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On the Classification of Hermitian Forms

IV. Adele Rings

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Although the algebraic K-theory of a product of two rings splits in a natural way as a product, this need not hold for an infinite product. In the first paragraph below we give sufficient conditions for such a result to hold, generalised to the case of restricted direct products. In the next, we verify these conditions for adele rings. Then § 3 checks corresponding results for categories of quadratic modules, and in § 4 we consider based quadratic modules, and then tabulate the L-groups of adele rings that we have obtained. As well as computing L-groups, we obtain cancellation and stability properties of these categories, which will be needed in the next paper.

In a final section we compare L-groups of the semisimple ring S with those of the adele ring $S_A = \hat{S} \oplus (S \otimes \mathbb{R})$. In general, $L_i(S) \to L_i(S_A)$ is injective—for i even, this is the classical Hasse principle for forms over fields, which we merely quote. We also systematically compute the co-kernel: this completes—and also checks—the results of [6].

§ 1. Projective Modules: General Theory

Suppose given an indexing set I and, for each $\alpha \in I$, a ring B_α and a subring C_α. For each finite $S \subset I$ we can form the direct product ring

$$A_S = \prod_S B_\alpha \times \prod_{I-S} C_\alpha.$$

The union, or direct limit of the A_S over (increasing) finite subsets S of I is called the restricted direct product $A = \prod_I (B_\alpha, C_\alpha)$. The same construction yields the restricted direct product of groups, and of other algebraic systems; moreover, it is not essential that the morphisms $C_\alpha \to B_\alpha$ be injective. The notion includes direct sums $\coprod_I B_\alpha = \prod_I (B_\alpha, 0)$ as well as direct products $\prod B_\alpha = \prod (B_\alpha, B_\alpha)$.

Clearly, a finitely generated projective A-module is induced from a projective A_S-module for some S, and this determines projective modules over C_α for almost all α (i.e. for all but finitely many—in this case, S is the finite exceptional set) and over B_α for all α, where the B_α-modules are almost all induced from C_α-modules. Conversely, if we assign projective C_α-modules for $\alpha \notin S$ and B_α-modules for $\alpha \in S$ we obtain a projective A_S-module which in turn induces an A-module. Analogous remarks hold also for morphisms. Unfortunately, it does not follow that if the C_α-modules are all finitely generated, so is the A-module: to see this, consider the case where I is the set of natural numbers, $B_\alpha = C_\alpha$ for all α, and we have the module $\Pi(C_n)^n$.

For any ring A, we write $\mathscr{P}(A)$ for the category of finitely generated projective A-modules. For any monoidal category $\mathscr{A}$, we write $k\mathscr{A}$ for the monoid of isomorphism classes of objects of $\mathscr{A}$, and $K_0\mathscr{A}$, $K_1\mathscr{A}$ for the usual algebraic K-theoretic groups of $\mathscr{A}$ [1, pp. 346, 348]. An element of $K_0\mathscr{A}$ is called *positive* if it is in the image of $k\mathscr{A}$.

Theorem 1.1. *Let* $A = \prod_I (B_\alpha, C_\alpha)$.

(i) *Suppose* $k\mathscr{P}(C_\alpha) \to k\mathscr{P}(B_\alpha)$ *injective for almost all* α. *Then* $k\mathscr{P}(A) \to \prod_I (k\mathscr{P}(B_\alpha), k\mathscr{P}(C_\alpha))$ *is injective. Hence if further* $k\mathscr{P}(B_\alpha)$ *is a cancellation semigroup for all* α, *so is* $k\mathscr{P}(A)$.

(ii) *Suppose* $K_0\mathscr{P}(C_\alpha) \to K_0\mathscr{P}(B_\alpha)$ *injective for almost all* α, *say for* $\alpha \notin R$. *Suppose also for some finite* $S \subset I$ *and some integer* n *(independent of* α*) that whenever* $\alpha \in (I - S)$ *and* P, P' *are finitely generated projective* C_α*-modules with the same class in* $K_0\mathscr{P}(C_\alpha)$, *there is an isomorphism* $P \oplus C_\alpha^n \to P' \oplus C_\alpha^n$. *Then the natural map*

$$\omega_0 : K_0\mathscr{P}(A) \to \prod_I (K_0\mathscr{P}(B_\alpha), K_0\mathscr{P}(C_\alpha))$$

is injective.

(iii) *Suppose, in addition to the hypotheses of* (ii), *that we have, for almost all* α *(say for* $\alpha \notin U$*), a collection of homomorphisms* $r_{\alpha\beta}$ *of* $k\mathscr{P}(B_\alpha)$ *into the additive semigroup of non-negative rational numbers with* $r_{\alpha\beta}(B_\alpha) = 1$ *for all* β, *and an integer* n *(independent of* α, β*) such that any* $x_\alpha \in K_0\mathscr{P}(C_\alpha)$ *with* $r_{\alpha\beta}(x_\alpha) \geqq n$ *for all* β *is positive. Then the image of* ω_0 *is the subgroup of* x *with* $\sup_{\alpha,\beta} |r_{\alpha\beta}(x)| < \infty$.

Proof. (i) Write $\mathscr{P}'(A)$ for the restricted direct product category $\prod(\mathscr{P}(B_\alpha), \mathscr{P}(C_\alpha))$: more explicitly, we can write

$$\mathscr{P}'(A_S) = \prod_S \mathscr{P}(B_\alpha) \times \prod_{I-S} \mathscr{P}(C_\alpha),$$

and $\mathscr{P}'(A)$ for the limit of the $\mathscr{P}'(A_S)$. Note that $\mathscr{P}(A)$ is a full subcategory of $\mathscr{P}'(A)$, for it is clear that $\mathscr{P}(A_S)$ is a full subcategory of $\mathscr{P}'(A_S)$, and the assertion follows on taking limits.

Clearly, we have a surjective homomorphism

$$k\mathscr{P}'(A) \to \prod_I (k\mathscr{P}(B_\alpha), k\mathscr{P}(C_\alpha)).$$

Now two objects of $\mathscr{P}'(A)$ with the same image admit isomorphisms over B_α for all α. Represent them as A_S-modules: then if the hypothesis of (i) holds, and we enlarge S to include the exceptional set, we have isomorphisms over C_α for $\alpha \notin S$. Hence we have an isomorphism in $\mathscr{P}'(A)$. Thus the above map is bijective. Since $k\mathscr{P}(A) \subset k\mathscr{P}'(A)$, the assertions of (i) follow.

(ii) It is again clear (e.g. by considering $\mathscr{P}'(A)$) that we have a natural homomorphism ω_0. Take any element ξ of the kernel, represented as the difference of the A_T-modules X and X', for some finite $T \subset I$. For $\alpha \notin (R \cup S \cup T)$, the C_α-modules X_α and X'_α must determine the same class in $K_0\mathscr{P}(C_\alpha)$, since by hypothesis this is a subgroup of $K_0\mathscr{P}(B_\alpha)$. By our other hypothesis, we have an isomorphism

$$X_\alpha \oplus C_\alpha^n \to X'_\alpha \oplus C_\alpha^n.$$

If now Y, Y' are the $A_{R \cup S \cup T}$-modules induced by X, X', we know for $\alpha \in R \cup S \cup T$ that Y_α, Y'_α determine the same class in $K_0\mathscr{P}(B_\alpha)$, hence for some $n(\alpha)$ we have an isomorphism

$$Y_\alpha \oplus B_\alpha^{n(\alpha)} \to Y'_\alpha \oplus B_\alpha^{n(\alpha)}.$$

If N is the largest of n and the (finitely many) $n(\alpha)$, we thus have an isomorphism

$$Y \oplus A_{R \cup S \cup T}^N \to Y' \oplus A_{R \cup S \cup T}^N.$$

Hence $\xi = 0 \in K_0\mathscr{P}(A)$.

(iii) Note that $r_{\alpha\beta}$ induces homomorphisms of $k\mathscr{P}(C_\alpha)$ and $k\mathscr{P}(A)$; also homomorphisms into $\mathbb{Q}^+$ of the universal groups K_0. We denote all of these by the same letter. Now for any direct summand X of A^N,

$$0 \leqq r_{\alpha,\beta}(X) \leqq N \qquad \text{for all } \alpha, \beta,$$

so the condition $\sup |r_{\alpha,\beta}(x)| < \infty$ is necessary.

Conversely, suppose this holds. Choose $N_0 \geqq 0$ so large that $r_{\alpha,\beta}(x) + N_0 \geqq n$ for all α, β. Choose V so that x comes from an element

$$x' \in \prod_V K_0\mathscr{P}(B_\alpha) \times \prod_{I-V} K_0\mathscr{P}(C_\alpha).$$

Then for $\alpha \notin (U \cup V)$, $x'_\alpha + N_0$ is positive by hypothesis. Choose $N_1 \geqq N_0$ such that $x'_\alpha + N_1$ is positive for all α. We thus have a projective A_V-module, hence an object X of $\mathscr{P}'(A)$ whose class is $x' + N_1$. Similarly we find an object Y whose class is $N_2 - x'$, for some N_2. Hence $X \oplus Y$ and $A^{N_1+N_2}$ have the same class in $\prod (K_0\mathscr{P}(B_\alpha), K_0\mathscr{P}(C_\alpha))$. But now the argument given above under (ii) shows that for some N_3, there is an

isomorphism
$$X \oplus Y \oplus A^{N_3} \to A^{N_1+N_2+N_3}.$$

Hence X is finitely generated, and the result follows.

The discussion of K_1 runs into even more serious difficulties concerning finiteness. Fortunately, the following rather crude result will suffice for our purposes.

Theorem 1.2. (i) *If* $A = \prod_I (B_\alpha, C_\alpha)$, *there is a natural homomorphism*
$$\omega_1: K_1\mathscr{P}(A) \to \prod_I \big(K_1\mathscr{P}(B_\alpha), K_1\mathscr{P}(C_\alpha)\big).$$

(ii) *If for some integer* n *and finite subset* $S \subset I$, $GL_n(C_\alpha)$ *maps onto* $K_1\mathscr{P}(C_\alpha)$ *for* $\alpha \notin S$, *then* ω_1 *is surjective. If also* $GL_n(B_\alpha)$ *maps onto* $K_1\mathscr{P}(B_\alpha)$ *for all* α, *the induced map of* $GL_n(A)$ *is surjective.*

(iii) *Suppose further that for* $\alpha \notin S$, $K_1\mathscr{P}(C_\alpha) \to K_1\mathscr{P}(B_\alpha)$ *is injective, and that for* $\alpha \notin S$ *and* $m \geqq n$, *any element of* $\operatorname{Ker}\big(GL_m(C_\alpha) \to K_1\mathscr{P}(C_\alpha)\big)$ *can be expressed as a product of at most* $f(m)$ *elementary matrices, where* $f(m)$ *is independent of* α. *Then* ω_1 *is an isomorphism.*

Proof. (i) For S finite, there is a natural map
$$K_1\mathscr{P}(A_S) \to K_1\mathscr{P}'(A_S) = \prod_S K_1\mathscr{P}(B_\alpha) \times \prod_{I-S} K_1\mathscr{P}(C_\alpha).$$

Since $GL_\infty(A) = \bigcup GL_\infty(A_S)$, and the same holds for the commutator subgroup, $K_1\mathscr{P}(A)$ is the limit of the $K_1\mathscr{P}(A_S)$, and maps to the limit on the right, viz. $\prod \big(K_1\mathscr{P}(B_\alpha), K_1\mathscr{P}(C_\alpha)\big)$.

(ii) Let $x \in \prod_I \big(K_1\mathscr{P}(B_\alpha), K_1\mathscr{P}(C_\alpha)\big)$: we may suppose x the image of
$$x' \in \prod_R K_1\mathscr{P}(B_\alpha) \times \prod_{I-R} K_1\mathscr{P}(C_\alpha).$$

For $\alpha \notin R \cup S$, x'_α is in the image of $GL_n(C_\alpha)$. For each $\alpha \in R \cup S$ we can find an element of $GL_{n(\alpha)}(B_\alpha)$, for some $n(\alpha)$, representing x_α. Then if $N = \max\big(n, n(\alpha)\big)$, we have an element of $GL_N(A_{R \cup S}) \subset GL_N(A)$ whose class is x. The last part follows by the same argument: we can take all $n(\alpha) = n$.

(iii) Let $x \in \operatorname{Ker} \omega_1$. Represent x by $\xi \in GL_m(A_T)$ for some m and finite $T \subset I$: we may suppose $S \subset T$ and $m \geqq n$. Then for $\alpha \notin T$, since $K_1\mathscr{P}(C_\alpha) \to K_1\mathscr{P}(B_\alpha)$ is injective, $\xi(\alpha)$ represents 0 in $K_1\mathscr{P}(C_\alpha)$. Since $\xi(\alpha)$ represents 0 in $K_1\mathscr{P}(B_\alpha)$ for $\alpha \in T$, and $K_1\mathscr{P}$ preserves finite products, to show that ξ represents 0 and hence that $x = 0$, it is enough to consider $\prod_{I-T} C_\alpha$.

An elementary matrix has the form $X_{ij}(r)$, equal to the identity matrix except in the (i, j) position, $i \neq j$, where we have r. By hypothesis, each $\xi(\alpha)$ for $\alpha \notin T$ is the product of at most $f(m)$ such matrices. Let F denote the set of sequences of $f(m)$ terms, each of which is a pair (i, j) with

$1 \leqq i, j \leqq m$, $i \neq j$. Then F is finite, and we can order F and by juxtaposition obtain a single finite sequence (i_r, j_r) which contains each element of F as a subsequence. Thus we can write

$$\xi(\alpha) = X_{i_1 j_1}(c_{\alpha,1}) X_{i_2, j_2}(c_{\alpha,2}) \ldots,$$

taking any superfluous $c_{\alpha,r} = 0$. But this gives an expression of $\{\xi(\alpha)\colon \alpha \in I - T\}$ as a finite product of elementary matrices over $\prod_{I-T} C_\alpha$, and so concludes the proof.

Corollary 1.3. *Under the conditions of the theorem, $GL_n(A)$ maps onto $K_1 \mathscr{P}(A)$.*

This follows from the final assertions in (ii) and (iii) above.

§ 2. Projective Modules over Adele Rings

Let $\mathbb{Z}$ denote the ring of integers, $\hat{\mathbb{Z}}$ its profinite completion – i.e. the inverse limit of its finite cyclic quotients. This is also the direct product of the pro-p-rings $\hat{\mathbb{Z}}_p$ of p-adic integers. Write $\mathbb{Q}$ for the field of rationals, $\hat{\mathbb{Q}} = \mathbb{Q} \otimes \hat{\mathbb{Z}}$ and $\hat{\mathbb{Q}}_p = \mathbb{Q} \otimes \hat{\mathbb{Z}}_p$. Then we have – as is easily seen – $\hat{\mathbb{Q}} = \prod (\hat{\mathbb{Q}}_p, \hat{\mathbb{Z}}_p)$. The ring $\mathbb{Q}_A$ of adeles is the product $\hat{\mathbb{Q}} \times \mathbb{R}$. Now let R be a ring whose additive group is finitely generated. We label its tensor products (over $\mathbb{Z}$) with the above rings systematically:

$$\begin{array}{llll} R(=R \otimes \mathbb{Z}) & \hat{R} = R \otimes \hat{\mathbb{Z}} & \hat{R}_p = R \otimes \hat{\mathbb{Z}}_p & T = R \otimes \mathbb{R} \\ S = R \otimes \mathbb{Q} & \hat{S} = R \otimes \hat{\mathbb{Q}} & \hat{S}_p = R \otimes \hat{\mathbb{Q}}_p & S_A = R \otimes \mathbb{Q}_A. \end{array}$$

We assume throughout that S is semisimple. In this paper we will study primarily $\hat{S}$: since $S_A = \hat{S} \times T$, results about it will follow. Observe that

$$\hat{R} = \prod \hat{R}_p \qquad \hat{S} = \prod (\hat{S}_p, \hat{R}_p).$$

Now $\hat{S}_p$ is also semisimple since S is, hence is a finite sum of matrix rings over division rings. The following facts are well known. For almost all p, $\hat{S}_p$ is unramified, i.e. a direct sum of matrix rings over fields $\hat{K}_p$. For almost all p, $\hat{R}_p$ is a maximal order in $\hat{S}_p$ [1, pp. 149, 154] and hence splits as a direct sum when $\hat{S}_p$ does. When both these conditions hold, we call $\hat{R}_p$ unramified. There is then [1, p. 162] an isomorphism $\hat{S}_p \to \oplus M_{r(\beta)}(\hat{K}_{p,\beta})$ under which $\hat{R}_p$ corresponds to the subring $\oplus M_{r(\beta)}(\hat{I}_{p,\beta})$, where $\hat{I}_{p,\beta}$ is the ring of integers in $\hat{K}_{p,\beta}$. Write P for the set of all primes, and B for the finite subset at which R is ramified. Then we have

Theorem 2.1. *With the above notation, all the hypotheses of* (1.1) *and* (1.2) *hold for $\hat{R}$ and for $\hat{S}$. Explicitly,*

(i) *Cancellation holds in $k\mathscr{P}(\hat{R}_p)$ and $k\mathscr{P}(\hat{S}_p)$.*

(ii) *If F denotes* $k\mathscr{P}$, $K_0\mathscr{P}$ *or* $K_1\mathscr{P}$, *then* $F(\hat{R}_p)\to F(\hat{S}_p)$ *is injective for* $p\in(P-B)$.

(iii) *For* $p\in P-B$, *and* $\hat{M}_p$ *a finitely generated* $\hat{S}_p$*-module, hence a sum* $\oplus\hat{M}_p(\beta)$ *corresponding to the decomposition of* $\hat{S}_p$, *set* $r_{p,\beta}(\hat{M}_p)=$ *dim* $\hat{M}_p(\beta)/\dim\hat{S}_p(\beta)$, where dimensions can be taken over $\hat{K}_{p,\beta}$ (*or* $\hat{\mathbb{Q}}_p$). *Then any* $x\in K_0\mathscr{P}(\hat{R}_p)$, *with* $r_{p,\beta}(x)\geqq 0$ *for all* β, *is positive.*

(iv) *For all* $p\in P$, $GL_1(\hat{R}_p)\to K_1\mathscr{P}(\hat{R}_p)$ *and* $GL_1(\hat{S}_p)\to K_1\mathscr{P}(\hat{S}_p)$ *are surjective.*

(v) *For* $p\in P-B$, *and any* $n\geqq 2$, *we can express any element of* $\operatorname{Ker}(GL_n(\hat{R}_p)\to K_1\mathscr{P}(\hat{R}_p))$ *as a product of at most* $\phi_n(R)$ *elementary matrices.*

Hence applying the results of the preceding section, we obtain at once

Corollary 2.2. (i) *Cancellation holds in* $k\mathscr{P}(\hat{R})$ *and* $k\mathscr{P}(\hat{S})$.

(ii) $K_0\mathscr{P}(\hat{R})$ *resp.* $K_0\mathscr{P}(\hat{S})$ *is the subgroup of bounded elements in* $\prod K_0\mathscr{P}(\hat{R}_p)$ *resp.* $\prod(K_0\mathscr{P}(\hat{S}_p), K_0\mathscr{P}(\hat{R}_p))$.

(iii) $GL_1(\hat{R})\to K_1\mathscr{P}(\hat{R})$, $GL_1(\hat{S})\to K_1\mathscr{P}(\hat{S})$ *are surjective.*

(iv) $K_1\mathscr{P}(\hat{R})\to\prod K_1\mathscr{P}(\hat{R}_p)$, $K_1\mathscr{P}(\hat{S})\to\prod(K_1\mathscr{P}(\hat{S}_p), K_1\mathscr{P}(\hat{R}_p))$ *are isomorphisms.*

Proof of (2.1). (i) Since $\hat{S}_p$ is semisimple, cancellation surely holds in the free abelian monoid $k\mathscr{P}(\hat{S}_p)$. As to $\hat{R}_p$, write $\bar{R}_p$ for the quotient by its radical. Then e.g. by [1, p. 90], $k\mathscr{P}(\hat{R}_p)\to k\mathscr{P}(\bar{R}_p)$ is an isomorphism, and we have cancellation for the semisimple ring $\bar{R}_p$.

(iv) This follows from [1, p. 266].

(ii) For $p\in P-B$, we have $\mathscr{P}(\hat{S}_p)\cong\prod\mathscr{P}(M_{r(\beta)}(\hat{K}_{p,\beta}))$, and by a Morita equivalence [1, pp. 65, 68], this is equivalent to $\prod\mathscr{P}(\hat{K}_{p,\beta})$; correspondingly,

$$\mathscr{P}(\hat{R}_p)\cong\prod\mathscr{P}(\hat{I}_{p,\beta}).$$

It thus suffices to consider the case where $\hat{S}_p=\hat{K}_p$ is a field and $\hat{R}_p=\hat{I}_p$ the ring of integers in it. But here the rank induces isomorphisms of $k\mathscr{P}(\hat{I}_p)\cong k\mathscr{P}(\hat{K}_p)\cong$ monoid of nonnegative integers. Also in this commutative case, determinant yields a left inverse to $GL_1(\hat{I}_p)\to K_1\mathscr{P}(\hat{I}_p)$ which is surjective by (iv), and similarly for $\hat{K}_p$. Thus

$$K_1\mathscr{P}(\hat{I}_p)=\hat{I}_p^\times\to\hat{K}_p^\times=K_1\mathscr{P}(\hat{K}_p)$$

is clearly injective.

(iii) As above, this reduces to the case $\hat{R}_p=\hat{I}_p$, $\hat{S}_p=\hat{K}_p$, but with $r_{p,\beta}=$ rank divided by $r(\beta)$ since rank is not invariant under equivalence of categories. It now follows at once from the above description of $k\mathscr{P}(\hat{R}_p)$.

(v) We first establish the result in the case $\hat{R}_p = \hat{I}_p$ and with $\phi_n = (n+2)(n-1)$. It suffices to describe an elementary reduction to the identity of an arbitrary unimodular matrix over a commutative local ring.

(a) Add some other column to the first to make $a_{n,1}$ a unit.

(b) Add $(1-a_{nn})\,a_{n,1}^{-1}$ times the first column to the last.

(c) Subtract $a_{n,i}$ times the last column from the i-th $(1 \leqq i < n)$.

(d) Subtract $a_{j,n}$ times the last row from the j-th $(1 \leqq j < n)$.

This reduces to an $(n-1)\times(n-1)$ matrix: continue by induction till we reach the 1×1 matrix of determinant 1. We observe that if we allow for all possibilities at (a), we achieve the result with $\frac{1}{2}(n-1)(3n+2)$ elementary matrices with the (i,j) in a fixed sequence.

The corresponding result also follows for the matrix ring $M_r(\hat{I}_p)$. For there are obvious isomorphisms $M_n(M_r(\hat{I}_p)) \cong M_{rn}(\hat{I}_p)$. An elementary matrix $X_{ij}(a)$ over $\hat{I}_p$ is also elementary over $M_r(\hat{I}_p)$ unless $rt+1 \leqq i,j \leqq r(t+1)$ for some $0 \leqq t < n$. In this case, provided $n \geqq 2$ we can choose $1 \leqq k < rn$ not in this subset: then the relation $X_{ij}(a) = [X_{ik}(a), X_{kj}(1)]$ expresses our matrix as the product of 4 elementary matrices over $M_r(\hat{I}_p)$. Thus here any element of the kernel is product of at most $4(nr+2)(nr-1)$ elementary matrices.

It remains to give a bound on r. But

$$r^2 = \dim_{\hat{K}_p} M_r(\hat{K}_p) \leqq \dim_{\hat{\mathbb{Q}}_p} M_r(\hat{K}_p) \leqq \dim_{\hat{\mathbb{Q}}_p}(\hat{S}_p) = \dim_{\mathbb{Q}} S.$$

We conclude this paragraph by giving a more effective description of $K_1\mathscr{P}(\hat{S})$. For any ring A, we denote its centre by $Z(A)$. Then if A is a semisimple algebra over some field, we have a reduced norm map which induces Nrd: $K_1\mathscr{P}(A) \to Z(A)^\times$. It is well known (references are given in [6]) that for $A = \hat{S}_p$, this is an isomorphism.

Proposition 2.3. *Reduced norm induces an isomorphism*

$$\text{Nrd}: K_1\mathscr{P}(\hat{S}) \to Z(\hat{S})^\times.$$

Proof. By (2.2).(iv), we have

$$K_1\mathscr{P}(\hat{S}) = \prod (K_1\mathscr{P}(\hat{S}_p), K_1\mathscr{P}(\hat{R}_p)),$$

and clearly

$$Z(\hat{S})^\times = \prod (Z(\hat{S}_p)^\times, Z(\hat{R}_p)^\times).$$

It will thus suffice to show that for almost all p, Nrd induces an isomorphism $K_1\mathscr{P}(\hat{R}_p) \to Z(\hat{R}_p)^\times$.

Suppose $p \notin B$. Now each of the above is compatible with products, and invariant (in an obvious way) on taking matrix rings, so as before we are reduced to the commutative case, $\hat{I}_p$. But here Nrd reduces to the ordinary determinant, and we have already noted that

$$\det: K_1 \mathscr{P}(\hat{I}_p) \to \hat{I}_p^{\times}$$

is an isomorphism.

§ 3. Quadratic Modules

The notions of antistructure (R, α, u) — essentially a ring with anti-involution — and the category $\mathscr{Q}(R, \alpha, u)$ of nonsingular quadratic modules were defined in [6] or in [8], as were the forgetful and hyperbolic functors

$$F: \mathscr{Q}(R, \alpha, u) \to \mathscr{P}(R), \qquad H: \mathscr{P}(R) \to \mathscr{Q}(R, \alpha, u):$$

see also the paper following this. Roughly speaking, $\mathscr{Q}$ is the category of nonsingular hermitian forms and isometries. In this paragraph we will indicate the modifications necessary for replacing $\mathscr{P}$ by $\mathscr{Q}$ in the preceding paragraphs.

Now suppose we have rings B_α with subrings C_α; the anti-automorphisms s_α of B_α leaves C_α invariant, and s_α^2 is the inner automorphism by $u_\alpha \in C_\alpha^\times$: moreover, s_α inverts u_α. Thus the s_α induce an anti-automorphism s of $A = \prod (B_\alpha, C_\alpha)$, and s^2 is induced by $u = \{u_\alpha\}$. Reference to the s_α, u_α, s and u will be suppressed in the sequel, as they may be supposed fixed.

Proposition 3.1. *The results* (1.1) *and* (1.2) *remain true if we make the following substitutions throughout: for $\mathscr{P}$ read $\mathscr{Q}$; for C_α^n*(as C_α-module) *read $H(C_\alpha^n)$; similarly for other rings: in particular, for $GL_n(C_\alpha) = \mathrm{Aut}(C_\alpha^n)$ read $U_n(C_\alpha) = \mathrm{Aut}(H(C_\alpha^n))$; for elementary matrices read elementary unitary matrices.*

Indeed the proofs, which do not depend deeply on the category $\mathscr{P}$, remain valid also for $\mathscr{Q}$. For example, [8, Lemma 4] shows that the group of elementary unitary matrices over any ring R is generated (for $n \geqq 3$) by subgroups (of matrices $E_{ij}(r)$ or $F_{ij}(r)$) isomorphic to the additive group R^+.

Now turn to the context and notation of § 2 above: an antistructure on R induces structures on all the $\hat{R}_p, \hat{S}_p, T, \hat{R}, \hat{S}, S_A$ which will again be suppressed in our notation, as we will not vary them.

Theorem 3.2. *Theorem* 2.1 *remains true with the following substitutions, in addition to those above: for $r_{p,\beta}$ read $\frac{1}{2} r_{p,\beta} \circ F$; for* 0 *resp.* 2 *in* (iii) *resp.* (v) *read* 1 *resp.* 3; *and for B read B'.*

Here, we define B' — the set of primes p where $(\hat{R}_p, \alpha)$ is unramified — as follows. If $p \in B$ or if $p = 2$, then $p \in B'$. Otherwise, we know that we have to consider fields $\hat{K}_{p,\beta}$ and α either leaves these invariant or interchanges them in pairs. We set $p \in B'$ if for some β, α acts on $\hat{K}_{p,\beta}$ which ramifies over the fixed field of α.

Proof of (3.2). (i) Cancellation certainly holds over division rings – see e.g. [2, 4.3] – hence over semisimple rings, such as $\hat{S}_p$ and $\bar{R}_p$. By [7, Lemma 2, Corollary 2], it holds also for $\hat{R}_p$.

(iv) This was verified in [7, Lemma 12].

(ii) This was explicitly verified in [7, Lemma 13] for $K_1\mathcal{Q}$ and for $\operatorname{Ker} K_0 F$: $K_0\mathcal{Q} \to K_0\mathcal{P}$, and the result for $K_0\mathcal{Q}$ follows. Now the assertion for $k\mathcal{Q}$ follows using cancellation.

(iii) We are given $x \in K_0\mathcal{Q}(\hat{R}_p)$, with $r_{p,\beta} F(x) > 0$ for all β. By the result for $\mathcal{P}$, $F(x)$ is positive: indeed, as before, we can reduce to the case $\hat{R}_p = \hat{I}_p$, or (if α interchanges two summands of $\hat{S}_p$) $\hat{R}_p = \hat{I}_p \oplus \hat{I}_p$. But in the latter case, $\mathcal{Q}(\hat{R}_p) \cong \mathcal{P}(\hat{I}_p)$ and the result follows; in the former, F: $K_0\mathcal{Q}(\hat{R}_p) \to K_0\mathcal{P}(\hat{R}_p)$ is injective save in the orthogonal case, where the kernel has order 2, and is detected by the determinant. Since, however, for $p \notin B'$ the module $\hat{I}_p$ already admits forms with both possible values of the determinant, so does any nonzero element of $k\mathcal{P}(\hat{I}_p)$, and the desired result follows.

(v) As in the case of $\mathcal{P}$, it suffices to prove the desired result for $\hat{I}_p$. Let k be the residue class field of $\hat{I}_p$, and write $x \to \bar{x}$ for reduction modulo the radical. We now consider the argument of [8, Theorem 5] applied to k: again we shall try to reduce an arbitrary element x of $U_n(\hat{I}_p)$ to the identity by elementary transformations[1]. The first step in reduction for $\bar{x}$ is to make $\bar{a} \neq 0$. Lifting the elementary transformation to $\hat{I}_p$, we have made a invertible. The next steps of the argument are valid over $\hat{I}_p$, and reduce x to $H(a) \oplus x_{n-1}$ with $x_{n-1} \in U_{n-1}(\hat{I}_p)$. In a bounded number of steps, we reach some $H(\alpha)$.

Now by the result for $\mathcal{P}$, we can reduce α to $\det \alpha \oplus I_{n-1}$ by a bounded number of elementary moves. Since [8] $H(X_{ij}(r))$ is a product of 4 elementary automorphisms, it remains only to consider the case $x = H(a)$, $a \in \hat{I}_p^{\times}$.

Now we may suppose x represents 0 in $K_1\mathcal{Q}(\hat{I}_p)$. Then either by [8, Prop. 11] with a stabilisation argument, or by inspection in this simple case, it follows that

$$b(e,e) = a, \qquad q(e) = \tfrac{1}{2} a$$

defines a nonsingular (α, $-u$)-quadratic form on the free module $e\hat{I}_p$: or equivalently (as we know a is invertible) $a^{\alpha} = -au$. The proof of [8, Prop. 11] now expresses $H(a \oplus 1)$ as the product of 6 elementary unitary matrices (the first 3 give Σ_{12}: the reader of [8] will easily supply the rest, but it does not seem worth developing the notation here).

Corollary 3.3. (2.2) *remains true on substituting as in* (2.1).

[1] The ε_i of [8, Lemma 6] are not strictly elementary, but each is (as there shown) the product of a bounded number of elementary transformations.

As in the preceding section, we can make these results more explicit in the case of $\hat{S}$. Write $\tilde{K}_0\mathcal{Q}(R)=\operatorname{Ker} F\colon K_0\mathcal{Q}(R)\to K_0\mathcal{P}(R)$.

Corollary 3.4. $\tilde{K}_0\mathcal{Q}(\hat{S})\cong\prod(\tilde{K}_0\mathcal{Q}(\hat{S}_p), \tilde{K}_0\mathcal{Q}(\hat{R}_p))$.

Now the discussion in [6, § 1] showed how to reduce calculations for arbitrary semisimple rings S to division rings D: the equivalences remain good on tensoring rings by $\hat{\mathbb{Q}}$, so it will suffice to study the K-theory of $\mathcal{Q}(\hat{D})$: again, the cases which occur are listed in [6, § 1].

Theorem 3.5. *If* (D, α, u) *has type* Sp, *then*

$$K_1\mathcal{Q}(\hat{D})=0, \qquad \tilde{K}_0\mathcal{Q}(\hat{D})=0.$$

If (D, α, u) *has type* U, E *is the centre of* D, *and* e *the fixed field of* α *on* E, *then*

$$K_1\mathcal{Q}(\hat{D})\cong\operatorname{Ker}(N\colon \hat{E}^\times\to\hat{e}^\times),$$

$$\tilde{K}_0\mathcal{Q}(\hat{D})\cong\hat{e}^\times/N\hat{E}^\times.$$

If (D, α, u) *has type* 0, *and* E *is the centre of* D, *then*

$$K_1\mathcal{Q}(\hat{D})\cong\hat{E}^\times/(\hat{E}^\times)^2\times\prod\{\pm 1\},$$

where the product is extended over those primes $\mathfrak{p}$ *of* E *for which* $\hat{D}_\mathfrak{p}$ *splits.*

Proof. In each case, we know that both sides of the equation are restricted direct products, so it suffices to verify the isomorphism over $\hat{S}_p$ for all p and over $\hat{R}_p$ for $p\notin B'$. But the calculations over $\hat{S}_p$ were listed in [6, p. 130 and 135] and for the unramified $(\hat{R}_p, \alpha)$ in the proof of [7, Lemma 13]. Note here that for type U, if $\hat{i}_p, \hat{I}_p$ denote the subrings of integers in $\hat{e}_p, \hat{E}_p$ then because the extension is unramified ($p\notin B'$), we have $N(\hat{I}_p^\times)=\hat{i}_p^\times$.

§ 4. Based Quadratic Modules

We recall from [8] (or [6]) that a based quadratic module over any ring R is a triple (M, q, v) where (M, q) is a quadratic module and where v is an equivalence class of free R-bases on M, where bases $\{e_i\}$ and $\{f_i=\Sigma e_j p_{ij}\}$ are equivalent if $P=(p_{ij})$ represents 1 in $K_1(R)$. We denote by $\mathcal{B}\mathcal{Q}(R, \alpha, u)$ the category of nonsingular based quadratic modules.

In the product situation envisaged in earlier sections, suppose (1.2) holds, so that

$$K_1\mathcal{P}(A)=\prod(K_1\mathcal{P}(B_\alpha), K_1\mathcal{P}(C_\alpha)).$$

Then—as for $\mathcal{P}(A)\subset\mathcal{P}'(A)$ and $\mathcal{Q}(A)$—the category $\mathcal{B}\mathcal{Q}(A)$ is a full subcategory of the restricted direct product $\prod(\mathcal{B}\mathcal{Q}(B_\alpha), \mathcal{B}\mathcal{Q}(C_\alpha))$, so the arguments of earlier paragraphs remain relevant. However, since by [8, Prop. 10] for any ring A,

$$K_1\mathcal{B}\mathcal{Q}(A)=\operatorname{Ker} F\colon K_1\mathcal{Q}(A)\to K_1\mathcal{P}(A),$$

there is no need for a special study of $K_1\mathscr{B}\mathscr{Q}(A)$. Also, the very definition of $\mathscr{B}\mathscr{Q}$ provides a unique rank homomorphism $r\colon K_0\mathscr{B}\mathscr{Q}(A)\to\mathbb{Z}$, given by counting the number of elements in a basis. We denote its kernel by $\tilde{K}_0\mathscr{B}\mathscr{Q}(A)$. The result corresponding to (1.1) is now

Lemma 4.1. (i) *Suppose* $k\mathscr{B}\mathscr{Q}(C_\alpha)\to k\mathscr{B}\mathscr{Q}(B_\alpha)$ *injective for almost all* α. *Then* $k\mathscr{B}\mathscr{Q}(A)\to\prod\big(k\mathscr{B}\mathscr{Q}(B_\alpha), k\mathscr{B}\mathscr{Q}(C_\alpha)\big)$ *is injective. Hence if further* $k\mathscr{B}\mathscr{Q}(B_\alpha)$ *is a cancellation semigroup for all* α, *so is* $k\mathscr{B}\mathscr{Q}(A)$.

(ii) *Suppose* $K_0\mathscr{B}\mathscr{Q}(C_\alpha)\to K_0\mathscr{B}\mathscr{Q}(B_\alpha)$ *injective for almost all* α. *Suppose also for some integer* n *and finite subset* $S\subset I$ *that whenever* $\alpha\in(I-S)$ *and* M, M' *are objects representing the same element of* $K_0\mathscr{B}\mathscr{Q}(C_\alpha)$, *there is an isomorphism* $M\oplus H(C_\alpha^n)\to M'\oplus H(C_\alpha^n)$. *Then the map*

$$\omega\colon \tilde{K}_0\mathscr{B}\mathscr{Q}(A)\to\prod\big(\tilde{K}_0\mathscr{B}\mathscr{Q}(B_\alpha), \tilde{K}_0\mathscr{B}\mathscr{Q}(C_\alpha)\big)$$

is injective.

(iii) *Suppose further that for some integer* n_1, *every element*

$$x\in K_0\mathscr{B}\mathscr{Q}(C_\alpha),\qquad \alpha\in I-S,$$

with $r(x)\geqq n_1$, *is positive. Then* ω *is an isomorphism.*

The extension of the arguments proving (1.1) to cover this case is routine; we leave it to the reader.

Lemma 4.2. *The hypotheses of* (4.1) *hold for* $\mathscr{B}\mathscr{Q}(\hat{R})$ *and* $\mathscr{B}\mathscr{Q}(\hat{S})$. *More precisely,*

(i) *Cancellation holds in* $k\mathscr{B}\mathscr{Q}(\hat{S}_p)$ *and, if* $p\in P-B'$, *in* $k\mathscr{B}\mathscr{Q}(\hat{R}_p)$. *For any* p, *if* M_2 *and* M_3 *are objects of* $\mathscr{B}\mathscr{Q}(\hat{R}_p)$ *with the same class in* $K_0\mathscr{B}\mathscr{Q}(\hat{R}_p)$, *there is an isomorphism* $M_2\oplus H(\hat{R}_p)\cong M_3\oplus H(\hat{R}_p)$; *if* $H(\hat{R}_p)$ *is a summand of* M_2, *then* $M_2\cong M_3$.

(ii) *If* F *denotes* $k\mathscr{B}\mathscr{Q}$ *or* $K_0\mathscr{B}\mathscr{Q}$, *then* $F(\hat{R}_p)\to F(\hat{S}_p)$ *is injective for* $p\in P-B'$.

(iii) *Any* $x\in K_0\mathscr{B}\mathscr{Q}(\hat{R}_p)$ *with rank* $\geqq 1$ *is positive.*

Proof. (ii) again follows from [7, Lemma 13].

(i) First consider any antistructure (R,α,u) such that cancellation holds in $\mathscr{Q}(R)$. Then given an isomorphism

$$\psi\colon M_1\oplus M_2\to M_1\oplus M_3\quad\text{ in }\mathscr{B}\mathscr{Q},$$

it follows that there exists a $\mathscr{Q}$-isomorphism $\phi\colon M_2\to M_3$. Let ϕ have determinant δ with respect to preferred bases of M_2 and M_3. Then $\psi^{-1}\circ(1\oplus\phi)$ is an isometry of $M_1\oplus M_2$ with determinant δ, hence $\delta\in\operatorname{Im}\big(K_1\mathscr{Q}(R)\xrightarrow{F}K_1\mathscr{P}(R)\big)$. If there is an isometry χ of M_2 with determinant δ, $\phi\circ\chi^{-1}$ gives the desired based isometry $M_2\to M_3$.

To apply this to our case, first observe that by (3.2).(iv), in all cases $M_2 \oplus H(R)$ certainly admits such an isometry, so $M_2 \oplus H(R) \cong M_3 \oplus H(R)$ in $\mathscr{B}\mathscr{Q}$. If M_2 admits $H(R)$ as a summand, then M_2 has enough isometries, so $M_2 \cong M_3$.

Next, we can reduce to the case when $\hat{S}_p$ is a division ring, or when $p \in (P-B')$ and $\hat{R}_p$ is the ring of integers in a field. Since if M_2 is the zero module, it follows by cancellation for $\mathscr{P}$ that $M_3=0$, we may assume that M_2 has rank $\geqq 1$. We now consider cases. For type GL or Sp, or for type 0 when $\hat{S}_p$ is not a field, $\mathrm{Im}(K_1\mathscr{Q}(R) \to K_1\mathscr{P}(R))$ is zero, and there is nothing to prove. For type 0 otherwise, the image is $\{\pm 1\}$, and a representative automorphism is $e_1 \to -e_1$, $e_i \to e_i$ for $i>1$, where $\{e_i\}$ is an orthogonal base (such exist in these cases—note [7, Lemma 2, Corollary 2]). Finally for type U, $\hat{S}_p$ is necessarily a field, $K_1\mathscr{P}(\hat{S}_p)=\hat{S}_p^{\times}$, the image consists of elements a with $aa^{\alpha}=1$, and we consider $e_1 \to e_1 a$, $e_i \to e_i$ for $i>1$ as above: the same argument goes for the ring of integers $\hat{R}_p$ since p is odd.

(iii) Here we have an element of $K_0\mathscr{B}\mathscr{Q}(\hat{R}_p)$ whose image in $K_0\mathscr{Q}(\hat{R}_p)$ is positive, by (3.2).(iii): say it is represented by M. Using cancellation in $k\mathscr{P}(\hat{R}_p)$, we see that M is free; by hypothesis, its rank is >0. Now since $GL_1(\hat{R}_p)$ maps onto $K_1\mathscr{P}(\hat{R}_p)$, we may vary the base by any element of $K_1\mathscr{P}(\hat{R}_p)$. The result now follows from exactness of

$$K_1\mathscr{P}(\hat{R}_p) \to K_0\mathscr{B}\mathscr{Q}(\hat{R}_p) \to K_0\mathscr{Q}(\hat{R}_p)$$

[8, Prop. 10].

Corollary 4.3.

(i) $\tilde{K}_0\mathscr{B}\mathscr{Q}(\hat{R}) \cong \prod_p \tilde{K}_0\mathscr{B}\mathscr{Q}(\hat{R}_p)$,

(ii) $\tilde{K}_0\mathscr{B}\mathscr{Q}(\hat{S}) \cong \prod_p (\tilde{K}_0\mathscr{B}\mathscr{Q}(\hat{S}_p), \tilde{K}_0\mathscr{B}\mathscr{Q}(\hat{R}_p))$.

Again this yields explicit computations. Let D be a division ring finite over $\mathbb{Q}$, E its centre, (D, α, u) an antistructure and, if $\alpha|E$ is nontrivial, e its fixed field.

Proposition 4.4. *For type Sp,*

$$\pi\colon\ \tilde{K}_0\mathscr{B}\mathscr{Q}(\hat{D}) \cong \hat{E}^{\times}.$$

For type U,

$$\delta\colon\ \tilde{K}_0\mathscr{B}\mathscr{Q}(\hat{D}) \cong \hat{e}^{\times}.$$

For type 0,

$$\delta\colon\ \tilde{K}_0\mathscr{B}\mathscr{Q}(\hat{D}) \to \hat{E}^{\times}$$

is surjective, with kernel isomorphic to

$$\coprod \{\pm 1\},$$

the direct sum being extended over all primes of E.

Proof. This follows as for (3.5): only the final case needs comment. Here, we know that

$$\delta\colon\ \tilde{K}_0\mathscr{B}\mathscr{Q}(\hat{D}_{\mathfrak{p}})\to\hat{E}_{\mathfrak{p}}^{\times}$$

has kernel of order 2, and if $p\in(P-B')$, we reduce to the case where $\hat{R}_p=\hat{I}_{\mathfrak{p}}$ is the ring of integers in $\hat{E}_{\mathfrak{p}}$: here (cf. [7, Lemma 13])

$$\delta\colon\ \tilde{K}_0\mathscr{B}\mathscr{Q}(\hat{I}_{\mathfrak{p}})\to\hat{I}_{\mathfrak{p}}^{\times}$$

is an isomorphism. It now follows from (4.3) that δ is surjective. Further, Ker δ is also a restricted direct product, and since – as just noted – the subgroups are almost all zero, this reduces to a direct sum.

We now recall the basic L-theory definitions of [8]. First, set

$$\begin{aligned} K_i(R)&=K_i\mathscr{P}(R),\\ \Lambda_0(R,\alpha,u)&=\tilde{K}_0\mathscr{B}\mathscr{Q}(R,\alpha,u),\\ \Lambda_1(R,\alpha,u)&=K_1\mathscr{Q}(R,\alpha,u),\\ \Lambda_{i+2}(R,\alpha,u)&=\Lambda_i(R,\alpha,-u)\qquad\text{for all } i\in\mathbb{Z}. \end{aligned}$$

We have determinant maps $\delta_i\colon \Lambda_i(R)\to K_1(R)$ (δ_1 is induced by F) and twisting maps $\tau_i\colon K_1(R)\to\Lambda_i(R)$ (τ_1 is induced by H), and by [8, Lemma 9] $\delta_i\circ\tau_i=1+(-1)^i T$, where T is the involution of $K_1(R)$ induced by α. Moreover, the sequence

$$\Lambda_{i+1}(R)\xrightarrow{\delta_{i+1}}K_1(R)\xrightarrow{\tau_i}\Lambda_i(R)$$

is exact, for all i. We set

$$\begin{aligned} L_i^S(R)&=\operatorname{Ker}\delta_i,\\ L_i^K(R)&=\operatorname{Coker}\tau_i. \end{aligned}$$

Apart from the obvious period 4, we noted in [6, p. 124] that over simple rings (and the same now follows for adele rings) we have period 2 for type U, and the interchange of $\pm u$ interchanges types 0 and Sp. The groups can thus be tabulated as follows.

Corollary 4.5. *Let (D,α,u) be an antistructure with D a division ring, finite over $\mathbb{Q}$, with centre E. If $\alpha|E$ is nontrivial, let e denote its fixed field. Write J for the set of primes of E, J' for the subset (almost all) at which D splits. Then the L-groups of (D,α,u) are as follows:*

Type	L_1^S	Λ_1	L_1^K	L_0^S	Λ_0	L_0^K
U	0	$\operatorname{Ker} N\colon \hat{E}^{\times}\to\hat{e}^{\times}$	0	0	$\hat{e}^{\times}$	$\coprod_J\{\pm1\}$
Sp	0	0	0	$\coprod_J\{\pm1\}$	$\hat{E}^{\times}$	0
0	$E^{\times}/(E^{\times})^2$	$E^{\times}/(E^{\times})^2\oplus\prod_{J'}\{\pm1\}$	$\prod_{J'}\{\pm1\}$	$\prod_J\{\pm1\}$	δ_0 surjective	?

We tabulated Λ_1 in (3.5) and Λ_0 in (4.4) with sufficient precision to determine the δ_i and τ_i except for τ_0 for type 0. The other results are immediate consequences, except the L_i^K for type U, where we have used class field theory (for the extension E/e) to simplify the result.

The table indicates also the corresponding results over the $\hat{D}_p$. It should be augmented by information at real primes (in [6], there is also a summary at the end of [8]) and the reminder that for type GL – i.e. a 'double ring' $R \oplus R^{\mathrm{op}}$, interchanged by α – we have $\Lambda_i(R \oplus R^{\mathrm{op}}) \cong K_1(R)$, $L_i^S = L_i^K = 0$.

§ 5. Global Fields and Adele Rings

In our previous paper [6] we gave a list of calculations of L-groups for the semisimple rings $\hat{S}_p$, T and S: summaries of these appeared as follows

p. 130 $K_1, K_1\mathcal{Q} = \Lambda_1$	p. 135 $\tilde{K}_0\mathcal{Q} \;= L_0^K$
p. 137 $K_1\mathcal{B}\mathcal{Q} \;= L_1^S$	p. 138 $\tilde{K}_0\mathcal{B}\mathcal{Q} = \Lambda_0$
p. 140 $\tilde{K}_0\mathcal{L} \;= L_0^S$,	

and the (easier) results for L_1^K are given by [8, Theorem 5].

The details of some of these calculations are fairly complicated. In this section we compare the K and L groups for the global algebra S with those for the adele ring

$$S_A = S \otimes \mathbb{Q}_A = S \otimes (\hat{\mathbb{Q}} \oplus R) = \hat{S} \oplus T, \quad \text{say}.$$

The arguments below will not use the full results of [6], and it is of interest to note what we really need: the following three items would suffice.

(1) Determination of the groups K_1, $K_1\mathcal{Q}$ (the Kneser-Tits conjecture). This is as well documented in [6] as anywhere.

(2) The Hasse principle for H^1 of the connected algebraic groups U, SU, Sp, SO – or the Hasse principle for forms of type U or Sp and for based forms of type 0. This is discussed in [4] and [5].

(3) The result that for G a simply connected algebraic group and K a global field,

$$H^1(K; G) \cong \Pi\, H^1(\hat{K}_v; G),$$

where v runs through the real places of K. This also is discussed in [4] and [5]: it really contains (2) above.

These are the deepest theorems: we deduce the rest, even the structure of $\operatorname{Ker}(L_0^K(S) \to L_0^K(S_A))$.

Lemma 5.1 *(Hasse principle). Let F denote any of the functors K_i, L_i^S, Λ_i, L_i^K $(i = 0, 1)$. Then $F(S) \to F(S_A)$ is injective unless $F = L_0^K$, some* *simple component of (S, α, u) has type OD, and the quaternion division ring D is ramified at more than two spots.*

Proof. This result is well-known in Galois cohomology. It also follows from [6] as, with the exception named, we gave invariants to detect the groups and all these can be locally detected. Indeed, this is obvious except perhaps when the invariant is a class modulo squares or modulo norms from a quadratic extension. However, it is known – see e.g. [3, pp. 180, 185] – that a number which is locally a square resp. norm from a quadratic extension is also one globally.

In the exceptional case, if S is a matrix ring over the division ring D with centre E, the kernel can be identified with the quotient $\Theta(D)$ of the subgroup of $Br_2(E)$ of algebras split at places where D is by the class of D itself. Thus if D is ramified at $2r$ spots, $\Theta(D)$ has 2-rank $2r-2$. This result is due to M. Kneser; a proof will be found in (5.6) below.

Our further efforts will be devoted to determining the cokernel $CF(S)$ of the natural map $F(S) \to F(S_A)$, for the listed functors F. The group K_0 being uninteresting, we begin with K_1. We may suppose S simple, with centre E. According to [6],

$$\mathrm{Nrd}\colon\ K_1(S) \to E^\times$$

is injective, with corresponding results for $\hat{S}_p$ and T, and by (2.3) also for $\hat{S}$ and S_A. Further, the map is surjective for $\hat{S}_p$ and $\hat{S}$: for T the image is the subset of $(E \otimes \mathbb{R})^\times$ of elements whose components at places where T ramifies are positive, and for S, S_A it is defined by this same condition on signs. Write E^*, E_A^* for the subgroups of E, E_A defined by this condition. Since the signs at real spots of elements of $E^\times$ are independent, $E^\times \cdot E_A^* = E_A^\times$. Hence

$$CK_1(S) = E_A^*/E^* \cong E_A^\times/E^\times,$$

the idele class group (*not* ideal class group) of E, which we will denote by C. In particular, this depends only on the centre, E of S.

For the other cases, the exact sequence

$$0 \to L_{i+1}^S \to \Lambda_{i+1} \to K_1 \to \Lambda_i \to L_i^K \to 0$$

of [8] will be useful. We have exact sequences for S and for S_A: what of the cokernel?

Lemma 5.2. *Let $f_*\colon A_* \to B_*$ be a map of exact sequences: let $f_n\colon A_n \to B_n$ have kernel X_n, image Y_n and cokernel Z_n. Then $H_n(Z) \cong H_{n-1}(Y) \cong H_{n-2}(X)$.*

Proof. We have exact sequences of chain complexes

$$0 \to X_* \to A_* \to Y_* \to 0$$
$$0 \to Y_* \to B_* \to Z_* \to 0,$$

which thus have exact homology sequences. Since A_*, B_* have zero homology groups, these yield the desired isomorphisms.

Corollary. *The sequence*

$$F_i(S):\ 0 \to CL^S_{i+1}(S) \to CA_{i+1}(S) \to CK_1(S) \to CA_i(S) \to CL^K_i(S) \to 0$$

is exact for S simple, unless (S, α, u) *has type OD (resp. Sp D) and* $i \equiv 0$ *(resp.* 2*)(mod* 4*) when the only nonzero homology group is*

$$\operatorname{Ker}\big(L^K_i(S) \to L^K_i(S_A)\big),$$

at the middle point.

This follows on applying the lemma as indicated, and using (5.1).

For the remaining discussion, we can split up (S, α, u) into its simple components. We have effectively discussed type GL already. Types 0 and Sp are interchanged by changing u to $-u$ [6, p.124], hence by adding 2 to i—we have already referred to this. Next we discuss type U. Write, as above, E for the centre of S. For any abelian group M on which $\mathbb{Z}/2\mathbb{Z}$ acts (by α), write

$$S^\varepsilon(M) = \{m \in M : m^\alpha = \varepsilon m\} \qquad \text{for } \varepsilon = \pm 1.$$

Proposition 5.3. *For S of type U.* $CA_0(S) = S^+(C)$, $CA_1(S) = S^-(C)$, $CL^S_0(S) = CL^S_1(S) = CL^K_1(S) = 0$, $CL^K_0(S) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. From [6] by inspection, or by the Galois cohomology for SU,

$$CL^S_0(S) = CL^S_1(S) = 0.$$

Thus $CA_i(S) \to CK_1(S)$ is injective. Since the composite $K_1 \to A_i \to K_1$ is $(1 + (-1)^i \alpha)$, by exactness of $F_{i-1}(S)$,

$$\begin{aligned} CA_i(S) &= \operatorname{Ker}\big(CK_1(S) \to CA_{i-1}(S)\big) \\ &= \operatorname{Ker}\big(1 + (-1)^{i-1}\alpha\big) = S^{(-1)^i}\, CK_1(S) = S^{(-1)^i}\, C. \end{aligned}$$

Now the cokernel

$$CL^K_i(S) = H^i(\mathbb{Z}/2\mathbb{Z};\ C)$$

and by class field theory, this has order 2 for i even, 1 for i odd. In fact, $CL^K_1(S) = 0$ by inspection.

Also by inspection, $CA_0(S)$ is the idele class group of the fixed field of α on E. It is easy to identify this directly with $S^+(C)$. We will find the cohomology of the idele class group coming up again later.

Before discussing the other cases, we describe an extension of the Pfaffian invariant, discovered too late to be included in [6]. This is not really needed for our results here, but helps in understanding the calculations in [6] of the relevant groups.

Recall that for K a field, the antistructure $(K, 1, 1)$ has $\Lambda_3 = L_2^K = 0$, so

$$\tau\colon\ K^\times = K_1(K) \to \Lambda_2(K)$$

is an isomorphism: we defined the Pfaffian π to be the inverse isomorphism, and observed that this definition extended in part to some other cases. We now discuss type Sp over a quaternion algebra D, with centre E. Let c be the canonical conjugation on D, so $(D, c, 1)$ has type Sp: we study this antistructure.

Proposition 5.4. *There is a natural homomorphism* $\pi\colon K_0\mathscr{B}\mathscr{Q}(D, c, 1) \to E^\times$ *such that*

i) $\pi \circ \tau = identity$.

ii) *The discriminant* $\delta = \pi^2$.

iii) *If* $D = \mathbb{H}$, *and* $x \in \tilde{K}_0\mathscr{B}\mathscr{Q}$ *has signature* 2ρ, *then* $(-1)^\rho \pi(x) > 0$.

Proof. By standard arguments, any form admits a (preferred) orthogonal base, and we can pass between two such bases by intermediate steps in each of which only two basis vectors are changed.

If $\{e_i\}$ is an orthogonal base for a form θ, set $a_i = b_\theta(e_i, e_i)$. Then $a_i \in E$, and $a_i \neq 0$ if θ is nonsingular. Define $\pi(\theta) = \Pi a_i$. To check that this is independent of choice of base it suffices (by the above) to consider the 2-dimensional case: this is a straightforward computation. It is then clear that we have a naturally defined homomorphism.

For (i), note that if e_1 is replaced by $e_1 d$ in a preferred base, this describes $\tau(\mathrm{Nrd}\, d)$, and a_1 is multiplied by $c(d)d = \mathrm{Nrd}\, d$. For (ii), it suffices to observe that $\delta(\theta) = \Pi(\mathrm{Nrd}\, a_i) = \pi(\theta)^2$. And (iii) follows as the sign of $\pi(\theta)$ is $(-1)^n$, where n is the number of negative a_i; if p is the number of positive ones, $p+n$ is the rank and $p-n$ the signature.

This proof does not exclude the case when D is split, when (i) shows that we have the same invariant as before. But (iii) shows that otherwise the image of π need *not* lie in $\mathrm{Nrd}\, D^\times = E^*$. From [6] with the above we see that in the global case, $\Lambda_0(D, c, 1)$ is detected by π and the signatures, and these are related only by (iii). This is a complete description, and implies the result for L_0^S.

After this, we return to the problem of determining the CL_i^S, $C\Lambda_i$ and CL_i^K for (S, α, u) simple of type 0.

Theorem 5.5. *We have the following table, for* (S, α, u) *of type* 0.

$i \pmod 4$	0	1	2	3
$CL_i^S(S)$	$\mathbb{Z}/2\mathbb{Z}$	C_2	${}_2C$	0
$C\Lambda_i(S)$	$\tilde{C}$	$C_2 \oplus \mathsf{X}(D)$	C	0
$CL_i^K(S)$		$\mathsf{X}(D)$	0	0

where there are exact sequences

$$0 \to {}_2C \to C \xrightarrow{2} C \to C_2 \to 0$$
$$0 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{C} \to C \to 0$$
$$0 \to \mathrm{X}(D) \to {}_2C \to \Theta'(D) \to 0,$$

and $\Theta'(D)=0$ *if* D *is commutative, and contains* $\Theta(D)$ *as a subgroup of index* 2 *otherwise.*

We determine $CL_0^K(S)$ in (5.6) below.

Observe the appearance of the cohomology groups C_2, ${}_2C$ of C.

Proof. We have $\Lambda_3(S_A)=0$, hence $C\Lambda_3(S)=0$ and $CL_3^S(S)=CL_3^K(S)=0$.

Next, by inspection of [6, p. 135] or by the Galois cohomology of Sp, $CL_2^K(S)=0$. The exact sequence $F_2(S)$ thus reduces to an isomorphism $C=CK_1(S)\to C\Lambda_2(S)$. But the composite $CK_1(S)\to C\Lambda_2(S)\to CK_1(S)$ is multiplication by 2. Putting this in $F_1(S)$, we see that $CL_2^S(S)\cong {}_2C$, and that

$$C_2 \cong \mathrm{Im}\big(CK_1(S)\to C\Lambda_1(S)\big)\cong \mathrm{Ker}\big(C\Lambda_1(S)\to CL_1^K(S)\big).$$

Now again by inspection, we see for S and for S_A, hence for the cokernel, that

$$\Lambda_1 = L_1^S \oplus L_1^K \text{ (naturally)}.$$

Thus $CL_1^S(S)\cong C_2$, and if we write $CL_1^K(S)=\mathrm{X}(D)$ we have $C\Lambda_1(S)\cong C_2\oplus \mathrm{X}(D)$.

For a division ring D over any field, $L_1^K(D)$ has order 2 or 1 according as D is commutative or not. Thus if $S=E$, $L_1^K(S_A)$ is a sum of groups of order 2, one for each place of E, which we can identify with ${}_2E_A^\times$. Factoring out the diagonal $L_1^K(S)\cong {}_2E^\times$ now gives ${}_2C$. For if the class of an idele x has order 2, $x^2=y$ with $y\in E^\times$, we see that y is locally, hence globally a square; say $y=z^2$, so our class contains the idele x/z of order 2. Suppose on the other hand $S=D+E$. Then

$$\mathrm{X}(D)=CL_1^K(S)\cong L_1^K(S_A)\cong \Pi\,\{\pm 1 : D \text{ split at } \mathfrak{p}\}.$$

This maps injectively to ${}_2C$, which is a direct product of $\{\pm 1\}$ at all $\mathfrak{p}$, modulo the diagonal: the quotient $\Theta'(D)$ is a sum of $\{\pm 1\}$ at primes where D ramifies, modulo the diagonal. This contains an obvious isomorph of $\Theta(D)$ as a subgroup of index 2.

Finally, $CL_0^S(S)\cong \mathbb{Z}/2\mathbb{Z}$ by the concluding result of [6], or by the Galois cohomology of Spin. If we now define $\tilde{C}=C\Lambda_0(S)$, our remaining exact sequence is given by $F_3(S)$.

We now investigate $F_0(S)$ which, in view of the splitting of $C\Lambda_1(S)$, we can replace by

$$0\to CL_1^K(S)\to CK_1(S)\xrightarrow{\tau} C\Lambda_0(S)\to CL_0^K(S)\to 0.$$

Since the composite $C = CK_1(S) \to C\Lambda_0(S) \to C$ is multiplication by 2, τ induces a map

$$\tau_0\colon {}_2C \to \mathbb{Z}/2\mathbb{Z} = CL_0^S(S) = \operatorname{Ker}(C\Lambda_0(S) \to C),$$

which we propose to determine. Now for any localisation $\hat{S}_L$ of S (including infinite primes) we have a commutative diagram

$$\begin{array}{ccc} \{\pm 1\} = {}_2\hat{E}_L^\times & \xrightarrow{\tau_L} & L_0^S(\hat{S}_L) \\ \downarrow & & \Vert \\ {}_2C & \xrightarrow{\tau_0} & CL_0^S(S), \end{array}$$

so it suffices to determine the τ_L. But since the image of $\Lambda_1(\hat{S}_L) \to K_1(\hat{S}_L)$ is isomorphic to $L_1^K(\hat{S}_L)$, hence nontrivial if and only if D splits L, we see that $\tau_L = 0$ if and only if D_L is split.

Hence, (as it must) τ_0 annihilates the subgroup $\mathsf{X}(D)$ of classes of elements $=1$ at spots where D ramifies. On the quotient $\Theta'(D)$, the kernel is the set of classes with an even number of components -1, viz $\Theta(D)$. This, with (5.2), Corollary, gives a computation of $\operatorname{Ker}(L_0^K(S) \to L_0^K(S_A))$. Also, $\tau_0 = 0 \Leftrightarrow D = E$: in the other cases, the cokernel CL_0^K of τ is $\cong C_2$, but in the commutative case, we have an extension of $\mathbb{Z}/2\mathbb{Z}$ by C_2. We can determine the extension by local considerations, as in the above calculation of τ (similarly we can determine $\tilde{C}$).

Addendum 5.6. *Let (S, α, u) have type* 0. *There is an epimorphism $CL_0^K(S) \to C_2$ with kernel of order* 1 *(type* 0*D) or* 2 *(type* 0*K).*

Our results are particularly neat for $CL_i^S(S)$: this is of exponent 2, and depends only on the centre of S. The statements for $CL_i^K(S)$ could probably be simplified by using relative groups: in the exact sequence

$$0 \to L_1^K(S) \to L_1^K(S_A) \xrightarrow{\alpha} L_1^K(S_A, S) \xrightarrow{\beta} L_0^K(S) \to L_0^K(S_A),$$

the image of α is $\mathsf{X}(D)$, the image of β is $\Theta(D)$. Presumably $L_1^K(S_A, S)$ could be identified with

$$\operatorname{Ker}\tau\colon {}_2C \to CL_0^S(S).$$

It hardly seems worth doing this, however. There are no relative Λ groups: the Λ_i belong to KU theory, and the relative KU groups fit in a rather different exact sequence.

To complete our results we should describe, with respect to the invariants listed, the epimorphisms $F(S_A) \to CF(S)$. This is not very difficult, and we leave it to the reader: in most cases, it involves using the discriminant (spinor norm, Pfaffian). The above is already more than adequate for the applications we have in mind.

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