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On the Classification of Hermitian Forms

V. Global Rings

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Introduction

The first half of this paper gives a reformulation of the standard number-theoretic technique for classifying forms over rings of algebraic integers (see [24] for the unitary case) to apply in the generality we require for integer group rings of finite groups. This technique consists essentially of using the strong approximation theorem for algebraic groups to pass from the local classification to the global, thus reducing the problem to a comparatively easy calculation over complete semilocal rings [26]. It can be axiomatised, and a convenient form of the axioms (cf. Sullivan [21]) groups together the completions at finite primes, thus giving the square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \hat{\mathbb{Z}} & \longrightarrow & \hat{\mathbb{Q}} \ (=\text{finite adeles}) \end{array}$$

Although we point out that our methods yield useful unstable results, the main theorems are formulated in terms of groups K_0 and K_1 of categories of hermitian forms, or more precisely in terms of algebraic L-theory. The definitions are reproduced in [27] and again below, but the reader may prefer to refer to [28], which gives an exposition of the foundational concepts: the alternative account [19] uses a slightly different notation.

Our main result (6.6) is as follows. Let (R, α, u) be an antistructure (i.e. essentially a ring with anti-automorphism) such that R is finitely generated as $\mathbb{Z}$-module and $S = R \otimes \mathbb{Q}$ is semisimple. Write $\hat{R} = R \otimes \hat{\mathbb{Z}}$, $\hat{S} = R \otimes \hat{\mathbb{Q}}$.

Use X to denote L groups with $K_1(\hat{S})$ as value group. Then we have a long exact sequence

$$\dots L_{i+1}^X(\hat{S}) \to L_i^X(R) \to L_i^X(\hat{R}) \oplus L_i^X(S) \to L_i^X(\hat{S}) \dots.$$

The technique mentioned above leads to a proof of exactness of the part of the sequence from $L_1^X(R)$ to $L_0^X(\hat{S})$: this may be regarded as a modified version of the "Hasse principle", which it replaces with advantage, since the sequence is always exact: the principle holds when the map $L_1^X(\hat{S}) \to L_0^X(R)$ is zero. The L-theoretic notation has the further advantage of suggesting the extension to higher values of i, which is invaluable for calculations.

These arguments admit some immediate generalisations. First, we can consider the "function field" type of local ring, with $\mathbb{Z}$ replaced throughout by $\mathbb{F}_p[t]$, a polynomial ring over a finite field of odd characteristic (some, but not all our results persist for p even). Second, we can replace $\mathbb{Z}$ by any localisation, at primes $p \in \pi$. These primes can then be omitted in forming the products $\hat{R}$ and $\hat{S}$. One can also replace $\mathbb{Z}$ by the ring I of integers in an algebraic number field: this does not increase the class of rings R in the original version, but does in the localised case. In all these cases, the arguments of this paper remain valid almost without change. We shall not spell out the details of such alterations: they would confuse the exposition, and the interested reader can supply them for himself. Moreover, Mr. Ranicki has communicated to the author a remarkable argument giving a sweeping generalisation of our main exact sequence (though not of our unstable results): this depends on more advanced algebraic L-theory.

Here is a more detailed description of the contents of the paper. We begin with a discussion of pullback categories *in abstracto*, quote an exact sequence of Bass [3], and formulate a list of seven axioms which imply, as well as this, results of classification, stability and cancellation of objects in the pullback category: results of standard type, but which gain in clarity from this approach.

The next three sections are devoted to verifying these axioms for the categories $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{B}\mathscr{Q}$ of projective, quadratic and based quadratic modules over $R, S, \hat{R}$ and $\hat{S}$. In § 2, we recall definitions, and show that various diagrams are pullbacks: this property holds rather generally. In § 3, we verify the axioms for categories $\mathscr{P}$ and $\mathscr{Q}$, and it is here that the strong approximation theorem is needed at a crucial point. In § 4 we consider based quadratic modules, and give the application to L-theory.

In all, several exact sequences are obtained knowledge of which, in combination with the unstable results, implies the answers to such questions as: which projective R-modules admit nonsingular forms? Which forms are in the same genus as a form on a free module? Which forms over S contain nonsingular forms over R? What are the discriminants of nonsingular forms over R? How do different classifications of types of nonsingular form compare? When does a form contain another as direct summand?

To define the boundary map $L_2^X(\hat{S}) \to L_1^X(R)$ and show that it fits into the exact sequence needs new ideas, not found in the classical literature, and here some acknowledgements are in order. Work of Petrie [17] and

Passmann and Petrie [16] showed how to use effectively the linking form method of [22] for odd dimensional surgery problems. Then Connolly [9] and Gough [11] studied when one could do surgery to get a rational homotopy equivalence (and thus reduce to a linking form problem). According to [22], surgery is possible if one can construct an (integral) skew-hermitian form with certain properties. It is natural to assume this possible locally at each prime, and try and fit together. Now we can change our local forms by adding something unimodular, and it is sufficient to fit the forms rationally, as we then have a lattice in the result. The obstruction is thus an element of the cokernel of

$$L_2(\hat{R}) \oplus L_2(S) \to L_2(\hat{S}),$$

represented by a (singular but nondegenerate) form over $\hat{R}$. There is a proof along these lines, using linking numbers, but the version below using formations involves fewer new concepts. We recall basic results on formations in § 5 and prove a pullback theorem, giving a bijection between S-split formations over R and $\hat{S}$-split formations over $\hat{R}$. The proof of (6.6) is then completed in § 6.

Finally in § 7 we reformulate our main exact sequence, and show that $L_i^X(R) \to L_i^S(R \otimes \mathbb{R})$ has finite kernel and cokernel. The cokernel has exponent 2, and the torsion subgroup of $L_i^X(R)$ exponent dividing 8. Corresponding results then also hold for surgery obstruction groups: these will be discussed in another paper.

§ 1. Abstract Theory

Given categories and functors $\mathscr{B} \xrightarrow{F} \mathscr{D} \xleftarrow{G} \mathscr{C}$, the *fibre product* (pullback) of F and G is [3, p. 358] the category $\mathscr{A}$ whose objects are triples (B, C, ϕ) with B, C objects of $\mathscr{B}, \mathscr{C}$ respectively and $\phi: F(B) \to G(C)$ an isomorphism. A morphism $(B, C, \phi) \to (B', C', \phi')$ in $\mathscr{A}$ is a pair (β, γ) where $\beta: B \to B'$, $\gamma: C \to C'$ are morphisms in $\mathscr{B}, \mathscr{C}$ respectively such that $\phi' \circ F(\beta) = G(\gamma) \circ \phi$, i.e. such that diagram (1) commutes

$$\begin{array}{ccc} F(B) & \xrightarrow{\phi} & G(C) \\ \downarrow{\scriptstyle F(\beta)} & & \downarrow{\scriptstyle G(\gamma)} \\ F(B') & \xrightarrow{\phi'} & G(C') \end{array} \qquad (1)$$

We also say that the diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{G'} & \mathscr{B} \\ \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} \\ \mathscr{C} & \xrightarrow{G} & \mathscr{D} \end{array} \qquad (2)$$

where F' and G' are the obvious forgetful functors, and ψ the natural equivalence $FG' \to GF'$, is a fibre product diagram, or that it is Cartesian. It is not essential that $\mathscr{A}$ be set theoretically as above: any equivalent category will do. In order to recognise such diagrams we have

Lemma 1.1. *A homotopy commutative square* (2) *is a fibre product diagram if and only if*

(A1) *Given objects B, C of $\mathscr{B}, \mathscr{C}$ and an isomorphism $\phi: F(B) \to G(C)$, there exist an object A of $\mathscr{A}$ and isomorphisms $\beta: G'(A) \to B$, $\gamma: F'(A) \to C$ such that $\phi \circ F(\beta) = G(\gamma) \circ \psi(A)$.*

(A2) *Given objects A, A' of $\mathscr{A}$ and morphisms $\beta: G'(A) \to G'(A')$, $\gamma: F'(A) \to F'(A')$ with $\psi(A') \circ F(\beta) = G(\gamma) \circ \psi(A)$, there is one and only one morphism $\alpha: A \to A'$ with $G'(\alpha) = \beta$, $F'(\alpha) = \gamma$.*

Proof. Let $\mathscr{S}$ denote the fibre product of F and G as defined above. We define a functor $H: \mathscr{A} \to \mathscr{S}$ by $H(A) = (G'(A), F'(A), \psi(A))$, $H(\alpha) = (G'(\alpha), F'(\alpha))$. Then (A1) above shows that for any object S of $\mathscr{S}$ there is an object A of $\mathscr{A}$ and an isomorphism $(\beta, \gamma): H(A) \to S$; and (A2) shows that if A, A' are objects of $\mathscr{A}$, then H induces a bijection $\mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{S}}(H(A), H(A'))$. Hence H is an equivalence of categories.

From now on, we will assume that we have a fibre product diagram (2). We define two equivalence relations on objects of $\mathscr{A}$. Objects A, A' are said to be in the same (isomorphism) *class* if there is an $\mathscr{A}$-isomorphism $A \to A'$, and in the same *genus* if we just have a $\mathscr{B}$-isomorphism $G'(A) \to G'(A')$ and a $\mathscr{C}$-isomorphism $F'(A) \to F'(A')$. Clearly objects in the same class are in the same genus; we next describe the classes in a genus. Denote (for $\mathscr{F} = \mathscr{A}, \mathscr{B}, \mathscr{C}$ or $\mathscr{D}$) by $\mathrm{Aut}(A, \mathscr{F})$ the group of $\mathscr{F}$-automorphisms of the image of A in $\mathscr{F}$ (i.e. A, $G'(A)$, $F'(A)$ or $GF'(A)$), and write $\mathrm{Aut}'(A, \mathscr{C}) = G(\mathrm{Aut}(A, \mathscr{C}))$ and

$$\mathrm{Aut}'(A, \mathscr{B}) = \psi(A) \circ F(\mathrm{Aut}(A, \mathscr{B})) \circ \psi(A)^{-1}$$

for the image subgroups of $\mathrm{Aut}(A, \mathscr{D})$.

Lemma 1.2. *There is a natural bijection between the set of classes in the genus of A and the set of double cosets in* $\mathrm{Aut}(A, \mathscr{D})$ *of the images* $\mathrm{Aut}'(A, \mathscr{B})$, $\mathrm{Aut}'(A, \mathscr{C})$.

Proof. Let A_1 be in the genus of A. Then we have isomorphisms

$$\beta_1: G'(A) \to G'(A_1), \qquad \gamma_1: F'(A) \to F'(A_1):$$

their images in $\mathscr{D}$ are isomorphisms. We let A_1 correspond to

$$\psi(A) F(\beta_1)^{-1} \psi(A_1)^{-1} G(\gamma_1) \in \mathrm{Aut}(A, \mathscr{D}).$$

Suppose A_2 (with similar notation) is in the class of A_1; let $\alpha: A_1 \to A_2$ be an isomorphism. Then

$$\beta = \beta_2^{-1} G'(\alpha)\, \beta_1 \in \operatorname{Aut}(A, \mathscr{B}), \qquad \gamma = \gamma_2^{-1} F'(\alpha)\, \gamma_1 \in \operatorname{Aut}(A, \mathscr{C}),$$

and so

$$\begin{aligned} \psi(A)\, F(\beta_1)^{-1} \psi(A_1)^{-1} G(\gamma_1) &= \psi(A) F(\beta_1)^{-1} F G'(\alpha)^{-1} \psi(A_2)^{-1} G F'(\alpha) G(\gamma_1) \\ &= \psi(A)\, F(\beta)^{-1}\, F(\beta_2)^{-1} \psi(A_2)^{-1} G(\gamma_2)\, G(\gamma) \end{aligned} \tag{3}$$

is in the same double coset as $\psi(A)\, F(\beta_2)^{-1} \psi(A_2)^{-1} G(\gamma_2)$. Thus we do have a map of the sets indicated.

Conversely, given A_2 in the genus of A, if it does determine the same double coset as A_1, then (3) holds for suitable β and γ. Hence

$$\psi(A_2)\, F(\beta_2 \beta \beta_1^{-1}) = G(\gamma_2 \gamma \gamma_1^{-1})\, \psi(A_1).$$

By Lemma 1.1 (ii), there exists $\alpha: A_1 \to A_2$ with $G'(\alpha) = \beta_2 \beta \beta_1^{-1}$, $F'(\alpha) = \gamma_2 \gamma \gamma_1^{-1}$. The same argument gives us an α^{-1}, and so shows that α is an isomorphism. So our map is injective.

Finally, given any $\delta \in \operatorname{Aut}(A, \mathscr{D})$, by Lemma 1.1 (i) there exist an object A_1 of $\mathscr{A}$ and isomorphisms $\beta_1: G'(A) \to G'(A_1)$, $\gamma_1: F'(A) \to F'(A_1)$ such that $\delta = \psi(A)\, F(\beta_1)^{-1} \psi(A_1)^{-1} G(\gamma_1)$. Thus the correspondence is surjective.

It is inconvenient to compute double cosets: for practical computation one needs something simpler. Moreover, we do not want to have a separate computation for each genus. Fortunately, the cases of interest to us have more structure.

(A3) *The categories in* (2) *are categories with product in the sense of* [3, *Chapter VII*, § 1], *and the functors are product-preserving functors.*

We are now aiming at the exact sequence of [3, Theorem 4.3]. However, our categories satisfy somewhat stronger stability conditions then are necessary to obtain this sequence: as these are no harder to check than Bass' axioms, and have further consequences, we present a modified version. We need some condition of cofinality: the following certainly implies that (F, G) is a cofinal pair of functors [3, p. 360].

(A4) *There is a cofinal sequence* $\{A_n\}$ *of objects of* $\mathscr{A}$ *whose images in* $\mathscr{B}, \mathscr{C}, \mathscr{D}$ *are cofinal in these categories.*

Finally we must include some version of Bass' "E-surjectivity". This is, indeed, the crux of the whole business as far as establishing the exact sequence for computing $K_0(\mathscr{A})$ is concerned, as can be seen from Lemma 1.2.

For A an object of $\mathscr{A}$ and $\mathscr{F} = \mathscr{A}, \mathscr{B}, \mathscr{C}$ or $\mathscr{D}$ we have the natural map

$$p(A, \mathscr{F}): \operatorname{Aut}(A, \mathscr{F}) \to K_1(\mathscr{F})$$

with kernel $E\,\mathrm{Aut}(A,\mathscr{F})$, say. We call A *stable* if $p(A,\mathscr{F})$ is surjective for $\mathscr{F}=\mathscr{B},\mathscr{C}$ and $\mathscr{D}$ (but not necessarily for $\mathscr{A}$). It is clear that if A is stable, so is $A\oplus A'$ for any A'. Also, if A is stable, so is anything in the same genus.

(A 5) *$\mathscr{A}$ contains a stable object.*

We call A *good* if

$$E\,\mathrm{Aut}'(A,\mathscr{B})\cdot E\,\mathrm{Aut}'(A,\mathscr{C})=E\,\mathrm{Aut}(A,\mathscr{D}),$$

and *very good* if $A\oplus A'$ is good for all A'. Good objects are "usually" very good, but the trivial object may also be good. If A is stable and A' is very good, then $A\oplus A'$ is both. Note that by the product $X\cdot Y$ of two subsets of a group, we mean $\{xy\colon x\in X \text{ and } y\in Y\}$.

(A 6) *$\mathscr{A}$ contains a very good object.*

We recall that on [3, p. 360] Bass defines the concept of E-surjectivity. The condition translates into our notation as follows: For any object A of $\mathscr{A}$ and element ε in the commutator subgroup of $\mathrm{Aut}(FG'A)$, we can find an object A' of $\mathscr{A}$, and elements $\varepsilon_1,\varepsilon_2$ in the commutator subgroups of $\mathrm{Aut}\,G'(A\oplus A')$, $\mathrm{Aut}\,F'(A\oplus A')$ respectively with

$$\varepsilon\oplus 1=\psi(A\oplus A')\,F\varepsilon_1\,\psi(A\oplus A')^{-1}\,G\varepsilon_2. \qquad (*)$$

The main difference of this from (A 6) is the change in emphasis from $E\,\mathrm{Aut}$ to commutator subgroups. However, by definition of $K_1\mathscr{F}$, we see that for any object X of $\mathscr{F}$ and $\xi\in\mathrm{Aut}\,X$ we have $\xi\in E\,\mathrm{Aut}\,X$ if and only if, for some object X' of $\mathscr{F}$, $\xi\oplus 1$ lies in the commutator subgroup of $\mathrm{Aut}(X\oplus X')$. Here we can of course choose X' to belong to a preassigned cofinal sequence of objects. Hence suppose A,ε as above. If A' is very good, we have $(*)$, but with $\varepsilon_1\in E\,\mathrm{Aut}'(A\oplus A',\mathscr{B})$, $\varepsilon_2\in E\,\mathrm{Aut}'(A\oplus A',\mathscr{C})$. Thus for n sufficiently large, ε_1 belongs to the commutator subgroup of $\mathrm{Aut}\,G'(A\oplus A'\oplus A_n)$; similarly for ε_2. This proves E-surjectivity.

Hence by [3, Chapter VII, Theorem 4.3], we have

Theorem 1.3. *If diagram* (2) *satisfies Axioms* (A 1)–(A 6), *there is a canonical exact sequence*

$$K_1(\mathscr{A})\to K_1(\mathscr{B})\oplus K_1(\mathscr{C})\to K_1(\mathscr{D})\to K_0(\mathscr{A})\to K_0(\mathscr{B})\oplus K_0(\mathscr{C})\to K_0(\mathscr{D}).$$

We can, however, go further and classify certain objects up to isomorphism. For this we need one further axiom (restrictive enough in principle, but true for our cases).

(A 7) *If $\mathscr{F}=\mathscr{B},\mathscr{C}$ or $\mathscr{D}$, cancellation holds in $\mathscr{F}$ – i.e. two objects of $\mathscr{F}$ determining the same element of $K_0(\mathscr{F})$ are isomorphic.*

Throughout this paper, if $\mathscr{F}$ is a category with product, we shall write $k(\mathscr{F})$ for the additive semigroup of isomorphism classes of objects of $\mathscr{F}$. There is a natural homomorphism of $k(\mathscr{F})$ to its universal group $K_0(\mathscr{F})$: (A7) states that this is injective if $\mathscr{F}=\mathscr{B},\mathscr{C}$ or $\mathscr{D}$. One cannot expect it to be surjective. An element of $K_0(\mathscr{F})$ is called *positive* if it is in the image of $k(\mathscr{F})$. Write $G=\operatorname{Im}(K_1(\mathscr{D})\to K_0(\mathscr{A}))=\operatorname{Ker}(K_0(\mathscr{A})\to K_0(\mathscr{B})\oplus K_0(\mathscr{C}))$, and let $k_A(\mathscr{A})$ denote the subset of $k(\mathscr{A})$ corresponding to objects in the genus of A. By subtracting the class of A, we obtain a map $\phi_A\colon k_A(\mathscr{A})\to G$.

Theorem 1.4. (i) *An element of $K_0(\mathscr{A})$ whose images in $K_0(\mathscr{B})$ and $K_0(\mathscr{C})$ are positive differs by an element of G from a positive element.*

(ii) *Two objects of $\mathscr{A}$ are in the same genus iff their classes in $K_0(\mathscr{B})$ and $K_0(\mathscr{C})$ coincide, hence iff their classes in $K_0(\mathscr{A})$ differ by an element of G.*

(iii) *If A is stable, ϕ_A is surjective.*

(iv) *If A is good and stable, ϕ_A is bijective.*

Corollary (Cancellation). *If A is good and stable and $A\oplus A_2\cong A_1\oplus A_2$, then $A\cong A_1$.*

For by (ii) A and A_1 are in the same genus; then by (iv) they are isomorphic.

Corollary (Stability). *Let A, A_1 be such that the image of A_1 in $\mathscr{B}$* (resp. $\mathscr{C}$) *is a summand of the image of A. Then there is an object A_2 of $\mathscr{A}$ with $A_1\oplus A_2$ in the genus of A. If A_2 is stable, and A is good we can choose $A_1\oplus A_2\cong A$.*

The first statement follows from (i) and the characterisation of genus in (ii). If A_2 is stable, by (iii) we can find A_2' so that $A_1\oplus A_2'$ has the same class as A in $K_0(\mathscr{A})$; the result then follows by (iv), since A is also stable (by an earlier remark).

Proof of Theorem 1.4. (i) The images in $K_0(\mathscr{B})$ and $K_0(\mathscr{C})$ are positive, so we have objects B and C; their images in $K_0(\mathscr{D})$ coincide so by (A7) (for $\mathscr{D}$), B and C become isomorphic in $\mathscr{D}$. By (A1) we obtain an object A whose image in $K_0(\mathscr{B})\oplus K_0(\mathscr{C})$ coincides with that of the given class.

(ii) If A, A' are in the same genus then by definition their images in $\mathscr{B}$ and in $\mathscr{C}$ are isomorphic. Conversely, if they determine the same element of $K_0(\mathscr{B})\oplus K_0(\mathscr{C})$ then by (A7) (for $\mathscr{B}$ and $\mathscr{C}$), their images in $\mathscr{B}$ and in $\mathscr{C}$ are isomorphic, i.e. they are in the same genus.

(iii) Clearly ϕ_A is closely related to the bijection of Lemma 1.2. If A is stable, $\operatorname{Aut}(A,\mathscr{D})$ maps onto $K_1(\mathscr{D})$ so the set $k_A(\mathscr{A})$ of isomorphism classes maps onto the image G of $K_1(\mathscr{D})$.

(iv) By Lemma 1.2 classes in the genus of A correspond to double cosets; if A is good, we can factor out $E\operatorname{Aut}(A, \mathscr{D})$ (and what is left is abelian), so they correspond to elements of the cokernel of

$$\operatorname{Aut}(A, \mathscr{B}) \oplus \operatorname{Aut}(A, \mathscr{C}) \to K_1(\mathscr{D}).$$

If A is stable, this coincides with the cokernel of

$$K_1(\mathscr{B}) \oplus K_1(\mathscr{C}) \to K_1(\mathscr{D}),$$

i.e. with G.

These arguments yield fairly good stability results for objects, but not for morphisms: such results appear to lie deeper. Nevertheless it is interesting to ask whether from the above or similar axioms one can conclude that for certain objects A of $\mathscr{A}$, $\operatorname{Aut}(A, \mathscr{A})$ maps onto $K_1(\mathscr{A})$ or even that sometimes $E\operatorname{Aut}(A, \mathscr{A})$ is the commutator subgroup of $\operatorname{Aut}(A, \mathscr{A})$, or is perfect. An easier problem, perhaps, is to decide whether good, stable objects are always very good.

§ 2. Pullback Categories of Projective and Quadratic Modules

We begin by introducing the rings in which we are interested, and some appropriate notation. Let $\mathbb{Z}$ denote the ring of integers; $\hat{\mathbb{Z}}$ its profinite completion – i.e. the inverse limit of the finite cyclic quotients. This is the direct product of the pro-p-rings $\hat{\mathbb{Z}}_p$ of p-adic integers. Write $\mathbb{Q}$ for the rationals, $\hat{\mathbb{Q}} = \mathbb{Q} \otimes \hat{\mathbb{Z}}$ and $\hat{\mathbb{Q}}_p = \mathbb{Q} \otimes \hat{\mathbb{Z}}_p$. The ring $\hat{\mathbb{Q}}$ is the "finite" part of the ring $\mathbb{Q}_A$ of adèles: the infinite part, i.e. $\mathbb{R}$, the field of real numbers, will be used later. $\hat{\mathbb{Q}}_p$ is the usual field of p-adic numbers. $\hat{\mathbb{Q}}$ is not the direct product of the $\hat{\mathbb{Q}}_p$, but rather the local product with respect to subrings $\hat{\mathbb{Z}}_p$: only a finite number of components may have non-trivial denominator. Using the inverse limit topology, we can regard $\hat{\mathbb{Z}}$ and each $\hat{\mathbb{Z}}_p$ as a compact topological ring, and $\hat{\mathbb{Z}} = \prod \hat{\mathbb{Z}}_p$ topologically. We also give $\hat{\mathbb{Q}}$ resp. $\hat{\mathbb{Q}}_p$ the topology in which $\hat{\mathbb{Z}}$ resp. $\hat{\mathbb{Z}}_p$ is an open subring.

R will be a ring whose underlying additive group is finitely generated. We label its tensor products (over $\mathbb{Z}$) with the above rings systematically:

$$\begin{array}{lll} R(=R \otimes \mathbb{Z}) & \hat{R} = R \otimes \hat{\mathbb{Z}} & \hat{R}_p = R \otimes \hat{\mathbb{Z}}_p \\ S = R \otimes \mathbb{Q} & \hat{S} = R \otimes \hat{\mathbb{Q}} & \hat{S}_p = R \otimes \hat{\mathbb{Q}}_p; \end{array}$$

and we will use corresponding notation for modules; e.g. if L is an R-module, we will write $\hat{L} = L \otimes \hat{\mathbb{Z}} = \prod \hat{L}_p$, an $\hat{R}$-module, and $M = L \otimes \mathbb{Q}$, an S-module. Later on we will also need the real completion $T = R \otimes \mathbb{R}$.

Similar comments to those above apply here. S is of course an algebra over $\mathbb{Q}$. In the traditional terminology (at least if R is torsion free) R is an order in S, L a lattice in M. There are several obvious inclusion relations among the above, and they acquire topologies with corresponding properties. Our guiding philosophy is to reduce questions about R to those about the other rings, which have simpler ring-theoretic properties e.g. S, $\hat{S}_p$ are algebras over fields; $\hat{R}_p$ is a complete semi-local ring.

For any ring P, we write $\mathcal{M}(P)$ for the category of finitely generated right P-modules; $\mathcal{P}(P)$ for the full subcategory of projective modules. These are categories-with-product. A ring morphism $f\colon P\to Q$ induces a product-preserving functor $\mathcal{M}(P)\to\mathcal{M}(Q)$ by $M\mapsto M\otimes_P Q$: similarly for $\mathcal{P}$. To simplify the statements of our next results, we will say that a covariant functor $\mathcal{F}$ from rings to categories is *Cartesian* for $(R, S;\, \hat{R}, \hat{S})$ if the diagram

$$\begin{array}{ccc} \mathcal{F}(R) & \longrightarrow & \mathcal{F}(S) \\ \downarrow & & \downarrow \\ \mathcal{F}(\hat{R}) & \longrightarrow & \mathcal{F}(\hat{S}) \end{array}$$

is a fibre product diagram.

Theorem 2.1. *The functors $\mathcal{M}, \mathcal{P}$ are Cartesian for $(R, S;\, \hat{R}, \hat{S})$.*

Lemma 2.2. *$\mathcal{M}$ is Cartesian for $(\mathbb{Z}, Q;\, \hat{\mathbb{Z}}, \hat{Q})$.*

Proof. We must check (A1) and (A2). Note for either that a finitely generated abelian group ($=\mathbb{Z}$-module) splits (not naturally) as direct sum of the torsion subgroup T (which is finite) and a free abelian group F. Tensoring with $\hat{\mathbb{Z}}$ (=profinite completion) gives $T\oplus\hat{F}$, and with $\mathbb{Q}$ gives $F\otimes\mathbb{Q}$. So T can be recovered from $\mathcal{M}(\hat{\mathbb{Z}})$ already; similarly for morphisms. We can thus assume $T=0$ from now on.

Now (A2) follows easily, for given finitely generated free $\mathbb{Z}$-modules L, L' and morphisms $\beta\colon L\otimes\mathbb{Q}\to L'\otimes\mathbb{Q}$, $\gamma\colon L\otimes\hat{\mathbb{Z}}\to L'\otimes\hat{\mathbb{Z}}$ – or equivalently, $\gamma_p\colon L\otimes\hat{\mathbb{Z}}_p\to L'\otimes\hat{\mathbb{Z}}_p$, such that for each p, β and γ_p coincide over $\hat{\mathbb{Q}}_p$, the existence of γ_p shows that the matrix of β expressed via bases of L, L' has denominators prime to p: this holds for all p, so we have an integer matrix, and β induces $\alpha\colon L\to L'$, clearly unique and inducing γ.

Now suppose given a finitely generated torsion free $\hat{\mathbb{Z}}$-module $\hat{L}$, a $\mathbb{Q}$-module M, and an isomorphism

$$\delta\colon \hat{L}\otimes\hat{\mathbb{Q}}\to M\otimes\hat{\mathbb{Q}}.$$

M is free of rank n; the existence of δ shows that $\hat{L}$ has constant rank, hence is free. Now for any $x\in\hat{\mathbb{Q}}$, we can find a non-zero integer k with $kx\in\hat{\mathbb{Z}}$. The same follows for matrices over $\hat{\mathbb{Q}}$. It follows that if $e_1, \ldots, e_n$ is a basis of M, spanning the $\hat{\mathbb{Z}}$-submodule $\hat{L}_1$ of $M\otimes\hat{\mathbb{Q}}$, then for some integer k,

$$k\hat{L}_1\subset\delta(\hat{L})\subset k^{-1}(\hat{L}_1).$$

Thus $L=\delta(\hat{L})\cap M$ is commensurable with $\hat{L}_1\cap M$, and so is a free abelian group which spans M. To complete the verification of (A1) it will suffice to show that

$$L\otimes\hat{\mathbb{Z}}=\delta(\hat{L}).$$

Clearly $L\otimes\hat{\mathbb{Z}}\subset\delta(\hat{L})$, and by the above,

$$\delta(\hat{L})\subset k^{-1}(L\otimes\hat{\mathbb{Z}}).$$

Let $x\in\hat{L}$, so $k\,\delta(x)\in L\otimes\hat{\mathbb{Z}}$. Choose $u\in L$, $v\in L\otimes\hat{\mathbb{Z}}$ such that

$$k\,\delta(x)=u+k\,v.$$

Then

$$k^{-1}\,u=\delta(x)-v\in(L\otimes\mathbb{Q})\cap\delta(\hat{L})=L,$$

so

$$\delta(x)=k^{-1}\,u+v\in L\otimes\hat{\mathbb{Z}}.$$

This completes the proof.

Proof of Theorem 2.1. The first part of the Theorem follows by regarding $\mathcal{M}(R)$, $\mathcal{M}(S)$ etc. as the category of objects of $\mathcal{M}(\mathbb{Z})$, $\mathcal{M}(\mathbb{Q})$ etc. with extra structure given by a ring homomorphism

$$R\to\operatorname{End}M.$$

For e.g. to verify (A1), suppose given an $\hat{R}$-module $\hat{L}$, an S-module M and an $\hat{S}$-isomorphism

$$\hat{L}\otimes\hat{\mathbb{Q}}=\hat{L}\otimes\hat{S}\to M\otimes\hat{S}=M\otimes\hat{\mathbb{Q}}$$

then by Lemma 2.2 we obtain a group L, defined as the pullback of $\hat{L}\to M\otimes\hat{\mathbb{Q}}\leftarrow M$, with $L\otimes\hat{\mathbb{Z}}=\hat{L}$, $L\otimes\mathbb{Q}=M$. But a pullback of R-module maps is again an R-module map.

The second part now follows from the well-known result [3, III, 6.6] that a finitely generated (in fact, presented) R-module L is projective iff all localisations iff all completions $L\otimes\hat{\mathbb{Z}}_p$ and M are.

Remarks. (i) One can reformulate the proof using the adjoint functors constructed by Bass [3, p. 483]. This does not give any real simplification, however.

(ii) By passage to the limit, it is now easy to obtain a pullback theorem for categories of arbitrary (not necessarily finitely generated) modules. Compare Fakhruddin [10].

We next consider quadratic modules: our basic definitions here were expounded in [23] and [28]. Let R be a ring, α an anti-automorphism and u a unit of R such that $u^\alpha=u^{-1}$ and $(x^\alpha)^\alpha=u\,x\,u^{-1}$ for all $x\in R$: we call (R,α,u) an antistructure. If M,N are (right) R-modules, we write $S_\alpha(M,N)$ for the additive group of R-sesquilinear (with respect to α)

maps $M \times N \to R$, $S_\alpha(M) = S_\alpha(M, M)$. We define

$$T_u: S_\alpha(M) \to S_\alpha(M)$$

by

$$T_u \phi(m, n) = \phi(n, m)^\alpha u$$

and observe that $T_u^2 = 1$. We then define the groups of reflexive and of quadratic forms on M by

$$R_{\alpha,u}(M) = \operatorname{Ker}(1 - T_u),$$

$$Q_{\alpha,u}(M) = \operatorname{Coker}(1 - T_u).$$

Note that multiplying by $1 + T_u$ gives a "bilinearisation" map

$$b: Q_{\alpha,u}(M) \to R_{\alpha,u}(M),$$

so that any quadratic form has an "underlying" reflexive (i.e., usually, hermitian) form. A form ϕ is nonsingular if $x \mapsto \phi(x, .)$ induces an isomorphism

$$A\phi: M \to \operatorname{Hom}_R(M, R);$$

a quadratic form is called nonsingular if its underlying reflexive form is so.

As an example, for any M we can form the dual module $M^* = \operatorname{Hom}_R(M, R)$ (with R-module structure induced using α) and define $H(M)$ to be $H \oplus M^*$ with quadratic form defined as the equivalence class of

$$\phi((m, f), (m', f')) = f(m').$$

If M is finitely generated projective, this is nonsingular.

We write $\mathscr{R}^0(R, \alpha, u)$ for the category whose objects are pairs (M, ϕ) with M a finitely generated R-module and $\phi \in R_{\alpha,u}(M)$. A morphism $(M_1, \phi_1) \to (M_2, \phi_2)$ is a homomorphism $f: M_1 \to M_2$ with $f^* \phi_2 = \phi_1$, i.e.

$$\phi_2(f(m), f(n)) = \phi_1(m, n) \quad \text{for all } m, n \in M_1.$$

Similarly write $\mathscr{Q}^0(R, \alpha, u)$ for the category of pairs (M, q) with M a finitely generated R-module, $q \in Q_{\alpha,u}(M)$. Here, a morphism $(M_1, q_1) \to (M_2, q_2)$ is a homomorphism $f: M_1 \to M_2$ such that for some (hence any) representatives $\phi_1 \in S_\alpha(M_1)$ of q_1, $\phi_2 \in S_\alpha(M_2)$ of q_2, there exists $\chi \in S_\alpha(M_1)$ with

$$f^* \phi_2 - \phi_1 = (1 - T_u) \chi.$$

The morphisms in either category will be referred to as *isometries*. An object (M, q) of $\mathscr{Q}^0(R, \alpha, u)$ with M projective is called a *quadratic module* over (R, α, u).

We shall be primarily concerned with the full subcategories $\mathscr{R}(R, \alpha, u)$, $\mathscr{Q}(R, \alpha, u)$ of the above whose objects are those pairs (M, ϕ) resp. (M, q) with M a finitely generated projective R-module and ϕ resp. q nonsingular.

Objects of the latter are called *nonsingular quadratic modules*. When there is no danger of confusion, we shall frequently omit reference to (α, u).

Given a morphism $(R_1, \alpha_1, u_1) \xrightarrow{\phi} (R_2, \alpha_2, u_2)$ – i.e. a ring homomorphism ϕ: $R_1 \to R_2$ with $\phi \circ \alpha_1 = \alpha_2 \circ \phi$ and $\phi(u_1) = u_2$ – there are natural induced functors of the categories $\mathscr{R}$, $\mathscr{Q}$. Further, these are categories with product (viz., orthogonal direct sum), and such functors preserve the product.

We now return to the square considered above. Given an antistructure (R, α, u) then tensoring with $\hat{\mathbb{Z}}$ etc. provides anti-automorphisms and units for the rings $\hat{R}$ etc.; we will denote them indiscriminately by α and u.

Lemma 2.3. *The functors* $\mathscr{R}^0$, $\mathscr{Q}^0$, $\mathscr{R}$ *and* $\mathscr{Q}$ *are Cartesian for* $(R, S; \hat{R}, \hat{S})$.

Proof. Since by Theorem 2.1 we have a fibre product diagram of categories $\mathscr{M}$, it suffices to consider structures on a given R-module L. Now the exact sequence of additive groups

$$0 \to \mathbb{Z} \to \hat{\mathbb{Z}} \oplus \mathbb{Q} \to \hat{\mathbb{Q}} \to 0$$

(here all the maps, except for a sign in $\mathbb{Q} \to \hat{\mathbb{Q}}$, are the natural inclusions) induces an exact sequence

$$0 \to S_\alpha(L) \to S_\alpha(\hat{L}) \oplus S_\alpha(M) \to S_\alpha(\hat{M}) \to 0$$

where $\hat{L} = L \otimes \hat{R}$, etc. Exactness is easy to show since L is finitely generated as abelian group, hence so is $S_\alpha(L)$: the rest are tensor products. Now T_u induces an endomorphism of this sequence. We can thus apply the "snake lemma" (see e.g. [7, 1.4]) to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_\alpha(L) & \longrightarrow & S_\alpha(\hat{L}) \otimes S_\alpha(M) & \longrightarrow & S_\alpha(\hat{M}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1-T_u} & & \downarrow{\scriptstyle 1-T_u} & & \downarrow{\scriptstyle 1-T_u} & & \\
0 & \longrightarrow & S_\alpha(L) & \longrightarrow & S_\alpha(\hat{L}) \oplus S_\alpha(M) & \longrightarrow & S_\alpha(\hat{M}) & \longrightarrow & 0
\end{array}$$

and obtain an exact sequence

$$\begin{aligned}
0 \to R_{(\alpha,u)}(L) \to R_{(\alpha,u)}(\hat{L}) \oplus R_{(\alpha,u)}(M) \to R_{(\alpha,u)}(\hat{M}) \\
\xrightarrow{\delta} Q_{(\alpha,u)}(L) \to Q_{(\alpha,u)}(\hat{L}) \oplus Q_{(\alpha,u)}(\hat{M}) \to Q_{(\alpha,u)}(\hat{M}) \to 0.
\end{aligned}$$

It remains to show that $\delta = 0$. But this is clear since $R_{(\alpha,u)}(\hat{M})$ is a rational vector space, whereas $Q_{(\alpha,u)}(L)$ is a finitely generated group.

This completes the proof for $\mathscr{R}^0$ and $\mathscr{Q}^0$. It follows from Theorem 2.1 that we still have pullback diagrams if all modules are required to be projective. Finally, the same goes for nonsingularity since this means that a certain map $L \to L^*$ is an isomorphism, and again by Theorem 2.1 this holds over R if it does so over $\hat{R}$ and S.

In respect of this proof we observe, first that the exact sequence represents the strong approximation theorem for the additive group; and second, that this is where the notion in [24] of arithmetic functor is subsumed in the present account.

§ 3. Verification of the Axioms

We now show that with suitable further restrictions, the pullback diagrams constructed in § 2 satisfy the remaining axioms of § 1. In fact, we now concentrate on the functors $\mathscr{P}$ and $\mathscr{Q}$: even (A4) fails for the others, though one could also treat similarly $\mathscr{R}$ and the intermediate generalisations considered by Bak [1] and Bass [4]. Our result will be the following.

Theorem 3.1. *Let R be a ring (resp. (R, α, u) an antistructure) with additive group R^+ finitely generated. Write $\hat{R} = R \otimes \hat{\mathbb{Z}}$, $S = R \otimes \mathbb{Q}$, $\hat{S} = R \otimes \hat{\mathbb{Q}}$. Then the diagram*

$$\begin{array}{ccc} \mathscr{P}(R) & \longrightarrow & \mathscr{P}(S) \\ \downarrow & & \downarrow \\ \mathscr{P}(R) & \longrightarrow & \mathscr{P}(\hat{S}) \end{array} \qquad \textit{resp.} \qquad \begin{array}{ccc} \mathscr{Q}(R) & \longrightarrow & \mathscr{Q}(S) \\ \downarrow & & \downarrow \\ \mathscr{Q}(\hat{R}) & \longrightarrow & \mathscr{Q}(\hat{S}) \end{array}$$

satisfies (A1)–(A4) *and* (A7). *The object R of $\mathscr{P}(R)$ resp. $H(R)$ of $\mathscr{Q}(R)$ is stable, so* (A5) *holds.*

If further S is semisimple (as algebra over $\mathbb{Q}$), then the object $R \oplus R$ of $\mathscr{P}(R)$ resp. $H(R \oplus R)$ of $\mathscr{Q}(R)$ is very good, so (A6) *also holds.*

One can give more precise statements about which objects are stable or good – for example, our proof shows that any object of $\mathscr{Q}(R)$, of rank $\geqq 3$, with a hyperbolic summand, is good – but it would clutter up the exposition unnecessarily to discuss these here; besides, my results are still not complete on this point.

Proof. Let us first deal with the easy points. Axioms (A1) and (A2) were verified in the preceding paragraph. It is clear from the definitions that we have categories with product and product-preserving functors, so (A3) holds. Now any projective module is a direct summand of a free module, so (A4) holds for $\mathscr{P}$, with $A_n = R^n$. And by [23, Theorem 3], for any ring A any object of $\mathscr{Q}(A)$ is a summand of a hyperbolic object $H(P)$, and hence of some $H(A^n)$. Hence (A4) also holds for $\mathscr{Q}$, taking $A_n = H(R^n)$.

Next, we discuss cancellation and stability. Cancellation is well-known over division rings, hence over semisimple rings S, and was verified for $\mathscr{P}(\hat{R})$ and $\mathscr{P}(\hat{S})$ in [27, 2.2i] and for $\mathscr{Q}(\hat{R})$ and $\mathscr{Q}(\hat{S})$ in [27, 3.3i]; thus (A7) holds. Also it is well known that for S semisimple, Aut S maps onto

$K_1\mathscr{P}(S)$ and Aut $H(S)$ onto $K_1\mathscr{Q}(S)$ – for a direct proof of the latter see, for example, [28, Theorem 5]. The corresponding results for $\mathscr{P}(\hat{R})$ and $\mathscr{P}(\hat{S})$ were shown in [27, 2.2 iii] and for $\mathscr{Q}(\hat{R})$ and $\mathscr{Q}(\hat{S})$ in [27, 3.3 iii]. Hence R is stable in $\mathscr{P}(R)$ and $H(R)$ in $\mathscr{Q}(R)$.

It remains to discuss goodness. In outline, the idea of the proof is to show that if L is a suitable quadratic module over R, $\hat{L} = L \otimes \hat{\mathbb{Z}}$, $M = L \otimes \mathbb{Q}$ and $\hat{M} = L \otimes \hat{\mathbb{Q}}$, then the image of $E\,\mathrm{Aut}(\hat{L})$ is an open subgroup of $E\,\mathrm{Aut}(\hat{M})$, and $E\,\mathrm{Aut}(M)$ is dense, which will clearly imply the desired result. We proceed to details. To save discussion of cases, observe that the result for $\mathscr{P}$ is a special case of that for $\mathscr{Q}$ – taking for R the sum of anti-isomorphic rings interchanged by α. It will thus suffice to discuss $\mathscr{Q}$.

Note that for arguments concerning $\hat{S}$ we can tolerate a finite number of exceptional primes. Since R^+ is finitely generated, it has p-torsion for only finitely many p, which may be largerly ignored below.

The automorphism group $\mathrm{Aut}(M)$ of a quadratic module M over the semisimple ring S is a product of algebraic groups of types GL, U, 0 or Sp. Assume that M has a hyperbolic summand isomorphic to $H(R)$. Then (e.g. by [28, Theorem 5]) this product maps onto $K_1\mathscr{Q}(S)$ and – by [25, § 2] – the kernel $E\,\mathrm{Aut}(M)$ is the corresponding product of groups of types SL, SU, Ω and Sp, where Ω denotes the image of Spin $\to SO$ (the kernel has order 2).

Now all of SL, SU, Spin and Sp are simple algebraic groups with the following low-dimensional exceptions:

SL_1, Spin_1 are trivial over a commutative field (but the latter is excluded by the hypothesis of a hyperbolic summand),

Spin_4 over a commutative field, Spin_2 over a quaternion ring (with M hyperbolic in each case) are products of two simple groups (this will not affect our argument),

Spin_2 over a commutative field, Spin_1 over a quaternion ring are abelian (the latter is again excluded by the hyperbolic summand: it is mainly to exclude the former that we restricted the statement to the case: $H(R^2)$ a summand).

Thus at least in each case we consider, $E\,\mathrm{Aut}(M)$ is a product of simple, simply-connected algebraic groups, modulo replacing some components Spin by Ω. The corresponding result now follows over each $\hat{S}_p$.

Hence, as in [26, Theorem 9], the commutator subgroup of an open subgroup of $E\,\mathrm{Aut}(\hat{M}_p)$ is open: in particular, the image of $E\,\mathrm{Aut}(\hat{L}_p)$ is open in $E\,\mathrm{Aut}(\hat{M}_p)$. But by [26, Lemma 13], for almost all p the map $K_1\mathscr{Q}(\hat{R}_p) \to K_1\mathscr{Q}(\hat{S}_p)$ is injective, so for L a lattice in M and almost all p,

$$E\,\mathrm{Aut}'(\hat{L}_p) = \mathrm{Aut}'(\hat{L}_p) \cap E\,\mathrm{Aut}(\hat{M}_p).$$

It now follows from the definition of restricted product topology that $E\,\mathrm{Aut}'(\hat{L}) = \prod_p E\,\mathrm{Aut}'(\hat{L}_p)$ is an open subgroup of $E\,\mathrm{Aut}(\hat{M})$.

For G a simple, simply-connected algebraic group over $\mathbb{Q}$, the strong approximation theorem [13] assures us that $G(\mathbb{Q})$ is dense in $G(\hat{\mathbb{Q}})$, provided that $G(\mathbb{R})$ is noncompact. Perhaps I should emphasise, since I merely quote this, that this is the key technical point of the entire argument: the strong approximation theorem is sophisticated, and quite difficult to prove. Now the only compact group of type SL is $SL_1(\mathbb{H})$: the group $E\,\mathrm{Aut}$ for a hyperbolic form is never compact, and only has a compact factor for $\mathrm{Spin}_2(\mathbb{H})$. Thus we have already excluded the cases where G is compact, so $E\,\mathrm{Aut}(M)$ is indeed dense in $E\,\mathrm{Aut}(\hat{M})$ in the stated cases. This concludes the proof.

We will not reformulate the cancellation and stability results of § 1 in this setting, since little is to be gained by doing so (the results are not new, only the proofs) but the following seems worth explicit mention.

Corollary 3.2. *Under the conditions of the theorem, there are exact sequences*

$$K_1\mathscr{P}(R) \to K_1\mathscr{P}(\hat{R}) \oplus K_1\mathscr{P}(S) \to K_1\mathscr{P}(\hat{S}) \to K_0\mathscr{P}(R) \to$$
$$\to K_0\mathscr{P}(\hat{R}) \oplus K_0\mathscr{P}(S) \to K_0\mathscr{P}(\hat{S})$$
$$K_1\mathscr{Q}(R) \to K_1\mathscr{Q}(\hat{R}) \oplus K_1\mathscr{Q}(S) \to K_1\mathscr{Q}(\hat{S}) \to K_0\mathscr{Q}(R) \to$$
$$\to K_0\mathscr{Q}(\hat{R}) \oplus K_0\mathscr{Q}(S) \to K_0\mathscr{Q}(\hat{S}).$$

Compare also Bak-Scharlau [2].

§ 4. Based Modules

We now extend the result to cover based modules. This extension may appear trivial, but based quadratic modules fit the terminology of [28]. However, the real reason for introducing bases is that this seems to be the only version of our method that leads to a classification of forms on free (as opposed to projective) modules.

A *based module* is a free module with an equivalence class of free bases, two bases being equivalent if the matrix representing change of basis has determinant 1 in $K_1(\hat{S})$. Note that we use $K_1(\hat{S})$ as value group for each of the rings $R, S, \hat{R}, \hat{S}$. More generally, following Milnor [15], we define a *stably based* R-module to be a module L with, for some r, a class of bases on $L \oplus R^r$. Note that there is a bijection between stable bases on L and ones on $L \oplus R$. Note also that though L must be projective, it need not be free.

Write $\mathscr{B}(R)$ (similarly $\hat{R}, S, \hat{S}$) for the category of stably based R-modules and based isomorphisms.

Lemma 4.1. *The functor $\mathscr{B}$ is Cartesian for $(R, S;\ \hat{R}, \hat{S})$. Moreover, if a stably based R-module is good for $\mathscr{P}$, the stable base is equivalent to a base.*

Proof. Since cancellation holds in $\mathscr{P}(S)$, $\mathscr{P}(\hat{R})$ and $\mathscr{P}(\hat{S})$ any stably based module is free and (if nontrivial) stable (for $\mathscr{P}$), so any stable base is equivalent to a base. Now for (A1) we know by Theorem 2.1 how to construct L, but wish to find a stable base for L. Adding R if necessary, we can suppose L good. But then the matrix in $SL_n(\hat{S})$ representing ϕ is the product of such matrices over $\hat{R}, S$; thus adjusting bases of $\hat{L}, M$ within their equivalence classes we can suppose ϕ the identity. It is now obvious that there is a base (not just a stable base) for L inducing the given bases of $\hat{L}$, M. For (A2), by Theorem 2.1 we get an isomorphism in $\mathscr{P}(R)$; it is based (by definition) since it becomes so in $\mathscr{P}(\hat{S})$.

Now—as in [28]—we write $\mathscr{B}\mathscr{Q}(R, \alpha, u)$ etc. for the category of (stably) based nonsingular quadratic modules and based isometries, and $\mathscr{L}(R, \alpha, u)$ for the full subcategory of forms (of even rank) with discriminant. Note that the discriminant of [28] is only defined for forms of even rank: it is 1 for hyperbolic forms with the standard basis.

Theorem 4.2. *With the hypotheses of Theorem* 3.1, *the conclusions apply to $\mathscr{B}$ (for $\mathscr{P}$) and $\mathscr{B}\mathscr{Q}$, $\mathscr{L}$ (for $\mathscr{Q}$): axioms* (A1)–(A6) *hold. As to* (A7), *it holds for $\mathscr{B}$ and cancellation holds for $\mathscr{B}\mathscr{Q}$ and $\mathscr{L}$ over S and $\hat{S}$: over $\hat{R}$, if $A_1 \oplus A_2 \cong A_1 \oplus A_3$ and A_2 has a hyperbolic summand, then $A_2 \cong A_3$.*

Proof. We have verified the pullback axioms (A1) and (A2) for $\mathscr{Q}$ and $\mathscr{B}$: they follow for $\mathscr{B}\mathscr{Q}$ (each structure pulls back uniquely). (A3) holds: again we have categories with product. (A4) holds by definition for $\mathscr{B}$, taking $A_n = R^n$ with the natural base. To obtain it for $\mathscr{B}\mathscr{Q}$, since the $H(R^n)$ are cofinal in $\mathscr{Q}(R)$ etc., it suffices to show that for any ring A, $H(A^n)$ with the "wrong" base (say after base change with matrix M) is a summand of $H(A^{2n})$ with the right base. But if we add $H(A^n)$ with basis changed by M^{-1}, the result is as desired.

As to (A5), (A6), it suffices to observe that being stable or good in $\mathscr{B}$ resp. $\mathscr{B}\mathscr{Q}$ is equivalent to (or weaker than) having the same property in $\mathscr{P}$ resp. $\mathscr{Q}$, so we can use the same examples as before. The cancellation axiom for $\mathscr{B}$ follows from the first sentence of the proof of (4.1). The results concerning cancellation for $\mathscr{B}\mathscr{Q}$ were obtained in [27, Lemma 4.2].

Finally, the result for $\mathscr{L}$ follows trivially from that for $\mathscr{B}\mathscr{Q}$, for since determinants were defined in $K_1(\hat{S})$, $\mathscr{L}(R)$, $\mathscr{L}(S)$, $\mathscr{L}(\hat{R})$ are the full subcategories defined as mapping to $\mathscr{L}(\hat{S})$.

Corollary. *We have an exact sequence*

$$K_1\mathscr{L}(R) \to K_1\mathscr{L}(\hat{R}) \oplus K_1\mathscr{L}(S) \to K_1\mathscr{L}(\hat{S}) \to K_0\mathscr{L}(R) \to$$
$$\to K_0\mathscr{L}(\hat{R}) \oplus K_0\mathscr{L}(S) \to K_0\mathscr{L}(\hat{S}).$$

We now translate this into a sequence of L-groups. For $A=R,\hat{R},S$ or $\hat{S}$ we write $X=X(A)=\operatorname{Ker}(K_1(A)\to K_1(\hat{S}))$ (for the sake of uniformity) and study $L_i^X(A)$. Then we have

$$L_0^X(A)=\tilde{K}_0\mathscr{L}(A)=\operatorname{Ker}(rk\colon K_0\mathscr{L}(A)\to\mathbb{Z})$$

and

$$L_1^X(A)=\operatorname{Coker}(H\colon X(A)\to K_1\mathscr{L}(A)).$$

Here, of course, rk denotes the rank homomorphism.

Theorem 4.3. *With hypotheses as above, we have an exact sequence*

$$L_1^X(R)\to L_1^X(\hat{R})\oplus L_1^X(S)\to L_1^X(\hat{S})\to L_0^X(R)\to L_0^X(\hat{R})\oplus L_0^X(S)\to L_0^X(\hat{S}).$$

Proof. We begin with the exact sequence of the above corollary. It is trivial to verify that exactness is not destroyed by replacing each $K_0\mathscr{L}(A)$ by the corresponding $\tilde{K}_0\mathscr{L}(A)=L_0^X(A)$.

Next, we have $X(\hat{S})=0$, by definition. Further, the exact sequence (for $\mathscr{P}$) of (5.3) yields an exact sequence

$$X(R)\to X(\hat{R})\oplus X(S)\to 0.$$

Finally, diagram chasing in the following diagram, where the upper two rows and all columns are exact, shows that the lower row is exact also.

$$\begin{array}{ccccccc}
X(R) & \longrightarrow & X(\hat{R})\oplus X(S) & \longrightarrow & 0 & & \\
\downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} & & \downarrow & & \\
K_1\mathscr{L}(R) & \longrightarrow & K_1\mathscr{L}(\hat{R})\oplus K_1\mathscr{L}(S) & \longrightarrow & K_1\mathscr{L}(\hat{S}) & \longrightarrow & \tilde{K}_0\mathscr{L}(R) \\
\downarrow & & \downarrow & & \| & & \| \\
L_1^X(R) & \longrightarrow & L_1^X(\hat{R})\oplus L_1^X(S) & \longrightarrow & L_1^X(\hat{S}) & \longrightarrow & L_0^X(R) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

This is still only an interim result. First since for any antistructure (A,α,u) and $i\in\mathbb{Z}$ we have by definition

$$L_{i+2}(A,\alpha,u)=L_i(A,\alpha,-u),$$

we have exact sequences as above with the suffix increased by any even integer. We will then see in the next section how to splice these to obtain a long exact sequence. I do not know whether the unmodified sequences of (3.2) can be extended to long exact sequences: this is a much deeper problem, and an affirmative answer would be very interesting.

§ 5. Formations

We begin by reviewing basic notions about formations. The term is due to Ranicki [19, 20]: we also introduced it in [28]. A *subkernel* in a quadratic module L over R is a subspace F whose identity map extends to an isometry $L \to H(F)$. A *formation* is a triple $(H; F, G)$ with F, G subkernels in H. We are mainly concerned here with the based case, with determinants in some value group V. Thus the definitions depend on the choice of V: in this paper, we take $V = K_1(\hat{S})$. We regard F, G as objects of $\mathscr{B}(R)$, H as one of $\mathscr{L}(R)$: the above isomorphism is to hold in $\mathscr{L}(R)$. We call the formation *split* if $H = F \oplus G$ as modules, and *simply split* if this holds in $\mathscr{B}(R)$: in the latter case, call F and G *complementary*. Thus a simply split formation is isomorphic to

$$\tau_0(F) = (H(F);\, F, F^*)$$

for a suitable object F of $\mathscr{B}(R)$. To classify up to isomorphism split formations in general, we need the determinant of this isomorphism $F \oplus G \to H$ with respect to the preferred bases, which we call the *torsion* of the formation.

For any $(R, \alpha, -u)$-quadratic module (L, θ), we define its *boundary* $\partial(L, \theta)$ to be the formation $(H(L);\, L, \Gamma)$ constructed as follows. $H(L)$ is the (R, α, u)-quadratic hyperbolic module on L. If θ has bilinearisation b_θ with associated map β: $L \to L^*$, Γ is the graph of β. By [28, Lemma 2, Corollary 1], these graphs give just the subkernels of $H(L)$ complementary to L^*, so any subkernel $(H; F, G)$ such that F, G have a common complement is of this shape. Observe that $\partial(L, \theta)$ is split if and only if θ is non-singular: when this is so, its torsion equals the discriminant of θ.

A formation $\Phi = (H; F, G)$ defines an element of $L_1^X(R)$, which we can regard (when F, G are free, hence isomorphic) as the class of an automorphism of H taking F to G. We showed in [28, Lemma 8] that $\Phi \sim 0$ if and only if, stably, F and G have a common complement – or equivalently, if and only if we can write

$$\Phi \oplus \tau_0(P) \cong \partial(L, \theta)$$

for a suitable based R-module P and based quadratic R-module (L, θ). In particular, simply split formations and boundaries define the zero class, and by [28, Theorem 4], $L_1^X(R)$ is isomorphic to the Grothendieck group (under $\oplus$) of formations modulo these: this is Ranicki's definition. Observe that a split formation need not represent 0, but its equivalence class is determined by its torsion. Also, Ranicki shows [19] – or it follows at once from our definition above – that

$$(H;\, E, F)\oplus(H;\, F, G)\sim(H;\, E, G).$$

Define the category $\mathscr{F}(R)$ of formations to have formations (as above) as its objects; a morphism $(H;\, F, G)\to(H';\, F', G')$ in $\mathscr{F}$ is to be an isometry $H\to H'$ (in $\mathscr{L}$) inducing isomorphisms (in $\mathscr{B}$) $F\to F'$, $G\to G'$.

Lemma 5.1. *With the notations and hypotheses of* § 2, *take* $K_1(\hat{S})$ *as value group for* $R, S, \hat{R}$ *and* $\hat{S}$. *Then* $\mathscr{F}$ *is Cartesian for* $(R, S;\, \hat{R}, \hat{S})$.

Proof. We must verify (A1) and (A2) for $\mathscr{F}$. For the former, suppose given formations $\Phi_1=(H_1;\, F_1, G_1)$ over S and $\Phi_2=(H_2;\, F_2, G_2)$ over $\hat{R}$, and an isomorphism $\alpha\colon \Phi_1\otimes\hat{S}\to\Phi_2\otimes\hat{S}$. Since $\mathscr{L}$ is Cartesian, $(H_1, H_2;\,\alpha)$ defines an object H of $\mathscr{L}(R)$; similarly as $\mathscr{B}$ is Cartesian we have objects F, G of $\mathscr{B}(R)$ mapping into H.

Next, observe that we can pull back $(H_1/F_1, H_2/F_2;\,\alpha)$ to an object M of $\mathscr{B}(R)$ with a morphism $\phi\colon H\to M$ which is surjective, since it is so locally. Thus $H\cong\operatorname{Ker}\phi\oplus M$: comparing with the above, we see that we can identify $F=\operatorname{Ker}\phi$. Clearly this is an isotropic submodule, and applying property (A2) for $\mathscr{B}$ shows that the induced map $M\to F^*$ is an isomorphism. Hence F (and similarly, G) is a (based) subkernel.

Finally, property (A2) for $\mathscr{F}$ follows at once on applying the corresponding property for $\mathscr{B}$ and for $\mathscr{L}$.

It follows from this result and (1.2) that the formations in a genus are classified by double cosets. In order to use this fact, we must compute some automorphism groups of formations.

We begin with a split formation $\Phi=(H;\, F, G)$. Any automorphism of Φ in $\mathscr{F}$ induces an automorphism α, say, of F in $\mathscr{B}$. The induced automorphism of $G\cong F^*$ is then α^{*-1}, and the effect on Φ is determined. Thus

$$\operatorname{Aut}_{\mathscr{F}}(\Phi)=\operatorname{Aut}_{\mathscr{B}}(F).$$

It follows, for example, that the formations in the genus of $\tau_0(F)$ are the $\tau_0(P)$ with P in the genus of F; similarly for other split formations.

Lemma 5.2. *Let* $R\subset S$; *let* $\Phi=(H(F);\, F, G)$ *be an* S*-split formation over* R (i.e. $\Phi\otimes S$ *splits*). *If* $\lambda\colon (F\otimes S)^*\to(F\otimes S)$ *is the map whose graph is* $G\otimes S$, *then*

$$\operatorname{Aut}_{\mathscr{F}}(\Phi)=\{\alpha\in\operatorname{Aut}_{\mathscr{B}}(F)\colon \exists\,\theta\colon F^*\to F,\ \alpha\lambda\alpha^*-\lambda=\theta+\theta^* u\}.$$

Proof. Clearly $\operatorname{Aut}_{\mathscr{F}}(\Phi)\subset\operatorname{Aut}_{\mathscr{F}(S)}(\Phi\otimes S)\cong\operatorname{Aut}_{\mathscr{B}(S)}(F\otimes S)$ by the above. To make this more explicit, if λ' is the endomorphism of $H(F\otimes S)$ with λ its only nonzero matrix component, then $(1+\lambda')$ is an isometry fixing $F\otimes S$ and taking $F^*\otimes S$ onto $G\otimes S$. Hence $\alpha\in\operatorname{Aut}_{\mathscr{B}(S)}(F\otimes S)$

corresponds to the automorphism $(1+\lambda')^{-1}H(\alpha)(1+\lambda')$ of $\Phi\otimes S$, i.e.

$$(u,v)\mapsto(\alpha(u)+\alpha\lambda(v)-\lambda\alpha^{*-1}(v),\alpha^{*-1}(v)).$$

For this to preserve F, we need $\alpha\in\mathrm{Aut}_{\mathscr{B}(R)}(F)$. It preserves the R-submodule H if and only if, in addition, $\alpha\lambda-\lambda\alpha^{*-1}$ maps F^* into F, or equivalently, if $\alpha\lambda\alpha^*-\lambda$ does: it then preserves $G=H\cap(G\otimes S)$ (and is simple on it, as it is on $G\otimes S$). Finally, it induces an isometry of H, hence an isomorphism of $\mathscr{F}$, if and only if it satisfies the condition stated.

We can now prove

Proposition 5.3. *Let R be as in § 3. Let $\hat{\Phi}$ be an $\hat{S}$-split formation over $\hat{R}$, such that the torsion of $\Phi\otimes\hat{S}$ is in the image of $K_1(S)$. Then there is an S-split formation Φ over R, unique up to isomorphism, with $\Phi\otimes\hat{R}\cong\hat{\Phi}$.*

Proof. $\hat{\Phi}\otimes\hat{S}$ is split, hence isomorphic to $(H(\hat{S}^r);\hat{S}^r,\hat{G})$ where $\hat{G}=(\hat{S}^r)^*$, but with a different preferred base. Since the torsion is in the image of $K_1(S)$, and $GL_1(S)$ maps onto $K_1(S)$, there is a split formation $\Psi=(H(S^r);S^r,G)$ over S with $\Psi\otimes\hat{S}\cong\hat{\Phi}\otimes\hat{S}$. Now by (5.1) there is a pullback formation Φ.

As to uniqueness, observe that the conditions determine the genus of Φ, and formations in the genus are classified by double cosets $\mathrm{Aut}\,\hat{\Phi}\backslash\mathrm{Aut}(\hat{\Phi}\otimes\hat{S})/\mathrm{Aut}\,\Psi$. Further, as these formations are split, $\mathrm{Aut}\,\Psi=SL_r(S)$ and $\mathrm{Aut}(\hat{\Phi}\otimes\hat{S})=SL_r(\hat{S})$. Now unless $\hat{\Phi}$, hence Φ, is trivial, $r\geqq 2$, so $SL_r(S)$ is dense in $SL_r(\hat{S})$ by the strong approximation theorem, as in § 3. But by (5.2), if R is torsion free $\mathrm{Aut}\,\Phi'$ is an open subgroup. Applying this result to R/torsion, we see that it holds generally. For if the torsion has exponent N, a group containing all automorphisms congruent to 1 mod K (mod torsion) contains all automorphisms congruent to 1 mod NK. Hence there is only one class in the genus.

The above proof seems clumsy: I do not feel one should need to invoke strong approximation here. But this is the crucial argument which by-passes my use of linking forms in an earlier version of this proof. Indeed, the underlying idea of the following arguments can be regarded as follows. There is a relative group $L_2(S,R)$ which can be computed by equivalence classes of S-split R-formations. The bijection of (5.3) thus induces an "excision" isomorphism $L_2(S,R)\to L_2(\hat{S},\hat{R})$. It now follows that there is an exact "Mayer-Vietoris" sequence. We do not present arguments in this form, since this would involve some development of the notion of relative group (which I prefer not just to do *ad hoc*), and also there is some trouble with the equivalence relation on S-split formations – the obvious procedure appears only to work if R is a Frobenius algebra over $\mathbb{Z}$. The reader may also prefer to regard (5.3) as a form of E-surjectivity for the categories of formations.

§ 6. Completion of the Proof

For any based quadratic module $(\hat{M}, \theta)$ over $\hat{S}$, we can find one, $\hat{L}$ say, over $\hat{R}$ with $\hat{L} \otimes \hat{S} \cong \hat{M}$. It suffices to take any basis $\{e_i\}$ of $\hat{M}$, and choose $k \in \mathbb{Z}$ with each $k\,\theta(e_i, e_j) \in \hat{R}$: then let $\hat{L}$ be the submodule with basis $\{k\,e_i\}$. If we take this as preferred basis, the induced isomorphism $\hat{L} \otimes \hat{S} \to \hat{M}$ is not simple, but has determinant $\delta(k)^{-r}$, where $\delta(k)$ is the class of k in $K_1(\hat{S})$ and r is the rank of $\hat{M}$: this is at least in the image of $K_1(S)$.

We now describe our basic construction. Let $(\hat{M}, \theta)$ be a based module over $\hat{S}$ with discriminant in $K_1(S)$. Choose $(\hat{L}, \theta)$ such that there is an isometry $(\hat{L}, \theta) \otimes S \to (\hat{M}, \theta)$ with torsion in $K_1(S)$. Take its boundary formation $\partial \hat{L}$. Since $\hat{L}$ becomes nonsingular over $\hat{S}$, $\partial \hat{L}$ is $\hat{S}$-split, and its torsion comes from $K_1(S)$. We then apply (5.3) to obtain a formation Φ over R: this determines an element of $L_1^X(R)$.

Lemma 6.1. *The above construction gives a well-defined map* $\delta: L_2^X(\hat{S}) \to L_1^X(R)$, *which vanishes on the image of* $L_2^X(S)$.

Proof. It will suffice to show that if $(\hat{M}, \theta)$ comes from a form over S, the resulting element of $L_1^X(R)$ vanishes. For as the construction is additive for direct sums, if $(\hat{M}, \theta)$ gave rise (by making different choices in the construction) to $x, x' \in L_1^X(R)$, then applying the result to $(\hat{M}, \theta) \oplus (\hat{M}, -\theta)$, which comes from a form over S, shows that for some y, $x+y=x'+y$ and so $x=x'$. Thus δ is well-defined on $K_0 \mathscr{L}(\hat{S})$, hence on $\tilde{K}_0 \mathscr{L}(\hat{S}) = L_2^X(\hat{S})$.

Now as $\mathscr{B}\mathscr{Q}$ is Cartesian by (4.2), if $(\hat{M}, \theta)$ comes from a form over S, then $(\hat{L}, \theta)$ comes from a based form (L, θ) over R. But then $\partial(L, \theta)$ itself is an S-split formation with $\partial(L, \theta) \otimes \hat{R} \cong \partial(\hat{L}, \hat{\theta})$, so $\Phi \cong \partial(L, \theta)$ by (5.3). Since Φ is a boundary, it represents 0 in $L_1^X(R)$.

Lemma 6.2. *The sequence*

$$L_2^X(\hat{R}) \oplus L_2^X(S) \to L_2^X(\hat{S}) \xrightarrow{\delta} L_2^X(R) \to L_1^X(\hat{R}) \oplus L_1^X(S)$$

has order 2.

Proof. An element of $L_2^X(\hat{R})$ is represented as the difference of two objects of $\mathscr{L}(\hat{R})$ of the same rank: say $\hat{L}_1, \hat{L}_2$. We can take these in turn for $\hat{L}$ in the basic construction. But $\partial \hat{L}_i$ is then simply split, and thus pulls back to a simply split formation Φ_i, so $\Phi_i \sim 0$.

Finally, for any Φ arising from the basic construction, $\Phi \otimes \hat{R} \cong \partial(\hat{L}, \theta)$, so represents 0 in $L_1^X(\hat{R})$ and $\Phi \otimes S$ is split, so its class in $L_1^X(S)$ depends only on the torsion. But if we start from an element of $L_2^X(\hat{S})$ represented as the difference of two nonsingular quadratic modules of the same rank, we can perform the construction with the same torsion for both, so the resulting formations $\Phi \otimes S$ are isomorphic, and have zero difference.

The remaining two parts of exactness are more demanding. Indeed, the next argument, a reformulation of results of Connolly [9] and Gough [11], inspired this whole section.

Proposition 6.3. (i) *Every element of* $\mathrm{Ker}(L_1^X(R) \to L_1^X(S))$ *can be represented by an S-split formation* Φ.

(ii) $L_2^X(\hat{S}) \to L_1^X(R) \to L_1^X(\hat{R}) \oplus L_1^X(S)$ *is exact.*

Proof. (i) Represent the element by a formation $\Phi = (H; F, G)$. Since it represents 0 in $L_1^X(S)$, $\Phi \otimes S \sim 0$: thus stably, $F \otimes S$ and $G \otimes S$ have a common complement. Adding $\tau_0(R^{2n})$ for n large enough to Φ, we can omit the "stably".

Seek a subkernel C complementary to F with $C \otimes S$ complementary (ignoring bases) to $G \otimes S$. If we identify H with $H(F)$, the complements to F are the graphs of the $\phi \in Q_{(\alpha,-u)}(F^*) = A$, say, and the complements to $F \otimes S$ are the graphs of the $\phi \in Q_{(\alpha,-u)}(F^* \otimes S) = A \otimes \mathbb{Q}$. The condition of not being complementary (ignoring bases) to $G \otimes S$ defines an algebraic set X (over $\mathbb{Q}$) in the vector space $A \otimes \mathbb{Q}$: since $F \otimes S$ and $G \otimes S$ do have a common complement, X is a proper algebraic subset of $A \otimes \mathbb{Q}$. It follows that X cannot contain every point of the integer lattice (the image of A). Choose any $\phi \in A$ with image not in X: then we can take $C = \mathrm{graph}\,\phi$. But now $(H; F, C)$ is simply split, so

$$\Phi = (H; F, G) \sim (H; F, C) \oplus (H; C, G) \sim (H; C, G),$$

which is S-split, as required.

(ii) Suppose $\Phi = (H; F, G)$ an S-split formation over R defining 0 in $L_1^X(\hat{R}) \oplus L_1^X(S)$. As before, by adding a suitable $\tau_0(R^{2n})$, we can write $\Phi \otimes \hat{R} \cong \partial(L_1, \theta_1)$, $\Phi \otimes S \cong \partial(L_2, \theta_2)$. Now the $(L_i, \theta_i) \otimes \hat{S} = (M_i, \theta_i)$ have the same rank and the same discriminant, viz. the torsion of the split formation $\Phi \otimes \hat{S} \cong \partial(M_i, \theta_i)$. Thus $(M_1, \theta_1) - (M_2, \theta_2)$ represents an element of $L_2^X(\hat{S})$, whose image by δ is the class of Φ.

For the final part, we need a "glueing" theorem of Ranicki [20], which holds over any ring.

Theorem 6.4. *Suppose given quadratic modules* (L_0, θ_0), (L_1, θ_1) *and a stable isomorphism* $\partial(L_0, \theta_0) \to \partial(L_1, \theta_1)$. *Then there exists a nonsingular quadratic module* (L, θ) *such that*

(i) *there is an isometric embedding* $i_0: (L_0, -\theta_0) \to (L, \theta)$,

(ii) *there is an isometric embedding* $i_1: (L_1, \theta_1') \to (L, \theta)$, *where* θ_1' *has the same bilinearisation as* θ_1,

(iii) *each* $i_r(L_r)$ *is additively a direct summand, so the composite* $L \to L^* \to L_r^*$ *is a split epimorphism; its kernel is* L_{1-r},

(iv) *there is an isometry*

$$(L_0, \theta_0) \oplus (L, \theta) \cong H(L_0) \oplus (L_1, \theta_1').$$

The proof goes by expressing the hypothesis in matrix terms, writing an explicit formula for θ on $L=L_0\oplus L_1^*$, and then verifying each conclusion. The necessity of replacing θ_1 by θ_1' can be understood since $\partial(L_1,\theta_1)$ depends only on the bilinearisation. The proof is repeated in [29], with an example to show that replacement really is necessary.

Although the statement does not mention bases, the proof includes the based version, and here the form (L,θ) has discriminant 1.

Lemma 6.5. *The sequence*

$$L_2^X(\hat{R})\oplus L_2^X(S)\to L_2^X(\hat{S})\xrightarrow{\delta} L_1^X(R)$$

is exact.

Proof. Let $(\hat{M}_1,\theta_1)-(\hat{M}_2,\theta_2)$ represent an element x of Ker δ. Adding $(\hat{M}_2,-\theta_2)$ to each if necessary, we may suppose $(\hat{M}_2,\theta_2)$ of the form $(M_2,\theta_2)\otimes\hat{S}$, where θ_2 is defined over S. Adjusting both bases by the same amount, we can also find quadratic modules $(\hat{L}_i,\theta_i)$ over $\hat{R}$ with $\hat{L}_i\otimes\hat{S}=\hat{M}_i$. Then as in (6.1), our construction performed on $\hat{M}_2$ leads to 0 in $L_1^X(R)$. Hence it also does on $\hat{M}_1$.

As usual, adding suitable hyperbolic modules throughout, we may suppose the S-split formation Φ over R defined by $\hat{M}_1$ a boundary: $\Phi\cong\partial(L_0,\theta_0)$. But $\Phi\otimes\hat{R}=\partial(\hat{L}_1,\theta_1)$. Applying (6.4) to $L_0\otimes\hat{R}$ and L_1, we receive an object $(\hat{L},\theta_3)$ of $\mathscr{L}(\hat{R})$ and an isometry

$$(L_0,\theta_0)\otimes\hat{R}\oplus(\hat{L},\theta_3)\cong H(L_0\otimes\hat{R})\oplus(\hat{L}_1,\theta_1').$$

Hence, tensoring with $\hat{S}$ where, as 2 is invertible, the difference between θ_1 and θ_1' disappears, an isometry

$$(L_0,\theta_0)\otimes\hat{S}\oplus(\hat{L},\theta_3)\otimes\hat{S}\cong H(L_0\otimes\hat{S})\oplus(\hat{M}_1,\theta_1).$$

Thus x is also represented by

$$\{(\hat{L},\theta_3)-H(L_0\otimes\hat{R})\}\otimes\hat{S}+\{(L_0,\theta_0)\otimes S-(M_2,\theta_2)\}\otimes\hat{S},$$

and here, the first term clearly comes from $L_2^X(\hat{R})$, the second – as the forms have the same discriminant – from $L_2^X(S)$.

Assembling our results, we see that this completes the proof of

Theorem 6.6. *Let (R,α,u) be an antistructure such that the additive group R^+ is finitely generated and $S=R\otimes\mathbb{Q}$ is semisimple. Write $\hat{R}=R\otimes\hat{\mathbb{Z}}$ for the profinite completion, $\hat{S}=\hat{R}\otimes\mathbb{Q}$. Then there is a long exact sequence*

$$\dots L_{i+1}^X(\hat{S})\to L_i^X(R)\to L_i^X(\hat{R})\oplus L_i^X(S)\to L_i^X(\hat{S})\dots,$$

of period 4 *in i. Here, X signifies that determinants are all to be evaluated in $K_1(\hat{S})$, i.e. by Nrd.*

§ 7. Approximate Calculation

In this section we first put our results into a form more suitable for calculation, and then make such deductions as are possible in the general case, leaving more detailed and explicit calculation for a subsequent paper.

We begin by recalling some results from [27]. We have the algebra S, its completion $\hat{S}$, $T=S\otimes\mathbb{R}$ and the adele ring $S_A=\hat{S}\oplus T$. By [27, 2.3], $K_1(S)\to K_1(\hat{S})$ is injective, so $L_i^X(S)=L_i^S(S)$. We split S into simple components or pairs of such, interchanged by α, whose types we label by the algebraic groups of automorphisms of quadratic modules: GL, U, 0 or Sp. If S is simple with centre E, the cokernel of $K_1(S)\to K_1(S_A)$ is the idele class group C of E. Define ${}_2C$, C_2 by the exact sequence

$$0\to {}_2C\to C\xrightarrow{2} C\to C_2\to 0.$$

Proposition 7.1. *The map $L_i^S(S)\to L_i^S(S_A)$ is injective. Its cokernel $CL_i^S(S)$ is described as follows, for S simple. For type GL or U, $CL_i^S(S)=0$. For type* 0, *we have $i=0$, $\mathbb{Z}/2\mathbb{Z}$; $i=1$, C_2; $i=2$, ${}_2C$; $i=3$, 0; also for $CL_{i+2}^S(S)$ of type Sp.*

This is part of [27, Theorem 5.5].

Theorem 7.2. *There is a long exact sequence*

$$\ldots CL_{i+1}^S(S)\to L_i^X(R)\to L_i^X(\hat{R})\oplus L_i^S(T)\to CL_i^S(S)\ldots.$$

Proof. We begin with the exact sequence of (6.6)

$$\ldots L_{i+1}^X(\hat{S})\to L_i^X(R)\to L_i^X(\hat{R})\oplus L_i^X(S)\to L_i^X(\hat{S})\ldots$$

where we have, as observed above,

$$L_i^X(\hat{S})=L_i^S(\hat{S}),\qquad L_i^X(S)=L_i^S(S).$$

Now the natural maps

$$L_i^X(R)\to L_i^X(S)=L_i^S(S)\to L_i^S(T)$$

induce a map of the above to the elementary exact sequence

$$\ldots 0\to L_i^S(T)=L_i^S(T)\to 0\ldots.$$

Hence the algebraic mapping cone is also exact. Since

$$L_i^S(\hat{S})\oplus L_i^S(T)=L_i^S(\hat{S}\oplus T)=L_i^S(S_A),$$

this can be written as

$$\dots L_{i+1}^S(S_A) \to L_i^X(R) \to L_i^X(\hat{R}) \oplus L_i^S(S) \oplus L_i^S(T) \to L_i^S(S_A) \dots .$$

Here the induced map $L_i^S(S) \to L_i^S(S_A)$ is the natural one; by [27, 5.1] it is injective. Thus exactness is preserved if we delete $L_i^S(S)$, and replace $L_i^S(S_A)$ by the cokernel, $CL_i^S(S)$. This gives the exact sequence of the theorem.

In using this sequence to compute $L_i^X(R)$, we need to recall the values of the other groups. $CL_i^S(S)$ is described in (7.1).

$L_i^S(T)$, for T simple, vanishes except in the following cases:

Type U: $L_0^S(\mathbb{C}) \cong L_2^S(\mathbb{C}) \cong 4\mathbb{Z}$,

Type 0: $L_0^S(\mathbb{R}) \cong 4\mathbb{Z}, L_1^S(\mathbb{R}) \cong \{\pm 1\}, L_2^S(\mathbb{R}) \cong L_2^S(\mathbb{C}) \cong \{\pm 1\}, L_2^S(\mathbb{H}) \cong 2\mathbb{Z}$.

$L_i^X(\hat{R})$ is the direct product of the $L_i^X(\hat{R}_p)$. We have an exact sequence [28, Theorem 3]

$$\dots L_i^X(\hat{R}_p) \to L_i^K(\hat{R}_p) \to H^i(K_1(\hat{R}_p)/X(\hat{R}_p)) \to \cdots,$$

where $K_1(\hat{R}_p)/X(\hat{R}_p) = \mathrm{Im}(K_1 \hat{R}_p \to K_1(\hat{S}_p)) = Nrd(\hat{R}_p^\times)$.

Further [26, Lemma 5], $L_i^K(\hat{R}_p) \cong L_i^K(\bar{R}_p)$, where $\bar{R}_p$ is the quotient of $\hat{R}_p$ by its radical, hence is semisimple. Thus this last is easily computed: for a finite simple ring, L_0^K and L_1^K have order 1 resp. 2 for types GL, U, Sp resp. 0, SPOT.

In the special case when p is odd and $\hat{R}_p$ has good reduction—and we saw in the proof of [26, Lemma 13] that this holds for almost all p—we can argue more simply that $X(\hat{R}_p)$ is a p-group, so $H^i(X(\hat{R}_p)) = 0$. The exact sequence of [28, Theorem 3] then shows $L_i^S(\hat{R}_p) \cong L_i^X(\hat{R}_p)$, and by [26, Theorem 10, Corollary 2], $L_i^S(\hat{R}_p) \cong L_i^S(\bar{R}_p)$. This vanishes for summands of types GL and U, and for those of type 0, $L_0^S \cong L_3^S \cong 0$, $L_1^S \cong L_2^S \cong \{\pm 1\}$. The main difficulty in calculation thus concerns the case $p = 2$, when we need information about $Nrd(\hat{R}_2^\times)$.

Theorem 7.3. *$L_i^X(R)$ is finitely generated; more precisely,* $\mathrm{Ker}(L_i^X(R) \to L_i^S(T))$ *is finite.*

Proof. This result is essentially known. One proof goes as follows: L_1^X (even Λ_1) is finitely generated, since by results of Bak [1], $U_3(R)$ maps onto Λ_1, and by a result of Borel and Harish-Chandra [6], $U_3(R)$ is finitely generated. Similarly, it follows e.g. from Corollary 2 to (1.4) that a form which becomes hyperbolic over T is the sum of a hyperbolic form and one of rank 4. But by classical reduction theory (e.g. [5, 6]) the number of classes of such forms is finite.

However, we can also prove the result by the methods developed above: we can simplify the calculations by ignoring finite groups. Thus we have (for a global field E) an exact sequence

$$0 \to \frac{\text{unit idèles}}{\text{units in } E} \to \frac{E_A^\times}{E^\times} \to \text{ideal class group} \to 0:$$

as the ideal class group is finite, we can replace C by the first group. As we are only interested in C_2 and ${}_2C$, we can also ignore the (finitely generated) unit group.

This reduces $CL_i^S(S)$: we next consider $L_i^X(\hat{R})$. As each $L_i^X(\hat{R}_p)$ is finite, we can ignore a finite number of primes p. We now refer to the proof of [26, Lemma 13]. For all but a finite number of primes p,

i) p is odd,

ii) $\hat{R}_p$ is a maximal order in $\hat{S}_p$,

iii) $\hat{S}_p$ is unramified, and

iv) α is unramified at p.

Split $\hat{S}_p$ up into simple components: by (ii), $\hat{R}_p$ splits correspondingly. Then we can reduce further to the case when $\hat{S}_p = \hat{K}_p$ is a field, $\hat{R}_p = \hat{I}_p$ the ring of integers in it. According to the calculations in [26, Lemma 13], the map $L_i(\hat{R}_p) \to L_i(\hat{S}_p)$ can now be described as follows. For types GL, U both groups are zero. For type 0,

$$L_0^S: 0 \to \mathbb{Z}/2\mathbb{Z} \qquad L_1^S: \hat{I}_p^\times/\hat{I}_p^{\times 2} \to \hat{K}_p^\times/\hat{K}_p^{\times 2}$$

$$L_2^S: \{\pm 1\} \to \{\pm 1\} \qquad L_3^S: 0 = 0.$$

Thus the cases when $L_i^S(\hat{R}_p)$ are nonzero are precisely those when $CL_i^S(S)$ is non-trivial (and not of order 2); moreover, the image in C_2 resp. ${}_2C$ in these cases is just the image of $(\hat{I}_p^\times)_2$ resp. ${}_2(\hat{I}_p^\times)$. In view of the reduction above, this completes the proof.

It follows from our exact sequences that since $K_0(R)$ and $K_1(R)$ are finitely generated (same argument as first proof above), $L_i^K(R)$, $\Lambda_i(R)$, $L_i^S(R)$ etc. are finitely generated. More generally, as Quillen has shown [18] that all $K_i(R)$ are finitely generated, we see (in the "synthesis" notation of [28]) that, modulo Karoubi's conjecture, all $KU_{p,n}(R)$ and hence all $L_{p,n-\frac{1}{2}}$ are finitely generated.

We conclude our approximate calculations by limiting this finite group.

Theorem 7.4. *The torsion subgroup of $L_i^X(R)$ has exponent dividing 8; the cokernel of $L_i^X(R) \to L_i^S(T)$ has exponent 1 or 2.*

Proof. In the exact sequence of (8.1), the terms $CL_i^S(S)$ have exponent 1 or 2. The result follows by exactness, once we know that the torsion subgroup of $L_i^X(\hat{R}) \oplus L_i^S(T)$ has exponent dividing 4. But this follows at once from the results quoted above.

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