Annals of Mathematics, **103** (1976), 1-80

Classification of hermitian forms. VI Group rings

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This paper contains the application of the techniques developed in the preceding papers of the series (listed separately in the references at the end) to calculate groups $L_i(\mathbf{Z}\pi)$ for finite groups π; thus is completed a programme which has occupied the author for a decade. There is no simple formula for the answer, but results are obtained for satisfyingly general classes of groups: we have a reasonable picture for any π with abelian Sylow 2-subgroup, also for π a 2-group.

The contents are listed below. In Chapter 1 we recapitulate definitions and results from earlier papers, and develop some illustrative calculations. Chapter 2 recounts representation theory in the form we need it. We give the calculation in Chapter 3 for abelian 2-groups; then in the long and difficult Chapter 4 for 2-hyperelementary groups with abelian Sylow 2-subgroups. In Chapter 5 we discuss miscellaneous further calculations and problems.

Here is a general summary of our results: First, precise calculations.

(1) $L'(\pi)$ can be computed from knowledge of L groups of hyperelementary subgroups of π, and their mappings (discussion in (2.1), example in (5.3)).

(2) If π is p-hyperelementary with p odd, then (2.4) the only torsion in $L'_r(\pi)$ is $\mathbf{Z}/2$ if $r = 2$ (the usual 'Arf invariant' element) and, for π of even order; (i) orientable case, $\mathbf{Z}/2$ when $r = 3$, mapping isomorphically to $L_3(\pi/\pi^2)$, (ii) nonorientable case, $w({}_2\pi) = 1$, $\mathbf{Z}/2 \oplus \mathbf{Z}/2$ when $r = 3$.

For 2-hyperelementary groups, matters are complicated.

(3) If π is abelian, the torsion in $L_*(\pi)$ is unaltered on replacing π by its Sylow 2-subgroup (2.4.2); it is computed explicitly in (3.3.2) (orientable case) and (3.4.5) and (3.5.1) (nonorientable case). If π merely has abelian Sylow 2-subgroup, the calculation is basically carried out in Section 4 and summarised in (4.7), but we have no explicit formula. See (5.3) for an example, rectifying the announcement in [L]. If π is a dihedral or quaternion 2-group, the L-groups are determined in (5.2).

Next, we have more general results.

(4) $L'_n(\pi)$ is finitely generated; the torsion subgroup has exponent dividing 8 [V]. I know of no example where 8 cannot be replaced by 4.

(5) $L'_{2k+1}(\pi)$ is finite [V]; we have a signature map from $L'_{2k}(\pi)$ with kernel and cokernel finite 2-groups (combine [V] with (2.2.1)). See (5.1) for a precise discussion and conjectures about the image of the signature map.

(6) There are simplifications of the general theory for the case when π is a 2-group (5.2), but again we have no explicit formula.

None of the calculations of Chapter 1 is claimed as original, and a few of the other results have already been obtained by other authors, particularly Bak [4] and Bass [6]. Some discussion of this is given in the final section.

CONTENTS

1. Algebraic L-theory

The purpose of this chapter is to familiarise the reader with our notation and techniques: no really new results will be obtained. In (1.1) we recall from [F], [17] the definitions and basic formal properties of the groups

$L_i(R, \alpha, u)$. Then in (1.2) we recapitulate the main results from the preceding papers [II], [III], [IV], [V] viz. the calculations of L-groups of fields which are finite or local, the comparison of L-groups of global fields and adele rings, and the corresponding results for semisimple rings; the theorem on reduction modulo the radical for complete semilocal rings, and the 'main exact sequence' for global rings.

The other sections of the chapter are devoted to some simple calculations, chosen to illustrate our techniques and to prepare the ground for later results. These groups are for the most part already known and may be regarded as a check on the accuracy and efficiency of our method. In (1.3) we look at the coordinate ring of a nonsingular curve over a finite field of odd characteristic; strictly speaking, this makes sense only for affine curves, but the methods admit an obvious generalisation (not yet justified) to arbitrary algebraic curves. This case is fruitful as it is simpler in many respects than our central problem, but presents the same key features. In (1.4) we consider the case $R = \mathbf{Z}$, perhaps the most basic of all. We then generalise R to be the ring of integers in any algebraic number field. The case of nontrivial involution is studied in (1.5), following [I], and of trivial involution in (1.6).

1.1. Recall of definitions

We begin by recalling the basic definitions and terminology of algebraic L-theory. An *antistructure* is a triple (R, α, u) where R is a ring (with unity), u a unit and α an anti-automorphism such that $u^\alpha = u^{-1}$ and $x^{\alpha\alpha} = uxu^{-1}$ for all $x \in R$. A *sesquilinear form* on a right R-module M is a map

$$\phi\colon M \times M \longrightarrow R$$

which is biadditive, R-linear in the second variable, and satisfies $\phi(mx, n) = x^\alpha\phi(m, n)$ for all $x \in R$ and $m, n \in M$. Write $S_\alpha(M)$ for the additive group of such maps. Call ϕ *nonsingular* (on the right) if the map

$$A\phi\colon M \longrightarrow \mathrm{Hom}_R(M, R)$$

defined by $A\phi(m)(n) = \phi(m, n)$ is an isomorphism. The *transposed form* $T_u\phi$ is given by

$$T_u\phi(m, n) = \phi(n, m)^\alpha u\ ;$$

then $T_u\colon S_\alpha(M) \to S_\alpha(M)$ is a homomorphism with $T_u^2 = 1$. The group $Q_{(\alpha,u)}(M)$ of *quadratic forms* is defined as the cokernel of $(1 - T_u)$. For such a form q, represented by $\phi \in S_\alpha(M)$, we define its *bilinearisation* $b_q = \phi + T_u\phi$; this does not depend on choice of representative. Call q nonsingular if b_q is.

We write $\mathcal{P}(R)$ for the category of finitely generated projective (right)

R-modules. A *quadratic module* is a pair (M, q) with M in $\mathcal{P}(R)$ and $q \in Q_{(\alpha,u)}(M)$; it is nonsingular if q is. Write $\mathfrak{Q}(R, \alpha, u)$ for the category whose objects are nonsingular quadratic modules, and morphisms $(M, q) \to (M', q')$ are *isometries*, i.e., isomorphisms $f\colon M \to M'$ such that if ϕ, ϕ' represent q, q' then

$$\phi' \circ (f \times f) - \phi = \chi + T_u \chi$$

for some $\chi \in S_\alpha(M)$.

Next, call a module M *based* if we have an isomorphism $f\colon M \to R^n$ for some n: we regard f, f' as defining the same class of bases (or volume element) v if $f' \circ f^{-1}$ has determinant 1 in $K_1(R) = K_1\mathcal{P}(R)$. Write $\mathcal{B}(R)$ for the category of based modules (M, v) and isomorphisms preserving preferred bases, and $\mathcal{B}\mathfrak{Q}(R, \alpha, u)$ for the category of based quadratic modules (M, q, v) and based isometries. Write $\tilde{K}_0\mathcal{B}\mathfrak{Q}(R, \alpha, u) = \operatorname{Ker} \operatorname{rank}\colon K_0\mathcal{B}\mathfrak{Q}(R, \alpha, u) \to \mathbf{Z}$.

There are [F] exact sequences

$$K_1\mathfrak{Q}(R, \alpha, u) \xrightarrow{F_*} K_1(R) \xrightarrow{\tau} \tilde{K}_0(\mathcal{B}\mathfrak{Q}(R, \alpha, u) ,$$
$$\tilde{K}_0\mathcal{B}\mathfrak{Q}(R, \alpha, -u) \xrightarrow{\partial} K_1(R) \xrightarrow{H} K_1\mathfrak{Q}(R, \alpha, u) ,$$

where F is the forgetful map, τ is induced by changing the preferred basis, ∂ is the discriminant and H induced by the hyperbolic map. We thus write

$$\Lambda_0(R, \alpha, u) = \tilde{K}_0\mathcal{B}\mathfrak{Q}(R, \alpha, u) ,$$
$$\Lambda_1(R, \alpha, u) = K_1\mathfrak{Q}(R, \alpha, u) , \quad \text{and for any } i \in \mathbf{Z} ,$$
$$\Lambda_{i+2}(R, \alpha, u) = \Lambda_i(R, \alpha, -u) .$$

The above exact sequences then take the form

$$\Lambda_{i+1}(R) \xrightarrow{\partial_{i+1}} K_1(R) \xrightarrow{\tau_i} \Lambda_i(R) .$$

Moreover, $\partial_i \circ \tau_i = 1 + (-1)^i \alpha'$, where α' is the automorphism of $K_1(R)$ induced by α. Since α^2 is inner, $\alpha'^2 = 1$.

We now define, for any α'-invariant subgroup X of $K_1(R)$,

$$L_i^X(R) = \partial_i^{-1}(X)/\tau_i(X) .$$

Of particular interest are the case $X = \{0\}$, when we write L_i^S, so that $L_i^S(R) = \operatorname{Ker} \partial_i$, and the case $X = K_1(R)$, when we write L_i^K, so that $L_i^K = \operatorname{Coker} \tau_i$. The groups for different values of X are related by exact sequences. If $X \subset Y$, and we write $H^i(Y/X)$ for the Tate cohomology of a group of order 2 acting on Y/X via α', we have an exact sequence:

$$\cdots L_i^X(R) \longrightarrow L_i^Y(R) \longrightarrow H^i(Y/X) \longrightarrow L_{i-1}^X(R) \longrightarrow L_{i-1}^Y(R) \cdots .$$

One can also give more direct definitions of the groups $L_i^X(R)$. For i

even, we generalise the notion of based quadratic module by using $K_1(R)/X$ instead of $K_1(R)$ as value group for the determinant function. For i odd, one can either spell out the above definition in words or discuss based formations as e.g. in [P].

1.2. Recall of calculations

Next, we recall calculations of L groups from earlier papers of this series. From [II] we recall that L_i is unchanged by Morita equivalence of antistructures where, in particular, $(R, \alpha, u) \sim (R, \alpha', u')$ if for some unit v in R, $u' = vv^{-\alpha}u$ and $x^{\alpha'} = vx^{\alpha}v^{-1}$, and $(R, \alpha, u) \sim (R_n, \alpha_n, u_n)$ where R_n is the ring of $(n \times n)$ matrices over R, for α_n we apply α to each element and transpose, and u_n is the scalar matrix with u on the diagonal. Now if R is semisimple Artinian, we can split (R, α, u) as a sum of simple components and L-groups split accordingly. In the case (type GL) when R is a sum of two simple components interchanged by α, all L groups vanish. Otherwise R is simple, and we may reduce by Morita equivalence to the case when R is a division ring and may even take $u = 1$, or else R commutative and $u = -1$. Assume now R simple, and of finite dimension n^2 over its centre, Z. We say (α, u) has *type U* if $\alpha \mid Z$ is not the identity, and *type O* resp. *type* Sp if $\alpha \mid Z = 1$, $u = \pm 1$, and the dimension over Z of the fixed set of α is $\frac{1}{2}(n^2 + un)$ resp. $\frac{1}{2}(n^2 - un)$: this is equivalent to extending the ground field and reducing by Morita equivalence to the commutative case. The type is that of the algebraic group of automorphisms of a nonsingular quadratic (R, α, u)-module. Since replacing u by $-u$ interchanges types O and Sp and also interchanges L_p and L_{p+2}, we can frequently omit mention of type Sp, and assume $u = 1$. We have the following calculations [II].

Finite fields F. L^S-groups vanish for type U. $L_p^S(F, 1, 1)$ has order 1 for $p \equiv 0, 3 \pmod 4$ and 2 for $p \equiv 1, 2 \pmod 4$, except if char $F = 2$, when all have order 2 (and $L^S = L^K$).

Continuous fields. Apart from type GL, we have only 4 essentially distinct cases: $(\mathbf{R}, 1)$, $(\mathbf{C}, 1)$, $(\mathbf{C}, c)$, $(\mathbf{H}, c)$ ($u = 1$ in each case), where c denotes conjugation in $\mathbf{C}$ and $\mathbf{H}$: these have respective types 0, 0, U, Sp. The signature induces isomorphisms of $L_0^S(\mathbf{H}, c)$ on $2\mathbf{Z}$ (i.e., the group of even integers), and of $L_0^S(\mathbf{R}, 1)$, $L_0^S(\mathbf{C}, c)$ and $L_2^S(\mathbf{C}, c)$ on $4\mathbf{Z}$. The groups $L_1^S(\mathbf{R}, 1)$, $L_2^S(\mathbf{R}, 1)$ and $L_2^S(\mathbf{C}, 1)$ have order 2. The others are trivial.

Local fields (of characteristic $\neq 2$). L^S-groups vanish for type U. If R is a division ring with centre K, and (R, α, u) has type O, $L_p^S(R, \alpha, u)$ has order 2 for $p \equiv 0, 2 \pmod 4$, is trivial for $p \equiv 3 \pmod 4$, and $L_1^S(R, \alpha, u) \cong$

$K^\times/(K^\times)^2$. We shall make repeated use of these results, and of the naturality of the invariants inducing the isomorphisms—in all the cases above except fields of characteristic 2, discriminant for L_1 and Pfaffian for L_2.

For almost all of the paper, R will be a $\mathbf{Z}$-order in a semisimple algebra S, finite over $\mathbf{Q}$, $\hat{R} = R \otimes \hat{\mathbf{Z}}$ its profinite completion, the direct product of the p-adic completions $\hat{R}_p$, $\hat{S} = \hat{R} \otimes \mathbf{Q}$ (and $\hat{S}_p = \hat{R}_p \otimes \mathbf{Q}$), and $T = S \otimes \mathbf{R}$. We shall use determinants calculated in $K_1(\hat{S})$. Since $K_1(\hat{S}_p)$, $K_1(S)$ inject in $K_1(\hat{S})$ this means we study $L_i^S(\hat{S}_p)$, $L_i^S(S)$, $L_i^S(\hat{S})$. Write $X = \operatorname{Ker}\big(K_1(R) \to K_1(S)\big)$; X is sometimes denoted $SK_1(R)$. We shall study $L_i^X(R)$; also (similarly) for $\hat{R}$ and $\hat{R}_p$. It will usually be the case that $\operatorname{Ker}\big(K_1(\hat{R}_p) \to K_1(\hat{S}_p)\big)$ is a finite p-group (we call this 'good reduction'). This implies, if p is odd, that $L_i^X(\hat{R}_p) = L_i^S(\hat{R}_p)$.

Recall from [III] that if $\hat{R}_p$ has radical J_p, then the J-adic and p-adic topologies coincide, and $\hat{R}_p$ is complete. So if $\bar{R}_p = \hat{R}_p/J_p$, the projection $\hat{R}_p \to \bar{R}_p$ induces isomorphisms of groups L_i^S if p is odd, and of L_i^K in any case. Since $\bar{R}_p$ is finite and semisimple, its L groups are among those tabulated above. In case $p = 2$, we 'know' $L_i^K(\bar{R}_2) = L_i^K(\hat{R}_2)$, and to deduce $L_i^X(\hat{R}_2)$ will rely on the exact sequence of the preceding section—thus we need independent calculation of $H^i\big(\operatorname{Im}\big(K_1(\hat{R}_2) \to K_1(\hat{S}_2)\big)\big)$.

We now have the "main exact sequence" from [V]

$$\cdots L_i^X(R) \longrightarrow L_i^X(\hat{R}) \oplus L_i^X(S) \longrightarrow L_i^X(\hat{S}) \longrightarrow L_{i-1}^X(R) \cdots .$$

Now $\hat{S} \oplus T$ is the adele ring, and we have the exact sequence (from [IV])

$$0 \longrightarrow L_i^S(S) \longrightarrow L_i^S(\hat{S}) \oplus L_i^S(T) \longrightarrow CL_i^S(S) \longrightarrow 0 .$$

Combining these, we have ([V, 7.2])

$$\cdots L_i^X(R) \longrightarrow L_i^X(\hat{R}) \oplus L_i^S(T) \longrightarrow CL_i^S(S) \longrightarrow L_{i-1}^X(R) \cdots .$$

Here, $L_i^X(\hat{R}) = \prod_p L_i^X(\hat{R}_p)$ is a product over completions at different primes, which were discussed above.

Finally, we have from [IV] that $CL_p^S(S)$ vanishes for summands of (S, α, u) of type GL or U, and that for a summand of type 0 we have

$$CL_0^S(S) \cong \mathbf{Z}/2 , \quad CL_1^S(S) \cong C_2 , \quad CL_2^S(S) \cong {}_2C , \quad CL_3^S(S) = 0 ,$$

where C is the idele class group of S, and we have the exact sequence

$$0 \longrightarrow {}_2C \longrightarrow C \xrightarrow{\;2\;} C \longrightarrow C_2 \longrightarrow 0 .$$

We shall also need to compute maps between several of the above groups. This will involve more detailed results from the earlier papers, but we may note some salient points. For any commutative field K,

$$L_2^S(K, 1, 1) = L_0^S(K, 1, -1) \cong \{\pm 1\} ,$$

for the skew-symmetric form admits a symplectic base, with respect to which it has determinant 1, and any two such bases are in the same class. The change of basis to our preferred basis thus has determinant x satisfying $x^2 = 1$. Most of the nonzero L_2^S above are computed by this method: maps between them can thus be deduced from the maps of the subgroup of units x with $x^2 = 1$. A similar analysis applies to L_1^S: here the hyperbolic form $H(R)$ admits auto-isometries $e \to ex$, $e^* \to e^* x^{-\alpha}$ for x a unit: the class of the isometry is zero if x is of the form y^2. Here, then, it is the group of units modulo squares which is important.

1.3. The case of curves over finite fields

We now give examples of calculations which, while different from (and much simpler than) the case of group rings, illustrate some of the difficulties to be encountered, and may help the reader to familiarise himself with our methods.

If the base ring $\mathbf{Z}$ is replaced by the polynomial ring $A = \mathbf{F}_p[t]$, p an odd prime, then [V, 6.6] still yields an exact sequence

$$\cdots L_i^X(R) \longrightarrow L_i^X(\hat{R}) \oplus L_i^S(S) \longrightarrow L_i^S(\hat{S}) \longrightarrow \cdots$$

for R an A-order, $\hat{R}$ the product of its completions at primes of A; S and $\hat{S}$ obtained by tensoring with the quotient field $\mathbf{F}_p(t)$ of A. However, the prime ideals of A yield all but one prime of $\mathbf{F}_p(t)$, and it is unnatural to omit this one from the calculation. We must take the completion at the remaining prime to be $\hat{R}_\infty = \hat{S}_\infty = R \otimes_A F_p((t^{-1}))$. For the rest of this section, the notations $\hat{R}$ and $\hat{S}$ will indicate that all primes are included.

If we take R to be the maximal A-order in a finite extension field K of $\mathbf{F}_p(t)$, this is naturally considered the coordinate ring of a nonsingular affine curve. Again, it is more natural not to restrict to affine curves. We will assume here that one can develop an L-theory for schemes (probably in terms of forms on locally free vector bundles), and that the natural extension of the above sequence remains exact: the point here is to show how to compute with it.

Now let K be a finite extension of $\mathbf{F}_p(t)$. Suppose R to be defined locally by maximal orders, except at a finite set E of places where we take $\hat{R}_p = \hat{K}_p$ for $p \in E$; this is considered geometrically as a nonsingular projective curve Γ, with the points of E removed.

First consider a nontrivial involution α. Now the primes $\mathfrak{p}$ of K correspond to algebraic points of the curve Γ, two such being identified if they are conjugate over Γ. Geometrically, α is an involution of Γ. It is now clear that for an algebraic point P, with corresponding prime $\mathfrak{p}$, the involution on $\bar{A}_\mathfrak{p}$ has type GL if and only if $\alpha(P)$ is not conjugate to P; type U if $\alpha(P) \neq P$, but is conjugate to it; and—for the antistructure $(K, \alpha, 1)$—type O if $\alpha(P) = P$. If K_0 is the fixed field of α on K, the third case yields precisely the primes ramified for K/K_0.

Now as $(K, \alpha, 1)$ has type U, $CL^S_*(K) = 0$. The main exact sequence thus yields an isomorphism of the L-group sought with

$$\textstyle\sum_{\mathfrak{p} \notin E} L^S_i(\hat{R}_\mathfrak{p}) \oplus \sum_{\mathfrak{p} \in E} L^S_i(\hat{K}_\mathfrak{p}) ,$$

where $\hat{K}_\mathfrak{p}$ is the completion at $\mathfrak{p}$, $\hat{R}_\mathfrak{p}$ the corresponding valuation ring. But $L^S_i(\hat{K}_\mathfrak{p}) = 0$ for since α is nontrivial, we have type GL or U. Note that, strictly speaking, places $\mathfrak{p} \neq \alpha(\mathfrak{p})$ must come paired in the summation (or we may sum over points of the orbit space Γ/α). Also, as p is odd, $L^S_i(\hat{R}_\mathfrak{p}) = L^S_i(\bar{R}_\mathfrak{p})$ vanishes for type GL or U; i.e., for all save those in the finite set F of ramified primes. Thus

PROPOSITION 1.3.1. *In the unitary case, with the notations above,*

$$L^S_i(R) = \textstyle\sum_{\mathfrak{p} \in (F-E)} L^S_i(\hat{R}_\mathfrak{p}) ,$$

and we have the table

i	0	1	2	3
$L^S_i\hat{R}_\mathfrak{p}$	0	$\hat{R}^\times_\mathfrak{p}/(\hat{R}^\times_\mathfrak{p})^2$ (*order* 2)	$\{\pm 1\}$	0 .

Now consider the case of quadratic forms, i.e., with R as above, the antistructure $(R, 1, 1)$. Observe first that the map $j\colon \hat{R}^\times \oplus S^\times \to \hat{S}^\times$, in the case in which E is empty, has as kernel the constants in S—for only these have no poles or zeros, so locally lie in $R^\times$. Write k for the field of constants. The cokernel of j is the group of divisor classes.

LEMMA 1.3.2. Cok $j \cong \mathbf{Z} \oplus J$, *where J is the group of k-rational points on the Jacobian curve of* Γ.

For this result, see, e.g., [22].

Now $L^S_i(S) \to L^S_i(\hat{S})$ is injective, as in the number field case, with the same cokernel $CL^S_i(S)$. We can tabulate these groups as follows:

i		0	1	2	3
$L^S_i(\hat{R}_\mathfrak{p})$	$\mathfrak{p} \notin E$	0	$\hat{R}^\times_\mathfrak{p}/(\hat{R}^\times_\mathfrak{p})^2$	$\{\pm 1\}$	0
$L^S_i(\hat{R}_\mathfrak{p}) = L^S_i(\hat{S}_\mathfrak{p})$	$\mathfrak{p} \in E$	$\mathbf{Z}/2$	$\hat{S}^\times_\mathfrak{p}/(\hat{S}^\times_\mathfrak{p})^2$	$\{\pm 1\}$	0
$CL^S_i(S)$		$\mathbf{Z}/2$	$\hat{S}^\times/S^\times \cdot (\hat{S}^\times)^2$	$(\prod_\mathfrak{p} \{\pm 1\})/\{\pm 1\}$	0 .

Here we have interpreted the terms C_2, ${}_2C$ involving the idele class group in explicit terms. We thus have, for the equivalent maps

$$\beta_i\colon L_i^S(\hat{R}) \oplus L_i^S(S) \longrightarrow L_i^S(\hat{S}) \ , \qquad \beta_i'\colon L_i^S(\hat{R}) \longrightarrow CL_i^S(S) \ ,$$
$$\operatorname{Ker} \beta_3 = \operatorname{Cok} \beta_3 = 0 \ .$$
$$\operatorname{Ker} \beta_2 = \{\pm 1\} \ , \quad \operatorname{Cok} \beta_2 = 0 \ .$$
$$\operatorname{Ker} \beta_0 \text{ is a sum of } |E| - 1 \text{ copies of } \mathbf{Z}/2$$
$$\text{(it vanishes for } E = \varnothing) \ .$$
$$\operatorname{Cok} \beta_0 \cong \mathbf{Z}/2\,(E = \varnothing) \quad \text{or} \quad 0\,(E \neq \varnothing) \ .$$

For β_0, β_2 are the obvious maps, as L_0 is given by the Hasse invariant in all cases, L_2 by the Pfaffian.

We must consider β_1 more carefully. It is the natural map

$$\hat{R}^\times/(\hat{R}^\times)^2 \longrightarrow \hat{S}^\times/S^\times\cdot(\hat{S}^\times)^2 \ .$$

Now an element of S is a square if it is so locally—i.e., $S^\times/(S^\times)^2 \to \hat{S}^\times/(\hat{S}^\times)^2$ is injective. Thus β_1 has the same kernel and cokernel as the more natural map

$$\hat{R}^\times/(\hat{R}^\times)^2 \oplus S^\times/(S^\times)^2 \longrightarrow \hat{S}^\times/(\hat{S}^\times)^2 \ .$$

But now $\hat{R}^\times/(\hat{R}^\times)^2 \to \hat{S}^\times/(\hat{S}^\times)^2$ is clearly injective, and its cokernel is D_E/D_E^2 where $D_E = \hat{S}^\times/\hat{R}^\times$ is the group of divisors modulo E. By the same argument, we have the same kernel and cokernel as for

$$S^\times/(S^\times)^2 \longrightarrow D_E/D_E^2 \ .$$

If $C_E = D_E/S^\times$ is the group of divisor classes (mod E), we see—as is clear anyway—that $\operatorname{Cok} \beta_1 = C_E/C_E^2$. Observe that $C_\phi = \mathbf{Z} \oplus J$, and $C_E \cong J$ if $|E| = 1$.

Write $S^{(2)}$ for the set of $x \in S^\times$ whose divisors (mod E) are squares. Then $\operatorname{Ker} \beta_1 = S^{(2)}/(S^\times)^2$. More useful for computation is the exact sequence

$$1 \longrightarrow R^\times/(R^\times)^2 \longrightarrow S^{(2)}/(S^\times)^2 \longrightarrow {}_2C_E \longrightarrow 1 \ .$$

For if the divisor $\langle x\rangle = Q^2$, Q defines an element of ${}_2C_E$. Replacing x by xy^2 changes Q to $Q\langle y\rangle$ and leaves its class unaltered. And if Q does define an element of ${}_2C_E$, Q^2 is principal. So we have a surjection $S^{(2)}/(S^\times)^2 \to {}_2C_E$. If x defines an element of the kernel, $\langle x\rangle = Q^2$ with Q principal, $Q = \langle y\rangle$, then xy^{-2} has unit divisor and belongs to $R^\times$. The rest of the proof is routine. One can argue alternatively using the exact sequences $1 \to R^\times \to S^\times \to P \to 1$, $1 \to P \to D_E \to C_E \to 1$ (P the group of principal divisors) and their cohomology exact sequences for a group of order 2 acting trivially. Note finally that since $S^{(2)}/(S^\times)^2$ has exponent 2, the exact sequence splits.

Finally, $L_i^X(R)$ is an extension of $\operatorname{Cok} \beta_{i+1}$ by $\operatorname{Ker} \beta_i$. Thus

THEOREM 1.3.3. *In the orthogonal case,*

$$\begin{aligned}&L_0^X(R) \textit{ is an extension of } C_E/C_E^2 \textit{ by } (|E|-1)\mathbf{Z}/2\ ,\\ &L_1^X(R) \textit{ is a split extension of } R^\times/(R^\times)^2 \textit{ by } {}_2C_E\ ,\\ &L_2^X(R) \cong \{\pm 1\}\ ,\\ &L_3^X(R) \cong \mathbf{Z}/2\,(E=\varnothing) \textit{ or } 0\,(E\neq\varnothing)\ .\end{aligned}$$

Here, L_3 (when nonzero) expresses reciprocity and L_2 is detected by the Pfaffian. For L_1, one can consider $R^\times/(R^\times)^2$ as a sort of spinor norm, but the full group is not easy to comprehend. The map of C_E to L_0 is induced by applying the hyperbolic functor to a projective (non-free) module, which certainly yields a free module, and by taking account of bases.

Let us illustrate these results by considering some cases of particular interest. First take $E=\varnothing$, so we have a projective curve. Then ${}_2C_E={}_2J$, and $R^\times=k^\times$ so $R^\times/(R^\times)^2$ has order 2. Here,

$$\begin{aligned}&L_0^X(R)\cong J/J^2\oplus \mathbf{Z}/2\ , \quad L_1^X(R) \text{ is a split extension of } \mathbf{Z}/2 \text{ by } {}_2J\ ,\\ &L_2^X(R)\cong\{\pm1\}\ ,\qquad L_3^X(R)\cong \mathbf{Z}/2\ .\end{aligned}$$

For example, if we take Γ as the projective line, $J=0$ and all four groups have order 2.

Next take Γ as an affine line. Then $J=0$, $|E|=1$, and $C_E=0$. Thus

$$L_0^X(\mathbf{F}_p[t])=0\ ,\qquad L_1^X(\mathbf{F}_p[t])\cong\mathbf{Z}/2\ ,\qquad L_2^X(\mathbf{F}_p[t])\cong\{\pm1\}\ ,\qquad L_3^X(\mathbf{F}_p[t])=0$$

and we obtain groups naturally isomorphic to $L_i^X(\mathbf{F}_p)$, as was to be expected.

Finally, take $R=\mathbf{F}_p[t,t^{-1}]$; Γ is the projective line with two points deleted. Again $J=0$, but now $|E|=2$, $C_E=0$ and the units of R have the form at^i ($a\in\mathbf{F}_p^\times$, $i\in\mathbf{Z}$). Thus

$$L_0^X\cong\mathbf{Z}/2\ ,\quad L_1^X\cong\mathbf{Z}/2+\mathbf{Z}/2\ ,\quad L_2^X\cong\{\pm1\}\ ,\quad L_3^X=0$$

agreeing with Ranicki's ([18]) calculation (note $\alpha=$ identity)

$$L_i^S(R)\cong L_i^S(\mathbf{F}_p)\oplus L_i^K(\mathbf{F}_p)\ .$$

1.4. The case $R=\mathbf{Z}$

We now illustrate our methods by considering $(R,\alpha,u)=(\mathbf{Z},1,1)$. The technique appears cumbersome in this case, but the details are not hard to follow, and we will find that most of the features of more general cases are already present here.

Since $\mathbf{Z}$ is a principal ideal domain, if I is the group of ideals, $\mathbf{Q}^\times\cong I\oplus\{\pm1\}$, $\hat{\mathbf{Q}}^\times\cong\hat{\mathbf{Z}}^\times\oplus I$ and $\mathbf{R}^\times\cong\mathbf{R}^*\oplus\{\pm1\}$. Thus the idele class group C is isomorphic to $\hat{\mathbf{Z}}^\times\oplus\mathbf{R}^*\cong\prod_p\hat{\mathbf{Z}}_p^\times\oplus\mathbf{R}^*$, and we have

$${}_2C\cong\textstyle\prod_p\{\pm1\}\ ,\qquad C_2\cong\prod_p(\hat{\mathbf{Z}}_p^\times/(\hat{\mathbf{Z}}_p^\times)^2)\ .$$

Since, for p odd, $L_i^S(\hat{\mathbf{Z}}_p,1,1)\cong 0$, $\hat{\mathbf{Z}}_p^\times/(\hat{\mathbf{Z}}_p^\times)^2$, $\{\pm1\}$, 0 for $i=0,1,2,3$ (respectively), we have the following table, in which $\prod$ denotes product over *odd* primes:

i	$\prod L_i^S(\hat{\mathbf{Z}}_p) \oplus L_i^S(\mathbf{R}) \longrightarrow CL_i^S(\mathbf{Q})$
3	$0 \oplus 0 \longrightarrow 0$
2	$\prod \{\pm 1\} \oplus \{\pm 1\} \longrightarrow \prod \{\pm 1\} \oplus \{\pm 1\}$
1	$\prod (\mathbf{Z}/p\mathbf{Z})^\times/((\mathbf{Z}/p\mathbf{Z})^\times)^2 \oplus \{\pm 1\} \longrightarrow \prod (\hat{\mathbf{Z}}_p^\times/(\hat{\mathbf{Z}}_p^\times)^2) \oplus \hat{\mathbf{Z}}_2^\times/(\hat{\mathbf{Z}}_2^\times)^2$
0	$0 \oplus 4\mathbf{Z} \longrightarrow \mathbf{Z}/2\,.$

We now determine the maps. The calculation for $CL_0^S(\mathbf{Q})$ involved the Brauer group: $\mathbf{Z}/2$ was essentially the cokernel of $\mathrm{Br}_2(\mathbf{Q}) \to \mathrm{Br}_2(\hat{\mathbf{Q}})$. Now $L_0^S(\mathbf{R}) \cong 4\mathbf{Z}$ maps onto $\mathrm{Br}_2(\mathbf{R}) \cong \mathbf{Z}/2\mathbf{Z}$ as is well-known, hence onto $CL_0^S(\mathbf{Q})$.

For $i = 1$, observe that projection $\hat{\mathbf{Z}}_p \to \mathbf{Z}/p$ induces a (split) surjection $\mathbf{Z}_p^\times \to (\mathbf{Z}/p)^\times$ with kernel a pro-p-group; hence for p odd, an isomorphism,

$$\hat{\mathbf{Z}}_p^\times/(\hat{\mathbf{Z}}_p^\times)^2 \longrightarrow (\mathbf{Z}/p)^\times/(\mathbf{Z}/p)^{\times 2}\,.$$

So the product over odd p maps isomorphically (by naturality of spinor norms, the map given by L-theory is the obvious one). Now in our decomposition of C, the image of $-1 \in \mathbf{R}^\times$ has -1 in each position, for $-1 \in \mathbf{Q}$ maps to zero, and if we adjust by this to obtain a positive adele, the assertion becomes clear. Thus our map for $i = 1$ is injective.

For $i = 2$, a similar (but easier) argument shows that the product term—and the whole group—map isomorphically. For $i = 3$ there is nothing to do.

Now we must evaluate $L_i^S(\hat{\mathbf{Z}}_2)$. We have

$$L_i^K(\hat{\mathbf{Z}}_2) \cong L_i^K(\mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}\,,$$

and since $\hat{\mathbf{Z}}_2^\times \cong \hat{\mathbf{Z}}_2 \oplus \mathbf{Z}/2\mathbf{Z}$ (with generators 5, -1), $H^i(\hat{\mathbf{Z}}_2^\times)$ has rank 2 for i even, 1 for i odd. The exact sequence to compute $L_i^S(\hat{\mathbf{Z}}_2)$ was discussed as a special case of [III, Theorem 11]; we have

$$L_0^S(\hat{\mathbf{Z}}_2) = 0\,, \quad L_1^S(\hat{\mathbf{Z}}_2) \cong \hat{\mathbf{Z}}_2^\times/(\hat{\mathbf{Z}}_2^\times)^2\,, \quad L_2^S(\hat{\mathbf{Z}}_2) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$$

and $L_3^S(\hat{\mathbf{Z}}_2)$ has order 4. In fact, this group is cyclic. Consider the automorphism τ of a hyperbolic plane over $(\hat{\mathbf{Z}}_2, 1, -1)$ given by

$$\tau(e) = f\,, \qquad \tau(f) = -e$$

so that $\tau^2 = -1$. I claim that the class of τ has order 4 in the group. For τ^2 is the image by the hyperbolic map of the class $H^0(\hat{\mathbf{Z}}_2^\times)$ represented by -1, and it suffices to show this is not in the image of $L_4^K(\hat{\mathbf{Z}}_2)$. But this group has order 2, with nonzero element represented by

$$\phi \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{so} \quad b_q \sim \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

with discriminant -3.

Since $L_i^S(\hat{\mathbf{Z}}) \oplus L_i^S(\mathbf{R}) \to CL_i^S(\mathbf{Q})$ is surjective for all i, $L_1^X(\mathbf{Z})$ is the kernel. Thus

PROPOSITION 1.4.1.

$$L_0^X(\mathbf{Z}) \cong 8\mathbf{Z}\ , \quad L_1^X(\mathbf{Z}) \cong \{\pm 1\}\ , \quad L_2^X(\mathbf{Z}) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2\ , \quad L_3^X(\mathbf{Z}) \cong \mathbf{Z}/4\ .$$

This slightly unexpected result is explained as follows. As $K_1(\mathbf{Z}) \cong \{\pm 1\}$, X is trivial, and it is more usual to consider $L_i^K(\mathbf{Z})$. In the exact sequence relating these, the boundary maps $\{\pm 1\} \to L_i^S(\mathbf{Z})$ are computed as follows:

$i = 0$	clearly trivial ;
$i = 3$	computed above (essentially): the image is τ^2 in $L_3^S(\mathbf{Z}) \cong L_3^S(\hat{\mathbf{Z}}_2)$;
$i = 2$	$H^1(\mathbf{Z}^\times) \cong H^1(\hat{\mathbf{Z}}_2^\times)$, $L_2^S(\mathbf{Z}) \cong L_2^S(\hat{\mathbf{Z}}_2)$ are isomorphisms induced by inclusion. The exact sequence for $\hat{\mathbf{Z}}_2$ is known: the map is injective .
$i = 1$	Here, $H^0(\hat{\mathbf{Z}}_2^\times) \longrightarrow L_1^S(\hat{\mathbf{Z}}_2)$ is an isomorphism and $\{\pm 1\} = H^0(\mathbf{Z}^\times) \longrightarrow H^0(\hat{\mathbf{Z}}_2^\times)$, $L_1^S(\mathbf{Z}) \longrightarrow L_1^S(\hat{\mathbf{Z}}_2)$ are injective. Thus $\{\pm 1\} \longrightarrow L_1^S(\mathbf{Z})$ is injective, hence bijective .

We deduce

$$L_0^K(\mathbf{Z}) \cong 8\mathbf{Z}\ , \quad L_1^K(\mathbf{Z}) \cong \mathbf{Z}/2\ , \quad L_2^K(\mathbf{Z}) \cong \mathbf{Z}/2\ , \quad L_3^K(\mathbf{Z}) \cong \mathbf{Z}/2\ .$$

To obtain surgery obstruction groups (as defined in [SCM]) for trivial fundamental groups, it only remains to kill the classes in L_1, L_3 of the automorphism σ (or τ). This yields the familiar sequence

$$8\mathbf{Z}\ , \quad 0\ , \quad \mathbf{Z}/2\ , \quad 0\ .$$

1.5. Integers in an algebraic number field: unitary case

We now consider the more difficult example, when $S = K$ is an algebraic number field and $R = A$ the ring of integers in it. The reader will observe the similarities—and differences—with the case considered in (1.3). First we recapitulate [I] by considering type U: here, α is a nontrivial involution with fixed field K_0 and integers A_0. We consider only the antistructure $(R, \alpha, 1)$. Since $CL_i^S(S) = 0$, the main exact sequence reduces to an isomorphism

$$L_i^X(A) \cong L_i^X(\hat{A}) \oplus L_i^S(K \otimes_{\mathbf{Q}} \mathbf{R})\ .$$

Moreover, as $SK_1(A) = 0$, $L_i^X(A) = L_i^S(A)$.

The second summand is easy: it vanishes for i odd, and has σ summands

$4\mathbf{Z}$ for i even. Here, if $K_0 \otimes \mathbf{R} \cong r_1\mathbf{R} \oplus r_2\mathbf{C}$ and $K \otimes \mathbf{R} \cong s_1\mathbf{R} \oplus s_2\mathbf{C}$ (so $s_1 + 2s_2 = 2(r_1 + 2r_2)$),

$$\sigma = s_2 - 2r_2 = \frac{1}{2}(2r_1 - s_1) .$$

If $\mathfrak{p}$ is a prime of A_0 unramified in A, then $\bar{A}_\mathfrak{p}$ has type U (inert case) or GL (decomposed case), and in either case $L_i^S(\hat{A}_\mathfrak{p}) = L_i^S(\bar{A}_\mathfrak{p}) = 0$ for all i. If $\mathfrak{p}$ is a ramified prime, $\bar{A}_\mathfrak{p}$ has type O. Thus if $\mathfrak{p}$ is odd, $L_i^S(\hat{A}_\mathfrak{p})$ has order 2 for $i = 1, 2$ and is trivial for $i = 0, 3$.

Finally, suppose $\mathfrak{p}$ even and ramified (or equivalently, wildly ramified). Then $L_i^K(\hat{A}_\mathfrak{p})$ has order 2 for all $\mathfrak{p}$. To compute $H^i(\hat{A}_\mathfrak{p}^\times)$, we use the exact sequence

$$1 \longrightarrow \hat{A}_\mathfrak{p}^\times \longrightarrow \hat{K}_\mathfrak{p}^\times \longrightarrow \mathbf{Z} \longrightarrow 1 .$$

Since $H^1(\hat{K}_\mathfrak{p}^\times) = 0$ (by Hilbert's Satz 90), and $H^0(\hat{K}_\mathfrak{p}^\times) \cong \mathbf{Z}/2$ by local class field theory, the cohomology exact sequence of the above is

$$H^1(\mathbf{Z}) = 0 \longrightarrow H^0(\hat{A}_\mathfrak{p}^\times) \longrightarrow \mathbf{Z}/2 \longrightarrow \mathbf{Z}/2 \longrightarrow H^1(\hat{A}_\mathfrak{p}^\times) \longrightarrow H^1(\hat{K}_\mathfrak{p}^\times) = 0 .$$

Here the middle map is 0 because, $\mathfrak{p}$ being ramified, invariants have even values. Next we compute

$$\mathbf{Z}/2 \cong L_i^K(\hat{A}_\mathfrak{p}) \longrightarrow H^i(\hat{A}_\mathfrak{p}^\times) \cong \mathbf{Z}/2 .$$

As we observed in [III], this is equivalent to Theorems 3 and 4 of [I]. We discuss the problem in our present terminology. As there is no fear of ambiguity, we drop the subscript $\mathfrak{p}$.

For i even, say $i = 2k$, choose $b \in \hat{A}$ with class $\beta \in \bar{A}$ not of the form $\xi + \xi^2$. Then the form over $\bar{A}$ with matrix $\begin{pmatrix} 1 & 1 \\ 0 & \beta \end{pmatrix}$ has nontrivial Arf invariant, so represents the nonzero element of $L_{2k}^K(\bar{A})$. We lift this to $\hat{A}$, lifting β to b; it has bilinearisation

$$\begin{pmatrix} 1 + (-1)^k & 1 \\ (-1)^k & b + (-1)^k\bar{b} \end{pmatrix}$$

and thus discriminant $1 - (-1)^k(1 + (-1)^k)(b + (-1)^k\bar{b})$. If $k = 1$, this is 1. Also it is 1 if $k = 0$ and $\bar{b} = -b$: now we can always choose $\bar{b} = b$ (since $\bar{A} = \bar{A}_0$) and if $\hat{K} = \hat{K}_0[\sqrt{d}]$, with d a unit, we can suppose $d \to 1 \in \bar{A}$ and then $\bar{d} = -d$ and we can choose bd for b above. Otherwise, $k = 0$, $b = \bar{b}$ and d is prime. The discriminant is $1 - 4b$, and as d is prime, $(1 - 4b, d) = -1$ so this is not a norm.

For i odd, say $i = 2k + 1$, consider the automorphism

$$\tau = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$$

of the hyperbolic plane. This has determinant $(-1)^{k+1}$ which is clearly trivial for k odd. For k even it is -1, hence of the form $\bar{u}/u$ (u a unit) precisely when d (above) is a unit.

Thus if $\hat{K} = \hat{K}_0[\sqrt{d}]$, with d a unit, all four maps $L_i^K(\hat{A}) \to H^i(\hat{A}^\times)$ are trivial, so each group $L_i^S(\hat{A})$ has order 4. Note here that $(\hat{A}, \alpha, 1) \sim (\hat{A}, \alpha, -1)$, a Morita equivalence, so L_i has period 2 in i. We now show that the extensions are all split. First consider L_1: the interchange τ of factors in a hyperbolic plane maps onto L_1^K, but $\tau^2 = 1$. The only other case we need discuss is L_2, and here we can use the argument of [III, Theorem 11, Corollary].

If, however, d is a prime, the exact sequences show $L_0^S(\hat{A}) = 0$, $L_1^S(\hat{A})(\cong H^2(\hat{A}^\times))$, and $L_3^S(\hat{A})(\cong L_3^K(\hat{A}))$ have order 2, and $L_2^S(\hat{A})$ has order 4. In this case too, the extension is split, for $H^1(\hat{A}^\times) = \{\pm 1\}$ as we saw above, and we can again argue as in [III, Theorem 11, Corollary].

1.6. Integers in an algebraic number field: orthogonal case

Retaining the notation of (1.5), we now turn to the orthogonal case, and consider the antistructure $(A, 1, 1)$. First we compute $L_i^S(\hat{A}_2)$; here we consider the g_2 summands $\hat{A}_\mathfrak{p}$, corresponding to the primes $\mathfrak{p}$ of K dividing 2, singly. By [III, Theorem 11, Corollary],

$$L_0^S(\hat{A}_\mathfrak{p}) = 0 , \quad L_1^S(\hat{A}_\mathfrak{p}) \cong \hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2 , \quad L_2^S(\hat{A}_\mathfrak{p}) \cong \mathbf{Z}/2 \oplus \{\pm 1\} ,$$

and there is an exact sequence

$$0 \longrightarrow \mathbf{Z}/2 \xrightarrow{(1-4b)} \hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2 \longrightarrow L_3^S(\hat{A}_\mathfrak{p}) \longrightarrow \mathbf{Z}/2 \longrightarrow 0 .$$

Here the usual interchange τ of basis elements in a hyperbolic plane represents the nonzero element of the final $\mathbf{Z}/2$, and $\tau^2 = -1$. Since $\hat{K}_\mathfrak{p}[\sqrt{(1-4b)}]$ is unramified and either -1 is a square or $\hat{K}_\mathfrak{p}[\sqrt{(-1)}]$ is ramified, we see that if -1 is a square in $\hat{K}_\mathfrak{p}$, the extension splits, but otherwise -1 and $(1-4b)$ are independent in $\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2$, and it doesn't split.

To compute the main exact sequence, we have now to consider the maps

$$\beta_i \colon L_i^S(\hat{A}) \oplus L_i^S(K \otimes_\mathbf{Q} \mathbf{R}) \longrightarrow CL_i^S(K) .$$

We first tabulate the calculations of the various types of summand that occur. Recall that for any abelian group G, ${}_2G = \{x \in G \colon x^2 = 1\}$ and $g(G)$ is the sum of g copies of G.

i	$L_i^S(\hat{A}_p)$ (p odd)	$L_i^S(\hat{A}_2)$	$L_i^S(\mathbf{R})$	$L_i^S(\mathbf{C})$	$CL_i^S(K)$
3	0	$L_3^S(\hat{A}_2)$	0	0	0
2	${}_2(\hat{A}_p^\times)$	${}_2(\hat{A}_2^\times) \oplus g_2(\mathbf{Z}/2)$	$\{\pm 1\}$	$\{\pm 1\}$	${}_2C$
1	$\hat{A}_p^\times/(\hat{A}_p^\times)^2$	$\hat{A}_2^\times/(\hat{A}_2^\times)^2$	$\{\pm 1\}$	0	C/C^2
0	0	0	$4\mathbf{Z}$	0	$\mathbf{Z}/2$

Here, three of the maps are easy. $\beta_3 = 0$ as the target is zero. If K is totally complex, $\beta_0 = 0$; otherwise, each summand $4\mathbf{Z}$ maps onto $\mathbf{Z}/2$ (for the Hasse symbol of a form of signature 4 is -1). Write Σ for the free abelian group Ker β_0. If x is an idele whose class has order 2, x^2 is principal, say $x^2 = y \in K^\times$. But y is locally, hence globally, a square: $y = z^2$, $z \in K^\times$. Thus xz^{-1} has order 2, and the same class as x. Hence ${}_2C = {}_2K_A^\times/{}_2K^\times$, where $K_A = \hat{K} \oplus (K \otimes_{\mathbf{Q}} \mathbf{R})$, is the adele ring of K. Now β_2 is seen to be surjective; its kernel is ${}_2K^\times \oplus g_2(\mathbf{Z}/2)$. Of course, ${}_2K^\times = \{\pm 1\}$.

As to β_1, we see, exactly as in (1.3), that it has cokernel Γ/Γ^2, where Γ is the ideal class group of K, and kernel $K^{(2)}/(K^\times)^2$, where $K^{(2)}$ is the subgroup of $K^\times$ of elements generating square ideals, or equivalently, with all $\mathfrak{p}$-adic values even. Moreover, there is a short exact sequence

$$1 \longrightarrow A^\times/(A^\times)^2 \longrightarrow K^{(2)}/(K^\times)^2 \longrightarrow {}_2\Gamma \longrightarrow 0 \ ,$$

necessarily split.

Now $L_i^S(A) = L_i^X(A)$ is an extension of Cok β_{i+1} by Ker β_i. Thus

$$\begin{aligned} L_0^S(A) &\cong \Gamma/\Gamma^2 \oplus \Sigma \ , \\ L_1^S(A) &\cong K^{(2)}/(K^\times)^2 \cong A^\times/(A^\times)^2 \oplus {}_2\Gamma \ , \\ L_2^S(A) &\cong \{\pm 1\} \oplus g_2(\mathbf{Z}/2) \ , \quad \text{and} \\ L_3^S(A) &= L_3^S(\hat{A}_2) \qquad (K \text{ not totally complex}) \ , \end{aligned}$$

$L_3^S(A)$ is an extension of $\mathbf{Z}/2$ by $L_3^S(\hat{A}_2)$ if K is totally complex. An explicit description of $L_3^S(\hat{A}_2)$ is given above, but I do not know how to determine the final extension in the totally complex case; in particular, whether the group has exponent 4 or 8 is an open question of some interest.

Of course, (1.4) is contained as a special case in the considerations above. Another special case, which will be important in (3.3), is the ring of Gaussian integers $A = \mathbf{Z}[i]$ where (as usual) $i^2 = -1$. Here we can read off, from the above, since $\Gamma = 0$,

i	0	1	2	3
$L_i^S(A)$	0	$A^\times/(A^\times)^2$	$\mathbf{Z}/2 \oplus \{\pm 1\}$	order 8
$L_i^S(\hat{A}_2)$	0	$\hat{A}_2^\times/(\hat{A}_2^\times)^2$	$\mathbf{Z}/2 \oplus \{\pm 1\}$	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$.

We shall be more interested below in the relative group which (see (3.1)) can be computed as the third term in an exact sequence including

$$L_i^s(\hat{A}_{\mathrm{odd}}) \oplus L_i^s(\mathrm{C}) \longrightarrow CL_i^s(\mathbf{Q}[i]) .$$

As there is just one 2-adic prime, this map is an isomorphism for $i = 2, 3$; for $i = 0$ it is injective, with cokernel $\mathbf{Z}/2$ and for $i = 1$ we can see directly that it is injective, with cokernel $\hat{A}_2^\times / A^\times \cdot (\hat{A}_2^\times)^2$, of rank 2. Indeed, $\hat{A}_2^\times$ is the direct sum of its torsion subgroup $A^\times$, which is cyclic of order 4, and the subgroup $(1 + 2\mathfrak{p}\hat{A}_2)^\times$, isomorphic by the 2-adic logarithm to $(2\mathfrak{p}\hat{A}_2)^+$, the ideal generated by $(2 + 2i)$, which we denote by L. Thus the relative groups are $\mathbf{Z}/2$, $L/2L$, 0, 0.

2. Representation theory

We described in the last chapter a general technique for computing L groups of orders. We are primarily interested in the case when these orders are group rings, and wish to make as explicit calculations as possible under rather general circumstances. This will depend on the exploitation of particular properties of group rings, and these properties depend on representation theory.

In (2.1) we summarise the induction theorems of Andreas Dress in the form in which we will apply them. This will permit us to restrict attention to the case of hyperelementary groups; and at the end of the chapter we perform the calculations for p-hyperelementary groups with p odd. In (2.2) we discuss real representation theory and signatures; and in (2.3) we apply modular representation theory. Finally in (2.4) we survey the application of representation theory techniques to our problem, and give a direct argument for p-hyperelementary groups with p odd.

2.1. Recall of induction theorems

We assume known the concept of *Green functor* G, usually thought of ([13]) as defined on subgroups of π, with rings as values, and with notions of restriction and induction satisfying the usual properties. A slicker formulation is given in [10]: consider G as a functor on the category $\Omega(\pi)$ of finite π-sets. We also have the notion of *module* M over a Green functor G.

There exists a collection $\mathfrak{D}(G)$ of subgroups of π (which is closed under conjugacy) called the *defect set* of G and characterised by:

(2.1.1) $G(X) \to G(\cdot)$ *is surjective if and only if each subgroup* $\sigma \in \mathfrak{D}(G)$ *of* π *has a fixed point in* X.

When this surjectivity holds, for any G module M, there is an exact sequence

$$0 \longrightarrow M(\cdot) \longrightarrow M(X) \longrightarrow M(X \times X) .$$

If we change to the other notation, where the functors are defined at subgroups of π, then if $\{\sigma_i\}$ runs through representatives of conjugacy classes in $\mathfrak{D}(G)$, the exact sequence appears as

$$0 \longrightarrow M(\pi) \longrightarrow \bigoplus_i M(\sigma_i) \longrightarrow \bigoplus_{i,j,x} M(\sigma_i^x \cap \sigma_j)$$

where x runs through representatives of double cosets of σ_i and σ_j. Thus we can compute $M(\pi)$ once we know the groups $M(\sigma)$ for all σ in the defect set (and all the relevant maps between them).

We shall need just two examples of Green functor, whose constructions are fairly similar.

For S a finite π-set write $\mathbb{S}$ for the category with object set S and morphism set $S \times \pi$, where

$$(s, g): s \longrightarrow sg .$$

For any category $\mathcal{C}$, write $[\mathbb{S}, \mathcal{C}]$ for the category whose objects are functors and morphisms natural transformations. In particular if R is a commutative ring and $\mathcal{C} = \mathcal{P}(R)$ the category of finitely generated projective R-modules we write $X_\pi(R)(S)$ for the Grothendieck group of objects of $[\mathbb{S}, \mathcal{P}(R)]$ modulo short exact sequences. We can regard $X_\pi(R)$ as contravariant functor using composition of functions, and as covariant functor using pullbacks: it is easy to verify that we have a Mackey functor, and further (using $\otimes_R$ to define a product), a Green functor.

Next we can take $\mathcal{C} = \mathcal{L}(R)$, the category of pairs (P, ϕ) where P is a finitely generated projective R-module and $\phi: P \times P \to R$ a nonsingular symmetric bilinear form; morphisms are isomorphisms of such pairs. A *sublagrangian* is a direct summand Q of P (as module) with $\phi(Q \times Q) = 0$. Write $Q^\perp$ for the submodule orthogonal to Q; then $Q^\perp \supset Q$, and ϕ induces a nonsingular form $[\phi]_Q$ on $Q^\perp/Q$. There are functors $F: \mathcal{L}(R) \to \mathcal{P}(R)$ (forget ϕ) and $H: \mathcal{P}(R) \to \mathcal{L}(R)$ defined by

$$H(Q) = \big(Q \oplus \operatorname{Hom}_R(Q, R), \phi_Q\big) ,$$
$$\phi_Q\big((q_1, f_1), (q_2, f_2)\big) = f_1(q_2) + f_2(q_1) .$$

Given a sublagrangian Q of (P, ϕ), there is an isomorphism of (P, ϕ) on $\big(Q^\perp/Q, [\phi]_Q\big) \oplus H(Q)$. We thus define $U_\pi(R)(S)$ to be the quotient of the Grothendieck group of $[\mathbb{S}, \mathcal{L}(R)]$ by the relations, for each sublagrangian $Q: \mathbb{S} \to \mathcal{P}(R)$ of $(P, \phi): \mathbb{S} \to \mathcal{L}(R)$,

$$\big(Q^\perp/Q, [\phi]_Q\big) + H(Q) - (P, \phi) .$$

Induced maps etc. are defined as in the case of X_π.

THEOREM 2.1.2. *For any commutative ring R, $X_\pi(R)$ and $U_\pi(R)$ are Green functors on $\Omega(\pi)$. The defect set of each is contained in the set of hyperelementary subgroups of π. For $X_\pi(R)$ or $U_\pi(R)$ localised at 2, we need only 2-hyperelementary subgroups.*

Note that by a result of Swan [5, p. 590]

$$X_\pi(R) \otimes \mathbf{Q} = X_\pi(R \otimes \mathbf{Q}) \otimes \mathbf{Q} ,$$

so the results on X_π follow from Artin's induction theorem. The assertions concerning U_π follow from the main result of [11].

Note that F, H induce transformations $U_\pi \underset{H_\pi}{\overset{F_\pi}{\rightleftarrows}} X_\pi$, with F_π a morphism of Green functors: thus any X_π module is a U_π-module. Note also that the category of U_π-modules is an abelian category, hence closed under formation of kernels and images. We now list important examples of modules to which the theory applies. Here we adopt a different notation and consider $M(\pi/\sigma)$ as functor of σ; since any finite π-set is a sum of orbits of this form, and M takes sums to products, this is enough to determine M.

PROPOSITION 2.1.3. *For any R-algebra A and $n = 0, 1$, $K_n\mathcal{P}(A\sigma)$ is an $X_\pi(R)$-module and $K_n\mathcal{Q}(A\sigma)$ a $U_\pi(R)$-module.*

Presumably the result holds for all n; also, the relative groups of F, H should be $U_\pi(R)$-modules and hence all the algebraic L-theory of $A\sigma$: certainly this holds in low dimensions, where the groups are kernels and cokernels of F and H on K_0 and K_1. We observe finally that if $X \subset K_1\mathcal{P}(A\sigma)$ is an $X_\pi(A)$-subfunctor, invariant under the involution α, then $L_n^X(A\sigma)$ is a $U_\pi(A)$-module. This follows at once from the definition and the remarks above.

2.2. Characters, real representations and signatures

Let K be a field of characteristic 0, π a finite group. To each $K\pi$-module V one associates its *character* $\chi_V\colon \pi \to K$ defined by taking the trace of the K-endomorphisms of V induced by elements of π. Clearly χ_V is constant on conjugacy classes. Write $R_K\pi$ for the Grothendieck group of such modules, and $\mathrm{Map}_c(\pi, K)$ for the group of class functions. Then the χ_V induce χ_K: $R_K\pi \to \mathrm{Map}_c(\pi, K)$. Standard representation theory (see e.g. [21]) yields the following:

If $K = \mathbf{C}$ (or any algebraically closed field), $\chi_\mathbf{C}$ is injective, and induces an isomorphism $R_\mathbf{C}\pi \otimes \mathbf{C} \to \mathrm{Map}_c(\pi, \mathbf{C})$. For any K (of characteristic 0), χ_K is injective. Moreover, $R_K\pi$ is a subgroup of finite index in

$$\bar{R}_K\pi = \chi_\mathbf{C}^{-1}(\mathrm{Map}_c(\pi, K)) .$$

The exponent of $\bar{R}_K\pi/R_K\pi$ is called the *Schur index* of $K\pi$.

If we consider $K\pi$ directly, it is a semisimple algebra, hence a sum of $(n \times n)$ matrix rings over division rings D which contain K in the centre. The irreducible modules are the D^n. If D has degree r^2 over its centre, it contributes r to the Schur index.

In the case $K = \mathbf{R}$, the only division rings D which may occur are $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$. Thus for each irreducible complex representation, with summand $M_n\mathbf{C}$ and character χ, either

(i) $\chi \neq \bar{\chi}$, and the summands of $\mathbf{C}\pi$ corresponding to χ and $\bar{\chi}$ yield a single summand $M_n\mathbf{C}$ of $\mathbf{R}\pi$,

(ii) $\chi = \bar{\chi}$, and the corresponding summand of $\mathbf{R}\pi$ is $M_n\mathbf{R}$,

(iii) $\chi = \bar{\chi}$, $n = 2m$, and we have a summand $M_m\mathbf{H}$.

For us, the group ring $\mathbf{R}\pi$ comes with the anti-involution $\alpha_0\colon g \to g^{-1}$. For any $\lambda = \sum a(g)$, $g \in \mathbf{R}\pi$, $\lambda \neq 0$, we have

$$\mathrm{Trace}_{\mathbf{R}\pi/\mathbf{R}}\, \lambda\, \lambda^{\alpha_0} = |\pi| \sum a(g)^2 > 0 ,$$

so α_0 is positive. Hence each simple summand of $\mathbf{R}\pi$ is preserved and (by the classification [26] of positive anti-involutions) the isomorphism onto a matrix ring over $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ can be chosen so that α_0 corresponds to taking the conjugate transposed matrix.

Now consider the antistructure $(\mathbf{R}\pi, \alpha_0, 1)$. We can decompose this into simple summands each of which, by the above, is equivalent to $(K, c, 1)$ with $K = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ and c the standard conjugation. Thus for L_0, the signature yields an isomorphism onto $4\mathbf{Z}$, $4\mathbf{Z}$ or $2\mathbf{Z}$ respectively; for L_2 we have $\{\pm 1\}$, $4\mathbf{Z}$ or 0.

This can be more invariantly formulated (cf. [2], [SCM]). Given a quadratic $(\mathbf{R}\pi, \alpha_0, 1)$-module (V, θ) we first contemplate the induced form $\theta_0 = \mathrm{Trace}_{\mathbf{R}\pi/\mathbf{R}}\,\theta$ over $\mathbf{R}$. Write V as the sum $U^+ \oplus U^-$ of π-invariant positive definite and negative definite subspaces for this induced form. The signature is then the class of U^+ minus that of U^- in $R_{\mathbf{R}}\pi$. Under the category equivalences of the paragraph above, one sees at once that the generator 1 of "$\mathbf{Z}$" corresponds to the irreducible $\mathbf{R}\pi$-module. (For a detailed version of this see Lewis [14], [15].) Thus the values of the signature as here defined are the elements of $R_{\mathbf{R}}\pi$ such that the coefficients of the irreducible representations of the three types are divisible by 4, 4, 2 respectively. Or, in the notation above,

$$L_0^S(\mathbf{R}\pi, \alpha_0, 1) = 4\bar{R}_{\mathbf{R}}\pi ,$$

since precisely in the third case the Schur index is 2.

A similar result holds for L_2. Here, θ_0 is skew-symmetric and we extend it to $V \otimes_{\mathbf{R}} \mathrm{C}$ as a skew-hermitian form: dividing by i yields a hermitian form. Now again, split into positive and negative definite π-invariant subspaces $U^+ \oplus U^-$, and take the class of U^+ minus that of U^- as the invariant. If the irreducible character χ_0 ($\chi_0 \neq \bar{\chi}_0$) corresponds to a form of signature N, then $\bar{\chi}_0$ corresponds to signature $-N$, and our invariant is $N(\chi_0 - \bar{\chi}_0)$. Thus the torsion free part of $L_2^S(\mathbf{R}\pi, \alpha_0, 1)$ is

$$4\chi_{\mathrm{C}}^{-1}(\mathrm{Map}_c(\pi, i\mathbf{R})) ,$$

the group of imaginary characters.

To conclude our discussion of α_0, observe that if we extend to $\mathrm{C}\pi$ as a conjugate linear map, it remains a positive anti-involution, and thus preserves each summand. Hence the C-linear extension interchanges the summands corresponding to χ and $\bar{\chi}$.

We now extend the argument to the nonorientable case. Let $w\colon \pi \to \{\pm 1\}$ have kernel π^+; choose T with $w(T) = -1$. For an irreducible complex representation of π, with character χ and corresponding $\mathrm{C}\pi$-module V, either

(1) $\chi(\pi - \pi^+) = 0$ or equivalently, $\chi(g) = w(g)\chi(g)$ for all g, and V is reducible over $\mathrm{C}\pi^+$ to $V_1 \oplus V_2$, or

(2) $\chi(\pi - \pi^+) \neq 0$ and V is irreducible over $\mathrm{C}\pi^+$.

Now $\chi^w(g) = \chi(g)w(g)$ is another irreducible character, and the anti-involution $\alpha\colon g \to w(g)g^{-1}$ interchanges the summands of $\mathrm{C}\pi$ corresponding to $\bar{\chi}$ and to χ^w.

It is not difficult to make a list of all possibilities occurring in the real case.

subtype	1a	1b	1c	1d	1e
Summand of $\mathbf{R}\pi$	$\mathbf{C}_{2n}$	$\mathbf{R}_{2n}$	$\mathbf{H}_{2n}$	$\mathbf{R}_{2n}$	$\mathbf{H}_n$
Summand of $\mathbf{R}\pi^+$	$\mathbf{C}_n + \mathbf{C}_n$	$\mathbf{R}_n + \mathbf{R}_n$	$\mathbf{H}_n + \mathbf{H}_n$	$\mathbf{C}_n$	$\mathbf{C}_n$
Real Lie group	U_{2n}	$O_{2n}(\mathbf{R})$	$\mathrm{Sp}_{2n}(\mathbf{H})$	$\mathrm{Sp}_{2n}(\mathbf{R})$	$O_n(\mathbf{H})$

subtype	2a	2b	2c	2d	2e
Summand of $\mathbf{R}\pi$	$\mathbf{C}_n + \mathbf{C}_n$	$\mathbf{R}_n + \mathbf{R}_n$	$\mathbf{H}_n + \mathbf{H}_n$	$\mathbf{C}_n$	$\mathbf{C}_{2n}$
Summand of $\mathbf{R}\pi^+$	$\mathbf{C}_n$	$\mathbf{R}_n$	$\mathbf{H}_n$	$\mathbf{R}_n$	$\mathbf{H}_n$
Real Lie group	$\mathrm{GL}_n(\mathbf{C})$	$\mathrm{GL}_n(\mathbf{R})$	$\mathrm{GL}_n(\mathbf{H})$	$O_n(\mathbf{C})$	$\mathrm{Sp}_{2n}(\mathbf{C})$.

Here $\mathbf{R}_n$ denotes a ring of $n \times n$ matrices over $\mathbf{R}$. The pair of algebras is a graded simple algebra in the sense of [25]: take I as the involution $g \to w(g)g$. We have the positive anti-involution $\alpha_0(g) = g^{-1}$ commuting with I, and study $\alpha = I\alpha_0$. The Lie group is $\{\lambda\colon \alpha(\lambda) = \lambda^{-1}\}$.

It follows that we need signatures for hermitian forms corresponding to types 1a, 1b, 1c and for skew-hermitian forms corresponding to types 1a, 1d, 1e. The cases can (in part) be recognised from the characters as follows:

$\chi, \bar{\chi}\, \chi^w\, \bar{\chi}^w$	all distinct	2a
$\chi = \bar{\chi} \neq \chi^w = \bar{\chi}^w$		2b, 2c
$\chi = \bar{\chi}^w \neq \bar{\chi} = \chi^w$		2d, 2e
$\chi = \chi^w \neq \bar{\chi} = \bar{\chi}^w$		1a
$\chi = \bar{\chi} = \chi^w = \bar{\chi}^w$		1b, 1c, 1d, 1e .

In particular, characters with signatures are of type 1; i.e., are those with $\chi = \chi^w$. Characters of type O or Sp are those with $\chi = \bar{\chi}^w$.

We turn to an invariant formulation. Given a form q on the $\mathbf{R}\pi$-module V, we consider V as an $\mathbf{R}\pi^+$-module and assign a signature to it as above to yield a character χ_q on π^+. Although the splitting $V = U^+ \oplus U^-$ can be taken π^+-invariant, we may suppose that each element T of $\pi - \pi^+$ interchanges U^+ and U^-. Hence for $A \in \pi^+$, $\chi_q(T^{-1}AT) = -\chi_q(A)$.

Conversely, each character χ of π^+ satisfying this condition vanishes at each summand corresponding to a representation of π^+ which extends to one of π—i.e., those of type 2, and subtypes 1d and 1e (whether the irreducible representation itself extends, or twice it, does not matter). At summands of subtypes 1a, 1b and 1c, the situation is almost as before, save that the two representations of π^+ which are interchanged by T must occur with opposite signs. Thus if we write $\mathrm{Map}_t\,(\pi^+, K)$ for the set of class functions $\chi\colon \pi^+ \to K$ satisfying the further condition $\chi(T^{-1}AT) = -\chi(A)$ for all $A \in \pi^+$, the torsion-free part of $L_0^S(\mathbf{R}\pi, \alpha, 1)$ can be identified via the signature with $4\chi_{\mathrm{C}}^{-1}(\mathrm{Map}_t\,(\pi^+, \mathbf{R}))$.

Finally, we consider L_2 in the nonorientable case. Again we define the signature by restricting to π^+; again it belongs to Map_t, and takes imaginary values only. Now for type 2, the irreducible complex character of π^+ extends to one of π, so cannot contribute to Map_t. For subtypes 1b and 1c, the characters are real. For subtypes 1d and 1e however, we have a representation of π with real character whose restriction to π^+ breaks up as $\chi \oplus \bar{\chi}$; the twisted characters are generated by $\chi - \bar{\chi}$. Now the least value of the signature corresponds (for types 1a, 1d, 1e) to the $\mathbf{R}\pi$-module $4\mathbf{C}^{2n}$, $4\mathbf{R}^{2n}$, $2\mathbf{H}^n$. These restrict to $\mathbf{R}\pi^+$-modules $4\mathbf{C}^n + 4\mathbf{C}^n$, $4\mathbf{C}^n$, $4\mathbf{C}^n$. Thus in each case, the twisted character is divisible precisely by 4.

We summarise as follows.

THEOREM 2.2.1. *In the orientable case* ($w = 1$), *we have*

$$L_0^S(\mathbf{R}\pi, \alpha_0, 1) \cong 4\chi_{\mathrm{C}}^{-1}\,(\mathrm{Map}_c\,(\pi, \mathbf{R}))\ ,$$
$$L_2^S(\mathbf{R}\pi, \alpha_0, 1)/\mathit{torsion} \cong 4\chi_{\mathrm{C}}^{-1}\,(\mathrm{Map}_c\,(\pi, i\mathbf{R}))$$

and otherwise

$$L_0^S(\mathbf{R}\pi,\, \alpha,\, 1)/torsion \cong 4\chi_{\mathbf{C}}^{-1}\left(\mathrm{Map}_t(\pi^+,\, \mathbf{R})\right),$$
$$L_2^S(\mathbf{R}\pi,\, \alpha,\, 1)/torsion \cong 4\chi_{\mathbf{C}}^{-1}\left(\mathrm{Map}_t(\pi^+,\, i\mathbf{R})\right),$$

where

$$\chi_{\mathbf{C}}\colon R_{\mathbf{C}}\pi \longrightarrow \mathrm{Map}_c(\pi,\, \mathbf{C}) \quad (\textit{similarly for } \pi^+)\ ;$$

Map_c *denotes class functions, and* Map_t *denotes those satisfying*

$$\chi(T^{-1}AT) = -\chi(A) \qquad \textit{for } A \in \pi^+\,, \quad T \in \pi - \pi^+\ .$$

When looking for signatures, we can restrict to summands of $\mathbf{R}\pi$ corresponding to characters χ satisfying $\chi = \chi^w$; those summands of type O or Sp have $\chi = \bar{\chi}^w$.

2.3. Modular representations

In order to compute the groups $L_i(\hat{\mathbf{Z}}_p\pi)$ we shall need some information on modular representations of π. First consider the (easy) case when p does not divide $|\pi|$. Then $\hat{\mathbf{Z}}_p\pi$ modulo its radical is just the group ring $\mathbf{F}_p\pi$, which is already semisimple. We can improve somewhat on this by the following argument.

Let B be a p-block of π, which we can consider as an algebraic direct summand of $\hat{\mathbf{Z}}_p\pi$. We need not assume $p \nmid |\pi|$, but only that B has trivial defect group (in the sense of Brauer: to conform to the terminology of Section 1 we should introduce the Green functor as in [13]). Then according to Green [13], B is projective as module over $\hat{\mathbf{Z}}_p(\pi \times \pi)$; or equivalently, over its enveloping ring $B^e = B^{\mathrm{op}} \otimes_{\hat{\mathbf{Z}}_p} B$. But then B is a separable algebra over $\hat{\mathbf{Z}}_p$ in the sense of Auslander and Goldman [3]. It follows from [3] that the centre C of B is the ring of integers in an unramified extension of $\hat{\mathbf{Q}}_p$, and from [3], [6], [2] that B is a matrix ring over C.

In particular, if $p \nmid |\pi|$, a decomposition of $\hat{\mathbf{Q}}_p\pi$ as direct sum induces one of $\mathbf{F}_p\pi$. Moreover, given an antistructure $(\mathbf{Z}\pi,\, \alpha,\, 1)$, corresponding summands have the same type. For as C is unramified over $\hat{\mathbf{Z}}_p$, if α is nontrivial on C, it also is on $\bar{C}$. In the case when α is trivial on C, if $(C,\, \alpha,\, u_c)$ is an antistructure then $u_c^2 = 1$, $u_c = \pm 1$ and clearly if $u_c = \pm 1$ then both $(B,\, \alpha,\, u)$ and $(\bar{C}, 1, \bar{u}_c)$ have type O; if $u_c = -1$, both have type Sp, provided p is odd.

In fact the above (with a few tricks) will suffice for most of the calculations in this paper, but we can give some results of a more general nature. For any modular representation of π, its Brauer character is an element χ of $\mathrm{Map}_c(\pi,\, \mathbf{C})$ satisfying the further condition that for $g \in \pi$ with p-regular part g_{reg}, $\chi(g) = \chi(g_{\mathrm{reg}})$. Equivalently, choose k such that if $|\pi| = p^a b$ with

$p \nmid b$, then $k \equiv 0 \pmod{p^a}$ and $k \equiv 1 \pmod{b}$. Then $g_{\mathrm{reg}} = g^k$ and $\chi(g_{\mathrm{reg}}) = \chi(g^k) = \Psi^k\chi(g)$ where Ψ^k is the Adams operation [1]. Our condition on χ can be rewritten $\Psi^k\chi = \chi$. Note that Ψ^k is idempotent on $\mathrm{Map}_c(\pi, \mathbf{C})$, hence on $R_{\mathbf{C}}\pi$.

We now introduce some notation. Write $R_{\mathbf{C}}(\pi, p)$ for the image of Ψ^k (k as above) in $R_{\mathbf{C}}(\pi)$. For $q = p^r$, let $\mathbf{F}_q$ be the finite field with q elements, $\hat{K}_q$ the (unramified) extension of $\hat{\mathbf{Q}}_p$ by $(q-1)^{\mathrm{st}}$ roots of unity, $\hat{A}_q$ the integers of $\hat{K}_q$. Now for any K of characteristic 0 we may regard $R_K(\pi) \subset \bar{R}_K(\pi) \subset R_{\mathbf{C}}(\pi)$. According to [Rat], we now have

$$R_{\mathbf{F}_q}(\pi) = R_{\hat{K}_q}(\pi) \cap R_{\mathbf{C}}(\pi, p) = \bar{R}_{\hat{K}_q}(\pi) \cap R_{\mathbf{C}}(\pi, p) .$$

We now extend to the unitary case the argument of [Rat], to prove the

THEOREM 2.3.1. *For p odd, $L_i^S(\hat{\mathbf{Z}}_p\pi) \to L_i^S(\hat{\mathbf{Q}}_p\pi)$ is injective.*

The reader may omit this section as an alternative proof may be given using induction theorems and the arguments of (4.1).

First, let S be a semisimple algebra with involution α over the field k. Write $R(S)$ for the Grothendieck group of finite S-modules, and $RO(S, \alpha)$, $R\mathrm{Sp}(S, \alpha)$ for the subgroup generated by those which admit α-hermitian resp. skew-hermitian forms. We call S *split* if it is a sum of matrix rings over k, and (S, α) split if in addition, for each α-invariant summand of S, α induces the identity of its centre (i.e., no summand has type U).

If S is split, there is a natural homomorphism

$$K_1(S) \cong \mathrm{Hom}\,(R(S), k^\times) ,$$

which arises as follows. If $s \in S$, and M is an S-module, then multiplication by s is a k-endomorphism of M, with determinant $\delta_M(s) \in k^\times$. The map $M \to \delta_M(s)$ induces a homomorphism $\delta(s)$: $R(S) \to k^\times$, and it is not now difficult to obtain the above description.

Now assume (S, α) split, and consider the homomorphism

$$H^*\colon \mathrm{Hom}\,(RO(S), k^\times) \longrightarrow \mathrm{Hom}\,(R(S), k^\times)^\alpha$$

induced by the hyperbolic map H: $R(S) \to RO(S)$. Here α acts trivially on $k^\times$: $H(x) = H(x^\alpha)$ for $x \in R(S)$, so the image of H^* is indeed α-invariant. To describe H^* in more detail, decompose (S, α) into simple summands.

Type GL. $R(S)$ is free abelian on two generators, x and x^α. $RO(S)$ is infinite cyclic, admitting a generator y with $H(y) = x + x^\alpha$. Thus H^* is an isomorphism.

Type Sp. $R(S)$ is infinite cyclic. $RO(S)$ is the subgroup of index 2. H is an isomorphism, hence H^* is also.

Type O. $RO(S) = R(S)$ is infinite cyclic (α is trivial). H is multiplication by 2. So H^* can be identified with the squaring map $k^\times \xrightarrow{2} k^\times$.

We define maps

$$\gamma_1 \colon L_1^S(S, \alpha) \longrightarrow \mathrm{Cok}\, H^* , \qquad \gamma_2 \colon L_2^S(S, \alpha) \longrightarrow \mathrm{Ker}\, H^*$$

by restricting to summands and taking spinor norm in the first case, Pfaffian in the second. As for $K_1(S)$ above, one sees that γ_1 and γ_2 are natural for automorphisms of (S, α) and for base field extensions. If k is a finite field, γ_1 and γ_2 are isomorphisms (recall, (S, α) is still split); if k is a (p-adic) local field, γ_1 is iso and γ_2 injective (but summands of type Sp may also contribute to L_2^S).

In the non-split case, choose a Galois extension l of k, with group G, which is a splitting extension of S resp., of (S, α). If k is finite or local, then

$$K_1(S) \cong Z(S)^\times \cong (Z(S \otimes_k l)^\times)^G \cong \mathrm{Hom}_G\left(R(S \otimes_k l), l^\times\right) .$$

For the unitary case we have similarly (but again ignoring in the local case any Hasse invariant $\mathbf{Z}/2$'s in L_2^S), that if S_0 is the sum of the simple summands of S of type O, $L_1^S(S)$, $L_2^S(S)$ are naturally isomorphic to the cokernel and kernel of the squaring map on $Z(S_0)^\times$, i.e., of

$$H^* \colon \mathrm{Hom}_G\left(RO(S \otimes_k l), l^\times\right) \longrightarrow \mathrm{Hom}_G\left(R(S \otimes_k l), l^\times\right)^\alpha .$$

We now begin the proof of the theorem. First suppose $k = \mathbf{F}_q$, $K = \hat{K}_q$ are splitting fields for π, so that

$$K_1(K\pi) \cong \mathrm{Hom}\,(R_K\pi, K^\times) , \qquad K_1(\overline{k\pi}) \cong \mathrm{Hom}\,(R_k\pi, k^\times) .$$

Then the composite

$$d^* \colon \mathrm{Hom}\,(R_k\pi, k^\times) \cong K_1(\overline{k\pi}) \cong {}_{p'}K_1(A\pi) \longrightarrow {}_{p'}K_1(K\pi) \cong \mathrm{Hom}\,(R_K\pi, {}_{p'}K^\times)$$

is ([Rat]) induced by the lifting isomorphism $k^\times \cong {}_{p'}K^\times$ and the decomposition map $d \colon R_K\pi \to R_k\pi$. Since d has a right inverse b induced (as above) by the Brauer character, d^* has a left inverse, so is injective.

Now in general we choose q so as to have splitting fields as above; then the commutative diagram

$$\begin{array}{ccc} K_1(\overline{\mathbf{F}_p\pi}) & \longrightarrow & K_1(\hat{K}_p\pi) \\ \downarrow & & \downarrow \\ K_1(\overline{\mathbf{F}_q\pi}) & \longrightarrow & K_1(\hat{K}_q\pi) , \end{array}$$

where the verticals are injective, tells us that the top map is injective, as desired.

For the unitary case, again choose r so that $k = \mathbf{F}_{p^r}$ and $K = \hat{K}_{p^r}$ are splitting fields for π, with A the integers in K. As p is odd, $L_i^S(A\pi) \cong L_i^S(\overline{k\pi})$. Consider the diagram:

$$\begin{array}{ccccccccc}
0 \to L_2^S(\overline{k\pi}) & \to & \mathrm{Hom}\,(RO(\overline{k\pi}),\, k^\times) & \to & \mathrm{Hom}\,(R(\overline{k\pi}),\, k^\times)^\alpha & \to & L_1^S(k\pi) \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to L_2^{S'}(K\pi) & \to & \mathrm{Hom}\,(RO(K\pi),\, K^\times) & \to & \mathrm{Hom}\,(R(K\pi),\, K^\times)^\alpha & \to & L_i^S(K\pi) \to 0\,.
\end{array}$$

Here the horizontal sequences were obtained above, and the vertical maps are defined by lifting to A in each case and then mapping into K. For a study of the decomposition map on RO to justify this, see Quillen [16]. As all our constructions were natural, the diagram commutes. Now replace $K^\times$ by $A^\times$ in the lower sequence. As $A^\times \cap (K^\times)^2 = (A^\times)^2$, the map $A^\times/(A^\times)^2 \to K^\times/(K^\times)^2$ is injective, so the cokernel of the middle homomorphism is a subgroup Λ of $L_1^S(K\pi)$. But now the obvious reduction homomorphism $A^\times \to k^\times$ is inverse to the inclusion described above. Again the Brauer lifting induces a one-sided inverse to the decomposition map. This induces a direct sum splitting of the diagram

$$\begin{array}{ccc}
\mathrm{Hom}\,(RO(\overline{k\pi}),\, k^\times) & \longrightarrow & \mathrm{Hom}\,(R(\overline{k\pi}),\, k^\times)^\alpha \\
\downarrow\uparrow & & \downarrow\uparrow \\
\mathrm{Hom}\,(RO(K\pi),\, A^\times) & \longrightarrow & \mathrm{Hom}\,(R(K\pi),\, A^\times)^\alpha\,.
\end{array}$$

If now G is the Galois group of $k = \mathbf{F}_{p^r}$ over $\mathbf{F}_p$, which we may identify with the group of $K = \hat{K}_{p^r}$ over $\hat{\mathbf{Q}}_p$, we have actions of G on $k^\times$ and $A^\times$ compatible with all the maps above. Thus we have an induced diagram of G-invariants, Hom_G. This induces embeddings of the kernel (resp. cokernel) of the upper horizontal map as a direct summand of the kernel (resp. cokernel) of the lower. But for the upper map, these are $L_2^S(\overline{\mathbf{F}_p\pi}) \cong L_2^S(\hat{\mathbf{Z}}_p\pi)$ and $L_1^S(\overline{\mathbf{F}_p\pi}) \cong L_1^S(\hat{\mathbf{Z}}_p\pi)$; for the lower, we have a direct summand of $L_2^S(\hat{\mathbf{Q}}_p\pi)$ and—by the same remark as in the split case above—a subgroup of $L_1^S(\hat{\mathbf{Q}}_p\pi)$.

This proves the theorem for $i = 1, 2$; for $i = 3, 0$ the result follows by interchanging O and Sp throughout. The same proof deals also with the nonorientable case, provided we extend Quillen's theory to show that b, d respect RO, when defined with respect to α in place of α_0.

2.4. L-theory of p-hyperelementary groups, p odd

According to the main exact sequence, the localisation of $L_n^X(\mathbf{Z}\pi)$ at odd primes coincides with that of $L_n^S(\mathbf{R}\pi)$, which is fully determined in (2.2). The localisation at 2 (and, in particular, the 2-torsion subgroup) satisfies induction with respect to 2-hyperelementary groups, by (2.1), so it suffices to do further calculations for such groups.

An alternative proof (i.e., not using the fact that the defect groups of the localisation at 2 of $U_\pi(\mathbf{Z})$ are 2-hyperelementary) can be given using a direct calculation of $L_n^X(\mathbf{Z}\pi)$ where π is p-hyperelementary with p odd. As this seems of some interest, we now give it.

We begin with a structure lemma.

LEMMA 2.4.1. *Let π be p-hyperelementary, p odd. Then π is the direct product of a cyclic 2-group and a group of odd order.*

Proof. By definition, π is an extension of a cyclic group by a p-group. Hence it has a normal cyclic Sylow 2-subgroup σ. Since σ has no automorphism of odd order, it is central; since its order is prime to its index the extension of σ by π/σ splits.

For our first arguments, we can generalise the hypothesis and consider any direct product $\sigma \times \rho$, with ρ of odd order. We will study L-theory of $A(\sigma \times \rho) = R$ relative to the subring $A\sigma = R_0$.

Over C the irreducible representations of $\sigma \times \rho$ are tensor products of ones of σ and ones of ρ: the representation is self-conjugate if and only if each factor is. But, by a well known theorem of Burnside [8, 222], the only self-conjugate representation of ρ is the trivial one. So all representations of $\sigma \times \rho$, other than those of σ, have type U or GL. Hence

$$CL_i^S(S, S_0) = 0 .$$

Thus from the main exact sequence follows

$$L_i^S(R, R_0) \cong L_i^S(\hat{R}, \hat{R}_0) \oplus L_i^S(T, T_0) .$$

It also follows, since modular representations can be 'embedded' in complex ones via the Brauer character (e.g., by (2.3) above), that the corresponding assertions hold there also. Hence

$$L_i^S(\hat{R}_p, \hat{R}_{p0}) = L_i^S(\bar{R}_p, \bar{R}_{p0}) = 0 \qquad (p \text{ odd}) ,$$

so that

$$L_i^S(R, R_0) = L_i^S(\hat{R}_2, \hat{R}_{20}) \oplus L_i^S(T, T_0) .$$

Further,

$$L_i^K(\hat{R}_2, \hat{R}_{20}) = L_i^K(\bar{R}_2, \bar{R}_{20}) = 0 ,$$

by the same argument.

We now restrict σ to be an abelian 2-group.

THEOREM 2.4.2. *Let σ be an abelian 2-group and ρ have odd order. Then*

$L_i^X(\mathbf{Z}(\sigma \times \rho)) = L_i^X(\mathbf{Z}\sigma) \oplus \tilde{L}_i\mathbf{R}(\sigma \times \rho)$ *and similarly for* L_i^Y. *The last summand is* 0 *for i odd; for i even, it is free abelian, and detected by signatures.*

Proof. If we can show $L_i^X(\hat{R}_2, \hat{R}_{20}) = 0$ (also L_i^Y), then the relative group $L_i^X(R, R_0)$ (resp. L_i^Y) maps isomorphically to $L_i^X = L_i^S(T, T_0)$. The description of this latter follows from (1.2), and is as described: it suffices to note that all these representations have type U or GL. Now since mapping each element of ρ to 1 gives a homomorphism $\mathbf{Z}(\sigma \times \rho) \to \mathbf{Z}\sigma$ compatible with the involution, the groups split as direct sums.

Now we use the exact sequence

$$\cdots H^{i+1}(K_1(\hat{R}_2)/X) \longrightarrow L_i^X(\hat{R}_2) \longrightarrow L_i^K(\hat{R}_2) \cdots$$

and the isomorphism

$$K_1(\hat{R}_2)/X \cong \mathrm{Nrd}\left(\hat{\mathbf{Z}}_2(\sigma \times \rho)^\times\right) .$$

For Y we factor out also the image of $\pm(\sigma \times \rho)$, but since ρ has odd order, this is now seen to be equivalent to the case of X. We have reduced to a question on units in group rings, to see that

$$\mathrm{Nrd}\,(\hat{\mathbf{Z}}_2\sigma)^\times \subset \mathrm{Nrd}\left(\hat{\mathbf{Z}}_2(\sigma \times \rho)\right)^\times$$

induces a mod 2 cohomology isomorphism.

But $\hat{\mathbf{Z}}_2(\sigma \times \rho) = \hat{\mathbf{Z}}_2\sigma \otimes \hat{\mathbf{Z}}_2\rho$, and since ρ has odd order, $\hat{\mathbf{Z}}_2\rho$ is unramified over $\hat{\mathbf{Z}}_2$, hence is a sum of matrix rings over finite unramified extensions A_i of $\mathbf{Z}_2$. Also the involution $g \to g^{-1}$ acts nontrivially on each A_i save the trivial representation. Since taking matrix rings does not change K_1 (or Nrd), it suffices to show that $(A\sigma)^\times$ is cohomologically trivial. But this follows from [UGR, (11.3)], as recalled in (3.2.3) below.

COROLLARY 2.4.3. *For ρ of odd order, $L_i^S(\rho) = L_i(1) \oplus \tilde{L}_i(\mathbf{R}\rho)$ is detected by signatures and ($i = 2$) the classical Arf invariant; it vanishes for i odd.*

An equally complete description for $L_i(\sigma \times \rho)$ must wait on the calculations of the next chapter.

The same result was obtained also by Bak [4].

3. Abelian 2-groups

Although the results obtained here are in principle contained in those of Chapter 4, the details are considerably simpler and the results more explicit. The reader is strongly advised to read this chapter before attempting the next.

In (3.1) we present a reformulation of our method which will be more convenient for the details of the calculation. Next, (3.2) is devoted to a description of the results of [UGR] which are needed for Chapters 3 and 4. Then in (3.3) we give the full calculation for the orientable case, following the pattern of (1.4)-(1.6), as modified in (3.1). The nonorientable case splits into two subcases which are treated respectively in (3.4) and (3.5): to complete the argument in (3.4) we also need an argument of L. R. Taylor [23].

3.1. Relative groups and notation for calculations

We now present a technique to allow the simplification of many calculations. This depends on the development of a relative L-theory by Ranicki [19]. We shall not need the details of this theory, just its existence. Given antistructures (R, α, u) and (R', α', u'), invariant subgroups $X \subset K_1(R)$ and $X' \subset K_1(R')$, and a ring homomorphism $f\colon R \to R'$ with $f(u) = u'$, $f \circ \alpha = \alpha' \circ f$ and $f_*(X) \subset X'$, we know there is a naturally induced map $f_*\colon L_p^X(R, \alpha, u) \to L_p^{X'}(R', \alpha', u')$. The relative theory yields groups $L_p(f)$ (depending on all the above) and an exact sequence

$$\cdots L_p(R) \longrightarrow L_p(R') \longrightarrow L_p(f) \longrightarrow L_{p-1}(R) \cdots ,$$

natural for morphisms preserving all the structure.

Denote our ring inclusions as follows:

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "i_R"] & S \arrow[r, "j"] \arrow[d, "i_S"] & T \\ \hat{R} \arrow[r, "\hat{i}"] & \hat{S} . \end{tikzcd}$$

Then our main exact sequence should be interpreted as an excision isomorphism $L_*(i) \cong L_*(\hat{i})$ or equivalently, $L_*(i_R) \cong L_*(i_S)$. We can also now identify $CL_*(S) = L_*\begin{pmatrix} i_S \\ j \end{pmatrix}$. We wish to study $R \to \hat{R}_2$. Denote the obvious projections by

$$p_R\colon \hat{R} \longrightarrow \hat{R}_2 , \qquad p_S\colon \hat{S} \longrightarrow \hat{S}_2 ,$$

and consider the exact sequence of the triple

$$R \xrightarrow{\binom{i_R}{ji}} \hat{R} \oplus T \xrightarrow{(p_R,\, 0)} \hat{R}_2 , \quad \text{viz} .$$

$$\cdots L_*\begin{pmatrix} i_R \\ ji \end{pmatrix} \longrightarrow L_*(p_R i_R) \longrightarrow L_*(p_R, 0) \xrightarrow{\partial} \cdots$$

(exactness follows from general principles; see e.g., [24]). Now our excision isomorphism shows that

$$L_*\begin{pmatrix} i_R \\ ji \end{pmatrix} = L_*\begin{pmatrix} i_S \\ j \end{pmatrix} = CL_*(S) ,$$

and as $(p_R, 0)$ is the projection of a direct sum,

$$L_*(p_R, 0) = L_{*-1}(\hat{R}_{\mathrm{odd}} \oplus T)$$

(writing $\hat{R}_{\mathrm{odd}}$ for Ker p_R). Thus our exact sequence reduces to

$$\cdots CL_*(S) \longrightarrow L_*(p_R i_R) \xrightarrow{\partial} L_*(\hat{R}_{\mathrm{odd}}) \oplus L_*(T) \xrightarrow{\gamma} CL_*(S) \cdots ,$$

so the relative groups $L_*(p_R i_R)$ of $R \to \hat{R}_2$ are extensions of Cok γ by Ker γ (in appropriate dimensions); if, in particular, Ker γ is torsion-free (as it normally is by (2.3.1)), hence free, we deduce

$$L_p(R \longrightarrow \hat{R}_2) \cong \mathrm{Cok}\, \gamma_p \oplus \mathrm{Ker}\, \gamma_{p-1} .$$

A more real advantage is the following. Suppose $R = \mathbf{Z}\pi$, and consider the diagram

$$\begin{array}{ccc} L^X(R) & \longrightarrow & L^{\hat{X}_2}(\hat{R}_2) \\ \downarrow & & \downarrow \\ L^Y(R) & \longrightarrow & L^{\hat{Y}_2}(\hat{R}_2) \end{array},$$

where $X = SK_1(R)$, $\hat{X}_2 = SK_1(\hat{R}_2)$ as usual and Y, $\hat{Y}_2$ are the sums of these with the image of $\pm\pi$, isomorphic to $\{\pm 1\} \oplus \pi/\pi'$ (π' the commutator subgroup). Then both verticals lie in exact sequences with the same third term, $H^*(\{\pm 1\} \oplus \pi/\pi')$, and so define the same relative groups. Therefore excision holds, and the horizontal maps also define the same relative groups. So $L_*(p_R i_R)$ above yields the relative group if we use X *or* if we use Y, and we can obtain $L^Y(R)$ directly, short-cutting the calculation via $L^X(R)$. Our procedure will thus involve computing γ_i, hence $L_i(R \to \hat{R}_2)$, then computing $L_i(\hat{R}_2)$ and finally ψ_i: $L_1(\hat{R}_2) \to L_i(R \to \hat{R}_2)$ and hence $L_i(R)$.

For our main calculations, we will determine $L^Y_*(R \to \hat{R}_2)$ by computing the homomorphisms γ_p; determine $L^Y_*(\hat{R}_2)$ from the exact sequence relating it to $L^K_*(\hat{R}_2)$; and then finally compute ψ_p: $L^Y_p(\hat{R}_2) \to L^Y_p(R \to \hat{R}_2)$ in order to determine $L^Y_p(R)$, at least up to extensions.

3.2. Recall of calculations of 2-adic units

We now need some of the calculations of units in 2-adic group rings, especially of abelian groups, made in [UGR]. It will be convenient to list here all the results of this kind which will be needed in later sections.

First we quote some results on cohomology of unit groups. These are copied from [UGR, § 11]. Let π be an abelian 2-group, A an unramified 2-ring—i.e., Galois extension of $\hat{\mathbf{Z}}_2$, or equivalently, ring of units in an unramified extension of $\hat{\mathbf{Q}}_2$. We have an involution α of $A\pi$, which may be trivial or nontrivial on A, and acts on elements of π by $g \to w(g)g^{-1}$ for some homomorphism $w: \pi \to \{\pm 1\}$. Then we are interested in computing $H^i(A\pi)^\times$. Note that here

$$K_1(A\pi) \cong K_1'(A\pi) \cong (A\pi)^\times .$$

Write ρ for the subgroup of elements of order 2 in π (sometimes also denoted ${}_2\pi$), $\mathbf{F} = A/2A$ for the finite residue field. Then $\mathbf{F}^\times$ has odd order, so is cohomologically trivial. We first consider $H^i(\mathbf{F}\pi)^\times$. This vanishes if α acts nontrivially on $\mathbf{F}$ (or equivalently, on A), so assume trivial action. Now using the exact sequence

$$1 \longrightarrow K_\rho \longrightarrow (\mathbf{F}\rho)^\times \xrightarrow{\varepsilon} \mathbf{F}^\times \longrightarrow 1$$

(ε the augmentation), we find

$$H^i(\mathbf{F}\rho)^\times \cong H^i(K_\rho) \cong K_\rho .$$

To state the next result, write π as a sum of cyclic groups with generators $T_1, \cdots, T_r$ where $T_1, \cdots, T_{r-s}$ have orders > 2 and the last s generators have order 2. For $1 \leqq i \leqq r - s$, let U_i be a power of T_i which has order 4. Finally observe that since $\mathfrak{P}: \mathbf{F} \to \mathbf{F}$ defined by $\mathfrak{P}x = x + x^2$ is an additive homomorphism with kernel $\{0, 1\}$, its cokernel also has order 2: choose β not in the image.

PROPOSITION 3.2.1. (i) *$K_\rho = H^0(\mathbf{F}\rho)^\times \to H^0(\mathbf{F}\pi)^\times$ is injective; a base for the cokernel is $\{1 + \beta(T_i + T_i^{-1}): 1 \leqq i \leqq r - s\}$.*

(ii) *The kernel of $K_\rho = H^1(\mathbf{F}\rho)^\times \to H^1(\mathbf{F}\pi)^\times$ is $\rho \cap \pi^2$; a base for the cokernel is $\{T_i, 1 + \beta(U_i + U_i^{-1}): 1 \leqq i \leqq r - s\}$.*

Now if π contains an element T of order 2 with $w(T) = -1$, we have $\pi = \pi_0 \oplus \pi_1$ where $\pi_0 = \operatorname{Ker} w$ and π_1 is generated by T. Then the composite $(A\pi)^\times \xrightarrow{T=1} (A\pi_0)^\times \to (\mathbf{F}\pi_0)^\times$ induces cohomology isomorphisms; so the calculation follows from the above proposition. We exclude this exceptional case in what follows.

PROPOSITION 3.2.2. *If $\alpha \mid A$ is nontrivial, $H^i(1 + 2A\pi)^\times = 0$. If it is trivial, $H^1(1 + 2A\pi)^\times \cong \{\pm 1\}$ and $H^0(1 + 2A\pi)^\times$ is a (split) extension of $H^0(1 + 2A)^\times$, of order 2, represented by $(1 + 4\beta)$, by $(\mathbf{F}\rho)^+$.*

PROPOSITION 3.2.3. *If $\alpha \mid A$ is nontrivial, $H^i(A\pi)^\times = 0$. If it is trivial, $H^1(A\pi)^\times \cong \{\pm 1\} \oplus \operatorname{Ker} w/\pi^2$ and $H^0(A\pi)^\times$ is a (split) extension of G (w trivial) or $G/\{\pm 1\}$ (w nontrivial) by $H^0(\mathbf{F}\pi)^\times$.*

Here, G is a group of exponent 2 with basis represented by $1 + 4\beta$, $1 + 2\beta T_i$ $(r - s < i \leqq r)$, and $1 + 2\alpha$, where α runs through an additive basis of $\mathbf{F}$.

In this last case, we may regard $H^0(A\pi)^\times$ as 'made up of':

> $H^0(A^\times)$ with basis $1 + 4\beta$ and the $1 + 2\alpha$, perhaps with the class of -1 factored out, $H^0(\mathbf{F}\rho)^\times \cong K_\rho$, and π/π^2, with basis $\{1 + \beta(T_i + T_i^{-1}); 1 \leqq i \leqq r\}$.

For our further results we obtain a much fuller description of the units, but under the extra condition that π is elementary ($\pi = \rho$). Here we can relax the condition that A be unramified: it can now denote the integers in any 2-adic field.

Again write $\{T_i, 1 \leqq i \leqq r\}$ for a basis of π; let $\hat{\pi} = \mathrm{Hom}\,(\pi, \{\pm 1\})$ be the dual group and $\{\chi_i, 1 \leqq i \leqq r\}$ the dual basis. Then the product $T_1 \cdots T_r$ induces the character ε of $\hat{\pi}$ with $\varepsilon(\chi_i) = -1$ for each i. Now each $\chi \in \hat{\pi}$ induces a ring homomorphism $\bar{\chi}$: $A\pi \to A$, and the collection of all the $\bar{\chi}$ gives an isomorphism of $K\pi$ with the product over $\hat{\pi}$ of copies of K; hence monomorphisms of $A\pi$ and of $(A\pi)^\times$.

To describe the image, we change coordinates. For $I \subset \{1, \cdots, r\}$, write $|I|$ for the cardinality of I and let σ_I be the subgroup of $\hat{\pi}$ generated by the χ_i, $i \in I$ and define ξ_I: $(A\pi)^\times \to A^\times$ by

$$\xi_I(u) = \prod \{\bar{\chi}(u)^{\varepsilon(\chi)} : \chi \in \sigma_I\} .$$

There are also simple formulae expressing the $\bar{\chi}$ in terms of the ξ_I: these may be considered as new coordinates on the product (over $\hat{\pi}$) of $A^\times$.

PROPOSITION 3.2.4. *The image of* $(A\pi)^\times$ *is the subgroup given by* $a_I \equiv 1 \pmod{2^{|I|}}$ *for all* $I \subset \{1, \cdots, r\}$. *And the unit is* $\equiv 1 \pmod{2^n A\pi}$ *if and only if each* $a_I \equiv 1 \pmod{2^{|I|+n}}$.

For $A = \hat{\mathbf{Z}}_2$, and generally if the square of any unit is $\equiv 1 \pmod 2$, we can omit ε in the above.

An application was also outlined in [UGR § 12]. Let π be any finite abelian 2-group, $\bar{\pi} = \pi/\pi^2$. Consider (α-) symmetric units in π; when $w = 1$, we have elements

$$\Sigma\{a(g)g : g^2 = 1\} + \Sigma\{b(g)(g + g^{-1})\}$$

whose images in $A\bar{\pi}$ are the elements $\Sigma\alpha(h)h$, where $\alpha(h)$ is even unless $h \in \mathrm{Im}\,({}_2\pi \to \bar{\pi}) = \sigma$, say. Such a unit of $A\bar{\pi}$ is the product of a unit of $A\sigma$ and a unit $\equiv 1 \pmod{2A\bar{\pi}}$. If we choose $T_1, \cdots, T_s$ above to be a base of σ, we have

COROLLARY 3.2.5. *The image in* $(A\bar{\pi})^\times$ *of symmetric units of* $A\pi$ *is characterised by*

$$a_I \equiv 1 \pmod{2^{|I|}} \quad \textit{for all } I ,$$
$$a_I \equiv 1 \pmod{2^{|I|+1}} \quad \textit{if } I \not\subset \{1, \cdots, s\} .$$

3.3. The orientable case

Let π have 2-rank r. Every irreducible complex representation of π has degree 1. Irreducible real representations have degree 1 or 2: those of degree 2 have type U, those of degree 1 type O. These form Hom $(\pi, \{\pm 1\})$, of order 2^r. Thus $S = \mathbf{Q}\pi$ has 2^r summands $\mathbf{Q}$; the rest have type U.

Now $L_i^S(\hat{S}) = 2^r L_i^S(\hat{\mathbf{Q}})$, and $L_i^S(T)$ is a sum of 2^r copies of $L_i^S(\mathbf{R})$ and (for i even) a group Σ of signatures, corresponding to representations of type U. Also, $CL_i^S(S)$ is a sum of 2^r copies of $CL_i^S(\mathbf{Q})$.

For p an odd prime, $\hat{\mathbf{Z}}_p\pi$ is unramified, so breaks into a sum of 2^r copies of $\hat{\mathbf{Z}}_p$ and summands of type U. Thus we can compute γ_i: $L_i^S(\hat{R}_{\text{odd}}) \oplus L_i^S(T) \to CL_i^S(S)$: it is a sum of 2^r copies of γ_i for $R = \mathbf{Z}$, together with Σ when i is even. So γ_i is injective for i odd; its kernel is Σ for $i = 2$, and is Σ plus 2^r copies of $8\mathbf{Z}$ for $i = 0$. And γ_i is surjective except for $i = 1$, when the cokernel is a sum of 2^r copies of $\mathbf{Z}/2$.

PROPOSITION 3.3.1. *$L_p^X(R \to \hat{R}_2) = L_p^Y(R \to \hat{R}_2) = 0$ $(p = 0, 2)$, Σ $(p = 3)$, and a sum of Σ with 2^r copies of* $(8\mathbf{Z} \oplus \mathbf{Z}/2)$ *if $p = 1$.*

Since π is a 2-group, $\bar{R}_2 = \overline{\mathbf{Z}_2\pi} = \mathbf{Z}/2$, so $L_p^K(\hat{R}_2) = L_p^K(\bar{R}_2) \cong \mathbf{Z}/2$ for all p. Now by (3.2.3), $H^1(K_1(\hat{R}_2)/Y) = 0$, and so $H^0(K_1(\hat{R}_2)/Y)$ has rank 2^r. We can now directly determine $L_p^X(\hat{R}_2)$ and $L_p^Y(\hat{R}_2)$. It is easiest to observe that (when $\pi \to 1$) $\hat{\mathbf{Z}}_2$ is a retract of $\hat{R}_2$, so that $L_p^Y(\hat{R}_2)$ is the direct sum of $L_p^Y(\hat{\mathbf{Z}}_2)$ and a relative group, which is thus an elementary 2-group of rank 0, resp. $2^r - 1$, for p even, resp. odd. Also, we recall from 1.4 that $L_p^Y(\hat{\mathbf{Z}}_2) \cong 0$, $\mathbf{Z}/2 + \mathbf{Z}/2$, $\mathbf{Z}/2$, $\mathbf{Z}/2$ for $p \equiv 0, 1, 2, 3 \pmod 4$.

Now we must compute the map ψ_1: $L_1^Y(\hat{R}_2) \to L_1^Y(R \to \hat{R}_2)$. We have the commutative diagram

$$\begin{array}{ccccccccc}
H^0(K_1(\hat{R}_2)) & \longrightarrow & L_1^S(\hat{R}_2) & \longrightarrow & L_1^S(\hat{S}_2) & \longrightarrow & CL_1^S(S) & \longrightarrow & L_1^X(R \longrightarrow \hat{R}_2) \\
\downarrow & & \downarrow & & & & & & \| \wr \\
H^0(K_1(\hat{R}_2)/Y) & \longrightarrow & L_1^Y(\hat{R}_2) & \longrightarrow & & \longrightarrow & & \longrightarrow & L_1^Y(R \longrightarrow \hat{R}_2)\ ,
\end{array}$$

and the map $L_1^S(\hat{S}_2) \to L_1^X(R \to \hat{R}_2)$ is computed as follows. Each group is a sum of 2^r components corresponding to the irreducible representations of π of type O. For each component,

$$L_1^S \cong \hat{\mathbf{Q}}_2^\times/(\hat{\mathbf{Q}}_2^\times)^2\ ,$$

and the map is given on the subgroup of units by

$$u \longrightarrow (-1)^{(u^2-1)/8}$$

(i.e., $u \equiv \pm 1 \bmod 8$ form the kernel).

Now the image of $H^0K_1(\hat{R}_2) = H^0(\hat{R}_2)^\times$, or equivalently of the symmetric units of $\hat{R}_2 = \hat{\mathbf{Z}}_2\pi$, is described by (3.2.5). If the notation is changed from the $\overline{\chi}(u)$ to the $\xi_I(u)$, the images of symmetric units are characterised by

$$\begin{aligned} a_I &\equiv 1 \bmod 2^{|I|} \quad \text{if } I \subset \mathbf{s}\ , \\ a_I &\equiv 1 \bmod 2^{|I|+1} \quad \text{otherwise}\ . \end{aligned}$$

Thus the value mod 8 may be nontrivial only if $|I| \leqq 1$ or $|I| = 2$ and $I \subset \mathbf{s}$, yielding $1 + r + \binom{s}{2}$ cases: this is the rank of the image of ψ_1. The exact sequence now shows

THEOREM 3.3.2. *For $R = \mathbf{Z}\pi$, π an abelian 2-group of rank r, orientable,*

$$\begin{aligned} L_0^Y(R) &\cong \Sigma \oplus 2^r(8\mathbf{Z}) \oplus \left(2^r - 1 - r - \binom{s}{2}\right)\mathbf{Z}/2\ , \\ L_2^Y(R) &\cong \Sigma \oplus \mathbf{Z}/2\ , \\ L_1^Y(R) &\cong \mathbf{Z}/2 \oplus \left(2^r - 1 - r - \binom{s}{2}\right)\mathbf{Z}/2\ , \\ L_3^Y(R) &\cong \mathbf{Z}/2 \oplus (2^r - 1)\mathbf{Z}/2\ . \end{aligned}$$

The groups announced in [L] are obtained by cancelling the summands $\mathbf{Z}/2$ in L_1 and L_3 generated by τ (or coming from $\mathbf{Z}$).

Some of these results have been independently obtained by H. Bass [6]; note that π is elementary if and only if $s = r$. Observe in particular the

COROLLARY 3.3.3. *If π is cyclic of order 2^n the surgery obstruction groups are given (orientable case) by*

$$L_0 \cong \Sigma \oplus 8\mathbf{Z} \oplus 8\mathbf{Z}\ , \quad L_1 = 0\ , \quad L_2 \cong \Sigma \oplus \mathbf{Z}/2\ , \quad L_3 \cong \mathbf{Z}/2$$

where Σ has $(2^{n-1} - 1)$ components, each isomorphic (via the signature) to $4\mathbf{Z}$.

Combining this with the results of the preceding chapter, we see that for a p-hyperelementary group π (p odd) of even order, $L_0(\pi^+)$ and $L_1(\pi^+)$ are torsion-free, the torsion subgroup of $L_2(\pi^+)$ has order 2, and is detected by the classical Arf invariant, and $L_3(\pi^+) \cong \mathbf{Z}/2$.

3.4. The nonorientable case

First observe that we can write $\pi = \pi_0 \oplus \pi_1$, where $w \mid \pi_0$ is trivial, and π_1 is generated by T of order 2^k, say, with $w(T) = -1$. For if π is expressed as a sum of cyclic groups with generators T_i, $w(T_i) = w(T_j) = -1$, and T_i has order at least that of T_j, we replace T_i by $T_i' = T_iT_j$; and so on by induction.

Complex representations coincide with the homomorphisms $\chi\colon \pi \to \mathbf{C}^\times$ which are their characters. By (2.2), characters of type O or Sp are those with $\chi = \bar{\chi}^w$, i.e., $\chi(g) = \chi(w(g)g^{-1})$, or equivalently $\chi(g^2) = w(g)$. Characters with signatures are those which vanish on $\pi - \pi^+$; clearly there can be none such. If $k = 1$, then $T^2 = 1$, $\chi(T^2) = 1$, and $w(T) = -1$, so there can be no characters of type O or Sp. We defer this exceptional case to the next section, and now suppose $k \geqq 2$.

For characters of type O or Sp, $\chi \mid \pi_0$ is a homomorphism into $\{\pm 1\}$ and $\chi(T) = \pm i$. The corresponding summands of $\mathbf{R}\pi$ (or $\mathbf{C}\pi$) all have type O— there are no skew-symmetric bilinear forms on a 1-dimensional vector space. Moreover, the corresponding summands of $\mathbf{R}\pi$ (resp. $\mathbf{Q}\pi$) are each isomorphic to $\mathbf{C}$ (resp. $\mathbf{Q}[i]$), and we can fix the isomorphism uniquely by requiring $\chi(T) = i$.

As in the orientable case, since $\hat{R}_p$ is unramified for p odd, $\hat{R}_p$ and $\hat{S}_p$ decompose following the decomposition of S, and the components not of type O do not contribute to the L-theory. If π has rank r, we have 2^{r-1} summands of type O, each isomorphic to $\mathbf{Q}[i]$. Thus following the calculation at the end of (1.6),

PROPOSITION 3.4.1. *$L_p^Y(R \to \hat{R}_2) = L_p^X(R \to \hat{R}_2)$ is a sum of 2^{r-1} copies of $L_p^X(A \to \hat{A}_2)$ (where $A = \mathbf{Z}[i]$), which is isomorphic to $\mathbf{Z}/2$, $L/2L$, 0, 0 for $p =$ 0, 1, 2, 3 respectively.*

As π is a 2-group, we again have $\bar{R}_2 \cong \mathbf{F}_2$, so $L_p^K(\hat{R}_2) \cong L_p^K(\bar{R}_2) \cong \mathbf{Z}/2$ for all p. Now by (3.2.3) (observe that we have excluded the exceptional case), $H^1(K_1(\hat{R}_2)/Y) = 0$ and $H^0(K_1(\hat{R}_2)/Y)$ has rank 2^r. The map

$$L_{2k}^K(\hat{R}_2) \longrightarrow H^0(K_1(\hat{R}_2)/Y)$$

is, as usual, computed by lifting a form with nonzero Arf invariant form $\bar{R}_2$ and evaluating the determinant: this is trivial for k odd, but not for k even. To compute extensions, observe (again as usual) that $L_{2k+1}^K(\hat{R}_2)$ is represented by the interchange τ of e and f in a hyperbolic plane. Now $\tau^2 = \pm 1$ represents in either case the trivial element of $H^0(K_1(\hat{R}_2)/Y)$ (here the result would be different if X replaced Y), so the extension splits.

PROPOSITION 3.4.2. *If $r = 1$, then $L_p^Y(\hat{R}_2) \cong 0$, $\mathbf{Z}/2 + \mathbf{Z}/2 + \mathbf{Z}/2$, $\mathbf{Z}/2$, $\mathbf{Z}/2 + \mathbf{Z}/2$ for $p = 0, 1, 2, 3$. In general, this is correct for p even; for p odd we add an elementary 2-group of rank $(2^r - 2)$.*

The final remark uses the splitting induced by the retraction of $\pi = \pi_0 \oplus \pi_1$ on π_1.

Now the map

$$\psi_p \colon L_p^Y(\hat{R}_2) \longrightarrow L_p^Y(R \longrightarrow \hat{R}_2)$$

is zero for $p \neq 1$ since one of the groups is zero. The same commutative diagram as in the orientable case shows that the image of ψ_1 is the image of the symmetric units of $\hat{R}_2$ via the map into $L_1^S(\hat{S}_2)$. We may compute this image by working in the group ring

$$\hat{\mathbf{Z}}_2[T/T^2 = -1][\pi_0/\pi_0^2] = \hat{A}_2[\pi_0/\pi_0^2] .$$

The image of the symmetric elements of $\hat{\mathbf{Z}}_2\pi$ is the set of elements

$$\Sigma\{(a_h + ib_h)h \colon h \in \pi_0/\pi_0^2\} ,$$

where b_h is even, and a_h is even unless h is the image of an element g of π_0 of order 2. Thus the image of the symmetric units is the product of the group of units of $\hat{A}_2[\pi_0/\pi_0^2]$ which are $\equiv 1 \pmod 2$ and the group of units of $\hat{\mathbf{Z}}_2\sigma$, where $\sigma = \operatorname{Im}({}_2\pi_0 \to \pi_0/\pi_0^2)$.

We now compute using (3.2.4). Take a base $\{\chi_i \colon 1 \leqq i \leqq r-1\}$ of $\operatorname{Hom}(\pi_0, \pm 1)$ such that $\{\chi_i \colon s < i \leqq r-1\}$ form a basis for the subgroup annihilating ${}_2\pi_0$ and define ξ_I as in [UGR, § 12]. Then the image of the group of units $\equiv 1 \pmod 2$ is given by

$$a_I \equiv 1 \bmod 2^{|I|+1} \quad \text{for all } I \subset \{1, \cdots, r-1\} .$$

The image of the group of units of $\hat{\mathbf{Z}}_2\sigma$ is given by

$$\begin{aligned} a_I &= 1 \quad \text{for } I \not\subset \{1, \cdots, s\} ; \\ a_I \in \hat{\mathbf{Z}}_2^\times,\ a_I &\equiv 1 \bmod 2^{|I|} \quad \text{for } I \subset \{1, \cdots, s\} . \end{aligned}$$

The class of a_I in $L/2L$ is determined by its congruence class mod $(4 + 4i)$, hence by its class (mod 8). Checking cases, we find that

$$\begin{cases} \text{the class is arbitrary only if } I = \varnothing ; \\ \text{we obtain the classes of 1, 5 if } |I| = 1 \text{ or if } |I| = 2 ; \\ I \subset \{1, \cdots, s\} ; \\ \text{the class is trivial otherwise} . \end{cases}$$

The image thus has rank $2 + (r-1) + \binom{s}{2} = 1 + r + \binom{s}{2}$.

We can now assemble our results, using the exact sequence

$$\cdots L_p^Y(R) \longrightarrow L_p^Y(\hat{R}_2) \xrightarrow{\psi_p} L_p^Y(R \longrightarrow \hat{R}_2) \longrightarrow L_{p-1}^Y(R) .$$

We have

$$\begin{aligned} &L_0^Y(R) \cong \operatorname{Cok} \psi_1 , \quad L_1^Y(R) \cong \operatorname{Ker} \psi_1 , \\ &L_2^Y(R) \cong L_2^Y(\hat{R}_2) \cong \mathbf{Z}/2 , \end{aligned}$$

and $L_3^Y(R)$ is an extension of $L_0^Y(R \to \hat{R}_2)$ by $L_3^Y(\hat{R}_2)$.

In [23], Taylor makes a very useful observation. The result can be appreciably sharpened by reformulating the argument in terms of our L-

theory, as introduced in [F] (see also (1.1) above).

THEOREM 3.4.3. *Let* $(R, \alpha, 1)$ *be an antistructure, and* u *a unit of* R *such that* $u^\alpha u = \varepsilon = \pm 1$. *Then inner automorphism* r_u *by* u *induces multiplication by* ε *on* $\Lambda_i(R, \alpha, 1)$ *and hence on any* $L_i^X(R, \alpha, 1)$.

Proof. For any based quadratic form (P, q, v) of rank n, there is an obvious isomorphism of $r_u(P, q)$ on $(P, \varepsilon q)$, whose determinant with respect to $r_u(v)$, v is $(u)^n$. Thus if (P, q) represents $\xi \in K_0\mathcal{B}\mathfrak{Q}(R, \alpha, \eta)$ (here $\eta = \pm 1$), $r_u(P, q)$ represents $n\tau(u) + \varepsilon\xi$. Since

$$\Lambda_{2k}(R, \alpha, 1) = \Lambda_0(R, \alpha, (-1)^k) = \tilde{K}_0\mathcal{B}\mathfrak{Q}(R, \alpha, (-1)^k)$$

corresponds to $n = 0$, the result follows for i even.

For the odd case, let λ_u be the element of $\mathrm{GL}_n(R)$ given by multiplication by u. Then for any automorphism $\Phi \in U(R^n)$, one can easily verify $r_u(\Phi) = H(\lambda_u)^{-1}\Phi_0 H(\lambda_u)$, where $\Phi_0 = \Phi$ if $\varepsilon = 1$, and if $\varepsilon = -1$, $\Phi_0 = \Phi' = i^{-1}\Phi i$ is obtained from Φ as in [F, Lemma 5]: here i is 1 on $\mathbf{R}^n$ and -1 on $(\mathbf{R}^n)^\alpha$. The effect on the commutator quotient $\Lambda_1(R, \alpha, \eta)$ of $\mathrm{GL}(R, \alpha, \eta)$ is thus the identity if $\varepsilon = 1$, and $\Phi \to \Phi'$, or equivalently (loc. cit.), change of sign, if $\varepsilon = -1$.

The following immediate consequence will be referred to in the sequel as 'Taylor's lemma'.

COROLLARY 3.4.4. *If* R *has a central unit* $u = -u^{-\alpha}$, $L_*^X(R, \alpha, 1)$ *has exponent* 2.

We can apply this to the above situation, taking $u = T$. It follows that the extension determining $L_3^Y(R)$ splits, hence

THEOREM 3.4.5. *Let* π *be an abelian* 2-*group of rank* r *with* s *summands of order* 2, $w: \pi \to \{\pm 1\}$ *nontrivial but* $w \mid {}_2\pi$ *trivial;* $R = \mathbf{Z}\pi$, α *the involution induced by* $g \to w(g)g^{-1}$, Y *the subgroup of* $K_1(R)$ *generated by* $\pm\pi$ *and* $SK_1(R)$. *Then* $L_p^Y(R)$ *is an elementary* 2-*group, of rank* $2^r - 1 - r - \binom{s}{2}$, $2^r - r - \binom{s}{2}$, 1, $2^r + 2^{r-1}$ *for* $p = 0, 1, 2, 3$.

The surgery obstruction groups are deduced by cancelling $\mathbf{Z}/2$ (generated by τ) when p is odd. The most interesting case is when π is cyclic (so $r = 1$, $s = 0$); the surgery obstruction groups are then 0, 0, $\mathbf{Z}/2$, $\mathbf{Z}/2 + \mathbf{Z}/2$.

3.5. The exceptional case

We return to the case excluded above, when $\pi_0 = \operatorname{Ker} w$ is a direct summand of π. As we observed in (3.3), there are then no characters of type O or Sp, hence no such summands occur in S, $\hat{S}_p$, or (for p odd) $\hat{R}_p$. Thus

$CL_*^S(S) = 0$ and by the main exact sequence,

$$L_*^X(R) \cong L_*^X(\hat{R}) \oplus L_*^S(T) .$$

Further, $L_*^S(\hat{R}_p) = 0$ for p odd and $L_*^S(T) = 0$ as there are, moreover, no signatures. So $L_*^X(R) \cong L_*^X(\hat{R}_2)$, and similarly for Y.

Now $\bar{R}_2 \cong \mathbf{F}_2$ so, as above, $L_*^K(\hat{R}_2) \cong \mathbf{Z}/2$. To deduce L^X and L^Y, we recall that by (3.2), the composite

$$(\hat{\mathbf{Z}}_2\pi)^\times \longrightarrow (\hat{\mathbf{Z}}_2\pi_0)^\times \longrightarrow (\mathbf{F}_2\pi_0)^\times$$

induces cohomology isomorphisms. Thus if π has 2-rank r, and s summands of order 2, $H^i(\hat{\mathbf{Z}}_2\pi)^\times$ has rank $2^{r-1} - 1 + r - s$, by (3.2.1).

First suppose π_0 trivial. Then $H^i(\hat{\mathbf{Z}}_2\pi_1)^\times = 0$. Thus

$$L_p^X(R) = L_p^Y(R) = L_p^X(\hat{R}_2) = L_p^Y(\hat{R}_2) \cong \mathbf{Z}/2 .$$

The groups for p even are detected by the Arf invariant. Those for p odd are generated by the class of τ: the corresponding surgery obstruction group vanishes.

In general, the retraction of π on π_1 with kernel π_0 is compatible with w, hence the L-theory splits. Thus

THEOREM 3.5.1. *Let π be an abelian 2-group of rank r with s summands of order 2, $w\colon \pi \to \{\pm 1\}$ such that $w(T) = -1$ for some $T \in \pi$ with $T^2 = 1$. Then $L_p^X(\mathbf{Z}\pi) \cong E \oplus \mathbf{Z}/2$, where $\mathbf{Z}/2$ is as above (for π_1) and E is an elementary 2-group of rank $2^{r-1} - 1 + r - s$. Also $L_p^Y(\mathbf{Z}\pi) \cong E' \oplus \mathbf{Z}/2$ similarly, where rank $E' = 2^{r-1} - s$.*

For the last clause we observe (as usual) that $\pi_0 \to H^1(\mathbf{F}_2\pi_0)^\times$ has kernel π_0^2, and that $\pm\pi_1$ maps to zero in $H^1(\hat{\mathbf{Z}}_2\pi)^\times$, so $E' = H^1\big(K_1(\hat{\mathbf{Z}}_2\pi)/Y\big)$ has rank $(r-1)$ less than that of $E = H^1\big(K_1(\hat{\mathbf{Z}}_2\pi)\big)$. A corresponding result must hold for H^0 by a Herbrand quotient argument (one may also compute directly). The above statement corrects the result announced in [L].

4. 2-hyperelementary groups

In the first section of this chapter, we follow an idea from [UGR] to prove a 'splitting theorem' which splits the calculation into a sum of other groups, each simpler to compute.

For the rest of the chapter we consider only the case where the Sylow 2-subgroup is abelian, and slog through the full calculation step by step. At the very end, we are caught in tricky questions involving class groups, where no simple formulae can be given.

More explicitly, in (4.2) we classify the types of summands that arise. In (4.3) we compute $L_i(\hat{R}_2)$ in each case, and in (4.4) we compute $L_i(R \to \hat{R}_2)$.

The map between these is discussed in (4.5); the technical part of the discussion is amplified in (4.6) after some number-theoretic preliminaries; (4.7) is an attempt at a conclusion. However, the results are so complicated that the reader who wishes to make effective use of them will need to understand the derivation, and not just a statement of results.

4.1. Splitting theorems

We have seen in the earlier chapters that the calculations can be essentially reduced to the case when π is 2-hyperelementary: an extension (necessarily split) of a cyclic group ρ of odd prime order n by a 2-group σ. The complexity of this case is alleviated by splitting theorems.

Write $p(n)$ for the set of prime divisors of n, and for $P \subset p(n)$, write $P' = p(n) - P$.

PROPOSITION 4.1.1. *Let F be any functor from groups to abelian groups. Then $F(\pi)$ is a direct sum indexed by subsets P of $p(n)$, whose P-summand depends only on the Hall $\{2, P\}$-subgroup of π.*

Proof. Choose a Sylow 2-subgroup σ of π. Let $f_P \colon \pi \to \pi$ be the unique homomorphism which is the identity on σ and on Sylow p-subgroups for $p \in P$, and takes Sylow q-subgroups for $q \notin P$ to $\{1\}$. Then, clearly, $f_P \circ f_Q = f_{P \cap Q}$. Applying F yields a commutative algebra of endomorphisms $F(f_P)$ of $F(\pi)$, which induce the desired splitting.

This can be seen explicitly as follows. For each $p \mid n$ set $A_p = F(f_{p'})$, $A_p = 1 - A'_p$ and then for $P \subset p(n)$ set

$$E_P = \prod\nolimits_{p \in P} A_p \prod\nolimits_{p \notin P} A'_p \,.$$

Then the E_P are orthogonal idempotents with sum 1 and yield the desired splitting. Moreover,

$$F(f_P) = \sum \{E_Q \colon Q \subset P\}$$

so the image of E_P is contained in that of the projection $F(f_P)$ of $F(\pi)$ on $F(f_P\pi)$, and depends only on the Hall subgroup $f_P\pi$ of π.

Applying this to L-theory, where $R = \mathbf{Z}\pi$, $\hat{R} = \hat{\mathbf{Z}}\pi$, $S = \mathbf{Q}\pi$, and $T = \mathbf{R}\pi$, we obtain a splitting of the entire exact sequence

$$\cdots L_i^X(R) \longrightarrow L_i^X(\hat{R}) \oplus L_i^S(T) \longrightarrow CL_i^S(S) \longrightarrow L_{i-1}^X(R) \cdots ;$$

also of

$$\cdots L_i^X(R) \longrightarrow L_i^Y(R) \longrightarrow H^i\{\pm\pi/\pi'\} \cdots .$$

However, the algebras S, $\hat{S}$, T, and $\hat{R}_q$ with $q \nmid n$ split (following—as in [UGR]—the decomposition of $\mathbf{Q}\rho$ as sum of fields) into summands—e.g., $S(d)$—

labelled by divisors d of n: the L-theory splits correspondingly, and this refines the above splitting since one sees immediately that, for example, $E_P L_*(S) = \Sigma\{L_*(S(d)) \colon p(d) = P\}$. We claim that this more refined splitting extends to the rest of the L-theory.

To illustrate the idea, consider the second sequence first. Since the functor $H^i(\pm\pi/\pi')$ takes the 'same' values for π and its Sylow 2-subgroup σ, the splitting concentrates this whole functor into the summand $P = \emptyset$. In fact this summand of the sequence is just the same sequence but with π replaced by σ. At the other summands we obtain $L^X = L^Y$, so it suffices to concentrate further study on L^X. Note that our more refined splitting will not further affect the summand $P = \emptyset$ (corresponding to $d = 1$).

Now consider the summand of the first sequence corresponding to the set P of primes. By the last clause in the proposition, it suffices to consider the case when P consists of all prime divisors of n. Let $p \in P$ (so p is odd), and write ρ_p for the Sylow p-subgroup of ρ (or of π). Then

$$L_i^X(\hat{R}_p) = L_i^X(\hat{\mathbf{Z}}_p\pi) \cong L_i^S(\overline{\mathbf{Z}_p\pi}) \cong L_i^S(\overline{\mathbf{Z}_p\pi/\rho_p}) \cong L_i^X(\hat{\mathbf{Z}}_p\pi/\rho_p)\ ,$$

where the third isomorphism holds since the augmentation ideal of ρ_p lies in the radical. But the composite isomorphism is precisely $F(f_{p'})$, where F is the functor $F(\pi) = L_i^X(\hat{\mathbf{Z}}_p\pi)$. Thus $E_P L_i^X(\hat{\mathbf{Z}}_p\pi) = 0$ for $p \in P$. This leads one to guess the following.

THEOREM 4.1.2. *For π as above, there is a natural splitting of $L_i^X(\mathbf{Z}\pi)$ into summands $L_i^X(\mathbf{Z}\pi)(d)$ labelled by divisors d of n. If $R(d) = \mathbf{Z}[T/\phi_d(T) = 0]\sigma$ denotes the corresponding quotient ring of $R = \mathbf{Z}\pi$ (and similarly for $S(d)$ etc.), we have an exact sequence*

$$\cdots\ L_i^X(\mathbf{Z}\pi)(d) \longrightarrow \textstyle\prod_{p \nmid d} L_i^X(\hat{R}_p(d)) \oplus L_i^S(T(d)) \longrightarrow CL_i^S(S(d)) \longrightarrow \cdots .$$

Proof. By the above, it suffices to show that $E_{p(n)} L_i^X(\mathbf{Z}\pi)$ further splits into summands corresponding to those $d \mid n$ with $p(d) = p(n)$. Let $\hat{R}'_p$ be the maximal order containing $\hat{R}_p$ (it is unique, since we must adjoin σ to the unique maximal order containing $\mathbf{Z}\rho$), and R' the intersection of S with $\hat{R}'$. Since $\hat{R}'_p = \hat{R}_p$ for $p \nmid n$, the usual 'main' exact sequence shows that

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \big\downarrow & & \big\downarrow \\ \prod_{p \mid n} \hat{R}_p & \longrightarrow & \prod_{p \mid n} \hat{R}'_p \end{array}$$

induces excision isomorphisms of relative L^X groups, hence also of the summands $E_{p(n)} L^X$. Thus

$$E_{p(n)} L_i^X(R) \cong E_{p(n)} L_{i+1}^X\big(R \longrightarrow \textstyle\prod_{p \mid n} \hat{R}_p\big)$$

by the result immediately preceding the statement of the theorem, and thus

$$\begin{aligned}&\cong E_{p(n)} L^X_{i+1}(R' \longrightarrow \textstyle\prod_{p|n} \hat{R}'_p)\\ &= \bigoplus \{L^X_{i+1}(R(d) \longrightarrow \textstyle\prod_{p|n} \hat{R}_p(d)) \colon d \mid n,\ p(d) = p(n)\}\end{aligned}$$

since $R' = \bigoplus_{d|n} R(d)$, and similarly for $\hat{R}'$. This yields the desired splitting; the exact sequence is now a formal consequence of the main exact sequence for $R(d)$.

4.2. Classification of types

We return to the notation of (4.1), but now further suppose the Sylow 2-subgroup σ abelian. By the splitting theorem (4.1), it is enough to study a single summand, and it thus suffices to consider the ring

$$R = \mathbf{Z}[T/\phi_n(T) = 0][\sigma/g^{-1}Tg = T^{\lambda(g)}]\ ,$$

where ϕ_n is the n^{th} cyclotomic polynomial, for some homomorphism $\lambda\colon \sigma \longrightarrow (\mathbf{Z}/n)^\times$. We write ζ for Ker λ, and C for the subring of $B = \mathbf{Z}[T/\phi_n(T) = 0]$ invariant under Im λ. Thus $C\zeta$ is the centre of R. Write P for the set of prime divisors of n, $Q = P \cup \{2\}$, and P' for the complementary set of primes. Recall that we are interested in computing the modified L-groups $L^X_i(R)(n)$.

Rings such as R were studied in [UGR, 8.2]. We showed there that the localisation $B_{P'}\sigma$ was Azumaya over its centre $C_{P'}\zeta$. Thus we obtain Azumaya algebras, on tensoring with $\mathbf{Q}$, $\hat{\mathbf{Q}}$, $\mathbf{R}$ and those $\hat{\mathbf{Z}}_p$ with $p \notin P$ (i.e., $p \nmid n$), and these are the only cases we need for the main exact sequence, split as in (4.1.2). We will be able to reduce the calculation of L-groups from the Azumaya algebra to its centre, and this reduces us to considering the ring $C\zeta$, which is not very different from the problem solved in Chapter 3.

There are, however, three main differences:

(i) Primes $p \in P$ are omitted from the term $L(\hat{R}) = \prod L(\hat{R}_p)$. This causes no extra problem—rather the reverse.

(ii) The base ring $\mathbf{Z}$ is replaced by C. This is no theoretical difficulty, but raises major problems (unit groups, class groups etc.) in numerical calculations, illustrated by the difference between (1.4) and (1.6) above.

(iii) The L-groups of an Azumaya algebra cannot be naively identified with those of the ground ring.

In this section we deal with (iii), and classify the cases arising.

Case 0. $-1 \notin \lambda(\sigma)$.

Since the standard involution takes T to T^{-1} in B, in this case it is nontrivial on C. But then it is nontrivial on every summand of $C\zeta \otimes \mathbf{Q}$, $C\zeta \otimes \hat{\mathbf{Q}}$, $C\zeta \otimes \hat{\mathbf{Z}}_p$ $(p \notin P)$. Thus all these summands have type GL or U.

Otherwise we may choose an element $g_0 \in \sigma$ with $\lambda(g_0) = -1$; thus $g_0 \notin \zeta$,

$g_0^2 \in \zeta$. In order to analyse types of summands in this case, we need a lemma.

PROPOSITION 4.2.1. *Let L be a Galois extension of K with group τ, E a crossed product of L and τ* (i.e., *a τ-graded algebra with base ring L and any unit of grade $T \in \tau$ induces T via inner automorphism on L).*

If α is an anti-involution of E such that grade $u = T$ implies grade $u^\alpha = T^{-1}$, then

(i) *if $\alpha \mid L \notin \tau$, $(E, \alpha, 1)$ has type U;*

(ii) *if $\alpha \mid L = T_0$, then $T_0^2 = 1 \in \tau$ and there exists $\varepsilon = \pm 1$ such that for all x of grade T_0, $x^\alpha = \varepsilon x$. $(E, \alpha, 1)$ has type O or* Sp *according to whether $\varepsilon = 1$ or -1.*

Proof. Observe that the hypothesis implies (see e.g., [UGR, § 3] for references) that E is central simple over K. By the definition of types of anti-involutions (1.1) or [II], it follows that $(E, \alpha, 1)$ has type U if $\alpha \mid K$ is nontrivial, and has type O or Sp if $\alpha \mid K$ is trivial. As $\alpha \mid K$ is trivial $\Leftrightarrow \alpha \mid L \in \tau$ (for α is an automorphism of the field L), (i) follows. As to (ii), if $\dim_K E = n^2$, and E^+, E^- as the ± 1-eigenspaces of α on E, we know that

$$\dim_K E^+ - \dim_K E^- = n\varepsilon' \qquad \text{for } \varepsilon' = \pm 1 ,$$

and that $(E, \alpha, 1)$ has type O or Sp according to whether $\varepsilon' = 1$ or -1.

It thus remains to prove $\varepsilon = \varepsilon'$. Now if $g \in \tau$, $g^2 \neq 1$ then α interchanges the subspaces of grades g, g^{-1} which thus contribute equally to E^+ and E^-. If $T \in \tau$, $T^2 = 1$, choose a unit $u \in E$ of grade T. Then a general element of this grade is lu ($l \in L$), so $u^\alpha = l_u u$ for some $l_u \in L$. Now

$$(lu)^\alpha = u^\alpha l^\alpha = l_u u l^\alpha = l_u l^{\alpha T} u .$$

In particular, $u = u^{\alpha\alpha} = (l_u u)^\alpha = l_u l_u^{\alpha T} u$, so $l_u l_u^{\alpha T} = 1$.

If $T \neq T_0$, αT is a nontrivial involution of L, and (by the 'Hilbert Satz 90') we can choose $l_0^{\alpha T} l_0^{-1} = l_u$. Write $v = l_0^{-1} u$; then $v^\alpha = v$ and $(lv)^\alpha = l^{\alpha T} v$. But the ± 1 eigenspaces of αT on L clearly have equal dimensions over K.

Finally, if $T = T_0$, αT is the identity, so $l_u^2 = 1$ and $l_u = \varepsilon = \pm 1$. Any element of grade T is of the form lu, and $(lu)^\alpha = \varepsilon lu$. Thus $\dim_K E^\varepsilon - \dim_K E^{-\varepsilon} = \dim_K L = n$, so $\varepsilon = \varepsilon'$ and the result follows.

We apply this result to the Azumaya algebra $\mathbf{Q} \otimes B\sigma$, with centre $\mathbf{Q} \otimes C\zeta$. This centre decomposes as a sum of fields, labelled by $\mathbf{Q} \otimes C$-equivalence classes of characters $\chi \colon \zeta \to \mathbf{C}^\times$, and there is a corresponding decomposition of the algebra $\mathbf{Q} \otimes B\sigma$.

COROLLARY 4.2.2. $(\mathbf{Q} \otimes B\sigma, \alpha, 1) \sim (\mathbf{Q} \otimes C\zeta, \alpha \mid C\zeta, w(g_0)g_0^2)$ *in the sense that corresponding summands have the same type.*

Recall that $\alpha \mid B$ is the identity by hypothesis and for $g \in \sigma$, $g^\alpha = w(g)g^{-1}$

for some homomorphism $w\colon \pi \to \{\pm 1\}$. Also, $g_0 \in \sigma$ satisfies $\lambda(g_0) = -1 \in (\mathbf{Z}/n)^\times$. The corollary follows by applying the proposition to each summand in turn. Each (relevant, irreducible) character χ of π gives a map of $\mathbf{Q} \otimes C\zeta$ to a field K; indeed, χ vanishes on $\sigma - \zeta$, and its restriction to ζ is a multiple of an irreducible character χ'. The corresponding value of ε is the image of g_0^α/g_0 (we take g_0 for u in the proposition), and this is inverse to (hence equivalent to) $w(g_0)g_0^2$.

Since $\mathbf{Q} \otimes C$ is unramified at 2 and ζ is a 2-group, the $\mathbf{Q} \otimes C$-equivalence classes of characters χ are the same as $\mathbf{Q}$-equivalence classes of χ'. Types O and Sp correspond to self-conjugate characters

$$\chi(g) = \bar{\chi}\big(w(g)g\big) = w(g)\chi(g^{-1}) , \quad \text{i.e.,} \quad \chi'(g)^2 = w(g) \qquad \text{for } g \in \zeta .$$

By the above we have type O if $w(g_0)\chi'(g_0^2) = 1$; of course, we always have $w(g_0^2) = 1$.

We deduce the behaviour at real places since in the orientable case ($w = 1$) only types $O(\mathbf{R})$, $U(\mathrm{C})$ and $\mathrm{Sp}(\mathbf{H})$ can appear ([15], [SCM]). Type U corresponds to $\chi \neq \bar{\chi}$; for the others, χ' is a homomorphism $\zeta \to \{\pm 1\}$, and the real ramified places (type $\mathbf{H}$) are those with $\chi'(g_0^2) = -1$. This can also of course be seen directly.

We now introduce notation for the cases that arise; there are two independent principles of classification. First, as in Chapter 3 and corresponding respectively to the exceptional, regular nonorientable and orientable cases of that chapter, we have

Case I. $w \mid {}_2\zeta \neq 1$, i.e., for some $T \in \zeta$ of order 2, $w(T) = -1$.
Thus ζ is the direct sum of $\zeta_0 = \operatorname{Ker} w$ and ζ_1, generated by T.

Case II. $w \mid {}_2\zeta = 1$ but $w \mid \zeta \neq 1$.

Case III. $w \mid \zeta = 1$.

As ζ is in the centre of π, Taylor's lemma is applicable to Cases I and II.

Now, however, we have a further classification into subcases (which will be written as suffixes).

Subcase 0. $-1 \notin \operatorname{Im} \lambda$ as discussed above.
Otherwise, choose $g_0 \in \sigma$ with $\lambda(g_0) = -1$. Then $g_0^2 \in \operatorname{Ker} \lambda = \zeta$. Observe that g_0 is unique up to multiplication by an element of ζ, hence g_0^2 is unique modulo ζ^2.

Subcase a. $g_0^2 \in \zeta^2$. Then g_0 can be rechosen so that $g_0^2 = 1$. In future, we will *always* suppose this done.

Subcase bc. $g_0^2 \neq \zeta^2$.
In our later and finer calculations, we will subdivide this further, as

Subcase b. $g_0^2 \in {}_2\zeta \cdot \zeta^2$ but $g_0^2 \notin \zeta^2$.

Subcase c. $g_0^2 \notin {}_2\zeta \cdot \zeta^2$.

There is even a further dichotomy, depending on the sign of $w(g_0)$, which is important in Cases II_a and III_{abc}. It is, however, clear from the preceding that a change of sign in $w(g_0)$ (everything else being the same) corresponds to interchanging L_i and L_{i+2} for each i. With this in mind, we assume henceforth that $w(g_0) = 1$.

4.3. The 2-adic calculation

Continuing the notation of (4.2), we will now compute the groups $L_i^K(\hat{R}_2)$. Now $\hat{R}_2$ is Azumaya, and indeed [UGR] a matrix ring over its centre $\hat{C}_2\zeta$. Thus the antistructure $(\hat{R}_2, \alpha, 1)$ is equivalent to an antistructure $(\hat{C}_2\zeta, \alpha|\hat{C}_2\zeta, u)$ for some $u \in (\hat{C}_2\zeta)^\times$: in fact we have the exact sequence of [12] (mentioned also in [II]),

$$H^1(\mathrm{Pic}\ A) \longrightarrow H^1(A^\times) \longrightarrow \mathrm{Br}_0(A, \mathbf{Z}/2^-) \longrightarrow H^0(\mathrm{Pic}\ A)\ ,$$

where Pic is the Picard group—trivial in the case of $\hat{C}_2\zeta$—and $\mathrm{Br}_0(A, \mathbf{Z}/2^-)$ the Brauer group of antistructures which are trivial in the usual Brauer group. The same holds of course for $\mathbf{Q} \otimes \hat{C}_2\zeta$, so by the preceding section $(\mathbf{Q} \otimes \hat{R}_2, \alpha, 1)$ is equivalent to $(\mathbf{Q} \otimes \hat{C}_2\zeta, \alpha, w(g_0)g_0^2)$. If $H^1(\hat{C}_2\zeta)^\times \to H^1(\mathbf{Q} \otimes \hat{C}_2\zeta)^\times$ is injective, we deduce $(\hat{R}_2, \alpha, 1) \sim (\hat{C}_2\zeta, \alpha, w(g_0)g_0^2)$. This is true (by results of [UGR] quoted in detail below), except in Case I. But the result can be shown here too. For let ι be the subgroup generated by T with $T^2 = 1$ and $w(T) = -1$. Then T is central in π, so $\pi = (\mathrm{Ker}\, w) \oplus \iota$. We can now apply the result just obtained for $\mathrm{Ker}\, w$, and tensor this Morita equivalence with the identity on $\mathbf{Z}\iota$. This completes the proof of

LEMMA 4.3.1. *There is a Morita equivalence of antistructures*

$$(\hat{R}_2, \alpha, 1) \sim (\hat{C}_2\zeta, \alpha, w(g_0)g_0^2)\ .$$

We may now begin to compute.

$$L_i^K(\hat{R}_2) \cong L_i^K(\hat{C}_2\sigma_2) \cong L_i^K(\bar{C}_2)\ .$$

Now in subcase 0, α is a nontrivial Galois automorphism of $\bar{C}_2$, so $L_i^K(\bar{C}_2) = 0$. Otherwise, α is the identity. If C has g_2 prime divisors of 2, $\bar{C}_2$ is a sum of g_2 fields so $L_i^K(\bar{C}_2)$ has rank g_2. Indeed, $\hat{C}_2$ itself is the sum of g_2 isomorphic completions $\hat{C}_\mathfrak{p}$ of C, each unramified over $\hat{\mathbf{Z}}_2$. To proceed to compute L_i^K, we next need the groups $H^i(\hat{C}_\mathfrak{p}\zeta)^\times$. These are given by (3.2.1) (Case I) and (3.2.3) (Cases II and III).

In subcase 0, these groups also vanish, hence $L_i^K(\hat{C}_2\zeta) = 0$. For the remaining cases, we must next compute the map $L_i^K \to H^{i+1}$ in the exact sequence for L_i^X.

PROPOSITION 4.3.2. *$L_i^K(\hat{C}_p\zeta) \to H^i(\hat{C}_p\zeta)^\times$ is injective except in the following cases:*

Case I, $i = 2k$. $g_0^2 \in {}_2\zeta \cdot \zeta^2$, $i = 2k+1$, $g_0^2 \in \zeta^2$.

Case II, III, $i = 2k$ *or* $2k+1$. $w(g_0)g_0^2 = (-1)^{k+1}w(g)g^2$ *for some* $g \in \zeta$.

Proof. First suppose $i = 2k$. Then the nonzero element of $L_i^K(\bar{C}_p)$ is represented by $\begin{pmatrix} 1 & 1 \\ 0 & \beta \end{pmatrix}$. Choose $b \in \hat{C}_p$ representing β. Then this lifts to the element of $L_i^K(\hat{C}_p\sigma_2)$ represented by $\begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix}$. Symmetrising yields

$$\begin{pmatrix} 1+u & 1 \\ u & b(1+u) \end{pmatrix},$$

where $u = (-1)^k w(g_0)g_0^2$. This has determinant $b(1+u)^2 - u$ and hence discriminant $\delta = 1 - b(u + 2 + u^{-1})$. Clearly this is 1 if $u = -1$, and is 1 (mod 2) if $u^2 = 1$; as $w(g_0)g_0^2$ is only determined modulo elements $w(g)g^2$, $g \in \zeta$, this proves the map zero in the cases stated.

Conversely, in Case I we have one of the listed basis elements provided g_0^2 (mod T) does not belong to ${}_2\bar{\zeta} \cdot \bar{\zeta}^2$; i.e., provided $g_0^2 \notin {}_2\zeta \cdot \zeta^2$. In the other cases, either (1) $g_0^2 \notin \zeta^2$, and we can find characters χ of ζ satisfying $\chi(g)^2 = w(g)$ and $\chi(g_0^2)$ taking both values ± 1 (recall that $w(g_0^2) = 1$); thus we may suppose $\chi(u) = 1$. Or (2) $g_0^2 \in \zeta^2$, and $\chi(u) = +1$ for all such χ, by hypothesis. The image of δ in the corresponding copy of $\hat{C}_p$ or $\hat{C}_p[i]$ is $1 - 4b$. This is not a square in $\hat{C}_p$: in fact $\mathbf{Q} \otimes \hat{C}_p[\sqrt{(1-4b)}]$ is the unramified quadratic extension of $\mathbf{Q} \otimes \hat{C}_p$. It is thus linearly disjoint from the ramified extension $\mathbf{Q} \otimes \hat{C}_p[i]$, and $1 - 4b$ is not a square here either. The projection of δ under $H^0(\hat{C}_p\zeta)^\times \xrightarrow{\chi} H^0(\hat{C}_p^\times)$ *or* $H^0(\hat{C}_p[i])^\times$ is nontrivial, hence the class of δ is already nontrivial.

Now let $i = 2k+1$. Here the nonzero element of $L_i^K(\bar{C}_p)$ is represented by interchange of two generators in a hyperbolic plane. This lifts to the corresponding automorphism of a plane over $(\hat{C}_p\zeta, \alpha, u)$, represented by the matrix $\begin{pmatrix} 0 & 1 \\ u^{-1} & 0 \end{pmatrix}$ with determinant u^{-1}. In this case we see at once from (3.2.1), resp. (3.2.3), that it represents the trivial class in $H^1(\hat{C}_p\sigma^2)^\times$ precisely in the cases listed.

We now determine the groups $L_i^X(\hat{R}_2, \alpha, 1) = L_i^S(\hat{C}_2\zeta, \alpha, w(g_0)g_0^2)$ (by (4.3.1)), treating the cases in turn. In each case, if C has g_2 even primes, the group decomposes as sum of g_2 isomorphic summands. In subcase 0, the groups vanish.

In Case I, we have $T \in \zeta$ of order 2 with $w(T) = -1$, and so $T^\alpha = -T = -T^{-1}$. It follows (since $T^\alpha = -T$) that scaling by T [Ax] induces isomorphisms of L_i on L_{i+2} for all i. By Taylor's lemma (3.4.4), the groups have exponent 2. We have subcases:

$$\begin{aligned}
&\mathrm{I_a}.\ g_0^2 \in \zeta^2 . && L_i \cong g_2\big(H^{i+1}(\hat{C}_p\zeta)^\times \oplus L_i^K(\bar{C}_p)\big) .\\
&\mathrm{I_b}.\ g_0^2 \in ({}_2\zeta\cdot\zeta - \zeta^2) . && L_0 \cong g_2\big(H^1(\hat{C}_p\zeta)^\times/\mathrm{Im}\, L_1^K(\hat{C}_p) \oplus L_0^K(\hat{C}_p)\big) ,\\
& && L_1 \cong g_2\big(H^0(\hat{C}_p\zeta)^\times\big) .\\
&\mathrm{I_c}.\ g_0^2 \notin ({}_2\zeta\cdot\zeta^2) . && L_i \cong g_2\big(H^{i+1}(\hat{C}_p\zeta)^\times/\mathrm{Im}\, L_{i+1}^K(\bar{C}_p)\big) .
\end{aligned}$$

In Case $\mathrm{II_a}$, we assume $g_0^2 = 1$, and $w(g_0) = 1$. The map is injective for $i = 2, 3$ but not for $i = 0, 1$. Taylor's lemma again applies. We thus have

$$L_0 \cong g_2\big(\mathrm{Ker}(w \mid \zeta)/\zeta^2\big) , \qquad L_1 \cong g_2\big(H^0(\hat{C}_p\zeta)^\times\big) ,$$
$$L_2 \cong g_2\big(\pm\mathrm{Ker}(w \mid \zeta)/\zeta^2 \oplus \mathbf{Z}/2\big) , \quad L_3 \cong g_2\big(H^0(C_p\zeta)^\times/\mathrm{Im}\, L_0^K(\bar{C}_p) \oplus \mathbf{Z}/2\big) .$$

In Case $\mathrm{III_a}$, the map is as in Case $\mathrm{II_a}$, but Taylor's lemma is not available. Again, suppose $g_0^2 = 1$ and $w(g_0) = 1$. We may obtain the extensions by noting that $\zeta \to 1$ induces a retraction of antistructures $(\hat{C}_2\zeta, \alpha, 1) \to (\hat{C}_2, 1, 1)$. Thus

$$L_i^X(\hat{R}_2) \cong L_i^S(\hat{C}_2, 1, 1) \oplus \mathrm{Ker}\big(H^{i+1}(\hat{C}_2\zeta)^\times \longrightarrow H^{i+1}(\hat{C}_2^\times)\big) .$$

The second summand is (a sum of g_2 copies, each) given by (3.2.3); for i even, we have ζ/ζ^2 and for i odd, a group (unnaturally) isomorphic to $K_\tau \oplus \zeta/\zeta^2$, where τ is the subgroup of ζ of exponent 2. The first summand is g_2 copies of $L_i^S(\hat{C}_p)$, and by (1.6),

$$L_0^S(\hat{C}_p) = 0 , \quad L_1^S(\hat{C}_p) \cong \hat{C}_p^\times/(\hat{C}_p^\times)^2 , \quad L_2^S(\hat{C}_p) \cong \{\pm 1\} \oplus \mathbf{Z}/2$$

and there is a non-split exact sequence

$$0 \longrightarrow \mathbf{Z}/2 \longrightarrow \hat{C}_p^\times/(\hat{C}_p^\times)^2 \longrightarrow L_3^S(\hat{C}_p) \longrightarrow \mathbf{Z}/2 \longrightarrow 0 .$$

This follows from (1.6), and the remark that -1 is not a square in $\hat{C}_p$.

Cases $\mathrm{II_{bc}}$ and $\mathrm{III_{bc}}$. The map is always injective, hence

$$L_i \cong g_2\big(H^{i+1}(\hat{C}_p\zeta)^\times/\mathrm{Im}\, L_{i-1}^K(\bar{C}_p)\big)$$

in all cases. Thus if $i = 2k$ is even, we have the quotient of $\{\pm 1 \oplus \mathrm{Ker}\, w \mid \zeta\}/\{w(g)g^2 \colon g \in \zeta\}$ by the class of $(-1)^{k+1}g_0^2$.

4.4. Calculation of γ_i and $L_i(R \to \hat{R}_2)$

We continue to suppose π is 2-hyperelementary with abelian Sylow 2-subgroup σ, and $w\colon \sigma \to \{\pm 1\}$, and continue the calculation of the summand of $L_i^X(\mathbf{Z}\pi)$ corresponding (in the splitting of (4.1)) to n. In this section, we describe the homomorphism

$$\gamma_i\colon \prod \{L_i^S(\hat{R}_p)\colon p \ \text{ odd}, \ p \nmid n\} \oplus L_i^S(T) \longrightarrow CL_i^S(S) .$$

Here, $R = \mathbf{Z}[T/\phi_n(T) = 0]\sigma$; the other notations are as usual.

As the $\hat{R}_p$ with p odd, $p \nmid n$ are maximal orders, any splitting of S

induces splittings of all these, as of T. It thus suffices for this section to consider summands of S separately. Now a summand of type GL contributes zero throughout. For one of type U, the corresponding $CL_i^S(S)$ vanish. So do the $L_i^S(\hat{S}_p)$, hence (as we have unramified orders, or by 'better reduction' (2.3i)) also the $L_i^S(\hat{R}_p)$. All that remains is $L_i^S(T)$, which is 0 for i odd and for i even is a free abelian group Σ defined, as usual, by signatures $\in 4\mathbf{Z}$ corresponding to summands of T (if any) of type U.

Now summands of $S = \mathbf{Q}[T/\phi_n(T) = 0]\sigma$ correspond to those of its centre $(\mathbf{Q} \otimes C)\zeta$. As ζ is a 2-group and C is unramified at 2, these correspond to summands of $\mathbf{Q}\zeta$, or to $\mathbf{Q}$-equivalence classes of characters χ' of ζ. Their types are given by (4.2.2): χ has type O or Sp if $\chi(g)^2 = w(g)$ for $g \in \zeta$ (type GL or U otherwise), and among these, type O corresponds to $w(g_0)\chi(g_0^2) = 1$.

Summands corresponding to signatures are (by (2.2)) those corresponding to characters χ of π with $\chi = \chi^w$, i.e., $\chi(g)(1 - w(g)) = 0$. As the representation is induced from $\rho\zeta$, this condition is equivalent to the same condition on the corresponding character $\chi' : \zeta \to \mathrm{C}^\times$ of ζ. As χ' never vanishes, this is equivalent to triviality of $w \mid \zeta$. Thus signatures appear for all χ in Case III and for no χ in Cases I, II. Observe how this fits Taylor's lemma, which tells us that the groups have exponent 2 in Cases I, II.

In Case I and in subcase 0, as already noted in (4.2), there are no summands of type O or Sp. In Cases $\mathrm{II_a}$ and $\mathrm{III_a}$, all characters χ of ζ with $\chi(g)^2 = w(g)$ for all $g \in \zeta$ have type O. In Cases $\mathrm{II_{bc}}$ and $\mathrm{III_{bc}}$, they have type O or Sp according to whether $\chi(g_0^2) = \pm 1$. Over $\mathbf{R}$, in Case III these have types $O(\mathbf{R})$ or $\mathrm{Sp}(\mathbf{H})$.

We now consider a single summand of type O: the corresponding result for type Sp can be obtained by interchanging i and $i + 2$. We are thus in Cases II or III. This summand corresponds to $\chi : \zeta \to \{\pm 1\}$ satisfying $\chi(g)^2 = w(g)$ for all $g \in \zeta$. Its centre, K, is $\mathbf{Q} \otimes C$ if $w \mid \zeta = 1$, i.e., in Case III, and $\mathbf{Q}[i] \otimes C$ if $w \mid \zeta \neq 1$—i.e., in Case II.

Write A for the ring of integers of K, C_K for its idele class group. We have already observed that the calculation of γ_i boils down to doing it for each summand. Now as $\hat{R}_p$ is a maximal order, it is a matrix ring over (hence is Morita equivalent to) $A \otimes \hat{\mathbf{Z}}_p$, and the equivalence is one of antistructures. We saw above that a corresponding equivalence holds for the real completion, except in Cases $\mathrm{III_{bc}}$ when $\chi(g_0^2) = -1$, when there is a matrix ring over $\mathbf{H}$ rather than one over $\mathbf{R}$. The calculation thus coincides with that of (1.6) except for this detail, and the fact that prime divisors of $2n$ are omitted.

We begin by listing the calculations of source and target for γ_i in tabular form.

i	$L_i^X(\hat{A}_\mathfrak{p})$	$L_i^S(\mathbf{R})$	$L_i^S(\mathbf{C})$	$L_i^S(\mathbf{H})$	$CL_i^S(S)$
3	0	0	0	0	0
2	$\{\pm 1\}$	$\{\pm 1\}$	$\{\pm 1\}$	$2\mathbf{Z}$	${}_2C_K^\times$
1	$\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2$	$\{\pm 1\}$	0	0	$C_K^\times/(C_K^\times)^2$
0	0	$4\mathbf{Z}$	0	0	$\mathbf{Z}/2$

Here, $\hat{A}_p$ splits as sum of rings $\hat{A}_\mathfrak{p}$ corresponding to primes $\mathfrak{p}$ of K over p (each is preserved by α). T also splits, and we have just listed the groups for each summand.

The map γ_3 is trivial. For $i = 0$, each summand $4\mathbf{Z}$ which occurs maps onto the $\mathbf{Z}/2$ (as in the case $A = \mathbf{Z}$). We denote the kernel by Σ. For $i = 2$, ${}_2C_K^\times$ is the product of groups $\{\pm 1\}$, one for each place of K, modulo the diagonal ± 1. The mapping γ_2 is just the inclusion of the factors corresponding to 'good' finite and infinite places; also by [IV, 5.4] it induces surjections $2\mathbf{Z} \to \{\pm 1\}$. It is injective, except that the kernel has a summand $4\mathbf{Z}$ corresponding to each infinite prime of type $\mathbf{H}$. The cokernel of γ_2 is the product of copies of $\{\pm 1\}$ corresponding to places of K dividing $2n$, modulo the diagonal $\{\pm 1\}$.

As usual, γ_1 is more complicated, but may be computed following (1.3). Certainly it is the natural homomorphism

$$\prod \{\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2 \colon \mathfrak{p} \text{ good}\} \longrightarrow C_K/C_K^2$$

where $\hat{A}_\mathfrak{p}$ is defined to be $\hat{K}_\mathfrak{p}$ if $\mathfrak{p}$ is an infinite place; the bad primes are the finite divisors of $2n$ and the infinite places of type $\mathbf{H}$, and other places are good. Now since, by the global square theorem, there is an exact sequence

$$1 \longrightarrow K^\times/(K^\times)^2 \longrightarrow \hat{K}^\times/(\hat{K}^\times)^2 \longrightarrow C_K/C_K^2 \longrightarrow 1 ,$$

γ_1 has the same kernel and cokernel as

$$K^\times/(K^\times)^2 \oplus \prod \{\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2 \colon \mathfrak{p} \text{ good}\} \longrightarrow \hat{K}^\times/(\hat{K}^\times)^2 .$$

But here we can cancel the second term from the source into the target. Observe, in fact, that for $\mathfrak{p}$ finite $\hat{K}_\mathfrak{p}^\times/\hat{A}_\mathfrak{p}^\times$ is infinite cyclic, so that $\hat{K}^\times$ is a (split) extension of $\hat{A}^\times$ by the group I of ideals. So γ_1 has the same kernel and cokernel as

$$K^\times/(K^\times)^2 \longrightarrow \prod \{\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2 \colon \mathfrak{p} \text{ bad}\} \oplus I/I^2 .$$

Now define $K^{(2)} = \operatorname{Ker}(K^\times \longrightarrow I/I^2)$ as the group of elements whose values at all places of K are even. The cokernel is Γ/Γ^2, where Γ is the ideal class group of K. So the kernel of γ_1 coincides with that of

$$\gamma_1' \colon K^{(2)}/(K^\times)^2 \longrightarrow \prod \{\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2 \colon \mathfrak{p} \text{ bad}\} ,$$

and its cokernel is a (split) extension of Coker γ_1' by Γ/Γ^2. Further, as in (1.3), there is a split short exact sequence

$$1 \longrightarrow A^\times/(A^\times)^2 \longrightarrow K^{(2)}/(K^\times)^2 \longrightarrow {}_2\Gamma \longrightarrow 1 .$$

Observe that we have now reduced to maps between finite groups, in contrast to the rather large groups above.

We deduce, for the summand considered,

PROPOSITION 4.4.1. *For a summand of S of type O, with the above notations, the contribution L_i^* to $L_i(R \to \hat{R}_2)$ is*

$L_0^* = \mathbf{Z}/2$ (*no places of type* $\mathbf{R}$), 0 (*otherwise*),

$L_3^* = \prod\{4\mathbf{Z}$: *places of type* $\mathbf{H}\}$,

and there is an exact sequence

$$\{\pm 1\} \longrightarrow \prod\{\pm 1\colon \mathfrak{p} \text{ bad, finite}\} \longrightarrow L_2^* \longrightarrow K^{(2)}/(K^\times)^2$$
$$\longrightarrow \prod\{\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2\colon \mathfrak{p} \text{ bad}\} \longrightarrow L_1^* \longrightarrow \Gamma/\Gamma^2 \oplus \Sigma \longrightarrow 0 .$$

Moreover, the extension determining L_1^* is split.

For convenience in the next section, we now integrate these into a complete statement, which is the formal result of this section.

Case I. $L_i(R \to \hat{R}_2) = 0$.

Case II. In subcase II_0, $L_i(R \to \hat{R}_2) = 0$. Otherwise, write $\zeta = \zeta_0 \oplus \zeta_1$, where $w \mid \zeta_0$ is trivial and ζ_1 is generated by T. Each element of $\hat{\zeta}_0 = \mathrm{Hom}(\zeta_0, \pm 1)$ extends to a character χ of ζ by taking T to i. Then in subcase II_a, $L_i(R \to \hat{R}_2)$ is a sum of copies of L_i^*, indexed by $\hat{\zeta}_0$. In subcase II_{bc}, we partition $\hat{\zeta}_0 = \hat{\zeta}_0^+ \cup \hat{\zeta}_0^-$ according to the sign of $\chi(g_0^2)$; then

$$L_i(R \longrightarrow \hat{R}_2) \cong \bigoplus \{L_i^*, \text{ indexed by } \hat{\zeta}_0^+\} \oplus \{L_{i+2}^*, \text{ indexed by } \hat{\zeta}_0^-\} .$$

In both subcases, all infinite places have type $\mathbf{C}$ (so none are bad) since $K = \mathbf{Q}[i] \otimes C$; so $L_0^* = \mathbf{Z}/2$, $L_3^* = 0$. By Taylor's lemma, everything has exponent 2; signatures are absent and sequences split.

Case III. In subcase III_0, $L_{2k+1}(R \to \hat{R}_2) \cong \bigoplus (4\mathbf{Z})$, indexed by characters χ of ζ, and $L_{2k}(R \to \hat{R}_2) = 0$. In subcase III_a, $L_i(R \to \hat{R}_2)$ is a sum of copies of L_i^*, indexed by $\hat{\zeta} = \mathrm{Hom}(\zeta, \pm 1)$, and of groups $4\mathbf{Z}$ (i odd), given by signatures at other characters. In subcase III_{bc}, partition $\hat{\zeta} = \hat{\zeta}^+ \cup \hat{\zeta}^-$ according to the sign of $\chi(g_0^2)$. Then

$$L_i(R \longrightarrow \hat{R}_2) \cong \bigoplus \{L_i^*, \text{ indexed by } \hat{\zeta}^+\} \oplus \{L_{i+2}^{**}, \text{ indexed by } \hat{\zeta}^-\} \oplus \Sigma' .$$

Here, $K = \mathbf{Q} \otimes C$; for L^* all infinite primes are of type $\mathbf{R}$, hence good and for L^{**} all are of type $\mathbf{H}$, hence bad; Σ' is a group of signatures, as above. Only $L_1(R \to \hat{R}_2)$ has an infinite contribution other than from Σ'.

4.5. Calculation of ψ_i

We retain the notation of the preceding section. Now as already observed in (3.3), ψ_i: $L_i(\hat{R}_2) \to L_i(R \to \hat{R}_2)$ factors through $L_i(\hat{S}_2)$. Let us first compute the homomorphism from $L_i(\hat{S}_2)$. Here we can afford, as in (4.4), to split S into its simple summands; and those not of type O or Sp can be ignored. For a single summand of S of type O, such that $\hat{S}_2$ is a sum of g_2' simple summands,

$$L_i^S(\hat{S}_2) \cong g_2'(\mathbf{Z}/2),\quad \hat{S}_2^\times/(\hat{S}_2^\times)^2,\quad g_2'(\pm 1),\quad 0$$

for $i = 0, 1, 2, 3$. Each $\mathbf{Z}/2$ in dimension 0 maps isomorphically to $CL_i^S(S)$ (cf. (1.3)), and the maps in dimensions 1 and 2 to $CL_i^S(S)$ are also the natural ones, as in (1.6) and the preceding section. However, by (1.6) even if $\hat{R}_2$ is maximal $L_0^S(\hat{R}_2)$ does not map onto $L_0^S(\hat{S}_2)$. Thus this $\mathbf{Z}/2$ is never in the image.

If we look at the conclusion (4.4.1), and observe that the image of $L_1^S(\hat{R}_2)$ is surely contained in $\hat{A}_2^\times/(\hat{A}_2^\times)^2$, we see that these components in dimensions 1 and 2 are naturally identified with the products of copies of $\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2$, resp. $\{\pm 1\}$, extended over *even* primes (which are all bad). Thus if we simply had an order whose 2-adic completion was maximal, the effect on (4.4.1) (at least for summands of type O) would just be to delete even primes from the list of bad primes.

We turn to the consideration of the map $L_i(\hat{R}_2) \to L_i(\hat{S}_2)$. Here it is natural to use the splitting of (4.3) into components $\hat{C}_\mathfrak{p}\zeta$ indexed by the g_2 even primes $\mathfrak{p}$ of C. Apart from the trivial subcase 0 (where $L_i(\hat{R}_2)$ vanishes), each summand $L_i^K(\hat{C}_\mathfrak{p}\zeta)$ has order 2, and thus the map $H^{i+1}(K_1(\hat{C}_\mathfrak{p}\zeta)) \to L_i^S(\hat{C}_\mathfrak{p}\zeta)$ is nearly an isomorphism. Thus 'most' of the structure will be given by computing the composite map $H^{i+1}(\hat{C}_\mathfrak{p}\zeta)^\times \to L_i^S(\hat{C}_\mathfrak{p}\zeta) \to L_i(\mathbf{Q} \otimes \hat{C}_\mathfrak{p}\zeta)$. This can now be done, as in Chapter 3, by invoking the characterisation of symmetric units.

Before starting all this, we clear the easy cases out the way. In Case I, $L_i(R \to \hat{R}_2) = 0$, so $L_i(R) = L_i(\hat{R}_2)$. In subcase 0, $L_i(\hat{R}_2) = 0$, so $L_{i+1}(R \to \hat{R}_2) = L_i(R)$. Thus

THEOREM 4.5.1. *We have the following results, for the summands of* $L_i(R)$ (*split by* (4.1)) *classified as in* (4.2):

Case I_0. $L_i(R) = 0$.

Case I_a. $L_i(R) \cong g_2(H^{i+1}(\hat{C}_\mathfrak{p}\zeta)^\times \oplus \mathbf{Z}/2)$.

Case I_b. $L_{2k}(R) \cong g_2(H^{2k+1}(\hat{C}_\mathfrak{p}\zeta)^\times/(g_0^2) \oplus \mathbf{Z}/2)$

$L_{2k+1}(R) \cong g_2(H^{2k+2}(\hat{C}_\mathfrak{p}\zeta)^\times)$

Case I_c. $L_i(R) \cong g_2(H^{i+1}(\hat{C}_\mathfrak{b}\zeta)^\times/\mathrm{Im}\, L_{i+1}^K(\hat{C}_\mathfrak{p}))$;

the image is generated by $1 + b(g_0^2 + g_0^{-2})$ *(i odd)*, g_0^2 *(i even)*.

Case II_0. $L_i(R) = 0$.

Case III_0. $L_{2k}(R) \cong \bigoplus \{4\mathbf{Z}$, *indexed by characters* χ *of* $\zeta\}$

$L_{2k+1}(R) = 0$.

We shall have no more to say about these cases in this section. There remain Cases II and III; subcases a, b and c of each.

We first consider ψ_2. Now $L_2(\hat{R}_2)$ was computed in (4.3); we have $\{\pm \operatorname{Ker}(w \mid \zeta)\}/\{w(g)g^2 \colon g \in \zeta\}$; to this, add $\mathbf{Z}/2$ in subcase a, but factor out $\{g_0^2\}$ in subcase bc; finally, we have a sum over the g_2 even primes $\mathfrak{p}$. As to $L_2(R \to \hat{R}_2)$, the summands L_0^* (or L_0^{**}) are irrelevant for ψ_2, so we have a sum of copies of L_2^* indexed by $\hat{\zeta}_0^+$ (Case II) or $\hat{\zeta}^+$ (Case III). Moreover, ψ_2 maps into this sum of L_2^* via its subgroup $\prod \{\pm 1\}$, the product over the g_2 even primes $\mathfrak{p}$. Now we can identify $\{\pm \operatorname{Ker}(w \mid \zeta)\}/\{w(g)g^2 \colon g \in \zeta\}$ with $\pm\zeta_0/\zeta_0^2$ (Case II) or $\pm\zeta/\zeta^2$ (Case III). For each $\mathfrak{p}$ we have, in subcase III_a, the natural inclusion $\pm\zeta/\zeta^2 \to \prod \{\pm 1$ over $\hat{\zeta}\}$; similarly (with ζ_0 for ζ) in subcase II_a. Noting that in subcases bc, $\hat{\zeta}_0^+$ (resp. $\hat{\zeta}^+$) is precisely the subgroup which kills g_0^2, we have corresponding injections in those cases also. It remains only to determine what happens to the summands $\mathbf{Z}/2$ in subcase a. For III_a, use the retraction of $\hat{C}_\mathfrak{p}\zeta$ on $\hat{C}_\mathfrak{p}$ to see that it suffices to consider the case ζ trivial. But then ψ_2 reduces to g_2 copies of $\{\pm 1\} \oplus \mathbf{Z}/2 \to \{\pm 1\}$, with the $\{\pm 1\}$ mapping isomorphically. We may thus suppose $\mathbf{Z}/2$ in the kernel. An analogous argument holds in subcase II_a, reducing to the case where ζ_0 is trivial ($\zeta = \zeta_1$).

We may deal similarly with ψ_0. By remarks above, this is zero in subcase a. For subcase bc, however, we may treat it exactly as above, but replacing g_0^2 by $-g_0^2$, and see that it is injective. To summarise,

LEMMA 4.5.2. *In Cases* II_a *and* III_a, ψ_0 *is trivial and the kernel of* ψ_2 *is* $g_2(\mathbf{Z}/2)$. *In Cases* II_{bc} *and* III_{bc}, *both* ψ_0 *and* ψ_2 *are injective.*

Observe that $\operatorname{Ker} \psi_i = \operatorname{Im}(L_i R \to L_i \hat{R}_2)$ is precisely that quotient of $L_i(R)$ which can be detected by reasonably simple invariants (apart from the signature).

Much as above, we see that $\psi_3 = 0$ in Cases II_a and III_a, and in the other cases it can be determined similarly to ψ_1; so we now discuss ψ_1. The image of ψ_1 (or ψ_3) is thus in all cases the same as that of $H^2(\hat{C}_\mathfrak{p}\zeta)^\times$, or equivalently, of the symmetric units of $\hat{C}_\mathfrak{p}\zeta$. As the only characters to be used, however, are those in $\hat{\zeta}_0^+$ (Case II) or $\hat{\zeta}^+$ (Case III), we may as well first map $\hat{C}_\mathfrak{p}\zeta$ to the quotient ring where $\zeta_0^2 = 1$, $T = i$ (Case II) or $\zeta^2 = 1$ (Case III) and moreover $g_0^2 = 1$. This is the group ring over $\hat{A}_\mathfrak{p} = \hat{C}_\mathfrak{p}[i]$ (Case II) or $\hat{C}_\mathfrak{p}$ (Case III) of an elementary abelian group ω of rank r', say. If the image ω_0 of ${}_2\zeta$ has rank s', the image of the symmetric elements of $\hat{C}_{\mathfrak{p}}\zeta$ are those elements of $\hat{A}_{\mathfrak{p}}\omega$ such that the coefficient of each element not in ω_0 is even. More precisely, the image of a symmetric unit is the product of a unit of $\hat{C}_{\mathfrak{p}}\omega_0$ and a unit of $\hat{A}_{\mathfrak{p}}\omega$ congruent to 1 mod 2.

The target is a sum of copies of L_1^* indexed by $\hat{\zeta}_0^+$ (Case II) or $\hat{\zeta}^+$ (Case III)—i.e., in either case, $\hat{\omega}$. Moreover, the maps to L_1^* factor through $\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2$, and the induced map $(\hat{A}_{\mathfrak{p}}\omega)^\times \rightarrow \prod \hat{A}_{\mathfrak{p}}^\times$, indexed by $\hat{\omega}$ is the natural one. It is thus determined by the results of (3.2). Following (3.2), we introduce new components $\hat{\xi}_I$, obtained by 'differencing' the χ's, to describe the image, which then becomes a direct product of subgroups of $\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2$. To determine the rank of ψ_1 we will need to compute the intersection (if any) of this image with that of

$$K^{(2)}/(K^\times)^2 \longrightarrow \prod \{\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2 \colon \mathfrak{p} \text{ bad}\} .$$

Let us conclude this section by tabulating some of these conclusions. Let ζ have rank r, and s summands of order 2. Then $r' = \operatorname{rank} \omega = r - 2 + \varepsilon_0 + \varepsilon_a$, and $s' = \operatorname{rank} \omega_0 = s - \varepsilon_b$, where $\varepsilon_a = 1$ (Case a), 0 (otherwise), and similarly for ε_b and $\varepsilon_0 = 0$ (Case II), 1 (Case III). Moreover the image subgroup in the I^{th} copy of $\hat{A}_{\mathfrak{p}}^\times$, where $I \subset \{1, \cdots, r'\}$ has i elements, is:

If $I \not\subset \{1, \cdots, s'\}$, all units of $\hat{A}_{\mathfrak{p}}^\times \equiv 1 \pmod{2^{i+1}}$.

If $I \subset \{1, \cdots, s'\}$, the product of this group and the group of units of $\hat{C}_{\mathfrak{p}} \equiv 1 \pmod{2^i}$.

This yields Im ψ_1; $\psi_3 = 0$ in Cases II_a, IIII_a; in Case II_{bc} ψ_3 is just like ψ_1; in Case III_{bc} it is like ψ_1, but with L_1^* replaced by L_1^{**}; i.e., infinite primes are bad.

4.6. Recall of algebraic number theory

It is clear from the preceding section that to push the calculation further will involve using properties of the field K. Here we will recall sufficient number theory to enable us to express our groups in terms of more-or-less standard number-theoretic invariants. Suitable general references are [7], [9].

Let K be any algebraic number field, A the integers in it. Let K have r_1 real and r_2 pairs of conjugate complex places; then $[K \colon \mathbf{Q}] = r_1 + 2r_2$. The Dirichlet unit theorem states that $A^\times$ is then the sum of a finite cyclic group (nontrivial, as it contains -1) and a free abelian group of rank $r_1 + r_2 - 1$. Thus $A^\times/(A^\times)^2$ has 2-rank $r_1 + r_2$.

If $\hat{K}_{\mathfrak{p}}$ is a local p-adic field, with integers $\hat{A}_{\mathfrak{p}}$ and residue class field $\mathbf{F}_{\mathfrak{p}}$, then $\operatorname{Ker}(\hat{A}_{\mathfrak{p}}^\times \rightarrow \mathbf{F}_{\mathfrak{p}}^\times)$ is a pro-p-group. Thus if p is odd, $\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2$ has the same

2-rank as $\mathbf{F}_\mathfrak{p}^\times/(\mathbf{F}_\mathfrak{p}^\times)^2$ viz. 1. For $p = 2$ we argue differently: the subgroup $\equiv 1$ (mod $2\mathfrak{p}$) is isomorphic by the 2-adic logarithm to $(2\mathfrak{p})^+$ (see e.g., [UGR]), so has rank equal to the degree $[\hat{K}_\mathfrak{p}: \hat{\mathbf{Q}}_\mathfrak{p}]$. Again, the torsion subgroup is cyclic of even order, so $\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2$ has 2-rank equal to $1 + [\hat{K}_\mathfrak{p}: \hat{\mathbf{Q}}_\mathfrak{p}]$.

To compute, for K global, the 2-rank of $\hat{A}_p^\times/(\hat{A}_p^\times)^2$, we must sum over prime divisors $\mathfrak{p}$ of p. For the cases of interest to us here, K is a Galois (in fact a cyclotomic) extension of $\mathbf{Q}$. Then we can write

$$[K: \mathbf{Q}] = e_p f_p g_p ,$$

where g_p is the number of prime divisors $\mathfrak{p}$ of p, and for each one, f_p is the degree of the residue field (i.e., $\mathbf{F}_\mathfrak{p}$ has order p^{f_p}) and e_p is the ramification index, i.e., the p-adic valuation $\hat{\mathbf{Z}}_p^\times \to \mathbf{Z}^+$ extends to a surjection $\hat{K}_\mathfrak{p}^\times \to (e_p^{-1}\mathbf{Z})^+$. Combining this with the preceding paragraph we see that

$$\text{2-rank } \hat{A}_p^\times/(\hat{A}_p^\times)^2 = g_p \quad (p \text{ odd}) ,$$
$$\text{2-rank } \hat{A}_2^\times/(\hat{A}_2^\times)^2 = g_2 + [K: \mathbf{Q}] .$$

For K a cyclotomic field, one can compute the numbers e_p, f_p, g_p using class field theory for $\mathbf{Q}$. First, consider the case when K is the field of n^{th} roots of unity, and let $p \nmid n$. Then $e_p = 1$, and f_p is the order of the class of p in $(\mathbf{Z}/n)^\times$, g_p the index of the subgroup it generates. If $n = p^r m$ with $p \nmid m$, then

$$[K: \mathbf{Q}] = \phi(n) = \phi(p^r)\phi(m), \text{ and } e_p = \phi(p^r) = p^r - p^{r-1} ,$$

f_p and g_p are the same as those with n replaced by m. Now let K be the subfield of the above fixed by the automorphism group $\lambda(\sigma) \subset \operatorname{Aut}(\mathbf{Z}/n) \cong (\mathbf{Z}/n)^\times$. The Galois group of K over $\mathbf{Q}$ is then Coker λ, and for $p \nmid n$, f_p and g_p are again the order and index of the class of p in this group. In general, g_p is the index in $(\mathbf{Z}/n)^\times$ of the subgroup generated by $\lambda(\sigma)$, p, and numbers $\equiv 1$ (mod m), f_p is the index in this of its subgroup generated by $\lambda(\sigma)$ and numbers $\equiv 1$ (mod m), and e_p is the index of $\lambda(\sigma)$ in the latter.

From units we proceed to class groups, but first we revise adelic notation. As well as the completion $\hat{K}$, we have the sum $K_\infty = K \otimes_\mathbf{Q} \mathbf{R}$ of completions at infinite primes; the adele ring $K_A = \hat{K} \oplus K_\infty$. Write K_∞^* for the subgroup of $K_\infty^\times$ of elements whose components in copies of $\mathbf{R}^\times$ are positive; thus $K_\infty^\times/K_\infty^* \cong \prod_{\mathfrak{p} \in \mathbf{R}} \{\pm 1\}$. Write A^*, K^* for the correspondingly defined subgroups of $A^\times$, $K^\times$. The approximation theorem shows that $K^\times/K^* \cong K_\infty^\times/K_\infty^*$.

Now the ideal group I of K can be defined as the quotient $\hat{\mathbf{K}}^\times/\hat{A}^\times$. The natural map $K^\times \to I$ has kernel $A^\times$; its cokernel Γ, called the class group, is finite. Γ is computable, but not very easily. We shall also have occasion to refer to the strict class group $\Gamma^* = \operatorname{Coker}(K^* \to I)$. It is easy to obtain an exact sequence

$$1 \longrightarrow A^* \longrightarrow A^\times \longrightarrow \prod_{\mathfrak{p} \in \mathbf{R}} \{\pm 1\} \longrightarrow \Gamma^* \longrightarrow \Gamma \longrightarrow 1$$

which summarises the available information (such as it is) on the relation of Γ^* to Γ. Indeed, another way to define Γ^* is to define the ∞-enriched ideal group $I^* = I \oplus \prod \{\pm 1\}$ and set $\Gamma^* = \operatorname{Coker}(K^\times \to I^*)$.

We also introduced in (1.6) and again above the subgroup $K^{(2)} = \operatorname{Ker}(K^\times \to I/I^2)$, and obtained a short exact sequence

$$1 \longrightarrow A^\times/(A^\times)^2 \longrightarrow K^{(2)}/(K^\times)^2 \longrightarrow {}_2\Gamma \longrightarrow 1 ,$$

where ${}_2\Gamma$ is the subgroup of elements of order 2 in Γ. Thus if Γ, hence also ${}_2\Gamma$, has 2-rank γ, the elementary 2-group $K^{(2)}/(K^\times)^2$ has 2-rank $r_1 + r_2 + \gamma$. We now turn to class field theory for quadratic extensions. The local Hilbert norm-residue symbol is defined by:

Let $x,\ y \in \hat{K}_\mathfrak{p}^\times$; then $(x, y)_\mathfrak{p} = 1$ or -1 according to whether $xa^2 + yb^2 = 1$ has or has not solutions $a,\ b \in \hat{K}_\mathfrak{p}$. Equivalently, $(x, y)_\mathfrak{p} = 1$ if and only if y is a norm from $\hat{K}_\mathfrak{p}[\sqrt{x}]$. Observe that this makes sense also for infinite completions. The symbol yields a symmetric bilinear pairing

$$\hat{K}_\mathfrak{p}^\times/(\hat{K}_\mathfrak{p}^\times)^2 \times \hat{K}_\mathfrak{p}^\times/(\hat{K}_\mathfrak{p}^\times)^2 \longrightarrow \{\pm 1\} ,$$

which can be shown to be nonsingular. It is not hard to compute it explicitly; for example, the annihilator of $\hat{A}_\mathfrak{p}^\times$ is for $\mathfrak{p}$ odd, $\hat{A}_\mathfrak{p}^\times$ itself and for $\mathfrak{p}$ even, the subgroup generated by $1 + 4\beta$, where $\beta \in \hat{A}_\mathfrak{p}$ has mod $\mathfrak{p}$ reduction not of the form $x + x^2$.

Taking the symbols together, as we may since almost all $\hat{A}_\mathfrak{p}^\times$ are self-orthogonal, yields a product pairing

$$K_A^\times/(K_A^\times)^2 \times K_A^\times/(K_A^\times)^2 \longrightarrow \{\pm 1\} .$$

By the Hilbert reciprocity theorem, $K^\times$ is self-orthogonal under this pairing —i.e., for $x,\ y \in K^\times$,

$$\prod_\mathfrak{p} (x, y)_\mathfrak{p} = 1 .$$

Thus if $C = K_A^\times/K^\times$ is the idele class group, we have an induced pairing

$$K^\times/(K^\times)^2 \times C/C^2 \longrightarrow \{\pm 1\} .$$

Now according to class field theory, this is a dual pairing, if we regard $K^\times$ as discrete and C as compact. There is thus a bijection between closed subgroups of index 2 in C (or epimorphisms $C \to \{\pm 1\}$) and nontrivial classes $x \in K^\times$ (mod squares), or equivalently with quadratic extensions $K[\sqrt{x}]$.

We illustrate this by considering unramified extensions. On one hand,

$K[\sqrt{x}]$ is unramified if and only if each $\hat{K}_\mathfrak{p}[\sqrt{x}]$ ($\mathfrak{p}$ finite) is either decomposed or inert, or equivalently, the subgroup of norms is either $\hat{K}_\mathfrak{p}^\times$ or $\hat{A}_\mathfrak{p}^\times \cdot (\hat{K}_\mathfrak{p}^\times)^2$. By the theorem, this is equivalent to saying that the corresponding homomorphism $\hat{K}_A^\times \to C \to \{\pm 1\}$ contains $\hat{A}^\times$ in its kernel. Thus it factors through

$$\hat{K}^\times \cdot K_\infty^\times / K^\times \cdot \hat{A}^\times \cdot K_\infty^* = \hat{K}^\times/(K^\times \cap K_\infty^*)\hat{A}^\times = I/K^* = \Gamma^* .$$

This shows that unramified extensions correspond bijectively to (nontrivial elements of) $\operatorname{Hom}(\Gamma^*, \{\pm 1\})$. A slight modification shows that strictly unramified extensions, where we exclude ramification at infinity (i.e., **C**/**R**) correspond to $\operatorname{Hom}(\Gamma, \{\pm 1\})$.

On the other hand, for $\mathfrak{p}$ odd, $\hat{K}_\mathfrak{p}[\sqrt{x}]$ is unramified if and only if the $\mathfrak{p}$-adic value of x is even. For $\mathfrak{p}$ even, this condition is necessary, but not sufficient; there are many square classes of units, but only one unramified extension, corresponding to $(1 + 4\beta)$. Thus $x \in K^{(2)}$, and we have a map

$$K^{(2)}/(K^\times)^2 \xrightarrow{\Phi} \hat{A}_2^\times/(\hat{A}_2^\times)^2 .$$

There is a subgroup G of rank g_2 on the right corresponding to unramified extensions. We conclude

$$\Phi^{-1}(G) \cong \operatorname{Hom}(\Gamma^*, \{\pm 1\}) .$$

For strictly unramifed extensions, we augment to

$$K^{(2)}/(K^\times)^2 \xrightarrow{\Phi'} \hat{A}_2^\times/(A_2^\times)^2 \cdot G \oplus \textstyle\prod_{\mathfrak{p} \in \mathbf{R}} \{\pm 1\}$$

with kernel $\operatorname{Hom}(\Gamma, \pm 1)$ and hence rank $r_1 + r_2$, since rank $\left(K^{(2)}/(K^\times)^2\right) = r_1 + r_2 + \gamma$. As moreover, rank $\left(\hat{A}_2^\times/(\hat{A}_2^\times)^2\right) = r_1 + 2r_2 + g_2$, we see that Coker Φ' also has rank $(r_1 + r_2)$. Returning to Φ, we observe that its kernel corresponds to quadratic extensions which decompose at even primes. By the duality above, this corresponds to homomorphisms $f: \Gamma^* \to \{\pm 1\}$ which annihilate (the classes of) even primes.

We are now ready to consider the homomorphism

$$\Phi: K^{(2)}/(K^\times)^2 \longrightarrow \prod \{\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2 : \mathfrak{p} \text{ bad}\}$$

arising in (4.4). Here, all even primes are bad; infinite primes are only so in one case. However, we see at once from the above considerations that $\operatorname{Ker} \Phi$ is isomorphic to the set of homomorphisms $\Gamma^* \to \{\pm 1\}$ which annihilate the classes of all bad primes. Thus

(4.6.1) *The rank of* $\operatorname{Ker} \Phi$ *equals that of the quotient of* $\Gamma^*/(\Gamma^*)^2$ (*or, if infinite primes are bad,* Γ/Γ^2) *by the classes of the finite bad primes.*

This can also be interpreted as the 2-rank of the (strict) class group of the Dedekind domain obtained by localising A at the bad primes, viz. $A[(2n)^{-1}]$.

We can also interpret it as the quotient of $\Gamma^*/(\Gamma^*)^2$ by the classes of all the bad primes.

For (4.5) we must be more careful. We begin by analysing more closely the structure of $\hat{A}_2^\times$ in the relevant cases.

First, corresponding to Case III, we have a subfield $\mathbf{Q} \otimes C$ of the field of n^{th} roots of unity, with n odd. Thus 2 is unramified in C, and in each $\hat{C}_\mathfrak{p}$, $\mathfrak{p}$ even. We have $[\mathbf{Q} \otimes \hat{C}_\mathfrak{p} \colon \hat{\mathbf{Q}}_2] = f_2$. Write $U^i(\hat{C}_\mathfrak{p})$ for the group of units of $\hat{C}_\mathfrak{p} \equiv 1 \pmod{\mathfrak{p}^i}$. Note that $\mathfrak{p} = \langle 2 \rangle$. Then

$$U^0(\hat{C}_\mathfrak{p})/U^1(\hat{C}_\mathfrak{p}) \cong \mathbf{F}_\mathfrak{p}^\times, \text{ of odd order.}$$
$$U^1(\hat{C}_\mathfrak{p})/U^{i+1}(\hat{C}_\mathfrak{p}) \cong \mathbf{F}_\mathfrak{p}^+ \qquad (i \geqq 1),$$

and the 2-adic logarithm gives an isomorphism of $U^2(\hat{C}_\mathfrak{p})$ onto $(4\hat{C}_\mathfrak{p})^+$. Now $(1 + 2\alpha)^2 = 1 + 4(\alpha + \alpha^2)$, so the composite

$$\mathbf{F}_\mathfrak{p}^+ \cong U^1(\hat{C}_\mathfrak{p})/U^2(\hat{C}_\mathfrak{p}) \xrightarrow{\text{square}} U^2(\hat{C}_\mathfrak{p})/U^3(\hat{C}_\mathfrak{p}) \cong \mathbf{F}_\mathfrak{p}^+$$

is $\mathfrak{P}$ ($\mathfrak{P}x = x + x^2$). This has kernel $\{0, 1\}$ corresponding to $\pm 1 \in \hat{C}_\mathfrak{p}^\times$, and cokernel $\{0, \beta\}$. A set of generators for $(\hat{C}_\mathfrak{p}^\times)/(\hat{C}_\mathfrak{p}^\times)^2$ is thus $\{1 + 2\alpha\colon \alpha$ represents a base of $\mathbf{F}_\mathfrak{p}^+$, $1 + 4\beta\}$; the group has rank $f_2 + 1$, as stated earlier. $U^1(\hat{C}_\mathfrak{p})$ maps onto the whole group, $U^2(\hat{C}_\mathfrak{p})$ onto the subgroup $(1 + 4\beta)$, $U^3(\hat{C}_\mathfrak{p})$ is killed.

Now, corresponding to Case II, we must consider also $A = C[i]$. The even primes of C (but no odd ones) ramify further in A, i.e., $e_2 = 2$ and $\langle 2 \rangle = \mathfrak{p}^2$. But f_2 is the same as before; we have the same $\mathbf{F}_\mathfrak{p}$; g_2 also is the same. Analogous to the above, is the following:

$$U^0(\hat{A}_\mathfrak{p})/U^1(\hat{A}_\mathfrak{p}) \cong \mathbf{F}_\mathfrak{p}^\times$$
$$U^i(\hat{A}_\mathfrak{p})/U^{i+1}(\hat{A}_\mathfrak{p}) \cong \mathbf{F}_\mathfrak{p}^+ \qquad \text{for } i \geqq 1\,,$$

and

$$U^3(\hat{A}_\mathfrak{p}) \cong (\mathfrak{p}^3\hat{A}_\mathfrak{p})^+ \,.$$

Here, squaring gives an isomorphism of U^1/U^2 on U^2/U^3, but induces a map $U^2/U^3 \to U^4/U^5$ described by $\mathfrak{P}$ (as above). Thus representatives of square classes can be chosen as representatives of U^1/U^2, of U^3/U^4, and $(1 + 4\beta)$. The group $\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2$ has rank $1 + 2f_2$; the image of the units $(U_2) \equiv 1 \pmod 2$ has rank $1 + f_2$, and the image of those $\equiv 1 \pmod 4$ has rank 1.

Consider finally the map $\hat{C}_\mathfrak{p}^\times/(\hat{C}_\mathfrak{p}^\times)^2 \to \hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2$. As the element $1 + 4\beta$ is the same in both cases, it maps injectively. In general, as $\hat{A}_\mathfrak{p} = \hat{C}_\mathfrak{p}[i]$, the kernel of the map is the subgroup generated by -1; so the image has rank f_2; clearly it is contained in the image of $U^1(\hat{C}_\mathfrak{p})$, hence in $U^2(\hat{A}_\mathfrak{p})$, and hence in the image of $U^3(\hat{A}_\mathfrak{p})$.

Now by (4.5) we can analyse the calculation of ψ_1 into components.

For each component, the target is the cokernel of

$$\Phi\colon K^{(2)}/(K^\times)^2 \to \prod\{\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2\colon \mathfrak{p} \text{ bad}\}$$

and the image is the image in this cokernel of one of the above described subgroups of $\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2$, $\mathfrak{p}$ even; more precisely, the product G over all even $\mathfrak{p}$. Observe that all groups now in question are elementary 2-groups.

The 2-rank of the image of ψ_1 at this component equals $l(\Phi^{-1}(G)) - l(\Phi^{-1}(0))$. We considered $l(\Phi^{-1}(0))$ in (4.6.1); we must now consider $l(\Phi^{-1}(G))$, case by case. First, we need some notation. Write $r(V)$ for the 2-rank of the finite abelian group V. Write $\Gamma(R)$ for the ideal class group of the Dedekind domain R; Γ^* for the strict group. Then by (4.6.1),

$$\begin{aligned} l(\Phi^{-1}(0)) &= r(\Gamma^*(A[(2n)^{-1}])) \text{ resp. } r(\Gamma(A[(2n)^{-1}])) \\ &= \gamma^*(A, 2n) \qquad\quad \text{resp. } \gamma(A, 2n) \end{aligned}$$

say, if infinite primes are good (resp. bad). This will not suffice below, however; we need to enrich the ideal group at even primes as well as at infinite ones. Define $I^{(2)} = \hat{K}^\times/\prod\{\hat{A}_\mathfrak{p}^\times\colon \mathfrak{p} \text{ odd}\}$, and

$$I^{*(2)} = K_A^\times/\prod\{\hat{A}_\mathfrak{p}\colon \mathfrak{p} \text{ odd}\} \times K_\infty^* \cong I^{(2)} \oplus \prod\{\pm 1\colon \mathfrak{p} \text{ real}\}\ ;$$

let $\Gamma^{(2)}$, $\Gamma^{*(2)}$ be the corresponding class groups obtained by factoring out $K^\times$. We have natural exact sequences

$$1 \longrightarrow \hat{A}_2^\times \longrightarrow I^{(2)} \longrightarrow I \longrightarrow 1\ , \qquad 1 \longrightarrow \hat{A}_2^\times \longrightarrow I^{*(2)} \longrightarrow I^* \longrightarrow 1\ .$$

We shall wish to factor $\Gamma^{(2)}$ or $\Gamma^{*(2)}$ by the prime divisors of n, giving $\Gamma^{(2)}(A, n)$ say; and further by a specified subgroup U of $\hat{A}_2^\times$, giving a group $\Gamma^{(2)}(A, n, U)$.

We proceed to cases.

Case III. $K = \mathbf{Q} \otimes C$ is totally real. G may be the whole group $\hat{A}_3^\times$ or the subgroup generated by the elements $1 + 4\beta$; infinite primes may be good or bad. The $(1 + 4\beta)$ case simply corresponds to having extensions unramified at 2, so here $l(\Phi^{-1}(G)) = \gamma^*(C, n)$ resp. $\gamma(C, n)$. In the other case, we have extensions $\mathbf{Q} \otimes C[\sqrt{x}]$ ramified at even primes, but with $v_\mathfrak{p}(x)$ even, or equivalently, $(x, 1 + 4\beta)_\mathfrak{p} = 1$. These correspond to homomorphisms $\Gamma^{*(2)} \to \{\pm 1\}$ which contain prime divisors of n in the kernel; also $U^2(\hat{C}_2)$ (i.e., units $\equiv 1 \pmod 4$). Thus

$$l(\Phi^{-1}(G)) = \gamma^{*(2)}(C, n, U^2) \text{ resp. } \gamma^{(2)}(C, n, U^2)\ .$$

Case II. $K = \mathbf{Q}[i] \otimes C$. Here infinite primes are all complex, so the question of their 'goodness' does not arise. G is the (product of g_2 copies of the) image of $U^2(\hat{A}_\mathfrak{p}^\times)$, of rank $1 + f_2$; of the image of $\hat{C}_\mathfrak{p}^\times$, of rank f_2; or of

$1 + 4\beta$, of rank 1. As above, the latter case is the simplest, and we have simply $l(\Phi^{-1}(G)) = \gamma(A, n)$.

In the other cases, we have those homomorphisms $\Gamma^{(2)} \longrightarrow \{\pm 1\}$ which kill the subgroup of $\hat{A}_2^\times$ dual (by the norm residue symbol) to G. Now $\hat{K}_\mathfrak{p}^\times/(\hat{K}_\mathfrak{p}^\times)^2$ has rank $2 + 2f_2$ and the symbol is nonsingular; $\hat{A}_\mathfrak{p}^\times/(\hat{A}_\mathfrak{p}^\times)^2$ (of rank $1 + 2f_2$) is dual to $1 + 4\beta$. According to [20, p. 237], the image of $U^2(\hat{A}_\mathfrak{p})$, of rank $1 + f_2$, is self-orthogonal and hence self-dual. And by class field theory, the dual of Im i_*: $\hat{C}_\mathfrak{p}^\times \longrightarrow \hat{A}_\mathfrak{p}^\times$ is Ker N: $\hat{A}_\mathfrak{p}^\times \longrightarrow \hat{C}_\mathfrak{p}^\times$ containing as well as $U^2(\hat{A}_\mathfrak{p}^\times)$, the class of i. Write T for the torsion subgroup of $\hat{A}_2^\times$. Then in the first two cases,

$$l(\Phi^{-1}(S)) = \gamma^{(2)}(A, n, U^2) \text{ resp. } \gamma^{(2)}(A, n, T\cdot U^2) .$$

4.7. Final calculations

Let us summarise the results of this chapter so far. In (4.1), we showed that the L-groups in question split as a sum, with a contribution from each divisor of $|\rho|$. In (4.2) we classified the situations arising into three cases, each divided into four subcases. Complete results for six of the subcases were listed in (4.5.1); there remain Cases II and III, subcases a, b and c of each. Now in Case II we know by Taylor's lemma that the L groups have exponent 2, so to complete the calculation it will suffice to compute the ranks. In Case III we succeeded in determining the extensions involved in (4.3), and many of those in (4.4)—if, as I suspect, L_2^* has exponent 2, this would complete the results in that case—but I see no method of determining the final extensions needed to describe $L_i(R)$. Indeed, this difficulty occurred already in (1.6). Thus here too we will have to content ourselves with computing the order of the torsion subgroup, and describing the values taken by the signature. To complete the picture, we will compute this order in all cases. Subcase 0 may, however, be omitted since $L_i(R)$ is then free.

All the torsion subgroups in question are finite 2-groups; most, but not all, have exponent 2. To simplify descriptions, we write $l(G) = a$, if the torsion subgroup of G has order 2^a. First, we compute the ranks involved in (4.3). Denote (as usual)

$$r = l(\zeta/\zeta^2) , \qquad s = l(\text{Im}\,({}_2\zeta \longrightarrow \zeta/\zeta^2)) .$$

Since 2 does not ramify in C, we have $\deg(\mathbf{Q} \otimes C/\mathbf{Q}) = f_2 g_2$, where f_2 is the degree of the residue field extension $\mathbf{F} = \bar{C}_\mathfrak{p}$ over $\mathbf{F}_2$, and g_2 is the number of even primes $\mathfrak{p}$ of C. We shall also use the parameters introduced in (4.5): $\varepsilon_a = 1$ (Case a), 0 (Case bc) and ε_b, ε_c similarly—as we exclude subcase 0, we have in fact $\varepsilon_a + \varepsilon_b + \varepsilon_c = 1$. Define also $\varepsilon_0 = 0$ (Case II), 1 (Case III).

First recall (3.2), with A replaced by $\hat{C}_{\mathfrak{p}}$ and π by ζ.

$$l(K) = l(\mathbf{F}({}_2\zeta)) - l(\mathbf{F}) = f_2(2^r - 1) \ .$$

By (3.2.1),

$$l(H^i(\mathbf{F}\zeta)^\times) = f_2(2^r - 1) + r - s \ .$$

In Case I,

$$l(H^i(\hat{C}_{\mathfrak{p}}\zeta)^\times) = l(H^i(\mathbf{F}\ \mathrm{Ker}\ w)^\times) = f_2(2^{r-1} - 1) + r - s \ .$$

Otherwise, we have

$$l(H^0(\hat{C}_{\mathfrak{p}}\zeta)^\times) = \varepsilon_0 + r\ , \qquad l(H^1(\hat{C}_{\mathfrak{p}}\zeta)^\times) = \varepsilon_0 + r + 2^r f_2 \ .$$

Now by (4.3) we have in Case I,

$$l(L_i(R)) = l(L_i(\hat{R}_2)) = g_2(f_2(2^{r-1} - 1) + r - s + \varepsilon_{\mathrm{a}} - \varepsilon_{\mathrm{c}}) \ .$$

Otherwise, we see by checking each subcase that

$$l(L_{2k+1}(\hat{R}_2)) = g(2^r f_2 + r - 1 + \varepsilon_0 + \varepsilon_{\mathrm{a}})\ ,$$
$$l(L_{2k}(\hat{R}_2)) = g_2(r - 1 + \varepsilon_0 + \varepsilon_{\mathrm{a}}(1 - (-1)^k)) \ .$$

From now on, we can exclude Case I as well as subcase 0. To express the results of (4.4), let us write $l_i^* = l(L_i^*)$, $l_i^{**} = l(L_i^{**})$. Recall from (4.5) that $r' = \mathrm{rank}\ \omega = r - 2 + \varepsilon_0 + \varepsilon_{\mathrm{a}}$. Again one can check that the formula

$$l(L_i(R \longrightarrow \hat{R}_2)) = 2^{r-1+\varepsilon_0} l_i^* + (1 - \varepsilon_{\mathrm{a}}) 2^{r-2+\varepsilon_0} (l_{i+2}^{**} - l_i^*)$$

covers all cases, where of course $l_i^{**} = l_i^*$ in Case II. Here, we have $0 = l_3^* = l_3^{**}$, $l_0^* = 1 - \varepsilon_0$, $l_0^{**} = 1$ (or more properly $l_0^{**} = 1 - \varepsilon_0\varepsilon_{\mathrm{a}}$, which gives the same result). The rest are to be computed from the exact sequence

$$\{\pm 1\} \longrightarrow \prod \{\pm 1 \colon \mathfrak{p} \text{ bad, finite}\} \longrightarrow L_2^* \longrightarrow K^{(2)}/(K^\times)^2$$
$$\xrightarrow{\Phi} \{\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2 \colon \mathfrak{p} \text{ bad}\} \longrightarrow L_1^* \longrightarrow \Gamma/\Gamma^2 \oplus \Sigma \longrightarrow 0\ ,$$

where the bad $\mathfrak{p}$ are the divisors of $2n$ and, for L^{**}, the infinite $\mathfrak{p}$.

Now by results in (4.6), and the notation introduced there, the ranks of the successive terms in the sequence are:

$$1,\ \Sigma\{g_p \colon p \mid 2n\},\ l_2^*,\ r_1 + r_2 + \gamma,\ \Sigma\{g_p \colon p \mid 2n\} + [K \colon \mathbf{Q}],\ l_1^*,\ \gamma;$$

at least when infinite primes are good. Let us write $G = \Sigma\{g_p \colon p \mid 2n\}$, and $m = f_2 g_2$. In Case II, $r_1 = 0$, $r_2 = m$ and $[K \colon \mathbf{Q}] = 2m$; infinite primes have type C, so cannot be bad. In Case III, $r_1 = m$, $r_2 = 0$ and $[K \colon \mathbf{Q}] = m$; in the case of l^{**}, bad primes contribute m to the fifth term. We conclude in both cases

Case III:
$$\begin{aligned} l_1^{**} &= m + G + \text{rk Ker } \Phi \ , \\ l_2^{**} &= G - 1 + \text{rk Ker } \Phi \ . \\ l_1^{*} &= G + \text{rk Ker } \Phi \ , \\ l_2^{*} &= G - 1 + \text{rk Ker } \Phi \ . \end{aligned}$$

As m and G are easily computable, the only awkward term here is rk Ker Φ, which must be calculated either directly or by using (4.6.1).

This completes our interpretation of (4.4); we now turn to (4.5). First we claim that the torsion-free quotients of L_iR and of $L_{i+1}(R \to \hat{R}_2)$ are the same, so that we may consider the torsion part independently too. To see this, it is enough to note that the calculation of the 'signature group' in (4.4) would be unaltered if we included $L_i\hat{R}_2$ there. It suffices to look at a single place of type O. Here we need to know that $L_0^S(\hat{A}_2) \to CL_0^S(\hat{K}_2)$ vanishes—which is true by (1.6)—and that $L_2^S(\hat{A}) \to CL_2^S(\hat{K}_2)$ does not contain the image of $L_2^S(\mathbf{H})$ which is clear since this group is the product over places $\mathfrak{p}$ of $\{\pm 1\}$, modulo the diagonal, but all places $\mathfrak{p} \mid n$ are omitted. (The case $n = 1$, where this argument fails is precisely the case already studied fully in Chapter 3; anyway there are then no summands of type **H**.)

Now the image of ψ_i is an elementary 2-group. Let us write $\psi_i = l(\text{Im } \psi_i)$. Then we have
$$l(L_i(R)) = l(L_{i+1}(R \longrightarrow \hat{R}_2)) + l(L_i(\hat{R}_2)) - \psi_i - \psi_{i+1} \ ,$$
so the calculation of ψ_i will complete our task.

When i is even, the result is given by (4.5.2): it does not seem worth while seeking a complicated formula to express so simple a conclusion.

For i odd, recall the notation at the end of (4.6): e.g., $\gamma(C, n)$ is the 2-rank of the quotient of the class group of C by the prime divisors of n. Write also, as before,
$$r' = r - 2 + \varepsilon_0 + \varepsilon_a \ , \qquad s' = s - \varepsilon_b \ .$$
Then in Case II:
$$\psi_1 = \gamma^{(2)}(A, n, U^2) + s'\gamma^{(2)}(A, n, T\cdot U^2) + \left(r' - s' + \binom{s'}{2}\right)\gamma(A, n) - \left(1 + r' + \binom{s'}{2}\right)\gamma(A, 2n) \ ;$$
in Case II_a, $\psi_3 = 0$ and in Case II_{bc}, $\psi_3 = \psi_1$. In Case III similarly,
$$\psi_1 = (1 + s')\gamma^{*(2)}(C, n, U^2) + \left(r' - s' + \binom{s'}{2}\right)\gamma^*(C, n) - \left(1 + r' + \binom{s'}{2}\right)\gamma^*(C, 2n) \ .$$

In Case $\mathrm{III_a}$, $\psi_3 = 0$ and in Case $\mathrm{III_{bc}}$, ψ_3 is like ψ_1 but with the asterisks omitted.

These numbers are not easy to compute; let us finally see which (if any) of our groups are. In subcases b and c, all ψ_{odd} are complicated, hence all $l(L_i)$. In subcase a, only ψ_1 is awkward, so we may expect simpler formulae for L_2 and L_3. We see in fact that

$$L_2(R) \cong g_2(\mathbf{Z}/2) \ \text{ in both } \mathrm{II_a} \text{ and } \mathrm{III_a}\ ;$$

$L_3(R)$ is an elementary abelian 2-group of rank

$$g_2(2^r f_2 + r) + 2^{r-1} \ \text{ in case } \mathrm{II_a}\ , \quad g_2(2^r f_2 + r + 1) \ \text{ in case } \mathrm{III_a}\ .$$

Of particular interest, since it is related to explicit invariants, is to know the image of $L_i^X(R) \to L_i^K(\bar{R}_2)$. Now $L_i^K(\bar{R}_2) = L_i^K(\hat{R}_2)$, and the image of $L_i^X(\hat{R}_2)$ is given by (4.3.2) viz., surjective in Cases $\mathrm{I_a}$ (all i), $\mathrm{I_b}$ (i even), $\mathrm{II_a}$, and $\mathrm{III_a}$ ($i = 2$ or 3), and zero otherwise. Now in Case I, $L_i^X(R) = L_i^X(\hat{R}_2)$. In Cases $\mathrm{II_a}$ and $\mathrm{III_a}$, by (4.5.2) $L_2^X(R)$ maps isomorphically to $L_2^K(\bar{R}_2)$, and $L_3^X(R)$ maps onto $L_3^X(\hat{R}_2)$ since the relative group vanishes. Thus

$$\operatorname{Im}\big(L_i^X(R) \longrightarrow L_i^K(\bar{R}_2)\big) = \operatorname{Im}\big(L_i^X(\hat{R}_2) \longrightarrow L_i^K(\bar{R}_2)\big)$$

in all cases. It would be interesting to know whether this is a general result.

If it were not for Case $\mathrm{I_b}$, we could summarise the above list of cases by noting that the following are equivalent:

(i) $L_{2k}^X(R) \to L_{2k}^K(\bar{R}_2)$ is nonzero, or surjective.

(ii) $L_{2k+1}^X(R) \to L_{2k+1}^K(\bar{R}_2)$ is nonzero, or surjective.

(iii) There exists $g_0 \in \sigma$ with $\lambda(g_0) = -1$, $g_0^2 = 1$, and $w(g_0) = -(-1)^k$.

However, only (ii) and (iii) are equivalent in general. Observe that this statement is corrected to be independent of the conventions introduced in (4.2).

5. Further calculations

5.1. Discussion of the general case

We first take up some questions arising in the computation of $L_i(R \to \hat{R}_2)$. In order to determine the extension arising, there is no problem if

$$\gamma_i\colon \textstyle\prod_{p\ \mathrm{odd}} L_i(\hat{R}_p) \oplus L_i(T) \longrightarrow CL_i(S)$$

has torsion-free (hence free) kernel, and it may be conjectured that this holds in general. The result of (2.3), that $L_i(\hat{R}_p) \to L_i(\hat{S}_p)$ is injective, lends support to this conjecture. When we turn to (4.4) to check whether this holds, we see that it does for γ_0, γ_2, and γ_3, but $\operatorname{Ker}\gamma_1$ is the kernel of

$$K^{(2)}/(K^\times)^2 \to \textstyle\prod \{\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2 \colon \mathfrak{p} \text{ bad}\}\ .$$

By (4.6.1), this is zero if and only if the bad primes generate $\Gamma^*/(\Gamma^*)^2$. However, this is not generally the case for cyclotomic fields. (Note that the bad primes are those ramified in the field of n^{th} roots of unity: they include those ramified in K, but we need have no more.) Indeed, we can still find counterexamples if $L_i(T)$ is omitted from the sum (so infinite primes count as bad). Although the method can be generalised (following (1.6); see also below), we must clearly find a better objective for it.

Secondly, we may try to give a computation of γ_i in the general case. It suffices, as noted in (2.4), to consider the case where $\pi = \rho\sigma$ is 2-hyper-elementary. Then the splitting Theorem (4.1.2) applies: we can replace $\mathbf{Z}\rho$ by a cyclotomic domain of d^{th} roots of unity, and then the contribution from $\hat{R}_p$ is taken as 0 if $p \mid d$. If, on the other hand, $p \nmid 2d$ then $\hat{R}_p$ is a maximal order in an unramified algebra, and the calculation of $L_i(\hat{R}_p) \to L_i(\hat{S}_p)$ is given by (1.5) and (1.6). Explicitly, for type U or GL both vanish and for type O we have the natural inclusion

i	0	1	2	3
$L_i(\hat{R}_p)$	0	$\hat{R}_p^\times/(\hat{R}_p^\times)^2$	$\{\pm 1\}$	0
$L_i(\hat{S}_p)$	$\mathbf{Z}/2$	$\hat{S}_p^\times/(\hat{S}_p^\times)^2$	$\{\pm 1\}$	0 .

Incidentally, this gives an alternative proof of (2.3). The further calculation of $L_i(\hat{S}) \oplus L_i(T) \to CL_i(S)$ now follows by using rational representation theory to describe the S which occur.

Our splittings were not entirely natural, so it would be useful to have an alternative technique to the above. Write S as a sum of simple algebras; let S' denote the sum of those of type O, and write R' for the projection of R on S' (*not* $R \cap S'$).

Conjecture 5.1.1. *The image of* $L_i(\hat{R}_p) \to L_i(\hat{S}_p)$ *is the image of* $K_1(\hat{R}'_p)$ *in* $L_1(\hat{S}'_p) \cong Z(\hat{S}'_p)^\times/\big(Z(\hat{S}'_p)^\times\big)^2$ $(i = 1)$ *and of the subgroup of exponent* 2 *in* $K_1(\hat{R}'_p)$ *if* $i = 2$.

I can only prove the conjecture under the hypothesis that $\hat{R}'_p$ has good reduction in the sense of [III] and [Rat].

For our next topic, recall that we have already seen in [V] that for any $\mathbf{Z}$-order R, $T = R \otimes \mathbf{R}$, the kernel and cokernel of $L_i^X(R) \to L_i^X(T)$ are finite 2-groups. We now further study $L_i^X(R)$ modulo torsion: we may suppose $i = 2k$, as $L_i^X(T)$ is finite for i odd.

In view of the main exact sequence, the image of $L_{2k}^X(R) \to L_{2k}^S(T)$ (which carries the torsion free quotient) is the preimage of $\operatorname{Im}\big(L_{2k}^X(\hat{R}) \to CL_{2k}^S(S)\big)$. Now S splits into simple summands and T splits correspondingly. For a summand S_1 of S of type GL,

$$0 = L^S_{2k}(T_1) = CL^S_{2k}(S_1) ;$$

there is thus no contribution.

For a summand S_2 of type U, we again have $CL^S_{2k}(S_2) = 0$, but if T has any corresponding summands of type U, we have a signature taking values in $4\mathbf{Z}$ for each. These summands must be in the image of $L^X_{2k}(R)$.

It remains to consider summands S_3 of type O. Then

$$CL^S_0(S_3) \cong \mathbf{Z}/2. \text{ Also, } L^S_0(\mathbf{R}) \cong 4\mathbf{Z}, \quad L^S_0(\mathbf{C}) = L^S_0(\mathbf{H}) = 0 .$$

Each component $4\mathbf{Z}$ (if there are any) maps onto $\mathbf{Z}/2$, as this is the sum of the local Hasse invariants. To complete this case, we need

PROPOSITION 5.1.2. *For any R as above, with S_3 a simple summand of type O of S, the composite*

$$L^X_0(\hat{R}) \longrightarrow L^S_0(\hat{S}) \rightarrow CL^S_0(S) \longrightarrow CL^S_0(S_3) \cong \mathbf{Z}/2$$

vanishes, except perhaps if S_3 is non-split at some even prime.

Proof. Let R_3 be the projection of R on S_3. This is an order, and the map factors through $L^X_0(\hat{R}_3)$. It thus suffices to prove the result with R replaced by R_3, i.e., we can suppose $S = S_3$ simple. Similarly, we may suppose R a maximal order in S.

We must show that for each prime $\mathfrak{p}$ of the centre, K of S, the composite $L^X_0(\hat{R}_\mathfrak{p}) \rightarrow L^S_0(\hat{S}_\mathfrak{p}) \rightarrow CL^S_0(S)$ is zero. If $\hat{S}_\mathfrak{p}$ is a matrix ring over $\hat{K}_\mathfrak{p}$ we can reduce (as in [II]) by Morita theory to the case $\hat{S}_\mathfrak{p} = \hat{K}_\mathfrak{p}$, $\hat{R}_\mathfrak{p}$ the integers in it. The result then follows since the Clifford algebra of the form over $\hat{R}_\mathfrak{p}$ is an Azumaya algebra over $\hat{R}_\mathfrak{p}$, which necessarily splits, so the Hasse invariant is trivial. Otherwise we reduce similarly to the case, $\hat{S}_\mathfrak{p}$ the quaternion division ring with centre $\hat{K}_\mathfrak{p}$, $\hat{R}_\mathfrak{p}$ the maximal order in it, with centre the integers $\hat{A}_\mathfrak{p}$ of $\hat{K}_\mathfrak{p}$. But then for $\mathfrak{p}$ odd,

$$L^S_0(\hat{R}_\mathfrak{p}) = L^S_0(\bar{R}_\mathfrak{p}) = 0 ,$$

since $\bar{R}_\mathfrak{p}$ is a finite field with involution of type U.

For $\mathfrak{p}$ even, the same argument shows $L^K_0(\hat{R}_\mathfrak{p}) = 0$. However, consider the commutative diagram

$$\begin{array}{ccccccc} L^K_1(\hat{R}_\mathfrak{p}) & \longrightarrow & H^1K_1(\hat{R}_\mathfrak{p}) & \longrightarrow & L^S_0(\hat{R}_\mathfrak{p}) & \longrightarrow & L^K_0(\hat{R}_\mathfrak{p}) = 0 \\ & & \downarrow & & \downarrow & & \\ 0 = L^S_1(\hat{S}_\mathfrak{p}) & \longrightarrow & H^1K_1(\hat{S}_\mathfrak{p}) & \longrightarrow & L^S_0(\hat{S}_\mathfrak{p}) & \longrightarrow & L^K_0(\hat{S}_\mathfrak{p}) = 0 . \end{array}$$

Since by [27, 5.7], $K_1(\hat{R}_\mathfrak{p}) \cong \hat{A}^\times_\mathfrak{p}$, $K_1(\hat{S}_\mathfrak{p}) \cong \hat{K}^\times_\mathfrak{p}$ and $H^1 = \{\pm 1\}$ in each case, we have here an isomorphism $L^S_0(\hat{R}_\mathfrak{p}) \rightarrow L^S_0(\hat{S}_\mathfrak{p})$.

Now consider $CL_2^S(S)$ (S still being supposed simple, of type O, with centre K). We know that this is the multiplicative group of idele classes of order 2, or equivalently, ${}_2K_A^\times/\{\pm 1\}$. For each real place $\mathfrak{p}$ of S where T ramifies, we have a summand of $L_2^S(T)$ isomorphic to $L_2^S(\mathbf{H}) \cong 2\mathbf{Z}$. The generator of this maps onto ${}_2K_\mathfrak{p}^\times = \{\pm 1\}$ by [IV, 5.4]. Now if, for some other prime $\mathfrak{p}_0$ of S, the completion $\hat{R}_{\mathfrak{p}_0}$ of [the projection on S of] R is such that

$$L_2^X(\hat{R}_{\mathfrak{p}_0}) \longrightarrow L_2^X(\hat{S}_{\mathfrak{p}_0}) \cong \{\pm 1\} = {}_2\hat{K}_{\mathfrak{p}_0}^\times$$

is *not* surjective, we can identify $CL_2^S(S)$ with $\prod\{\pm 1\colon \mathfrak{p} \neq \mathfrak{p}_0\}$ and deduce that the image of $L_2^X(R)$ has as torsion free part a sum of components $4\mathbf{Z}$; otherwise, the signatures need not be divisible by 4; in the contrary case, each will be $\equiv 2 \pmod 4$. We conjecture that in fact the signatures are always divisible by 4.

We now consider group rings. As no copies of $\mathbf{H}$ can occur in p-hyperelementary groups with p odd, let $\pi = \rho\sigma$ be 2-hyperelementary, with $|\rho| = n$ as usual. By (4.1), the L-theory splits corresponding to divisors d of n, and in computing the d-summand, we replace $L^X(\hat{R}_p)$ by 0 for prime divisors p of d. Thus the above condition holds (taking $\mathfrak{p}_0$ to divide such a p). It remains only to consider the case $d = 1$, which is the same as taking $\pi = \sigma$ as a 2-group.

Now suppose π a 2-group. As $\hat{\mathbf{Z}}_p\pi$ is a maximal order for p odd, the corresponding maps $L_2^S(\hat{R}_\mathfrak{p}) \to L_2^S(\hat{S}_\mathfrak{p})$ will be surjective. We must thus consider the 2-adic completion. Moreover, as our argument will be inconclusive anyway, we may as well restrict to the orientable case.

We have

$$L_i^K(\hat{\mathbf{Z}}_2\pi) \cong L_i^K(\overline{\mathbf{Z}_2\pi}) \cong L_i^K(\mathbf{F}_2) \cong \mathbf{Z}/2\ .$$

For $i = 2$ we know if π is trivial that this lifts to a summand of $L_i^Y(\hat{\mathbf{Z}}_2\pi)$ which maps to zero in $L_i^S(\hat{\mathbf{Q}}_2\pi)$. The same follows in general, since the groups split as direct sums, with the trivial case a summand. For $i = 0$, similarly, this is not in the image of $L_i^X(\hat{\mathbf{Z}}_2\pi)$ or $L_i^Y(\hat{\mathbf{Z}}_2\pi)$.

The image of $L_{2k}^Y(\hat{\mathbf{Z}}_2\pi) \to CL_{2k}^S(\mathbf{Q}\pi)$ thus coincides with that of $H^{2k+1}(K_1(\hat{\mathbf{Z}}_2\pi)/Y)$. Now by [UGR, Theorem 4.1], $\{\pm 1\} \oplus \pi/\pi'$ is the 2-torsion subgroup of $K_1'(\hat{\mathbf{Z}}_2\pi)$. Thus, factoring out a group of odd order which does not contribute to cohomology, we have $H^{2k+1}(Wh'(\hat{\mathbf{Z}}_2\pi))$. Recall that in [UGR, (11.4)] we conjectured that this group is trivial: this was verified in that paper for π abelian (11.3), dihedral or quaternion (10.3). For reference below, we reiterate this:

Conjecture 5.1.3.
For π a 2-group, in the orientable case, $H^1(Wh'(\hat{\mathbf{Z}}_2\pi)) = 0$.

When π satisfies this, the conjecture above also holds, in the orientable case. One can also argue similarly in the nonorientable case, but the situation is considerably more complicated, and we have not formulated an appropriate conjecture.

We are led to formulate explicitly further conjectures.

Conjecture 5.1.4. (i) *Signatures at summands of T of type $U(\mathbf{C})$ are divisible by* 4.

(ii) *Signatures at summands of T of type $O(\mathbf{R})$ are divisible by* 4, *and the sum of all those corresponding to a summand of S is divisible by* 8.

(iii) *Signatures at summands of T of type* $\mathrm{Sp}(\mathbf{H})$ *are divisible by* 4.

(iv) *There are no further relations among the signatures.*

Indeed, we already know (i) and (iv). It follows from (5.1.2) that (ii) holds unless, perhaps, some even localisation of one of the corresponding summands S is a matrix ring over a quaternion division ring. And we just proved (iii) in the orientable case subject to the 2-subgroups of π satisfying Conjecture (5.1.3).

We leave the reader to formulate this in the representation-theoretic terminology of (2.2).

Observe finally that the essential difficulty of conjecture (5.1.3) is its non-abelian character. Otherwise, if J is the radical of $\hat{\mathbf{Z}}_2\pi$, we could use the 2-adic logarithm to deduce from $H^i(J^+) = 0$ that $H^i(1 + 2J)^\times = 0$ (or perhaps $H^i(1 + 4J)^\times = 0$), and would then be within striking distance of a solution.

When π has abelian Sylow 2-subgroup, all the conjectures hold as we saw in Chapter 4. This follows easily from the above.

5.2. General calculation for π a 2-group

Although we have seen in a sense that the essential complication in computing L-groups comes from the Sylow 2-subgroup of π, Chapter 3 above is considerably simpler than Chapter 4. In this section, we will see that when π is a 2-group, striking simplifications can be made to the general calculation. We restrict to the orientable case, and will obtain $L_*^Y(\hat{R}_2)$ and $L_*^Y(R \rightarrow \hat{R}_2)$ in full detail, and compute ψ_i when π is dihedral or quaternionic. The section concludes with a few remarks about the nonorientable case.

In a companion paper on computation of surgery obstructions we will see that the problem essentially reduces to the case when π is a 2-group, so the case studied here has general theoretical interest also.

We first perform the 2-adic computation. As usual,

$$L_i^K(\hat{\mathbf{Z}}_2\pi) = L_i^K(\overline{\mathbf{Z}_2\pi}) = L_i^K(\mathbf{F}_2) \cong \mathbf{Z}/2$$

for all i. We will assume conjecture (5.1.3) for π: that $H^1(K_1(\hat{\mathbf{Z}}_2\pi)/Y) = 0$ —recall we are considering the orientable case. Now in [UGR, §2] we produced, using the 2-adic logarithm, an isomorphism between subgroups of finite index in

$$K_1(\hat{\mathbf{Z}}_2\pi)/Y = K_1'(\hat{\mathbf{Z}}_2\pi) \quad \text{(by [UGR, 4.1])} \quad \text{and} \quad Z(\hat{\mathbf{Z}}_2\pi)^+ .$$

Thus if κ is the number of self-inverse conjugacy classes of π (i.e., containing with each element its inverse), then $Z(\hat{\mathbf{Z}}_2\pi)^+$, hence also $K_1(\hat{\mathbf{Z}}_2\pi)/Y$, has Herbrand quotient 2^κ. Thus $H^0(K_1(\hat{\mathbf{Z}}_2\pi)/Y)$ has rank κ. Now the sequence relating L_*^Y and L_*^K for $\hat{\mathbf{Z}}_2\pi$ splits, with the sequence for $\hat{\mathbf{Z}}_2$ (which we know) as a summand. It follows that

LEMMA 5.2.1. *If π satisfies* (5.1.3), *$L_i^Y(\hat{\mathbf{Z}}_2\pi)$ has exponent* 2 *and rank* 0, $\kappa + 1$, 1, κ *for* $i = 0, 1, 2, 3$.

Next, to obtain $L_i^X(R \to \hat{R}_2)$ we must compute $\gamma_i\colon L_i^X(\hat{R}_{\text{odd}}) \oplus L_i^X(T) \to CL_i^S(S)$. But as $\hat{R}_p$ is a maximal order for p odd (π being a 2-group), it splits as a sum of rings corresponding to the decomposition of S. We may thus consider the summands of S separately (as usual).

Now $S = \mathbf{Q}\pi$ is a sum of simple algebras E with centres K. But K is generated over $\mathbf{Q}$ by values of the corresponding character of π, which are in turn sums of eigenvalues each a $(2^r)^{\text{th}}$ root of unity for some r. Thus for some n, K is a subfield of the field L_n of $(2^{n+1})^{\text{st}}$ roots of unity. As, moreover, we are considering the orientable case, E has type U if K is totally complex, type O or type Sp if K is totally real (only these cases are possible). Moreover, when K has type U, $L_i^X(\hat{R}_{\text{odd}})$ and $CL_i^S(S)$ vanish so the contribution $L_{i+1}^X(R \to \hat{R}_2) = L_i^S(T)$ is 0 for i odd, and a sum of copies of $4\mathbf{Z}$, one for each complex completion of K, for i even.

The major simplifications come in the remaining case. L_n is abelian, and it is easy to list the real subfields: there is just one, K_{n-1} of index 2 and any real subfield is a K_p for some p. Let us write then $K = K_n$. Now K_n is a very nice field, in particular its strict class group Γ^* has odd order (this is an old result, well-known to those who know these things, but too deep to be included in the textbooks. A suitable reference is [H. Hasse, Uber die Klassenzahl abelscher Zahlkorper]). Equivalently, the regular class group has odd order, and the map

$$\phi_0\colon A^\times/(A^\times)^2 \longrightarrow \textstyle\prod_{p\in\mathbf{R}} \{\pm 1\}$$

is surjective. Now in the terminology of (4.6), $r_2 = 0$ and $r_1 = 2^n$; both the above groups have rank 2^n, and the map is an isomorphism. Thus $A^* = (A^\times)^2$.

We can reinterpret this after (4.6): K_n has no extensions unramified at finite primes. We now consider extensions $K_n[\sqrt{x}]$ ramified only at even primes. As the extension is unramified at infinity, x is totally positive (i.e., positive at each real prime). Now 2 is totally ramified in L_{n+1}, hence in K_n ($e_2 = 2^n$, $f_2 = g_2 = 1$), so there is only one even prime $\mathfrak{p}$. We may suppose (multiplying by an appropriate unit) that $\mathfrak{p}$ is generated by the totally positive number q. Then x equals 1 or q multiplied by a totally positive element of $K^{(2)}$. But since ${}_2\Gamma = 0$ and $A^* = (A^\times)^2$, $K^{(2)*} = (K^\times)^2$. Then $K_n[\sqrt{q}]$ is the only extension ramified only at 2; it must therefore be K_{n+1}. Moreover, it follows that $\Gamma^{(2)}$ has order 2. But this is the cokernel of

$$\phi_2\colon A^\times/(A^\times)^2 \longrightarrow \hat{A}_2^\times/(\hat{A}_2^\times)^2 ,$$

and these groups have respective ranks 2^n, $2^n + 1$. Hence ϕ_2 is injective. The image of ϕ_2 cannot contain (-3), else we would have a quadratic extension ramified at no finite prime. In fact the image is the subgroup orthogonal (under the norm residue symbol) to q—so every unit of K_n is a norm from K_{n+1}. For if u is a global unit,

$$(u, q)_2 = \prod\nolimits_{\mathfrak{p}\neq 2} (u, q)_{\mathfrak{p}}$$

by the product formula. Now $(u, q)_{\mathfrak{p}} = 1$ for $\mathfrak{p}$ finite, since u and q are both units in $\hat{K}_{\mathfrak{p}}$, and for $\mathfrak{p}$ infinite, since q is positive. The product is thus 1.

We can now compute γ_i at the summand E. First let E have type O: then it is unramified at infinity. As it is also unramified at odd $\mathfrak{p}$, it follows by the product formula for Hasse invariants that E is a matrix ring over K_n; however, we do not need this explicitly. Comparing with (1.6), we see

$\gamma_3\colon 0 \to 0$ is an isomorphism.

γ_2 is also an isomorphism, since just one $\mathfrak{p}$ (the even one) is omitted from the domain.

γ_1 has the same kernel and cokernel as

$$K^{(2)}/(K^\times)^2 \longrightarrow \hat{A}_2^\times/(\hat{A}_2^\times)^2 .$$

But as Γ has odd order, $K^{(2)}/(K^\times) = A^\times/(A^\times)^2$, so we have ϕ_2; this is injective, with cokernel of order 2, by the above.

$$\gamma_0\colon \bigoplus \{4\mathbf{Z}\colon \mathfrak{p} \text{ real}\} \longrightarrow \mathbf{Z}/2$$

is surjective on each component; write Σ_E for the kernel. The contribution to $L_i(R \to \hat{R}_2)$ is thus zero except for $i = 1$, where we have $\Sigma_E \oplus \mathbf{Z}/2$.

The situtION is similar if E has type Sp; here the real places of E have type **H**, and are the only ramified ones (by the product formula again). Here, γ_3 has the same kernel and cokernel as

$$A^\times/(A^\times)^2 \longrightarrow \hat{A}_2^\times/(\hat{A}_2^\times)^2 \oplus \{\{\pm 1\}: \mathfrak{p} \text{ real}\}$$

so is injective, with cokernel $\hat{A}_2^\times/(\hat{A}_2^\times)^2$, of rank $2^n + 1$.

γ_2: $0 \to \mathbf{Z}/2$ is the unique such homomorphism; so is

γ_1: $0 \to 0$.

γ_0 (like γ_2 in the previous case) is surjective; its kernel is now $\bigoplus\{4\mathbf{Z}: \mathfrak{p}$ real$\} = \Sigma_E$, say.

THEOREM 5.2.2.

$L_3^X(R \to \hat{R}_2)$ is a sum of $\hat{A}_2^\times/(\hat{A}_2^\times)^2$, over summands of type Sp,

$L_2^X(R \to \hat{R}_2)$ is a sum of $\mathbf{Z}/2$, over summands of type Sp,

$L_1^X(R \to \hat{R}_2)$ is a sum of $\mathbf{Z}/2$, over summands of type O,

and the kernel Σ of the epimorphism

$\bigoplus\{4\mathbf{Z}$: *summands of* $\mathbf{R}\pi\} \to \bigoplus\{\mathbf{Z}/2$: *summands type O of* $\mathbf{Q}\pi\}$; $L_0^X(R \to \hat{R}_2) = 0$.

We can write (5.2.1) and (5.2.2) in a form which makes it easier to compute ranks while obscuring somewhat the genesis of the respective groups. Let us denote by a_0 (resp. b_0) the number of summands of $\mathbf{Q}\pi$ (resp. $\mathbf{R}\pi$ or $\mathbf{C}\pi$) of type O; a_{Sp} (resp b_{Sp}) for type Sp. Then the number κ above equals $b_0 + b_{Sp}$. Apart from the term Σ, then, we have elementary 2-groups with ranks as follows:

i	0	1	2	3
rank $L_i^Y(\hat{R}_2)$	0	$b_0 + b_{Sp} + 1$	1	$b_0 + b_{Sp}$
rank $L_i^X(R \to \hat{R}_2)$	0	a_0	a_{Sp}	$a_{Sp} + b_{Sp}$.

To complete the calculation, it remains mainly to determine ψ_1 and ψ_3, for $\psi_0 = 0$ and $\psi_2 = 0$ also, since the generator of $L_2^Y(\hat{\mathbf{Z}}_2\pi)$ is in the image of $L_2^Y(\mathbf{Z}) = L_2^Y(\hat{\mathbf{Z}}_2)$, hence also of $L_2^Y(\mathbf{Z}\pi)$. This argument shows also that $L_2^Y(\mathbf{Z}\pi)$ is a split extension of Coker ψ_3 by $\mathbf{Z}/2$, so the only extension in doubt is that giving $L_1^Y(\mathbf{Z}\pi)$.

To illustrate the above, we first reconsider the abelian case. Here we know (5.1.3) so all the above applies. No representation has type Sp, so $a_{Sp} = b_{Sp} = 0$ and $\psi_3 = 0$. We have $a_0 = b_0 = 2^r$. We need the calculation of rank $\psi_1 = 1 + r + \binom{s}{2}$ as in Section 3.3; the rest can then be read off.

Next we consider dihedral groups; here again (5.1.3) is known and no representation has type Sp. I claim that the image of ψ_1 is the torsion subgroup. Certainly it is contained in this; we prove surjectivity by induction. Write

$$D^{r+1} = \{x, y/x^{2^r} = y^2 = 1, y^{-1}xy = x^{-1}\} .$$

In the case $r = 1$, our assertion follows from the abelian result above. Now assume the result for D^{r+1}. We have

$$\mathbf{Q}D^{r+2} = \mathbf{Q}D^{r+1} \oplus \mathbf{Q}\{x, y/x^{2^r} = -1, y^2 = 1, y^{-1}xy = x^{-1}\}$$

with the second summand a matrix ring $M_2(K_{r-1})$. We have to show that units in $\hat{\mathbf{Z}}_2D^{r+2}$ map surjectively (via Nrd) to the product over these summands of $\hat{A}_2^\times/A^\times(\hat{A}_2^\times)^2$. As $(\hat{\mathbf{Z}}_2D^{r+2})^\times \to (\hat{\mathbf{Z}}_2D^{r+1})^\times$ is clearly surjective, for the kernel of $\hat{\mathbf{Z}}_2D^{r+2} \to \hat{\mathbf{Z}}_2D^{r+1}$ is contained in the radical so any lift of a unit is a unit, it suffices to find a unit mapping to 1 in $\hat{\mathbf{Z}}_2D^{r+1}$ and to a generator of $\hat{A}_2^\times/A^\times(\hat{A}_2^\times)^2$ at the last place. But consider $1 + (1 - x^{2^r})y$: it has the first property, and maps to $1 + 2y$ in the matrix ring; and $\mathrm{Nrd}(1 + 2y) = 1 - 4y^2 = -3$ does generate the quotient.

Since for D^{r+1}, $a_0 = r + 3$ and $b_0 = 2^{r-1} + 3$, we have

THEOREM 5.2.3.

$$\begin{aligned} L_0^Y(\mathbf{Z}D^{r+1}) &\cong \Sigma , \\ L_1^Y(\mathbf{Z}D^{r+1}) &\cong (2^{r-1} - r + 1)(\mathbf{Z}/2) , \\ L_2^Y(\mathbf{Z}D^{r+1}) &\cong \mathbf{Z}/2 , \\ L_3^Y(\mathbf{Z}D^{r+1}) &\cong (2^{r-1} + 3)(\mathbf{Z}/2) \end{aligned}$$

with the notation above.

Recall as usual that the ranks of the odd groups must be diminished by 1 to yield surgery obstruction groups.

Now we look at quaternion groups

$$Q^{r+2} = \{x, y/x^{2^{r+1}} = 1, y^2 = x^{2^r}, y^{-1}xy = x^{-1}\} .$$

As for D^{r-2} above, we observe that

$$\mathbf{Q}Q^{r-2} = \mathbf{Q}D^{r+1} \oplus \mathbf{Q}\{x, y/x^{2^r} = y^2 = -1, y^{-1}xy = x^{-1}\}$$

and the last summand is a quaternion division ring over K_{r-1}, ramified at infinite primes only. It thus follows from the previous calculation that the image of ψ_1 is still the torsion subgroup.

I now claim that ψ_3 is surjective. This means that we can find a 2-adic integer polynomial $P(x) + yQ(x)$ whose norm $P(x)P(x^{-1}) + Q(x)Q(x^{-1})$ is an arbitrarily chosen 2-adic unit of K_{r-1}. Now since powers of x form an integral base of L_r, $P(x)$ is an arbitrary 2-adic unit of L_r. By local class field theory, $N\hat{L}_{r,2}$ has index 2 in $\hat{K}_{r-1,2}$ and the corresponding assertion holds for units.

For later reference, we determine this norm map more explicitly. Let x generate the torsion subgroup of $L_r^\times$, of order 2^{r+1}; then $1 - x$ is a prime and its norm is a totally positive prime of K_{r-1}, so may be labelled as q. If U^i is the group of units of $\hat{K}_{r-1,2}$ congruent to 1 mod q^i, then U^i/U^{i+1} has order 2 for $i \geqq 1$. For $i = 2j + 1$ $(j \geqq 1)$ we have $1 + q^j(1 - x)$ with norm

$$1 + q^j(2 - x - x^{-1}) + q^{2j+1} = 1 + q^{j+1} + q^{2j+1} .$$

Thus the subgroup of norms of units contains, hence equals, U^2. In particular, if we cannot find a $P(x)$ on taking $Q(x) = 0$ then we can if we take $Q(x) = 1 - x$: so ψ_3 is surjective.

We now count constants as before, and obtain

THEOREM 5.2.4. *$L_0^Y(\mathbf{Z}Q^{r+2}) \cong \Sigma$, $L_1^Y(\mathbf{Z}Q^{r+2})$ is an extension of $\mathbf{Z}/2$ by $(2^r - r + 1)(\mathbf{Z}/2)$, $L_2^Y(\mathbf{Z}Q^{r+2}) \cong \mathbf{Z}/2$, $L_3^Y(\mathbf{Z}Q^{r+2}) \cong (2^{r-1} + 2)(\mathbf{Z}/2)$.*

It would clearly not be difficult to make more such calculations. The above is considerably more powerful than the author's previous attempts: observe that I have not even used the messy calculations of [UGR, § 12] which were included for this very purpose.

We conclude by considering also dihedral and quaternion groups in the nonorientable case. These are relatively simple since (as we have seen) the fields K_n are the only ones which appear—whereas even for π cyclic of order 4 we had $L_1 = \mathbf{Q}[i]$ as a summand of type O.

We first determine the types of the various summands of $\mathbf{Q}\pi$ which appear. These are given by

$$x^{2^r} = -1, \quad y^2 = \varepsilon \ (= \pm 1), \quad y^{-1}xy = x^{-1}$$

for some value of r. We thus have an additive basis $\{x^i, x^iy\colon 0 \leqq i < 2^r\}$. Now

$$\alpha(x^i) = w(x^i)x^{-i}, \quad \alpha(x^iy) = w(x^iy)(x^iy)^{-1} = \varepsilon w(x^iy)x^iy .$$

If $w(x) = -1$, we have the same number of x^iy fixed as changing sign; $x^0 = 1$ is fixed, but $x^{2^{r-1}}$ changes sign (unless $r = 1$), so we have type U; if $r = 1$, x is fixed too and we have type O. If, however, $w(x) = 1$, the contributions from x^0 and $x^{2^{r-1}}$ always cancel but those of the x^iy mount up; we have type O or Sp according to whether $\varepsilon w(y) = +1$ or -1.

Collecting these observations, we see that if

$$\pi = \{x, y / x^{2^r} = 1, \ y^2 = 1 \text{ or } x^{2^{r-1}}, \ y^{-1}xy = x^{-1}\}$$

is of type D^{r+1} or Q^{r+1}; then if $w(x) = -1$ there is only one self-conjugate representation, which corresponds to $x^2 = -1$ and has type O. The Herbrand quotient for $Z(\hat{\mathbf{Z}}_2\pi)^+$ is thus 2; hence also for $K_1(\hat{\mathbf{Z}}_2\pi)/Y$. If, however, $w(x) = 1$, all the representations of degree 2 are self-conjugate; as we are supposing π nonorientable, $w(y) = -1$, and all have the opposite type to the orientable case.

We first consider the case $w(x) = 1$, $w(y) = -1$; we begin with the 2-adic calculation. The torsion subgroup $\pm 1 \oplus \pi/\pi'$ of $K_1'(\hat{\mathbf{Z}}_2\pi)$ is "the same" (rank 3) in all cases, and if we define $K_1''(\hat{\mathbf{Z}}_2\pi)$ to be the subgroup whose projections at the four degree 1 representations are $\equiv 1 \pmod 4$, the projection $K_1''(\hat{\mathbf{Z}}_2\pi) \to K_1(\hat{\mathbf{Z}}_2\pi)/Y$ is an isomorphism. Now the projection on these four representations is

$$\{(a, b, c, d)\colon a \equiv b \equiv c \equiv d \equiv 1 \pmod 4\}\ .$$

In all the nonorientable cases, these are permuted in pairs by α, so are all type GL; the projection is an induced module over the group ι of order 2 generated by α, hence cohomologically trivial. Thus if K_1''' is the kernel of this projection, $H^i\big(K_1(\hat{\mathbf{Z}}_2\pi)/Y\big) = H^i(K_1''')$.

Now consider D^{r+1} and Q^{r+1} in the case $w(x) = 1$, $w(y) = -1$. It follows from the above that ι acts trivially on K_1''', so $H^1 = 0$ and H^0 has rank $2^{r-1} - 1$. The usual calculation shows that the map

$$L_{2i}^K(\hat{\mathbf{Z}}_2\pi) \longrightarrow H^{2i}(K_i''')$$

is zero, and indeed that, for all i,

$$(*) \qquad L_i^Y(\hat{\mathbf{Z}}_2\pi) \cong L_i^K(\hat{\mathbf{Z}}_2\pi) \oplus H^{i+1}(K_1''')\ .$$

Also, $L_*(R \to \hat{R}_2)$ is the same as in the orientable case, but with contributions of 1-dimensional representations missing, and all shifted by 2 dimensions. Thus in computing ψ_i we must only use units which are trivial at the 1-dimensional places, i.e., are congruent to 1 modulo the ideal $\langle 1 - x^2\rangle = \langle 1 - x^2, 1 - y^2\rangle$. This does not affect our inductive proof that in the dihedral case ψ_3 (corresponding to ψ_1 before) maps onto the torsion subgroup. In the quaternion case, however, we must have $Q(x)$ divisible by $1 - x^2$, hence $Q(x)Q(x^{-1})$ by q^2, so the image is contained in U^2; indeed our previous argument now shows that it coincides with U^2, so has index 2. We deduce

THEOREM 5.2.5. *In the case $w(x) = 1$, $w(y) = -1$,*

$L_0(\mathbf{Z}D^{r+1}) \cong \mathbf{Z}/2$, $L_1(\mathbf{Z}D^{r+1}) \cong 2^{r-1}(\mathbf{Z}/2)$, $L_2(\mathbf{Z}D^{r+1}) \cong \Sigma \oplus \mathbf{Z}/2$ and $L_3(\mathbf{Z}D^{r+1}) \cong (2^{r-1} - r + 1)(\mathbf{Z}/2)$.

$L_0(\mathbf{Z}Q^{r+1})$ has order 4, $L_1(\mathbf{Z}Q^{r+1}) \cong 2^{r-2}(\mathbf{Z}/2)$, $L_2(\mathbf{Z}Q^{r+1}) \cong \Sigma \oplus \mathbf{Z}/2$ and $L_3(\mathbf{Z}Q^{r+1})$ is an extension of $\mathbf{Z}/2$ by $(2^{r-1} - r + 2)(\mathbf{Z}/2)$.

It remains to consider the case when $w(x) = -1$.

Now if π is quaternion of order 8, there are automorphisms permuting x, y, and xy so we may assume $w(x) = 1$. In the remaining cases, there is just one self-conjugate representation, which corresponds to a quotient group of π which is dihedral of order 8. The other representations make zero contributions. Thus the projection $\pi \to D^3$ induces an isomorphism of $L_*(R \to \hat{R}_2)$.

PROPOSITION 5.2.6. *In the case* $w(x) = -1$, *the groups* $L_i^Y(\mathbf{Z}D^3)$ *are isomorphic for* $i = 0, 1, 2, 3$ *to*:

$$\mathbf{Z} \oplus \mathbf{Z}/2, \quad \mathbf{Z}/2, \quad \mathbf{Z}/2, \quad \mathbf{Z}/2 \oplus \mathbf{Z}/2\ .$$

For other dihedral or quaternion groups, projection on D^3 *induces a surjection of* $L_i^Y(\mathbf{Z}\pi)$. *The kernel is an elementary 2-group, with the same order as* $H^1(K_1(\hat{\mathbf{Z}}_2\pi)/Y)$.

I conjecture that these kernels vanish; this is an example of a nonorientable analogue of (5.1.3). It should not be too difficult to resolve this conjecture by a direct inductive calculation.

Proof (outline). For D^3, we have already identified $K_1(\hat{\mathbf{Z}}_2\pi)/Y$ with K_1'', and its cohomology with that of K_1''', which is isomorphic to $\hat{\mathbf{Z}}_2^+$. Thus H^1 vanishes here, and H^0 has order 2. The usual calculation shows again that the map $\hat{L}_{2i}^K(\mathbf{Z}_2\pi) \to H^{2i}$ is zero and that $(*)$ holds for all i. We have just seen that $L_i^Y(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi)$ is the same as for the trivial group, viz. zero, for $i \neq 1$, and isomorphic to $\mathbf{Z} \oplus \mathbf{Z}/2$ for $i = 1$. Now the image of ψ_1 is the torsion subgroup as we see directly or via the proof of (5.2.3) and the first assertion follows.

For the second, the map of $H^0(K_1(\hat{\mathbf{Z}}_2\pi)/Y)$ is clearly surjective; its kernel has the same order as H^1 by a Herbrand quotient argument. Now $(*)$ again holds, and the conclusion follows on applying the Five Lemma to the map of the exact sequences including ψ_i induced by $\pi \to D^3$.

We conclude with two observations. The first is that the calculations for the two nonorientable cases of D^3 differ only by a dimension shift of 2; it would be interesting to see a direct proof that the result differs only by such a shift. Secondly, we recall that the surgery obstruction groups are obtained from these by factoring out a subgroup of order 2 in odd dimensions (1 in even!). The calculations extend those announced in [L] which were, I am sorry to admit, quite inaccurate in the quaternion case (though correct in the dihedral).

5.3. Examples

This section is included to illustrate the results obtained by examining some cases numerically, though I will not give much actual calculation.

We begin with the dihedral groups D_{2n} and D_{4n}, and the quaternion group Q_{4n}, where n is odd. These are 2-hyperelementary, and the L-groups are direct sums indexed by divisors d of n, e.g.,

$$L_*(D_{2n}) = \bigoplus \{\tilde{L}_*(D_{2n};\, d) \colon d \mid n\}$$

where $\tilde{L}_*(D_{2n};\, d)$ depends only on d, not on n, so will be denoted from now on by $\tilde{L}_*(D_{2d})$. The decomposition was first obtained for L^X, but we have seen that it remains valid for L^Y, with the same $\tilde{L}$; again, the same holds for surgery obstruction groups—only the summand $d = 1$ needs modification.

From Chapter 3, we can obtain these groups when $d = 1$: they are given by the table

π	L_0	L_1	L_2	L_3
$D_2^+ = (\mathbf{Z}/2)^+$	$8\mathbf{Z} \oplus 8\mathbf{Z}$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$
$D_2^- = (\mathbf{Z}/2)^-$	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$	0
$D_4^+ = (\mathbf{Z}/2 \oplus \mathbf{Z}/2)^+$	$4(8\mathbf{Z})$	0	$\mathbf{Z}/2$	$3(\mathbf{Z}/2)$
$D_4^- = (\mathbf{Z}/2 \oplus \mathbf{Z}/2)^-$	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$	0
$Q_4^+ = (\mathbf{Z}/4)^+$	$8\mathbf{Z} \oplus 8\mathbf{Z} \oplus 4\mathbf{Z}$	0	$4\mathbf{Z} \oplus \mathbf{Z}/2$	$\mathbf{Z}/2$
$Q_4^- = (\mathbf{Z}/4)^-$	0	0	$\mathbf{Z}/2$	$2(\mathbf{Z}/2)$

For the rest, we may suppose $d = n > 1$. First consider orientability. The homomorphism $w\colon \pi \longrightarrow \{\pm 1\}$ may be trivial, or it may have kernel the cyclic subgroup of index 2 in π. Our calculations for these two cases differ only by a dimension shift of 2 (though I have no *a priori* proof of this), so we may ignore the second case. For D_{2n} and Q_{4n} these are the only cases; for D_{4n} there are two more, which are equivalent under an outer automorphism of π. This last belongs to Case $\mathrm{I_a}$ in the notation of (4.2); according to (4.4) we then have $L_*(R) = L_*(\hat{R}_2)$, and the value of this is given by (4.3). We will tabulate it with the others below.

It remains, then, to consider the orientable case. Then D_{2n}, D_{4n}, and Q_{4n} have types $\mathrm{II_a}$, $\mathrm{III_a}$, $\mathrm{III_b}$ respectively. The group ζ has order 1, 2, 2. Write

$$K = \mathbf{Q}[\exp(2\pi i/n)] \cap \mathbf{R}\ ;$$

then the centre of the summand of the rational group ring is K, $K \oplus K$, $K \oplus K$. Write A for the integers of K, Γ for its class group, Σ for the sum of $m = \frac{1}{2}\phi(n) = [K\colon \mathbf{R}]$ copies of $4\mathbf{Z}$, one for each real place of K, and $\Sigma' = \operatorname{Ker}(\Sigma \longrightarrow \mathbf{Z}/2)$.

The group $L_*(\hat{R}_2)$ is again given by (4.3); the relative group $L_i(R \longrightarrow \hat{R}_2)$ is equal respectively to L_i^* for D_{2n}, $L_i^* \oplus L_i^*$ for D_{4n}, and $L_i^* \oplus L_{i+2}^{**}$ for Q_{4n}.

We have

$$L_0^* = 0\ ,\quad L_0^{**} = \mathbf{Z}/2\ ,\quad L_3^* = 0\ ,\quad L_3^{**} = \Sigma\ ,$$

an exact sequence

$$\begin{aligned}0 &\longrightarrow \{\pm 1\} \longrightarrow \bigoplus \{\pm 1 \colon \mathfrak{p} \mid 2n\} \longrightarrow L_2^* \\ &\longrightarrow K^{(2)}/(K^\times)^2 \xrightarrow{\Phi} \bigoplus \{(\hat{A}_\mathfrak{p}^\times)/(\hat{A}_\mathfrak{p}^\times)^2 \colon \mathfrak{p} \mid 2n\} \longrightarrow L_1^* \longrightarrow \Gamma/\Gamma^2 \oplus \Sigma' \longrightarrow 0\ ,\end{aligned}$$

and a similar one for L_2^{**}, L_1^{**} but with Σ' omitted and $\bigoplus \{\pm 1 \colon \mathfrak{p} \text{ real}\}$ added to the target of Φ.

To compute ranks, we introduce the decomposition numbers g_p: if n_p is the largest factor of n prime to p, this equals the index in $(\mathbf{Z}/n_p)^\times$ of the subgroup generated by p and -1. Write $G = \Sigma\{g_p \colon p \mid 2n\}$. According to (4.7), the orders of the torsion subgroups are given by

$$\begin{aligned} l_2^* &= G - 1 + \operatorname{rk}\operatorname{Ker}\Phi\ , & l_1^* &= 1 + l_2^*\ , \\ l_2^{**} &= G - 1 + \operatorname{rk}\operatorname{Ker}\Phi^*\ . & l_1^{**} &= m + 1 + l_2^{**}\ .\end{aligned}$$

Now from (4.5) we find that for D_{2n} and D_{4n}, $\psi_0 = \psi_3 = 0$ and the image of ψ_2 is $g_2(\mathbf{Z}/2)$; for Q_{4n}, ψ_0 and ψ_2 are injective. For the rest, the images of ψ_1 (D_{2n} or Q_{4n}) and ψ_3 (Q_{4n}) are the natural images of $\hat{A}_2^\times$ and since $(\hat{A}_2\zeta_2)^\times$ has odd index in $\hat{A}_2^\times \oplus \hat{A}_2^\times$ and ι acts trivially, the image of ψ_1 for D_{4n} is the sum of two copies of that of $\hat{A}_2^\times \to L_1^*$.

To pursue the general case a little further, we conclude by observing that although our methods do not determine group extensions in general, the orders of the torsion subgroups can now all be written down in terms of those of the *cokernels* of ψ_1 and ψ_1^*.

From (4.3), $L_i^Y(\hat{R}_2)$ is given by the table:

π \ i	0	1	2	3
D_{2n}	0	$(m + g_2)(\mathbf{Z}/2)$	$2g_2(\mathbf{Z}/2)$	$(m - g_2)(\mathbf{Z}/2) \oplus g_2(\mathbf{Z}/4)$
D_{4n}	$g_2(\mathbf{Z}/2)$	$(2m + 2g_2)(\mathbf{Z}/2)$	$3g_2(\mathbf{Z}/2)$	$(2m)(\mathbf{Z}/2) \oplus g_2(\mathbf{Z}/4)$
Q_{4n}	$g_2(\mathbf{Z}/2)$	$(2m + g_2)(\mathbf{Z}/2)$	$g_2(\mathbf{Z}/2)$	$(2m + g_2)(\mathbf{Z}/2)$
D_{4n}, exc	$g_2(\mathbf{Z}/2)$	$g_2(\mathbf{Z}/2)$	$g_2(\mathbf{Z}/2)$	$g_2(\mathbf{Z}/2)$

THEOREM 5.3.1. *For n odd, in the exceptional case,*

$$\widetilde{L}_i(D_{4n}) \cong g_2(\mathbf{Z}/2)\ ,$$

and in the orientable case,

$$\tilde{L}_2(D_{2n}) \cong g_2(\mathbf{Z}/2)\,, \qquad \tilde{L}_2(D_{4n}) \cong 2g_2(\mathbf{Z}/2)\,,$$
$$\tilde{L}_3(D_{2n}) \cong L_3(\hat{R}_2) \cong (m - g_2)(\mathbf{Z}/2) \oplus g_2(\mathbf{Z}/4)\,,$$
$$\tilde{L}_3(D_{4n}) \cong L_3(\hat{R}_2) \cong (2m)(\mathbf{Z}/2) \oplus g_2(\mathbf{Z}/4)\,,$$

and $\tilde{L}_1$, $\tilde{L}_2$, $\tilde{L}_3$ are finite for D_{2n}, D_{4n}, and Q_{4n}, while $\tilde{L}_0$ has free part Σ', $\Sigma' \oplus \Sigma'$ resp. $\Sigma' \oplus \Sigma$. If the cokernels of ψ_1, ψ_1^* have orders 2^{ϕ}, 2^{ϕ^*} then the torsion subgroups of the groups $\tilde{L}$ have orders 2^r with r given by the table

$\pi \backslash i$	0	1	2	3
D_{2n}	ϕ	$\phi + m - 1$	g_2	$m + g_2$
D_{4n}	$2\phi + g_2$	$2\phi + 2m - 2 + g_2$	$2g_2$	$2m + 2g_2$
Q_{4n}	ϕ	$\phi + 2m$	ϕ^*	$\phi^* + m - 1$

This follows from the assertions above by chasing round exact sequences, which may be left to the reader. These also of course give some information about extensions—e.g., L_1^* has exponent 2, hence so does the torsion subgroup of L_0 for D_{2n} and Q_{4n}. Similarly if Φ is injective L_2^* has exponent 2; and it always has exponent 2 or 4.

Now the cokernel of ψ_1 lies in the (split) exact sequence

$$K^{(2)}/(K^{\times})^2 \xrightarrow{\Phi'} \bigoplus\{\hat{A}_{\mathfrak{p}}^{\times}/(\hat{A}_{\mathfrak{p}}^{\times})^2 : \mathfrak{p} \mid n\} \longrightarrow \operatorname{Cok} \psi_1 \longrightarrow \Gamma/\Gamma^2 \oplus \Sigma' \longrightarrow 0\,,$$

so we should compute rk Φ' (by-passing separate calculations for ranks of Φ and ψ_1). By (4.6), Ker Φ' is dual to the quotient of the $(\infty, 2)$-enriched class group $\Gamma^{*(2)}$ by $U^2(\hat{A}_2)$ and prime divisors of n. However, we may also compute directly. For example, suppose n a power of a prime p. Then p is totally ramified in K, and $g_p = 1$. There is thus only one $\mathfrak{p} \mid n$, and its residue field is $\mathbf{F}_p$. But if $\zeta = \exp 2\pi i/n$, then $(\zeta^r - \zeta^{-r})/(\zeta - \zeta^{-1})$ is a unit, with residue $r \in \mathbf{F}_p$ $(p \nmid r)$, so Φ' is surjective and ϕ equals the 2-rank of Γ. This is 0, for example, if $n = 3$, 5, or 7.

As to ϕ^*, I assert that if Γ^* has odd order, then

$$\phi^* = G - g_2 = \Sigma\{g_p : p \mid n\}\,.$$

For when Γ has odd order, we can simplify the sequence to

$$A^{\times}/(A^{\times})^2 \longrightarrow \bigoplus\{\hat{A}_{\mathfrak{p}}^{\times}/(\hat{A}_{\mathfrak{p}}^{\times})^2 : \mathfrak{p} \mid n\} \oplus \{\pm 1 : \mathfrak{p} \text{ real}\} \longrightarrow \operatorname{Coker} \psi_1^* \longrightarrow 0\,.$$

But the two groups

$$A^{\times}/(A^{\times})^2 \longrightarrow \bigoplus\{\pm 1 : \mathfrak{p} \text{ real}\}$$

have the same 2-rank m; this map is surjective since Γ^* is odd, hence bijective. The cokernel is thus as stated. In particular, for $n = 3$, 5, or 7, $\phi^* = 1$.

Observe in conclusion that the announcement [L] was somewhat premature: "ϕ" there equals the g_2 above, so L_0 and L_2 were correct; as to L_3, $\phi + p - 1$ was a misprint for $\phi + (p-1)/2$ and the determination of the extension was wrong. The calculation of L_1 was more seriously in error; the subtleties above were discovered later.

One primary source of motivation for writing this paper was the desire to determine the surgery obstruction groups for groups with periodic cohomology. After (2.4), it suffices to consider 2-hyperelementary groups with cyclic or quaternionic Sylow 2-subgroup. The cyclic case is covered in principle in Chapter 4: further details can be computed as above. For the other case, if Ker $\lambda: \sigma \to (\mathbf{Z}/n)^\times$ is abelian, then following [UGR] one sees that at least a method exists to attack the problem; it is likely to be even more complicated. For example, if

$$\sigma = \{x, y / x^4 = 1, y^2 = x^2, y^{-1}xy = x^{-1}\}$$

acts on $\mathbf{Q}[\omega/\omega^2 + \omega + 1 = 0]$ by $\omega^x = \omega$, $\omega^y = \omega^{-1}$, then we will be reduced to studying the ring R whose $\mathbf{Z}$-basis consists of

$$1, x^2, \omega x + \omega^2 x^{-1}, \omega^2 x + \omega x^{-1}: R \otimes \mathbf{Q} \cong \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q}[\sqrt{3}]\,.$$

To be a little more specific, let us consider the interesting group $\mathrm{SL}_2(q) = \mathrm{SL}_2(\mathbf{F}_q)$, of order $q(q^2-1)$, with q odd. The conjugacy classes are either

(i) $\pm I$, central;

(ii) diagonal, with distinct eigenvalues λ, λ^{-1};

(iii) semisimple; diagonalisable over $\mathbf{F}_{q^2}$ with eigenvalues λ, λ^{-1} where $\lambda^{1+q} = 1$;

(iv) $\pm T$ where T is upper unitriangular.

A 2-hyperelementary subgroup normalises an element of odd order. For an element of type (ii), the normaliser is contained in extension with index 2 of $\mathbf{F}_q^\times \oplus \mathbf{F}_q^\times$ (diagonal) generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For one of type (iii), it is the product of $\mathbf{F}_{q^2}^\times$ by an element of order 4 inducing the Frobenius automorphism $x \to x^q$. Note that only $-I$ has order 2. Finally, the normaliser of the cyclic group generated by $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is the group of matrices $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ such that $a^{-2}x$ is a multiple of x, i.e., $a^2 \in \mathbf{F}_p$, where q is a power of p, and hence $a \in \mathbf{F}_p^\times$. The 2-hyperelementary subgroups arising from cases (ii) and (iii) are quaternion groups, each contained in (a conjugate of) one maximal one, of order $2(q-1)$ resp. $2(q+1)$. For case (iv) we have an extension of $\mathbf{Z}/p$ by the Sylow 2-subgroup of $\mathbf{F}_p^\times$, acting by the square of its natural action. This is abelian if $p \equiv 3 \pmod 4$, and quaternionic if $p \equiv 5 \pmod 8$. All these groups are covered by Chapter 4, and most by the case Q_{4n} detailed above.

We conclude with one simple example showing how to reconstruct $L_*(\pi)$ from the 2-hyperelementary subgroups of π. If π is the tetrahedral group (alias the alternating group A_4), the maximal such subgroups are $\mathbf{Z}/2 \oplus \mathbf{Z}/2$ and $\mathbf{Z}/3$, and the inverse limit over the category of them is thus the pullback of

$$\begin{array}{ccc} & & L_*(\mathbf{Z}/3) \\ & & \downarrow \\ (L_*(\mathbf{Z}/2 \oplus \mathbf{Z}/2))^{\mathbf{Z}/3} & \longrightarrow & L_*(1) \ , \end{array}$$

where the affix $\mathbf{Z}/3$ indicates invariants under the automorphism induced by action of $\mathbf{Z}/3$. Thus

$$L_*(A_4) \cong (L_*(\mathbf{Z}/2 \oplus \mathbf{Z}/2))^{\mathbf{Z}/3} \oplus \tilde{L}_*(\mathbf{Z}/3) \ .$$

Similarly for the binary tetrahedral group $\mathrm{SL}_2(3)$ the pullbacks of the above form a cofinal subcategory, so

$$L_*(\mathrm{SL}_2(3)) \cong (L_*(Q_8))^{\mathbf{Z}/3} \oplus \tilde{L}_*(\mathbf{Z}/6) \ ,$$

where $\tilde{L}_*(\mathbf{Z}/6)$ denotes (as above) Ker $(L_*(\mathbf{Z}/6) \to L_*(\mathbf{Z}/2))$.

We conclude by computing these groups in the orientable case. In each case there is a natural splitting into free and torsion parts (e.g., $L_0(Q_8)$ is free). As to the free part, $\mathbf{Z}/3$ acts by permuting certain summands. We therefore have

$$(L_0(\mathbf{Z}/2 \oplus \mathbf{Z}/2))^{\mathbf{Z}/3} \cong 8\mathbf{Z} \oplus 8\mathbf{Z} \ , \quad (L_0(Q_8))^{\mathbf{Z}/3} \cong 8\mathbf{Z} \oplus 8\mathbf{Z} \oplus 4\mathbf{Z}$$

whereas by (2.4.2),

$$\begin{aligned} \tilde{L}_0(\mathbf{Z}/3) &\cong \tilde{L}_2(\mathbf{Z}/3) \cong 4\mathbf{Z} \ , \\ \tilde{L}_0(\mathbf{Z}/6) &\cong \tilde{L}_2(\mathbf{Z}/6) \cong 4\mathbf{Z} \oplus 4\mathbf{Z} \ , \end{aligned}$$

and these groups vanish in odd dimensions.

For the torsion, we begin by observing that if V is an elementary 2-group on which $\mathbf{Z}/3$ acts, we can split $V = V_0 \oplus V_1$ where V_0 is the subgroup of invariants, and V_1 can be regarded as a vector space over $\mathbf{F}_4$, with $\mathbf{Z}/3$ acting as $\mathbf{F}_4^\times$. Now $L_2(R)$ has order 2 so the action must be trivial. The action on the summand of $L_{2i+1}(\hat{R}_2)$ coming from the trivial group is trivial; what is left is

$$\mathrm{Ker}\ (H^0(\hat{R}_2^\times/Y) \longrightarrow H^0(\hat{\mathbf{Z}}_2^\times/\{\pm 1\})) \ .$$

This is a group V as above, and we see at once that $\dim_{\mathbf{F}_4} V_1 = 1$ in each case. But for $L_1(R)$, V_1 maps injectively to the relative group, so $\mathbf{Z}/3$ acts trivially (in the quaternion case, we observe that if $L_1(R)$ is a nontrivial extension, $\mathbf{Z}/3$ cannot act nontrivially on it). For $L_3(R)$, on the other hand, its image does contain V_1. To summarise, we have

THEOREM 5.3.2. *In the orientable case, surgery obstruction groups are given by*:

$L_0(A_4) \cong 8\mathbf{Z} \oplus 8\mathbf{Z} \oplus 4\mathbf{Z}$, $L_1(A_4) = 0$, $L_2(A_4) \cong 4\mathbf{Z} \oplus \mathbf{Z}/2$, $L_3(A_4) = 0$,
$L_0(\mathrm{SL}_2(3)) \cong 8\mathbf{Z} \oplus 8\mathbf{Z} \oplus 8\mathbf{Z} \oplus 4\mathbf{Z} \oplus 4\mathbf{Z}$, $L_1(\mathrm{SL}_2(3))$ *has order* 4,
$L_2(\mathrm{SL}_2(3)) \cong 4\mathbf{Z} \oplus 4\mathbf{Z} \oplus \mathbf{Z}/2$, *and* $L_3(\mathrm{SL}_2(3)) = 0$.

5.4. Variant forms of obstruction groups

In (1.1) above we defined groups $L_i^X(R, \alpha, u)$ for any α-invariant subgroup X of $K_1(R)$, and agreed to write L_i^S when $X = 0$ and L_i^K when $X = K_1(R)$. For the case $R = \mathbf{Z}\pi$ of primary interest to us we have concentrated throughout on $X = SK_1(\mathbf{Z}\pi) = \mathrm{Ker}\big(K_1(\mathbf{Z}\pi) \to K_1(\mathbf{Q}\pi)\big)$. Write also U for the subgroup of $K_1(\mathbf{Z}\pi)$ generated by the images of $\pm g$ $(g \in \pi)$, and $Y = X + U$. Then our groups are related to the surgery obstruction groups of [SCM], [L] by

$$L_{2k}^s(\pi) = L_{2k}^U(\mathbf{Z}\pi)\ , \quad L_{2k}'(\pi) = L_{2k}^Y(\mathbf{Z}\pi)\ , \quad L_{2k}^h(\pi) = L_{2k}^K(\mathbf{Z}\pi)\ ,$$

and correspondingly in odd dimensions except that we factor out the class of $\tau = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$. This was shown in [F]; the L' groups are intermediate in the sense of Cappell.

The interrelations between these depend on an analysis of the group $K_1(\mathbf{Z}\pi)$. This is a finitely generated abelian group (Bass [5, X, § 3]) whose rank is given in the notation of (2.2) above by

$$\mathrm{rk}\ K_1(\mathbf{Z}\pi) = \mathrm{rk}\ R_{\mathbf{R}}\pi - \mathrm{rk}\ R_{\mathbf{Q}}\pi\ .$$

As to the torsion subgroup, by the main result of [UGR, 6.5] we have

$$\begin{aligned} \mathrm{Tor}\ K_1(\mathbf{Z}\pi) &= \{\pm 1\} \oplus \pi/\pi' \oplus SK_1(\mathbf{Z}\pi) \\ &= U \oplus SK_1(\mathbf{Z}\pi)\ . \end{aligned}$$

The Whitehead group $\mathrm{Wh}(\pi)$ is defined to be $K_1(\mathbf{Z}\pi)/U$; we set also

$$K_1'(\mathbf{Z}\pi) = K_1(\mathbf{Z}\pi)/SK_1(\mathbf{Z}\pi) \text{ and } \mathrm{Wh}'(\pi) = \mathrm{Wh}(\pi)/SK_1(\mathbf{Z}\pi)\ ,$$
the torsion-free quotient.

The variant forms of L-groups are related by exact sequences, as given in (1.1). We first compare L' with L^h. The relative groups here are $H^*(\mathbf{Z}/2; \mathrm{Wh}'(\pi))$, and they vanish if $K_1(\mathbf{Z}\pi)$ has zero rank—i.e., if all real-valued characters of π are rational-valued, or equivalently if for any $x \in \pi$, any power of x generating the same cyclic subgroup is conjugate to x or to x^{-1}. This is the case, for example, if π is abelian of exponent 4 or 6, or if π is isomorphic to the symmetric group of n letters (any n) or to the alternating group ($n = 3$, 4, 7, 8, 9 or 12). In these cases, $L_n^h\pi = L_n'\pi$. In general, in the orientable case $\mathbf{Z}/2$ acts trivially on $\mathrm{Wh}'(\pi)$ and ([UGR, 7.4]) we have an exact sequence

$$0 \longrightarrow L'_{2k}(\pi) \longrightarrow L^h_{2k}(\pi) \longrightarrow \mathrm{Wh}'(\pi) \otimes \mathbf{Z}/2 \xrightarrow{\delta} L'_{2k-1}(\pi) \longrightarrow L^h_{2k-1}(\pi) \longrightarrow 0 .$$

Now if π has odd order, $L'_{2k-1}(\pi) = 0$ (2.4.3), so $\delta = 0$ and $L^h_{2k-1}(\pi) = 0$. More generally, suppose that π is p-hyperelementary with p odd, with Sylow 2-subgroup σ which we can also by (2.4.1) regard as a quotient. Then the composite $\pi \to \sigma \to \sigma/2\sigma$ induces a commutative diagram

$$\begin{array}{ccc} L'_{2k-1}(\pi) & \xrightarrow{f_1} & L^h_{2k-1}(\pi) \\ \Big\downarrow{\scriptstyle f_2} & & \Big\downarrow{\scriptstyle f_3} \\ L'_{2k-1}(\sigma/2\sigma) & \xrightarrow{f_4} & L^h_{2k-1}(\sigma/2\sigma) . \end{array}$$

Here, f_2 is an isomorphism by the calculation (3.3.2), and f_4 by the remarks above (applied to $\sigma/2\sigma$). Since f_1 is surjective, all four maps are isomorphisms. Hence $\delta = 0$. Recall from (3.3.3) that—assuming σ nontrivial—the groups L_{2k-1} have order 1 (k odd) or 2 (k even).

More interesting, perhaps, is to compare L' with L^s. The relative groups here are $H^*(\mathbf{Z}/2; SK_1(\mathbf{Z}\pi))$. Now if π has odd order—and more generally [UGR, 9.2] if the order of π is not divisible by 4—$SK_1(\mathbf{Z}\pi)$ has odd order, so these cohomology groups vanish and $L^s_n(\pi) = L'_n(\pi)$. The same conclusion holds [5] if π is abelian, and the Sylow 2-subgroup is either cyclic or elementary, also (Keating) for dihedral 2-groups (but probably in few other cases). In the contrary case, Bak has observed ("The involution on Whitehead torsion", to appear; see also Bass [6]) that, at least if π has Schur index 1 (i.e., $\bar{R}_{\mathbf{Q}}(\pi) = R_{\mathbf{Q}}(\pi)$), $\mathbf{Z}/2$ acts trivially on $SK_1(\mathbf{Z}\pi)$ so that if this has even order, so has $H^r(\mathbf{Z}/2; SK_1(\mathbf{Z}\pi))$, and L^s will no longer conicide with L'.

We can now compare our calculations with those of other authors: we begin with those of H. Bass [6]. For π an elementary abelian 2-group, $\mathrm{Wh}(\pi)$ vanishes [5], so $L^s = L' = L^h$. In the orientable case, for π of rank r, (3.3.2) shows that L_1, L_3 are elementary abelian of respective ranks $2^r - 1 - r - \binom{r}{2}$, $2^r - 1$. This coincides with (1.5) and (1.3) of [6], on taking into account the calculation [UGR, 12.9] that $\tilde{K}_0(\mathbf{Z}\pi)$ (denoted $\mathrm{Pic}(\mathbf{Z}\pi)$ by Bass) is a sum of $\binom{r}{i}$ cyclic groups of order 2^{i-2} for $3 \leqq i \leqq r$ (confirming Bass' calculation [6, 3.7.2] of its order).

In general (for π abelian, orientable case), Bass' results are incomplete, depending on the determination of certain subgroups $G_0(\pi) \subset \mathbf{F}_2[\pi/\pi^2]$ and $G_1(\pi) \subset \mathbf{F}_2[{}_2\pi]^\times$. In view of the fact that $L_3(\mathbf{Z}\pi) \cong L_3(\hat{\mathbf{Z}}_2\pi)$, and $H^0(\hat{\mathbf{Z}}_2\pi)^\times \to L_3(\hat{\mathbf{Z}}_2\pi)$ is 'almost an isomorphism', it is tempting to identify Bass' sequence

$$0 \longrightarrow \mathbf{F}_2[\pi/\pi^2]/\mathbf{F}_2\cdot 1 + G_0(\pi) \longrightarrow L_{-1}(\pi) \longrightarrow \mathbf{F}_2[{}_2\pi]^\times/G_1(\pi)G_2(\pi) \longrightarrow 0$$

(where $G_2(\pi) = {}_2\pi$ for L^s and $q(B(\pi)^\times)$ for L^h) with the sequences of (3.2):

$$H^0(1 + 2\hat{\mathbf{Z}}_2\pi)^\times \longrightarrow H^0(\hat{\mathbf{Z}}_2\pi)^\times \longrightarrow H^0(\mathbf{F}_2\pi)^\times$$

where the first group is a (split) extension of a group of order 2 by $\mathbf{F}_2[{}_2\pi]^+$, and the last is an extension of $H^0(\mathbf{F}_2[{}_2\pi]^\times)$ by a group of small order. Indeed, if any direct comparison can be made between these, it could lead to new information about $K_1(\mathbf{Z}\pi)$.

Next, we observe that [4], dealing with groups of odd order obtains the same results as above; indeed, we have already observed that in this case, $L^s = L' = L^h$. Finally, Bak has made further calculations for abelian groups (starting from his paper with Scharlau in Inventiones Math. **23** (1974) 207-240). Unfortunately, I do not have the details available at the time of writing, but here too a careful comparison of results should be fruitful.

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(Received January 22, 1975)