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On the Classification of Hermitian Forms

II. Semisimple Rings

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The object of this series of papers is to perform the classification in question over integer group rings of finite groups, by the methods illustrated in [21]: reduction to rings with better algebraic properties. In writing out and lecturing on this reduction I became aware that the classifications over semisimple rings were, although known and in the literature, somewhat scattered and unfamiliar to potential readers, so I decided to collect the results in a convenient form. This has proved a difficult task (the version below is completely different from a draft I wrote a year ago): the results in the literature are usually not stated in a form convenient for my application, and although the results given are usually more than strong enough to yield the desired calculations, some effort on my part has been needed also. However, to keep the paper in the form of an (I hope) readable summary of known results, I have suppressed these calculations, and given only the results and key references. Indeed, translating results from Galois cohomology into the results quoted here is trivial for those familiar with the subject, and to explain things to others one would need a full account of the subject. I recommend [16].

I believe that this survey will be of interest to the connoisseur as well as to those unfamiliar with these matters: points which have caused me particular trouble, and seem to have some novelty are commutator quotients of unitary groups over division rings, classification of orthogonal-type forms over quaternion rings, and virtually all of the last chapter, on based modules, where I have tried (as elsewhere) to be as explicit as possible. This distinguishes this paper from an otherwise similar recent survery by Fröhlich [10], which gives more proofs, but less complete results in a more general situation. I am grateful to Ali Fröhlich for constructive criticism of the first draft of this paper.

1. Categories of Quadratic Modules

We proceed by analogy with the simpler and more familiar case of categories of projective modules, so first recall the relevant definitions here. For any ring R, consider finitely generated (right) R-modules M: such a module is projective if it is isomorphic to a direct summand of a free module R^n. We define $\mathscr{P}(R)$ to be the category whose objects are the finitely generated projective R-modules, and morphisms their isomorphisms. The notion of direct sum of modules defines a functor $\oplus: \mathscr{P}(R)\times\mathscr{P}(R)\to\mathscr{P}(R)$ giving $\mathscr{P}(R)$ the structure of "category-with-product": one sees easily that this product is commutative and associative up to (coherent) natural equivalences.

Before defining quadratic modules, we introduce the notion of anti-structure.

By *antistructure* we mean a triple (R,α,u) where α is an anti-automorphism of the ring R, u a unit of R such that

$$u^\alpha = u^{-1},$$

and

$$x^{\alpha\alpha} = u\,x\,u^{-1} \quad \text{for all } x\in R.$$

Such triples were introduced in our paper [20] to formalise a general notion of quadratic form. Note the important special case when $u=1$: α is then an (anti-) involution of R.

For M a (right) R-module, write $\mathrm{Sesq}_\alpha(M)$ for the additive group of maps

$$\phi\colon M\times M\to R$$

satisfying the identities

$$\phi(m, n_1 r_1 + n_2 r_2) = \phi(m,n_1)\,r_1 + \phi(m,n_2)\,r_2$$
$$\phi(m_1 r_1 + m_2 r_2, n) = r_1^\alpha\,\phi(m_1,n) + r_2^\alpha\,\phi(m_2,n).$$

Define a transposition operator

$$T\colon \mathrm{Sesq}_\alpha(M)\to\mathrm{Sesq}_\alpha(M)$$

by

$$T\phi(m,n) = \phi(n,m)^\alpha\,u.$$

Then $T^2=1$; we define $R_{\alpha,u}(M)=\mathrm{Ker}(1-T)$, and $Q_{\alpha,u}(M)=\mathrm{Coker}(1-T)$; the elements of the latter group are called *quadratic forms* on M; a pair (M,q), $q\in Q_{\alpha,u}(M)$ is a *quadratic module.* Multiplication by $1+T$ induces a map (bilinearisation)

$$b\colon Q_{\alpha,u}(M)\to R_{\alpha,u}(M)$$

which is an isomorphism when 2 is invertible in R – this is generally the case in this paper, though it will not be in the sequel. A map $\phi\in\mathrm{Sesq}_\alpha(M)$ will be called *nonsingular* if the associated homomorphism

$$A\phi\colon M\to\mathrm{Hom}_R(M,R)$$

defined by

$$A\phi(m)(n)=\phi(m,n)$$

is an isomorphism; a quadratic form is called nonsingular if its bilinearisation is. We define $\mathcal{Q}(R,\alpha,u)$ to be the category whose objects are pairs (M,ϕ): M a finitely generated projective R-module, ϕ a nonsingular quadratic form on M; and whose morphisms are isometries – module isomorphisms which preserve the quadratic form. There is a natural notion of (orthogonal) direct sum of forms, giving $\mathcal{Q}(R,\alpha,u)$ the structure of category-with-product, as for $\mathcal{P}(R)$.

A role analogous to that of free modules in $\mathcal{P}(R)$ is played here by hyperbolic modules (see [20]). Generally, we define the hyperbolic functor H: $\mathcal{P}(R)\to\mathcal{Q}(R,\alpha,u)$ as follows: for P an object of $\mathcal{P}(R)$, define P^α as the dual module $P^\alpha=\mathrm{Hom}_R(P,R)$ with R-action given by $q\,a(p)=a^\alpha q(p)$ for $a\in R$, $p\in P$, $q\in P^\alpha$ and $H(P)=P\oplus P^\alpha$ with the form defined as the equivalence class of h, where

$$h((p,q),(p',q'))=q(p').$$

A map f: $P\to Q$ induces a dual f^α: $Q^\alpha\to P^\alpha$ and hence, if f is an isomorphism, $H(f)=f\oplus(f^\alpha)^{-1}$, which is easily checked to be an isometry. Now by [20, Theorem 3], for any quadratic module (M,θ) we have an isometry $(M,\theta)\oplus(M,-\theta)\cong H(M)$: the functor H is cofinal (even naturally cofinal in the sense of [12]). Further, since any projective module is a direct summand of a free module, any object of $\mathcal{Q}(R,\alpha,u)$ is a direct summand of some $H(R^n)$.

Conversely, an object with $H(R^n)$ as a summand is said to be of *index* $\geq n$. In discussions of stability in categories $\mathcal{Q}$, large enough index plays a role analogous to that of large enough rank in categories $\mathcal{P}$.

We now discuss equivalences of categories. Again we start with the case of $\mathcal{P}(R)$. If R is a direct sum of rings, e.g. $R=R_1\oplus R_2$, there is a natural equivalence of $\mathcal{P}(R)$ with the product category $\mathcal{P}(R_1)\times\mathcal{P}(R_2)$; thus problems about projective modules over R are reduced to the simpler cases over R_1 and R_2.

Next, suppose R,S two rings with an $R-S$-bimodule M such that M is faithfully projective as R-module and S is the endomorphism ring $\mathrm{End}_R(M)$. Then ("Morita theory") there is an $S-R$-bimodule N so that

$$M\otimes_S N\cong R,\qquad N\otimes_R M\cong S$$

and so tensoring by M and N gives inverse equivalences between the categories $\mathcal{P}(R)$ and $\mathcal{P}(S)$. The simplest case is $M=R^n$, $S=\mathrm{End}_R(R^n)$, i.e. the matrix ring R_n.

We apply this to the case of semisimple rings. A semisimple ring is [5, 5.3] a direct sum of simple rings: by the first paragraph above, it suffices to consider these. If we restrict (it suffices for later applications) to the case of algebras of finite dimension over a (commutative) field k, then [5, 5.4] a simple ring S is a matrix ring over a division ring D, finite over its centre K (which is of course a finite extension of k). By the second paragraph above, we have an equivalence $\mathscr{P}(S) \cong \mathscr{P}(D)$, so have now reduced to the case of division rings. This argument goes one stage further. Consider all simple rings S with centre K: they form an abelian monoid under $\otimes_K$. The submonoid of matrix rings K_n is cofinal in this: the quotient group is called the Brauer group $Br(K)$. Two simple rings determine the same element of $Br(K)$ if and only if they are Morita equivalent, if and only if they come from the same division ring, so we have effectively one category $\mathscr{P}(s)$ for each $s \in Br(K)$.

We now consider categories $\mathscr{Q}(R, \alpha, u)$. Again if R splits as direct sum of two rings, and each is invariant under α, $R = R_1 \oplus R_2$ with $u = (u_1, u_2)$, we have

$$\mathscr{Q}(R, \alpha, u) \cong \mathscr{Q}(R_1, \alpha | R_1, u_1) \oplus \mathscr{Q}(R_2, \alpha | R_2, u_2).$$

If, however, α interchanges R_1 and R_2, and thus induces an anti-isomorphism between them, then I claim

$$\mathscr{Q}(R, \alpha, u) \cong \mathscr{P}(R_1).$$

Note here that – whether 2 is invertible or not – the bilinearisation map is an isomorphism. We will talk in terms of $R_{\alpha, u}(M)$. Also in this case, we can write an R-module M as $M_1 \oplus M_2$ with M_1 an R_1-module, M_2 an R_2-module; a sesquilinear map $M \times M \to R$ amounts to a pair of maps $M_1 \times M_2 \to R_2$ and $M_2 \times M_1 \to R_1$; the symmetry condition states that either of these determines the other, and the nonsingularity condition that we have dual pairings, so can identify M_2 with the dual of M_1. Conversely, given M_1, we define M_2 as its dual and $M_2 \times M_1 \to R_1$ as the dual pairing, and can reconstruct the rest. It is easy now to see that we have an equivalence of categories.

There is also a "Morita theory" in this case, which is due to Fröhlich and McEvett [11]. Suppose (M, ϕ) a quadratic R-module, or indeed a module supporting a nonsingular reflexive form: which is faithfully projective as module, and we give $S = \mathrm{End}_R(M)$ an antistructure by using the adjoint map for ϕ, then we get, as above, an equivalence of categories

$$\mathscr{Q}(R, \alpha, u) \cong \mathscr{Q}(S, \beta, v).$$

The simplest special case of this is

$$\mathscr{Q}(R, \alpha, u) \cong \mathscr{Q}(R_n, \alpha_n, u_n),$$

where R_n is the $n \times n$ matrix ring, α_n conjugate (for α) transpose, and u_n the diagonal scalar matrix $u I_n$. Another is "scaling" (see [20]); for any unit $v \in R^\times$ we have

$$\mathscr{Q}(R, \alpha, u) \cong \mathscr{Q}(R, \beta, v v^{-\alpha} u),$$

where $x^\beta = v x^\alpha v^{-1}$ for all $x \in R$.

We now turn again to semisimple rings. If (R, α, u) is an antistructure, with R semisimple of finite dimension over k, first write $R = \bigoplus_{i \in I} R_i$ as sum of simple rings. Since α^2 is an inner automorphism, it preserves each summand R_i. So the summands are preserved by α or interchanged in pairs; correspondingly, $\mathscr{Q}(R, \alpha, u)$ is a product category with one factor for each orbit of α on I. Moreover, if α interchanges R_i and R_j, we can interpret the corresponding factor as $\mathscr{P}(R_i)$. We are thus reduced to the case when R is simple.

Consider antistructures (S, α, u), where S is simple and of finite dimension over its centre K. We will fix not only K but also the restriction (c, say) of α to K. Since K is the centre, $c(K) = K$, and $c^2 = 1_K$: we may or may not have c the identity. The isomorphism classes of such triples (S, α, u) form an abelian monoid with a product induced by $\otimes_K$, and those Morita equivalent to $(K, c, 1)$ a cofinal submonoid. The quotient group is the Brauer group $Br(K, c)$ of [19]: two triples define the same element of it if and only if they are Morita equivalent. (For further details, see Fröhlich and Wall [13].)

Thus for each field K, involution c of K (possibly $c = 1$), and $x \in Br(K, c)$ we have a category $\mathscr{Q}(x)$ of quadratic modules. We next describe the determination of the groups $Br(K, c)$. Sometimes here—and later—we refer without comment to natural maps of these groups and to natural product-preserving functors between the underlying categories induced by change of base ring: these are defined as follows. Given a map $f: R \to S$ of commutative rings we have, for each central simple R-algebra A, a central simple S-algebra $B = A \otimes_R S$, and for each projective A-module (and hence also R-module) L, a projective B-module $L \otimes_R S$. For quadratic modules we have essentially the same, provided the involutions c_R, c_S of R, S satisfy $f \circ c_R = c_S \circ f$.

Unlike the general theory preceding, the structure of the Brauer group depends critically on the particular field under consideration. We shall be interested in this paper in four classes of fields, which we label as follows:

Finite finite fields (including characteristic 2).
Real real and complex fields.
Local (p-adic) local fields, and
Global algebraic number fields (finite extensions of $\mathbb{Q}$).

Corresponding results hold for the other local and global fields, but this paper is long enough as it is. We first give results for the ordinary Brauer group: the computation of such groups is not easy; we shall merely quote the results.

Finite. $Br(K)=0$: every finite division ring is a field [5, 11.1].
Real. $Br(\mathbb{C})=0$ [5, 7.4]. $Br(\mathbb{R})$ has order 2 [5, 11.2]: the division rings are $\mathbb{R}$ and the ring $\mathbb{H}$ of ordinary quaternions.
Local. Here [16, p. 200] [24, p. 222] [7, p. 130] there is an isomorphism

$$\operatorname{inv}: Br(K) \cong \mathbb{Q}/\mathbb{Z}.$$

For $K=\mathbb{R}$ one extends the notation, setting $\operatorname{inv}(\mathbb{H})=\frac{1}{2}$.
Global. If we let $\mathfrak{p}$ run through the places of K, we obtain [24, p. 264] [7, p. 188] an exact sequence

$$0 \to Br(K) \to \oplus Br(K_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

where the first map is induced by the inclusions of K in its localisations $K_{\mathfrak{p}}$, and the second by adding up the local invariants.

We compute $Br(K, c)$ by comparison with $Br(K)$. Note that c induces an involution c_* of $Br(K)$. Since taking the opposite algebra gives inverse in $Br(K)$, and an algebra in $Br(K, c)$ is c-isomorphic to its opposite algebra, the image of the forgetful map $Br(K, c) \to Br(K)$ is contained in the kernel $Br(K)^c$ of $1+c_*$. Indeed, the general theory of [13] gives for this case an exact sequence

$$0 \to H^1(\mathbb{Z}_2; K^\times) \xrightarrow{\lambda} Br(K, c) \to Br(K)^c \xrightarrow{\theta} H^2(\mathbb{Z}_2; K^\times), \qquad (*)$$

where the action of $\mathbb{Z}_2$ on $K^\times$ is that of c. Here, if $a \in K^\times$ satisfies $a^c a=1$ and (A, α, u) is an antistructure, to add $\lambda(a)$ means replacing u by au. Also, θ is defined as follows: if A is an algebra representing an element of $Br(K)^c$, choose a c-anti-automorphism α of A (i.e. $\alpha|K=c$). Then α^2 is an automorphism of A over its centre K hence, by the Skolem-Noether theorem [5, 10.1], inner. Choose $u \in A$ with $x^{\alpha\alpha}=u x u^{-1}$ for all $x \in A$. Then $u^\alpha u \in K$, and is invariant by c: its class modulo elements $d^\alpha d$ $(d \in K^\times)$ represents $\theta(A)$.

There are now several distinct cases. If c acts trivially on K ("involutions of the first kind"), $Br(K)^c$ is the subgroup $Br_2(K)$ of elements of order 2 in $Br(K)$. By a result of Albert [1, p. 161], the map θ is zero. The group $H^1(\mathbb{Z}_2; K^\times)$ consists essentially of the elements $\pm 1 \in K^\times$, so has order 2 (unless K has characteristic 2, when it is trivial). Thus adding an element of the kernel will interchange $\pm u$ in (D, α, u). Moreover, the extension is split for if $\bar{K}$ is an algebraic (or separable) closure of K, $Br(K, 1)$ maps onto $Br(\bar{K}, 1) \cong \{\pm 1\}$. Antistructures corresponding to $+1$ here are of *orthogonal type*, or type O; the others of *symplectic* *type*, or type *Sp*: this being the type of algebraic group of automorphisms of objects of the category. In characteristic 2, we refer to type *SPOT*.

If c acts non-trivially on K ("involutions of the second kind"), we have antistructures of *unitary type* or type U. We always denote the fixed field of c acting on K by k. Here $H^1(\mathbb{Z}_2; K^\times)$ vanishes, but θ need not. For completeness, we also refer to *linear type* or type GL for the categories $\mathscr{P}(R)$.

We now consider the algebras in more detail. The proof of exactness of (*) shows, in fact, that $\theta(A)$ is the obstruction to A itself supporting an antistructure; thus if A does, so does any equivalent algebra, and hence also the underlying division ring D: more precisely, any antistructure on A is equivalent to one on D. We can even normalise this antistructure to some extent.

Lemma. *Let (D, α, u) be an antistructure, D a division ring. Except in the case $(K, 1, -1)$, there exists v with $u = v v^{-\alpha}$, so scaling by v takes u to 1 and α to an (anti-)involution.*

If $u \neq -1$, take $v = 1 + u$. If $u = -1$ and we are not in the excluded case, we can first scale by any $v \neq v^\alpha$ to make $u \neq -1$.

The analogous result follows with D replaced by any equivalent simple algebra D_n. In the excluded case $A = K_n$, α = transposition, $u = -I$ we seek a nonsingular skew-symmetric matrix v: this exists if and only if n is even. Indeed even when R is not semisimple, any antistructure (R, α, u) is equivalent to (R_2, α_2, u_2) and we can reduce u_2 to 1 by scaling. For the rest of this paper, we will always have $u = \pm 1$: this is the traditional notation.

It remains to compute θ. In the finite and real cases, there is nothing to do. In the local case, θ is an isomorphism: this follows from another result of Albert [1, p. 161][1]. So in these cases, for type U, the underlying division ring is commutative.

For a number field we use the fact (true for quadratic extensions by [7, p. 199]) that a number is a norm if it is everywhere locally. Thus for $\xi \in Br(K)$, $\theta(\xi) = 0$ if and only if for each place $\mathfrak{p}$ of K, either

$$c_*(\mathfrak{p}) \neq \mathfrak{p} \quad \text{and} \quad \operatorname{inv}_{c^*(\mathfrak{p})}(\xi) = -\operatorname{inv}_{\mathfrak{p}}(\xi)$$

or

$$c_*(\mathfrak{p}) = \mathfrak{p} \quad \text{and} \quad \operatorname{inv}_{\mathfrak{p}}(\xi) = 0,$$

i.e. if and only if ξ has the form $c_*(\eta) - \eta$.

Indeed, over any ring R, $\theta(c_* \eta - \eta)$ is always trivial, for if A represents η, A^{op} is the opposite algebra, and A^α the same, but with the

[1] Since $Br\,k$ maps onto $Br\,K$, $c_* = 1$, so the only nonzero element of $(Br\,K)^c$ has invariant $\frac{1}{2}$, and is represented by a quaternion algebra. If it came from a quaternion algebra over k, it would have invariant $2 \times \frac{1}{2} \equiv 0$. See also [15, p. 40].

embedding of K twisted by c. We can represent $c_* \eta - \eta$ by $A^\alpha \otimes A^{\mathrm{op}}$: interchanging the factors here gives an anti-automorphism α, inducing c on the centre, and with $\alpha^2 = 1$.

Summary. $Br(K, 1) \cong \{\pm 1\} \times Br_2(K)$, except if $\mathrm{char}^c K = 2$, when we just have $Br_2(K)$. For $c \neq 1$, $Br(K, c)$ is a subgroup of $Br(K)$ lying between $\{c_* \eta - \eta \colon \eta \in Br\, K\}$ and $\{\xi \in Br\, K \colon c_* \xi = -\xi\}$: it coincides with the former in all cases of interest (possibly for any field).

2. Commutator Quotient Groups and K_1

Given a category-with-product $\mathscr{C}$, Bass [4, p. 348] defines a group $K_1(\mathscr{C})$ as the commutator quotient group of a direct limit of groups of automorphisms of the objects of the category. It is not always necessary to proceed to a limit: if C is an object of $\mathscr{C}$ such that, for any object D, the inclusion

$$\mathrm{Aut}\, C \subset \mathrm{Aut}\, C \times \mathrm{Aut}\, D \subset \mathrm{Aut}(C \oplus D)$$

induces an isomorphism of commutator quotients, then the commutator quotient group of Aut C is already isomorphic to $K_1(\mathscr{C})$. As this will be the case for all categories considered in this series of papers, we will refer to such an object C as a *superstable* object of $\mathscr{C}$.

For categories $\mathscr{P}(R)$ in general, a stability theorem has been obtained by Serre and Bass [4, Chapter V], but in the case when R is a division ring D, the result goes back to Dieudonné [8]: every nonzero object is superstable, and $K_1(\mathscr{C})$ is the commutator quotient group of $D^\times$.

If $R = K$ is commutative—i.e. a field—this result $K_1 \mathscr{P}(K) = K^\times$ is simple enough, and the class of an automorphism of a module (necessarily $\cong K^n$) is just the determinant of its matrix. In general, however, one needs a more effective computation.

Suppose the division ring D has finite degree over its centre, K: choose a finite Galois extension L of K which splits D and an L-isomorphism $\phi \colon D \otimes_K L \to L_n$ (this is, of course, unique up to inner automorphism). Now we have, for any matrix ring D_m over D, the diagram

$$\begin{array}{ccc} D_m \longrightarrow D_m \otimes_K L & \xrightarrow{\ \phi\ } & L_{mn} \\ \Big\downarrow{\scriptstyle \mathrm{Nrd}} & & \Big\downarrow{\scriptstyle \det} \\ K & \xrightarrow{\qquad\qquad} & L \end{array}$$

where the image of D_m lies in K since it is invariant under the Galois group, so the "reduced norm" Nrd is defined; one can show it independent of all choices [24, p. 168]. So this gives an alternative determinant map Nrd: $GL_m(D) \to K^\times$. We can regard this as the composite of the Dieudonné determinant and the factor δ in

$$\mathrm{Nrd}: GL_1(D) = D^\times \to D^\times/[D^\times, D^\times] \xrightarrow{\delta} K^\times.$$

For D a quaternion ring, it is easy to see that δ is injective. The argument was extended by Wang [23] to arbitrary division rings over local or global fields; this covers all cases of interest to us, though it is natural to speculate that δ is always injective. We define $SL_m(D)$ to be the kernel of Nrd: when δ is injective, this coincides with the kernel of Dieudonné's determinant. By results in [9, Chapter 2] we can deduce when δ is injective (except when $m=2$ and D has cardinality 2 or 3) that if $m \geqq 2$ this group is generated by transvections, it is perfect, and its quotient $PSL_m(D)$ by the centre (consisting of the intersection of $SL_m(D)$ with the group $K^\times$ of scalar matrices) is simple. As similar situations will arise again below, we will call a subgroup A of $GL_m(D)$ *quasi-simple* if A is perfect and the quotient PA of A by $A \cap K^\times$ is simple. Clearly if A is normal in another group B, is perfect, and B/A is abelian, then A is the commutator subgroup of B. It follows that (with sole exception of $GL_2(F_2)$). $SL_m(D)$ is the commutator subgroup of $GL_m(D)$ for all $m \geqq 1$ and the quotient is isomorphic to $D^\times/[D^\times, D^\times] \cong \mathrm{Nrd}\, D^\times$.

It remains to describe the image of

$$\mathrm{Nrd}: D^\times \to K^\times.$$

Clearly the map is an isomorphism if $D=K$ is commutative. It is also surjective if K is a non-Archimedean local field (see e.g. [24, p. 195]). If $K=\mathbb{R}$ and $D=\mathbb{H}$, then of course the image is just the set of positive reals. Finally for K a number field it was shown by Eichler (see [24, p. 206]) that the image is the set of $x \in K$ such that for each embedding (if any) $f: K \to \mathbb{R}$ where D ramifies (i.e. $\mathbb{R} \otimes_K D$ is a matrix ring over $\mathbb{H}$, not $\mathbb{R}$), $f(x) > 0$. But for K a Laurent series field $k((x))$, and $D = d((x))$ for d a quaternion division ring over k, the image of Nrd cannot be described by orderings of K (I am indebted to M. Kneser for this example).

For categories $\mathscr{Q}(R, \alpha, u)$ in general a stability result has been obtained by Bak [3], but our case is much simpler. Here, the general result is due to G.E. Wall [22]. Suppose $R=D$ a division ring (with centre K, and of finite degree over K), $u=-1$. Then for forms f of index $\geqq 1$, if $U(D,f)$ is the corresponding unitary group, $T(D,f)$ the subgroup generated by transvections, apart from one exception (see below)

$$U(D,f)/T(D,f) \cong D^\times/\Sigma[D^\times, \Omega] \qquad (**)$$

where Σ is the subgroup generated by the α-symmetric elements of $D^\times$, and Ω depends on f, but is certainly $D^\times$ if f has index $\geqq 2$. Further [9, Chapter 2], we usually have $T(D,f)$ quasi-simple. Note that the categories *not* equivalent to one of the above form are the $\mathscr{Q}(K, 1, 1)$ with K commutative.

For a fuller discussion, it is necessary to separate types.

Type Sp. Here, $\Sigma = D^\times$ if D is commutative; it is also easy to check this if D is a quaternion ring, and it is known [9] to hold in general. Since D has order (at most) 2 in $Br\,K$, it is indeed a quaternion ring in the cases considered in this paper. This does not always hold, however: see e.g. [1, Chapter XI]. Thus, provided f has index $\geqq 1$, $Sp(D,f) = T(D,f)$ is quasi-simple in all cases except rank 2, D of cardinality 2 or 3 (for $Sp_2 K = SL_2 K$ for any commutative field K) and rank 4, D of cardinality 2 ($Sp_4 F_2 \cong S_6$ has a simple subgroup of index 2). Thus K_1 is trivial for this type, and except for small fields, every form of index $\geqq 1$ (over commutative fields, every form is hyperbolic, so this includes every non-trivial case) is superstable.

Type U. This is more like the case of type GL above: we are interested in $SU(D,f) = SL_m(D) \cap U(D,f)$ (where f has rank m), and need to suppose that f has index $\geqq 1$. We continue to write K for the centre of D; $\alpha | K$ is now non-trivial; write k for its fixed field. First suppose $D = K$ commutative. Then it is immediate that the determinant, Δ of a unitary matrix satisfies $\Delta^\alpha \Delta = 1$. Here, the right hand side of (**) reduces to $K^\times / k^\times$, and if an element here is represented by $x \in K^\times$, an easy calculation shows that $\Delta = x/x^\alpha$. But $x \mapsto x/x^\alpha$ induces an isomorphism of $K^\times/k^\times$ on $\{\Delta : \Delta^\alpha \Delta = 1\}$, so $SU(D,f) = T(D,f)$, as is noted on [9, p. 49]; there is one exception (where (**) also fails), the case of $SU_3(F_4)$ (which is soluble). Otherwise (loc. cit) $SU(K,f)$ is then quasi-simple save (in view of the isomorphism $SU_2(K) \cong SL_2(k)$) the usual two exceptions: it follows in non-exceptional cases that SU is the commutator subgroup of U, and $K_1(K, \alpha, -1)$ isomorphic to the group of Δ, i.e. to $\operatorname{Ker} N$: $K^\times \to k^\times$.

When D is noncommutative, we combine this argument with the results for type GL. Suppose

(i) $\delta : D^\times / [D^\times, D^\times] \to K^\times$ is injective,
(ii) $\operatorname{Nrd} \Sigma = k^\times \cap \operatorname{Nrd} D^\times$, and
(iii) $U(D,f)/T(D,f)$ is abelian. Then

$$\begin{aligned} U(D,f)/T(D,f) &\cong D^\times / \Sigma [D^\times, D^\times] && \text{by } (**) \\ &\cong \operatorname{Nrd} D^\times / \operatorname{Nrd} \Sigma && \text{by (i)} \\ &\cong \operatorname{Nrd} D^\times / k^\times \cap \operatorname{Nrd} D^\times && \text{by (ii)}, \end{aligned}$$

so the map $x \mapsto x/x^\alpha$ maps this injectively to $K^\times$. Since the composite $U(D,f) \to K^\times$ is the reduced norm, its kernel $T(D,f)$ coincides with $SU(D,f)$. For us, this case only arises over global fields, and here (i) holds and we can prove (ii) by the same method ([25], c.f. [15, 5.7]). By [22] (see also [9, p. 49]), (iii) holds whenever $n \geqq 3$ and index $f \geqq 1$. Hence these forms f are all stable (except $n = 3$, D of order 4), hence also superstable.

And $K_1(D, \alpha, -1) \cong \{x/x^\alpha : x \in \mathrm{Nrd}\, D^\times\}$
$\cong \{y \in \mathrm{Nrd}\, D^\times : y^\alpha y = 1\}$,
as one easily verifies.

Type O (and SPOT). This is the most difficult case: let us first describe the situation when $D = K$ is commutative: we will refer to this in future as type OK. Then determinant takes values ± 1 on $O(K, f)$, and we denote the kernel by $SO(K, f)$. In characteristic 2, the definition of SO is a little different, but the result is the same: note here that n is even. Now using the Clifford algebra of f one constructs another algebraic group $\mathrm{Spin}(K, f)$ and an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(K, f) \to SO(K, f) \to K^\times/(K^\times)^2,$$

where the last map is surjective if index $f \geqq 1$. Then it is $\mathrm{Spin}(K, f)$ which one expects to be quasi-simple and the quotient of $O(K, f)$ by its image is $\{\pm 1\} \oplus (K^\times/(K^\times)^2)$.

If $n = 1$, $SO(K, f)$ is trivial and $\mathrm{Spin}(K, f)$, $O(K, f) \cong \{\pm 1\}$.
If $n = 2$, $SO(K, f)$ and $\mathrm{Spin}(K, f)$ are abelian; if index $f = 1$, both are isomorphic to $K^\times$.
If $n = 3$ and index $f = 1$, $\mathrm{Spin}(K, f) \cong SL_2(K)$, so is quasi-simple if $|K| \geqq 4$.
If $n = 4$, index $f = 1$, f has discriminant Δ and $L = K[\sqrt{\Delta}]$,
$\mathrm{Spin}(K, f) \cong SL_2(L)$ is quasi-simple
index $f = 2$, $\mathrm{Spin}(K, f) \cong SL_2(K) \times SL_2(K)$.
If $n \geqq 5$, index $f \geqq 1$, then Spin (K, f) is quasi-simple.

These isomorphisms are well-known: see e.g. [9, Chapter 2], which contains all these results.

If D is non-commutative, we can consider D to be a quaternion ring (discussion as for case Sp). We will call this type OD. Then first, $SO(D, f) = O(D, f)$ [9, Chapter 2]. Next, we can apply (**): here it is easy to see that $\Omega = D^\times$ and $\Sigma = K^\times$, so the right hand side is isomorphic (via Nrd) to the image of

$$\mathrm{Nrd}\, D^\times \subset K^\times \to K^\times/(K^\times)^2,$$

and we can define a double covering $\mathrm{Spin}(D, f)$ of $T(D, f)$.

Now if $n = 1$, $\mathrm{Spin}(D, f)$ is abelian
if $n = 2$, index $f = 1$, $\mathrm{Spin}(D, f) \cong SL_1(D) \times SL_2(K)$
if $n \geqq 3$, index $f \geqq 1$, $\mathrm{Spin}(D, f)$ is quasi-simple

Summary. Apart from a handful of exceptions over very small fields, any form of index $\geqq 1$ and rank $\geqq 3$ is superstable (for $\mathscr{P}$, any module of rank $\geqq 1$). The groups K_1 are:

Type GL. $K^\times$, except for $D=\mathbb{H}$, only positive reals, and for D global, the subgroup taking positive values at ramified Archimedean spots ($\mathrm{Nrd}\, D^\times$)
Type Sp. $\{1\}$
Type U. $\mathrm{Ker}\, N: K^\times \to k^\times$ except for D global, the subgroup as above
Type OK. $\{\pm 1\} \oplus K^\times/(K^\times)^2$
Type OD. The image of $\mathrm{Nrd}\, D^\times$ in $K^\times/(K^\times)^2$.

Further, for type GL any module of rank $\geqq 1$, in the other cases, any form of index $\geqq 1$ has automorphism group mapping surjectively to this.

We have proved this for finite, local and global fields, but the essential results only depend on

(a) $\delta: D^\times/[D^\times, D^\times] \to K^\times$ is injective (GL, U)
(b) $\mathrm{Nrd}\, \Sigma = k^\times \cap \mathrm{Nrd}\, D^\times$ (U)
(c) Any division ring of order 2 in $Br(K)$ is quaternionic (O).

It appears that (a) and (b) have recently been proved by Platonov for arbitrary fields: (c) does not hold in general, but this fact is unlikely to affect our conclusion. These results are equivalent to the "Kneser-Tits conjecture" for classical groups, that an isotropic simple simply-connected algebraic group (over a field) is generated by its unipotent subgroups.

3. Classification of Quadratic Modules and K_0

We now seek to classify the objects in the categories $\mathcal{Q}(R, \alpha, u)$ described above; by § 2, it is enough to consider the case $R = D$ a division ring, $u = \pm 1$. For any category-with-product $\mathscr{C}$, we write $k(\mathscr{C})$ for the abelian monoid of isomorphism classes of objects of $\mathscr{C}$, $K_0(\mathscr{C})$ for its universal group; and say that cancellation holds in $\mathscr{C}$ if it holds in $k(\mathscr{C})$, i.e. if the natural map $k(\mathscr{C}) \to K_0(\mathscr{C})$ is injective. Now cancellation holds for $\mathscr{P}(R)$ by the theory of dimension, and for $\mathcal{Q}(D, \alpha, u)$ by a mild extension of the theorem of Witt: see [6, 4.3]. Thus not much information is lost by confining ourselves to computing $K_0(\mathscr{C})$.

Any nonsingular form is a direct sum of forms on 1-dimensional vector spaces—i.e. admits an orthogonal basis—except for type Sp or $SPOT$ over a commutative field. The easy proof is in [6, 6.1]. In these cases we can instead choose a symplectic basis $\{e_i, f_i: 1 \leqq i \leqq r\}$ such that for $\phi = b(x)$,

$$\phi(e_i, e_j) = \phi(f_i, f_j) = 0$$
$$\phi(e_i, f_j) = 0\ (i \neq j) \quad 1\ (i = j):$$

this is even easier to prove. For type Sp, over a commutative field, this already gives a complete classification of forms. Note that for type GL we also have orthogonal bases: this really only amounts to saying that a module over a division ring has a basis.

Bilinearisation b is an isomorphism, except for type $SPOT$. Let K be the centre of D. The result is clear if K does not have characteristic 2, since $\mathrm{Sesq}_\alpha(M)$ is a vector space over K, and if k is the fixed field of $\alpha|K$, $\mathrm{Sesq}_\alpha(M)$ splits over k as the direct sum of $+1$ and -1 eigenspaces for T_u. We have already noted the result for type GL: there remains the case when $k \neq K$ has characteristic 2. For any $c \in K-k$, $c^\alpha + c$ is a nonzero element of k; thus $a=(c^\alpha+c)^{-1} c$ satisfies $a^\alpha + a = 1$. Now our assertion is equivalent to showing that

$$\mathrm{Sesq}_\alpha(M) \xrightarrow{1+T} \mathrm{Sesq}_\alpha(M) \xrightarrow{1+T} \mathrm{Sesq}_\alpha(M)$$

is exact (note: signs disappear in characteristic 2). But for any $x \in \mathrm{Sesq}_\alpha(M)$ with $Tx=x$, we have $x=(1+T)(x\,a)$.

We now start constructing our invariants.

Rank. The most obvious invariant of a quadratic module is the dimension of the underlying vector space. This defines a homomorphism rank: $K_0\,\mathcal{Q}(D,\alpha,u) \to \mathbb{Z}$ whose image (by the above remark about orthogonal bases) is $\mathbb{Z}$ except for $D=K$, types Sp and $SPOT$, when it is $2\mathbb{Z}$. If we restrict the rank to be even, this is split by considering hyperbolic spaces $H(D^n)$ (of rank $2n$). We will be primarily interested in the kernel $\tilde{K}_0\,\mathcal{Q}$ of the above, and in the even-dimensional case, will try to find invariants which respect the above splitting—i.e. are trivial on hyperbolic forms. For example, we already know that $\tilde{K}_0\,\mathcal{Q}=0$ for type GL or type Sp over a commutative field.

Arf Invariant (Type $SPOT$). We clear this out of the way first, as it is rather different from the other cases (though à posteriori one can see an analogy to the discriminant). Consider type $SPOT$, with $D=K$ commutative (e.g. finite): take a symplectic base $\{e_i, f_i\}$ of the quadratic module defined by the equivalence class of θ: $V \times V \to K$. The bilinearisation $b=\theta+T\theta$ does not determine the numbers $q(x)=\theta(x,x)$ which are, however, determined by the equivalence class of θ (c.f. [20]). The class of

$$c(q)=\Sigma_i\, q(e_i)\, q(f_i)$$

modulo $\mathfrak{P}K=\{y+y^2\colon y\in K\}$ does not depend on the choice of symplectic basis, and is zero on hyperbolic spaces [2]. Moreover, if K is perfect (e.g. if it is finite), it is easily shown that $c(q)$ determines the quadratic module up to isomorphism. The Arf invariant c is additive for direct sums, so gives an isomorphism

$$c\colon \tilde{K}_0(K,1,1) \cong K^+/\mathfrak{P}K^+:$$

it is surjective since the $q(e_i)$, $q(f_i)$ can be chosen to have any desired values. For K finite, the cokernel of $\mathfrak{P}$: $K \to K$ has the same order (2) as the kernel. Let S: $K \to \mathbb{Z}/2\mathbb{Z}$ have kernel $\mathfrak{P}K$. Then $Sc(q) \in \mathbb{Z}/2\mathbb{Z}$ is the most convenient form of the Arf invariant.

Signature (Real Fields). There are ten categories to be discussed for real fields, but we have already dismissed $\mathscr{P}(\mathbb{R})$, $\mathscr{P}(\mathbb{C})$, $\mathscr{P}(\mathbb{H})$, $\mathscr{Q}(\mathbb{R}, 1, -1)$ and $\mathscr{Q}(\mathbb{C}, 1, -1)$ ($\tilde{K}_0 = 0$): there remain $\mathscr{Q}(\mathbb{R}, 1, 1)$ (type $O(\mathbb{R})$), $\mathscr{Q}(\mathbb{C}, 1, 1)$ (type $O(\mathbb{C})$), $\mathscr{Q}(\mathbb{C}, c, 1)$ (type $U(\mathbb{C})$), $\mathscr{Q}(\mathbb{H}, c, 1)$ (type $Sp(\mathbb{H})$) and $\mathscr{Q}(\mathbb{H}, c, -1)$ (type $O(\mathbb{H})$) where, in the last 3 cases, c is the standard conjugation map. In these cases, each form has an orthogonal base: let us classify 1-dimensional forms. Suppose b is the reflexive form; e a basis vector: then possible values of $b(e, e) = \beta$ are in the 5 cases

$$\mathbb{R}^\times, \mathbb{C}^\times, \mathbb{R}^\times, \mathbb{R}^\times, \quad \{x \in \mathbb{H}: \bar{x} = -x, x \neq 0\}.$$

Writing $e = f\lambda$ has the effect of replacing β by $\bar{\lambda}\beta\lambda$. Hence the number of equivalence classes is 1 for types $O(\mathbb{C})$, $O(\mathbb{H})$ and 2 for the other types. It follows for types $O(\mathbb{C})$ and $O(\mathbb{H})$ that two forms of the same rank are isomorphic, and $\tilde{K}_0 \mathscr{Q} = 0$.

In the remaining 3 cases—types $O(\mathbb{R})$, $U(\mathbb{C})$ and $Sp(\mathbb{H})$—any form has an orthogonal base $\{e_i\}$ with each $b(e_i, e_i) = \pm 1$. Suppose $+1$ occurs p times and -1 q times, then $\sigma = p - q$ is called the signature of the form. In the cases $O(\mathbb{R})$, $U(\mathbb{C})$, this is well-known to be independent of the choice of orthogonal basis. We can see this also for type $Sp(\mathbb{H})$ as follows: if V is a vector space over $\mathbb{H}$, b: $V \times V \to \mathbb{H}$ then we can write $b = b_0 + b_1 j$, where b_i: $V \times V \to \mathbb{C}$. Now b_0 is a hermitian form (in the usual sense) on V, considered as complex vector space via the inclusion $\mathbb{C} \subset \mathbb{H}$, and its signature is twice that of b (note $\dim_{\mathbb{C}} V = 2 \dim_{\mathbb{H}} V$). In effect, we have constructed a functor $\mathscr{Q}(\mathbb{H}, c, 1) \to \mathscr{Q}(\mathbb{C}, c, 1)$. For a hyperbolic space $H(D^n)$, $p = q = n$, so $\sigma = 0$.

Summary. $\tilde{K}_0 \mathscr{Q} = 0$, except for the cases $O(\mathbb{R})$, $U(\mathbb{C})$ and $Sp(\mathbb{H})$. In these cases, σ induces an isomorphism of $\tilde{K}_0(\mathscr{Q})$ on $2\mathbb{Z}$ (for on $\tilde{K}_0$, $p + q = 0$, so $\sigma = 2p$).

Discriminant. Let (V, b) be a quadratic module in $\mathscr{Q}(D, \alpha, U)$, where b is a reflexive form on V (the bilinearisation): choose a base $\{e_i\}$ of V over D, and form the matrix $B = \{b(e_i, e_j)\}$. This is nonsingular as b is; we define $\det b$ to be its reduced norm. If B^* denotes the conjugate (by α) transpose of B, then $B = B^* u$, so $\det b = (\det b)^\alpha (\operatorname{Nrd} u)^n$: in particular, if $u = 1$, $\det b$ is α-symmetric. If we make a change of basis of V, with matrix A, B is replaced by $A^* B A$, so $\det b$ is multiplied by $c^\alpha c$, where $c = \operatorname{Nrd} A$. In particular if A is the matrix of an automorphism, $A^* B A = B$ so $c^\alpha c = 1$: a fact already noted above, where we also discussed the converse.

The hyperbolic form of rank 2 has determinant $\mathrm{Nrd}(-u)$. For a quadratic module of rank $2n$, we define the discriminant by

$$\mathrm{dis}(q)=\mathrm{Nrd}(-u)^{-n}\det(q)$$

to be trivial on hyperbolic forms. Since $u^{\alpha}=u^{-1}$, we find that $(\mathrm{dis}\, q)^{\alpha}=\mathrm{dis}\, q$ in all cases. Like $\det q$, $\mathrm{dis}\, q$ is only defined up to multiplying by a factor $c^{\alpha} c$, $c\in \mathrm{Nrd}\, D^{\times}$.

Clifford Algebra. Consider type O, firstly over a commutative field K: thus the category $\mathscr{Q}(K,1,1)$. Given a quadratic module (V,q) there is a standard way to construct from it an algebra $C(V,q)$ called its Clifford algebra [6, § 9]. If $\dim V$ is even, this is central simple over K, so determines an element of $Br(K)$, which can be shown to be of order 2. We will follow rather [18], in which $C(V,q)$ is regarded as a graded algebra $C_0\oplus C_1$, and a further invariant (which can be recognised as the discriminant) obtained by considering the centraliser of C_0. (The Arf invariant, too, can be so defined.)

It is shown in [18] that (whether $\dim V$ is even or odd), $C(V,q)$ is graded central simple over K, and that classes of such algebras form a group $G\,Br(K)$: also this group has a series

$$G\,Br(K)\supset G\,Br^{+}(K)\supset Br(K)\supset 1$$

with successive quotients $\{\pm 1\}$, $K^{\times}/(K^{\times})^2$, $Br(K)$. Moreover, the extensions are computed by assigning to a class in $G\,Br(K)$ a triple (ε, a, D) of elements of these factor groups, and obtaining the product formulae

$$(+,a,D)\cdot(+,a',D')=(+,a\,a',DD'\langle a,a'\rangle)$$
$$(+,a,D)\cdot(-,a',D')=(-,a\,a',DD'\langle a,-a'\rangle)$$
$$(-,a,D)\cdot(-,a',D')=(+,-a\,a',DD'\langle a,a'\rangle)$$

where $\langle a,a'\rangle$ denotes a quaternion algebra. It follows in particular that if we restrict D to lie in the subgroup of $Br(K)$ generated by quaternion algebras (which is "usually" the subgroup $Br_2(K)$ of elements of order 2) we obtain a subgroup $Qr(K)$ of $G\,Br(K)$ which contains all Clifford algebras.

Now if we add quadratic modules, we multiply elements of $Qr(K)$, and a hyperbolic module determines the trivial class, so we have a homomorphism $C\colon \tilde{K}_0\mathscr{Q}(K,1,1)\to Qr(K)$ which "contains" the discriminant.

For type O over a quaternion division ring D, one would like a similar invariant, but we will argue below that there is none. The best that can be achieved by direct construction is an analogue of the even Clifford algebra C_0: see e.g. [17]. An alternative trick to find invariants is to choose an extension L of K which splits D: tensoring with L now gives a functor

$$\mathcal{Q}(D, c, -1) \to \mathcal{Q}(L \otimes D, c, -1) \cong \mathcal{Q}(L_2, c, -1) \cong \mathcal{Q}(L, 1, 1),$$

and we can construct a Clifford algebra over L. This, too, we discuss below.

Invariants and Category Equivalences. Given a category of quadratic modules over a semisimple ring R, we saw above how to reduce its study to that of a category $\mathcal{Q}(D, \alpha, u)$ with D a division ring, and that we could further suppose $u = \pm 1$, and have now seen in this case how to construct invariants. There remains the question, what happens to the invariants if we choose a different equivalence of categories? Clearly, the only situation we need consider in detail is that of equivalences of $\mathcal{Q}(D, \alpha, \varepsilon)$ with itself; and such equivalences all arise from scaling by some v, central in D, with $v^\alpha = v$. We consider our invariants in turn.

The rank is clearly unaltered. For signature, we can choose $v \in \mathbb{R}^\times$ (each time), and the signature is multiplied by the sign of v. The determinant of a form of rank m is multiplied by $(\operatorname{Nrd} v)^m$—this is true even without restricting v to be central and satisfy $v^\alpha = v$—and discriminant by $(\operatorname{Nrd} v \operatorname{Nrd} v^\alpha)^n$ if $m = 2n$: thus its equivalence class is unchanged. For Arf invariant, if $\{e_i, f_i\}$ is a symplectic base for q, then $\{e_i, f_i v^{-1}\}$ is a symplectic base for $q' = q\,v$, so

$$\begin{aligned} c(q') &= \Sigma\, q'(e_i)\, q'(f_i v^{-1}) \\ &= \Sigma\, q(e_i)\, q(f_i v^{-1})\, v^2 \\ &= \Sigma\, q(e_i)\, q(f_i) = c(q) \end{aligned}$$

and the Arf invariant is unaltered by scaling.

It remains to consider the Clifford algebra. It is fairly easy here to see that the even Clifford algebra C_0 is unaffected by scaling (an isomorphism is given by multiplying a monomial of degree $2r$ in elements of V by v^{-r}), but it turns out that this gives the only information unaltered by scaling. Indeed, in the notation above, scaling acts on Clifford algebras by

$$\begin{aligned} (-, a, D)^v &= (-, a\,v, D) \\ (+, a, D)^v &= (+, a, D\langle a, v\rangle): \end{aligned}$$

see e.g. [10]. So for $\dim V$ odd, only the class of D (=that of C_0) is invariant; for $\dim V$ even, we have the class of a (and $K[\sqrt{a}]$ is the centre of C_0) and that of D modulo the subgroup of $Br(K)$ consisting of quaternion algebras $\langle a, v\rangle$. But this is just the kernel of $Br(K) \to Br(K[\sqrt{a}])$, and $D \otimes K[\sqrt{a}]$ is Brauer equivalent to C_0. This, I think, is the essential reason why only C_0 can be defined for quadratic forms over a quaternion algebra.

We are now ready to give the classification of quadratic modules—or more precisely, the calculation of groups $\tilde{K}_0\mathscr{Q}(R,\alpha,u)$. In the case when $R=K$ is a field, these results are fairly well known: for proofs when $R=D$ is a division ring, one can argue in terms of Galois cohomology, using the basic results of Kneser. See [14] for the local case and [15] for a full exposition with references. The above calculations of groups K_1 allow us to pass between these and the desired classifications.

In listing the $\tilde{K}_0(\mathscr{Q})$ we will list which invariants suffice to define a monomorphism of the group; their values, and any relation between them. We write Sc for Arf invariant, δ for discrimininant, σ for signature, C for Clifford algebra.

Type SPOT. (finite fields only): $Sc\in\mathbb{Z}/2\mathbb{Z}$.

Type Sp. O for finite fields, local fields, and real fields except $Sp(\mathbb{H})$, where we have $\sigma\in 2\mathbb{Z}$. For global fields, over a division ring D, we have $\sigma_{\mathfrak{p}}\in 2\mathbb{Z}$ for each real place $\mathfrak{p}$ of K at which D is ramified.

Type U. Finite fields: O.
Real fields (type $U(\mathbb{C})$): $\sigma\in 2\mathbb{Z}$, $\delta=(-1)^{\sigma/2}$.
Local fields: $\delta\in k^{\times}/NK^{\times}\cong\mathbb{Z}_2$.
Global fields: $\delta\in k^{\times}/NK^{\times}$, $\sigma_{\mathfrak{p}}\in 2\mathbb{Z}$ for each real place $\mathfrak{p}$ of k with $K\otimes_k\mathbb{R}\cong C$. These are related by: $(-1)^{\sigma_{\mathfrak{p}}/2}f_{\mathfrak{p}}(\delta)>0$ for each $f_{\mathfrak{p}}\colon k\to\mathbb{R}$ as above $f_{\mathfrak{p}}(\delta)>0$ for each $f_{\mathfrak{p}}\colon k\to\mathbb{R}$ where D ramifies.

Type OK. Finite fields: $\delta\in K^{\times}/(K^{\times})^2\cong\mathbb{Z}/2\mathbb{Z}$.
Real fields: O for $O(\mathbb{C})$, $\sigma\in 2\mathbb{Z}$ for $O(\mathbb{R})$.
Local fields: $C\in Qr^{+}(K)$. We recall that this is a non-trivial extension of $Br_2(K)$ (here of order 2) by $K^{\times}/(K^{\times})^2$—here C yields δ—which has order 4 if p is odd. (Recall that ε takes the value $+1$ for even-dimensional forms.)
Global fields: $C\in Qr^{+}(K)$, $\sigma_{\mathfrak{p}}\in 2\mathbb{Z}$ for each real place $\mathfrak{p}$ of K. These are related by having the $\sigma_{\mathfrak{p}}$ determined mod $8\mathbb{Z}$ by the image of C in $Qr^{+}(K\otimes\mathbb{R})$.

Type OD. Real fields: O.
Local fields: $\delta\in K^{\times}/(K^{\times})^2$.
Global fields: No effective classification is known.

We have invariants $\delta\in K^{\times}/(K^{\times})^2$, $\sigma_{\mathfrak{p}}\in 2\mathbb{Z}$ for each real place $\mathfrak{p}$ of K where D is *not* ramified. These are related only by $(-1)^{\sigma_{\mathfrak{p}}/2}f_{\mathfrak{p}}(\delta)>0$ for such $\mathfrak{p}$. The subgroup of $\tilde{K}_0\mathscr{Q}$ corresponding to trivial invariants is isomorphic to the quotient by the class of D of the subgroup $Br_{20}(K)$ of $Br_2(K)$ corresponding to division rings unramified at all real places of K where D is unramified. Tits' [17] invariant $C_0 \in Br_2(K\sqrt{\delta})$ is of no use for detecting this. One can get closer by applying our other suggestion to all $\mathfrak{p}$-adic completions $\hat{K}_\mathfrak{p}$ which split D, but if there are s completions which do not, we still have as kernel an elementary 2-group of rank $(s-2)$. Local equivalence does not imply global here for $s \geqq 3$ [15, p. 138].

4. Categories of Based Modules

In order to have discriminants well-defined, to give a more effective approach to the difficult case $O(D)$ above, and to obviate the trouble about scaling and Clifford algebras, we now refine the concept of quadratic module to that of based quadratic module. This leads us to reconsider all the preceding. Further, this step will be decisive for obtaining explicit calculations in subsequent papers.

A *based R-module* is one with an equivalence class of free bases, two bases being equivalent if the matrix of the basis change has determinant (i.e. Nrd) 1. A based isomorphism takes a preferred basis to another. We write $\mathscr{B}(R)$ for the category of based R-modules and based isomorphisms; $\mathscr{B}\mathscr{Q}(R, \alpha, u)$ for the category of based quadratic modules and based isometries.

We will not discuss equivalences of these categories in quite the same detail as for the $\mathscr{P}$, $\mathscr{Q}$; but we still do need equivalences. For example, let R_n denote the matrix ring over R: then in the standard equivalence $\mathscr{P}(R_n) \to \mathscr{P}(R)$, R_n itself corresponds to R^n, and if we let the base I of R_n correspond to the standard base of R^n, and correspondingly for sums and isomorphic modules, we get an embedding $\mathscr{B}(R_n) \to \mathscr{B}(R)$ whose image consists of the free modules with rank divisible by n. There are of course other equivalences. If we change the base of a free R-module of rank r by a matrix with determinant λ^r (for $\lambda \in K_1\mathscr{P}(R)$), this defines an equivalence of $\mathscr{B}(R)$ with itself. Applying the same idea to R_n as n varies we find the group $K_1\mathscr{P}(R) \otimes \mathbb{Q}$ intervening (c.f. Bass [4, p. 519]). Fortunately these equivalences will not affect elements of $\tilde{K}_0\mathscr{P}(R)$.

We can regard a category $\mathscr{B}\mathscr{Q}$ as pullback of a diagram

$$\begin{array}{ccc} \mathscr{B}\mathscr{Q}(R, \alpha, u) & \longrightarrow & \mathscr{B}(R) \\ \downarrow & & \downarrow \\ \mathscr{Q}(R, \alpha, u) & \longrightarrow & \mathscr{P}(R): \end{array}$$

properties of the $\mathscr{B}\mathscr{Q}$ come from combining those of the other categories. When we have equivalences of categories $\mathscr{Q}$, since we are concerned only with matrix rings over division rings in this paper, the above discussion of bases is adequate. In particular, there exist equivalences (or at least, full embeddings) of categories $\mathscr{B}\mathscr{Q}$ corresponding to those of the $\mathscr{Q}$: the bases are subject to some variation, but these choices will not affect elements of $\tilde{K}_0(\mathscr{B}\mathscr{Q})$.

Morphisms in a category $\mathscr{B}\mathscr{Q}$ are just those in $\mathscr{Q}$ which preserve preferred bases: automorphisms are those with Nrd$=1$ with respect to a given base. Thus we can read off the groups $K_1(\mathscr{B}\mathscr{Q})$ for the cases of interest to us from the calculations before, viz:

Calculation of Groups $K_1(\mathscr{B}\mathscr{Q})$. $K_1(\mathscr{B})$ and $K_1(\mathscr{B}\mathscr{Q})$ are zero for types GL, U and Sp. For type O, the spinor norm gives an isomorphism of $K_1(\mathscr{B}\mathscr{Q})$ on the image of Nrd $D^\times$ in $K^\times/(K^\times)^2$—thus if $D=K$ or in the local case, on $K^\times/(K^\times)^2$ itself.

For a based quadratic module the discriminant, defined using a preferred basis, has a uniquely determined value. Moreover suppose given two based quadratic modules, isomorphic as unbased modules, and with the same discriminant. Then if the given isomorphism has determinant $\Delta \in \mathrm{Nrd}\, D$, we must have $\Delta^\alpha \Delta = 1$. If Δ is the determinant of an automorphism of one of the modules then composing with the inverse of this we get a based isomorphism. We can use this principle to compare the classifications: it amounts to an exact sequence

$$K_1\mathscr{Q}(D,\alpha,u) \to \{\Delta \in \mathrm{Nrd}\, D\colon\ \Delta^\alpha \Delta = 1\} \to \tilde{K}_0\mathscr{B}\mathscr{Q}(D,\alpha,u)$$
$$\to \tilde{K}_0\mathscr{Q}(D,\alpha,u) \oplus \{\delta \in \mathrm{Nrd}\, D\colon\ \delta^\alpha = \delta\}.$$

Here, the first map is the determinant. The second is defined by changing the base by an automorphism with determinant Δ. Exactness at $\tilde{K}_0\mathscr{B}\mathscr{Q}$ was proved above, and at the previous point follows (by stabilising) from the remark that if a base change with determinant Δ leaves unaltered the based isomorphism class of φ, then φ has an automorphism with determinant Δ.

Now by results above, this first map is surjective for types GL and U in the cases under consideration: thus no new invariant is needed in these cases. The same holds for type O when D is commutative; but if D is a quaternion ring here, or for type Sp, the group of Δ's is $\{\pm 1\} \cap \mathrm{Nrd}\, D$ and the image of $K_1\mathscr{Q}$ is trivial; thus we are liable to need a further invariant for classification.

We will now give the classification of based quadratic modules. This, too, is derived from Galois cohomology—roughly speaking, in place of $H^1(U)$, $H^1(Sp)$ and $H^1(O)$ which we had to consider before, we now have $H^1(SU)$, $H^1(Sp)$ and $H^1(SO)$—and for based modules, $H^1(SL)$ in place of $H^1(GL)$. One must, however, make two reservations. First, if we interpret $H^1(SL)$ carefully, we find that it corresponds not to based modules but to a slightly more general notion which we can call pseudo-based modules. One can define a pseudo-base of a D-vector space V as a function which assigns to each D-base $\{e_i\}$ of V an element $\phi\{e_i\}\in K^\times$ such that if $e_i=\Sigma f_j a_{ij}$ then $\phi\{e_i\}=\phi\{f_j\}\,\Delta$, where Δ is the reduced norm of the matrix (a_{ij}). If $\phi\{e_i\}\in \operatorname{Nrd} D^\times$ for some base, then it is for all, and those with $\phi\{e_i\}=1$ give V the structure of based module. Thus we must select based modules out of the pseudo-based ones given by our Galois cohomology. The second point to watch is that for the theory to apply, we need to know that forms (of given type and rank) all become equivalent over the algebraic (or separable) closure. It follows at once from the preceding paragraph that, provided we consider forms of a fixed discriminant, this holds *except* for type Sp.

Thus we need a new invariant here. We define it as follows for type Sp or $SPOT$ over a field $D=K$: let π be the determinant of a change of basis from the given base to a symplectic base. Since a symplectic automorphism has determinant 1, this is uniquely defined. It coincides with the Pfaffian of the skew-symmetric matrix representing our (reflexive) form with respect to the given basis. Since, for a symplectic base, the discriminant $\delta=1$ we have in general $\delta=\pi^2$. Of course π depends entirely on the choice of base. The same is valid if $D\neq K$ but K is a p-adic local field, for here again all unbased forms are hyperbolic.

We now list the groups $\tilde{K}_0(\mathscr{B}\mathscr{Q})$ for the various cases. In each, we specify the possible values of the relevant invariants δ (discriminant), σ (signature), C (Clifford algebra), Sc (Arf invariant), π (Pfaffian); and any relations between them.

Type SPOT, finite fields. $Sc\in\mathbb{Z}/2\mathbb{Z}$, $\pi\in K^\times$.

Type Sp. Local fields, and all cases when $D=K$: $\pi\in K^\times$ $(\delta=\pi^2)$.
Real fields, case $D=\mathbb{H}$: $\delta\in\mathbb{R}^*$, $\sigma\in 2\mathbb{Z}$.
Global fields. If there are no real places of K where D is ramified, all unbased forms are hyperbolic, and we have $\pi\in K^\times$ as before. If there are some, $-1\notin \operatorname{Nrd} D^\times$, so the invariants $\sigma_\mathfrak{p}\in 2\mathbb{Z}$ (D ramifies at $\mathfrak{p}$), $\delta\in (K^\times)^2$ suffice.

Type U. We have $\delta\in k^\times$ in all cases; $\sigma\in 2\mathbb{Z}$ for type $U(\mathbb{C})$, and in the global case $\sigma_\mathfrak{p}\in 2\mathbb{Z}$ for real places $\mathfrak{p}$ of k becoming complex for K. Relations as for $\tilde{K}_0(\mathscr{Q})$.

Type OK. Finite fields: $\delta\in K^\times$.
Real fields: $\delta\in K^\times$ and, for $O(\mathbb{R})$, $\sigma\in 2\mathbb{Z}$ with $(-1)^{\sigma/2}\delta>0$.
Local fields: $C\in Qr^+(K)$, $\delta\in K^\times$. These determine the same class in $K^\times/(K^\times)^2$.
Global fields: $C\in Qr^+(K)$, $\delta\in K^\times$ and, for each real place $\mathfrak{p}$ of K, $\sigma_\mathfrak{p}\in 2\mathbb{Z}$. C and δ are related as above; the image of C in $Qr^+(K\otimes_\mathfrak{p}\mathbb{R})$ is given by $\sigma_\mathfrak{p}\,(\mathrm{mod}\ 8\mathbb{Z})$.

Type OD. Real fields (type $O(\mathbb{H})$): $\delta \in \mathbb{R}^*$.
Global fields: As for $\tilde{K}_0(\mathscr{Q})$, the kernel of the map of $\tilde{K}(\mathscr{B}\mathscr{Q})$ given by δ and the $\sigma_\mathfrak{p}$ can be identified with $Br_{20}(K)/\{D\}$. But now we have the Hasse principle: if $\mathfrak{p}$ runs through all places of K (including infinite ones) and we write $\hat{K}_\mathfrak{p}$ for the $\mathfrak{p}$-adic completion, $\hat{D}_\mathfrak{p} = D \otimes_K \hat{K}_\mathfrak{p}$, the natural map

$$\tilde{K}_0 \mathscr{B}\mathscr{Q}(D, c, -1) \to \Pi_\mathfrak{p} \tilde{K}_0 \mathscr{B}\mathscr{Q}(\hat{D}_\mathfrak{p}, c, -1)$$

in injective. Discussion of the cokernel is best left till the result is reformulated in adèle language. We thus see that for effective invariants, it suffices to discuss in detail the one case remaining:
Local fields: As for type OK, we have an exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{K}_0 \mathscr{B}\mathscr{Q} \xrightarrow{\delta} K^\times \to \{1\}.$$

Now $K^\times$ is isomorphic to the sum of the finite cyclic group $\mu(K)$ of units in K, $\mathbb{Z}$, and a free module over the (usual) p-adic integers, hence

$$\operatorname{Ext}(K^\times, \mathbb{Z}/2\mathbb{Z}) \cong \operatorname{Ext}(\mu(K), \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$

and to decide whether the extension is trivial or not it suffices to choose $u \in \tilde{K}_0(\mathscr{B}\mathscr{Q})$ with discriminant -1 and decide whether or not $2u = 0$. In fact $2u \neq 0$, so the extension does not split. Moreover, suppose one can define a Clifford algebra invariant

$$C\colon\ \tilde{K}_0(\mathscr{B}\mathscr{Q}) \to Br(K) \cong Q/\mathbb{Z}$$

with

$$C(x+y) = C(x) \cdot C(y) \cdot \langle \operatorname{dis} x, \operatorname{dis} y \rangle.$$

Then for $p \neq 2$ if $\operatorname{dis} x, \operatorname{dis} y \in \mu(K)$ we have $\langle \operatorname{dis} x, \operatorname{dis} y \rangle = 0$, so C is a homomorphism; thus if C is nontrivial on $\mathbb{Z}/2\mathbb{Z}$, and 2^r divides $|\mu(K)|$, the image of C must have order at least 2^{r+1}: it seems clear that no such definition can be natural.

On the other hand, if we restrict to forms which are hyperbolic as unbased forms we can, as in type Sp, define an invariant $\pi \in K^\times$ with $\delta = \pi^2$. Indeed, if -1 is a square in K, this is enough to prove non-splitting of the sequence. We can generalise this as follows to obtain an explicit invariant for all forms. First, we must make an arbitrary choice of one, say θ, of the two forms of rank 3 with discriminant 1: then $2\theta = 3H$ where H is hyperbolic with discriminant 1; working modulo the subgroup of $K_0(\mathscr{B}\mathscr{Q})$ generated by H and θ, every form not of the above type is equivalent to a unique 1-dimensional form. But for such a form q, the value $q(e)$ for e a preferred basis element has $\operatorname{Nrd} q(e) = \delta(q)$, so $q(e)^2 = -\delta(q)$. Changing to another preferred base replaces $q(e)$ by $u^{-1} q(e) u$ with $\operatorname{Nrd} u = 1$. But there are exactly 2 classes of solutions of

$x^2 = -\delta$ under this relation: fixing one of the two has given our desired invariant.

The above is fairly explicit, and can be made more so. If, for example, p is odd and -1 is square in K, each equivalence class above contains just one representative of the form λ, $\lambda \underline{i}$, $\lambda \underline{j}$ or $\lambda \underline{k}$ ($\lambda \in K^\times$). If $\underline{i}^2 = u$ and $\underline{j}^2 = \pi$, we can give explicit addition formulae: for example,

$$\{\lambda \underline{i}\} + \{\mu \underline{i}\} = \{(u, \lambda\mu)\, \lambda \mu u\}; \qquad \{\lambda \underline{i}\} + \{\mu \underline{j}\} = \{-(u, \lambda)\, (\pi, \mu)\, \lambda \mu \underline{k}\};$$

where (a, b) denotes the Hilbert symbol, ± 1 according as the quaternion algebra $\langle a, b \rangle$ over K is or is not split.

The preceding discussion is to a large extent dominated by the discriminant. We define $\mathscr{L}(R, \alpha, u)$ to be the full subcategory of $\mathscr{B}\mathscr{Q}(R, \alpha, u)$ of based forms with discriminant 1. This is cofinal in $\mathscr{B}\mathscr{Q}$, so has the same groups K_1. The groups $\tilde{K}_0$ here are given by:

Type $SPOT$: $Sc \in \mathbb{Z}/2\mathbb{Z}$.

Type Sp (commutative D, local fields, or global fields with D unramified at infinity): $\pi = \pm 1$.
Real field, $D = \mathbb{H}$: $\sigma \in 2\mathbb{Z}$.
Other global fields: $\sigma_{\mathfrak{p}} \in 2\mathbb{Z}$ for each real place $\mathfrak{p}$ of K where D ramifies. The $\sigma_{\mathfrak{p}}$ are congruent to each other mod $4\mathbb{Z}$.

Type U (finite and local fields): O.
$U(\mathbb{C})$: $\sigma \in 4\mathbb{Z}$.
Global fields: $\sigma_{\mathfrak{p}} \in 4\mathbb{Z}$ for each real place $\mathfrak{p}$ of k which becomes complex for K.

Type OK. Finite fields: O.
Real fields: O if $K = \mathbb{C}$. $\sigma \in 4\mathbb{Z}$ if $K = \mathbb{R}$.
Local fields: $C = 0$ or $\frac{1}{2} \in Br_2(K)$.
Global fields: $C \in Br_2(K)$, $\sigma_{\mathfrak{p}} \in 4\mathbb{Z}$ for each real place of K; the image of C in $Br_2(K \otimes \mathbb{R})$ is $\sigma_{\mathfrak{p}}/8$ (mod 1) for such a place. In view of the calculation of $Br_2(K)$, we can reformulate this by noting that there is an exact sequence

$$0 \to \tilde{K}_0 \mathscr{L}(K, 1) \to \oplus_{\mathfrak{p}} \tilde{K}_0 \mathscr{L}(\hat{K}_{\mathfrak{p}}, 1, 1) \to \mathbb{Z}_2 \to 0.$$

Type OD
Real fields ($O\mathbb{H}$): O.
Local fields: $\pi = \pm 1$.
Global fields: Again there is an exact sequence as above.

In conclusion, for each type the natural map from a global group $\tilde{K}_0(\mathscr{L})$ to the direct product of the localisations is injective, and the cokernel has exponent 2. In subsequent papers, this will be the key corollary from this whole paper, though it will be modified by considering the group $\tilde{K}_0 \mathscr{L}$ defined over the adèle ring of K.

References

1. Albert, A. A.: Structure of algebras. Amer. Math. Soc. 1939.
2. Arf, C.: Untersuchungen über quadratische Formen in Körpern der Charakteristik 2. J. reine angew. Math. **183**, 148–167 (1941).
3. Bak, A.: The stable structure of quadratic modules. Preprint, Princeton, 1970.
4. Bass, H.: Algebraic *K*-theory. New York: W.A. Benjamin Inc. 1968.
5. Bourbaki, N.: Algèbre, Ch. 8: Modules et anneaux semi-simples. Paris: Hermann 1958.
6. Bourbaki, N.: Algèbre, Ch. 9: Formes sesquilinéaires et formes quadratiques. Paris: Hermann 1959.
7. Cassels, J. W. S., Fröhlich, A. (eds.): Algebraic number theory. New York: Academic Press 1967.
8. Dieudonné, J.: Les déterminants sur un corps non commutatif. Bull. Soc. Math. France **71**, 27–45 (1943).
9. Dieudonné, J.: La géometrie des groupes classiques (2nd edition). Berlin-Göttingen-Heidelberg: Springer 1963.
10. Fröhlich, A.: Orthogonal and symplectic representations of groups. Proc. London Math. Soc. **24**, 470–506 (1972).
11. Fröhlich, A., McEvett, A. M.: Forms over rings with involution. J. Algebra **12**, 79–104 (1969).
12. Fröhlich, A., Wall, C. T. C.: Foundations of equivariant algebraic *K*-theory. Pp. 12–27. In: Algebraic *K*-theory and its geometric applications. Lecture Notes in Mathematics **108**. Berlin-Heidelberg-New York: Springer 1969.
13. Fröhlich, A., Wall, C. T. C.: Generalisations of the Brauer group I. Preprint, University of Liverpool, 1971.
14. Kneser, M.: Galois-Kohomologie halbeinfacher algebraischer Gruppen über *p*-adischen Körpern. Math. Z. **88**, 40–47 (1965).
15. Kneser, M.: Galois cohomology of classical groups. Tata Institute, Bombay 1972.
16. Serre, J.-P.: Corps Locaux. Paris: Hermann 1962.
17. Tits, J.: Formes quadratiques, groupes orthogonaux et algèbres de Clifford. Inventiones math. **5**, 19–41 (1968).
18. Wall, C. T. C.: Graded Brauer groups. J. reine angew. Math. **213**, 187–199 (1963).
19. Wall, C. T. C.: Graded algebras, anti-involutions, simple groups and symmetric spaces. Bull. Amer. Math. Soc. **74**, 198–202 (1968).
20. Wall, C. T. C.: On the axiomatic foundations of the theory of Hermitian forms. Proc. Camb. Phil. Soc. **67**, 243–250 (1970).
21. Wall, C. T. C.: On the classification of Hermitian forms I. Rings of algebraic integers. Comp. Math. **22**, 425–451 (1970).
22. Wall, G. E.: The structure of a unitary factor group. Publ. math. I.H.E.S. **1** (1959).
23. Wang, S.: On the commutator group of a simple algebra. Amer. J. Math. **72**, 323–334 (1950).
24. Weil, A.: Basic number-theory. Berlin-Heidelberg-New York: Springer 1967.
25. Wall, C.T.C.: On the commutator subgroups of certain unitary groups. (To appear.)

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