$e_i^*(e_j) = \delta_{ij} \quad \text{(Kronecker delta).}$$

If we identify this with $M^{\alpha^{-1}}$ and with $M^{\alpha}$, however, the isomorphism $\lambda_u$ does not preserve the class of this basis. I thus declare that for $n = 2k$, a preferred base of $M^{\alpha}$ shall be $e_1^*, e_2^* u^{-1}, \ldots, e_{2k-1}^*, e_{2k}^* u^{-1}$, and one of $M^{\alpha^{-1}}$ is $e_1^* u, e_2^*, \ldots, e_{2k-1}^* u, e_{2k}^*$. Then $\lambda_u$ preserves preferred bases and so (up to equivalence) does $\omega_{M,\alpha}$. For the case $n$ odd, we do not define the concept of dual preferred base: ad hoc definitions can be found in special cases, but are not invariant under Morita equivalences (c.f. discussion in [18, II]).

We now define the hyperbolic functor $H : \mathcal{B}_0(R) \to \mathcal{B}\mathcal{Q}(R, \alpha, u)$ : it suffices to describe the case of rank 2. If $e_1, e_2$ is a base of $M$ and $e_1^*, e_2^*$ as above, we find

$$b_\theta(e_1, e_1^*) = b_\theta(e_2, e_2^*) = u , \ b_\theta(e_1^*, e_1) = b_0(e_2^*, e_2) = 1 ,$$

$b_0$ vanishes on other pairs of basis elements and $q_0$ on all basis elements. We will usually use the base $f_1 = e_1^* u^{-1}, f_2 = e_2^* u^{-1}$, but our preferred base (for $M^{\alpha}$) is $e_1^*, e_2^* u^{-1}$.

We now call $(M, v)$ a <u>based subkernel</u> of $H(M, v)$ and $M^{\alpha}$ (with the above base) a <u>complementary based subkernel</u> (we see, as in the unbased case, that it is a based subkernel). As before, there is a recognition principle: if $E$, $F$ are complementary subkernels of $(N, \theta)$, bases for $E$, $F$ are complementary iff (i) they are dual in the above sense and (ii) they combine to give a preferred base of $N$.

If $(M, \theta)$ is an object of $\mathcal{B}_0\mathcal{Q}(R, \alpha, u)$, the homomorphism $Ab_\theta : M \to M^\alpha$ associated to $b_\theta$ is now a map of based modules, hence has a well defined determinant in $K_1(R)$: we call this the discriminant of $(M, \theta)$, $\delta(M, \theta)$. We write $\mathcal{Z}(R, \alpha, u)$ for the full subcategory of forms with zero discriminant.

This completes our list of categories and functors: the algebraic K-theory of these categories is (roughly) what I mean by algebraic L-theory. Before establishing the basic relations between them, singling out the important ones and fixing notation, we next give some computations with unitary automorphisms, which will be needed for the proofs.

## §2 The elementary unitary group

We begin by recalling those results from the linear case which we wish to imitate, and fixing notation. All modules will be finitely generated projective right R-modules; maps also are written on the right. For $M$ a module, $GL(M)$ is the group of R-automorphisms of $M$. There are natural injections

$$GL(M) \subset GL(M) \times GL(N) \subset GL(M \oplus N)$$

which we regard as inclusions. We write $GL_n$ for $GL(R^n)$, and $GL_\infty$ for the union of the $GL_n$, with inclusions defined by

$$R^n \subset R^n \oplus R = R^{n+1} \quad .$$

Similar notations will apply below for other groups defined as functors of M. Since the $R^n$ are cofinal in $\mathcal{P}(R)$, $K_1(R)$ is the commutator quotient group of $GL_\infty$. For any M, we write SL(M) for the kernel of the determinant map so there is an exact sequence

$$1 \to SL(M) \to GL(M) \xrightarrow{det} K_1(R).$$

Elements of SL(M) are called <u>simple automorphisms</u> of M.

Let $e_1, \ldots, e_n$ denote the standard base of $R^n$. For

$$r \in R, \quad 1 \leq i, j \leq n, \quad i \neq j,$$

let $X_{ij}(r)$ be the automorphism which leaves each $e_k (k \neq i)$ fixed, and takes $e_i$ to $e_i + e_j r$. We call the $X_{ij}(r)$ <u>elementary transvections</u>, and write $E_n$ for the subgroup of $GL_n$ which they generate, and $E_\infty$ for the union of the $E_n$. The $X_{ij}$ can be expressed as commutators $[x, y] = xyx^{-1}y^{-1}$; in fact, if i, j, k are distinct,

$$[X_{ij}(r), X_{jk}(s)] = X_{ik}(sr).$$

Thus for $n \geq 3$, $E_n$ is contained in the commutator subgroup of $GL_n$, and a fortiori in $SL_n$: indeed, it is perfect. Stably, the converse holds.

<u>Lemma 3</u> <u>(Whitehead's lemma)</u>

<u>$E_\infty$ is the commutator subgroup of</u> $GL_\infty$.

<u>Proof</u> We show, in fact, that the commutator subgroup of $GL_n$ lies in $E_{2n}$. It is convenient to use matrix notation, with blocks of $n \times n$ matrices. Then the matrices of the form

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$$

form a group, whose product is given by addition of matrices A. If A has only one nonzero element, we have an elementary transvection. Hence all such matrices belong to $E_{2n}$; and similarly if the positions of 0 and A are interchanged. Now

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ A^{-1} & I \end{pmatrix}\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}\begin{pmatrix} I & I \\ 0 & I \end{pmatrix},$$

and hence belongs to $E(R^{2n})$ and, finally, so does

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & (AB)^{-1} \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}$$

since each factor is of the above type.

There is, of course, more to be said, but the above seems the essential basis for understanding the functor $K_1(R)$. We now undertake the corresponding study in the unitary case; we work at a similar depth, but there is more to do: the results are much richer. We supposed fixed an antistructure $(R, \alpha, u)$ in what follows.

For $(N, \theta)$ a quadratic module - i.e. object of $\mathcal{Q}(R, \alpha, u)$ - write Aut $(N, \theta)$ for its group of automorphisms in this category. However, we write $U(M)$ for Aut $H(M)$. Write also $GI(M)$ for the subgroup of $U(M)$ of automorphisms leaving the subkernel $M$ of $H(M)$ invariant. The subgroups of $GI(M)$ where the restriction to $M$ of the automorphism belongs to $SL(M)$, $E(M)$ (when $M = R^n$) or is trivial are denoted respectively by

SI(M), EI(M) and I(M). For the corresponding subgroups leaving $M^\alpha$ invariant, we use J in place of I ; also if $M = R^n$ we use a suffix n, and have $GI_\infty$ etc. for the appropriate limits. The hyperbolic functor induces a monomorphism $GL(M) \to GI(M)$ ; in fact GI(M) is the semidirect product of GL(M) with the normal subgroup I(M) ; and correspondingly for SI, EI .

The subgroup EU(M) of elementary automorphisms of H(M) is that generated by I(M) and J(M) . As in the linear case, we can also give explicit generators. If $e_1, \ldots, e_n$ is (again) the standard base of $R^n$, we extend by $f_1, \ldots, f_n$ to a base of $H(R^n)$ such that $e_i \cdot f_j = \delta_{ij}$ (the $b_\theta$ for a hyperbolic module is denoted by a dot ; we write q for $q_\theta$) and $q(e_i) = q(f_i) = 0$. For $i \neq j$, $1 \leq i, j \leq n$, $r \in R$ , we define $E_{ij}(r)$ to be the identity on all basis vectors except

$$e_i \to e_i + f_j r \qquad\qquad e_j \to e_j - f_i r^\alpha u$$

and $F_{ij}(r)$ on all save for

$$f_i \to f_i + e_j r \qquad\qquad f_j \to f_j - e_i u^{-1} r^\alpha ,$$

so that $E_{ij}(r) \in J_n$ , $F_{ij}(r) \in I_n$ . Then for i, j, k distinct,

$$[E_{ij}(r), F_{jk}(s)] = H(X_{ik}(-sr)).$$

Since the $X_{ij}(-sr)$ generate $E_n$ , it follows that for $n \geq 3$ ,

$$EI_n \subset EU_n .$$

We also write

$$\Sigma_{ij} = E_{ij}(1)F_{ij}(u^{-1})E_{ij}(1) :$$

under $\Sigma_{ij}$,

$$e_i \to f_j \to - e_i \quad \text{and} \quad e_j \to - f_i u \to - e_j .$$

Next we observe that

$$[H(X_{12}(1)),\ E_{12}(r)]$$

acts as the identity on all basis elements except

$$e_1 \to e_1 + f_1(r - r^{\alpha}u).$$

<u>Lemma 4</u> $EU_n$ <u>is generated (for</u> $n \geq 3$) <u>by the</u> $E_{ij}(r)$ <u>and</u> $F_{ij}(r)$ <u>for</u> $1 \leq i, j \leq n, r \in R$ .

<u>Proof</u> It suffices (by symmetry) to show that $J_n$ is contained in the subgroup with these generators. By corollary 2 to lemma 2, each element of $J_n$ is of the form

$$e_i \to e_i + \Sigma_j f_j (b_{ij} - b^{\alpha}_{ji}\, u),\ f_i \to f_i$$

for some matrix $(b_{ij})$; Since composition in this group corresponds to matrix addition, it is enough to consider matrices with only one nonzero entry $b_{ij}$. If $i \neq j$, this is $E_{ij}(b_{ij})$; the case $i = j$ is dealt with by the calculation preceding this lemma.

<u>Corollary</u> <u>For</u> $n \geq 3$, $EU_n$ <u>is perfect</u>.

For, as with $E_n$, its generators are commutators :

$$E_{ij}(r) = [H(X_{jk}(1)),\ E_{ik}(r)] .$$

Our next objective is to show that $EU_\infty$ is the commutator subgroup of $U_\infty$ .

<u>Lemma 5</u> <u>If</u> $A \in U(M)$ <u>we can find</u> $A' \in U(M')$ <u>with</u> $A \oplus A' \in EU(M \oplus M')$.

<u>Proof</u> Since $A \oplus A^{-1} \in E(H(M) \oplus H(M))$, it is enough to prove the result under the extra hypothesis that $A \in E(H(M))$ . By adding further modules, we may suppose $M$ free of even rank.

Define $A'$ to be the conjugate of $A$ by the (non-unitary) automorphism $\iota$ which is 1 on $M$ and $-1$ on $M^\alpha$ : then also $A' \in U(M) \cap E(H(M))$. Now $A \oplus A'$ leaves invariant the subkernel $\Delta$ of $H(M) \oplus H(M)$ defined as the graph of $\iota$, and the induced automorphism of $\Delta$ belongs to $E(\Delta)$, since $A \in E(H(M))$. If we can find $\mu \in EU(M \oplus M)$ taking $M \oplus M$ isomorphically onto $\Delta$ , it will follow that

$$\mu(A \oplus A')\mu^{-1} \in EI(M \oplus M) ,$$

whence $A \oplus A' \in EU(M \oplus M)$, as desired.

It suffices to find $\mu$ for $M = R^2$ (we can then take direct sums for other cases. A suitable element of $U_4$ is, in fact,

$$\mu = \Sigma_{34}\, H(X_{13}(1)\; X_{24}(1)).$$

<u>Theorem 1</u> <u>$EU_\infty$ is the commutator subgroup of</u> $U_\infty$

<u>Proof</u> Since each $EU_n$ is perfect, so is $EU_\infty$, so it is contained in the commutator subgroup. Conversely, let $A, B \in U_\infty$ : say $A, B \in U_m$. By the lemma, there exist (for some $r$) $A', B' \in U_r$ with $A \oplus A', B \oplus B' \in EU_{m+r}$. Hence $EU_{m+2r}$ contains $A \oplus A' \oplus 1, B \oplus 1 \oplus B'$ , hence their commutator $[A,B]$.

It is interesting to note that, as on [17, p. 65] the crux of the proof is the construction of $\mu$ (but here we have avoided the matrix identity). Note that $\mu$ carries $e_1, e_2, e_3, e_4$ respectively to $e_1 + e_3, e_2 + e_4, -(f_2 - f_4)$ , $(f_1 - f_3)u$ which, by our definitions, is indeed a preferred base of $\Delta$ .

Our second basic result is a sort of normal form, analogous to the Bruhat decomposition, for elements of $E_\infty$ . This result was first mooted in [17, 6.6] and the first formal proof is due to Sharpe [15].

Lemma 6 Let $x \in U(M)$. Then

$x \in SI(M)SJ(M)SI(M)$

if and only if M and Mx have a common based complement in H(M).

Proof If $x = uvw$ is of this form, $M = Mw$ is based complementary to $M^{\alpha}w = M^{\alpha}vw$, and so is $Mx = Muvw = Mvw$. Conversely, if F is the common based complement, we can find $u \in SI(M)$ with $M^{\alpha}u^{-1} = Fx^{-1}$, since SI(M) is transitive on based complements to M (c.f. Lemma 2); and also $w \in SI(M)$ with $M^{\alpha}w = F$. Then $v = u^{-1}xw^{-1}$ preserves the based subkernel $M^{\alpha}$, so lies in SJ(M).

Remark Our construction of $\Sigma_{12} \in U_2$ was by such a product, and $\Sigma_{12}$ interchanges $\{e_1, e_2\}$ and the complementary based subkernel $\{f_1u, f_2\}$. We deduce the

Corollary In $H(R^{2n})$, two complementary based subkernels always have a common based complement.

Lemma & (Ranicki) Suppose given based subkernels $K_i$ $(1 \leq i \leq 4)$ in $(N, \theta)$ with $K_i$ based complementary to $K_i+1$ $(i = 1,2,3)$. Then $K_1 \oplus K_3^{\alpha}$, $K_4 \oplus K_3$ have a common based complement in $(N, \theta) \oplus H(K_3)$.

Proof Up to based isomorphism, we can identify $(N, \theta)$ with $H(K_3)$ and $K_2^{\alpha}$ with $K_3^{\alpha}$ (by definition of based complements). To avoid confusion, we then use primes to indicate the second copy of $H(K_3)$. We now claim that the twisted diagonal $\Delta = \Delta(H(K_3))$, defined as in the proof of Lemma 5, is a common based complement.

For by that lemma, it is a based sbukernel. Hence it suffices to show that it is additively a based complement to $K_1 \oplus K_2'$. We change bases by a series of elementary moves. First, change $\Delta = \Delta(K_3) \oplus \Delta(K_2)$ modulo $K_2' \subset K_1 \oplus K_2'$ to obtain $\Delta(K_3) \oplus K_2$. Next change $K_1 \oplus K_2'$ by

$K_2 \subset \Delta(K_3) \oplus K_2$ to obtain $K_3 \oplus K_2'$. Finally, change $\Delta(K_3) \oplus K_2$ by $K_3 \subset K_3 \oplus K_2'$ to obtain $K_3' \oplus K_2$. But by hypothesis, $K_3 \oplus K_2'$ and $K_3' \oplus K_2$ are based complements.

For based subkernels $K_1$, $K_2$ of $(N, \theta)$ we define $K_1 \sim K_2$ if we can find a based complementary pair $(L_1, L_2)$ such that $K_1 \oplus L_1$, $K_2 \oplus L_2$ have a common based complement.

<u>Lemma 8</u> <u>$\sim$ is an equivalence relation.</u>

<u>Proof</u> It is clearly reflexive and symmetric. Suppose, then, that $K_1 \sim K_2 \sim K_3$; that $K_1 \oplus L_1$ and $K_2 \oplus L_2$ both have based complement $C_1$, and that $K_2 \oplus M_2$, $K_3 \oplus M_3$ have based complement $C_2$. Applying lemma 7 to $(K_1 \oplus L_1 \oplus M_2,\ C_1 \oplus M_3,\ K_2 \oplus L_2 \oplus M_2.\ C_2 \oplus L_1)$, we find a based complementary pair $(N_1, N_2)$ such that $K_1 \oplus L_1 \oplus M_2 \oplus N_1$, $C_2 \oplus L_1 \oplus N_2$ have based complement $C_3$. By the corollary to lemma 6, $N_1$ and $N_2$ have a common based complement $N_3$. Now apply lemma 7 to

$(K_1 \oplus L_1 \oplus M_2 \oplus N_1,\ C_3,\ C_2 \oplus L_1 \oplus N_2,\ K_3 \oplus L_2 \oplus M_3 \oplus N_3)$,

and we obtain the desired conclusion.

<u>Theorem 2</u> <u>For all $x \in EU_n$, we can find $\Sigma \in EU_m$ interchanging the based subkernels $R^m$, $(R^m)^\alpha$, such that</u>

$$x \oplus \Sigma \in SI_{m+n}\, SJ_{m+n}\, SI_{m+n}.$$

<u>Proof</u> By lemma 6, the conclusion holds if $R^{m+n}$ and $R^n x \oplus (R^m)^\alpha$ have a common based complement; by definition, this holds if $R^n \sim R^n x$. Now $EU_n$ is generated by $I_n$ and $J_n$. The result holds for $x \in I_n$ since $R^n x = R^n$ and for $x \in J_n$ since $R^n$ and $R^n x$ have common based complement $(R^n)^\alpha$. If it holds for $x$ and for $y$, then

$R^n \sim R^n x$, so $R^n y \sim R^n xy$, and $R^n \sim R^n y$; so $R^n \sim R^n xy$. The result in general now follows.

Corollary We can improve the conclusion to

$$x \oplus \Sigma \in H(SL_{m+n}) \circ I_{m+n} \cdot J_{m+n} \cdot I_{m+n} \cdot$$

This follows on using the equations

$$SI_r = H(SL_r).I_r = I_r \cdot H(SL_r) \; .$$

## §3 $K_0$ and $K_1$ of categories of quadratic modules

In §1 we defined the categories $\mathcal{P}(R)$, $\mathcal{Q}(R, \alpha, u)$ and $\mathcal{BQ}(R, \alpha, u)$. We now obtain some exact sequences relating their algebraic K-groups. In addition to the maps induced by the functors F, H and the forgetful functor $G : \mathcal{BQ}(R, \alpha, u) \to \mathcal{Q}(R, \alpha, u)$, these involve two further maps: the discriminant map and one which we now define.

Suppose $(N, \theta, v)$ a based quadratic module, an object of $\mathcal{BQ}(R, \alpha, u)$ with class $y \in K_0 \mathcal{BQ}(R, \alpha, u)$ and $\alpha$ an automorphism of N (as R-module), with determinant $x \in K_1(R)$. Then applying $\alpha$ to a preferred base of M gives another base, whose equivalence class $v'$ depends only on x and v. If $(N, \theta, v')$ has class $y'$, we define $\tau(x) = y' - y$. If we replace $(N, \theta, v)$ by its direct sum with any $(N_2, \theta_2, v_2)$, then $(N, \theta, v')$ is affected in the same way, so we obtain the same value for $\tau(x)$. Hence $\tau$ is well defined. It is defined for any x, since we can apply an automorphism of M to H(M). Hence we have

$$\tau : K_1(R) \to K_0 \mathcal{BQ}(R, \alpha, u)$$

<u>Lemma 9</u> <u>The composite</u>

$$K_1(R) \overset{H_*}{\rightarrow} K_1\mathfrak{Q}(R, \alpha, u) \overset{F_*}{\rightarrow} K_1(R)$$

<u>is</u> $1 - T$ ; <u>the composite</u> $\delta \circ \tau = 1 + T$, <u>where</u> $T$ <u>is the involution of</u> $K_1(R)$ <u>induced by</u> $(\alpha -)$ <u>duality</u>.

In matrix terms, $T$ comes from the anti-automorphism of $GL_n$ which sends $A = (a_{ij})$ to $A^* = (a_{ji}^{\alpha})$: note that its square is an inner automorphism, hence induces the identity on $K_1(R)$.

<u>Proof</u> If $x \in K_1(R)$ is represented by the matrix $A$, $H(x)$ has matrix $\begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix}$, so the first assertion is clear. As to the second, given a (based) quadratic form with matrix $B$, and change of base with matrix $P$, the form with its new base has matrix $P^*BP$, and this result also is immediate.

<u>Note</u> Our description of $\tau$ was perhaps vague as to sign: we can take the above as normalising this (unimportant) choice.

<u>Proposition 10</u> <u>The following sequence is exact:</u>

$$0 \rightarrow K_1\mathcal{B}\mathfrak{Q}(R, \alpha, u) \overset{G_*}{\rightarrow} K_1\mathfrak{Q}(R, \alpha, u) \overset{F_*}{\rightarrow} K_1(R) \overset{\tau}{\rightarrow} K_0\mathcal{B}\mathfrak{Q}(R, \alpha, u) \overset{G_*}{\rightarrow} K_0\mathfrak{Q}(R, \alpha, u) .$$

<u>Proof</u> We first show that the sequence has order two. An automorphism in $\mathcal{B}\mathfrak{Q}$ must preserve preferred bases by definition, hence is mapped to 0 by $F_*$. If $x$ is the determinant of an automorphism $A$ of $(N, \theta)$, where we may suppose $M$ free since such are cofinal, then we can assign $N$ a preferred base $v$. Changing this by $A$, though, gives an isomorphic object of $\mathcal{B}\mathfrak{Q}(R, \alpha, u)$, so $\tau(x) = 0$. Finally, if we refer to the definition

$\tau(x) = y' - y$ of $\tau$, we see at once that y, y' have the same image in $K_0\mathcal{L}(R, \alpha, u)$.

Conversely, let $y \in K_0\mathcal{BL}(R, \alpha, u)$ be in Ker $G_*$. Let u be the difference of the classes of $(N_1, \theta_1, v_1)$ and $(N_2, \theta_2 v_2)$. Then $(N_1, \theta_1)$ and $(N_2, \theta_2)$ are stably isomorphic in $\mathcal{L}(R, \alpha, u)$: since the $H(R^n)$ are cofinal, we can suppose (adding this to each of $M_1$, $M_2$) that they are already isomorphic. If A is an isomorphism, and has determinant x with respect to $v_1$, $v_2$, it follows from the definition that $\tau(x) = y$.

Next let $\tau(x) = 0$. Stabilising as before, we can suppose $(N, \theta, v)$ and $(N, \theta, v')$ isomorphic. But then x is the determinant of an automorphism in $\mathcal{L}(R, \alpha, u)$ of $(N, \theta)$. Exactness at $K_1\mathcal{L}(r, \alpha, u)$ holds by definition of $\mathcal{BL}$. Finally, $G_*$ is injective, since $EU_\infty$ is the commutator subgroup of $U_\infty$ and, being perfect, also of the subgroup with determinant 0.

Writing $S^{\varepsilon}(K_1(R)) = \{x \in K_1(R) : \bar{x} = \varepsilon x\}$ for $\varepsilon = \pm$, we have

<u>Corollary</u> <u>There is an exact sequence</u>

$$K_1\mathcal{L}(R, \alpha, u) \xrightarrow{\delta} S^-(K_1(R)) \xrightarrow{\tau} K_0\mathcal{BL}(R, \alpha, u) \to \tilde{K}_0\mathcal{L}(R, \alpha, u) \oplus S^+(K_1(R)).$$

This follows at once by diagram-chasing, taking due note of Lemma 9. It is sometimes a more convenient form for calculations.

The above is reasonably straightforward and not unexpected. The following exact sequence, though to some extent it plays a symmetrical rôle below, appears to lie deeper. There is a natural forgetful map

$$K_0\mathcal{BL}(R, \alpha, u) \to K_0\mathcal{B}(R) \xrightarrow{\varepsilon} \mathbb{Z},$$

where $\varepsilon$ counts the number of elements in a preferred basis. We write $K_0\mathcal{B}(R, \alpha, u)$ for the kernel.

<u>Proposition 11</u> The sequence

$$K_0\mathcal{BL}(R, \alpha, -u) \xrightarrow{\delta} K_1(R) \xrightarrow{H} K_1\mathcal{L}(R, \alpha, +u)$$

is exact.

Note the change here here u to -u.

Proof We will describe Ker H. Let x be an automorphism of a free module M of even rank, representing $\xi \in K_1(R)$. Then H(x) represents H($\xi$), and so does H(x)$\Sigma$, if $\Sigma$ interchanges the based subkernels M and $M^\alpha$. Then H($\xi$) = 0 if and only if this is (stably) in $EU_\infty$, so we can apply the corollary to Theorem 2: replacing x (if necessary) by its direct sum with an identity matrix, we get

$$H(x)\Sigma = H(x_0)uvw$$

with $x_0 \in SL(M)$, u, $w \in I(M)$ and $v \in J(M)$.

By Lemma 1 (c.f. lemma 4), there is a unique ($\alpha$, -u) - quadratic form $\theta$ on M such that for $m \in M$,

$$mv = m + Ab_\theta(m)\ M \oplus M^\alpha.$$

Since also m = mu, w induces the identity on the submodule M and the quotient module $M^\alpha$, and muvw $M^\alpha$, we deduce

$$muvw = Ab_\theta(m).$$

It follows that $Ab_\theta$ is an isomorphism, hence $\theta$ nonsingular. Next, we see by computing determinants that $\xi = \det(Ab_\theta) = \delta(\theta)$. Thus Ker H $\subset$ Im $\delta$.

We can prove the converse using the same identity as for the Whitehead lemma. Alternatively, if v, $\Sigma$ are defined as above, Mv is complementary (unbased) to M as well as to $M^\alpha = M\Sigma$ so we can find $w \in I(M)$ with $Mvw = M^\alpha$ and then $x \in J(M)$ such that vwx interchanges M and $M^\alpha$. Then vwx has the form H(a)$\Sigma$ , where

det a = det Ab($\theta$) = $\delta(M, \theta)$ and $\Sigma \in EU(M)$. Hence $H\delta(M, \theta) = 0$.

We observe various simple corollaries of the last three results - most of which can easily be proved independently.

Proof We will describe Ker H . Let $x$ be an automorphism of a free module $M$ of even rank, representing $\xi \in K_1(R)$. Then $H(x)$ represents $H(\xi)$, and so does $H(x)\Sigma$, if $\Sigma$ interchanges the based subkernels $M$ and $M^{\alpha}$. Then $H(\xi) = 0$ if and only if this is (stably) in $EU_{\infty}$, so we can apply the corollary to Theorem 2 : replacing $x$ (if necessary) by its direct sum with an identity matrix, we get

$$H(x)\Sigma = H(x_0)\,uvw$$

with $x_0 \in SL(M)$, $u, w \in I(M)$ and $v \in J(M)$ .

By Lemma 1 (c.f. lemma 4), there is a unique $(\alpha, -u)$ - quadratic form $\theta$ on $M$ such that for $m \in M$ ,

$$mv = m + Ab_{\theta}(m) \in M \oplus M^{\alpha} .$$

Since also $m = mu$, $w$ induces the identity on the submodule $M$ and the quotient module $M^{\alpha}$, and $muvw \in M^{\alpha}$ , we deduce

$$muvw = Ab_{\theta}(m) .$$

It follows that $Ab_{\theta}$ is an isomorphism, hence $\theta$ nonsingular. Next, we see by computing determinants that $\xi = \det(Ab_{\theta}) = \delta(\theta)$. Thus Ker $H \subset$ Im $\delta$ .

We can prove the converse using the same identity as for the Whitehead lemma. Alternatively, if $v$, $\Sigma$ are defined as above, $Mv$ is complementary (unbased) to $M$ as well as to $M^{\alpha} = M\Sigma$ so we can find $w \in I(M)$ with $Mvw = M^{\alpha}$ and then $x \in J(M)$ such that $vwx$ interchanges $M$ and $M^{\alpha}$. Then $vwx$ has the form $H(a)\Sigma$, where $\det a = \det Ab(\theta) = \delta(M, \theta)$ and $\Sigma \in EU(M)$. Hence $H\delta(M, \theta) = 0$.

We observe various simple corollaries of the last three results - most of which can easily be proved independently.

Corollary

$$\mathrm{Im}(1 - T) \subset \mathrm{Ker}\ \tau = \mathrm{Im}\ F_* \subset \mathrm{Ker}\ (1 + T)$$
$$\mathrm{Im}(1 + T) \subset \mathrm{Ker}\ H_* = \mathrm{Im}\ \delta \subsetneq \mathrm{Ker}\ (1 - T) .$$

To conclude this section, we recall the category $\mathcal{L}(R, \alpha, u)$ of based forms of discriminant 0 . This is a full, cofinal subcategory of $\mathcal{B}\mathcal{Q}(R, \alpha, u)$ - as is clear from the above. Hence we have the easy

Lemma 12 $\quad K_1\mathcal{L}(R, \alpha, u) = K_1\mathcal{B}\mathcal{Q}(R, \alpha, u)$ .

$$\tilde{K}_0\mathcal{L}(R, \alpha, u) = \mathrm{Ker}\ \delta : \tilde{K}_0\mathcal{B}\mathcal{Q}(R, \alpha, u) \to K_1(R).$$

## §4 Definitions of the L - groups

We have already drawn attention to the symmetry between Propositions 10 and 11. We now develop a notation to make the most of this. First, write

$$\Lambda_0(R, \alpha, u) = \tilde{K}_0\mathcal{B}\mathcal{Q}(R, \alpha, u)$$
$$\Lambda_1(R, \alpha, u) = K_1\ \mathcal{Q}(R, \alpha, u)$$

and, for any $i \in \mathbb{Z}$,

$$\Lambda_{i+2}(R, \alpha, u) = \Lambda_i(R, \alpha, -u)$$

so that $\Lambda_i$ is periodic with period 4 in $i$ . Since the transition from $u$ to $-u$ is now dealt with in our suffix, we can write $\Lambda_i(R)$ for the rest of this section without risk of confusion. Now we have exact sequences

$$\Lambda_{i+1}(R) \xrightarrow{\delta_{i+1}} K_1(R) \xrightarrow{\tau_i} \Lambda_i(R) ,$$

where $\delta_{i+1}$ means $F_*$ or $\delta$, and $\tau_i$ is $H_*$ or $\tau$, according to the parity of $i$, and by Lemma 9,

$$\delta_i \circ \tau_i = 1 + (-1)^i T.$$

Let $X$ be any subgroup of $K_1(R)$ such that $T(X) = X$. Then we define

$$L_i^X(R) = L_i^X(R, \alpha, u) = \delta_i^{-1}(X)/\tau_i(X).$$

The most obvious (and important) examples are $X = \{0\}$: we will write $L_i^S$ for these. We have

$$L_0^S(R) = \text{Ker } \delta : \tilde{K}_0 \mathcal{B} \mathcal{Q}(R) \to K_1(R) = \tilde{K}_0 \mathcal{L}(R)$$
$$L_1^S(R) = \text{Ker } F_* : K_1 \mathcal{Q}(R) \to K_1(R) = K_1 \mathcal{B}\mathcal{Q}(R) = K_1 \mathcal{L}(R),$$

so these are essentially the K groups of the category $\mathcal{L}(R)$. Next we can take $X = K_1(R)$, and write $L_i^K$ for these groups:

$$L_0^K(R) = \text{Coker } \tau : K_1(R) \to \tilde{K}_0 \mathcal{B} \mathcal{Q}(R) = \tilde{K}_0 \mathcal{Q}(R)$$
$$L_1^K(R) = \text{Coker } H_* : K_1(R) \to K_1 \mathcal{Q}(R).$$

Although these are the main examples, we will have occasion in other papers to consider: $X = \text{Ker}(K_1(R) \to K_1(S))$ for a ring homomorphism $R \to S$ (of antistructures) and, if $R$ is the integer group ring $\mathbb{Z}\pi$ of a group $\pi$ also $X$ = the image in $K_1(R)$ of the $1 \times 1$ matrices $\pm g$, $g \in \pi$. The latter is the important case for topological applications (c.f. [17]). The idea of defining all the $L_i^X$ was suggested by S. Cappell.

These groups are related by exact sequences. If $G$ is a group with involution $T$ of order 2 - e.g. $K_1(R)$ or $X$ above - we write $H^i(G)$ for the Tate cohomology groups of the action:

$$H^{2i}(G) = \{x \in G : Tx = x\}/\{y + Ty : y \in G\},$$
$$H^{2i+1}(G) = \{x \in G : Tx = -x\}/\{y - Ty : y \in G\}.$$

<u>Theorem 3</u> <u>If</u> $X \subset Y$ <u>are</u> $T$-<u>invariant subgroups of</u> $K_1(R)$, <u>there is an exact sequence</u>

$$\ldots L_i^X(R) \xrightarrow{j} L_i^Y(R) \xrightarrow{d} H^i(Y/X) \xrightarrow{t} L_{i-1}^X(R) \xrightarrow{j} L_{i-1}^Y(R) \ldots$$

<u>Proof</u> It will be convenient to use the temporary notation

$$\Lambda_i^{X,Y} = \delta_i^{-1}(Y)/\tau_i(X).$$

Then $\delta_i$ induces a map

$$\delta_i' : \Lambda_i^{X,Y} \to Y/\delta_i\tau_i(X) \to Y/X$$

whose kernel is the set of equivalence classes of elements mapping by $\delta_i$ to $X$, i.e. is $L_i^X(R)$. Similarly, $\tau_i$ induces a map

$$\tau_i' : Y/X \to \tau_i(Y)/\tau_i(X) \to \Lambda_i^{X,Y}$$

whose cokernel equals that of $\tau_i : Y \to \delta_i^{-1}(Y)$, i.e. is $L_i^Y(R)$. Since $\tau_i \; \delta_{i+1} = 0$, $\tau_i' \; \delta_{i+1}' = 0$. Conversely, if $\tau_i'(y + X) = 0$, $\tau_i(y) \in \tau_i(X)$ so for some $x \in X$, $y - x \in \operatorname{Ker} \tau_i = \operatorname{Im} \; \delta_{i+1}$. It follows that $y + X \in \operatorname{Im} \; \delta_{i+1}'$.

We thus have exact sequences

$$0 \to L_{i+1}^X(R) \xrightarrow{j_i} \Lambda_{i+1}^{X,Y} \xrightarrow{\delta_{i+1}'} Y/X \xrightarrow{\tau_i'} \Lambda_i^{X,Y} \xrightarrow{q_i} L_i^Y(R) \to 0 .$$

Also, the relation $\delta_i' \circ \tau_i' = 1 + (-1)^i T$ follows from the corresponding result for $\delta_i \circ \tau_i$. The result thus follows formally from the following elementary lemma, whose proof we leave to the reader.

<u>Lemma 13</u> <u>Given a sequence of exact sequences</u>

$$A_{i+1} \xrightarrow{a_{i+1}} B_i \xrightarrow{b_i} A_i$$

<u>write</u> $H_i(B)$ <u>for the homology of the complex</u>

$$\ldots B_{i+1} \xrightarrow{a_{i+1}b_{i+1}} B_i \xrightarrow{a_i b_i} B_{i-1} \quad \ldots .$$

<u>Then there is an exact sequence</u>

$$\ldots \operatorname{Ker} a_{i+1} \to \operatorname{Coker} b_{i+1} \to H_i(B) \to \operatorname{Ker} a_i \to \operatorname{Coker} b_i \ \ldots$$

Corollary  There is an exact sequence

$$L_i^S(R) \to L_i^K(R) \to H^i(K_1(R)) \to L_{i-1}^S(R) \to L_{i-1}^K(R) \ .$$

A special case of this is due to Rothenberg (see [14]); the general case is also proved in [12].

The above definition makes the $L_i^X$ appear somewhat unnatural. We conclude our discussion by giving a more directly geometrical definition which is, moreover, one which we shall need to refer back to.

Let $K_1(R) \to V$ be a homomorphism with kernel $X$, equivariant with respect to an action of $\mathbb{Z}/2$ by $\alpha$. We can, for example, take $V = K_1(R)/X$, but it is sometimes more convenient to let $V = K_1(S)$ with the map induced by a ring map $R \to S$. We use this map to calculate determinants with values in $V$. Now define the category $\mathcal{B}(R)$ as in §1, but referring to determinant in $V$. With no further change, we obtain definitions of $\mathcal{B}\mathfrak{Q}(R, \alpha, u)$ a forgetful functor $F : \mathcal{B}\mathfrak{Q}(R, \alpha, u) \to \mathcal{B}(R)$, and a hyperbolic functor $H : \mathcal{B}_0(R) \to \mathcal{B}\mathfrak{Q}(R, \alpha, u)$. The discussion of based subkernels, complementarity, and discriminant at the end of §1 is also unaltered, and we have a category $\mathcal{L}(R, \alpha, u)$. It is sometimes possible to define dual bases and hence $H$, on free modules of odd rank: for example, if we require (as is often done) $u = \pm 1$, and that the determinant (in $V$) of $-1 \in R^X$ is zero. As we restrict ourselves to the case when the rank is formally zero, this point is unimportant for us.

It is immediately clear that $L_0^X(R, \alpha, u) = K_0\mathcal{L}(R, \alpha, u)$: forms representing objects in $\mathcal{L}(R, \alpha, u)$ admit free bases: the discriminant in our original sense is restricted to lie in $X \subset K_1(R)$, and the basis is free to change by (the image under $\tau$ of) $X$.

More interesting is the case of $L_1^X$. We refer to the proof of Theorem 2 (starting with Lemma 6): note that $SI(M)$ now has a new meaning, i.e. automorphisms of $H(M)$ which leave $M$ invariant and induce an automorphism of $M$ with determinant $0 \in V$. To avoid confusion with our earlier notation, let us write $S'I(M)$ for this, $E'U(M)$ for the group generated by $S'I(M)$ and $S'J(M)$ (we make no bones here about listing elementary matrices ), $S'U(M)$ for elements of $U(M)$ with determinant $0 \in V$, and conventions as before when $M = R^n$. I claim first that $L_1^X(R, \alpha, u) = S'U_\infty/E'U_\infty$: this is indeed simply a matter of referring back to the definition. We seek, however, a more directly geometric form of the def[illegible]on.

[illegible]e (as we see directly) $S'I(M)$ acts transitively on the based complements to $M$, the proof of Lemma 6 remains valid; so of course does the corollary (the new form is a weaker version than the old). The proof of Lemma 7 remains valid without alteration, and if we define a relation $\sim$ on subkernels as there, we see as before that it is an equivalence relation. Now $S'U_n$ acts (and it clearly acts transitively) on the based subkernels in $H(R^n)$. The given proof of Theorem 2 shows that for $x \in S'U_n$,

$x \in E'U_n$ implies that $R^n \sim R^n x$ and hence for $\Sigma$ as before

$$x \oplus \Sigma \in SI_{m+n}\, SJ_{m+n}\, SI_{m+n}.$$

But this in turn implies $x \otimes 1$, $x \otimes \Sigma \in E'U_{m+n}$. The relation $\sim$ between subkernels thus detects neatly the group we want, however, as subkernels are abstractly isomorphic we seek a more intrinsic invariant.

Following Ranicki we define a <u>formation</u> to consist of a triple $(H; F, G)$, where $H$ is a nonsingular based $(R, \alpha, u)$ - quadratic module and $F$, $G$ are based subkernels in $H$. A formation is <u>trivial</u> (or split) if $F$ and $G$ are based complements; we define stable equivalence $\approx$ between formations to mean that they can be made isomorphic by adding trivial pairs.

Definition Two formations $(H, F, G)$, $(H', F', G')$ are equivalent $(\sim)$ if, after replacing if necessary by stably equivalent formations, we can find a based isomorphism $H \to H'$ taking $F$ to $F'$ and $G$ to $G''$ with $G'' \sim G'$.

Theorem Equivalence classes of formations form an abelian group under $\oplus$. This group is isomorphic to $L_1^X(R, \alpha, u)$. The isomorphism is induced by taking the class of a based automorphism $\alpha$ of $H$ which takes $F$ to $G$ (as based subkernel).

Proof Any element of $S'I_n$ defines $0 \in L_1^X(R, \alpha, u)$. It follows (transporting by an isomorphism) that so does any automorphism which preserves a based subkernel. Since $\alpha$ is unique up to left and right composition with such automorphisms, its class $\xi \in L_1^X(R, \alpha, u)$ is determined by $(H; F, G)$. Clearly, the sum of two formations has the sum of their invariants. It remains to show that two formations with the same invariant are equivalent.

Up to stable isomorphism, we can identify the formations with $(H(R^n), R^n, R^n x)$ and $(H(R^n), R^n, R^n y)$ where $xy^{-1} \in E'U_\infty$. We seek to show $R^n x \sim R^n y$, or equivalently, $R^n xy^{-1} \sim R^n$. But this was done above.

Note that - absorbing more in the stable equivalence - we can modify $\sim$ to require that $G'$ and $G''$ have a common complement. Also, $(H, F, G) \sim 0 \Leftrightarrow F \sim G \Leftrightarrow$ stably, $F$ and $G$ have a common complement.

The return from automorphisms to pairs of subkernels brings us closer to the geometry in [17, Chapter 6]. It also now follows that the L-groups of [17] can be described in our present terms as follows. Take $V = \mathrm{Wh}\,\pi$, so $X$ is the image in $K_1(\mathbb{Z}\pi)$ of $\{\pm\pi \in (\mathbb{Z}\pi)^\times\}$. Then

$$L_{2k}(\pi) = L_0^X(\mathbb{Z}\pi, \alpha, (-1)^k)$$
$$L_{2k+1}(\pi) = L_1^X(\mathbb{Z}\pi, \alpha, (-1)^k)/\text{class of } \sigma = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix},$$

where $\alpha$ is the anti-involution given by

$$\alpha(g) = w(g)\, g^{-1} \qquad \text{for} \quad g \in \pi$$

($w$ the orientation homomorphism). These identifications are now immediate on comparing the definitions. Similarly, we obtain the surgery obstruction groups $L^h$ for homotopy equivalence as above, but taking $X = \{0\}$.

## §5 Further remarks

Although I regard the above as moreorless in final form it is, in some important respects, incomplete. In this section I discuss desirable generalisations, and compare with the work of other authors.

First, there is the problem of dealing with reflexive bilinear, rather than quadratic forms. The work of Bak [1] [2] has suggested that we should generalise, and consider the concept of 'unitary ring' as formulated in Bass [5]. For $(A, \alpha, u)$ an antistructure, we consider an additive subgroup $\Lambda$ of $A$ satisfying

(i) $S_{-u}(A) = \{a - a^{\alpha}u : a \in A\} \subseteq \Lambda \subseteq S^{-u}(A) = \{a \in A : a = -a^{\alpha}u\}$

(ii) $a^{\alpha}ra \in \Lambda$ for all $a \in A$, $r \in \Lambda$ .

Then a $(-u)$-reflexive form over the unitary ring $(A, \alpha, u, \Lambda)$ is a $(-u)$-reflexive form $\phi$ over $(A, \alpha, u)$ with $\phi(x, x) \in \Lambda$ for all $x$. The module of $u$-quadratic forms is the quotient of the group of sesquilinear forms by the subgroup of $(-u)$-reflexives. Bass gives generalisations of all our results up to Theorem 1 to quadratic forms in this sense. However, this does not really solve the problem of giving a good account of reflexives. Nor does it seem possible to proceed to analogues of Theorem 2 and its corollaries (which really constitute our main theme), as there is no natural choice of 'dual category' (as we had $\mathcal{L}(A, \alpha, u)$ and $\mathcal{L}(A, \alpha, -u)$).

It seems to me that if there is a common generalisation of our two approaches, it should go somewhat as follows. We choose $\Lambda_-(\subset S^{-u}(A))$ and $\Lambda_+(\subset S^u(A))$ independently and then seek (e.g. using some modified version of Witt vectors) a more general notion of form where $\phi(x, x)$ can take any value in $\Lambda_+$, and is to be interpreted as $y + y^\alpha u$ where $y$ is defined mod $\Lambda_-$. Here, $y$ cannot be assumed to take values in $A$ : we need a larger group. A nontrivial example is Brown's notion of quadratic forms over the field $A$ of 2 elements, taking integers mod 4 as values.

Next, there is the question of higher (and lower!) K (or L) groups. The most suggestive work here has been done by Karoubi, and the ideas can be expressed as follows. We start with the forgetful and hyperbolic functors

$$F : \mathcal{Q}(A, \alpha, u) \to \mathcal{P}(A) \qquad H : \mathcal{P}(A) \to \mathcal{Q}(A, \alpha, u) .$$

Following Quillen and others, from the monoidal category we construct a topological infinite loop space

$$\mathcal{K} = \Omega B|\mathcal{P}(A)| \simeq K_0(A) \times B(GL(A))^{ab} ,$$

$$\Omega B|\mathcal{Q}(A,\alpha,u)| \simeq K_0\mathcal{Q}(A, \alpha, u) \times B(U(A, \alpha, u))^{ab};$$

and F, H induce maps between these; in fact, infinite loop maps. Write $\mathcal{U}(A, \alpha, u)$, $\mathcal{V}(A, \alpha, u)$ for the mapping fibres of H, F respectively.

<u>Main conjecture</u> (<u>Karoubi</u>) <u>There is a natural homotopy equivalence</u>

$$\Omega\, \mathcal{U}(A, \alpha, u) \to \mathcal{V}(A, \alpha, -u) .$$

It has been shown by Karoubi [8] that if Quillen's higher K-groups are replaced by those of Karoubi-Villamayor type [9], the corresponding result holds. Also, Sharpe [15] has shown that there is an isomorphism of $\pi_1$. We will now describe the periodicity situation which would result from the conjecture.

Write $\mathcal{L}^0 = \Omega B\mathcal{Q}(A, \alpha, u)$, $\mathcal{L}^1 = \mathcal{U}(A, \alpha, u)$,

$\mathcal{L}^2 = \Omega B\mathcal{Q}(A, \alpha, -u)$, $\mathcal{L}^3 = \mathcal{U}(A, \alpha, -u)$,

and regard the p in $\mathcal{L}^p$ as taking values integers mod 4. Then for each p, we have a fibering (up to homotopy)

$$\mathcal{L}^p \to \mathcal{K} \to \mathcal{L}^{p+1} .$$

Define $KU_{p,n} = KU_{p,n}(A, \alpha, u) = \pi_n(\mathcal{L}^{p-n})$. This has period 4 in p, and we have exact sequences

$$\ldots KU_{p+1,n} \to K_n \to KU_{p,n} \to KU_{p,n-1} \to K_{n-1} \to \ldots$$

Now define $L_{p,n-\frac{1}{2}} = \mathrm{Im}(KU_{p,n} \to KU_{p,n-1})$. This can of course also be defined as a kernel or as a cokernel.

<u>Consequences</u> <u>The composite</u> $K_n \to KU_{p,n} \to K_n$ <u>is</u> $1 + (-1)^p\alpha$ .

<u>There are exact sequences</u>

$$\ldots L_{p,n+\frac{1}{2}} \to L_{p,n-\frac{1}{2}} \to H^p(K_n) \to L_{p-1,n+\frac{1}{2}} \ldots$$

The first should be a simple verification; the second will then follow from Lemma 13. As in §4, we will then also be able to define intermediate L-groups between $L_{p,n-\frac{1}{2}}$ and $L_{p,n+\frac{1}{2}}$ for each $\alpha$-invariant subgroup X of $K_n$.

To illustrate this pattern, here are two simple consequences. First, for any $(A, \alpha, u)$, tensor all K and L groups by $\mathbb{Z}[\frac{1}{2}]$ . Then $\bar{L}_p = L_{p,n-\frac{1}{2}} \otimes \mathbb{Z}[\frac{1}{2}]$ is independent of n, and we have canonical splittings

$$KU_{p,n} \otimes \mathbb{Z}[\tfrac{1}{2}] = S_{(-1)^p}(K_n \otimes \mathbb{Z}[\tfrac{1}{2}]) \oplus \bar{L}_p .$$

Next suppose A the sum of two anti-isomorphic rings R, S interchanged by $\alpha$ . Then $\mathcal{Q}(A, \alpha, u) \cong \mathcal{P}(R)$ , whence

$$KU_{p,n}(A, \alpha, u) \cong K_n(R) \qquad L_{p,n-\frac{1}{2}}(A, \alpha, u) = 0 .$$

Although this development is still conjectural, the description has been justified for low values of n - e.g. the above exact sequence is valid for n = 1 (Theorem 3) and n = 0 (this and the case n = 1 are in Ranicki [12]), thus answering the problems raised in [17, §17D]. Ranicki has also considered the case $n < 0$ where there is a definition analogous to that of Bass [4] for $K_n$. One would hope here for a spectrum, as Gersten [7] obtains for algebraic K theory.

The above notation illustrates well the difference between what I have described as KU-theory and L-theory. In the former (as studied by the Bass school) the natural spaces are $\mathcal{L}^0$ (and, to lesser degree, other $\mathcal{L}^i$) and the natural sequence of groups is $\pi_n(\mathcal{L}^0)$. In the latter, the natural sequences are the periodic sequences with n fixed, and Ranicki [13] has succeeded in constructing (by simplicial sets) periodic spaces $\mathcal{L}_n$ with $\pi_p(\mathcal{L}_n) = L_{p,n}$ $(n = -\frac{1}{2}, \frac{1}{2}, 1\frac{1}{2})$.

I hope this paper will help explain the viewpoint of L - theory as opposed to KU - theory.

Since we have spaces, relative groups can be defined as homotopy groups of mapping fibres. Algebraic definitions of the relative KU groups in low dimensions are also given by Bass [5]. In general, the relative L groups cannot be very closely related to the relative KU groups: the theory here is clearly susceptible of improvement.

Products have been studied to some extent by Karoubi [8]. It seems, for example, that if A is commutative, $KU_{p,n}(A, \alpha, 1)$ should be a bigraded ring. Again, the complete situation is obscure.

The development likely to be of most value for topological applications would be a definition replacing modules by chain complexes throughout. For the case when 2 is invertible in A, this was achieved by Miscenko [10]. See also the discussion in [17, §17G].

To conclude, we give a dictionary of notations: I will compare others with the systematic notation [S] of this paragraph.

<u>L - theorists</u>

| | | | | | | |
|---|---|---|---|---|---|---|
| [11] [12] [13] | $U_p$ | | $V_p$ | | $W_p$ | |
| [17, §17D] | $L^A_p$ | $L^F_p$ | $L^B_p = L^h_p$ | $L^C_p$ | $L^E_p = L^S_p$ | $L^D_p$ |
| This paper | | | $L^K_p$ | $\Lambda_p$ | $L^S_p$ | |
| [S] | $L_{p,-\frac{1}{2}}$ | $KU_{p,0}$ | $L_{p,\frac{1}{2}}$ | $KU_{p,1}$ | $L_{p,1\frac{1}{2}}$ | $\mathrm{Im}(L_{p,1\frac{1}{2}} \to L_{p,\frac{1}{2}})$ |

The identifications are not quite precise: the notation of [17] was provisional, but referred to determinants in $Wh(\pi)$, not $K_1(\mathbb{Z}\pi)$; also, the automorphism $\sigma$ is factored out in the groups of the top two rows.

Karoubi

| | | | | |
|---|---|---|---|---|
| [8] [9] | ${}_1L_n$ | ${}_{-1}V_{n-1} \cong {}_1U_n$ | ${}_1W_n$ | ${}_1L'_n$ |
| [S] | $KU_{n,n}$ | $KU_{n+1,n}$ | $L_{n,n-\frac{1}{2}}$ | $L_{n,n+\frac{1}{2}}$ |

Changing the prefix from 1 to -1 has the effect of changing the first suffix in the lower row by 2 also. Karoubi also has 'homotopical' versions ${}_1L^{-n}$ etc.

KU - theorists

| | | | | | | |
|---|---|---|---|---|---|---|
| [3] | $KF_n(A, \lambda, \Lambda)$ | $KU_n^\lambda(A)$ | $KQ_n^\lambda(A)$ | $W_n(A,\lambda,\Lambda)$ | $W_n^\lambda(A)$ | $WQ_n^\lambda(A)$ |
| [5] | $KU_n^\lambda(A, \Lambda)$ | $KU_n^\lambda(A, S^\lambda(A))$ | $KU_n^\lambda(A, S_\lambda(A))$ | $W_n^\lambda(A, \Lambda)$ | $W_n^\lambda(A, S^\lambda(A))$ | $W_n^\lambda(A, S_\lambda(A))$ |
| [S] | | | $KU_{n,n}(A,\alpha,\lambda)$ | | | $L_{n,n-\frac{1}{2}}(A,\alpha,\lambda)$ |

where $n = 0$ or $1$ (usually $0$); $\alpha$ is understood.

## §6 L - theory of division rings

By way of a simple illustration to the preceding, we now give one calculation. It is not really original: see e.g. [6] . We begin by introducing a new type of elementary matrix.

In $H(R) \oplus (N, \theta)$ we define $\epsilon^1(y, \lambda)$ for $y \in N$, $\lambda \in q_\theta(y)$ by

$$e \mapsto e - f\lambda + y \qquad f \mapsto f$$

and for, $x \in N$, $x \mapsto x - f\, b_\theta(y, x)$.

Then a simple calculation shows that

$$\epsilon^1(y, \lambda)\, \epsilon^1(z, \mu) = \epsilon^1(y + z, \lambda + \mu + b_\theta(y, z)).$$

In the case $(N, \theta) = H(R^n)$, we have

$$\epsilon^1(f_i r, 0) = E_{1i}(r)$$
$$\epsilon^1(e_i r, 0) = \Sigma_{ji}^{-1} E_{1j}(ur) \Sigma_{ji}$$
$$\epsilon^1(0, \mu - \mu^\alpha u) = [\epsilon^1(e_1, 0), \epsilon^1(f_1\mu, 0)]$$

so for $n \geq 2$, all $\epsilon^1(y, \lambda) \in EU_{n+1}$. The same applies, similarly, to $\epsilon^2(y, \lambda)$ defined by

$$e \mapsto e \qquad f \mapsto -e\lambda^\alpha + f + y$$

and for $x \in N$ $\qquad x \mapsto x - e u^{-1} b_\theta(y, x)$

<u>Theorem 5</u> , <u>Let</u> R <u>be a division ring. Then</u> $L_1^K(R, \alpha, u) = 0$ <u>unless</u> R <u>is commutative,</u> $\alpha$ <u>is the identity and</u> $u = 1$, <u>in which case the group has order 2.</u>

<u>Proof</u> We consider a general automorphism of $H(R^n)$, and seek to modify it by elementary and hyperbolic transformations till we obtain a normal form. The argument proceeds by induction on n . Let p be an automorphism of $H(R) \oplus (N, \theta)$, and write

$$ep = ea + fb + x \qquad a, b \in R, \quad x \in N .$$

Suppose $a \neq 0$. Then as p is an isometry, $0 = q(e_1) = q(e_1\, p\, H(a^{-1})) = \lambda + q(x)$, so $\epsilon^1(-x, ba^\alpha)$ is defined, and $p' = p\, H(a^{-1})\epsilon^1(-x, ba^\alpha)$ leaves e fixed. Write $fp' = ec + fd + y$. Since p' preserves the inner product $b_\theta(e,f)$, $d = 1$. Then $p'' = p'\epsilon^2(-y, c^{\alpha^{-1}})$ leaves e and f fixed, and thus can be regarded as an automorphism of $(N, \theta)$.

Apart from the need for supposing $a \neq 0$ , the argument shows by induction that p is the product of elementary and hyperbolic transformations, which is what we are trying to prove. It thus remains to see whether we can always multiply p by an elementary transformation to ensure $a \neq 0$. Now the coefficient of e in $ep\,\epsilon^2(y, \lambda)$ is minus

$$\lambda^\alpha b + u^{-1} b_\theta(y, x) .$$

If $b = 0$, then $x = ep \neq 0$, so we can choose $b_\theta(y, x) \neq 0$ by nonsingularity. If $b \neq 0$, first try to choose $y = 0$, $\lambda = \mu - \mu^\alpha u \neq 0$. This is possible unless $u = 1$ and $\mu = \mu^\alpha$ for all $\mu$, so $\alpha$ = identity, an antiautomorphism, and R is commutative: we are in the exceptional case. Finally, in this case, $\lambda = q_\theta(y)$ is determined by $y$. If now $q_\theta(y) b + b_\theta(y, x)$ vanishes for all $y \in N$, the quadratic form $q_\theta$ is additive in $y$, hence

$$0 = q_\theta(y + z) - q_\theta(y) - q_\theta(z) = b_\theta(y, z)$$

for all $y, z$. Since our form is nonsingular, it follows that $N = 0$.

These results prove that $L_1^K = 0$ save in the exceptional case, and that in that case any nonzero element of $L_1^K$ can be represented by an automorphism of a hyperbolic plane

$$e \mapsto ea + fb$$

$$f \mapsto ec + fd$$

where, moreover, $a = 0$. Since, moreover, we have an isometry of quadratic forms over R it follows that $d = 0$ and $c = b^{-1}$. Multiplying by a hyperbolic automorphism, we reduce to the 'interchange' $\sigma$ :

$$e\sigma = f \qquad f\sigma = e .$$

It remains to show that $\sigma$ does not give $0 \in L_1^K(R, 1, 1)$.

If K does not have characteristic 2, this is easy: any elementary or hyperbolic automorphism has determinant (in the naïve sense) + 1, whereas $\det \sigma = -1$. Another proof, which includes the characteristic 2 case, runs as follows. Form the Clifford algebra $C = C_0 \oplus C_1$ of the quadratic form; let Z be the centraliser in C of $C_0$. Then Z is a quadratic Galois extension of R : either $R \oplus R$ or a field. Any automorphism of the form induces automorphisms of C, $C_0$ and Z over R. It is now easily shown that any elementary or hyperbolic automorphism induces the identity on Z, whereas $\sigma$ induces the nontrivial automorphism.

The above theorem follows from those quoted in [18,II] , but this direct proof seems in the spirit of L - theory.

Our argument also yields an unstable result, but since better results are known [2], [5], it does not seem worth pursuing this point. Other L groups for fields were computed in [18,II], and it seems appropriate to quote them here, except for global fields where a better formulation will be given in [18 , V] .

Suppose R a division ring with centre K . If $\alpha|K$ is not the identity, but has fixed field k (type U), our groups $L_n$ have period 2 in n, and vanish for n odd. The exact sequence of Theorem 3, Corollary thus reduces to

$$0 \to H^1(K_1(R)) \to L_0^S(R) \to L_0^K(R) \to H^0(K_1(R)) \to 0 .$$

For R finite, all groups are zero. For R local, the first two are zero; the latter two isomorphic to $k^\times/Nk^\times$, hence of order 2 . For $R = \mathbb{C}$, we have

$$0 \to 0 \to 4\mathbb{Z} \to 2\mathbb{Z} \to \{\pm 1\} \to 0 .$$

Next let $\alpha$ be trivial on K, which has characteristic 2 (type SPOT). We suppose R finite (then R = K). Then $L_i^S = L_i^K$ has order 2 and $H^i(K_1(R)) = 0$ for all i .

Finally suppose $\alpha$ trivial on K , of characteristic $\neq 2$ . We suppose $L_1(R)$ the commutator quotient of a group which (as algebraic group) is orthogonal (not symplectic); otherwise replace u by -u. We give the table of groups

$$L_3^S \to L_3^K \to H^3(K_1) \to L_2^S \to L_2^K \to H^2(K_1)$$
$$\to L_1^S \to L_1^K \to H^1(K_1) \to L_0^S \to L_0^K \to H^0(K_1)$$

with the convention that 1 denotes a group of order 1, 2 a group of order 2, $G = K^\times/(K^\times)^2$ and $d\mathbb{Z}$ is the subgroup of $\mathbb{Z}$ generated by d .

| | | | |
|---|---|---|---|
| R finite (G = 2) | 1 - 1 - 2 = 2 - 1 - G<br>G - 2 = 2 - 1 - G = G | $R = \mathbb{R}$ | 1 - 1 - 2 = 2 - 1 - 2<br>2 - 2 = 2 - 4$\mathbb{Z}$ - 2$\mathbb{Z}$ - 2 |
| R local commutative | 1 - 1 - 2 = 2 - 1 - G<br>G - 2 = 2 - 2 - $L_0^K$ - G | $R = \mathbb{C}$ | 1 - 1 - 2 = 2 - 1 - 1<br>1 - 2 = 2 - 1 - 1 - 1 |
| R local non-commutative | 1 - 1 - 2 = 2 - 1 - G<br>G - 1 - 2 = 2 - G = G | $R = \mathbb{H}$ | 1 - 1 - 1 - 2$\mathbb{Z}$ = 2$\mathbb{Z}$ - 1<br>1 - 1 - 1 - 1 - 1 - 1 |

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