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On the Classification of Hermitian Forms

III. Complete Semilocal Rings

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Like [7], this paper was originally written as a tool for [8]. After a year's reflection, and some useful comments from Martin Kneser, I succeeded in improving the results. Also, I noticed that a more systematic use of ideas from algebraic K-theory led to a great gain in conceptual simplicity, by treating the groups L_0, L_1 on the same footing. Indeed, it was in course of rewriting this paper for the sixth time that I was led to sort out the foundations in [5], a paper which I now use as basic reference.

This paper is organised as follows. In § 1 we recall basic lifting theorems for projective and quadratic modules over a ring R complete in the I-adic topology for some ideal I. Then § 2 gives the reformulation of these results in the terminology of algebraic L-theory. In § 3 we study the special case of orders in semisimple p-adic algebras; we first prove two basic results about $K_1(R)$, which at once yield our main theorems for p odd. The case $p=2$ is illustrated by giving the calculations in the unramified case. In § 4 we mention related questions: we outline proofs of some simple stability results, and discuss further calculation of $K_1(R)$.

Some results similar to ours have also been obtained by Bak and Scharlau.

§ 1. Lifting Theorems

We suppose throughout that R is a ring (associative, with unit), and I a 2-sided ideal in R such that R is conplete in the I-adic topology. Write $\bar{R}$ for the quotient ring R/I; similarly for an R-module P, write $\bar{P}=P/PI=P\otimes_R \bar{R}$, etc. In this section we obtain the basic relations between the algebraic K- and L-theory of R and $\bar{R}$. We first discuss objects (where the results are well known), then morphisms.

Lemma 1. *The quotient functor $\mathscr{P}(R)\to\mathscr{P}(\bar{R})$ is full, and induces an isomorphism $k\mathscr{P}(R)\to k\mathscr{P}(\bar{R})$. A morphism f in $\mathscr{P}(R)$ is surjective resp. bijective if and only if $\bar{f}$ is.*

For the proof, see Swan [4, Theorem 2.26] or Bass [2, II.12].

Corollary. *We have an isomorphism $K_0\mathscr{P}(R)\to K_0\mathscr{P}(\bar{R})$ and an epimorphism $K_1\mathscr{P}(R)\to K_1\mathscr{P}(\bar{R})$.*

Now let (R, α, u) be an antistructure, with $\alpha(I)=I$. There is then an induced antistructure on $\bar{R}$. We suppress mention of these antistructures, which will be the same throughout § 1.

Lemma 2. *For M a finitely generated projective R-module, the natural map* $Q(M)\to Q(\bar{M})$, $x\to\bar{x}$ *say, is surjective; and x is nonsingular if and only if* $\bar{x}$ *is. If* $x_i\in Q(M_i)$ $(i=1,2)$ *are nonsingular, any isometry* $\bar{\lambda}: \bar{M}_1\to\bar{M}_2$ *of* $\bar{x}_1$ *on* $\bar{x}_2$ *lifts to an isometry* λ *of* x_1 *on* x_2.

The first assertion is contained in [6, Lemma 1]; the rest is a restatement of [6, Theorem 2].

Corollary 1. *We have isomorphisms* $k\mathscr{Q}(R)\to k\mathscr{Q}(\bar{R})$, $K_0\mathscr{Q}(R)\to K_0\mathscr{Q}(\bar{R})$, *and an epimorphism* $K_1\mathscr{Q}(R)\to K_1\mathscr{Q}(\bar{R})$.

Corollary 2. *If* $\bar{x}$ *has an orthogonal base, or a hyperbolic summand, so has x. If cancellation holds in* $\mathscr{Q}(\bar{R})$ *– e.g. if* $\bar{R}$ *is semisimple – it holds in* $\mathscr{Q}(R)$.

For the first assertion, note that a set of elements of M, projecting to a base of $\bar{M}$, is a base by Lemma 1. The rest is clear. Since we have the ability to lift bases, we obtain

Corollary 3. *The map* $K_0\mathscr{B}\mathscr{Q}(R)\to K_0\mathscr{B}\mathscr{Q}(\bar{R})$ *is surjective.*

We now turn to automorphisms: again we begin with the category $\mathscr{P}(R)$. Since R is complete in the I-adic topology, an element $r\in R$ is a unit if and only if it is a unit modulo I; hence the additive coset $(1+I)$ is a multiplicative group.

Proposition 3. *There is an exact sequence*

$$(1+I)^{\times}\to K_1\mathscr{P}(R)\to K_1\mathscr{P}(\bar{R})\to 0.$$

Proof. The first map is the composite

$$(1+I)^{\times}\subset R^{\times}=GL_1(R)\subset GL(R)\to K_1\mathscr{P}(R):$$

it is clear that the image maps to $0\in K_1\mathscr{P}(\bar{R})$. Conversely, let $A\in GL(R)$ determine 0 in $K_1\mathscr{P}(\bar{R})$. Then the image $\bar{A}\in GL(\bar{R})$ belongs to the commutator subgroup, and can be expressed as a product of elementary matrices:

$$\bar{A}=\bar{e}_1\,\bar{e}_2\ldots\bar{e}_n.$$

As $R\to\bar{R}$ is surjective, there exist elementary matrices e_i over R covering the $\bar{e}_i$. Now A represents the same element of $K_1\mathscr{P}(R)$ as does

$$B=e_n^{-1}\ldots e_1^{-1}A, \quad \text{and} \quad \bar{B}=1.$$

Since B is congruent (mod I) to the identity matrix, and elements of $1+I$ are invertible, we can perform elementary column operations on B

to reduce it to a diagonal matrix (i. e. multiply B on the right by elementary matrices $X_{rs}(\lambda)$, $\lambda \in I$), with elements of $1+I$ on the diagonal. The class in $K_1 \mathscr{P}(R)$ is thus the product of the classes of these elements.

Corollary. *If* $(\bar{R})^\times \to K_1 \mathscr{P}(\bar{R})$ *is surjective, so is* $R^\times \to K_1 \mathscr{P}(R)$.

This is, in particular, the case when $\bar{R}$ is semisimple.

We will customarily write V for the image of $(1+I)^\times$, i.e. (by the above),

$$V = \operatorname{Ker}\{K_1 \mathscr{P}(R) \to K_1 \mathscr{P}(\bar{R})\}.$$

There is little one can say in general about the structure of V.

We come now to the quadratic case, which is the one of real interest to us; we follow the pattern of the above proof.

Proposition 4. *There is an exact sequence*

$$V \xrightarrow{H} K_1 \mathscr{Q}(R) \to K_1 \mathscr{Q}(\bar{R}) \to 0.$$

Here, H denotes the hyperbolic functor.

Proof. Following the same argument as in the preceding proof, we see that it suffices to consider an automorphism B of a hyperbolic space, with $\bar{B}=1$. The rest of the proof now follows that of [5, Theorem 4]: we use the same notation.

B is an automorphism of $H(R) \oplus (N, \theta)$, say; let

$$eB = ea + fb + x, \qquad x \in N.$$

Then $a \in 1+I$, so is a unit. Then

$$B' = BH(a^{-1})\, \varepsilon^1(-x, b\, a^\alpha)$$

leaves e fixed; composing with a suitable ε^2, we obtain a B'' which also leaves f fixed, so can be regarded as an automorphism of (N, θ). It follows by induction that B is a product of elementary transformations with certain $H(a)$, $a \in 1+I$; and this implies the result.

§ 2. *L*-Theory of Complete Rings

We retain the notation and assumptions of the preceding paragraph and proceed to reformulate the main results in the terminology of [5].

Lemma 5. *The natural map* $L_i^K(R) \to L_i^K(\bar{R})$ *is an isomorphism for all i.*

Proof. We have $L_0^K(R) = \ker F\colon K_0 \mathscr{Q}(R) \to K_0 \mathscr{P}(R)$; but by the corollaries to Lemmas 1 and 2, $K_0 \mathscr{P}(R) \cong K_0 \mathscr{P}(\bar{R})$ and $K_0 \mathscr{Q}(R) \cong K_0 \mathscr{Q}(\bar{R})$. Hence $L_0^K(R) \cong L_0^K(\bar{R})$.

Now $L_1^K(R) = \operatorname{Coker} H\colon K_1\mathscr{P}(R) \to K_1\mathscr{Q}(R)$. It thus follows from the exact sequences (Proposition 4)

$$0 \to V \to K_1\mathscr{P}(R) \to K_1\mathscr{P}(\bar{R}) \to 0$$
$$V \xrightarrow{H} K_1\mathscr{Q}(R) \to K_1\mathscr{Q}(\bar{R}) \to 0$$

that $L_1^K(R) \to L_1^K(\bar{R})$ is an isomorphism. The result for L_i for other values of i now follows from the definition.

From now on, we abbreviate $K_i\mathscr{P}(R)$ to $K_i(R)$.

Theorem 6. *Let $\bar{X}$ be an α-invariant subgroup of $K_1(\bar{R})$; let X be its preimage in $K_1(R)$. Then we have isomorphisms*

$$L_i^X(R) \to L_i^{\bar{X}}(\bar{R}).$$

Proof. Since the map $K_1(R)/X \to K_1(\bar{R})/\bar{X}$ is an isomorphism, the result follows by applying the Five Lemma to the map of exact sequences [5, Theorem 3]

$$\begin{array}{ccccccccc} \cdots \longrightarrow & L_i^X(R) & \longrightarrow & L_i^K(R) & \longrightarrow & H^i(K_1(R)/X) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & L_i^{\bar{X}}(\bar{R}) & \longrightarrow & L_i^K(\bar{R}) & \longrightarrow & H^i(K_1(\bar{R})/\bar{X}) & \longrightarrow \cdots \end{array}$$

and using Lemma 5.

Corollary. *There is an exact sequence*

$$\cdots L_i^S(R) \to L_i^S(\bar{R}) \to H^i(V) \to L_{i-1}^S(R) \to \cdots.$$

For, by the theorem, $L_i^S(\bar{R}) = L_i^V(R)$; we now refer to [5, Theorem 3].

It is also of interest to compute the "large" groups $\Lambda_i(R)$:

$$\Lambda_0(R) = \tilde{K}_0\mathscr{B}\mathscr{Q}(R) \quad \text{and} \quad \Lambda_1(R) = K_1\mathscr{Q}(R).$$

Here, there are two natural invariants: one in $\Lambda_i(\bar{R})$, and the discriminant (i even) or determinant (i odd), which takes values in $K_1(R)$, and satisfies a symmetry condition. We recall from [5] the exact sequences

$$\Lambda_{i+1}(R) \xrightarrow{\delta_{i+1}} K_1(R) \xrightarrow{\tau_i} \Lambda_i(R) \to L_i^K(R) \to 0$$

and that

$$\delta_i \circ \tau_i = 1 + (-1)^i T.$$

Lemma 7. *The following sequence is exact:*

$$V \xrightarrow{\tau_i'} \Lambda_i(R) \to \Lambda_i(\bar{R}) \to 0.$$

Proof. By Corollaries 1 and 3 to Lemma 2, $\Lambda_i(R)$ maps onto $\Lambda_i(\bar{R})$. Exactness for i odd is given by Proposition 4, and for i even by observing that given based quadratic modules over R, and a based isomorphism over $\bar{R}$, we can lift this and obtain an isomorphism over R with determinant in V. We can also deduce the result from Lemma 5 by diagram chasing.

For X any module over a group of order 2, acting by T, write

$$S_i X = \{x \in X : Tx = (-1)^i x\}$$

for the subgroup of "i-symmetric" elements of T. Then

$$\operatorname{Im} \delta_i = \operatorname{Ker} \tau_{i-1} \subset \operatorname{Ker}(\delta_{i-1} \tau_{i-1}) = \operatorname{Ker}(1 - (-1)^i T) = S_i K_1(R).$$

We thus have a commutative diagram

$$\begin{array}{ccc} \Lambda_i(R) & \longrightarrow & S_i K_1(R) \\ \downarrow & & \downarrow \\ \Lambda_i(\bar{R}) & \longrightarrow & S_i K_1(\bar{R}), \end{array}$$

which we regard as defining a map from $\Lambda_i(R)$ to the pullback P_i of

$$\Lambda_i(\bar{R}) \to S_i K_1(\bar{R}) \leftarrow S_i K_1(R).$$

Proposition 8. *There is an exact sequence*

$$L_{i+1}^S(\bar{R}) \to H^{i+1}(V) \to \Lambda_i(R) \to P_i \to H^i(V) \to 0.$$

Proof. The first map is as in Theorem 6, corollary; the second is the composite $H^{i+1}(V) \to L_i^S(R) \subset \Lambda_i(R)$; the third is as above. Exactness at $H^{i+1}(V)$ is clear.

By definition, $\operatorname{Ker}(\Lambda_i(R) \to S_i K_1(R)) = L_i^S(R)$. Hence

$$\operatorname{Ker}(\Lambda_i(R) \to P_i) = \operatorname{Ker}(L_i^S(R) \to L_i^S(\bar{R})),$$

which equals the image of $H^{i+1}(V)$ by Theorem 6, corollary.

Given $(\bar{a}, b) \in P_i \subset \Lambda_i(\bar{R}) \oplus S_i K_1(R)$, lift $\bar{a}$ to $a \in \Lambda_i(R)$. Then $\delta_i(a)$ and b have the same image in $K_1(\bar{R})$, so $c = \delta_i(a) - b$ is an (i-symmetric) element of V. Any lift of $\bar{a}$ has a form $a + \tau_i(v)$, $v \in V$ (by Lemma 7); this maps onto b if and only if $\delta_i \tau_i(v) = -c$. Thus the class of c in $H^i(V)$ is determined, and vanishes if and only if $(\bar{a}, b) \in \operatorname{Im} \Lambda_i(R)$. Finally, $0 \oplus S_i V \subset P_i$ maps onto $H^i(V)$.

Special Cases

(i) Suppose $H^i(V) = 0$ for all i. Then the corollary to Theorem 6 gives isomorphisms $L_i^S(R) \cong L_i^S(\bar{R})$. Proposition 8 gives an isomorphism

$$\Lambda_i(R) \cong P_i.$$

Thus we have a pullback diagram of epimorphisms

$$\begin{array}{ccc} \Lambda_i(R) & \longrightarrow & S_i K_1(R) \\ \downarrow & & \downarrow \\ \Lambda_i(\bar{R}) & \longrightarrow & S_i K_1(\bar{R}). \end{array}$$

(ii) Suppose $L_i^S(\bar{R})=0$ for all i. Then our exact sequences yield isomorphisms

$$L_i^K(R)\cong L_i^K(\bar{R})\cong H^i(K_1(\bar{R})) \quad \text{and} \quad L_i^S(R)\cong H^{i+1}(V).$$

Also, the exact sequence

$$0\to \Lambda_i(\bar{R}) \xrightarrow{\delta_i} K_1(\bar{R}) \xrightarrow{\tau_{i-1}} \Lambda_{i-1}(\bar{R})$$

shows that

$$\Lambda_i(\bar{R})\cong \operatorname{Ker}\tau_{i-1}=\operatorname{Ker}\delta_{i-1}\tau_{i-1}=\operatorname{Ker}(1+(-1)^{i-1}T)=S_i K_1(\bar{R}).$$

Thus the pullback $P_i\cong S_i K_1(R)$. Finally, the map $H^{i+1}(V)\to\Lambda_i(R)$ of Prop. 8 is induced by τ_i': $V\to\Lambda_i(R)$; since it is injective, it follows that $\operatorname{Ker}\tau_i'=\{x-(-1)^i Tx\colon x\in V\}$.

§ 3. *p*-Adic Rings

We now suppose that S is a finite semisimple algebra over the field $\hat{\mathbb{Q}}_p$ of p-adic rationals, and R an order in it – i.e. that R is free (and finitely generated) as module over the ring $\hat{\mathbb{Z}}_p$ of p-adic integers, and that

$$R\otimes\hat{\mathbb{Q}}_p=S.$$

We give these rings, and related modules, matrix rings etc., the p-adic topology. Then R is a compact open subring of S. We first study $K_1(R)$.

Theorem 9. *The kernel of $K_1(R)\to K_1(S)$ is finite.*

Proof. By a theorem of Borel [3, p. 523] if G is an analytic group over $\hat{\mathbb{Q}}_p$ with simple Lie algebra, any normal subgroup of an open subgroup is open (or discrete): in particular, the commutator subgroup is open. The corresponding result follows for the semisimple case.

Now for $n\geqq 2$, $SL_n(S)$ is a semisimple analytic group, and

$$W_n=GL_n(R)\cap SL_n(S)$$

an open subgroup of it. Hence the commutator subgroup of W_n is open. Since also W_n is compact (since closed and bounded in the matrix ring), its commutator quotient group is finite.

Now $SL_n(S)=\operatorname{Ker}(GL_n(S)\to K_1(S))$ by the definition of $SL_n(S)$ and the computation of $K_1(S)$ (see e.g. [7]), so

$$W_n=\operatorname{Ker}(GL_n(R)\to K_1(S)).$$

By Corollary 1 to Lemma 3, $GL_n(R)$ maps onto $K_1(R)$ for $n\geqq 1$, so the theorem follows from the preceding paragraph.

We write $J=J(R)$ for the (Jacobson) radical of R, $\bar{R}=R/J$. Clearly $p\in J$, so $pR\subset J$. Conversely, since R/pR is finite, its radical J/pR is nilpotent. Thus for some positive integer N, $(J/pR)^N=0$, i.e. $J^N\subset pR$. Hence the J-adic and p-adic topologies on R coincide; in particular, R is complete in the J-adic topology. Thus the results of the preceding sections (with J in place of I) are applicable.

Theorem 10. *The kernel V of $K_1(R)\to K_1(\bar{R})$ is a pro-p-group.*

Proof. By Proposition 3, V is a quotient of $(1+J)^\times$. Now $(1+J)^\times$ is a pro-p-group, for the topology is defined by the subgroups $(1+J^r)^\times$, and

$$(1+J^r)^\times/(1+J^{r+1})^\times\cong J^r/J^{r+1},$$

an additive group of exponent p. But the proof of the preceding theorem shows (for any n) that

$$\operatorname{Ker}(GL_n(R)\to K_1(R))=\operatorname{Ker}(W_n\to K_1(R))$$

is open (hence closed) in W_n, hence closed in $GL_n(R)$; thus the quotient inherits the topology. The theorem follows.

Corollary 1. $K_1(R)=V\oplus K_1(\bar{R})$.

For $\bar{R}$ is a semisimple finite ring of characteristic p, hence isomorphic to a direct sum of matrix rings over fields $F(p^i)$. Thus $K_1(\bar{R})$ is a sum of cyclic groups of orders (p^i-1), hence is a finite group of order prime to p. Any extension of the pro-p-group V by $K_1(\bar{R})$ thus splits, uniquely.

It follows that $S_i K_1(R)=S_i V\oplus S_i K_1(\bar{R})$, so the pullback P_i of §2 reduces to $\Lambda_i(\bar{R})\oplus S_i(V)$.

Corollary 2. *If p is odd, $H^i(V)=0$ for all i. Hence*

$$L_i^S(R)\cong L_i^S(\bar{R}),$$
$$\Lambda_i(R)\cong\Lambda_i(\bar{R})\oplus S_i(V).$$

When $p=2$, things are less simple. The main step needed to obtain calculations over R is to compute the maps

$$L_i^S(\bar{R})\to H^i(V).$$

The groups $L_i^S(\bar{R})$ are known [7]: a summand of $\bar{R}$ of type GL or U contributes zero; a summand of type $SPOT$ contributes $\mathbb{Z}/2\mathbb{Z}$ to each group. We can even drop the index S, for as $K_1(\bar{R})$ has odd order, all $L_i^X(\bar{R})$ coincide. One can regard some of the difficulties here as due to the fact that to compute $K_1\mathscr{Q}$ one needs spinor norms as well as determinants.

To illustrate that this really is a problem, we now give the calculation for the case when $p=2$ and the order R is unramified – i.e. S is a direct sum of matrix rings over fields (not division rings) and R is a maximal order. Then the direct sum splitting of S induces one of R. If α interchanges two components here, they contribute nothing to the L-theory (cf. [5]). So it suffices to consider one component – i.e. suppose S simple.

Let $S=M_n(K)$ be the matrix ring over a field K; A be the ring of integers in K. Then R is (conjugate to) $M_n(A)$. As in [7], we may now reduce by Morita theory to the case $R=A$: now the anti-involution α may or may not be trivial (on A, as on K); if α is trivial, $u=\pm 1$.

The case when α is not trivial was discussed fully in [6]: we have the "wildly ramified" case of that paper. Each group $L_i(\bar{A})$, $H^i(V)$ has order 2. Two cases arise, depending on the class of u in $H_1(\bar{A}^\times)$. [6, Theorem 3] shows that the map $L_0(\bar{A})\to H^0(V)$ is an isomorphism in the good case; zero in the bad, and [6, Theorem 4] that the same holds for $L_1(\bar{A})\to H^1(V)$. For $i=2, 3$ we must replace u by $-u$: whether $u\sim -u$ or not depends on (K,α). Both cases can arise.

If α is trivial, $u=\pm 1$, and $\bar{A}$ certainly has type $SPOT$: K has type 0 or Sp according as $u=1$ or $u=-1$. For this case, $H^1(V)=H^1(A^\times)=\{\pm 1\}$, while $H^0(V)=H^0(A^\times)\cong(A^\times)/(A^\times)^2$. The latter can be regarded as a subgroup (of index 2) of $K^\times/(K^\times)^2$.

Theorem 11. *Let A be the ring of integers in a 2-adic field K; α the identity. Then*

$$\begin{aligned} L_0(\bar{A}) &\to H^0(A^\times) && \textit{is injective}\\ L_1(\bar{A}) &\to H^1(A^\times) && \textit{is an isomorphism}\\ L_2(\bar{A}) &\to H^2(A^\times) && \textit{is zero}\\ L_3(\bar{A}) &\to H^3(A^\times) && \textit{is zero.}\end{aligned}$$

Proof. The nonzero element of $L_i(\bar{A})$ (i even) is represented by a plane with nonzero Arf invariant: we can regard this as the equivalence class of the form with matrix

$$\begin{pmatrix}1 & 1\\ 0 & \bar{b}\end{pmatrix},$$

where $\bar{b}\in\bar{A}$ is not of the form x^2+x. If $b\in B$ maps to $\bar{b}$, we lift this in the obvious way to a form on A. Its bilinearisation has matrix

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}(u=-1) \quad\text{or}\quad \begin{pmatrix}2 & 1\\ 1 & 2b\end{pmatrix}(u=1)$$

and so discriminant 1 ($u=-1$) resp. 1–4b ($u=1$). In the former case (corresponding to $i=2$), our map is zero; in the latter (corresponding to $i=0$) it is not, since 1–4b is not a square in K. For if it is the square of $1+2x$, say, then $x^2+x=-b$ so x is integral, and taking its class in $\bar{A}$ contradicts the choice of b.

For i odd, the nonzero class of $L_i(\bar{A})$ is represented by the automorphism of a hyperbolic plane which interchanges the factors. We lift this to the automorphism $e \to f \to e u$ ($u=\pm 1$); this has determinant $-u$. Since $u=+1(-1)$ corresponds to $i=1(3)$, this completes the proof.

Note that applying the same argument to [6, Theorem 4] shows that our "general nonsense" here has replaced the lemma.

Corollary.

$$L_0^S(A)=0, \quad L_1^S(A) \cong A^\times/(A^\times)^2$$
$$L_2^S(A) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

and there is an exact sequence

$$0 \to \mathbb{Z}/2 \to A^\times (A^\times)^2 \to L_3^S(A) \to \mathbb{Z}/2 \to 0.$$

Proof. The only result which does not follow at once by combining the above with Theorem 6, Corollary is that $L_2^S(A)$ is a trivial extension. Now it suffices to choose a plane with nonzero Arf invariant, and show that it has order 2 in $L_2^S(A)$. Suppose then $q_\theta(e')=1$, $q_\theta(f')=b$, $b_\theta(e',f')=1$ and similarly for the (orthogonal) space spanned by e'', f''. Then $\{e', f'-f''\}$ is easily seen to be a hyperbolic plane with standard basis (after adjusting e' suitably modulo $(f'-f'')$). Its orthogonal complement has preferred basis $e'''=e'+e''$, f'' and $q_\theta(e''')=2$, $q_\theta(f'')=b$. Now by Hensel's lemma we can find λ with $q_\theta(e'''+f''\lambda)=0$, so take $e=e'''+f''\lambda$. Finally adjust f'' modulo e (this is now easy) to make q_θ vanish here too.

§ 4. Complements

The preceding results can be sharpened in several ways, some of which are clear. Lemma 2 shows that the actual (not merely stable) classification of forms over R is the same as over $\bar{R}$. Now we obtain a corresponding classification of based forms over ϕ, provided Aut ϕ maps onto $K_1 \mathcal{Q}(R)$.

Lemma 12. *Suppose R complete semilocal. Then*

(i) $R^\times = GL_1(R) \to K_1(R)$ *is surjective.*

(ii) Aut $H(R) \to K_1 \mathcal{Q}(R)$ *is surjective.*

Proof. The first assertion follows from the Corollary to Proposition 3, as $\bar{R}^\times \to K_1(\bar{R})$ is surjective since $\bar{R}$ is semisimple.

It follows that $H(R^\times)$ maps onto

$$H(K_1(R)) = \mathrm{Ker}(K_1 \mathcal{Q}(R) \to L_1^K(R)).$$

Finally, $L_1^K(R) \cong L_1^K(\bar{R})$ is a sum of groups of order 2, one for each of certain summands of $\bar{R}$. Clearly, the automorphism group of a hyperbolic plane maps onto this (it suffices to note this for simple summands). The result follows since, by Lemma 2, we can lift automorphisms.

We thus obtain a classification of based forms over ϕ whenever ϕ has a hyperbolic summand. By Lemma 2, Corollary 2 this is equivalent to $\bar{\phi}$ having a hyperbolic summand; by the classification of forms over finite fields, this holds whenever $\bar{\phi}$ (or equivalently, ϕ) has rank $\geqq 3$. If we note also that any element of $K_0 \mathcal{Q}(\bar{R})$ of rank $\geqq 2$ corresponds to a quadratic module unique up to isomorphism (and hence that the same holds for $K_0 \mathcal{Q}(R)$), we conclude

Corollary. *Suppose further that $\bar{R}$ is finite. Then any element of $K_0 \mathcal{B}\mathcal{Q}(R)$ of rank $\geqq 3$ corresponds to a based quadratic module, unique up to isomorphism.*

This allows a certain amount of cancellation, for example. Of course, it is possible by the same methods but with more effort to obtain results also for lower ranks, but these will be complicated and depend on cases.

These stability results for K_0 can be matched by some for K_1. The surjectivity is already noted in Lemma 12. An injectivity statement should take the form: for all $n \geqq r$,

$$EU_n(R) = \mathrm{Ker}(U_n(R) \to K_1 \mathcal{Q}(R)). \qquad (*)$$

We cannot give this in a sharp form due to our restricted list of generators of EU_n – for example, if R is a division ring, the kernel is always generated by transvections when $n \geqq 2$ (see discussion in [7], for example), but the arguments in [5] only work conveniently for $n \geqq 3$, and would indeed prove (*) for $n \geqq 3$, R a division ring. Since a stronger result has already been obtained by Bak [1]. I will confine myself to outlining an argument that a corresponding result holds for R semilocal (it is essentially the same argument as would apply for a division ring).

Since we may assume the result for the semisimple ring $\bar{R}$, it suffices to consider automorphisms congruent modulo I to the identity. It follows from our proof of Proposition 4 that we can reduce one of these to a direct sum of hyperbolic automorphisms $H(x)$, $x \in 1 + I$. Conjugating by the elementary automorphisms $\sum_{ij}$ of [5] permutes the summands, so we may assume only one summand non-trivial. Then x determines an element ξ of $V \subset K_1 \mathcal{P}(R)$ which maps to 0 in $K_1 \mathcal{Q}(R, \alpha, u)$, hence is in the image of $\tilde{K}_0 \mathcal{B}\mathcal{Q}(R, \alpha, -u)$. By the previous corollary, we can find a corresponding form θ of any rank $n \geqq 4$. Now by the arguments of [5, Lemma 6 and Prop. 11] we can find a product of elementary matrices of the form $H(x')$, where x' also has class ξ. But then xx'^{-1} determines $0 \in K_1(R)$, so is a product of elementary matrices in E_3, and $H(xx'^{-1})$ is thus also a product of elementary matrices.

For purposes of computation, it is easier (if possible) to replace $K_1\mathscr{P}(R)$ by its image $K_1'\mathscr{P}(R) \subset K_1\mathscr{P}(S)$. By Theorem 9, the kernel of this projection is finite. If also this kernel is a p-group, an analogue to Theorem 10 holds for $K_1'\mathscr{P}(R)$. We say in this case that R has *good reduction.* Clearly if R is commutative, it has good reduction. More generally, so does any Morita-equivalent ring. In particular, if R is unramified (i.e. a maximal order in a sum of matrix rings over fields) then R has good reduction.

An example of an R which does not have good reduction is a maximal order in a quaternion division algebra – e.g.

$$R = \hat{\mathbb{Z}}_3[i, j/i^2 = 3, j^2 = -1, ji = -ij].$$

Here, $J(R)$ is generated by 3 and i, and $\bar{R}$ is the field of order 9. We have $\mathrm{Nrd}\, j = 1$, but the image of j in $\bar{R}$ is not 1, and equals its reduced norm.

An example (due to Kneser) of a ring R with good reduction, but $K_1\mathscr{P}(R) \neq K_1'\mathscr{P}(R)$ is the subring $\hat{\mathbb{Z}}_p + pM_n(\hat{\mathbb{Z}}_p)$ of $M_n(\hat{\mathbb{Z}}_p)$ for n prime to $p-1$: the elementary matrix $X_{1,n}(p)$ has determinant 1, but is not 1 in $K_1\mathscr{P}(R)$.

Even $K_1'\mathscr{P}(R)$ is not so easy to compute: the image by the reduced norm need not lie in the centre of R (though e.g. by Swan [4, 5.4] this is the case if R is a maximal order). An example here is the ring of matrices over $\hat{\mathbb{Z}}_p$ of shape

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$$

with $a - b \equiv d - c \equiv e \pmod p$. These remarks apply à fortiori to the $K_1\mathscr{Q}(R)$.

We conclude with one result which we shall need in extending our calculations to adèles.

Lemma. *Let F denote one of the functors* L_i^S, Λ_i, L_i^K, K_0, K_1. *Let R be a* $\mathbb{Z}$*-order in a semisimple* $\mathbb{Q}$*-algebra S. Write* $\hat{R}_p = R \otimes \hat{\mathbb{Z}}_p$, $\hat{S}_p = R \otimes \hat{\mathbb{Q}}_p = S \otimes \hat{\mathbb{Q}}_p$. *Then for almost all primes p,* $F(\hat{R}_p) \to F(\hat{S}_p)$ *is injective.*

Proof. Each of the following assertions holds for all but a finite number of primes p.

(i) p is odd

(ii) $\hat{R}_p$ is a maximal order in $\hat{S}_p$

(iii) $\hat{S}_p$ is unramified.

Property (iii) implies that $\hat{S}_p$ is a direct sum of matrix rings over fields $\hat{K}_p$, and property (ii) that we can identify $\hat{R}_p$ with the corresponding direct sum of matrix rings over the rings $\hat{I}_p$ of integers in $\hat{K}_p$. It thus suffices to consider the case: $\hat{R}_p = \hat{I}_p$, $\hat{S}_p = \hat{K}_p$.

We first consider K_0, K_1. Clearly $K_0(R) = \mathbb{Z}$ for $R = \hat{I}_p$ or $\hat{K}_p$: the result is trivial. Also for any commutative semilocal ring R, $R^\times \to K_1(R)$ is surjective by Prop. 3, Corollary; since the determinant provides a left inverse, this map is an isomorphism. Now $\hat{I}_p^\times \to \hat{K}_p^\times$ is clearly injective.

The remaining functors F depend on an involution α of $\hat{K}_p$. There are two main cases, according as α is the identity or not. If not, we call α *unramified* (at p) if $\hat{K}_p$ is unramified over the field of α: for the above $\hat{S}_p$, we require this condition for each summand. This is equivalent to saying that α does not induce the identity on the finite residue field F_p of $\hat{I}_p$. Observe that for all but a finite number of primes p,

(iv) α is unramified at p.

We will prove the assertion for primes p satisfying (i)–(iv).

From the calculations in [7] (for example), we have the following: If $\alpha = 1$ (and $u = 1$),

$$\Lambda_0(F_p) \overset{\delta}{=} F_p^\times, \quad \Lambda_1(F_p) \cong \{\pm 1\} \times F_p^\times/(F_p^\times)^2, \quad \Lambda_2(F_p) \overset{\tau}{=} F_p^\times, \quad \Lambda_3(F_p) = 0.$$

If $\alpha \neq 1$, and f_p is its fixed field on F_p,

$$\Lambda_{2i}(F_p) \overset{\delta}{=} f_p^\times, \quad \Lambda_{2i+1}(F_p) \overset{\delta}{=} \operatorname{Ker} N\colon F_p^\times \to f_p^\times.$$

From Theorem 10, Corollary 2, we lift these to

If $\alpha = 1$, $u = 1$,

$$\Lambda_0(\hat{I}_p) \overset{\delta}{=} \hat{I}_p^\times, \quad \Lambda_1(\hat{I}_p) \cong \{\pm 1\} \times F_p^\times/(F_p^\times)^2 \cong \{\pm 1\} \times \hat{I}_p^\times/(\hat{I}_p^\times)^2,$$
$$\Lambda_2(\hat{I}_p) \overset{\tau}{=} \hat{I}_p^\times, \quad \Lambda_3(\hat{I}_p) = 0.$$

If $\alpha \neq 1$, and $\hat{i}_p$ is its fixed subring on $\hat{I}_p$,

$$\Lambda_{2i}(\hat{I}_p) \overset{\delta}{=} \hat{i}_p^\times, \quad \Lambda_{2i+1}(\hat{I}_p) \overset{\delta}{=} \operatorname{Ker} N\colon \hat{I}_p^\times \to \hat{i}_p^\times.$$

Injectivity now follows since each invariant (determinant, discriminant, spinor norm, Pfaffian) is already defined over $\hat{K}_p$. In fact, the results of [7] yield also:

If $\alpha = 1$, $u = 1$,
$$\Lambda_0(\hat{K}_p) \xrightarrow{\delta} \hat{K}_p^\times$$

is surjective, with kernel $Br_2(\hat{K}_p)$ of order 2 (we can describe the extension using Clifford algebras)

$$\Lambda_1(\hat{K}_p) \cong \{\pm 1\} \times \hat{K}_p^\times/(\hat{K}_p^\times)^2, \quad \Lambda_2(\hat{K}_p) \overset{\pi}{\cong} \hat{K}_p^\times, \quad \Lambda_3(\hat{K}_p) = 0.$$

If $\alpha \neq 1$, and $\hat{k}_p$ is its fixed subfield,

$$\Lambda_{2i}(\hat{K}_p) \stackrel{\delta}{=} \hat{k}_p^\times, \quad \Lambda_{2i+1}(\hat{K}_p) \stackrel{\delta}{=} \operatorname{Ker} N\colon \hat{K}_p^\times \to \hat{k}_p^\times.$$

The cokernels in the 6 cases are thus:

$$\mathbb{Z} \times \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}, 0, \mathbb{Z}, 0.$$

Since $L_i^S \subset \Lambda_i$, it follows at once that the result also holds for $F = L_i^S$. Finally, we deduce from Lemma 5 that $L_i^K(\hat{I}_p) = L_i^K(F_p)$, so

if $\alpha = 1$, $u = 1$,

$$L_0^K(\hat{I}_p) \cong \hat{I}_p^\times/(\hat{I}_p^\times)^2, \quad L_1^K(\hat{I}_p) \cong \{\pm 1\}, \quad L_2^K(\hat{I}_p) = L_3^K(\hat{I}_p) = 0.$$

If $\alpha \neq 1$, $L_i^K(\hat{I}_p) = 0$.

Since the only nonzero groups are detected by the global δ, the result again follows. In fact,

if $\alpha = 1$, $u = 1$, $\quad L_0^K(\hat{K}_p) \to \hat{K}_p^\times/(\hat{K}_p^\times)^2$

is surjective with kernel of order 2,

$$L_1^K(\hat{K}_p) \cong \{\pm 1\}, \quad L_2^K(\hat{K}_p) = L_3^K(\hat{K}_p) = 0.$$

If $\alpha \neq 1$, $L_{2i}^K(\hat{K}_p) = \hat{k}_p/N\hat{K}_p$ (of order 2), $L_{2i+1}^K(\hat{K}_p) = 0$.

References

1. Bak, A.: On modules with quadratic forms, pp. 55–66, in: Algebraic K-theory and its geometrical applications. Lecture Notes in Mathematics. **108**. Berlin-Heidelberg-New York: Springer 1969. See also preprint, Princeton: 1970.
2. Bass, H.: Algebraic K-theory. New York: W.A. Benjamin Inc. 1968.
3. Riehm, C.: The norm 1 group of a p-adic division algebra. Amer. J. Math. **92**, 499–523 (1970).
4. Swan, R.G.: K-theory of finite groups and orders. Lecture Notes in Mathematics. **149**. Berlin-Heidelberg-New York: Springer 1970.
5. Wall, C.T.C.: Foundations of algebraic L-theory. Proceedings of the conference on algebraic K-theory, Seattle, 1972. Lecture Notes in Mathematics (in press).
6. Wall, C.T.C.: On the classification of Hermitian forms. I. Rings of algebraic integers. Comp. Math. **22**, 425–451 (1970).
7. Wall, C.T.C.: On the classification of Hermitian forms. II. Semisimple rings. Inventiones math. **18**, 119–141 (1972).
8. Wall, C.T.C.: On the classification of Hermitian forms. IV. Global rings. To appear.

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