Poincaré complexes: I

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Recent developments in differential and PL-topology have succeeded in reducing a large number of problems (classification and embedding, for example) to problems in homotopy theory. The classical methods of homotopy theory are available for these problems, but are often not strong enough to give the results needed. In this paper we attempt to develop a branch of homotopy theory applicable to the classification problem for compact manifolds.

A Poincaré complex is (approximately) a finite CW-complex which satisfies the Poincaré duality theorem. A precise definition is given in §1, together with a discussion of chain complexes. In Chapter 2, we give a *cutting and gluing* theorem, define connected sum, and give a theorem on product decompositions. Chapter 3 is devoted to an account of the *tangential* properties first introduced by M. Spivak (Princeton thesis, 1964). We then start our classification theorems; in Chapter 4, for dimensions up to 3, where the dominant invariant is the fundamental group; and in Chapter 5, for dimension 4, where we obtain a classification theorem when the fundamental group has prime order. It is complicated to use, but allows us to construct two interesting examples.

In the second part of this paper, we intend to classify highly connected Poincaré complexes; to show how to perform surgery, and give some applications; by constructing handle decompositions and computing some cobordism groups.

This paper was originally planned when the only known fact about topological manifolds (of dimension >3) was that they were Poincaré complexes. Novikov's proof [30] of topological invariance of rational Pontrjagin classes and subsequent work in the same direction has changed this, but we can still easily summarize the basis of the relation of Poincaré complexes to smooth and PL-manifolds. The first point of difference lies in the structure group of the normal bundle; G, O, or PL in the three cases. If this group be appropriately reduced (from G to O or PL), surgery can be performed as in [13] to try to construct a manifold; certain algebraic obstructions arise in the middle dimension (see [13] for the simply-connected and [26] for the general case). These algebraic structures provide the second point of difference; one in which we are particularly interested. Chapter 5 was originally written for the purpose of constructing examples to illustrate these (5.4.1 and 5.4.2). A fuller discussion of these points is planned to appear as a sequel to [26].

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CHAPTER 1. Definitions and chain complexes

Let X be a connected CW-complex, dominated by a finite complex. Suppose given

(i) a homomorphism $w: \pi_1(X) \to \{\pm 1\}$, defining a Λ-module structure $\mathbf{Z}^t$ on $\mathbf{Z}$,

(ii) an integer n and a class $[X] \in H_n(X; \mathbf{Z}^t)$ such that

(iii) for all integers r, cap product with $[X]$ induces an isomorphism

$$[X] \frown : H^r(X; \Lambda) \longrightarrow H_{n-r}(X; \Lambda \otimes \mathbf{Z}^t) ,$$

then we call X a *connected Poincaré complex*, $[X]$ a fundamental class, and n the formal dimension.

For the most part, we will use the notation of [25], but the *twisted* module structure on $\mathbf{Z}^t$ leads us to make some modifications of the definitions there used, which will be more convenient when studying Poincaré complexes. If C_* is (as in [25]) the chain complex of $\tilde{X}$, and B a left Λ-module, we retain the form of the definitions

$$H_r(X; B) = H_r(C_* \otimes_\Lambda B) , \qquad H^r(X; B) = H_{-r}(\mathrm{Hom}_\Lambda (C_*, B)) ,$$

but will reinterpret them. Both definitions refer to the given left module structure on B, and the second also refers to the natural left module structure on C_*. However, the first uses a derived right module structure on C_*, which we redefine, using the anti-automorphism of Λ in which

$$\sum_{g \in \pi} n(g) g \longrightarrow \sum_{g \in \pi} w(g) n(g) g^{-1} .$$

Note that this differs from the obvious definition by insertion of the signs $w(g)$. It is thus clear that the right module structure with this definition coincides with the right module structure on $\Lambda \otimes_{\mathbf{Z}} \mathbf{Z}^t$ with the old definition. To avoid overmuch confusion, we write $H_r^t(X; B)$ for the homology group under the new definition. Note however that $H_r^t(X; \Lambda) = H_r(X; \Lambda)$. In fact this is just the homology of C_*, as $\Lambda \otimes_\Lambda$ is equivalent to the identity functor.

With w fixed, our definition now requires a fundamental class $[X] \in H_n^t(X; \mathbf{Z})$ and isomorphisms

$$[X] \frown : H^r(X; \Lambda) \longrightarrow H_{n-r}^t(X; \Lambda) .$$

The following result is due to Milnor, at least in the orientable case.

LEMMA 1.1. *Suppose X a connected Poincaré complex. Then for any integer r and left Λ-module B,*

$$[X] \frown : H^r(X; B) \longrightarrow H_{n-r}^t(X; B)$$

is an isomorphism.

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PROOF. We represent $[X]$ by a cycle $\xi \in C_n$. Now cap products can be defined at the chain level, by taking a cellular approximation $d: X \to X \times X$ to the diagonal map, and writing, for a chain x and cochain y,

$$y \frown x = y \backslash d_*(x) ,$$

where $d_*: C_* \to C_* \otimes_{\mathbf{Z}} C_*$ is induced by d, and $\backslash$ is Steenrod's slant product. (I am indebted to A. Dold for pointing this out to me.) Thus the choice of ξ and d defines a chain map $\xi \frown: \mathrm{Hom}_\Lambda(C_*, \Lambda) \to C_*$ (of degree n). Now this induces the homology map $[X] \frown$, which is by hypothesis an isomorphism. But $\mathrm{Hom}_\Lambda(C_*, \Lambda)$ and C_* are both finitely generated and projective, so (by a result of J.H.C. Whitehead) $\xi \frown$ is a chain homotopy equivalence.

Now take the tensor product over Λ with a Λ-module B, and use the isomorphism (valid since C_* is free) $\mathrm{Hom}_\Lambda(C_*, \Lambda) \otimes B \cong \mathrm{Hom}_\Lambda(C_*, B)$. We deduce that

$$\xi \frown: \mathrm{Hom}_\Lambda(C_*, B) \longrightarrow C_* \otimes_\Lambda B$$

is a homotopy equivalence, and so induces homology isomorphisms.

COROLLARY 1.1.1. $[X]$ *is unique up to sign.*

For it is a generator of $H_n^t(X; \mathbf{Z}) \cong H^0(X; \mathbf{Z}) \cong \mathbf{Z}$. In general, a Poincaré complex is a finite disjoint union of connected Poincaré complexes of the same formal dimension. We will sometimes use the notation adapted for the connected case. Then Λ is to be interpreted as the direct sum[1] of the integral group rings of the fundamental groups of the components, and $H^r(X; \Lambda)$ as a direct sum likewise. $[X]$ will denote the sum of fundamental classes of the components, and the above isomorphisms are then true without change. Note that the finiteness obstruction for such a disconnected space X continues to be represented by a projective Λ-module.

The Poincaré duality theorem (in a strong form) asserts that a closed topological manifold has the homotopy type of a Poincaré complex. We next proceed to the homotopy analogue of a compact manifold with boundary.

Let Y be a connected CW-complex dominated by a finite complex, X a subcomplex which is a Poincaré complex. Write $\pi = \pi_1(Y)$, and Λ for the integral group ring of π. We call (Y, X) a *connected Poincaré pair* if we are given a homomorphism $w: \pi_1(Y) \to \{\pm 1\}$, [inducing the given homomorphisms $\pi_1(X_i) \to \{\pm 1\}$ for the various components X_i of X], and a class

$$[Y] \in H_{n+1}^t(Y, X; \mathbf{Z}) \qquad [\text{with } \partial_*[Y] = [X] \in H_n^t(X; \mathbf{Z})]$$

such that

[1] The free product would be more appropriate in some respects, but is not altogether convenient to use.

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$$[Y]\frown : H^{r+1}(Y; \Lambda) \longrightarrow H^t_{n-r}(Y, X; \Lambda)$$

is an isomorphism for all integers r.

We conjecture that the two requirements enclosed in square brackets are redundant; however, we will have to impose them.

LEMMA 1.2. *For any integer r and Λ-module B, $[Y]\frown$ induces isomorphisms*

$$H^{r+1}(Y; B) \longrightarrow H^t_{n-r}(Y, X; B), \quad H^{r+1}(Y, X; B) \longrightarrow H^t_{n-r}(Y; B) .$$

PROOF. The proof of the first is essentially the same as that of Lemma 1.1. For the second, we use the following diagram, which has exact rows and squares commutative up to sign

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^r(X; i^*B) & \longrightarrow H^{r+1}(Y, X; B) & \longrightarrow & H^{r+1}(Y; B) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle [X]\frown} & \downarrow{\scriptstyle [Y]\frown} & & \downarrow{\scriptstyle [Y]\frown} & \\
\cdots \longrightarrow & H^t_{n-r}(X; i^*B) \longrightarrow & H^t_{n-r}(Y; B) & \longrightarrow H^t_{n-r}(Y, X; B) & \longrightarrow \cdots .
\end{array}$$

Here i^*B denotes the induced module over $\pi_1(X)$ (or correspondingly if X is disconnected). The Five Lemma permits us to conclude that the middle map is an isomorphism, as asserted.

Although we shall not study them in detail, we give here also the definition of Poincaré triad (cf. [26]). Suppose $(Y; X_+, X_-)$ a CW-triad such that $(Y, X_+ \cup X_-)$, $(X_+, X_+ \cap X_-)$, and $(X_-, X_+ \cap X_-)$ are Poincaré pairs; the homomorphism $w: \pi_1(Y) \to \{\pm 1\}$ induces the corresponding homomorphisms for [the components of] $X_+ \cup X_-$, X_+, X_-, and $X_+ \cap X_-$; and, finally, that

$$\partial_*[Y] = [X_+] - [X_-] .$$

Then we call $(Y; X_+, X_-)$ a *Poincaré triad.* The comments made about the definition of Poincaré pair are pertinent here also. Note in addition the special case $X_+ \cap X_- = \varnothing$, which has particular importance. In this case we call Y a *cobordism* of X_- to X_+.

Since this chapter is mainly concerned with chain complexes, we give here a duality theorem for our finiteness obstruction. This has been obtained independently by J. Milnor. In order to state it, we observe that the standard module operations tensor and Hom preserve projectives and lead to operations on $K^0(\Lambda)$. In particular, $P \to \mathrm{Hom}_\Lambda(P, \Lambda)$ (where the latter has the module structure induced by the canonical anti-automorphism of Λ) defines an involution on $K^0(\Lambda)$, which we will denote by $*$.

THEOREM 1.3. *Suppose X a Poincaré complex of formal dimension n Then $\sigma(X) = (-1)^n \sigma(X)^*$.*

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PROOF. The proof of Lemma 1.1 provides us with a chain homotopy equivalence of degree n

$$[X]\frown: \mathrm{Hom}_\Lambda(C_*, \Lambda) \longrightarrow C_* .$$

We may assume by [25, Th. 6] that $C_*(\tilde{X})$ has already been replaced by a homotopy equivalent finitely generated projective complex. But (as is there observed) the generalised Euler characteristic is an invariant of chain homotopy type. The result now follows immediately:

$$\begin{aligned}\sigma(X) &= \sigma(C_*) = \sigma\big(\mathrm{Hom}_\Lambda(C_*, \Lambda)\big)\\ &= \textstyle\sum_{n-N}^{n}(-1)^i\{\mathrm{Hom}_\Lambda(P_{n-i}, \Lambda)\}\\ &= \textstyle\sum_{n-N}^{n}(-1)^i\{P_{n-i}\}^* = \sum_0^N(-1)^{n-j}\{P_j\}^* = (-1)^n\big(\sigma(X)\big)^* .\end{aligned}$$

It is interesting to compare (1.3) with Milnor's duality theorem for Reidemeister torsion [16]. The two results correspond in the analogy between the functors K^0 and K^1; however, the result concerning K^0 appears to be much weaker.

Note that the statement and proof of (1.3) are valid when X is disconnected, e.g., when X is the second member of a Poincaré pair. We now relativise our result. Let (Y, X) be a [connected] Poincaré pair, Λ_y the integral group ring of $\pi_1(Y)$, and Λ_x the corresponding ring for X (maybe summed over the components X_i). The inclusion $X \subset Y$ induces homomorphisms $\Lambda_{x_i} \to \Lambda_y$, hence $K^0(\Lambda_{x_i}) \to K^0(\Lambda_y)$, and hence $i_*: K^0(\Lambda_x) = \bigoplus_i K^0(\Lambda_{x_i}) \to K^0(\Lambda_y)$.

THEOREM 1.4. *Suppose (Y, X) a connected Poincaré pair of formal dimension $(n+1)$. Then $\sigma(Y) = i_*\sigma(X) + (-1)^{n+1}\big(\sigma(Y)\big)^*$.*

Note that since i_* commutes with the involution $*$, if we apply $*$ to the above equation, and subtract, the result follows from (1.3) by applying i_*.

PROOF. The argument already used provides us with a chain homotopy equivalence of degree $(n+1)$

$$[Y]\frown: \mathrm{Hom}_\Lambda\big(C_*(Y), \Lambda\big) \longrightarrow C_*(Y, X; \Lambda) .$$

We replace all chain complexes by homotopy equivalent finitely generated projective ones. Arguing as for (1.3), we see that the Euler characteristic of the above complexes is $(-1)^{n+1}\big(\sigma(Y)\big)^*$.

Now the homomorphism $K^0(\Lambda_{x_i}) \to K^0(\Lambda_y)$ is induced by tensor products: $P \to \Lambda_y \otimes_{\Lambda_{x_i}} P$, and $i_*\big(\sigma(X_i)\big)$ is the Euler characteristic of $C_*(X_i; \Lambda)$, and $i_*\sigma(X)$ of $C_*(X; \Lambda)$. Now we have the short exact sequence

$$0 \longrightarrow C_*(X; \Lambda) \longrightarrow C_*(Y; \Lambda) \longrightarrow C_*(Y, X; \Lambda) \longrightarrow 0 ;$$

using additivity of Euler characteristics (the sequence splits since all modules are projective), the result follows.

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It is not the case that the image of $\sigma(X)$ in the projective class group necessarily vanishes for Poincaré complexes X. The following is a fairly comprehensive counter-example (but can be somewhat improved). I am indebted to J. Milnor for an improvement on my original proof.

THEOREM 1.5. *Given $n \geqq 4$, a finite* CW*-complex K, and $\kappa \in K^0(\pi_1(K))$, there exists a Poincaré complex X^n, having the same $(\frac{1}{2}n - 1)$-type as K, and $\sigma(X) \equiv \kappa + (-1)^n\kappa^*$ (modulo free modules).*

PROOF. Let $n = 2k$ or $2k + 1$. We first replace K by its k-skeleton, embed this (or a homotopy equivalent finite simplicial complex) in $\mathbf{R}^{n+1}$, and take the boundary of a smooth regular neighbourhood. This is a smooth n-manifold N^n (hence Poincaré complex), with the same $(k - 1)$-type as K.

Let P be a projective module over the integral group ring of $\pi_1(K) = \pi_1(N) = \pi$, say, representing $(-1)^k \kappa$ (modulo free modules). Let F be a free module of the form $F = P \oplus Q$; let F have rank r. Form the connected sum of N with r copies of $S^k \times S^k$ or $S^k \times S^{k+1}$. The effect on $C_*(N)$ is to add F to C_k and a further summand F', say, to C_k or to C_{k+1}. We now intend to add further cells so as to reduce this F to P and F' to $\mathrm{Hom}_\Lambda(P, \Lambda)$. If we can do this so as to preserve Poincaré duality, then we are done; the resulting complex satisfies all our requirements.

Now cap product with the fundamental homology class of our manifold induces isomorphisms of $\mathrm{Hom}_\Lambda(F', \Lambda)$ on F and of $\mathrm{Hom}_\Lambda(F, \Lambda)$ on F'. Thus if we *kill* the summands Q of F and $\mathrm{Hom}_\Lambda(Q, \Lambda)$ of F', duality will hold as required.

The argument concludes just as in the proof of [24, Th. F]. The only point that perhaps needs comment is that, in our previous paper, the spheres to be killed were attached as a wedge. Here the situation is essentially the same, only an n-cell is attached to our spheres. This does not affect the argument.

We now consider the above result. Of course, any σ of the form $\kappa + (-1)^n\kappa^*$ satisfies $(-1)^n\sigma^* = \sigma$. However, the converse is not the case, and we shall see in (5.4.2) that σ need not be of the form $\kappa + (-1)^n\kappa^*$. The best result is not yet known.

As regards the *modulo free modules* condition, we write $K^0(\Lambda) = \mathbf{Z} \oplus \tilde{K}^0(\Lambda)$ in the usual way. Then $*$ preserves the split and induces the identity on the first component. Thus if n is odd, only elements of $\tilde{K}^0(\Lambda)$ need be considered anyway. If $n = 2k$, we can add $(-1)^k \cdot 2 \in \mathbf{Z}$ to the class of σ by taking the connected sum of the manifold with an extra $S^k \times S^k$. Using $S^{k-1} \times S^{k+1}$ will subtract this, but will of course alter the $(k - 1)$-type.

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CHAPTER 2. Geometry and cell decompositions

We first give an elementary result which is an analogue to *cutting and gluing* arguments on manifolds. Next we shall discuss dual cell decompositions, and obtain some more exact results on the decompositions possible, particularly in top dimensions. This leads on to a study of the connected sum operation. Finally, we shall discuss product decompositions.

For the first result, suppose $Y \cup Y' = Z$, $Y \cap Y' = X$ are four CW-complexes dominated by finite complexes, with Y and Y' connected, and let $w\colon \pi_1(Z) \to \{\pm 1\}$ define twistings for all four. Let $[Z] \in H^t_m(Z; \mathbf{Z})$ have image $[Y] + [Y']$ in $H^t_m(Z, X; \mathbf{Z}) \cong H^t_m(Y, X; \mathbf{Z}) \oplus H^t_m(Y', X; \mathbf{Z})$.

THEOREM 2.1. (i) *If (Y, X) and $(Y' X)$ are Poincaré pairs with fundamental classes $[Y]$ and $[Y']$, Z is a Poincaré complex with fundamental class $[Z]$.*

(ii) *If Z is a Poincaré complex with fundamental class $[Z]$, and (Y, X) a Poincaré pair with fundamental class $[Y]$, and if every coefficient bundle over Y' extends over Z, then (Y', X) is a Poincaré pair with fundamental class $[Y']$.*

PROOF. Use the exact commutative diagram (due to Browder (Cap products and Poincaré duality), mimeographed, Cambridge University, 1964).

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^q(Y, X) & \longrightarrow & H^q(Z) & \longrightarrow & H^q(Y') & \longrightarrow H^{q+1}(Y, X) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle [Y]\frown} & & \downarrow{\scriptstyle [Z]\frown} & & \downarrow{\scriptstyle [Y']\frown} & \downarrow{\scriptstyle [Y]\frown} & \\
\cdots \longrightarrow & H_{m-q}(Y) & \longrightarrow & H_{m-q}(Z) & \longrightarrow & H_{m-q}(Y', X) & \longrightarrow H_{m-q-1}(Y) & \longrightarrow \cdots .
\end{array}$$

If two out of three vertical maps are isomorphisms, then so (by the Five Lemma) is the third. We have omitted the coefficient bundle, but that this can be arbitrary is evident in (i), and true by hypothesis in (ii).

REMARK. As it is sufficient to prove the case when the group ring is used to define the coefficient bundle, the extra hypothesis in (ii) can be weakened to:

The right action of $\pi_1(Y')$ on its integer group ring can be extended to an action of $\pi_1(Z)$.

It is sufficient, for example, that $\pi_1(Y')$ be a retract of $\pi_1(Z)$, but this is not necessary; another example is with $\pi_1(Z)$ the symmetric group on 3 symbols, and $\pi_1(Y')$ the alternating group.

There are several variants of the above theorem. For example, we can extend it to Poincaré triads as follows.

THEOREM 2.1, ADDENDUM. *If $(Y_0; X_0, X_1)$ and $(Y_1; X_1, X_2)$ are Poincaré triads, with $Y_0 \cap Y_1 = X_1$, then $(Y_0 \cup Y_1; X_0, X_2)$ is a Poincaré triad.*

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The proof of this involves no new arguments. We leave it (and any other case we may need) to the reader.

The next result is a consequence of [25, Th. 5].

THEOREM 2.2. *Suppose X a Poincaré complex of formal dimension $n \neq 2$. Then X is homotopy equivalent to an n-dimensional complex, which can be chosen finite if $\sigma(X) \in \mathbf{Z}$. Suppose (Y, X) a connected Poincaré pair of formal dimension $(n+1) \geqq 4$, with $X \neq \emptyset$. Then Y has the homotopy type of an n-dimensional complex.*

PROOF. By hypothesis, X (resp. Y) is dominated by a finite CW-complex, so by [24, Th. E] we can suppose it of finite dimension. Now, by duality, we have

$$H^i(X; \Lambda) \cong H^t_{-(i-n)}(X; \Lambda) = 0 \qquad \text{if } i > n ,$$
$$H^i(Y; \Lambda) \cong H^t_{n+1-i}(Y, X; \Lambda) = 0 \qquad \text{if } i > n+1 ,$$

and in the second case $i = n+1$ gives $H^t_0(Y, X; \Lambda)$ which vanishes since Y is connected and X non-empty. The result now follows (if $n \geqq 3$) by appealing to [25, Cor. 5.1]. The case $n \leqq 1$ will be dealt with in Chapter 4.

We next seek an analogue of the dual cell decompositions of combinatorial manifolds which provided the original proof of the Poincaré duality theorem. Let X be a Poincaré n-complex (from now on we shall often not use the term *formal dimension*) with chain complex C_*. This is, we know, chain homotopy-equivalent to $\mathrm{Hom}_\Lambda(C_*, \Lambda)$ shifted by n-dimensions; call the latter complex D_*. We seek a cell decomposition of a space homotopy equivalent to X, with chain complex D_*. Such decompositions are provided by [25, Th. 4].

THEOREM 2.3. *Let X be a Poincaré n-complex, of dimension $n \geqq 3$, C the chain complex of $\tilde{X}$, and $\hat{C}$ the dual chain complex. Then we can find Y, homotopy equivalent to X, such that $C(\tilde{Y})$ is the same as $\hat{C}$ in dimensions >3; also in dimension 3, modulo a subcomplex Y_0 satisfying* (D2).

We have already discussed this application. Now for some special cases.

COROLLARY 2.3.1. *With the hypotheses above, we can suppose for $n > 3$ that Y has only one n-cell, and for $n = 3$ that Y is obtained by attaching a 3-cell to a complex satisfying* (D2).

For as X is connected, we can always suppose that X has only one 0-cell.

COROLLARY 2.3.2. *If $n \geqq 4$, we can suppose that Y is obtained from Z (of dimension $n-2$, or satisfying* (D2)*) by attaching along its boundary a smooth manifold H, obtained from D^n by adding 1-handles.*

PROOF. The 1-skeleton of X consists of a collection of loops, each with

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both ends at the single vertex. Thus each $(n-1)$-cell of Y is incident just twice with the n-cell. We can normalize the attaching map of S^{n-1} (for the n-cell) so that a collection of disjoint $(n-1)$-discs are mapped on the $(n-1)$-cells (two on each); the remainder into the $(n-2)$-skeleton. The complex Y is then as described above.

This result is of crucial importance for performing low dimensional surgery on Poincaré complexes. We can improve it somewhat as follows. First, replace Z by the mapping cylinder Z' of the attaching map $\partial H \to Z$. This has the same homotopy type (thus still satisfies $D(n-2)$). We now have $Y = Z' \cup H$, $\partial H = Z' \cap H$. As Z' contains the $(n-2) \geqq 2$-skeleton of Y, we have $\pi_1(Z') = \pi_1(Y)$. It follows at once from Theorem 2.1 (ii) that $(Z', \partial H)$ is a Poincaré pair. Now note that, as $n \geqq 4$ and H is a handlebody, $\pi_1(\partial H) \cong \pi_1(H)$. The following additional result is less obvious, but very useful.

ADDENDUM 2.3.3. *The map $\pi_1(H) \to \pi_1(Y)$ induced by inclusion is surjective.*

PROOF. We use the construction of Z and H *via* dual cell decomposition. In the original complex, there was one 0-cell e_0, and the 1-cells e_1^i satisfying $\partial e_1^i = g_i e_0 - e_0$, where the elements g_i of $\pi_1(Y)$ generate it (see e.g., [25, discussion preceding Th. 5]). In the dual complex, then, we have one n-cell e_n and $(n-1)$-cells e_{n-1}^i with $\partial e_n = \sum_i (\pm g_i e_{n-1}^i - e_{n-1}^i)$, where the sign is $w(g_i)$. It follows that the loop round the i^{th} handle of the handlebody H has homotopy class g_i. These generate $\pi_1(Y)$, so $\pi_1(H) \to \pi_1(Y)$ is surjective as asserted.

It seems that X is also *manifold-like* in codimension 2, but not in codimension 3. We will return to this below when we come to consider tangential properties.

It is also interesting to observe that as we can always suppose X finite except in dimensions $(n-1)$ and n, in (2.3) Y mod Y_0 is finite, but Y_0 may need an infinite number of cells added to X^2.

Corollary (2.3.1) can be sharpened, to give a sort of *disc theorem* for Poincaré complexes.

THEOREM 2.4. *Let X be a Poincaré complex,* $\dim[X] = n \geqq 3$. *Then there exists a complex K satisfying $D(n-1)$, a map $f: S^{n-1} \to K$,* and a *homotopy equivalence $Y = K \cup_f e^n \to X$. The pair (K, f) is unique up to homotopy and orientation. If we replace f by an inclusion, (K, S^{n-1}) is a Poincaré pair.*

PROOF. The first sentence follows from (2.3.1). If f is an inclusion, we have $Y = K \cup D^n$, $S^{n-1} = K \cap D^n$, and (2.1) (ii) applies to show that (K, S^{n-1}) is a Poincaré pair.

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It remains to prove uniqueness. Suppose $h: K_1 \cup_{f_1} e^n \to K_2 \cup_{f_2} e^n$ a homotopy equivalence between two such, where each K_i satisfies $D(n-1)$. We suppose h a cellular map; write $X_i = K_i \cup_{f_i} e^n$. Then we have a diagrum

$$\begin{array}{ccccccc} H^{n-1}(K_2) & \longrightarrow & H^n(X_2, K_2) & \longrightarrow & H^n(X_2) & \longrightarrow & H^n(K_2) = 0 \\ \downarrow h^* & & \downarrow h^* & & \downarrow h^* & & \\ H^{n-1}(K_1) & \xrightarrow{\delta^*} & H^n(X_1, K_1) & \longrightarrow & H^n(X_1) & \longrightarrow & H^n(K_1) = 0 \end{array}$$

(with coefficients Λ throughout), and $H^n(X_i; \Lambda) \cong H_0^t(X_i; \Lambda) = H_0(X_i; \Lambda) \cong \mathbf{Z}$, $H^n(X_i, K_i; \Lambda) \cong \Lambda$. The induced map of $\mathbf{Z}$ can only be the identity up to sign. Since $H^n(X_2, K_2)$ is free, there is a map

$$\delta: H^n(X_2, K_2) \longrightarrow H^{n-1}(K_1) ,$$

such that $(\delta^*\delta + h^*): H^n(X_2, K_2) \to H^n(X_1, K_1)$ induces the identity map of Λ. The image of $1 \in \Lambda$ defines a cohomology class in K_1; pick a representative cocycle $z: C_{n-1}(K_1) \to \Lambda$.

Now we use z to define a homotopy of h. We keep the $(n-2)$-skeleton of K_1 fixed, and move each $(n-1)$-cell θ by a cellular homotopy defining the chain $\{z(\theta)\}e^n$. (Such a homotopy is easy to construct, starting from the homotopy represented by a homeomorphism $I^{n-1} \times I \to I^n$.) Extend the homotopy over K_1. Now the chain map of the homotopy is a chain homotopy between the initial and final induced chain maps. It follows from the choice of z that the final map $H^n(X_2, K_2) \to H^n(X_1, K_1)$ corresponds to the identity map of Λ. Thus the induced map of the dual module

$$\pi_n(X_1, K_1) \cong H_n(X_1, K_1; \Lambda) \longrightarrow H_n(X_2, K_2; \Lambda) \cong \pi_n(X_2, K_2)$$

is also the identity. Hence the image of the homotopy class of f_1 is that of f_2. So we have a map of Poincaré pairs $(K_1, f_1 S^{n-1}) \to (K_2, f_2 S^{n-1})$, which has degree 1 since the original map had. Our map is evidently a homotopy equivalence in dimensions $\leqq (n-2)$. It follows that we have a homotopy equivalence of K_1 on K_2. For by Whitehead's theorem, it suffices to check that the induced maps $H^r(K_1; \Lambda) \leftarrow H^r(K_2; \Lambda)$ are isomorphisms. We have already proved this for $r \neq n-1$, and since our map has degree 1, the remaining case follows from the commutative diagram

$$\begin{array}{ccc} H^{n-1}(K_1; \Lambda) & \longleftarrow & H^{n-1}(K_2; \Lambda) \\ \downarrow \cong & & \downarrow \cong \\ H_1^t(K_1, f_1 S^{n-1}; \Lambda) & \cong & H_1^t(K_2, f_2 S^{n-1}; \Lambda) . \end{array}$$

We can now define the *connected sum* of Poincaré complexes of dimension $\geqq 3$; write $X_1 = K_1 \cup_{f_1} e^n$, $X_2 = K_2 \cup_{f_2} e^n$, and define $X_1 \# X_2 = (K_1 \vee K_2) \cup_g e^n$,

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where the homotopy class of g is the sum of those of f_1 and f_2.

COROLLARY 2.4.1. *The operation $\sharp$ is well determined on homotopy types (with the usual reservations about orientation).*

Note that the uniqueness of (2.4) includes a statement that changing the element of $\pi_{n-1}(K)$ by an operation of an element of $\pi_1(K)$ gives a result equivalent to the first by a base-point-preserving equivalence.

Of course, $\sharp$ is a special case of surgery; other cases will be considered below.

THEOREM 2.5. (i) *If (Y_1, X_1) and (Y_2, X_2) are Poincaré pairs, then so is $(Y_1 \times Y_2, X_1 \times Y_2 \cup Y_1 \times X_2)$.*

(ii) *If $X = A \times B$ is a Poincaré complex, then so are A and B.*

PROOF. We have $C_*(Y_1 \times Y_2) = C_*(\tilde{Y}_1) \otimes_{\mathbf{Z}} C_*(\tilde{Y}_2)$, and similarly in the relative case. Set $[Y_1 \times Y_2] = [Y_1] \otimes [Y_2]$. The result (i) is now trivial.

For (ii) we may suppose X connected. We have

$$\pi_1(A) \times \pi_1(B) = \pi_1(X) \xrightarrow{w} \{\pm 1\}$$

defining twistings on X, A, and B which are compatible for what follows. Let n be the formal dimension of X. By the Künneth theorem (over $\mathbf{Z}$) we have a split short exact sequence

$$\begin{aligned} 0 \longrightarrow \textstyle\sum_{r+s=n} H_r^t(A; \mathbf{Z}) \otimes_{\mathbf{Z}} H_s^t(B; \mathbf{Z}) &\longrightarrow H_n^t(X; \mathbf{Z}) \\ \longrightarrow \textstyle\sum_{i+j=n-1} \operatorname{Tor}_1^{\mathbf{Z}}\left(H_i^t(A; \mathbf{Z}), H_j^t(B; \mathbf{Z})\right) &\longrightarrow 0 \ . \end{aligned} \tag{2.6}$$

But the central term is infinite cyclic, generated by $[X]$. Since an infinite cyclic group is torsion-free and indecomposable, there is a unique pair of values (r, s), with $r + s = n$, such that

$$H_r^t(A; \mathbf{Z}) \otimes_{\mathbf{Z}} H_s^t(B; \mathbf{Z}) \cong \mathbf{Z} \ .$$

Thus $H_s^t(B; \mathbf{Z})$ has torsion-free rank 1, and torsion-free quotient isomorphic to $\mathbf{Z}$, hence also a direct summand isomorphic to $\mathbf{Z}$. So $H_r^t(A; \mathbf{Z})$ is a direct summand of $H_r^t(A; \mathbf{Z}) \otimes_{\mathbf{Z}} H_s^t(B; \mathbf{Z}) \cong \mathbf{Z}$, hence is infinite cyclic. Now, by the Künneth theorem again, since $H_{r+j}^t(X; \mathbf{Z}) = 0$ for $r + j > n$, we deduce

$$0 = H_r^t(A; \mathbf{Z}) \otimes_{\mathbf{Z}} H_j^t(B; \mathbf{Z}) \cong H_j^t(B; \mathbf{Z}) \qquad \text{for } j > s \ .$$

The same considerations apply with A and B interchanged. But now the last term in (2.6) vanishes, since if $H_i^t(A; \mathbf{Z})$ and $H_j^t(B; \mathbf{Z})$ are both non-zero, one of them is isomorphic to $\mathbf{Z}$. Thus (2.6) reduces to an isomorphism

$$H_r^t(A; \mathbf{Z}) \otimes_{\mathbf{Z}} H_s^t(B; \mathbf{Z}) \cong H_n^t(X; \mathbf{Z}) \ .$$

We choose $[A]$ and $[B]$ so that $[A] \otimes [B] = [X]$.

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Write Λ_1, Λ_2 for the integral group rings of the fundamental groups of A and B. Then $\Lambda_1 \otimes_{\mathbf{Z}} \Lambda_2 = \Lambda$, and similarly for the chain and cochain complexes. Now consider

$$\begin{array}{ccccc} H^{r-i}(A;\Lambda_1) \otimes_{\mathbf{Z}} H^{s-j}(B;\Lambda_2) & \longrightarrow & H^{n-i-j}(X;\Lambda) \\ \downarrow [A]\frown & & \downarrow [B]\frown & & \downarrow [X]\frown \\ H_i^t(A;\Lambda_1) \otimes_{\mathbf{Z}} H_j^t(B;\Lambda_2) & \longrightarrow & H_{i+j}^t(X;\Lambda) \end{array} \tag{2.7}$$

which commutes since $[A] \otimes [B] = [X]$. Also, the right hand map is an isomorphism. We wish to deduce that the left hand maps are isomorphisms. Now if $i < 0$ or $j < 0$, the lower left term vanishes, hence also the map across the top. But the Künneth theorem gives a split short exact sequence

$$\begin{aligned} 0 \longrightarrow \textstyle\sum_{i+j=n} H^i(A;\Lambda_1) \otimes_{\mathbf{Z}} H^j(B;\Lambda_2) &\longrightarrow H^n(X;\Lambda) \\ \longrightarrow \textstyle\sum_{i+j=n+1} \operatorname{Tor}_1^{\mathbf{Z}}\left(H^i(A;\Lambda_1),\ H^j(B;\Lambda_2)\right) &\longrightarrow 0\ . \end{aligned}$$

The central term being infinite cyclic, we deduce that $H^r(A;\Lambda_1) \otimes_{\mathbf{Z}} H^s(B;\Lambda_2)$ also is, and hence (as above) that $H^r(A;\Lambda_1)$ and $H^s(B;\Lambda_2)$ are infinite cyclic, $H^i(A;\Lambda_1)$ vanishes for $i > r$, $H^j(B;\Lambda_2)$ vanishes for $j > s$, and, finally, that we have an isomorphism

$$H^r(A;\Lambda_1) \otimes H^s(B;\Lambda_2) \longrightarrow H^n(X;\Lambda)\ .$$

Now take tensor products over $\mathbf{Z}$ with a field k. Then the Künneth formula gives isomorphisms

$$\begin{aligned} \textstyle\sum_{i+j=h} H^i(A;\Lambda_1 \otimes_{\mathbf{Z}} k) \otimes_k H^j(B;\Lambda_2 \otimes_{\mathbf{Z}} k) &\xrightarrow{\cong} H^h(X;\Lambda \otimes_{\mathbf{Z}} k)\ , \\ \textstyle\sum_{i+j=h} H_i^t(A;\Lambda_1 \otimes_{\mathbf{Z}} k) \otimes_k H_j^t(B;\Lambda_2 \otimes_{\mathbf{Z}} k) &\xrightarrow{\cong} H_h^t(X;\Lambda \otimes_{\mathbf{Z}} k)\ . \end{aligned}$$

Using the commutativity of (2.7), we infer that the maps

$$H^i(A;\Lambda_1 \otimes_{\mathbf{Z}} k) \otimes_k H^j(B;\Lambda_2 \otimes_{\mathbf{Z}} k) \xrightarrow{[A]\frown \otimes [B]\frown} H_{r-i}^t(A;\Lambda_1 \otimes_{\mathbf{Z}} k) \otimes_k H_{s-j}^t(B;\Lambda_2 \otimes_{\mathbf{Z}} k)$$

are all isomorphisms. Now in the case $i = 0$, we have $H_0^t(A;\Lambda_1) \cong \mathbf{Z}$; we have seen that $H^r(A;\Lambda_1) \cong \mathbf{Z}$, and that $[A]$ induces an isomorphism. Tensoring with k we still have an isomorphism, as both groups are isomorphic to k. Substituting in the above, we obtain isomorphisms

$$H^j(B;\Lambda_2 \otimes_{\mathbf{Z}} k) \longrightarrow H_{s-j}^t(B;\Lambda_2 \otimes_{\mathbf{Z}} k)$$

for all j and fields k; similarly if A and B are interchanged.

Now write K for the cokernel of the map

$$H^i(A;\Lambda_1) \xrightarrow{[A]\frown} H_{r-i}^t(A;\Lambda_1)\ . \tag{2.8}$$

For p prime, consider the commutative exact diagram

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$$\begin{array}{ccccccccc} H^{i-1}(A;\Lambda_1\otimes_{\mathbf{Z}}\mathbf{Z}_p) & \longrightarrow & H^i(A;\Lambda_1) & \xrightarrow{p} & H^i(A;\Lambda_1) & \longrightarrow & H^i(A;\Lambda_1\otimes_{\mathbf{Z}}\mathbf{Z}_p) & \longrightarrow & H^{i+1}(A;\Lambda_1) \\ \downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong & & \downarrow \\ H_{r-i+1}(A;\Lambda_1\otimes_{\mathbf{Z}}\mathbf{Z}_p) & \longrightarrow & H_{r-i}(A;\Lambda_1) & \xrightarrow{p} & H_{r-i}(A;\Lambda_1) & \longrightarrow & H_{r-i}(A;\Lambda_1\otimes_{\mathbf{Z}}\mathbf{Z}_p) & \longrightarrow & H_{r-i-1}(A;\Lambda_1) \end{array}$$

where the vertical maps are induced by cap product with $[A]$. A short diagram chase shows that the induced map $K \xrightarrow{p} K$ is monomorphic. Since this holds for all primes p, K is torsion-free. However, we now take $k = \mathbf{Q}$ in the above, and deduce that (2.8) is an isomorphism modulo torsion groups, hence that K is a torsion group. Thus K vanishes, and (2.8) is epimorphic.

Now consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sum_{i+j=k} H^i(A;\Lambda_1)\otimes_{\mathbf{Z}} H^j(B;\Lambda_2) & \longrightarrow & H^k(X;\Lambda) & \longrightarrow & \sum_{i+j=k-1}\operatorname{Tor}_1^{\mathbf{Z}}\left(H^i(A;\Lambda_1),\, H^j(B;\Lambda_2)\right) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle [A]\frown\otimes[B]\frown} & & \downarrow{\scriptstyle [X]\frown} & & \downarrow{\scriptstyle \operatorname{Tor}_1^{\mathbf{Z}}([A]\frown,\,[B]\frown)} & & \\ 0 & \longrightarrow & \sum H^i_{r-i}(A;\Lambda_1)\otimes_{\mathbf{Z}} H^t_{s-j}(B;\Lambda_2) & \longrightarrow & H^t_{n-k}(X;\Lambda) & \longrightarrow & \sum \operatorname{Tor}_1^{\mathbf{Z}}\left(H^t_{r-i}(A;\Lambda_1),\, H^t_{s-j}(B;\Lambda_2)\right) & \longrightarrow & 0 \end{array}$$

in which $[X]\frown$ is an isomorphism, and we have just proved $[A]\frown\otimes[B]\frown$ surjective. It follows that $[A]\frown\otimes[B]\frown$ is bijective, hence so is each component map

$$H^i(A;\Lambda_1)\otimes_{\mathbf{Z}} H^j(B;\Lambda_2) \xrightarrow{[A]\frown\otimes[B]\frown} H^t_{r-i}(A;\Lambda_1)\otimes_{\mathbf{Z}} H^t_{s-j}(B;\Lambda_2)\ .$$

Taking $i = r$, and using the known isomorphism

$$H^r(A;\Lambda_1) \cong \mathbf{Z} \xrightarrow{[A]\frown} H^t_0(A;\Lambda_1) \cong \mathbf{Z}\ ,$$

we deduce that the map

$$H^j(B;\Lambda_2) \xrightarrow{[B]\frown} H^t_{s-j}(B;\Lambda_2)$$

is an isomorphism for all j. Thus B is a Poincaré complex; similarly for A. This completes the proof.

CHAPTER 3. Tangential properties

Since we are effectively only considering spaces up to homotopy equivalence, the reader may well wonder how there can possibly be any tangential properties to consider. We must admit that we use (or rather, abuse) the word for properties, due mostly to M. Spivak (Princeton thesis, 1964),* which bear a close formal relation to tangential properties of smooth manifolds, but whose genesis is somewhat different.

The vector bundles arising in the study of smooth manifolds are replaced here by spherical fibrations. We will begin with a general discussion of these. A map $\pi: E \to B$ will be called an $(n-1)$-spherical fibration if it is a fibration in the sense of Dold [3] (i.e., satisfies his W.C.H.P.), and if the fibres are

* Added in proof: See Topology, 6 (1967), pp. 77-102.

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homotopy equivalent to S^{n-1}. By a result of Stasheff [21], these fibrations have a structural monoid, the monoid G_n of all self-homotopy equivalences of S^{n-1} (i.e., maps of degree ± 1; we will write SG_n for the set of those of degree $+1$). Moreover, there exists a classifying space BG_n, so that G_n is homotopy equivalent (as H-space) to ΩBG_n, and such that spherical fibrations over B are classified by maps $B \to BG_n$, at least if B has the homotopy type of a CW-complex; or, more generally, if B has a numerable covering over each set of which π is fibre homotopically equivalent to the projection of a product.

It is convenient to consider, in conjunction with the G_n, the submonoids F_n of G_{n+1} consisting of base-point preserving maps. Clearly, F_n can be identified with the loop space $\Omega^n S^n$ (the multiplication, however, is inequivalent to loop multiplication, see [17]). We have $F_n \subset G_{n+1}$, and suspension (or reduced suspension) gives an inclusion $G_n \subset F_n$, both inclusions being H-maps. It is easy to deduce from the definitions that there is a fibration $G_{n+1} \to S^n$ with fibre F_n, so $\pi_r(G_{n+1}, F_n) \cong \pi_r(S^n)$. The homotopy properties of the other inclusion are less perspicuous, but it is known (James [9]) that (F_n, G_n) is $(2n-4)$-connected. All we shall use of this is the deduction that, if $n \geqq \dim B$, the suspension map from (fibre homotopy equivalence classes of) $(n-1)$-spherical fibrations over B to n-spherical fibrations is surjective; if $n > \dim B$, it is bijective. We will write G for the limit of the G_n under inclusion.

There is also a construction for the join of two fibrations $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$. Let E be the subspace of $E_1 * E_2$ consisting of segments whose ends both lie over the same point of B, $\pi : E \to B$ the projection. Then if π_1 (resp. π_2) is $(m-1)$-(resp. $(n-1)$-) spherical, π is known to be an $(m+n-1)$-spherical fibration, which we call the sum of the others. *Suspension* consists in adding the trivial 0-spherical fibration. Addition is commutative and associative. If we write BG for the limit of the BG_n, BG acquires the structure of homotopy-commutative and homotopy-associative H-space [17]. Using these notions instead of vector bundles, we now construct the universal group of the semi-group of fibre homotopy equivalence classes of spherical fibrations over B under addition, and denote it by $KG(B)$. The usual arguments show that $KG(B)$ splits as $\mathbf{Z} \oplus \tilde{K}G(B)$, and that if $n > \dim B$, each element of $\tilde{K}G(B)$ is represented by an $(n-1)$-spherical fibration over B, unique up to fibre homotopy equivalence. We will call such fibrations stable.

Let $\pi : E \to B$ be an $(n-1)$-spherical fibration over a CW-complex B. The mapping cone of π, $B \cup_\pi CE$ is also called the *Thom complex* of π, and denoted B^π.

LEMMA 3.1. *B^π is homotopy equivalent to a complex which, apart from a*

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base point, has just one cell (of dimension $n+r$) for each cell (of dimension r) of B.

PROOF. We will give the proof when B is finite, by induction on the number of cells of B. The infinite case then follows by taking a direct limit over finite subcomplexes.

Suppose then $B = C \cup e^r$, and the result established for C. (The induction basis: B a point p is trivial, for as $E \simeq S^{n-1}$, we have $B^\pi \simeq S^n$). The fibration is fibre homotopically trivial over a contractible set, so E is homotopy equivalent to $\pi^{-1}(C)$, with $D^r \times S^{n-1}$ attached along $S^{r-1} \times S^{n-1}$, and we can choose the equivalence to preserve fibres. This gives a homotopy equivalence of B^π on C^π, with $D^r \times S^n$ attached along $S^{r-1} \times S^n$, and $D^r \times \{1\}$ shrunk to a point. The result now follows.

Next assume B of dimension b, and with only one 0-cell and b-cell. Then B^π has dimension $(b+n)$; the 0-cell of B gives an inclusion $i\colon S^n \subset B^\pi$, and shrinking all but the top cell to a point a projection $j\colon B^\pi \to S^{b+n}$. We call B^π *reducible* if j has a right (homotopy) inverse, *coreducible* if i has a left inverse. In fact, since i resp. j is induced by the inclusion of the base point in B, resp. the shrinking map $B \to S^b$, we do not need such stringent hypotheses to define these terms, which make sense (in particular) if B is a Poincaré complex.

The relevance of spherical fibrations to the study of Poincaré complexes lies in the following result, due to M. Spivak.

THEOREM 3.2. *Let M^m be a Poincaré complex. Then there exists an element ν of $\tilde{K}G(M)$ such that a corresponding fibration over M has reducible Thom complex M^ν. Similarly, if $(M, \partial M)$ is a Poincaré pair, there is a $\nu \in \tilde{K}G(M)$ such that $M^\nu/(\partial M)^\nu$ is reducible; $(\partial M)^\nu$ is then also reducible.*

Spivak's proof of this result assumes that M is finite. The general case follows, however, on noting that, by a result of M. Mather [29], $M \times S^1$ is homotopy equivalent to a finite complex N, also a Poincaré complex. Then there exists a spherical fibration ν', say, over $M \times S^1$ with reducible Thom space. Let ν be the induced fibration over $M \times \{1\}$. If the fibres over $M \times \{-1\}$ are shrunk to a point in $(M \times S^1)^{\nu'}$, we obtain the suspension of M^ν (cf. proof of (3.7) below). Hence $M^{\nu+\varepsilon}$ is reducible. An analogous argument is valid in the case of Poincaré pairs.

The uniqueness is also due to Spivak, but we will repeat the proof, since it is closely related to some of the arguments to appear later. We shall only be concerned with stable properties for a while, so will omit reference to the fibre dimension of spherical fibrations, and will even confound these with elements of $\tilde{K}G(M)$.

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Suppose ν as above, and let $f_!: S^{m+N} \to M^\nu$ be the homotopy right inverse to j. Such maps f are called *of degree* 1. Let $\alpha \in \tilde{K}G(M)$; consider the product fibration $\alpha \times (\nu - \alpha)$ over $M \times M$. The diagonal Δ induces ν over M. Now consider the map induced on Thom spaces by Δ. We have

$$M^\nu \xrightarrow{\Delta} (M \times M)^{\alpha \times (\nu - \alpha)} = M^\alpha \wedge M^{\nu - \alpha} .$$

Composing with f gives a map $F: S^{m+N} \to M^\alpha \wedge M^{\nu-\alpha}$.

Similarly, if $(M, \partial M)$ is a Poincaré pair, note that Δ above carries ∂M into $M \times \partial M$. We obtain eventually

$$F: S^{m+N} \longrightarrow M^\alpha \wedge M^{\nu-\alpha}/(\partial M)^{\nu-\alpha} .$$

THEOREM 3.3. *The maps F above are duality maps in the sense of Spanier* [20]. *Hence if M is a Poincaré complex and M^ν reducible, M^α and $M^{\nu-\alpha}$ are S-dual. Similarly if $(M, \partial M)$ is a Poincaré pair and $M^\nu/(\partial M)^\nu$ reducible, M^α is S-dual to $M^{\nu-\alpha}/(\partial M)^{\nu-\alpha}$ for $\alpha \in \tilde{K}G(M)$.*

PROOF. Let us first recall some homological algebra [2, IV. 6]. Given chain-complexes A, B there is an external product

$$\alpha' : H(\mathrm{Hom}\,(A, B)) \longrightarrow \mathrm{Hom}\,(H(A), H(B)) .$$

Suppose X and Y finite CW-complexes. We take

$$A = C^*(X; \mathbf{Z}) \qquad B = C_*(Y; \mathbf{Z}) .$$

(In any case we know that there exist equivalent chain complexes of finite type, and use these). Then $\mathrm{Hom}\,(A, B) = C_*(X \times Y; \mathbf{Z})$. Thus we have

$$\alpha': H_*(X \times Y; \mathbf{Z}) \longrightarrow \mathrm{Hom}\,(H^*(X; \mathbf{Z}), H_*(Y; \mathbf{Z})) .$$

Similarly with base points and reduced homology we have

$$\alpha': \tilde{H}_*(X \wedge Y; \mathbf{Z}) \longrightarrow \mathrm{Hom}\,(\tilde{H}^*(X; \mathbf{Z}), \tilde{H}_*(Y; \mathbf{Z})) .$$

Now according to Spanier, a map $F: S^N \to X \wedge Y$ determines an S-duality between X and Y if its homology class $[F]$ is such that

$$\alpha'[F] : \tilde{H}^r(X; \mathbf{Z}) \longrightarrow \tilde{H}_{N-r}(Y; \mathbf{Z})$$

is an isomorphism for all r.

We will now use the Thom isomorphism. Let $\mathbf{Z}^a$, $\mathbf{Z}^b$ and $\mathbf{Z}^a \otimes \mathbf{Z}^b = \mathbf{Z}^c$ be the twisted integer coefficient bundles determined by α, $\nu - \alpha$, and ν. The Thom isomorphism gives a diagram (when M is a Poincaré complex)

$$\begin{array}{ccc} H_*(M \times M; \mathbf{Z}^c) & \xrightarrow{\alpha'} & \mathrm{Hom}\,(H^*(M; \mathbf{Z}^a), H_*(M; \mathbf{Z}^b)) \\ \downarrow \Phi & & \downarrow \mathrm{Hom}\,(\Phi, \Phi) \\ \tilde{H}_*(M^\alpha \wedge M^{\nu-\alpha}; \mathbf{Z}) & \xrightarrow{\alpha'} & \mathrm{Hom}\,(\tilde{H}^*(M^\alpha; \mathbf{Z}), \tilde{H}_*(M^{\nu-\alpha}; \mathbf{Z})) . \end{array}$$

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The diagram is commutative since all maps are induced by cup products. Now by assumption, $\Phi^{-1}[F] = D$ is the homology class of the diagonal. But $\alpha'(D)$ is an isomorphism since M is a Poincaré complex. The first result follows. The proof of the second is precisely similar.

COROLLARY 3.4. *The ν in Theorem* 3.2 *is unique.*

PROOF. Suppose that α is such that $M^\alpha/(\partial M)^\alpha$ is reducible. By Theorem 3.3, $M^{\nu-\alpha}$ is S-dual to it, hence is co-reducible. Take a particular spherical fibration representing $\nu - \alpha$, and suspend each fibre to obtain E. Then $M^{\nu-\alpha}$ is obtained by identifying all the suspension points in E. Since $M^{\nu-\alpha}$ is co-reducible, E retracts onto a fibre. By a theorem of Dold [3] it is fibre homotopically trivial. Hence $\alpha = \nu$.

The above argument is essentially due to Atiyah [1]; it was adapted by Spivak to prove the above. The following result, however is (as far as I know) new.

THEOREM 3.5. *Let $(M, \partial M)$ be a Poincaré pair, ν a stable spherical fibration over M, $f\colon S^{m+N} \to M^\nu/(\partial M)^\nu$ of degree* 1. *Then $h \to h \circ f$ induces a bijective correspondence between the set of fibre homotopy classes of automorphisms of ν over the identity of M, and the set of elements of $\pi_{m+N}(M^\nu/(\partial M)^\nu)$ of degree* 1.

PROOF. We denote the set of homotopy classes of maps $A \to B$ by $[A\colon B]$, and of S-homotopy classes of S-maps by $\{A : B\}$. Then for any A,

$$[A : G] = \lim_{N\to\infty} [A : \Omega^N S^N] = \lim_{N\to\infty} [S^N A : S^N] = \{A : S^0\}$$

and by S-duality, $\{M^0 : S^0\} = \{S^{m+N} : M^\nu/(\partial M)^\nu\}$, where maps of degree 1 on the right correspond to maps of co-degree 1 on the left; i.e., to classes of maps $S^N M \vee S^N \to S^N$ which are the identity on S^N, hence (omitting S^N as irrelevant) to $[S^N M : S^N] = [M : G]$. Now by definition, $[M : G]$ determines the homotopy classes of fibre homotopy automorphisms of a trivial stable spherical fibration over M. It is now easy to see that the triviality condition is unnecessary. For if the fibration is stable, the automorphisms will (by stability) depend only on the stable equivalence class in $\tilde{K}G(M)$. Also, an inclusion as a Whitney summand induces (taking the sum with the identity) a map of automorphisms. The rest is formal: If $A(\eta)$ is the group of automorphism classes of η, inclusions as summands give maps (if $\eta \oplus \xi = \varepsilon^N$ is trivial)

$$A(\eta) \longrightarrow A(\varepsilon^N) \longrightarrow A(\varepsilon^N \oplus \eta) \longrightarrow A(\varepsilon^{2N}) \ .$$

The composite of two is bijective, hence so is the central map, and thus also the others.

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We have obtained a bijection; it remains to show that there is one induced by $h \to h \circ f$. We claim that the above is induced by $h \to h^{-1} \circ f$, which is good enough. For consider $\nu \times \varepsilon^N$ over $M \times M$, and subject the factors to h^{-1}, h respectively. The result gives a map over $\Delta(M)$ homotopic to the identity, and hence compatible with the fixed S-duality defined before. Thus $S^N M \xrightarrow{h} S^N M$ is S-dual to $M^\nu/(\partial M)^\nu \xrightarrow{h^{-1}} M^\nu/(\partial M)^\nu$. It is clear that the bijection of $[M : G]$ on $\{M : S^0\}$ is induced by composition with h; our claim (and the Theorem) now follows by S-duality.

COROLLARY 3.6. *Given a Poincaré pair $(M, \partial M)$, there exist a spherical fibration ν over M and a map $f: S^{m+N} \to M^\nu/(\partial M)^\nu$ of degree* 1. *(ν, f) is unique up to suspension and equivalence.*

The remainder of this chapter prepares for our work on highly connected Poincaré complexes (and pairs). For much of this, the hypothesis $M^m = SK \cup e^m$ is adequate. Note that this is automatically satisfied if M is $(r-1)$-connected, with $m < 3r$. We now study the homotopy type of the Thom space.

PROPOSITION 3.7. *Let α be a spherical fibration over SK, with fibre dimension r. Then the reduced Thom space $(SK)^\alpha/*^\alpha$ can be identified with $S^{r+1}K$. If $\chi: K \to G$ is the characteristic map of α, the corresponding S-map $K \to S^0$, or rather $S^r K \to S^r$, is the attaching map in $(SK)^\alpha$.*

PROOF. Write SK as the union of cones C_-K and C_+K. Then α is fibre homotopically trivial over each. The first assertion now follows from

$$(SK)^\alpha/*^\alpha \cong (SK)^\alpha/(C_-K)^\alpha = (C_+K)^\alpha/K^\alpha$$
$$\cong \frac{C_+K \times S^r}{C_+K \times *} \Big/ \frac{K \times S^r}{K \times *} = \frac{C_+K}{K} \wedge S^r = S^{r+1}K .$$

As to the second, we have $(C_-K)^\alpha = *^\alpha \cong S^r$; SK^α is obtained by attaching $(C_+K)^\alpha$ along K^α. We identify K^α with the suspension of $K \times S^{r-1}/(K \times *)$; then inclusions in $(C_-K)^\alpha$, $(C_+K)^\alpha$ correspond to projections $K \times S^{r-1} \to S^{r-1}$. These differ by a twisting of $K \times S^{r-1}$, which is defined precisely by χ. The corresponding S-map $S^r \times K \to S^r K \to S^r$ is obtained by precisely the construction above. The Proposition follows.

COROLLARY 3.8. *Let $M = SK \cup_f e^m$ be a Poincaré complex, with normal fibration ν. Then K is S-dual to itself, and the S-dual to f is associated to the characteristic map of $\nu \mid SK$.*

PROOF. M^0 is S-dual to M^ν. Removing the 0-cell and the m-cell, and applying the Proposition now gives our first assertion. The S-dual to f is the attaching map of the bottom cell of M^ν; the second assertion now also follows from (3.7).

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Remark. Corollary (3.8) clears up a point which has caused the author trouble for some time. On [23, p. 182] we refer to a result of Milnor [25]. In fact, the reference contains only the statement of the result, and refers to yet another paper (which was never published) for the proof. This proof, then, as also the generalised form of it used in [23], appears here for the first time. The connection between these two problems will be treated more fully in Chapter 6 below.

CHAPTER 4. Low dimensional classifications

We now start our series of classifications up to homotopy type of Poincaré complexes of various kinds. These are useful in acquiring familiarity, and also as sources of counter-examples. We will find several different reasons why Poincaré complexes differ from manifolds, for example.

It will be convenient to argue with homology and cohomology of the universal cover $\tilde{K}$ of K. Now we have $H_i(\tilde{K}; \mathbf{Z}) = H_i(K; \Lambda)$. But $H^i(K; \Lambda)$ is calculated with finite cochains: If K is compact, this gives the cohomology of $\tilde{K}$ with compact supports, $\mathcal{H}^i(\tilde{K}; \mathbf{Z})$. As we can always replace K by a homotopy equivalent complex with finite skeletons, and argue on these, it is safe for us to identify $H^i(K; \Lambda) = \mathcal{H}^i(\tilde{K}; \mathbf{Z})$. The coefficient group $\mathbf{Z}$ is understood from here on.

The results we need are, first, that $\mathcal{H}^0(\tilde{K}) \neq 0$ if and only if $\tilde{K}$ is compact; i.e., $\pi_1(K)$ finite. If this is not the case, there is an exact sequence

$$0 \longrightarrow H^0(\tilde{K}) \longrightarrow H^0_e(\tilde{K}) \longrightarrow \mathcal{H}^1(\tilde{K}) \longrightarrow 0 ,$$

where $H^0_e(\tilde{K})$ is a free abelian group, whose rank is the number of ends of $\pi_1(K)$. This number of ends is 1, 2, or ∞; it is 1 for a direct product of infinite groups, ∞ for a free product other than $\mathbf{Z}_2 * \mathbf{Z}_2$, and 2 if and only if $\pi_1(K)$ has an infinite cyclic subgroup of finite index. These results are due to Freudenthal. A convenient reference is Epstein [6]. We sharpen the last of them as follows.

LEMMA 4.1. *Suppose that π has 2 ends. Then there is a finite normal subgroup F of π, such that the quotient group is isomorphic to $\mathbf{Z}$ or to $\mathbf{Z}_2 * \mathbf{Z}_2$.*

PROOF. We know that π has an infinite cyclic subgroup of finite index. The intersection A of its conjugates has the same properties and is, moreover, normal. Let H be the centraliser of A. Since A is normal, and has only two automorphisms, the index of H in π is at most 2.

The centre of H contains A, so has finite index in H. By a result of Schur, (see e.g., W.R. Scott, Group Theory, Prentice-Hall, 1964, § 15.1.13), it follows that the commutator subgroup H' is finite (hence disjoint from A).

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The abelian group H/H' has $A/A \cap H' \cong A$ as a subgroup of finite index, hence as a finitely generated abelian group it must be the direct sum of a finite subgroup T and an infinite cyclic group.

The inverse image F of T in H is now a finite normal subgroup, with $H/F \cong \mathbf{Z}$. So F is even fully invariant in H, as the set of all elements of finite order.

In the case $H = \pi$, the proof is complete. Otherwise H has index 2, hence is normal in π; as F is characteristic in H, it too is normal in π. The quotient π/F has a subgroup of index 2 isomorphic to $\mathbf{Z}$; the quotient group induces the non-trivial automorphism of it (for the projection $A \to \mathbf{Z}$ is monomorphic), so the square of any element of the coset $(\pi/F) - \mathbf{Z}$ both commutes with that element and is transformed into its inverse, hence is the identity. This is enough to characterise π/F as isomorphic to $\mathbf{Z}_2 * \mathbf{Z}_2$.

THEOREM 4.2. *Suppose* (Y, X) *a connected Poincaré pair*, $\dim[Y] = n$.

(i) *If* $n = 0$, *then* $(Y, X) \cong (D^0, \varphi)$.

(ii) *If* $n = 1$, *then* $(Y, X) \cong (D^1, S^0)$ *or* (S^1, φ).

(iii) *If* $n = 2$, π *is finite, and* $X \neq \varphi$, *then* $(Y, X) \cong (D^2, S^1)$.

(iv) *If* $n = 2$, π *is finite, and* $X = \varphi$, *then* $\tilde{Y} \cong S^2$.

(v) *If* $n = 2$, π *is infinite, then* X *is a union of circles and* Y *a* $K(\pi, 1)$.

(vi) *If* $n = 3$, π *is finite, and* $X = \varphi$, *then* $\tilde{Y} \cong S^3$.

PROOF. $\tilde{Y}$ is connected and simply-connected in all cases. By duality, $H_2(\tilde{Y}) = H_2(Y; \Lambda) \cong H^{n-2}(Y, X; \Lambda)$ which vanishes in all cases except (iv), when it is infinite cyclic. Similarly, $H_3(\tilde{Y})$ vanishes except in case (vi), when it $\cong \mathbf{Z}$. Thus $\tilde{Y}$ is contractible, except in (iv), when it is a homotopy S^2, and (vi), when it is a homotopy S^3.

If $n < 0$, all homology vanishes, so Y is the empty set. So if $n = 0$, $X = \varphi$. But then $\mathcal{H}^0(\tilde{Y}) \cong H^0(Y; \Lambda) \cong H_0(Y; \Lambda) \cong H_0(\tilde{Y}) \cong \mathbf{Z}$, so π is finite. If $\pi \neq \{1\}$, it has non-zero homology in arbitrarily high dimensions [7]; or we may note, more simply, that the covering space of Y corresponding to a non-trivial cyclic subgroup of π has such homology. Thus $\pi = \{1\}$ and Y is contractible.

In (ii), first suppose Y orientable. Let X have r components (which may be supposed points). We have $H_1(Y, X) \cong H^0(Y) \cong \mathbf{Z}$. If π is finite, as above it is trivial: Y is contractible, so $X \neq \varphi$. The sequence

$$0 \longrightarrow H_1(Y, X) \longrightarrow H_0(X) \longrightarrow H_0(Y) \longrightarrow 0 ,$$

shows $r = 2$. Thus $(Y, X) \cong (D^1, S^0)$. If π is infinite, $0 = H^0(Y; \Lambda) = H_1(Y, X; \Lambda)$. The sequence

$$0 \longrightarrow H_0(X; \Lambda)(\cong r\Lambda) \longrightarrow H_0(Y; \Lambda)(\cong \mathbf{Z})$$

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shows that $r = 0$: $X = \varphi$. The sequence

$$\mathcal{H}^0(\tilde{Y})(=0) \longrightarrow H^0(\tilde{Y}) \longrightarrow H_e^0(\tilde{Y}) \longrightarrow \mathcal{H}^1(\tilde{Y}) \longrightarrow H^1(\tilde{Y})(=0)$$

shows that $H_e^0(\tilde{Y})$ has rank 2, hence that $\tilde{Y}$, and so $\pi_1(Y)$, has 2 ends. Also $\pi_1(Y)$ has no element of finite order (else the covering space corresponding to a finite cyclic subgroup would provide a contradiction, as above). It follows from (4.1) that $\pi_1(Y) \cong \mathbf{Z}$: As $\tilde{Y}$ is contractible, $Y \simeq S^1$.

If Y were non-orientable in (ii), its orientable double cover would have to be in one of the two cases above. In the first, this shows $Y \simeq K(\mathbf{Z}_2, 1)$, hence infinite dimensional, a contradiction. In the second, π has a subgroup $\mathbf{Z}$ of index 2 (hence normal), and the other coset reverses orientation in Y, and hence induces the non-trivial automorphism of $\mathbf{Z}$. So $\pi \cong \mathbf{Z}_2 * \mathbf{Z}_2$ and has torsion, again a contradiction.

It remains only to prove (iii). But $\tilde{Y}$ is contractible and π finite, hence trivial. If X consists of r copies of S^1, duality now shows that $r = 1$. This completes the proof of the theorem.

Our result is very unsatisfactory. For example in case (v), if $X \neq \varphi$, $H^2(Y; S) = 0$ for all coefficient bundles S. We conjecture that this implies π free (cf. Eilenberg & Ganea [4]). Also in case (v), if $X = \varphi$, we can show that the cohomology (with simple coefficients) of Y is the same as for a unique closed 2-manifold, and even that the fundamental groups are similar to some extent (one can argue modulo the ω-term of the lower central series; also with associated profinite groups). But we cannot even prove in the case $H_1(Y) = 0$ that $\pi_1(Y)$ must vanish. However, we can give a complete result for cases (iv) and (vi).

THEOREM 4.3. *Let Y be a Poincaré n-complex, finitely covered by a homotopy S^n. If n is even, then $Y \simeq S^n$ or $Y \simeq P_n(\mathbf{R})$. If n is odd, then Y is orientable and $\pi = \pi_1(Y)$ has period $(n + 1)$; in fact, the first k-invariant of Y is a generator $g \in H^{n+1}(\pi; \mathbf{Z})$. Given π, and $g \in H^{n+1}(\pi; \mathbf{Z})$ of order $|\pi|$, there exists a complex $Y(g)$ as above, and $Y(g_1)$ and $Y(g_2)$ are homotopy equivalent if and only if there is an isomorphism $e: \pi_1 \to \pi_2$ with $e^*(g_2) = g_1$.*

PROOF. Let X be the minimal orientable covering of Y. By [2, p. 358], $\pi_1(X)$ has period $(n + 1)$, and the first k-invariant of X is a generator of $H^{n+1}(\pi_1(X); \mathbf{Z})$. If n is even, this implies that $\pi_1(X)$ is trivial, so $X \simeq S^n$, and Y is at most doubly covered by X; moreover, if $\pi_1(Y) \cong \mathbf{Z}_2$, the first k-invariant of Y is the non-zero element of $H^{n+1}(\pi_1(Y); \mathbf{Z}^t)$.

Next we show that if n is odd, Y is orientable. Suppose not; then some covering X with cyclic fundamental group will also be non-orientable. We

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compare X with a lens space L which has the same dimension and fundamental group. Since we can form a $K(\pi_1(X), 1)$ from either X or L by attaching cells of dimension $\geqq(n+1)$, X and L have the same $(n-1)$-type. Hence, using duality for both, and observing that L is orientable, we have isomorphisms

$$H_1(L; \mathbf{Z}^t) \longrightarrow H^{n-1}(L; \mathbf{Z}^t) \longrightarrow H^{n-1}(X; \mathbf{Z}^t) \longrightarrow H_1(X; \mathbf{Z}) \longrightarrow H_1(L; \mathbf{Z}) .$$

But in fact $H_1(L; \mathbf{Z}^t) = 0 \neq H_1(L; \mathbf{Z})$, which provides the required contradiction.

The construction of the examples is given in full by Swan [22]. We will repeat it here, using the results of our previous paper [25]; however, this is not a new proof, just a reformulation of the old one. By [22, Th. 4.1], π has a periodic projective resolution of period $(n+1)$. This gives us an exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbf{Z} \longrightarrow 0$$

defining an element g_0 of $\mathrm{Ext}_\pi^{n+1}(\mathbf{Z}, \mathbf{Z}) = H^{n+1}(\pi; \mathbf{Z})$, which is a generator. By [22, Lem. 7.4], if r is any integer prime to $|\pi|$, we can modify the above resolution to replace g_0 by rg_0, so can obtain any generator g of $H^{n+1}(\pi; \mathbf{Z})$. By splicing many copies of the sequence, we obtain a projective resolution of $\mathbf{Z}$ over π, which is thus chain-homotopy equivalent to the chain complex of a $K(\pi, 1)$. By adding in elementary complexes $0 \to F \to F \to 0$ with F free of countable dimension, we can suppose the P_i free.

Now apply [25, Th. 4]. Then there exists a $K(\pi, 1)$-space whose chain complex in dimensions $\geqq 4$ is the given one. The n-skeleton of this space $(n \geqq 4)$ has chain complex equivalent to the one above, hence is a space of the required type.

As to the last clause of the theorem, the necessity of the condition is evident; its sufficiency follows from an argument of C.B. Thomas using obstruction theory (Cambridge thesis, 1965). The same argument proves that in the non-orientable case, $Y \simeq P_n(\mathbf{R})$.

We will not attempt to go further, and list all cases of the theorem, though even this appears not altogether impossible; the main difficulty in a detailed classification is the study of the operation on $H^{n+1}(\pi; \mathbf{Z})$ of the group of outer automorphisms of π.

Part of the interest of the above classification stems from the fact, studied in detail by Swan (loc. cit.) that the complexes $Y(g)$ do not automatically have the homotopy type of finite complexes (though no specific counter-example is at present known to the author). The finiteness obstruction $\chi \in K_0(\pi)$ will, by the above, depend on g only. Swan showed that $\chi(g^r) = r\chi(g)$, and made a fairly detailed study of $K_0(\pi)$, showing eventually that the obstruction must vanish in certain cases.

Another (probably related) observation is that according to a theorem of

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Milnor [14], many groups with periodic cohomology can not operate fixed-point free on any (topological) manifold which is a homotopy sphere. For example, π contains at most one element of order 2, so the non-abelian group of order 6 is excluded (here we can take $n = 3$). Thus some of the above Poincaré complexes do not have the same homotopy type as any topological manifold.

We now continue our study of Poincaré 3-complexes Y. We write $\pi = \pi_1(Y)$, $G = \pi_2(Y)$, $e =$ the number of ends of π. Then $e = 0$ is equivalent to π being finite: Theorems (4.2) (vi) and (4.3) give us the classification in this case. Otherwise, π is infinite, $\tilde{Y}$ non-compact, and $e = 1, 2$, or ∞. We have

$$H_0(\tilde{Y}) \cong \mathbf{Z} \qquad H_1(\tilde{Y}) = 0$$
$$H_2(\tilde{Y}) \cong \mathcal{H}^1(\tilde{Y}) = G \qquad H_3(\tilde{Y}) \cong \mathcal{H}^0(\tilde{Y}) = 0 .$$

Since G is $\mathcal{H}^1$ of a space, G is free abelian. The exact sequence

$$\begin{array}{ccccccccc} \mathcal{H}^0(\tilde{Y}) & \longrightarrow & H^0(\tilde{Y}) & \longrightarrow & H_e^0(\tilde{Y}) & \longrightarrow & \mathcal{H}^1(\tilde{Y}) & \longrightarrow & H^1(\tilde{Y}) \\ \| & & \| & & & & \| & & \| \\ 0 & & \mathbf{Z} & & & & G & & 0 \end{array}$$

shows, since $H_e^0(\tilde{Y})$ is free abelian of rank e, that

if $e = 1$, $G = 0$, so $\tilde{Y}$ is contractible,

if $e = 2$, $G \cong \mathbf{Z}$, so $\tilde{Y} \simeq S^2$,

if $e = \infty$, G is free abelian, of countably infinite rank.

If $e = 1$, Y is a $K(\pi, 1)$. We note that this occurs if π is a direct product of two infinite groups, $\pi \cong A \times B$. In this case, $Y \simeq K(A, 1) \times K(B, 1)$. By Theorem 2.5 (ii), we may say that $K(A, 1)$ is a Poincaré 1-complex, and $K(B, 1)$ a Poincaré 2-complex, so by Theorem 4.2 (ii), $A \cong \mathbf{Z}$, and $K(A, 1) \simeq S^1$. This result is due to Epstein [5] in the case when Y is a 3-manifold.

If $e = \infty$, we have the easy results

(a) $M = M_1 \# M_2$ implies $\pi_1(M)$ a free product,

(b) $\pi_1(M)$ a free product $\neq \mathbf{Z}_2 * \mathbf{Z}_2$ implies $e = \infty$.

In the case of manifolds, the implication (a) can be reversed (Whitehead [27], [28]). Also, if M is orientable, the sphere theorem shows that (b) can be reversed [27]. It would be very interesting to decide whether either or both of these results carry over to Poincaré complexes. We make no conjectures, but observe that the converse of (a) looks more probable than that of (b).

We have complete results for the case $e = 2$. Write P_i for i-dimensional real projective space.

THEOREM 4.4. *Let Y be a Poincaré 3-complex such that $\pi_1(Y)$ has 2 ends. Then Y is homotopy equivalent to one of $P_3 \# P_3$, $S^1 \times P_2$, or the trivial or non-trivial S^2-bundle over S^1.*

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PROOF. Suppose $x \in \pi_1(Y)$ has finite order n. Let X be the covering space of Y corresponding to the subgroup generated by x. Then X is finite dimensional, $\pi_1(X) \cong \mathbf{Z}_n$, and $\tilde{X} = \tilde{Y} \simeq S^2$. As in the proof of (4.3), it follows that $n = 2$ and $X \simeq P_2$; in particular, the covering transformation determined by x reverses the orientation of S^2.

Now suppose $\pi_1(Y)$ acts trivially on $\pi_2(Y) \cong H_2(\tilde{Y})$. By the above, $\pi_1(Y)$ has no element of finite order. By (4.1), $\pi_1(Y) \cong \mathbf{Z}$. Y is homotopy equivalent to a fibre space, with fibre $\tilde{Y} \simeq S^2$ and base $K(\pi_1(Y), 1) \simeq S^1$; moreover, $\pi_1(S^1)$ operates trivially on $\pi_2(S^2)$. Hence $Y \simeq S^1 \times S^2$.

Otherwise $\pi_1(Y)$ acts non-trivially. A subgroup of index 2 acts trivially: By the above, this is isomorphic to $\mathbf{Z}$, and determines a double covering of Y homotopy equivalent to $S^1 \times S^2$. By (4.1), $\pi_1(Y)$ is isomorphic to one of $\mathbf{Z}$, $\mathbf{Z} \times \mathbf{Z}_2$, $\mathbf{Z}_2 * \mathbf{Z}_2$. If we have $\mathbf{Z}$, then Y is again homotopy equivalent to a fibre space with fibre S^2 and base S^1, only this time $\pi_1(S^1)$ acts non-trivially on $\pi_2(S^2)$, so we have the non-trivial fibering. If $\pi_1(Y) \cong \mathbf{Z} \times \mathbf{Z}_2$, Y has a regular covering X (with group $\mathbf{Z}$) with fundamental group $\mathbf{Z}_2$ and hence $\simeq P_2$. So Y is homotopic to a fibre space with fibre P_2 and base S^1; since every homotopy equivalence of P_2 is homotopic to the identity, $Y \simeq S^1 \times P_2$.

Finally, suppose $\pi \cong \mathbf{Z}_2 * \mathbf{Z}_2$. Then Y is homotopy equivalent to a fibre space with fibre S^2 and base $K(\mathbf{Z}_2 * \mathbf{Z}_2, 1) = K(\mathbf{Z}_2, 1) \vee K(\mathbf{Z}_2, 1)$. The inverse image of each $K(\mathbf{Z}_2, 1)$ can only be $\simeq P_2$, and the inclusion of the base point in $K(\mathbf{Z}_2, 1)$ is covered by a map homotopy equivalent to the double covering $S^2 \to P_2$. Thus Y is homotopy equivalent to the space obtained by attaching to S^2 two copies of the mapping cylinder of the double covering $S^2 \to P_2$. But this space is precisely $P_3 \sharp P_3$.

CHAPTER 5. Poincaré 4-complexes with fundamental group of prime order

We now turn to 4-complexes. Here we are interested less in the fundamental group π (indeed, any finitely presented group is already the fundamental group of a smooth closed 4-manifold) than in duality phenomena in the middle dimension.

We shall assume from now on that Y is a Poincaré 4-complex, with fundamental group $\pi = \pi_1(Y)$ of prime order p (hence cyclic; let T be a generator), and group ring Λ. We use the notations $\zeta = \exp(2\pi i/p)$, $\Lambda_0 = \mathbf{Z}$, $\Lambda_1 = \mathbf{Z}[\zeta]$, and the ring homomorphism $\alpha: \Lambda \to \Lambda_0 \oplus \Lambda_1$ is defined by $\alpha(T) = (1, \zeta)$. We have an exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{\alpha} \Lambda_0 \oplus \Lambda_1 \xrightarrow{\beta} \mathbf{Z}_p \longrightarrow 0$$

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where $\beta(r, \sum_0^{p-2} s_i\zeta^i) = r - \sum s_i \pmod p$. (This notation follows [26, Ch. 4]).

We shall also need the classification [18] of torsion-free Λ-modules or, more precisely, of Λ-modules which are free as $\mathbf{Z}$-modules. Any such module M can be written as a direct sum $M_0 \oplus M_1 \oplus M_2$ where M_0 is a projective Λ_0-module, M_1 a projective Λ_1-module, M_2 a projective Λ-module. Write r_i for the rank of M_i (over the appropriate ring). Then (r_0, r_1, r_2) depends only on M, not on the splitting. If also $r_1 + r_2$ is finite, consider the class c of the projective Λ_1-module $M \otimes_\Lambda \Lambda_1 \cong M_1 \oplus (M_2 \otimes_\Lambda \Lambda_1)$ in the projective class group of Λ_1 (= ideal class group of $\mathbf{Q}[\zeta]$); this too depends only on M.

Then M and M' are isomorphic if and only if $(r_0, r_1, r_2) = (r_0', r_1', r_2')$ and, if $r_1 + r_2$ is finite, $c = c'$. The r_i are clearly independent non-negative integers; if $r_1 + r_2 = 0$, c is trivial; if $0 < (r_1 + r_2) < \infty$, c can take any value.

LEMMA 5.1. *Let* $0 \to A \to C \to B \to 0$ *be an exact sequence of* ($\mathbf{Z}$-*free*) Λ-*modules, with* C *free. Then* $r_0(A) = r_1(B)$, $r_1(A) = r_0(B)$.

PROOF. If $0 \to A \to C \to B \to 0$ and $0 \to A' \to C' \to B \to 0$ are two exact sequence as above, then (by Schanuel's theorem [22, p. 270]) $A \oplus C' \cong A' \oplus C$, and so $r_0(A) = r_0(A')$, $r_1(A) = r_1(A')$. Thus it is enough to choose a convenient epimorphism of a projective module onto B, and compute its kernel. For this we split B (as above) into a direct sum. For M_2 we take

$$0 \longrightarrow 0 \longrightarrow M_2 \xrightarrow{1} M_2 \longrightarrow 0 ,$$

where the result is trivial. For M_0 take

$$0 \longrightarrow M_0 \otimes_{\mathbf{Z}} \Lambda_1 \longrightarrow M_0 \otimes_{\mathbf{Z}} \Lambda \longrightarrow M_0 \longrightarrow 0$$

(where $\mathbf{Z}$ acts trivially on Λ); here, $M_0 \otimes_{\mathbf{Z}} \Lambda_1$ is a free Λ_1-module, with the same rank as that of M_0 over $\mathbf{Z}$. We write M_1 as a direct sum of projective modules B of rank 1; now we know that for any such, there is a projective Λ-module C of rank 1 with $B = C \otimes_\Lambda \Lambda_1$. The kernel of the obvious map $C \to B$ is isomorphic to Λ_0, and the lemma now follows.

LEMMA 5.2. *Let* Y *be a Poincaré* 4-*complex, with* $\pi_1(Y)$ *as above; set* $G = H_2(\tilde{Y}) \cong \pi_2(Y)$. *If* Y *is orientable,* $r_0(G) = 0$, $r_1(G) = 2$. *If* Y *is non-orientable* (*and so* $p = 2$), $r_0(G) = r_1(G) = 1$ *or* 0.

(We observe that elementary duality for $\tilde{Y}$ shows that G is torsion-free.)

PROOF. Let

$$0 \longleftarrow C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} C_3 \xleftarrow{d_4} C_4 \longleftarrow 0$$

be the chain complex of $\tilde{Y}$ (which we may suppose 4-dimensional by (2.2)); each C_i is a free Λ-module, perhaps infinitely generated. The only non-

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vanishing homology groups are $H_0 \cong \Lambda_0$, $H_4 \cong \Lambda_0$ or, in the non-orientable case, $H_4 \cong \Lambda_1$, and $H_2 = G$.

A repeated use of (5.1) shows that $\operatorname{Ker} d_2$ has $(r_0, r_1) = (0, 1)$, and that $\operatorname{Im} d_3$ has $(r_0, r_1) = (1, 0)$; or, in the non-orientable case, $=(0,1)$. Now consider the exact sequence

$$0 \longrightarrow \operatorname{Im} d_3 \longrightarrow \operatorname{Ker} d_2 \longrightarrow H_2 \longrightarrow 0 .$$

Here we need a slight refinement of (5.1). We can choose an exact sequence

$$0 \longrightarrow \Lambda_0 \longrightarrow C' \longrightarrow \operatorname{Ker} d_2 \longrightarrow 0$$

by the proof of that result. If J is the kernel of the composite epimorphism $C' \longrightarrow H_2$, we have an exact sequence

$$0 \longrightarrow \Lambda_0 \longrightarrow J \longrightarrow \operatorname{Im} d_3 \longrightarrow 0 .$$

In the orientable case, $\operatorname{Im} d_3$ has $(r_0\, r_1) = (1, 0)$, and it is now easy to see that this sequence must split. Then J has $(r_0, r_1) = (2, 0)$, and so H_2 has $(r_0, r_1) = (0, 2)$ by (5.1).

In the non-orientable case, the argument above shows that for G, $(r_0, r_1) = (1, 1)$ (if the sequence splits), or $(0, 0)$ (otherwise). This proves the lemma.

We now construct a cell complex having the same homotopy type as Y. This will give a *normal form* for Y. Let $*$ be a point, and φ_0 map $*$ to a point of Y (the base point). Now attach a 1-cell, giving a circle K^1, and extend φ_0 to $\varphi_1 \colon K^1 \longrightarrow Y$ representing $T \in \pi_1(Y)$. The group $\pi_2(\varphi_1)$ lies in the exact sequence

$$0 \longrightarrow \pi_2(Y) \longrightarrow \pi_2(\varphi_1) \longrightarrow \pi_1(K^1) \longrightarrow \pi_1(Y) \longrightarrow 0 ,$$

and so in an exact sequence of Λ-modules

$$0 \longrightarrow G \longrightarrow \pi_2(\varphi_1) \longrightarrow \Lambda_0 \longrightarrow 0 .$$

We assert that $\pi_2(\varphi_1)$ has $(r_0, r_1) = (0, 1)$, or in the non-orientable case, $(1, 0)$. In fact, as we can regard φ_1 as the inclusion of the 1-skeleton, this follows as above using (5.1).

It follows that there is a free Λ-module F and a projective Λ_1-module P (or take $P = \Lambda_0$ in the non-orientable case) such that $\pi_2(\varphi_1) \cong F \oplus P$. The easiest case now is when $c(G) = 0$, whence it follows that $c(\pi_2(\varphi_1)) = 0$, and P is free. In this case, we use the construction of [24, § 1] to kill $\pi_2(\varphi_1)$, using an epimorphism $F' \longrightarrow F \oplus P$ with kernel Λ_0, (in the non-orientable case, $c = 0$ necessarily; the kernel here is Λ_1). If $c(G) \neq 0$, we attach an infinite number of 2-cells, using an epimorphism $F' \longrightarrow F \oplus P$, F' free of countable rank, with kernel isomorphic to $F' \oplus \Lambda_0$.

We are constructing inductively complexes K^n, and n-connected maps

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φ_n: $K^n \longrightarrow Y$, and will end with a homotopy equivalence $K \longrightarrow Y$. Thus we can identify $\pi_{n+1}(\varphi_n)$ with $\pi_{n+1}(K, K^n)$ and hence, for $n > 1$, with $H_{n+1}(\tilde{K}, \tilde{K}^n)$, which is the quotient of the group of $(n+1)$-chains by boundaries. We deduce an exact sequence

$$0 \longrightarrow H_{n+1}\tilde{Y} \longrightarrow \pi_{n+1}(\varphi_n) \longrightarrow C_n \longrightarrow \pi_n(\varphi_{n-1}) \longrightarrow 0 ,$$

valid for $n \geqq 3$, and for $n = 2$ if $\pi_2(\varphi_1)$ is replaced by $H_2(\tilde{\varphi}_1)$. Thus given $\pi_n(\varphi_{n-1})$ we choose a set of Λ-generators, defining an epimorphism to it of the free Λ-module C_n; use these generators to attach n-cells, giving K^n and an extension φ_n of φ_{n-1} to K^n; and then the above sequence shows how to compute $\pi_{n+1}(\varphi_n)$.

In our case, $\pi_2(\varphi_1) = H_2(\tilde{\varphi}_1)$, and the sequence is valid for $n \geqq 2$. As $H_3(\tilde{Y}) = 0$, we deduce that $\pi_3(\varphi_2)$ is the kernel of the map $C_2 \longrightarrow \pi_2(\varphi_1)$, which we chose above to be Λ_0 (in the non-orientable case Λ_1) or $F' \bigoplus \Lambda_0$. Choose C_3 correspondingly to be Λ or $F' \bigoplus \Lambda$; the obvious epimorphism to $\pi_3(\varphi_2)$ then has kernel Λ_1 (in the non-orientable case Λ_0). The choice of C_3 determines K^3 and φ_3: $K^3 \longrightarrow Y$, up to homotopy. Now $\pi_4(\varphi_3)$ lies in the exact sequence above, which reduces to

$$0 \longrightarrow \Lambda_0 \longrightarrow \pi_4(\varphi_3) \longrightarrow \Lambda_1 \longrightarrow 0$$

(in the non-orientable case, Λ_0 and Λ_1 are interchanged). But by [24, Lem. 2.1], $\pi_4(\varphi_3)$ is projective. As it clearly has rank 1, and $c = 0$ (tensoring with Λ_1, we obtain Λ_1), it follows that $\pi_4(\varphi_3) \cong \Lambda$. We can then attach a single 4-cell, giving K^4 and a homotopy equivalence φ_4: $K^4 \longrightarrow Y$.

Note that if $c(Y) = 0$, K has only one cell each of dimensions 0, 1, 3, 4 (though we may need several 2-cells). Of course, if $c(Y) \neq 0$, we need infinitely many 2- and 3-cells, but still one 4-cell is enough, as we know already from (2.3.1).

Next observe that the homotopy type of K^3 can be described very simply. Indeed, in the easy case $c(Y) = 0$, the 1-skeleton is a circle; we then attach one 2-cell by degree p, and further ones at the base point; and then attach a 3-cell. This last attachment is determined by an element of $\pi_2(K^2) \cong \pi_2(\tilde{K}^2) \cong H_2(\tilde{K}_2)$, the group of 2-cycles of $\tilde{K}$, and hence by pure homology theory. In fact we had $H_2(\tilde{K}) = H_2(\tilde{Y}) \cong r\Lambda + 2\Lambda_1$ by (5.1). (The non-orientable case is similar; we leave discussion to the reader). Then $C_2\tilde{K}$ was chosen as $(r+2)\Lambda$, and the boundary d_2 sent the first generator e_0^2 to $(1 + T + \cdots + T^{p-1})e^1$ (where e^1 is the 1-cell), and the others to 0. We had the exact sequence

$$0 \longrightarrow \pi_3(\varphi_2) \longrightarrow C_2 \longrightarrow \pi_2(\varphi_1) \longrightarrow 0 ,$$

with $\pi_3(\varphi_2)$ isomorphic to Λ_0. Thus we may choose the base $e_0^2, e_1^2, \cdots, e_r^2, e_\infty^2$ of C_2

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(determining the 2-cells) with $de^3 = (1 + T + \cdots + T^{p-1})e^2_\infty$. The complex K^3 is now determined by the integer r; we have the bouquet of a standard complex K_0 with r copies of S^2. In the non-orientable case, we have a wedge of a collection of 2-spheres with $P_3(\mathbf{R})$ or the 3-skeleton of $S^2 \times P_2(\mathbf{R})$.

If $c \neq 0$, the result is analogous. First suppose $G \cong \Lambda_1 \oplus \Lambda_1 \oplus P$, for some projective Λ-module P. Then the cells $e^0, e^1, e^2_0, e^2_\infty, e^3$ are fixed as before; we now take a bouquet of this with an infinite collection of 2-spheres, which replaces the second homotopy group by its direct sum with F'. But $F' \cong F' \oplus P$; we now add 3-cells to kill the first summand F'. Thus the 3-skeleton is determined by P. A similar (but slightly more complicated) description is valid if G does not have the above form, in which case we have necessarily $G \cong \Lambda_1 \oplus P_1$, P_1 an ideal (hence projective) in Λ_1.

LEMMA 5.3. *Suppose Y as in* (5.2). *Then Y has the homotopy type of a* CW*-complex K, obtained from K^3 by attaching a single* 4*-cell. The homotopy type of K^3 is determined by G.*

Thus the classification will be determined by a study of attaching maps of the 4-cell. So we next consider $\pi_3(K^3) \cong \pi_3(\tilde{K}^3)$. Since $\tilde{K}^3$ is a simply-connected 3-complex, with $H_2(\tilde{K}^3) = G$ a free abelian group, $\tilde{K}^3$ has the homotopy type of a bouquet of 2-spheres and 3-spheres. This can be seen, for example, by repeating the above construction for $\tilde{K}^3$. The details become trivial in the simply-connected case. We can count the numbers of spheres by looking at the homology groups: $H_2(\tilde{K}^3) \cong G$, and $H_3(\tilde{K}^3)$ is the group of 3-cycles (or of 3-boundaries) of $\tilde{K}$, hence isomorphic to Λ_1 (in the non-orientable case, Λ_0).

We compute the homotopy group $\pi_3(\tilde{K}^3)$ using the simplest non-trivial case of the Hilton-Milnor theorem [8]. Suppose given a bouquet of 2-spheres S^2_i and 3-spheres S^3_a, then $\pi_3(\bigvee_i S^2_i \bigvee_a S^3_a)$ can be expressed as a sum of components $\pi_3(S^2_i) \cong \mathbf{Z}$ (each i), $\pi_2(S^2_i) \otimes \pi_2(S^2_j) \cong \mathbf{Z}$ (all pairs $i < j$) injected by the Whitehead product, and $\pi_3(S^3_a) \cong \mathbf{Z}$ (all a). We seek to algebraise this result.

If we attach a 4-sphere to a bouquet of 2-spheres, we obtain a cup-product $H^2 \otimes H^2 \to H^4 \cong \mathbf{Z}$ for the result. In this case, we can interpret the above result as saying that $\pi_3(\bigvee_i S^2_i)$ is isomorphic to the group of symmetric bilinear forms on $H^2(\bigvee_i S^2_i)$. We have written $H_2 = G$, and will now write $\hat{G}$ for the dual cohomology group, and $S_2(\hat{G})$ for the group of symmetric bilinear forms (with integer values) on $\hat{G}$. Then the computation yields a split short exact sequence

$$0 \longrightarrow S_2(\hat{G}) \longrightarrow \pi_3(K^3) \longrightarrow H_3(\tilde{K}^3) \longrightarrow 0 .$$

But we know the image in $H_3(\tilde{K}^3)$ of the attaching map of e^4 (this is determined by homology considerations). What remains to distinguish two such attaching

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maps is an element of $S_2(\hat{G})$.

In a sense, this gives our classification. Unfortunately, our element of $S_2(\hat{G})$ only represents the difference of two attaching maps, rather than determining one; and it is necessary to make a choice of a basis of G. The details of such algebra appear rather uninteresting. We shall ignore them. Also, not all elements of $S_2(\hat{G})$ will give Poincaré complexes. This, we must investigate. Note that as K is a Poincaré complex if and only if $\tilde{K}$ satisfies duality; the only remaining condition here is that the cup product pairing $\hat{G} \otimes \hat{G} \to H^4(\tilde{K}) \cong \mathbf{Z}$ be non-singular. Now $\tilde{K}$ is obtained from $\tilde{K}^3$ by attaching p 4-cells, which cover the single 4-cell of K. We can regard $(p-1)$ of these as *killing* the $(p-1)$ 3-spheres of $\tilde{K}^3$ (corresponding to $\mathbf{Z}$-generators of $\Lambda_1 = H_3(\tilde{K})$), and the remaining one as attached to the 2-skeleton, giving the top class in $\tilde{K}$. More precisely, the multiple $(1 + T + T^2 + \cdots + T^{p-1})e^4$ (or, in the non-orientable case, $(1 - T)e^4$) is attached by a map zero in homology, and hence is directly attached to the 2-skeleton. The attaching map of this linear combination of cells induces the cup product for $\tilde{K}$.

To perform our calculation, we must determine the action of π on $\pi_3(\tilde{K}^3)$. In our exact sequence, there are obvious actions of π on $S_2(\hat{G})$ (since π operates on G) and on $H_3(\tilde{K}^3) \cong \Lambda_1$ (or Λ_0), but these do not determine the action of π on the extension. (It is a split extension of $\mathbf{Z}$-modules; but not, as it turns out, of Λ-modules).

Instead of examining the complex directly, we shall construct a corresponding Poincaré 4-complex; in fact, a closed smooth manifold, and investigate cup products there. In the non-orientable cases, use $S^2 \times P_2(\mathbf{R})$ and $P_4(\mathbf{R})$. In the orientable case, start with the framed manifold $M_- = S^1 \times S^3$ and perform framed surgery as in [13] or [26] to kill p times the generator of the fundamental group. Write N for the cobordism, and M_+ for the other component of ∂N. Then a short calculation shows that the sequence

$$0 \longrightarrow H_3(\tilde{N}, \partial\tilde{N}) \longrightarrow H_2(\tilde{M}_+) \longrightarrow H_2(\tilde{N}) \longrightarrow 0$$

is exact, and each outside module isomorphic to Λ_1; or dually, we have an exact sequence

$$0 \longrightarrow H^2(\tilde{N}) \longrightarrow H^2(\tilde{M}) \longrightarrow H^3(\tilde{N}, \partial\tilde{N}) \longrightarrow 0 \ .$$

Cup products clearly vanish on $H^2(\tilde{N})$. The sequence splits, since $\mathrm{Ext}^1_\Lambda(\Lambda_1, \Lambda_1)$ vanishes, e.g., by the classification of modules, or since it lies in the exact sequence

$$0 = \mathrm{Hom}_\Lambda(\Lambda_0, \Lambda_1) \longrightarrow \mathrm{Ext}^1_\Lambda(\Lambda_1, \Lambda_1) \longrightarrow \mathrm{Ext}^1_\Lambda(\Lambda, \Lambda_1) = 0 \ ,$$

Since cup products are non-singular, the induced pairing of $H^2(\tilde{N})$ and

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$H^3(\tilde{N}, \partial\tilde{N})$ is a dual pairing. Thus if x is a free Λ_1-generator of $H^2(\tilde{N})$, we can choose a free generator y of $H^3(\tilde{N}, \partial\tilde{N})$ such that

$$xT^i \smile yT^j = \begin{cases} 1 & i \equiv j \pmod{p} \\ -1 & i \equiv j+1 \pmod{p} \\ 0 & \text{otherwise.} \end{cases}$$

Now lift y to an element (still denoted by y) of $H^2(\tilde{M})$, and write

$$a_i = y \smile yT^i = yT^j \smile yT^{i+j} .$$

Let the suffixes i, j run over integers mod p. Then symmetry shows $a_i = a_{-i}$, and since $\sum_0^{p-1} T^i$ acts as zero on Λ_1, we have $\sum_0^{p-1} a_i = 0$. Thus it is enough to consider $a_1, \cdots, a_r$, where $r = \frac{1}{2}(p-1)$. Set $z = y + x\left(\sum_1^r \lambda_i T^i\right)$. Then for $1 \leqq j \leqq r$, we have $z \smile zT^j = a_j + \lambda_j - \lambda_{1+j}$ (where $\lambda_{1+r} = 0$). Now choose $\lambda_i = -\sum_i^r a_j$. Then $z \smile zT^j = 0$ for all j, so in terms of the free Λ_1-basis $\{x, z\}$ we have a precise description of cup products.

We thus arrive at our final result.

THEOREM 5.4. *Suppose Y a Poincaré 4-complex, $\pi_1(Y)$ cyclic of order p. Set $G = \pi_2(Y)$; G is determined in* (5.2). *The homotopy type of Y is determined by G and by a symmetric, $\mathbf{Z}$-bilinear map $f: H^2(\tilde{Y}) \times H^2(\tilde{Y}) \to \mathbf{Z}$. Cup products in Y are given by the map $f_0 + (1 + T + \cdots + T^{p-1})f = m$ [or, in the non-orientable case, $f_0 + (1 - T)f$] where f_0 is the multiplication in the example above. Any map f can be chosen such that the corresponding m is non-singular.*

To arrive at a precise classification, one should permit $\pi_1(Y)$ and $\pi_2(Y)$ to vary by automorphisms, and compute what happens to f. We are not at present able to do this except in very special cases. The above is, however, sufficiently precise to provide us with examples.

COROLLARY 5.4.1. *For any p, there exist orientable Poincaré complexes Y, as above, with the signature $\sigma(\tilde{Y}) \neq p\sigma(Y)$.*

PROOF. Set $G = \Lambda_1 \oplus \Lambda_1 \oplus F$, where F is a free Λ-module of rank 8. We choose f to vanish except on the summand $F \times F$. Thus we must choose a symmetric $\mathbf{Z}$-bilinear map $f: F \times F \to \mathbf{Z}$, form $m = (1 + T + \cdots + T^{p-1})f$, and calculate the signatures of the quadratic forms induced on F (for $\tilde{Y}$) and on $F \otimes_\Lambda \Lambda_0$ (for Y). Note that the summand $\Lambda_1 \oplus \Lambda_1$ with f_0 contributes nothing.

In fact we shall choose a Λ-map m, rather than choose f. Since F is free, m will automatically have the form required. Now recalling the exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Lambda_0 \oplus \Lambda_1 \longrightarrow \mathbf{Z}_p \longrightarrow 0 ,$$

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and tensoring with F, we see that to define m it is equivalent to define forms on $F \otimes_\Lambda \Lambda_0$ and on $F \otimes_\Lambda \Lambda_1$, and check that the results agree mod p on $F \otimes_\Lambda \mathbf{Z}_p$. On $F \otimes_\Lambda \Lambda_1$ we choose a sum of four hyperbolic planes (one such is a sum of two isotropic copies of Λ_1, dually paired to $\mathbf{Z}$). The induced form on $F \otimes_\Lambda \mathbf{Z}_p$ to $\mathbf{Z}_p$ is then also a sum of hyperbolic planes. On $F \otimes_\Lambda \Lambda_0$, we choose a positive definite unimodular even quadratic form (in rank 8, this exists, and is unique up to isomorphism). It is easily verified that the induced form on $F \otimes_\Lambda \mathbf{Z}_p$ is again a sum of hyperbolic planes, hence can be identified with the above (if we choose the right form on $F \otimes \Lambda_0$).

The two now combine to give a form on F; the resulting Y is such that $\sigma(\tilde{Y}) = \sigma(Y) = 8$.

In order to state the next corollary, we first observe that $\tilde{K}_0(\pi) = \tilde{K}_0(\Lambda) \cong \tilde{K}_0(\Lambda_1)$ as already noted, and by a theorem of Rim [19], $\tilde{K}_0(\Lambda_1)$ can be identified with the group of ideal classes in the Dedekind ring Λ_1. Let Λ_2 be the subring of Λ_1 left fixed by our involution $\zeta \to \zeta^{-1}$; since this coincides with complex conjugation, Λ_2 is the real subring of Λ_1. The norm map from Λ_1 to Λ_2 (which it extends with degree 2) induces a homomorphism of projective class groups $\tilde{K}_0(\Lambda_1) \xrightarrow{N} \tilde{K}_0(\Lambda_2)$. Denote the kernel of N by $K_0^*(\Lambda_1)$.

A theorem of Kummer [10] states that N is always surjective. Denote the orders of $K_0^*(\Lambda_1)$ and $\tilde{K}_0(\Lambda_2)$ by h_1, h_2 respectively; then $\tilde{K}_0(\Lambda_1)$ has order $h_1 h_2$, and h_1 and h_2 are the so-called first and second factors of the class number of Λ_1. It is known [11] [12] that $h_1 = 1$ if $p < 23$ and $h_1 > 1$ for $23 \leqq p \leqq 163$, and tends to infinity with p.

COROLLARY 5.4.2. *Let $\kappa \in \tilde{K}(\pi)$ determine an element of $K_0^*(\Lambda_1)$. Then there exists a Poincaré 4-complex Y with finiteness obstruction $\chi(Y) = \kappa$.*

PROOF. Represent the image of κ in $\tilde{K}_0(\Lambda_1)$ by an ideal P in Λ_1. Since $N(P)$ is a principal ideal, $P\bar{P}$ is generated by a real number b. Define $\varphi: P \times P \to \Lambda_1$ by

$$\varphi(x, y) = \bar{x} y b^{-1} ,$$

then φ is clearly hermitian and non-singular. Reducing φ mod $\zeta - 1$ induces $\varphi_p: P_p \times P_p \to \mathbf{Z}_p$, where P_p has order p. Now $\varphi_p(x, x)$ is either a residue or a non-residue. In the first case, φ_p is isomorphic to the restriction mod p of the form over $\Lambda_0 \cong \mathbf{Z}$ with matrix (1). Choose an isomorphism, and denote by Q the kernel of the difference map $P \oplus \Lambda_0 \to \mathbf{Z}_p$. There is an induced self-pairing of Q to $\Lambda_1 \oplus \Lambda_0$. It follows from the definition of Q that the pairing actually takes values in Λ, and then is non-singular. The desired result now follows as in the preceding Corollary.

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Next suppose $\varphi_p(x, x)$ a non-residue. We will change it to a residue by multiplying b by an appropriate unit of Λ_2. Since $(\sum_{-r\leq i\leq r} \zeta^i)$ is such a unit (provided $2r + 1 \not\equiv 0 \bmod p$) and induces $(2r + 1)$ which is an arbitrary element of $\mathbf{Z}_p$, this can be done.

The last corollary suggests the conjecture that, for any Poincaré $2k$-complex with fundamental group π of order p, the finiteness obstruction lies in $K_0^*(\Lambda_1)$.

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