PFAFFIAN SUBSCHEMES

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Abstract. A subscheme $X \subset \mathbb{P}^{\ltimes+\nVdash}$ of codimension 3 is *Pfaffian* if it is the degeneracy locus of a skew-symmetric map $f : \mathcal{E}^\vee(-\sqcup) \longrightarrow \mathcal{E}$ with $\mathcal{E}$ a locally free sheaf of odd rank on $\mathbb{P}^{\ltimes+\nVdash}$. It is shown that a codimension 3 subscheme $X \subset \mathbb{P}^{\ltimes+\nVdash}$ is Pfaffian if and only if it is locally Gorenstein, subcanonical (i.e. $\omega_X \cong \mathcal{O}_\mathcal{X}(\updownarrow)$ for some integer l), and the following parity condition holds: if $n \equiv 0 \pmod 4$ and l is even, then $\chi(\mathcal{O}_\mathcal{X}(\updownarrow/\in))$ is also even.

The paper includes a modern version of the Horrocks correspondence, stated in the language of derived categories. A local analogue of the main theorem is also proved.

One method of constructing a codimension 3 subscheme $X \subset \mathbb{P}^{\ltimes+\nVdash}$ is to consider a *Pfaffian subscheme*. (This is discussed in a recent paper of Okonek [O].) That is, one considers the degeneracy locus of a skew-symmetric map $f : \mathcal{E}^\vee(-\sqcup) \longrightarrow \mathcal{E}$ such that $\mathcal{E}$ is a locally free sheaf of odd rank $2p+1$ on $\mathbb{P}^{\ltimes+\nVdash}$, and f is generically of rank $2p$ and degenerates to rank $2p-2$ in the expected codimension 3. The scheme X then has a locally free resolution of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{\ltimes+\nVdash}}(-\sqcup-\in\int) \xrightarrow{\langle} \mathcal{E}^\vee(-\sqcup-\int) \xrightarrow{\{} \mathcal{E}(-\int) \xrightarrow{\}} \mathcal{O}_{\mathbb{P}^{\ltimes+\nVdash}} \longrightarrow \mathcal{O}_\mathcal{X} \qquad (1)$$

where $s = c_1(\mathcal{E}) + \sqrt{}\sqcup$, and where g and $h = g^\vee(-t-2s)$ are given locally by the Pfaffians of order $2p$ of f. This resolution is just a patching together of the local version studied in [BE2]. The self-duality of the resolution (1) implies that X is locally Gorenstein with canonical sheaf $\omega_X \cong \mathcal{O}_\mathcal{X}(\sqcup+\in\int-\setminus-\triangle)$. Thus Pfaffian subschemes are always locally Gorenstein of codimension 3 in $\mathbb{P}^{\ltimes+\nVdash}$ and are subcanonical, i.e. they satisfy $\omega_X \cong \mathcal{O}_\mathcal{X}(\updownarrow)$ for some integer l.

It is now natural to consider the following question asked by Okonek [O]: Are all locally Gorenstein subcanonical subschemes of codimension 3 in $\mathbb{P}^{\ltimes+\nVdash}$ Pfaffian? Arithmetically Gorenstein subschemes of codimension 3 certainly are Pfaffian because of a structure theorem of Buchsbaum and Eisenbud ([BE2] Theorem 2.1). We will show that in conjunction with a certain number of other ideas, their method can be adapted to yield the following result:

Theorem 0.1. *Let k be a field not of characteristic 2. Suppose $X \subset \mathbb{P}^{\ltimes+\nVdash}_\urcorner$ is a locally Gorenstein subscheme of equidimension $n > 0$ such that $\omega_X \cong \mathcal{O}_\mathcal{X}(\updownarrow)$ for some integer l. Then X is a Pfaffian subscheme if and only if the following parity condition holds: if $n \equiv 0 \pmod 4$ and l is even, then $\chi(\mathcal{O}_\mathcal{X}(\updownarrow/\in))$ is also even.*

As Okonek pointed out, the Barth-Lefschetz theorems imply that every smooth subvariety of codimension 3 in $\mathbb{P}^{\nleftarrow}$ and $\mathbb{P}^{\nrightarrow}$ is subcanonical. Hence we have a corollary:

1991 *Mathematics Subject Classification.* Primary 14M05, 14F05, 13C05.

Key words and phrases. Codimension 3, subcanonical, Pfaffian.

The author was supported in part by NSA research grant MDA904-92-H-3009.

Corollary 0.2. *Every smooth subvariety of codimension* 3 *in* $\mathbb{P}^{\nleftarrow}$ *and* $\mathbb{P}^{\nrightarrow}$ *is a Pfaffian subscheme.*

Of course the corollary holds for $\mathbb{P}^{\mathbb{N}}$ with $N \geq 10$ as well (with certain conditions if $N \equiv 3 \pmod 4$). But in this range all smooth subvarieties of codimension 3 in $\mathbb{P}^{\mathbb{N}}$ are supposed to be complete intersections according to Hartshorne's Conjecture.

The parity condition in Theorem 0.1 may be deduced as follows. Suppose X is Pfaffian and n and l are both even. We may twist the resolution (1) by $l/2$ to get

$$0 \longrightarrow \omega_{\mathbb{P}}(-l/2) \longrightarrow \mathcal{E}^{\vee} \otimes \omega_{\mathbb{P}}(\int - \updownarrow/\in) \longrightarrow \mathcal{E}(\updownarrow/\in - \int) \longrightarrow \mathcal{O}_{\mathbb{P}}(\updownarrow/\in) \longrightarrow \mathcal{O}_{\mathcal{X}}(\updownarrow/\in) \longrightarrow \prime.$$

From this sequence and Serre duality on $\mathbb{P}^{\ltimes+\nVdash}$, it now follows that:

$$\chi(\mathcal{O}_{\mathcal{X}}(\updownarrow/\in)) = \in\chi(\mathcal{O}_{\mathbb{P}}(\updownarrow/\in)) - \in\chi(\mathcal{E}(\updownarrow/\in - \int)) \equiv \prime \pmod{\in}.$$

The main theorem has the following local analogue. Call an unmixed ideal I of height 3 in a regular local ring $(R, \mathfrak{m}, \mathfrak{k})$ *Pfaffian* if R/I has a resolution of the form

$$0 \longrightarrow R \xrightarrow{g^{\vee}} E^{\vee} \xrightarrow{f} E \xrightarrow{g} R \longrightarrow R/I$$

with E is a reflexive R-module of odd rank such that $E_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$, and f skew-symmetric. The canonical module of any unmixed ideal I of height 3 is by definition $\omega_{R/I} = \operatorname{Ext}^3_R(R/I, R)$. It is always saturated. We prove the following:

Theorem 0.3. *Let* $(R, \mathfrak{m}, \mathfrak{k})$ *be a regular local ring of dimension* $n > 4$ *with residue field not of characteristic* 2*. Let* I *be an unmixed ideal of* R *of height* 3*. Then* I *is Pfaffian if and only if the following three conditions hold*:

(*a*) $(R/I)_{\mathfrak{p}}$ *is Gorenstein for all prime ideals* $\mathfrak{p} \neq \mathfrak{m}$,

(*b*) $\omega_{R/I} \cong (R/I)^{\mathrm{sat}}$, *and*

(*c*) *if* $n \equiv 0 \pmod 4$, *then* $H^{n/2}_{\mathfrak{m}}(I)$ *is of even length.*

In both theorems we have stated the parity condition only for $n \equiv 0 \pmod 4$ rather than for all even n. This is because the graded commutativity built into cohomology rings causes the perfect pairing of Serre duality

$$H^{n/2}(\mathcal{O}_{\mathcal{X}}(\updownarrow/\in)) \times \mathcal{H}^{\setminus/\in}(\mathcal{O}_{\mathcal{X}}(\updownarrow/\in)) \longrightarrow \mathcal{H}^{\setminus}(\mathcal{O}_{\mathcal{X}}(\updownarrow)) \cong \| \tag{2}$$

(or its analogue in local duality) to be $(-1)^{n/2}$-symmetric. Hence if $n \equiv 2 \pmod 4$, then $H^{n/2}(\mathcal{O}_{\mathcal{X}}(\updownarrow/\in))$ admits a non-degenerate skew-symmetric bilinear form and so is of even dimension. Thus subcanonical varieties with $n \equiv 2 \pmod 4$ and l even automatically have

$$\chi(\mathcal{O}_{\mathcal{X}}(\updownarrow/\in)) \equiv \langle^{\setminus/\in}(\mathcal{O}_{\mathcal{X}}(\updownarrow/\in)) \equiv \prime \pmod{\in}.$$

In characteristic 2 the perfect pairing (2) and its local analogue apparently need not be alternating even if $n \equiv 2 \pmod 4$. Thus one cannot expect Theorem 0.1 or 0.3 to be valid in characteristic 2 unless the phrase "if $n \equiv 0 \pmod 4$" is replaced by the phrase "if n is even." However, we will show that with this modification, both theorems are valid in characteristic 2.

Outline of the Paper. In the first section we review the proof of the local version of Theorem 0.1 given by Buchsbaum and Eisenbud ([BE2] Theorem 2.1). We show that their proof will work for us if we can replace their minimal projective resolution by a locally free resolution of $\mathcal{O}_{\mathcal{X}}$ which satisfies two properties (Proposition 1.2).

The rest of the paper is devoted to finding a locally free resolution of $\mathcal{O}_{\mathcal{X}}$ which satisfies these properties.

Our main tool for constructing this locally free resolution is the Horrocks correspondence of [Ho]. In the second section of the paper, we give a modern description of this correspondence using derived categories. This point of view is not identical to Horrocks', so we have felt it prudent to include a full proof of Horrocks' principal result (Theorem 2.4) from this point of view. However, the derived categories viewpoint is useful because it permits us to further develop Horrocks' ideas so as to obtain a method for transfering a portion of the cohomology of the coherent sheaf $\mathcal{O}_{\mathcal{X}}$ to a locally free sheaf in a controlled way (Proposition 2.8). This is critical for our construction.

In the third section we apply the Horrocks correspondence to construct a particular locally free resolution of the form (1). The basic idea is to cut in half the cohomology of the subscheme X by using truncations of $\mathbf{R}\Gamma_*(\mathcal{I}_{\mathcal{X}})$. Our results on the Horrocks correspondence then permit us to find a vector bundle $\mathcal{F}_\infty$ whose intermediate cohomology is one of the halves of the cohomology of $\mathcal{O}_{\mathcal{X}}$. Moreover, there is a natural morphism from this $\mathcal{F}_\infty$ to $\mathcal{I}_{\mathcal{X}}$. This more or less gives the right half of the resolution, and the left half comes from the conventional methods of the Serre correspondence. We then show that if the cohomology of $\mathcal{O}_{\mathcal{X}}$ was cut in half properly (viz. if the subcomplex carries an "isotropic" half of the cohomology), then the resolution is self-dual in a very strong way: i.e. any chain map from the resolution to its dual which extends the identity on $\mathcal{O}_{\mathcal{X}}$ is necessarily an isomorphism of complexes. This is one of the properties required of the locally free resolution in order to make the Buchsbaum-Eisenbud proof work.

In the fourth section we show that our locally free resolution of $\mathcal{O}_{\mathcal{X}}$ can be endowed with a commutative differential graded algebra structure. This is a matter of calculating the obstruction to the lifting of a certain map. This is the second property required of the locally free resolution in order for the Buchsbaum-Eisenbud proof to work. This will complete the proof of Theorem 0.1.

In the fifth section we consider Theorem 0.1 in characteristic 2. Essentially, certain lemmas in the fourth section fail in characteristic 2 and must be replaced by analogues which are slightly different.

In the sixth section we consider the results for regular local rings. Theorem 0.1 concerning projective spaces has an obvious analogue (Theorem 6.1) for the punctured spectrum of a regular local ring. We show that this analogue is equivalent to Theorem 0.3.

Acknowledgment . The author would like to thank R. M. Mirò-Roig who brought the problem to his attention and with whom he had several discussions concerning it. The paper was written in the context of the Space Curves group of Europroj.

1. The Buchsbaum-Eisenbud Proof

In this section we review Buchsbaum and Eisenbud's proof of the local version of Theorem 0.1. In particular, we describe the two conditions that a locally free resolution of $\mathcal{O}_{\mathcal{X}}$ must satisfy in order for their proof to show that a subcanonical subscheme $X \subset \mathbb{P}^{\ltimes+\nvDash}$ is Pfaffian (Proposition 1.2).

Theorem 1.1 ([BE2] Theorem 2.1)**.** *Let R be a regular local ring and I an ideal of R of height 3 such that R/I is a Gorenstein ring. Then I has a minimal projective* *resolution of the form*

$$0 \longrightarrow R \xrightarrow{g^\vee} F^\vee \xrightarrow{f} F \xrightarrow{g} R \longrightarrow R/I$$

such that F of odd rank $2p+1$, the map f is skew-symmetric, and g is composed of the Pfaffians of order $2p$ of f.

Sketch of Buchsbaum and Eisenbud's proof of Theorem 1.1. One considers a minimal projective resolution of R/I. Since R/I is Gorenstein, it is of the form

$$\mathbf{P}^* : \qquad \mathbf{0} \longrightarrow \mathbf{R} \xrightarrow{\mathbf{d_3}} \mathbf{F_2} \xrightarrow{\mathbf{d_2}} \mathbf{F_1} \xrightarrow{\mathbf{d_1}} \mathbf{R}$$

We now seek to find a way of identifying $F_2 \cong F_1^\vee$ so that d_2 becomes skew-symmetric.

The first step is to endow $\mathbf{P}^*$ with the structure of a commutative associative differential graded algebra ([BE2] pp. 451–453). To define the multiplication, they define $S_2(\mathbf{P}^*) = (\mathbf{P}^* \otimes \mathbf{P}^*)/\mathbf{M}^*$ where M^* is the graded submodule of $\mathbf{P}^* \otimes \mathbf{P}^*$ generated by

$$\{a \otimes b - (-1)^{(\deg a)(\deg b)} b \otimes a \mid a, b \text{ homogeneous elements of } \mathbf{P}^*\}.$$

Using universal properties of projective modules, they then construct a map of complexes $\Phi : S_2(\mathbf{P}^*) \longrightarrow \mathbf{P}^*$ which extends the multiplication $R/I \otimes R/I$ and which is the identity on the subcomplex $R \otimes \mathbf{P}^* \subset \mathbf{S_2}(\mathbf{P}^*)$. This makes $\mathbf{P}^*$ into a commutative differential graded algebra. The associativity of this algebra follows from the fact that it of length 3, i.e. $P_n = 0$ for $n \geq 4$.

The next step (p. 455) is to note that the multiplication $F_i \otimes F_{3-i} \longrightarrow F_3 = R$ induces maps $s_i : F_i \longrightarrow F_{3-i}^\vee$ and a commutative diagram:

$$\begin{tikzcd}
\mathbf{P}^* : & 0 \arrow[r] & R \arrow[r, "d_3"] \arrow[d, equal] & F_2 \arrow[r, "d_2"] \arrow[d, "s_2"] & F_1 \arrow[r, "d_1"] \arrow[d, "s_1"] & R \arrow[d, equal] \\
(\mathbf{P}^*)^\vee : & 0 \arrow[r] & R \arrow[r, "d_1^\vee"] & F_1^\vee \arrow[r, "-d_2^\vee"] & F_2^\vee \arrow[r, "d_3^\vee"] & R
\end{tikzcd} \tag{3}$$

This map of complexes is an extension of the Gorenstein duality isomorphism $R/I \cong \omega_{R/I} = \operatorname{Ext}^3_R(R/I, R)$ to the minimal projective resolutions of R/I and $\omega_{R/I}$. Since any map between minimal projective resolutions which extends an isomorphism in degree 0 must be an isomorphism, it follows that the s_i are all isomorphisms.

We can therefore use the identification $s_2 : F_2 \cong F_1^\vee$. A very simple computation (p. 465) shows that with this identification, the commutativity and associativity of the differential graded algebra structure on $\mathbf{P}^*$ imply the skew-symmetry of d_2. In particular d_2 must have even rank (say $2p$), and F_2 must have odd rank $2p+1$. The identification of d_1 and d_3 with the vectors of Pfaffians of order $2p$ of d_2 is a lengthy but unproblematic computation (pp. 458–464). $\square$

Now let X be a locally Gorenstein subcanonical subscheme of codimension 3 in $\mathbb{P}^{\ltimes+\nVdash}$ with $\omega_X \cong \mathcal{O}_{\mathcal{X}}(\updownarrow)$. We wish to repeat the proof we have just sketched only with $\mathbf{P}^*$ replaced by a locally free resolution of $\mathcal{O}_{\mathcal{X}}$:

$$\mathcal{P}^* : \qquad \prime \longrightarrow \mathcal{L} \xrightarrow{\lceil_\ni} \mathcal{F}_\in \xrightarrow{\lceil_\in} \mathcal{F}_\infty \xrightarrow{\lceil_\infty} \mathcal{O}_{\mathbb{P}^{\ltimes+\nVdash}} \tag{4}$$

where we will write $\mathcal{L}$ in place of $\omega_{\mathbb{P}^{\ltimes+\nVdash}}(-l)$ in order to simplify our diagrams.

A careful reading yields only two places where the fact that $\mathbf{P}^*$ is a minimal projective resolution of R/I was used in a way that does not immediately carry over to the locally free resolution $\mathcal{P}^*$. The first place was in the definition of the map $\Phi : S_2(\mathbf{P}^*) \to \mathbf{P}^*$ which made $\mathbf{P}^*$ into a commutative differential graded algebra. Therefore we will need to show directly the existence of a map of complexes

$$\begin{tikzcd}
S_2(\mathcal{P}^*): & \cdots \arrow[r] & \mathcal{L} \oplus [\mathcal{F}_{\in} \otimes \mathcal{F}_{\infty}] \arrow[r, "\sigma"] \arrow[d, "\phi_3"] & \mathcal{F}_{\in} \oplus \Lambda^{\in}\mathcal{F}_{\infty} \arrow[r] \arrow[d, "\phi_2"] & \mathcal{F}_{\infty} \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^{\ltimes+\not\models}} \arrow[d, equal] \\
\mathcal{P}^*: & 0 \arrow[r] & \mathcal{L} \arrow[r, "d_3"] & \mathcal{F}_{\in} \arrow[r, "d_2"] & \mathcal{F}_{\infty} \arrow[r, "d_1"] & \mathcal{O}_{\mathbb{P}^{\ltimes+\not\models}}
\end{tikzcd}$$

The critical problem in defining the morphism of complexes is the following. Let $\psi : \Lambda^2\mathcal{F}_{\infty} \to \ker(\lceil_{\infty})$ be defined by $\psi(a \wedge b) = d_1(a)b - d_1(b)a$. We then must lift

$$\begin{tikzcd}
 & & & \Lambda^2\mathcal{F}_{\infty} \arrow[d, "\psi"] & \\
0 \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{F}_{\in} \arrow[r] & \ker(d_1) \arrow[r] & 0
\end{tikzcd} \tag{5}$$

to a $\phi \in \mathrm{Hom}(\Lambda^2\mathcal{F}_{\infty}, \mathcal{F}_{\in})$. Once that is done, the rest of the chain map follows. For one may define $\phi_2 = (1_{\mathcal{F}_{\in}}, \phi)$. Then

$$\phi_2 \circ \sigma(\mathcal{L} \oplus [\mathcal{F}_{\in} \otimes \mathcal{F}_{\infty}]) \subset \ker(\lceil_{\in}) = \mathcal{L}.$$

So $\phi_2 \circ \sigma$ factors through $\mathcal{L}$, allowing one to define ϕ_3. Thus one can put a commutative associative differential graded algebra structure on $\mathcal{P}^*$ provided ψ can be lifted. The obstruction to lifting ψ lies in $\mathrm{Ext}^1(\Lambda^2\mathcal{F}_{\infty}, \mathcal{L}) \cong \mathcal{H}^{\setminus+\in}(\Lambda^{\in}\mathcal{F}_{\infty}(\updownarrow))^*$.

Once we have the commutative differential graded algebra structure on $\mathcal{P}^*$, we may use it to define maps $s_i : \mathcal{F}_{\rangle} \to \mathcal{F}^{\vee}_{\ni-\rangle} \otimes \mathcal{L}$ and a commutative diagram analogous to (3):

$$\begin{tikzcd}
\mathcal{P}^*: & 0 \arrow[r] & \mathcal{L} \arrow[r, "d_3"] \arrow[d, equal] & \mathcal{F}_{\in} \arrow[r, "d_2"] \arrow[d, "s_2"] & \mathcal{F}_{\infty} \arrow[r, "d_1"] \arrow[d, "s_1"] & \mathcal{O}_{\mathbb{P}^{\ltimes+\not\models}} \arrow[d, equal] \\
(\mathcal{P}^*)^{\vee}: & 0 \arrow[r] & \mathcal{L} \arrow[r, "d_1^{\vee}"] & \mathcal{F}_{\infty}^{\vee} \otimes \mathcal{L} \arrow[r, "-d_2^{\vee}"] & \mathcal{F}_{\in}^{\vee} \otimes \mathcal{L} \arrow[r, "d_3^{\vee}"] & \mathcal{O}_{\mathbb{P}^{\ltimes+\not\models}}
\end{tikzcd} \tag{6}$$

The vertical maps extend the isomorphism $\mathcal{O}_{\mathcal{X}} \cong \omega_{\mathcal{X}}(-\updownarrow) = \mathcal{E}\S\sqcup^{\ni}(\mathcal{O}_{\mathcal{X}}, \mathcal{L})$. We now run into the second problem with locally free resolutions. Namely, a morphism of locally free resolutions which extends an isomorphism in degree 0 is not automatically an isomorphism between the resolutions. But we reach the conclusion:

Proposition 1.2. *Suppose X is a locally Gorenstein subcanonical subscheme of codimension 3 in $\mathbb{P}^{\ltimes+\not\models}$ with $\omega_X \cong \mathcal{O}_{\mathcal{X}}(\updownarrow)$. Then X will be a Pfaffian scheme if $\mathcal{O}_{\mathcal{X}}$ has a locally free resolution $\mathcal{P}^*$ as in (4) satisfying the following two conditions:*

(a) Any morphism of complexes $\mathcal{P}^* \to (\mathcal{P}^*)^{\vee}$ as in (6) which extends the identity of $\mathcal{O}_{\mathcal{X}}$ is an isomorphism of complexes, and

(b) The morphism ψ of (5) lifts to a map $\phi \in \mathrm{Hom}(\Lambda^2\mathcal{F}_{\infty}, \mathcal{F}_{\in})$.

We will now construct locally free resolutions $\mathcal{P}^*$ satisfying the conditions of the proposition. Our method involves the Horrocks correspondence.

2. The Horrocks Correspondence

In this section we give a modern description of the Horrocks correspondence of [Ho] using derived categories. We include a full proof of the principal properties of the correspondence from this point of view (Theorem 2.4). Taking advantage of the greater flexibility of the derived category viewpoint, we develop a technique which allows us to transfer a prescribed portion of the cohomology of $\mathcal{O}_{\mathcal{X}}$ to prescribed parts of a locally free resolution (Proposition 2.8).

Notation and Generalities. We first recall some generalities about complexes. If $\mathfrak{A}$ is an abelian category, let $C(\mathfrak{A})$ (resp. $K(\mathfrak{A})$, $D(\mathfrak{A})$) denote the category (resp. homotopy category, derived category) of complexes of objects of $\mathfrak{A}$, and let $C^b(\mathfrak{A})$, $C^-(\mathfrak{A})$, $C^+(\mathfrak{A})$, etc., denote the corresponding complexes of bounded (resp. bounded above, bounded below) complexes of objects of $\mathfrak{A}$. When speaking of complexes, we will generally reserve the word "isomorphism" for isomorphisms in $C(\mathfrak{A})$. Isomorphisms in $K(\mathfrak{A})$ (resp. $D(\mathfrak{A})$) are referred to as homotopy equivalences (resp. quasi-isomorphisms).

If r is an integer, then any complex C^* of objects of $\mathfrak{A}$ has two *canonical truncations* at r and a *naive truncation*:

$$\begin{array}{llcccccccccc}
\tau_{\leq r}(C^*): & \cdots \to C^{r-2} \to C^{r-1} \to & \ker(\delta^r) & \to & 0 & \to & 0 & \to \cdots, \\
\tau_{>r}(C^*): & \cdots \to \quad 0 \quad \to \quad 0 \quad \to & C^r/\ker(\delta^r) & \to & C^{r+1} & \to & C^{r+2} & \to \cdots. \\
\sigma_{\geq r}(C^*): & \cdots \to \quad 0 \quad \to \quad 0 \quad \to & C^r & \to & C^{r+1} & \to & C^{r+2} & \to \cdots.
\end{array}$$

All the truncations are functorial in $C(\mathfrak{A})$. The canonical truncations are functorial in $K(\mathfrak{A})$ and $D(\mathfrak{A})$ as well. We will often find it more convenient to write $\tau_{<r+1}$ instead of $\tau_{\leq r}$.

Suppose now that $\mathfrak{A}$ has enough projectives. Every bounded above complex C^* of objects in $\mathfrak{A}$ admits a *projective resolution*, i.e. a quasi-isomorphism $P^* \to C^*$ with P^* a complex of projectives ([Ha] Proposition I.4.6). The projective resolution of a complex is unique up to homotopy equivalence. If C^* and E^* are bounded above complexes of objects in $\mathfrak{A}$, and if $P^* \to C^*$ is a projective resolution of C^*, then there is a natural isomorphism $\mathrm{Hom}_{D^-(\mathfrak{A})}(C^*, E^*) \cong \mathrm{Hom}_{K^-(\mathfrak{A})}(P^*, E^*)$. In particular if $\mathfrak{P}$ denotes the full subcategory of projective objects of $\mathfrak{A}$, then the natural functor $K^-(\mathfrak{P}) \to \mathfrak{D}^-(\mathfrak{A})$ is an equivalence of categories ([Ha] Proposition I.4.7). This can be refined to the following statement:

Lemma 2.1. *Let $\mathfrak{A}$ be an abelian category with enough projectives, and let $\mathfrak{P}$ be the full subcategory of projective objects of $\mathfrak{A}$. Suppose $A \subset D^-(\mathfrak{A})$ and $P \subset K^-(\mathfrak{P})$ are full subcategories such that $\mathrm{ob}(P) \subset \mathrm{ob}(A)$ and every object of A has a projective resolution belonging to P. Then the natural functor $P \to A$ is an equivalence of categories.*

Let $S = k[X_0, \ldots, X_N]$ be the homogeneous coordinate ring of $\mathbb{P}^{\mathbb{N}}$, and let $\mathfrak{m} = (\mathfrak{X}_{\mathfrak{o}}, \ldots, \mathfrak{X}_{\mathfrak{N}})$ be its irrelevant ideal. Let $\mathrm{Mod}_{S,\mathrm{gr}}$ be the category of graded S-modules. Then $\mathrm{Mod}_{S,\mathrm{gr}}$ has enough projectives, namely the free modules. We will call a complex P^* of projectives in $\mathrm{Mod}_{S,\mathrm{gr}}$ a *minimal* if all its objects P^i are free of finite rank and its differential δ^* satisfies $\delta^i(P^i) \subset \mathfrak{m}\mathfrak{P}^{\mathfrak{i}+\mathfrak{1}}$ for all i. If C^* is a bounded above complex of objects in $\mathrm{Mod}_{S,\mathrm{gr}}$ whose cohomology modules $H^i(C^*)$ are all finitely generated, then C^* has a *minimal projective resolution*, i.e. a projective resolution by a minimal complex of projectives. The next lemma, which is a well known consequence of Nakayama's lemma, says that minimal projective resolutions are unique up to isomorphism and not merely up to homotopy equivalence:

Lemma 2.2. *Let $\phi : P^* \to Q^*$ be a homotopy equivalence between* minimal *complexes of free graded S-modules of finite rank. Then ϕ is an isomorphism.*

Let $\mathrm{Mod}_{\mathcal{O}}$ be the category of sheaves of $\mathcal{O}_{\mathbb{P}^{\mathbb{N}}}$-modules. For $\mathcal{E}$ a sheaf of $\mathcal{O}_{\mathbb{P}^{\mathbb{N}}}$-modules, let $\Gamma_*(\mathcal{E}) = \bigoplus_{\sqcup\in\mathbb{Z}} \Gamma(\mathcal{E}(\sqcup))$. Then Γ_* defines a left exact functor from $\mathrm{Mod}_{\mathcal{O}}$ to $\mathrm{Mod}_{S,\mathrm{gr}}$. It has a right derived functor $\mathbf{R}\Gamma_* : \mathbf{D^b}(\mathrm{Mod}_{\mathcal{O}}) \to \mathbf{D^b}(\mathrm{Mod}_{S,\mathrm{gr}})$ whose cohomology functors we denote $H^i_*(\mathcal{E}) = \bigoplus_{\sqcup\in\mathbb{Z}} \mathcal{H}^{\rangle}(\mathcal{E}(\sqcup))$. The functor Γ_* has an exact left adjoint $\tilde{\ }$, the functor of associated sheaves.

Let $\Gamma_{\mathfrak{m}} : \mathrm{Mod}_{S,\mathrm{gr}} \to \mathrm{Mod}_{S,\mathrm{gr}}$ be the functor associating to a graded S-module M the maximal submodule $\Gamma_{\mathfrak{m}}(M) \subset M$ supported at the origin 0 of $\mathbb{A}^{\mathbb{N}+\not{\Bbbk}}$. This functor is also left exact and has a right derived functor $\mathbf{R}\Gamma_{\mathfrak{m}} : \mathbf{D^b}(\mathrm{Mod}_{S,\mathrm{gr}}) \to \mathbf{D^b}(\mathrm{Mod}_{S,\mathrm{gr}})$. Its cohomology functors are denoted $H^i_{\mathfrak{m}}$.

Lemma 2.3. *Let P^* be a bounded complex of free graded S-modules of finite rank where $S = k[X_0, \ldots, X_N]$. If P^* is minimal, then*

$$\begin{aligned} \max\{i \mid P^i \neq 0\} &= \max\{i \mid H^i(P^*) \neq 0\}, \\ \min\{i \mid P^i \neq 0\} &= \min\{i \mid H^i_{\mathfrak{m}}(P^*) \neq 0\} - N - 1. \end{aligned}$$

Proof. The assertion about maxima is a simple and well-known application of the minimality condition and Nakayama's Lemma. The assertion about minima, which is essentially the Auslander-Buchsbaum theorem, reduces to the assertion about maxima by Serre duality. □

The Horrocks Correspondence. We now begin to describe the components of the Horrocks correspondence. Let $\mathfrak{B}$ be the full subcategory of $\mathrm{Mod}_{\mathcal{O}}$ of locally free sheaves of finite rank, and let $\mathfrak{Z}$ denote the full category of $D^b(\mathrm{Mod}_{S,\mathrm{gr}})$ of complexes C^* such that $H^i(C^*)$ is of finite length for $0 < i < N$ and $H^i(C^*)$ vanishes for all other i.

The Horrocks correspondence consists of a functor $\zeta : \mathfrak{B} \to \mathfrak{Z}$ and a map $\mathcal{H} : \mathrm{ob}(\mathfrak{Z}) \to \mathrm{ob}(\mathfrak{B})$ in the opposite direction. The functor ζ is simply $\tau_{>0}\tau_{<N}\mathbf{R}\Gamma_*$. For $\mathcal{E}$ a vector bundle on $\mathbb{P}^{\mathbb{N}}$, the cohomology of $\zeta(\mathcal{E})$ is of course:

$$H^i(\zeta(\mathcal{E})) = \begin{cases} H^i_*(\mathcal{E}) & \text{if } 0 < i < N, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{E}$ is locally free of finite rank, $H^i_*(\mathcal{E})$ is of finite length for $0 < i < N$. So $\zeta(\mathcal{E}) \in \mathrm{ob}(\mathfrak{Z})$.

We now define $\mathcal{H}$. Any $C^* \in \mathrm{ob}(\mathfrak{Z})$ has a minimal projective resolution $P^* \to C^*$. We define $\mathcal{H}(C^*)$ to be the kernel of the differential $\widetilde{\delta}^0 : \widetilde{P}^0 \to \widetilde{P}^1$. Then $\mathcal{H}(C^*)$ is a vector bundle because it fits into an exact complex of vector bundles

$$\cdots \to 0 \to \mathcal{H}(\mathcal{C}^*) \to \widetilde{\mathcal{P}}' \to \widetilde{\mathcal{P}}^{\infty} \to \cdots \to \widetilde{\mathcal{P}}^{\mathcal{N}-\infty} \to \prime \to \cdots . \tag{7}$$

Note that $\mathcal{H}(\mathcal{C}^*)$ is well-defined up to isomorphism because the minimal projective resolution P^* of C^* is unique up to isomorphism because of Lemma 2.2. However, $\mathcal{H}$ is not a functor.

The principal results of Horrocks' paper [Ho] can be described in the following way:

Theorem 2.4 (Horrocks)**.** *Let $\mathfrak{B}$ be the category of locally free sheaves of finite rank on $\mathbb{P}^{\mathbb{N}}$, and let $\mathfrak{Z}$ be the full subcategory of $D^b(\mathrm{Mod}_{S,\mathrm{gr}})$ of complexes C^* such that $H^i(C^*)$ is of finite length if $0 < i < N$, and $H^i(C^*) = 0$ for all other i. Let*

$\zeta = \tau_{>0}\tau_{<N}\mathbf{R}\Gamma_* : \mathfrak{B} \to \mathfrak{Z}$, and let $\mathcal{H} : \mathrm{ob}(\mathfrak{Z}) \to \mathrm{ob}(\mathfrak{B})$ be the map defined as in (7) above.

(a) If $\mathcal{E} \in \mathrm{ob}(\mathfrak{B})$, then $\mathcal{E} \cong \mathcal{H}\zeta(\mathcal{E}) \oplus \bigoplus_\rangle \mathcal{O}_{\mathbb{P}^\mathbb{N}}(\backslash_\rangle)$ for some integers n_i.

(b) If $C^* \in \mathrm{ob}(\mathfrak{Z})$, then $\zeta\mathcal{H}(\mathcal{C}^*) \simeq \mathcal{C}^*$.

(c) If $\mathcal{E}, \mathcal{F} \in \mathrm{ob}(\mathfrak{B})$, then $\mathrm{Hom}_\mathfrak{Z}(\zeta(\mathcal{E}), \zeta(\mathcal{F})) \cong \mathrm{Hom}(\mathcal{E}, \mathcal{F})/\mathrm{Hom}_\Phi(\mathcal{E}, \mathcal{F})$ where $\mathrm{Hom}_\Phi(\mathcal{E}, \mathcal{F})$ is the set of all morphisms which factor through a direct sum of line bundles.

The theorem may be read as saying the following. Call two vector bundles $\mathcal{E}$ and $\mathcal{F}$ *stably equivalent* if there exist sets of integers $\{n_i\}$ and $\{m_j\}$ such that $\mathcal{E} \oplus \bigoplus_\rangle \mathcal{O}_{\mathbb{P}^\mathbb{N}}(\backslash_\rangle) \cong \mathcal{F} \oplus \bigoplus_| \mathcal{O}_{\mathbb{P}^\mathbb{N}}(\Updownarrow_|)$. Then the theorem says that ζ and $\mathcal{H}$ induce a one-to-one correspondence between stable equivalence classes of vector bundles on $\mathbb{P}^\mathbb{N}$ and quasi-isomorphism classes of complexes in $\mathfrak{Z}$.

For Horrocks' proof of the theorem, see [Ho] Lemma 7.1 and Theorem 7.2 and the discussion between them. However, Horrocks' definition of the category $\mathfrak{Z}$ and the functor ζ are different from ours, and demonstrating the equivalence of the definitions is somewhat tedious. So instead of referring the reader to Horrocks' paper, we give a new proof. The first step is the following lemma:

Lemma 2.5. *(a) Suppose*

$$P^* : \qquad \cdots \to 0 \to P^0 \to P^1 \to \cdots \to P^{N-1} \to 0 \to \cdots$$

is a complex of free graded S-modules of finite rank such that $H^i(P^*)$ is a module of finite length for $0 < i < N$. Let $\mathcal{E} = \mathcal{H}'(\mathcal{P}^*)^\sim$. Then P^* is quasi-isomorphic to $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$.

(b) Conversely, if $\mathcal{E}$ is a vector bundle on $\mathbb{P}^\mathbb{N}$, then the minimal projective resolution of $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$ is of the above form.

Proof. (a) Note that the complex $\widetilde{P}^*$ of coherent sheaves on $\mathbb{P}^\mathbb{N}$ has vanishing cohomology in degrees different from 0. So it is quasi-isomorphic to $H^0(\widetilde{P}^*) = \mathcal{E}$. Hence the triangle of functors of [W] Proposition 1.1:

$$\mathbf{R}\Gamma_\mathfrak{m} \to \mathrm{Id} \to \mathbf{R}\Gamma_* \circ^\sim \to \mathbf{R}\Gamma_\mathfrak{m}[\mathbf{1}],$$

when applied to P^*, yields a triangle

$$\mathbf{R}\Gamma_\mathfrak{m}(\mathbf{P}^*) \to \mathbf{P}^* \xrightarrow{\beta} \mathbf{R}\Gamma_*(\mathcal{E}) \to \mathbf{R}\Gamma_\mathfrak{m}(\mathbf{P}^*)[\mathbf{1}]. \tag{8}$$

By Lemma 2.3, we have $H^i_\mathfrak{m}(P^*) = 0$ for $i \le N$. So $H^i(\beta) : H^i(P^*) \to H^i_*(\mathcal{E})$ is an isomorphism for $i < N$. Therefore β induces a quasi-isomorphism of P^* onto $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$.

(b) Conversely, if $\mathcal{E}$ is a vector bundle on $\mathbb{P}^\mathbb{N}$, then $H^i_*(\mathcal{E})$ is finitely generated for $i < N$. Hence $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$ has a minimal projective resolution P^*. For $0 < i < N$ the module $H^i(P^*) = H^i_*(\mathcal{E})$ is of finite length because $\mathcal{E}$ is locally free. By construction $H^i(P^*) = H^i(\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})) = \prime$ for $i \ge N$. So we have $P^i = 0$ for $i \ge N$ by Lemma 2.3. Looking again at the triangle (8), we see by the construction of P^* that $H^i(\beta)$ is an isomorphism for $i < N$ and an injection for $i = N$. So $H^i_\mathfrak{m}(P^*) = 0$ for $i \le N$. So by Lemma 2.3 we see that $P^i = 0$ for $i \le -1$. Thus P^* has the form asserted by the lemma. ☐

We now wish to functorialize the previous lemma. Let $B \subset K^b(\mathrm{Mod}_{S,\mathrm{gr}})$ be the full subcategory of complexes of the form

$$\cdots \to 0 \to P^0 \to P^1 \to \cdots \to P^{N-1} \to 0 \to \cdots \tag{9}$$

such that the P^i are free of finite rank for all i, the modules $H^i(P^*)$ are of finite length for $0 < i < N$ and the differentials satisfy $\delta^i(P^i) \subset \mathfrak{m}\mathfrak{P}^{\mathfrak{i}+\mathfrak{1}}$ for all i. For any vector bundle $\mathcal{E}$ on $\mathbb{P}^{\mathbb{N}}$ we now define $P^*(\mathcal{E})$ as the minimal projective resolution of $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$. By Lemma 2.5, $P^*(\mathcal{E})$ is always an object of B.

Lemma 2.6. *The functor $P^* : \mathfrak{B} \longrightarrow \mathfrak{B}$ which associates to an $\mathcal{E} \in \mathrm{ob}(\mathfrak{B})$ the minimal projective resolution of $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$ is an equivalence of categories with inverse given by $C^* \mapsto H^0(C^*)^\sim$.*

Proof. Since the functor $\tau_{<N}\mathbf{R}\Gamma_* : \mathfrak{B} \longrightarrow \mathfrak{D}^-(\mathrm{Mod}_{S,\mathrm{gr}})$ has a left inverse $H^0(-)^\sim$, it induces an equivalence between $\mathfrak{B}$ and the full subcategory $A \subset D^-(\mathrm{Mod}_{S,\mathrm{gr}})$ of complexes quasi-isomorphic to complexes in the image of $\tau_{<N}\mathbf{R}\Gamma_*$. But by Lemma 2.5, the full subcategory $B \subset K^-(\mathrm{Mod}_{S,\mathrm{gr}})$ has the properties that $\mathrm{ob}(B) \subset \mathrm{ob}(A)$ and that the minimal projective resolution of every object of A belongs to B. Hence the natural functor $B \longrightarrow A$ is also an equivalence of categories by Lemma 2.1. Since P^* is exactly the composition of the equivalence $\tau_{<N}\mathbf{R}\Gamma_* : \mathfrak{B} \longrightarrow \mathfrak{A}$ with the inverse of the equivalence $B \longrightarrow A$, it is an equivalence. The inverse of P^* remains the same as that of $\tau_{<N}\mathbf{R}\Gamma_*$, namely $H^0(-)^\sim$. $\square$

Now the graded module associated to a vector bundle $\mathcal{E}$ on $\mathbb{P}^{\mathbb{N}}$ has a minimal projective resolution:

$$0 \longrightarrow Q^{-(N-1)} \longrightarrow \cdots \longrightarrow Q^{-1} \longrightarrow Q^0 \longrightarrow \Gamma_*(\mathcal{E})$$

For any $\mathcal{E}$ we now define the following complexes in addition to the $P^*(\mathcal{E})$ defined above. First we set:

$$Q^*(\mathcal{E}) : \qquad \cdots \longrightarrow / \longrightarrow \mathcal{Q}^{-(\mathcal{N}-\infty)} \longrightarrow \cdots \longrightarrow \mathcal{Q}^{-\infty} \longrightarrow \mathcal{Q}' \longrightarrow / \longrightarrow \cdots .$$

We then let $R^*(\mathcal{E})$ be the natural concatenation of $Q^*(\mathcal{E})$ with $P^*(\mathcal{E})$ induced by the composition $Q^0 \twoheadrightarrow \Gamma_*(\mathcal{E}) \hookrightarrow \mathcal{P}'$:

$$R^*(\mathcal{E}) : \qquad \cdots \longrightarrow / \longrightarrow \mathcal{Q}^{-(\mathcal{N}-\infty)} \longrightarrow \cdots \longrightarrow \mathcal{Q}' \longrightarrow \mathcal{P}' \longrightarrow \cdots \longrightarrow \mathcal{P}^{\mathcal{N}-\infty} \longrightarrow / \longrightarrow \cdots$$

Thus $R^i(\mathcal{E}) = \mathcal{P}^{\rangle}(\mathcal{E})$ for $i \geq 0$, and $R^i(\mathcal{E}) = \mathcal{Q}^{\rangle+\infty}(\mathcal{E})$ for $i < 0$. Note that although the projective complexes $P^*(\mathcal{E})$ and $Q^*(\mathcal{E})$ are minimal, $R^*(\mathcal{E})$ may not be minimal, because there may be a direct factor of $Q^0(\mathcal{E})$ which is mapped isomorphically onto a direct factor of $P^0(\mathcal{E})$. However, one may write $R^*(\mathcal{E})$ as the direct sum of a minimal complex of projectives $R^*_{\min}(\mathcal{E})$

$$R^*_{\min}(\mathcal{E}) : \qquad \cdots \longrightarrow \mathcal{Q}^{-\in} \longrightarrow \mathcal{Q}^{-\infty} \longrightarrow \mathcal{Q}'_{\min} \longrightarrow P^0_{\min} \longrightarrow P^1 \longrightarrow P^2 \longrightarrow \cdots$$

and of an exact complex of projectives

$$\cdots \longrightarrow 0 \longrightarrow L \xrightarrow{\mathrm{Id}} L \longrightarrow 0 \longrightarrow \cdots . \tag{10}$$

The complexes $Q^*(\mathcal{E})$, $R^*(\mathcal{E})$, and $R^*_{\min}(\mathcal{E})$ are all functorial (in the homotopy category) in $\mathcal{E}$. Moreover, we may use the identification between the categories $\mathfrak{B}$ and B to define complexes $Q^*(P^*)$, $R^*(P^*)$, and $R^*_{\min}(P^*)$ for P^* in B. Namely, $Q^*(P^*)$ is the minimal projective resolution of $H^0(P^*)$, $R^*(P^*)$ is the concatenation of $Q^*(P^*)$ with P^*, etc.

We now define a homotopy category of complexes of type $R^*_{\min}$. More formally, let $Z \subset K^b(\mathrm{Mod}_{S,\mathrm{gr}})$ be the full subcategory of minimal complexes of projective modules of finite rank of the form

$$\cdots \longrightarrow 0 \longrightarrow R^{-N} \longrightarrow \cdots \longrightarrow R^{-1} \longrightarrow R^0 \longrightarrow \cdots \longrightarrow R^{N-1} \longrightarrow 0 \longrightarrow \cdots \tag{11}$$

such that the cohomology modules $H^i(R^*)$ are of finite length for $0 < i < N$ and vanish for all other i.

We need one more lemma before proving Theorem 2.4.

Lemma 2.7. *The natural functor $Z \to \mathfrak{Z}$ is an equivalence of categories.*

Proof. Let R^* be the minimal projective resolution of an object C^* of $\mathfrak{Z}$. Since $H^i(R^*) = H^i(C^*) = 0$ for $i \geq N$, we have $R^i = 0$ for $i \geq N$ by Lemma 2.3. Moreover, all the $H^i(C^*)$ are of finite length, so $H^i_{\mathfrak{m}}(C^*) = H^i(C^*)$ for all i. In particular, $H^i_{\mathfrak{m}}(R^*) = H^i_{\mathfrak{m}}(C^*) = 0$ for $i \leq 0$. So $R^i = 0$ for $i \leq -N-1$ by Lemma 2.3. Thus the minimal projective resolution of any object of $\mathfrak{Z}$ is in Z. The lemma now follows from Lemma 2.1. $\square$

Proof of Theorem 2.4. Lemma 2.6 permits us to identify a vector bundle $\mathcal{E}$ with the complex $P^*(\mathcal{E})$ of B. Since $P^*(\mathcal{E})$ is already quasi-isomorphic to $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$, the complex $\zeta(\mathcal{E}) = \tau_{>/}\tau_{<\mathcal{N}}\mathbf{R}\Gamma_*(\mathcal{E})$ is quasi-isomorphic to the complex

$$\cdots \to 0 \to \Gamma_*(\mathcal{E}) \to \mathcal{P}' \to \mathcal{P}^\infty \to \cdots \to \mathcal{P}^{\mathcal{N}-\infty} \to / \to \cdots$$

and hence to the complexes $R^*(\mathcal{E})$ and $R^*_{\min}(\mathcal{E})$. Hence the object $\zeta(\mathcal{E})$ in $\mathfrak{Z}$ is quasi-isomorphic to the object $R^*_{\min}(\mathcal{E})$ of Z. Hence after identifying $\mathfrak{B}$ and $\mathfrak{Z}$ with B and Z by Lemmas 2.6 and 2.7, the functor ζ may be identified with the functor from B to Z which associates to any complex P^* in B the corresponding complex $R^*_{\min}$ as described earlier.

Similarly, given any object C^* of $\mathfrak{Z}$ with minimal projective resolution R^*, the definitions say that $P^*(\mathcal{H}(\mathcal{C}^*)) = \sigma_{\geq /}(\mathcal{R}^*)$, the naive truncation. Thus the map $\mathcal{H} : \mathrm{ob}(\mathfrak{Z}) \to \mathrm{ob}(\mathfrak{B})$ may be identified with $\sigma_{\geq 0} : \mathrm{ob}(Z) \to \mathrm{ob}(B)$. Note that since all objects of Z and B are minimal complexes of projective modules, homotopy equivalence classes of objects of Z and B coincide with isomorphism classes. Hence the map $\sigma_{\geq 0} : \mathrm{ob}(Z) \to \mathrm{ob}(B)$ preserves homotopy equivalence. Since Z and B are subcategories of the homotopy category, this means that $\sigma_{\geq 0}$ is well-defined on objects of Z. However, $\sigma_{\geq 0}$ and hence $\mathcal{H}$ are not well-defined on morphisms of Z.

(a) The above identifications now say if $\mathcal{E} \in \mathrm{ob}(\mathfrak{B})$, then $\mathcal{H}\zeta(\mathcal{E})$ is the object of $\mathfrak{B}$ corresponding to the complex $\sigma_{\geq 0}(R^*_{\min}(\mathcal{E}))$:

$$\sigma_{\geq 0}(R^*_{\min}(\mathcal{E})) : \qquad \cdots \to / \to \mathcal{P}'_{\min} \xrightarrow{\mu} \mathcal{P}^\infty \to \cdots \to \mathcal{P}^{\mathcal{N}-\infty} \to / \to \cdots .$$

By Lemma 2.6, the sheaf $\mathcal{H}\zeta(\mathcal{E})$ is $\ker(\mu)^\sim$. So $\mathcal{E} = \mathcal{H}\zeta(\mathcal{E}) \oplus \widetilde{\mathcal{L}}$ where L is the projective module of (10). Since $\widetilde{L}$ is now a direct sum of line bundles, (a) follows.

(b) If C^* is an object of $\mathfrak{Z}$ with minimal projective resolution R^* in Z of the form (11), then the above computations identify $\mathcal{H}(\mathcal{C}^*)$ in $\mathfrak{B}$ with $P^*(\mathcal{H}(\mathcal{C}^*)) = \sigma_{\geq /}(\mathcal{R}^*)$ in B. Thus $\zeta\mathcal{H}(\mathcal{C}^*)$ becomes identified with $R^*_{\min}(\mathcal{H}(\mathcal{C}^*))$ which is just R^* again. Since R^* is quasi-isomorphic to C^*, we have $\zeta\mathcal{H}(\mathcal{C}^*) \simeq \mathcal{C}^*$ as desired.

(c) After identifying $\mathfrak{B}$ with B and $\mathfrak{Z}$ with Z, assertion (c) becomes the statement: For any pair of objects E^* and F^* in B, the natural map

$$\mathrm{Hom}_B(E^*, F^*) \to \mathrm{Hom}_Z(R^*_{\min}(E^*), R^*_{\min}(F^*)) \tag{12}$$

is surjective and its kernel is the subspace of morphisms which factor through an object of B of the form

$$\cdots \to 0 \to L \to 0 \to \cdots \tag{13}$$

with L a free graded S-module of finite rank appearing in degree 0.

We first prove surjectivity. Suppose $\phi \in \mathrm{Hom}_Z(R^*_{\min}(E^*), R^*_{\min}(F^*))$. Since Z is a homotopy category, ϕ is actually a homotopy equivalence class of maps in $C(\mathrm{Mod}_{S,\mathrm{gr}})$. So we may choose a chain map f in the class ϕ. Then f may be extended to a chain map $\overline{f} : R^*(E^*) \longrightarrow R^*(F^*)$ by defining it to be 0 on the exact factor of the type (10). Then $\sigma_{\geq 0}\overline{f}$ maps E^* to F^*, and its homotopy class in B has image ϕ in Z. This proves surjectivity.

We now compute the kernel of (12). First if $\alpha \in \mathrm{Hom}_B(E^*, F^*)$ factors through a complex L^* of the form (13), then $R^*_{\min}(\alpha)$ factors through $R^*_{\min}(L^*) = 0$ and so vanishes. So the kernel of (12) contains all morphisms which factor through complexes of the form (13).

Conversely, suppose α is in the kernel of (12). Since α is a morphism in B, it is a homotopy class of chain maps from which we may choose a member β. We may complete β to a chain map $\rho : R^*(E^*) \longrightarrow R^*(F^*)$.

$$\begin{array}{ccccccccccccccc}
R^*(E^*) & & \cdots \to & 0 & \to & \overline{E}^{-N} & \to \cdots \to & \overline{E}^{-1} & \to & E^0 & \to \cdots \to & E^{N-1} & \to & 0 & \to \cdots \\
\big\downarrow{\scriptstyle\rho} & & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\beta} & & \big\downarrow & \\
R^*(F^*) & & \cdots \to & 0 & \to & \overline{F}^{-N} & \to \cdots \to & \overline{F}^{-1} & \to & F^0 & \to \cdots \to & F^{N-1} & \to & 0 & \to \cdots
\end{array}$$

The homotopy class of ρ is the image of α under R^* and so must vanish by hypothesis. (Note that R^* and $R^*_{\min}$ are homotopy equivalent.) Thus ρ is homotopic to 0. Thus if we write δ^i for the differentials of $R^*(E^*)$, and ϵ^i for the differentials of $R^*(F^*)$, then there is a chain homotopy $h = (h^i)$ such that $\rho^i = h^{i+1}\delta^i + \epsilon^{i-1}h^i$ for all i. Now restrict h to a chain homotopy $\widehat{h} = (\widehat{h}^i)$ with $\widehat{h}^i : E^i \longrightarrow F^{i-1}$ defined by defined by $\widehat{h}^i = h^i$ for all $i \geq 1$, and $\widehat{h}^i = 0$ for all $i \leq 0$. Then β is homotopic to a morphism whose components are

$$\beta^i - (\widehat{h}^{i+1}\delta^i + \epsilon^{i-1}\widehat{h}^i) = \begin{cases} \rho^i - (h^{i+1}\delta^i + \epsilon^{i-1}h^i) = 0 & \text{if } i \geq 1, \\ \rho^0 - h^1\delta^0 = \epsilon^{-1}h^0 & \text{if } i = 0, \\ 0 & \text{if } i \leq -1. \end{cases}$$

Hence the homotopy class α of β factors through the complex

$$\cdots \longrightarrow 0 \longrightarrow \overline{F}^{-1} \longrightarrow 0 \longrightarrow \cdots$$

of type (13). So the kernel of (12) is as asserted. This completes the proof of the theorem. $\square$

We will use three further results concerning the Horrocks correspondence. The first will permit us to use the Horrocks correspondence to constuct locally free resolutions of coherent sheaves.

Proposition 2.8. *Let $\mathcal{Q}$ be a quasi-coherent sheaf on $\mathbb{P}^{\mathbb{N}}$, let $C^* \in \mathrm{ob}(\mathfrak{Z})$, and let $\beta : C^* \longrightarrow \tau_{>0}\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{Q})$ be a morphism in $D^b(\mathrm{Mod}_{S,\mathrm{gr}})$. Then there exists a morphism of quasi-coherent sheaves $\widetilde{\beta} : \mathcal{H}(\mathcal{C}^*) \longrightarrow \mathcal{Q}$ such that $\beta = \tau_{>0}\tau_{<N}\mathbf{R}\Gamma_*(\widetilde{\beta})$. In particular, the induced morphisms $H^i_*(\mathcal{H}(\mathcal{C}^*)) \longrightarrow \mathcal{H}^{\rangle}_*(\mathcal{Q})$ are the same as $H^i(\beta)$ for $1 \leq i \leq N-1$.*

Proof. Let R^* be a minimal projective resolution of C^*, and let $\mathcal{I}^*$ be an injective resolution of $\mathcal{Q}$. Then β may be identified with an actual chain map

$$\begin{tikzcd}
\cdots \arrow[r] & R^{-2} \arrow[r] \arrow[d] & R^{-1} \arrow[r] \arrow[d] & R^0 \arrow[r, "\lambda"] \arrow[d] & R^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & R^{N-2} \arrow[r] \arrow[d] & R^{N-1} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & \Gamma_*(\mathcal{Q}) \arrow[r] & \Gamma_*(\mathcal{I}') \arrow[r, "\mu"] & \Gamma_*(\mathcal{I}^\infty) \arrow[r] & \cdots \arrow[r] & \Gamma_*(\mathcal{I}^{\mathcal{N}-\in}) \arrow[r] & \ker(\delta^{N-1}) \arrow[r] & 0
\end{tikzcd}$$

Thus β induces a morphism $\widetilde{\beta}$ from $\mathcal{H}(\mathcal{C}^*) = \ker(\lambda)^\sim$ to $\mathcal{Q} = \ker(\mu)^\sim$.

We now need to calculate $\mathbf{R}\Gamma_*(\widetilde{\beta})$. Consider the complex

$$P^* : \qquad \cdots \to 0 \to R^0 \to R^1 \to \cdots \to R^{N-1} \to 0 \to \cdots .$$

The previous diagram induces a new commutative diagram

$$\begin{tikzcd}
\widetilde{P}^*: \arrow[d, "\overline{\beta}"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \widetilde{R}^0 \arrow[r] \arrow[d] & \widetilde{R}^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \widetilde{R}^{N-2} \arrow[r] \arrow[d] & \widetilde{R}^{N-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\mathcal{I}^*: & \cdots \arrow[r] & 0 \arrow[r] & \mathcal{I}' \arrow[r] & \mathcal{I}^\infty \arrow[r] & \cdots \arrow[r] & \mathcal{I}^{\mathcal{N}-\in} \arrow[r] & \mathcal{I}^{\mathcal{N}-\infty} \arrow[r] & \mathcal{I}^{\mathcal{N}} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

between resolutions of $\mathcal{H}(\mathcal{C}^*)$ and $\mathcal{Q}$ extending $\widetilde{\beta}$. Let $\gamma : P^* \longrightarrow J^*$ be an injective resolution of P^*. Then $\overline{\beta}$ factors through $\widetilde{\gamma}$ as $\widetilde{P}^* \longrightarrow \widetilde{J}^* \longrightarrow \mathcal{I}^*$. Applying Γ_* now gives a factorization

$$P^* \longrightarrow \Gamma_*(\widetilde{J}^*) \longrightarrow \Gamma_*(\mathcal{I}^*). \tag{14}$$

Now $\overline{\beta}$ is a map between resolutions of $\mathcal{H}(\mathcal{C}^*)$ and $\mathcal{Q}$, respectively, which extends $\widetilde{\beta} : \mathcal{H}(\mathcal{C}^*) \longrightarrow \mathcal{Q}$, while $\widetilde{\gamma}$ is a quasi-isomorphism. So the map $\widetilde{J}^* \longrightarrow \mathcal{I}^*$ is a map between injective resolutions of $\mathcal{H}(\mathcal{C}^*)$ and $\mathcal{Q}$ extending $\widetilde{\beta}$. So by definition, the second arrow of (14) is $\mathbf{R}\Gamma_*(\widetilde{\beta}) : \mathbf{R}\Gamma_*(\mathcal{H}(\mathcal{C}^*)) \longrightarrow \mathbf{R}\Gamma_*(\mathcal{Q})$. On the other hand, the proof of Lemma 2.5(a) shows that the first arrow of (14) can be identified with the truncation $\tau_{<N}(\mathbf{R}\Gamma_*(\mathcal{H}(\mathcal{C}^*))) \to \mathbf{R}\Gamma_*(\mathcal{H}(\mathcal{C}^*))$ because it induces isomorphisms $H^i(P^*) \cong H^i(\Gamma_*(\widetilde{J}^*)) = H^i_*(\mathcal{H}(\mathcal{C}^*))$ for $i < N$. Hence $\Gamma_*(\overline{\beta}) : P^* \to \Gamma_*(\mathcal{I}^*)$ can be identified with the composition of the truncation $\tau_{<N}(\mathbf{R}\Gamma_*(\mathcal{H}(\mathcal{C}^*))) \to \mathbf{R}\Gamma_*(\mathcal{H}(\mathcal{C}^*))$ with $\mathbf{R}\Gamma_*(\widetilde{\beta})$. Thus $\tau_{<N}\mathbf{R}\Gamma_*(\widetilde{\beta})$ may be identified with the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & R^0 \arrow[r] \arrow[d] & R^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & R^{N-2} \arrow[r] \arrow[d] & R^{N-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \Gamma_*(\mathcal{I}') \arrow[r] & \Gamma_*(\mathcal{I}^\infty) \arrow[r] & \cdots \arrow[r] & \Gamma_*(\mathcal{I}^{\mathcal{N}-\in}) \arrow[r] & \ker(\delta^{N-1}) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

induced by β. Truncating on the left, we reach a diagram equivalent to the first diagram of the proof of the proposition. So $\beta = \tau_{>0}\tau_{<N}\mathbf{R}\Gamma_*(\widetilde{\beta})$. $\square$

We now need two homological criteria for maps of vector bundles to be isomorphisms.

Lemma 2.9. *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $\mathbb{P}^{\mathbb{N}}$ with neither containing a line bundle as a direct factor. If $\alpha : \mathcal{E} \longrightarrow \mathcal{F}$ is a map such that $H^i_*(\alpha) : H^i_*(\mathcal{E}) \longrightarrow \mathcal{H}^\rangle_*(\mathcal{F})$ is an isomorphism for $0 < i < N$, then α is an isomorphism.*

Proof. We use the notation of the proof of Theorem 2.4. Let $E^* = P^*(\mathcal{E})$ and $F^* = P^*(\mathcal{F})$, and let $\overline{\alpha} : E^* \longrightarrow F^*$ be the map induced by α. The hypothesis $\mathcal{E} = \mathcal{H}\zeta(\mathcal{E})$ implies that E^* is homotopy equivalent to $\sigma_{\geq 0}R^*_{\min}(E^*)$, or equivalently that

$R^*(E^*)$ is a minimal complex of projectives. Similarly, $R^*(F^*)$ is a minimal complex of projectives. The hypothesis on α implies that $\zeta(\alpha) : \zeta(\mathcal{E}) \to \zeta(\mathcal{F})$ is a quasi-isomorphism. This in turn translates into $R^*_{\min}(\overline{\alpha})$ being a homotopy equivalence. But because of the earlier hypotheses, this means that $R^*(\overline{\alpha}) : R^*(E^*) \to R^*(F^*)$ is a homotopy equivalence between the minimal complexes of projectives. Hence by Lemma 2.2 $R^*(\overline{\alpha})$ is actually an isomorphism of complexes. So its naive truncation $\sigma_{\geq 0}R^*(\overline{\alpha}) = \overline{\alpha}$ is also an isomorphism. Therefore α is an isomorphism. $\square$

We will also need a slight generalization of the previous lemma.

Lemma 2.10. *Let $\mathcal{E} = \mathcal{H}\zeta(\mathcal{E}) \oplus \bigoplus \mathcal{O}_{\mathbb{P}^{\mathbb{N}}}(\backslash_\rangle)$ and $\mathcal{F}$ be vector bundles on $\mathbb{P}^{\mathbb{N}}$, and let $\mathcal{Q}$ be a coherent sheaf on $\mathbb{P}^{\mathbb{N}}$. Suppose that there exist morphisms $\alpha : \mathcal{E} \to \mathcal{F}$ and $\beta : \mathcal{F} \to \mathcal{Q}$ such that*

(i) $H^i_*(\alpha) : H^i_*(\mathcal{E}) \to \mathcal{H}^\rangle_*(\mathcal{F})$ is an isomorphism for $0 < i < N$,

(ii) $\beta\alpha$ takes the generators of the factors $S(n_i)$ of $\Gamma_*(\mathcal{E})$ onto a minimal set of generators of the module $\overline{Q} := \Gamma_*(\mathcal{Q})/\beta\alpha(\Gamma_*(\mathcal{H}\zeta(\mathcal{E})))$,

(iii) $\mathcal{E}$ and $\mathcal{F}$ have the same rank.

Then α is an isomorphism.

Proof. Write $\mathcal{F} = \mathcal{H}\zeta(\mathcal{F}) \oplus \bigoplus \mathcal{O}_{\mathbb{P}^{\mathbb{N}}}(\Updownarrow_|)$. The splittings of $\mathcal{E}$ and of $\mathcal{F}$ into direct factors are not canonical. But choosing such splittings gives an injection $\mathcal{H}\zeta(\mathcal{E}) \hookrightarrow \mathcal{E}$ and a projection $\mathcal{F} \twoheadrightarrow \mathcal{H}\zeta(\mathcal{F})$. Then the composition

$$\overline{\alpha} : \quad \mathcal{H}\zeta(\mathcal{E}) \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \to \mathcal{H}\zeta(\mathcal{F})$$

is, like α, an isomorphism on H^i_* for $0 < i < N$. So $\overline{\alpha}$ is an isomorphism by Lemma 2.9. Hence by identifying $\mathcal{H}\zeta(\mathcal{F})$ with $\alpha(\mathcal{H}\zeta(\mathcal{E})) \subset \mathcal{F}$, we see that α induces a morphism of diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{H}\zeta(\mathcal{E}) & \longrightarrow & \mathcal{E} & \longrightarrow & \bigoplus \mathcal{O}_{\mathbb{P}^{\mathbb{N}}}(\backslash_\rangle) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_1} & & \\
0 & \longrightarrow & \mathcal{H}\zeta(\mathcal{F}) & \longrightarrow & \mathcal{F} & \longrightarrow & \bigoplus \mathcal{O}_{\mathbb{P}^{\mathbb{N}}}(\Updownarrow_|) & \longrightarrow & 0
\end{array} \tag{15}$$

The morphisms α and β therefore induce maps

$$\bigoplus S(n_i) \xrightarrow{\Gamma_*(\alpha_1)} \bigoplus S(m_j) \xrightarrow{\overline{\beta}} \overline{Q} = \Gamma_*(\mathcal{Q})/\beta\alpha(\Gamma_*(\mathcal{H}\zeta(\mathcal{E}))).$$

The composition is a surjection corresponding to a minimal set of generators of $\overline{Q}$ by hypothesis (ii). Hence the righthand map $\overline{\beta}$ must be a surjection corresponding to a set of generators of $\overline{Q}$. However, the two free modules have the same rank by hypothesis (iii). Hence $\overline{\beta}$ also corresponds to a minimal set of generators, and $\Gamma_*(\alpha_1)$ must be an isomorphism. So returning to diagram (15), α_1 and hence α are isomorphisms. $\square$

3. The Self-Dual Resolution

Let $X \subset \mathbb{P}^{\Bbbk+\not\models}$ be a locally Gorenstein subcanonical subscheme of equicodimension 3 satisfying the parity condition. In this section we use the Horrocks correspondence and especially Proposition 2.8 to construct a locally free resolution of $\mathcal{O}_{\mathcal{X}}$. We then use Lemma 2.10 to show that the resolution satisfies condition (a) of Proposition 1.2.

In the course of the construction we will need a more refined variant of the canonical truncation. Namely, suppose D^* is a complex of objects in an abelian category with differentials $\delta^i : D^i \to D^{i+1}$. Suppose r is an integer, and $W \subset H^r(D^*)$ a subobject. Then W may be pulled back to a $\overline{W}$ satisfying

$$\operatorname{im}(\delta^{r-1}) \subset \overline{W} \subset \ker(\delta^r) \subset D^r$$

We then define:

$$\tau_{\leq r,W}(D^*): \qquad \cdots \to D^{r-2} \to D^{r-1} \to \overline{W} \to 0 \to 0 \to \cdots \quad .$$

The cohomology of this complex is given by

$$H^i(\tau_{\leq r,W}(D^*)) = \begin{cases} H^i(D^*) & \text{if } i < r, \\ W & \text{if } i = r, \\ 0 & \text{if } i > r. \end{cases}$$

We will also use the following conventions. If $\mathcal{E}$ is a coherent sheaf on $\mathbb{P}^{\mathbb{N}}$ and $\alpha, \beta \in \mathbb{Q}$ are not both integers, then we define $H^\alpha(\mathcal{E}(\beta)) = /$. Also if D^* is a complex and $\alpha \in \mathbb{Q}$, we define $\tau_{\leq\alpha}(D^*) = \tau_{\leq[\alpha]}(D^*)$.

Definition of the Locally Free Resolution. Suppose $X \subset \mathbb{P}^{\ltimes+\not\models}$ is a locally Gorenstein subscheme of equidimension $n > 0$ such that $\omega_X \cong \mathcal{O}_{\mathcal{X}}(\updownarrow)$ for some integer l and such that $h^{n/2}(\mathcal{O}_{\mathcal{X}}(\updownarrow/\in))$ is even.

Let $\nu = n/2$ and $l' = l/2$. By hypothesis $H^\nu(\mathcal{O}_{\mathcal{X}}(\updownarrow'))$ is an even-dimensional vector space (zero if n or l is odd) equipped with a nondegenerate $(-1)^\nu$-symmetric bilinear form

$$H^\nu(\mathcal{O}_{\mathcal{X}}(\updownarrow')) \times \mathcal{H}^\nu(\mathcal{O}_{\mathcal{X}}(\updownarrow')) \to \mathcal{H}^\setminus(\mathcal{O}_{\mathcal{X}}(\updownarrow)) \cong \|.$$

Let $U \subset H^\nu(\mathcal{O}_{\mathcal{X}}(\updownarrow'))$ be an isotropic subspace of maximal dimension $h^\nu(\mathcal{O}_{\mathcal{X}}(\updownarrow'))/\in$. Let

$$W = U \oplus \bigoplus_{t>l'} H^\nu(\mathcal{O}_{\mathcal{X}}(\sqcup)) \subset \mathcal{H}^\nu_*(\mathcal{O}_{\mathcal{X}}) \tag{16}$$

We begin the construction of the locally free resolution with the short exact sequence

$$0 \to \mathcal{I}_{\mathcal{X}} \to \mathcal{O}_{\mathbb{P}^{\ltimes+\not\models}} \to \mathcal{O}_{\mathcal{X}} \to /. \tag{17}$$

Since $H^i_*(\mathcal{O}_{\mathcal{X}}) \cong \mathcal{H}_*^{\rangle+\infty}(\mathcal{I}_{\mathcal{X}})$ for $0 < i < n+2$, we have $W \subset H_*^{\nu+1}(\mathcal{I}_{\mathcal{X}})$.

Now since X is locally Cohen-Macaulay of equidimension n, the modules $H^i_*(\mathcal{I}_{\mathcal{X}})$ are of finite length for $0 < i < n+1$. Hence the truncated complex $C^*_X = \tau_{>0}\tau_{\leq\nu+1,W}\mathbf{R}\Gamma_*(\mathcal{I}_{\mathcal{X}})$ has cohomology modules $H^i(C^*_X)$ of finite length for $0 < i \leq \nu+1$, while $H^i(C^*_X) = 0$ for all other i. Hence C^*_X is in $\mathfrak{Z}$.

The definition of C^*_X as a truncation means that it is endowed with a natural map $\beta : C^*_X \to \tau_{>0}\tau_{<n+3}\mathbf{R}\Gamma_*(\mathcal{I}_{\mathcal{X}})$. By Proposition 2.8 this map induces a morphism $\widetilde{\beta} : \mathcal{H}(\mathcal{C}^*_{\mathcal{X}}) \to \mathcal{I}_{\mathcal{X}}$. Let Q be the cokernel

$$H^0_*(\mathcal{H}(\mathcal{C}^*_{\mathcal{X}})) \xrightarrow{\mathcal{H}'_*(\widetilde{\beta})} \mathcal{H}'_*(\mathcal{I}_{\mathcal{X}}) \to \mathcal{Q} \to /.$$

Let $d_1, \ldots, d_r$ be the degrees of a minimal set of generators of Q. These generators lift to $H^0_*(\mathcal{I}_{\mathcal{X}})$, allowing us to define a surjection

$$\gamma : \mathcal{F}_\infty := \mathcal{H}(\mathcal{C}^*_{\mathcal{X}}) \oplus \bigoplus \mathcal{O}_{\mathbb{P}^{\ltimes+\not\models}}(-\lceil_\rangle) \twoheadrightarrow \mathcal{I}_{\mathcal{X}}. \tag{18}$$

By construction, $\mathcal{F}_\infty$ is locally free.

Let $\mathcal{K} = \ker(\gamma)$. We may then attach the short exact sequence $0 \to \mathcal{K} \to \mathcal{F}_\infty \to \mathcal{I}_\mathcal{X} \to \prime$ to the short exact sequence (17) to get an exact sequence

$$0 \to \mathcal{K} \to \mathcal{F}_\infty \to \mathcal{O}_{\mathbb{P}^{\ltimes+\not\asymp}} \to \mathcal{O}_\mathcal{X} \to \prime. \tag{19}$$

The construction described above leads immediately to the following conclusions about the cohomology of $\mathcal{F}_\infty$ and about the induced morphisms $H^i_*(\gamma) : H^i_*(\mathcal{F}_\infty) \to \mathcal{H}^{\rangle}_*(\mathcal{I}_\mathcal{X})$ (cf. Proposition 2.8).

- $H^i_*(\gamma)$ is surjective (resp. an isomorphism) for $i = 0$ (resp. $0 < i < \nu + 1$).
- $H^{\nu+1}_*(\gamma) : H^{\nu+1}_*(\mathcal{F}_\infty) \cong \mathcal{W} \hookrightarrow \mathcal{H}^{\nu+\infty}_*(\mathcal{I}_\mathcal{X})$ is injective.
- $H^i_*(\mathcal{F}_\infty) = \prime$ for $\nu + 1 < i < n + 3$.

One may now draw the following conclusions about the cohomology of $\mathcal{K}$.

- $H^i_*(\mathcal{K}) = \prime$ for $0 < i < \nu + 2$.
- $H^{\nu+2}_*(\mathcal{K}) \cong \mathcal{H}^\nu_*(\mathcal{O}_\mathcal{X})/\mathcal{W}$.
- $H^i_*(\mathcal{K}) \cong \mathcal{H}^{\rangle-\in}_*(\mathcal{O}_\mathcal{X})$ for $\nu + 2 < i < n + 3$.

To finish the definition of the locally free resolution, consider the isomorphisms

$$\mathrm{Ext}^1(\mathcal{K}, \omega_{\mathbb{P}^{\ltimes+\not\asymp}}(-\updownarrow)) \cong \mathcal{H}^{\backslash+\in}(\mathcal{K}(\updownarrow))^* \cong \mathcal{H}^{\backslash}(\mathcal{O}_\mathcal{X}(\updownarrow))^* \cong \mathcal{H}'(\mathcal{O}_\mathcal{X}).$$

The extension class corresponding to $1 \in H^0(\mathcal{O}_\mathcal{X})$ gives a short exact sequence

$$0 \to \omega_{\mathbb{P}^{\ltimes+\not\asymp}}(-l) \to \mathcal{F}_\in \to \mathcal{K} \to \prime \tag{20}$$

which we may attach to (19) to get a complex of the type (4) resolving $\mathcal{O}_\mathcal{X}$

$$\mathcal{P}^* : \qquad \prime \to \omega_{\mathbb{P}^{\ltimes+\not\asymp}}(-\updownarrow) \to \mathcal{F}_\in \to \mathcal{F}_\infty \to \mathcal{O}_{\mathbb{P}^{\ltimes+\not\asymp}}. \tag{21}$$

Lemma 3.1. *The sheaves $\mathcal{F}_\infty$ and $\mathcal{F}_\in$ in the resolution* (21) *satisfy* $H^i_*(\mathcal{F}_\in) \cong \left(\mathcal{H}^{\backslash+\ni-\rangle}_*(\mathcal{F}_\infty)\right)^*(\updownarrow)$ *for* $0 < i < n + 3$.

Proof. If $0 < i < \nu + 2$, then $H^i_*(\mathcal{F}_\in) \cong \mathcal{H}^{\rangle}_*(\mathcal{K}) = \prime$ and $H^{n+3-i}_*(\mathcal{F}_\infty) = \prime$. So the lemma holds for these values of i.

If $i = \nu + 2$, then $H^{\nu+2}_*(\mathcal{F}_\in) \cong \mathcal{H}^{\nu+\in}_*(\mathcal{K}) \cong \mathcal{H}^\nu_*(\mathcal{O}_\mathcal{X})/\mathcal{W}$, while $H^{\nu+1}_*(\mathcal{F}_\infty) \cong \mathcal{W}$. However, the submodule $W \subset H^\nu_*(\mathcal{O}_\mathcal{X})$ has been constructed so that it is an isotropic submodule with respect to the perfect pairing of Serre duality

$$H^\nu_*(\mathcal{O}_\mathcal{X}) \times \mathcal{H}^\nu_*(\mathcal{O}_\mathcal{X}) \to \mathcal{H}^{\backslash}_*(\mathcal{O}_\mathcal{X}) \xrightarrow{\mathrm{tr}} \|(-\updownarrow).$$

Moreover the length of W is half the length of $H^\nu_*(\mathcal{O}_\mathcal{X})$. Hence $W = W^\perp$, and the duality isomorphism $H^\nu_*(\mathcal{O}_\mathcal{X}) \cong (\mathcal{H}^\nu_*(\mathcal{O}_\mathcal{X}))^*\,(\updownarrow)$ carries the submodule W onto $(H^\nu_*(\mathcal{O}_\mathcal{X})/\mathcal{W})^*\,(l)$.

If $\nu + 2 < i < n + 2$, then $H^i_*(\mathcal{F}_\in) \cong \mathcal{H}^{\rangle}_*(\mathcal{K}) \cong \mathcal{H}^{\rangle-\in}_*(\mathcal{O}_\mathcal{X})$, while $H^{n+3-i}_*(\mathcal{F}_\infty) \cong \mathcal{H}^{\backslash+\ni-\rangle}_*(\mathcal{I}_\mathcal{X}) \cong \mathcal{H}^{\backslash+\in-\rangle}_*(\mathcal{O}_\mathcal{X})$. The asserted duality is then simply the Serre duality pairing

$$H^{i-2}_*(\mathcal{O}_\mathcal{X}) \times \mathcal{H}^{\backslash+\in-\rangle}_*(\mathcal{O}_\mathcal{X}) \to \mathcal{H}^{\backslash}_*(\mathcal{O}_\mathcal{X}) \xrightarrow{\mathrm{tr}} \|(-\updownarrow).$$

Finally if $i = n + 2$, we have an exact sequence

$$0 \to H^{n+2}_*(\mathcal{F}_\in) \to \mathcal{H}^{\backslash+\in}_*(\mathcal{K}) \to \mathcal{H}^{\backslash+\ni}_*(\omega_{\mathbb{P}^{\ltimes+\not\asymp}}(-\updownarrow)).$$

Now $H^{n+2}_*(\mathcal{K}) \cong \mathcal{H}^{\backslash}_*(\mathcal{O}_\mathcal{X})$. Moreover, the fact that the extension class defining $\mathcal{F}_\in$ corresponded under the Serre duality identifications to $1 \in H^0_*(\mathcal{O}_\mathcal{X}) \cong$

$\left(\mathcal{H}_*^{\setminus+\in}(\mathcal{K})\right)^*(\updownarrow)$ implies that the last exact sequence dualizes to

$$H^0_*(\mathcal{O}_{\mathbb{P}^{\ltimes+\nVdash}}) \xrightarrow{\infty} \mathcal{H}'_*(\mathcal{O}_\mathcal{X}) \to \left(\mathcal{H}_*^{\setminus+\in}(\mathcal{F}_\in)\right)^*(\updownarrow) \to \prime.$$

Hence $\left(H^{n+2}_*(\mathcal{F}_\in)\right)^*(l) \cong H^1_*(\mathcal{I}_\mathcal{X}) \cong \mathcal{H}^\infty_*(\mathcal{F}_\infty)$. Dualizing now gives the last of the asserted isomorphisms. $\square$

Corollary 3.2. *The coherent sheaf* $\mathcal{F}_\in$ *in the resolution* (21) *is locally free.*

Proof. Since $\mathcal{F}_\infty$ is locally free, $H^i_*(\mathcal{F}_\infty)$ is of finite length for $0 < i < n+3$. So by the lemma, $H^i_*(\mathcal{F}_\in)$ is also of finite length for $0 < i < n+3$. But this implies that $\mathcal{F}_\in$ is locally free. $\square$

Proposition 3.3. *The locally free resolution* (21) *satisfies condition* (*a*) *of Proposition 1.2.*

Proof. We write $\mathcal{L} = \omega_{\mathbb{P}^{\ltimes+\nVdash}}(-\updownarrow)$. We have to show that if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \arrow[r, "d_3"] \arrow[d, equal] & \mathcal{F}_\in \arrow[r, "d_2"] \arrow[d, "s_2"] & \mathcal{F}_\infty \arrow[r, "d_1"] \arrow[d, "s_1"] & \mathcal{O}_{\mathbb{P}^{\ltimes+\nVdash}} \arrow[d, equal] \\
0 \arrow[r] & \mathcal{L} \arrow[r, "d_1^\vee"] & \mathcal{F}_\infty^\vee \otimes \mathcal{L} \arrow[r, "-d_2^\vee"] & \mathcal{F}_\in^\vee \otimes \mathcal{L} \arrow[r, "d_3^\vee"] & \mathcal{O}_{\mathbb{P}^{\ltimes+\nVdash}}
\end{tikzcd} \tag{22}$$

such that the vertical maps extend the identity on $\mathcal{O}_\mathcal{X}$, then s_1 and s_2 are isomorphisms.

By exactness, the image of $d_3^\vee$ is $\mathcal{I}_\mathcal{X}$. We will show that s_1 is an isomorphism by applying Lemma 2.10 to the composition

$$\mathcal{F}_\infty \xrightarrow{\int_\infty} \mathcal{F}_\in^\vee \otimes \mathcal{L} \twoheadrightarrow \mathcal{I}_\mathcal{X}.$$

Note that this composition is exactly the surjection $\gamma : \mathcal{F}_\in \twoheadrightarrow \mathcal{I}_\mathcal{X}$ of (18). Hence the composition

$$H^i_*(\mathcal{F}_\infty) \xrightarrow{\mathcal{H}^\rangle_*(\int_\infty)} \mathcal{H}^\rangle_*(\mathcal{F}_\in^\vee \otimes \mathcal{L}) \to \mathcal{H}^\rangle_*(\mathcal{I}_\mathcal{X})$$

is injective for $0 < i < n+3$. A fortiori, $H^i_*(s_1)$ is also injective for $0 < i < n+3$.

However, by Serre duality $H^i_*(\mathcal{F}_\in^\vee \otimes \mathcal{L}) \cong \left(\mathcal{H}_*^{\setminus+\ni-\rangle}(\mathcal{F}_\in)\right)^*(\updownarrow)$ for all i. So by Lemma 3.1, we have $H^i_*(\mathcal{F}_\in^\vee \otimes \mathcal{L}) \cong \mathcal{H}^\rangle_*(\mathcal{F}_\infty)$ for $0 < i < n+3$. Hence for each $0 < i < n+3$, the morphism $H^i_*(s_1)$ is an injection of modules of the same finite length. Hence $H^i_*(s_1)$ is an isomorphism for $0 < i < n+3$. Thus condition (i) of Lemma 2.10 holds.

Condition (ii) of Lemma 2.10 holds because of the method of construction of $\mathcal{F}_\infty$ and of the surjection γ in (18). Finally exactness in the resolution implies that $\mathcal{F}_\infty$ and $\mathcal{F}_\in$ have the same rank. Hence $\mathcal{F}_\infty$ and $\mathcal{F}_\in^\vee \otimes \mathcal{L}$ also have the same rank, which is condition (iii) of Lemma 2.10. Hence all three conditions of Lemma 2.10 hold, and we may conclude that s_1 is an isomorphism.

The map s_2 must now also be an isomorphism by the five-lemma. This completes the proof of the proposition. $\square$

4. The Differential Graded Algebra Structure

In this section we finish the proof of Theorem 0.1 by showing that the locally free resolution (21) defined in the previous section satisfies condition (b) of Proposition 1.2. That is to say, we show that the locally free resolution (21) admits a commutative, associative differential graded algebra structure.

Throughout this section we assume that the characteristic is not 2.

We recall what needs to be proven. In the previous section we defined a locally free resolution (21) of $\mathcal{O}_{\mathcal{X}}$

$$0 \to \mathcal{L} \xrightarrow{\lceil_\ni} \mathcal{F}_\in \xrightarrow{\lceil_\in} \mathcal{F}_\infty \xrightarrow{\lceil_\infty} \mathcal{O}_{\mathbb{P}^{\ltimes+\not\models}} \to \mathcal{O}_{\mathcal{X}}.$$

Let $\mathcal{K} = \ker(\lceil_\infty)$. We then had a morphism $\psi : \Lambda^2\mathcal{F}_\infty \to \mathcal{K}$ defined by $\psi(a \wedge b) = d_1(a)b - d_1(b)a$. We also have a long exact sequence

$$\cdots \to \mathrm{Hom}(\Lambda^2\mathcal{F}_\infty, \mathcal{F}_\in) \to \mathrm{Hom}(\Lambda^\in\mathcal{F}_\infty, \mathcal{K}) \to \mathrm{Ext}^\infty(\Lambda^\in\mathcal{F}_\infty, \mathcal{L}) \to \cdots.$$

According to diagram (5), the problem is to lift $\psi \in \mathrm{Hom}(\Lambda^2\mathcal{F}_\infty, \mathcal{K})$ to a $\phi \in \mathrm{Hom}(\Lambda^2\mathcal{F}_\infty, \mathcal{F}_\in)$. The obstruction to doing this is simply the image of ψ in

$$\mathrm{Ext}^1(\Lambda^2\mathcal{F}_\infty, \mathcal{L}) = \mathcal{H}^{\setminus+\in}(\Lambda^\in\mathcal{F}_\infty(\updownarrow))^*.$$

Our first goal will therefore be to compute $H^{n+2}(\Lambda^2\mathcal{F}_\infty(\updownarrow))$. We begin by considering a complex of locally free sheaves on $\mathbb{P}^\mathbb{N}$.

$$\mathcal{G}^* : \qquad \prime \to \mathcal{G}' \to \mathcal{G}^\infty \to \cdots \to \mathcal{G}^\nabla \to \prime. \tag{23}$$

There is an involution

$$\begin{aligned} T: \quad \mathcal{G}^* \otimes \mathcal{G}^* &\to \mathcal{G}^* \otimes \mathcal{G}^* \\ a \otimes b &\mapsto (-1)^{(\deg a)(\deg b)} b \otimes a \end{aligned}$$

interchanging the factors of $\mathcal{G}^* \otimes \mathcal{G}^*$. Since the characteristic is not 2, the complex $\mathcal{G}^* \otimes \mathcal{G}^*$ splits into a direct sum of subcomplexes on which T acts as multiplication by ± 1, viz. $\mathcal{G}^* \otimes \mathcal{G}^* = \mathcal{S}_\in(\mathcal{G}^*) \oplus \Lambda^\in(\mathcal{G}^*)$. The complex $\Lambda^2(\mathcal{G}^*)$ is of the form

$$\Lambda^2(\mathcal{G}^*) : \qquad \prime \to \mathcal{H}' \to \mathcal{H}^\infty \to \cdots \to \mathcal{H}^{\in\nabla} \to \prime \tag{24}$$

where (cf. [BE2] p. 452)

$$\mathcal{H}^\rangle \cong \bigoplus_{\amalg < \rangle/\in} \left(\mathcal{G}^\amalg \otimes \mathcal{G}^{\rangle-\amalg}\right) \oplus \begin{cases} 0 & \text{if } i \text{ is odd,} \\ \Lambda^2(\mathcal{G}^{\rangle/\in}) & \text{if } i \equiv 0 \pmod 4, \\ S_2(\mathcal{G}^{\rangle/\in}) & \text{if } i \equiv 2 \pmod 4. \end{cases} \tag{25}$$

Lemma 4.1. *Suppose $\mathcal{G}^*$ is a complex of locally free sheaves on $\mathbb{P}^\mathbb{N}$ as in* (23) *which is exact except in degree* 0*. Let $\mathcal{E} = \mathcal{H}'(\mathcal{G}^*)$. Then $\Lambda^2(\mathcal{G}^*)$ is an exact sequence of locally free sheaves which is exact except in degree* 0*, and $H^0(\Lambda^2(\mathcal{G}^*)) = \Lambda^\in\mathcal{E}$.*

Proof. The standard spectral sequences of the double complex $\mathcal{G}^* \otimes \mathcal{G}^*$ degenerate to show that the simple complex $\mathcal{G}^* \otimes \mathcal{G}^*$ is exact except in degree 0, and $H^0(\mathcal{G}^* \otimes \mathcal{G}^*) = \mathcal{E} \otimes \mathcal{E}$. Thus the augmented complex $0 \to \mathcal{E} \otimes \mathcal{E} \to \mathcal{G}^* \otimes \mathcal{G}^*$ is exact, and consequently its direct factor $0 \to \Lambda^2\mathcal{E} \to \Lambda^\in(\mathcal{G}^*)$ is also exact. $\square$

Lemma 4.2. *Suppose $\mathcal{E}$ is a locally free sheaf on $\mathbb{P}^N$. Let $r < N/2$ be an integer. Suppose that $H^i_*(\mathcal{E}) = 0$ for $r < i < N$. Then*

(a) $H^i_*(\Lambda^2\mathcal{E}) = 0$ for $2r < i < N$,

(b) $H^{2r}_*(\Lambda^2\mathcal{E}) \cong \mathcal{S}_2(\mathcal{H}^r_*(\mathcal{E}))$ if r is odd, and $H^{2r}_*(\Lambda^2\mathcal{E}) \cong \Lambda^2(\mathcal{H}^r_*(\mathcal{E}))$ if r is even.

(c) If $H^r(\mathcal{E}(t)) = 0$ for $t < q$ for some integer q, then $H^{2r}(\Lambda^2\mathcal{E}(t)) = 0$ for $t < 2q$, while $H^{2r}(\Lambda^2\mathcal{E}(2q)) \cong \mathcal{S}_2(\mathcal{H}^r(\mathcal{E}(q)))$ if r is odd, and $H^{2r}(\Lambda^2\mathcal{E}(2q)) \cong \Lambda^2(\mathcal{H}^r(\mathcal{E}(q)))$ if r is even.

Proof. By Lemma 2.5(b), the minimal projective resolution of P^* of the truncation $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$ is a complex of free graded S-modules such that $P^i = 0$ unless $0 \le i \le N-1$. Indeed, since $H^i(\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})) = 0$ for all $i > r$, Lemma 2.3 indicates that $P^i = 0$ unless $0 \le i \le r$, i.e. P^* is of the form

$$P^* : \qquad 0 \to P^0 \to \cdots \to P^{r-1} \to P^r \to 0.$$

We now consider the complex of free graded S-modules

$$\Lambda^2(P^*) : \qquad 0 \to \Lambda^2 P^0 \to \cdots \to P^{r-1} \otimes P^r \to T_2(P^r) \to 0.$$

where $T_2(P^r) = \Lambda^2(P^r)$ if r is even, and $T_2(P^r) = S_2(P^r)$ if r is odd (cf. (24) and (25)). According to Lemma 2.5, the complex of sheaves $\widetilde{P}^*$ associated to P^* is exact except in degree 0 where the homology is $\mathcal{E}$. So Lemma 4.1 implies that the complex of sheaves $\Lambda^2(\widetilde{P}^*)$ is also exact except in degree 0 where the homology is $\Lambda^2\mathcal{E}$. The complex $\Lambda^2(P^*)$ of graded S-modules therefore has homology of finite length except in degree 0. Moreover, the complex $\Lambda^2(P^*)$ vanishes except in degrees between 0 and $2r < N$, and the coefficients of its differentials lie in $\mathfrak{m}$ because it those of P^* and therefore $P^* \otimes P^*$ do. It now follows from Lemma 2.5(a) that $\Lambda^2(P^*)$ is the minimal projective resolution of $\tau_{<N}\mathbf{R}\Gamma_*(\Lambda^2\mathcal{E})$.

Therefore $H^i_*(\Lambda^2\mathcal{E}) \cong \mathcal{H}^i(\Lambda^2(\mathcal{P}^*))$ for all $i < N$. In particular, since $\Lambda^2(P^*)$ is concentrated in degrees between 0 and $2r$ by (24), we see that $H^i_*(\Lambda^2\mathcal{E}) = 0$ for $2r < i < N$. This is part (a) of the lemma.

For (b) note that $H^r_*(\mathcal{E})$ and $H^{2r}_*(\Lambda^2\mathcal{E})$ has respective presentations

$$\begin{aligned} P^{r-1} \quad &\xrightarrow{\delta} \quad P^r \quad \to \quad H^r_*(\mathcal{E}) \quad \to 0, \\ P^{r-1} \otimes P^r &\xrightarrow{\delta_1} T_2(P^r) \to H^{2r}_*(\Lambda^2\mathcal{E}) \to 0, \end{aligned}$$

where $\delta_1(e \otimes f) = \delta(e)f \in T_2(P^r)$. But since the presentation of $T_2(H^r_*(\mathcal{E}))$ is of exactly this form, we see that $H^{2r}_*(\Lambda^2\mathcal{E}) \cong \mathcal{T}_2(\mathcal{H}^r_*(\mathcal{E}))$, as asserted by the lemma.

For (c) write $H = H^r(\mathcal{E}(q))$. The hypothesis that $H^r(\mathcal{E}(t)) = 0$ for $t < q$ implies that $P^r = (H \otimes_k S(-q)) \oplus F$ with $F = \bigoplus S(-n_i)$ for some $n_i > q$. Then $T_2(P^r) = (T_2 H \otimes_k S(-2q)) \oplus G$ with $G = (H \otimes_k F(-q)) \oplus T_2 F = \bigoplus S(-m_j)$ for some $m_j > 2q$. Since the presentation of $H^{2r}_*(\Lambda^2\mathcal{E})$ given above has the property that no direct factor of $P^{r-1} \otimes P^r$ is mapped surjectively onto a factor of $T_2(P^r)$, it now follows that $H^{2r}(\Lambda^2\mathcal{E}(t)) = 0$ for $t < 2q$, and $H^{2r}(\Lambda^2\mathcal{E}(2q)) \cong \mathcal{T}_2\mathcal{H}$. $\square$

Corollary 4.3. *Let n, l, and $X \subset \mathbb{P}^{n+l}$ be as in Theorem 0.1. Suppose that $U \subset H^{n/2}(\mathcal{O}_X(l/2))$ is the maximal isotropic subspace defined in (16), and that*

$\mathcal{F}_\infty$ *is the locally free sheaf defined in* (18)*. Then*

$$H^{n+2}(\Lambda^2\mathcal{F}_\infty(\updownarrow)) \cong \begin{cases} 0 & \text{if } n \text{ or } l \text{ is odd,} \\ S_2U & \text{if } l \text{ is even, and } n \equiv 0 \pmod 4, \\ \Lambda^2 U & \text{if } l \text{ is even, and } n \equiv 2 \pmod 4. \end{cases}$$

Proof. If n is odd, then $H^i_*(\mathcal{F}_\infty) = \prime$ for $(n+1)/2 < i < n+3$. So Lemma 4.2(a) applies with $r = (n+1)/2$. Therefore $H^i_*(\Lambda^2\mathcal{F}_\infty) = \prime$ for $n+1 < i < n+3$, i.e. $H^{n+2}(\Lambda^2\mathcal{F}_\infty(\sqcup)) = \prime$ for all t.

If n is even but l is odd, then Lemma 4.2(c) applies with $r = (n+2)/2$ and $q = (l+1)/2$. Then $H^{n+2}(\Lambda^2\mathcal{F}_\infty(\sqcup)) = \prime$ for all $t < l+1$.

If l and n are even, then Lemma 4.2(c) applies with $r = (n+2)/2$ and $q = l/2$. Since $H^{(n+2)/2}(\mathcal{F}_\infty(\updownarrow/\in)) \cong \mathcal{U}$, it follows that $H^{n+2}(\Lambda^2\mathcal{F}_\infty(\updownarrow)) \cong \Lambda^\in\mathcal{U}$ if r is even, and $H^{n+2}(\Lambda^2\mathcal{F}_\infty(\updownarrow)) \cong \mathcal{S}_\in\mathcal{U}$ if r is odd. The corollary follows. $\square$

Lemma 4.4. *If $\mathcal{F}_\infty$ is the locally free sheaf defined in* (18)*, then the image of the map ψ of* (5) *in* $\mathrm{Ext}^1(\Lambda^2\mathcal{F}_\infty, \mathcal{L}) \cong \mathcal{H}^{\backslash+\in}(\Lambda^\in\mathcal{F}_\infty(\updownarrow))^*$ *vanishes.*

Proof. If n or l is odd, then $H^{n+2}(\Lambda^2\mathcal{F}_\infty(\updownarrow)) = \prime$ according to Corollary 4.3, so the image of ψ is evidently zero.

If n and l are even, then we claim that the image of ψ in $H^{n+1}(\Lambda^2\mathcal{F}_\infty(\updownarrow))^*$ is the map

$$\left\{S_2U \text{ or } \Lambda^2U\right\} \to k$$

which is the restriction to U of the pairing $H^{n/2}(\mathcal{O}_\mathcal{X}(\updownarrow/\in)) \times \mathcal{H}^{\backslash/\in}(\mathcal{O}_\mathcal{X}(\updownarrow/\in)) \to \|$ of (2). Since U was chosen isotropic, this map vanishes.

In order to prove the claim, we consider the diagonal $i : \mathbb{P}^{\ltimes+\not\Vdash} = \Delta \subset \mathbb{P}^{\ltimes+\not\Vdash} \times \mathbb{P}^{\ltimes+\not\Vdash}$. Then there is a natural inclusion $i(X) \subset X \times X$ which corresponds to a restriction map

$$\mathcal{O}_{\mathcal{X}\times\mathcal{X}} \to \rangle_*\mathcal{O}_\mathcal{X}. \tag{26}$$

This map is essentially the multiplication $\mathcal{O}_\mathcal{X} \otimes \mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{X}$. In any case applying $\mathbf{R}\Gamma_*$ to (26) gives the cup product map

$$\mathbf{R}\Gamma_*(\mathcal{O}_\mathcal{X}) \otimes_\| \mathbf{R}\Gamma_*(\mathcal{O}_\mathcal{X}) \to \mathbf{R}\Gamma_*(\mathcal{O}_\mathcal{X}). \tag{27}$$

Now consider the "resolution" of $\mathcal{O}_\mathcal{X}$ given in (19)

$$\mathcal{K}^* : \qquad \prime \to \mathcal{K} \to \mathcal{F}_\infty \xrightarrow{\lceil_\infty} \mathcal{O}_{\mathbb{P}^{\ltimes+\not\Vdash}} \to \prime.$$

The complex $\mathcal{K}^*$ is quasi-isomorphic to $\mathcal{O}_\mathcal{X}$. Hence the restriction to the diagonal map (26) corresponds to a morphism in the derived category

$$p_1^*\mathcal{K}^* \otimes \sqrt{}^*_\in\mathcal{K}^* \to \rangle_*\mathcal{K}^*.$$

In fact this morphism in the derived category is represented by an actual map of complexes of sheaves

$$\begin{array}{ccccccc} \cdots \to p_1^*\mathcal{K} \oplus (\sqrt{}^*_\infty\mathcal{F}_\infty \otimes \sqrt{}^*_\in\mathcal{F}_\infty) \oplus \sqrt{}^*_\in\mathcal{K} & \to & p_1^*\mathcal{F}_\infty \oplus \sqrt{}^*_\in\mathcal{F}_\infty & \to & \mathcal{O}_{\mathbb{P}\times\mathbb{P}} & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to i_*\mathcal{K} & \to & i_*\mathcal{F}_\infty & \to & i_*\mathcal{O}_\mathbb{P} & \to & 0 \end{array} \tag{28}$$

All the vertical maps are straightforward restrictions to the diagonal except for the component $p_1^*\mathcal{F}_\infty \otimes \sqrt{}^*_{\in}\mathcal{F}_\infty \to \rangle_*(\mathcal{K})$ which is defined (like ψ of (5)) by noting that the composition

$$\begin{array}{ccccc} p_1^*\mathcal{F}_\infty \otimes \sqrt{}^*_{\in}\mathcal{F}_\infty \to & p_1^*\mathcal{F}_\infty \oplus \sqrt{}^*_{\in}\mathcal{F}_\infty & \to & i_*\mathcal{F}_\infty \\ p_1^*(a)\otimes p_2^*(b) & & \mapsto & i_*\left(d_1(a)b - d_1(b)a\right) \end{array} \tag{29}$$

is contained in the kernel of $i_*\mathcal{F}_\infty \to \rangle_*\mathcal{O}_{\mathbb{P}}$.

Now since $\mathcal{K}^*$ is quasi-isomorphic to $\mathcal{O}_{\mathcal{X}}$, if we apply $\mathbf{R}\Gamma_*$ to (28) we get a morphism $\mathbf{R}\Gamma_*(\mathbf{p_1^*}\mathcal{K}\otimes \sqrt{}^*_{\in}\mathcal{K}) \to \mathbf{R}\Gamma_*(\mathbf{i}_*\mathcal{K}^*)$ in $D^b_{\mathrm{Mod}_{S\otimes S,\mathrm{gr}}}$ which is quasi-isomorphic to (27). In particular, the maps of hypercohomology are quasi-isomorphic to the cup product

$$H^n_*(p_1^*\mathcal{O}_{\mathcal{X}} \otimes \sqrt{}^*_{\in}\mathcal{O}_{\mathcal{X}}) \cong \bigoplus_{\rangle} \mathcal{H}^{\rangle}_*(\mathcal{O}_{\mathcal{X}}) \otimes_{\|} \mathcal{H}^{\setminus - \rangle}_*(\mathcal{O}_{\mathcal{X}}) \to \mathcal{H}^{\setminus}_*(\mathcal{O}_{\mathcal{X}}).$$

The hypercohomology $H^n_*(\mathcal{K}^*) \cong \mathcal{H}^{\setminus}_*(\mathcal{O}_{\mathcal{X}})$ is of course the same as the H^n of the total complex of the double complex

$$0 \to \mathbf{R}\Gamma_*(\mathcal{K}) \to \mathbf{R}\Gamma_*(\mathcal{F}_\infty) \to \mathbf{R}\Gamma_*(\mathcal{O}_{\mathbb{P}}) \to \prime.$$

According to the calculations at the beginning of the previous section, this H^n_* is all attributable to $\mathcal{K}$, i.e. the truncation $\mathcal{K}^* \to \mathcal{K}[\in]$ induces an isomorphism $H^n_*(\mathcal{O}_{\mathcal{X}}) \cong \mathcal{H}^{\setminus}_*(\mathcal{K}^*) \cong \mathcal{H}^{\setminus+\in}_*(\mathcal{K})$.

Similarly, the hypercohomology $H^n_*(p_1^*\mathcal{K}^* \otimes \sqrt{}^*_{\in}\mathcal{K}^*)$ is the same as the H^n of the total complex of $\mathbf{R}\Gamma_*$ of the first row of (28). The submodule $W \otimes_k W \subset H^n_*(p_1^*\mathcal{O}_{\mathcal{X}} \otimes \sqrt{}^*_{\in}\mathcal{O}_{\mathcal{X}})$ is attributable as the H^{n+2}_* of the factor $p_1^*\mathcal{F}_\infty \otimes \sqrt{}^*_{\in}\mathcal{F}_\infty$ in the first row of (28). Therefore H^{n+2}_* of the vertical map $p_1^*\mathcal{F}_\infty \otimes \sqrt{}^*_{\in}\mathcal{F}_\infty \to \mathcal{K}$ is simply the cup product map $W \otimes_k W \to H^n_*(\mathcal{O}_{\mathcal{X}})$.

Now the fact that $i_*(\mathcal{K})$ is supported on Δ, plus the symmetry of the product map imply that the vertical map of (28) factors as

$$\begin{array}{ccccccc} p_1^*\mathcal{F}_\infty \otimes \sqrt{}^*_{\in}\mathcal{F}_\infty & \to & i_*(\mathcal{F}_\infty \otimes \mathcal{F}_\infty) & \to & i_*(\Lambda^2\mathcal{F}_\infty) & \xrightarrow{i_*(\psi)} & i_*(\mathcal{K}). \\ p_1^*(a)\otimes p_2^*(b) & \mapsto & i_*(a\otimes b) & \mapsto & i_*(a\wedge b) & \mapsto & d_1(a)b - d_1(b)a \end{array} \tag{30}$$

We wish to calculate H^{n+2}_* of the above morphisms. Let

$$P^*: \qquad 0 \to P^0 \to \cdots \to P^{(n+2)/2} \to 0$$

be a minimal projective resolution of $\tau_{<n+3}\mathbf{R}\Gamma_*(\mathcal{F}_\infty)$ (cf. Lemma 2.5). Then if one applies $\tau_{<n+3}\mathbf{R}\Gamma_*$ to the first two morphisms of (30), one gets the natural maps

$$P^* \otimes_k P^* \to P^* \otimes_S P^* \to \Lambda^2(P^*)$$

(cf. the proof of Lemma 4.2). All three complexes are supported in degrees between 0 and $n+2$, and applying H^{n+2} gives surjections

$$W \otimes_k W \twoheadrightarrow W \otimes_S W \twoheadrightarrow \left\{S_2W \text{ or } \Lambda^2 W\right\}.$$

It therefore follows that $H^{n+2}_*(\psi): H^{n+2}_*(\Lambda^2\mathcal{F}_\infty) \to \mathcal{H}^{\setminus+\in}_*(\mathcal{K})$ is isomorphic to the cup product map $\left\{S_2U \text{ or } \Lambda^2 U\right\} \to H^n_*(\mathcal{O}_{\mathcal{X}})$. In particular, in degree l the morphism $H^{n+2}(\mathcal{F}_\infty(\updownarrow)) \to \mathcal{H}^{\setminus+\in}(\mathcal{K}(\updownarrow))$ is the same as $\left\{S_2U \text{ or } \Lambda^2 U\right\} \to H^n(\mathcal{O}_{\mathcal{X}}(\updownarrow))$.

We now have to consider the extension of (20)

$$0 \to \omega_{\mathbb{P}}(-l) \to \mathcal{F}_{\in} \to \mathcal{K} \to \imath.$$

(Recall $\mathcal{L} = \omega_{\mathbb{P}}(-\updownarrow)$.) In the associated long exact sequence of cohomology

$$\cdots \to H^{n+2}(\mathcal{F}_{\in}(\updownarrow)) \to \mathcal{H}^{\backslash+\in}(\mathcal{K}(\updownarrow)) \xrightarrow{\mathrm{tr}} \mathcal{H}^{\backslash+\ni}(\omega_{\mathbb{P}^{\ltimes+\nVdash}}) \cong \|,$$

the differential is the element of $H^{n+2}(\mathcal{K}(\updownarrow))^* \cong \mathrm{Ext}^{\infty}(\mathcal{K}(\updownarrow), \omega_{\mathbb{P}^{\ltimes+\nVdash}})$ corresponding to the extension class. So by construction the differential is the trace map $\mathrm{tr} \in H^{n+2}(\mathcal{K}(\updownarrow))^* \cong \mathcal{H}^{\backslash}(\mathcal{O}_{\mathcal{X}}(\updownarrow))^*$ which corresponds under Serre duality to $1 \in H^0(\mathcal{O}_{\mathcal{X}})$.

Now the image of $\psi \in \mathrm{Ext}^1(\Lambda^2\mathcal{F}_{\infty}(\updownarrow), \omega_{\mathbb{P}}) \cong \mathcal{H}^{\backslash+\in}(\Lambda^{\in}\mathcal{F}_{\infty}(\updownarrow))^*$ is exactly the composition

$$H^{n+2}(\Lambda^2\mathcal{F}_{\infty}(\updownarrow)) \xrightarrow{\mathcal{H}^{\backslash+\in}(\psi)} \mathcal{H}^{\backslash+\in}(\mathcal{K}(\updownarrow)) \xrightarrow{\mathrm{tr}} \mathcal{H}^{\backslash+\ni}(\omega_{\mathbb{P}^{\ltimes+\nVdash}}) \cong \|.$$

By our previous calculations, this is the composition of the cup product map $\{S_2U \text{ or } \Lambda^2U\} \to H^n(\mathcal{O}_{\mathcal{X}}(\updownarrow))$ with the trace map $H^n(\mathcal{O}_{\mathcal{X}}(\updownarrow)) \to \|$. Therefore this composition is the restriction to S_2U or Λ^2U of the Serre duality pairing $H^{n/2}(\mathcal{O}_{\mathcal{X}}(\updownarrow/\in)) \otimes \mathcal{H}^{\backslash/\in}(\mathcal{O}_{\mathcal{X}}(\updownarrow/\in)) \to \|$. This is what was claimed at the beginning of the proof of the lemma. Since U was chosen isotropic, this composition vanishes, i.e. the image of ψ in $\mathrm{Ext}^1(\mathcal{K}, \mathcal{L})$ vanishes. ☐

Proof of Theorem 0.1. According to Proposition 1.2, in order to prove Theorem 0.1 it suffices to find a locally free resolution

$$0 \to \mathcal{L} \to \mathcal{F}_{\in} \to \mathcal{F}_{\infty} \to \mathcal{O}_{\mathbb{P}^{\ltimes+\nVdash}} \to \mathcal{O}_{\mathcal{X}}$$

which satisfies two conditions. But the locally free resolution defined in (21) was shown to satisfy the first of these conditions was shown in Proposition 3.3. Moreover, this resolution was just shown to satisfy the second condition in Lemma 4.4. Hence Theorem 0.1 holds. ☐

5. Characteristic 2 Computations

In the introduction, we asserted that Theorem 0.1 also holds in characteristic 2 provided the phrase "$n \equiv 0 \pmod 4$" in the parity condition is replaced by the phrase "n is even." In this section we justify that assertion by proving analogues of Lemmas 4.1 and 4.2 and Corollary 4.3 in characteristic 2. These were the only steps in the proof of Theorem 0.1 where we used the assumption that the characteristic is not 2.

Throughout this section we assume that the characteristic is 2.

We recall certain simple facts from modular representation theory. Let R be a commutative algebra over a field of characteristic 2, and let V be a free R-module. Let $t \in \mathrm{End}(V \otimes V)$ be the endomorphism $t(a \otimes b) = a \otimes b - b \otimes a$. Set $D_2V = \ker(t)$, and $\Lambda^2V = \mathrm{im}(t)$, and $S_2V = \mathrm{coker}(t)$. Since $t^2 = 0$ in characteristic 2, there are inclusions

$$0 \subset \Lambda^2V \subset D_2V \subset V \otimes V$$

and corresponding surjection of quotients of $V \otimes V$

$$V \otimes V \twoheadrightarrow S_2V \twoheadrightarrow \Lambda^2V \to 0.$$

The subquotient D_2V/Λ^2V is $F(V)$, the Frobenius pullback of V. It is a free module of the same rank as V. This $F(V)$ is also the kernel of the surjection $S_2V \twoheadrightarrow \Lambda^2V$.

Note that the natural map from S_2V to $V \otimes V$ given by $xy \mapsto x \otimes y + y \otimes x$ is not injective because any $x^2 \mapsto 0$. The map is the composition $S_2V \twoheadrightarrow \Lambda^2 V \hookrightarrow V \otimes V$ with kernel $F(V)$.

The operations D_2, Λ^2, S_2, and F are all functorial. Therefore we may define $D_2\mathcal{E}$, $\Lambda^2\mathcal{E}$, $S_2\mathcal{E}$, and $F(\mathcal{E})$ for any locally free sheaf $\mathcal{E}$ on any scheme X over a field of characteristic 2.

In some ways F has better properties than the others. If $M = (m_{ij}) : R^n \to R^m$ is a morphism of free R-modules, then $F(M) = (m_{ij}^2) : R^n \to R^m$. One may use this formula together with the Buchsbaum-Eisenbud exactness criterion [BE1] to show that F of an exact sequence of locally free sheaves is exact. As a result of this we get the following lemma.

Lemma 5.1. *Let $\mathcal{E}$ be a locally free sheaf on $\mathbb{P}^{\mathbb{N}}$. If for some integer r one has $H^i_*(\mathcal{E}) = \prime$ for $r < i < N$, then $H^i_*(F(\mathcal{E})) = \prime$ for $r < i < N$ also.*

Proof. Let P^* be the minimal projective resolution of $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$. By Lemmas 2.5 and 2.3, P^* has the form

$$P^* : \qquad 0 \to P^0 \to P^1 \to \cdots \to P^r \to 0.$$

Moreover, P^* is exact except in degree 0 away from the irrelevant ideal $\mathfrak{m} \subset \mathfrak{S}$, and $H^0(\widetilde{P}^*) = \mathcal{E}$.

The functoriality and exactness of F now imply that

$$F(P^*) : \qquad 0 \to F(P^0) \to F(P^1) \to \cdots \to F(P^r) \to 0$$

is exact except in degree in degree 0 away from the irrelevant ideal, and has $H^0(F(\widetilde{P}^*)) = F(\mathcal{E})$. Applying Lemma 2.5 again, we conclude that $F(P^*)$ is the minimal projective resolution of $\tau_{<N}\mathbf{R}\Gamma_*(\mathbf{F}(\mathcal{E}))$.

So if $r < i < N$, then $H^i_*(F(\mathcal{E})) = \mathcal{H}^{\rangle}(\mathcal{F}(\mathcal{P}^*)) = \prime$ since $F(P^*)$ vanishes in degrees greater than r. $\square$

Corollary 5.2. *Let $\mathcal{E}$ be a locally free sheaf on $\mathbb{P}^{\mathbb{N}}$ over a field of characteristic 2 such that $H^{N-1}_*(\Lambda^2\mathcal{E}) = \prime$. Suppose $r < N$ is an integer such that $H^i_*(\mathcal{E}) = \prime$ for $r < i < N$. Then $H^i_*(S_2\mathcal{E}) \cong \mathcal{H}^{\rangle}_*(\Lambda^{\in}\mathcal{E})$ for $r < i < N$.*

Proof. We consider the exact sequence $0 \to F(\mathcal{E}) \to \mathcal{S}_{\in}\mathcal{E} \to \Lambda^{\in}\mathcal{E} \to \prime$ and the associated long exact sequence

$$\cdots \to H^i_*(F(\mathcal{E})) \to \mathcal{H}^{\rangle}_*(\mathcal{S}_{\in}\mathcal{E}) \to \mathcal{H}^{\rangle}_*(\Lambda^{\in}\mathcal{E}) \to \mathcal{H}^{\rangle+\infty}_*(\mathcal{F}(\mathcal{E})) \to \cdots$$

The hypothesis $H^i_*(\mathcal{E}) = \prime$ for $r < i < N$ implies also $H^i_*(F(\mathcal{E})) = \prime$ for $r < i < N$ by Lemma 5.1. Hence the long exact sequence implies that $H^i_*(S_2\mathcal{E}) \cong \mathcal{H}^{\rangle}_*(\Lambda^{\in}\mathcal{E})$ for $r < i < N-1$ and that $H^{N-1}_*(S_2\mathcal{E}) \hookrightarrow \mathcal{H}^{\mathcal{N}-\infty}_*(\Lambda^{\in}\mathcal{E})$ is injective. But by hypothesis $H^{N-1}_*(\Lambda^2\mathcal{E}) = \prime$, so $H^{N-1}_*(S_2\mathcal{E}) = \prime$ as well. $\square$

We now prove the analogue of Lemma 4.2.

Lemma 5.3. *Suppose $\mathcal{E}$ is a locally free sheaf on $\mathbb{P}^{\mathbb{N}}$ over a field of characteristic 2. Let $0 < r < N/2$ be an integer such that $H^i_*(\mathcal{E}) = \prime$ for $r < i < N$. Then*

(a) $H^i_*(\Lambda^2\mathcal{E}) = \prime$ for $2r < i < N$,

(b) $H^{2r}_*(\Lambda^2\mathcal{E}) \cong \mathcal{S}_{\in}(\mathcal{H}^{\nabla}_*(\mathcal{E}))$.

(c) If $H^r(\mathcal{E}(\sqcup)) = \prime$ for $t < q$ for some integer q, then $H^{2r}(\Lambda^2\mathcal{E}(\sqcup)) = \prime$ for $t < 2q$, while $H^{2r}(\Lambda^2\mathcal{E}(\in\amalg)) \cong \mathcal{S}_{\in}(\mathcal{H}^{\nabla}(\mathcal{E}(\amalg)))$.

Proof. Let P^* be the minimal projective resolution of $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$

$$P^*: \qquad 0 \to P^0 \xrightarrow{\delta^0} P^1 \xrightarrow{\delta^1} P^2 \to \cdots \to P^r \to 0$$

Let $\mathcal{P} = \widetilde{\mathcal{P}'}$, and let $\mathcal{F} = \widetilde{\ker(\delta^\infty)}$. Then we have an exact sequence $0 \to \mathcal{E} \to \mathcal{P} \to \mathcal{F} \to '$ such that $\mathcal{P}$ is a direct sum of line bundles, $H^i_*(\mathcal{F}) = \mathcal{H}^{\rangle+\infty}_*(\mathcal{E})$ for $0 < i < N-1$, and $H^{N-1}_*(\mathcal{F}) = '$.

It is easy to see that there is a natural exact complex

$$0 \to \Lambda^2\mathcal{E} \to \Lambda^\in\mathcal{P} \to \mathcal{P}\otimes\mathcal{F} \to \mathcal{S}_\in\mathcal{F} \to '. \tag{31}$$

We now prove parts (a) and (b) of the lemma by induction on r. If $r = 1$, then $\mathcal{F} = \widetilde{\mathcal{P}}^\infty$ is a direct sum of line bundles, and the complex (31) is just the augmented complex

$$0 \to \Lambda^2\mathcal{E} \to \Lambda^\in(\widetilde{\mathcal{P}}^*)$$

which is still exact in this case. So we may conclude just as in Lemma 4.2 that $H^i_*(\Lambda^2\mathcal{E}) = '$ for $2 < i < N$, and that $H^2_*(\Lambda^2\mathcal{E}) = \mathcal{S}_\in(\mathcal{H}^\infty_*(\mathcal{E}))$.

If $r > 1$, then $H^i_*(\mathcal{F}) = '$ for $r-1 < i < N$. So by induction $H^i_*(\Lambda^2\mathcal{F}) = '$ for $2r-2 < i < N$, and also $H^{2r-2}_*(\Lambda^2\mathcal{F})) \cong \mathcal{S}_\in(\mathcal{H}^{\nabla-\infty}_*(\mathcal{F})) \cong \mathcal{S}_\in(\mathcal{H}^\nabla_*(\mathcal{E}))$. It now follows from Corollary 5.2 that $H^i_*(S_2\mathcal{F}) \cong \mathcal{H}^\rangle_*(\Lambda^\in\mathcal{F}) = '$ for $r-1 < i < N$. So in particular $H^i_*(S_2\mathcal{F}) = '$ for $2r-2 < i < N$ and that $H^{2r-2}_*(S_2\mathcal{F}) \cong \mathcal{S}_\in(\mathcal{H}^\nabla_*(\mathcal{E}))$. Now since $\mathcal{P}$ is a direct sum of line bundles, we have $H^i_*(\Lambda^2\mathcal{P}) = '$ for $0 < i < N$, and $H^i_*(\mathcal{P}\otimes\mathcal{F}) = '$ for $r-1 < i < N$. So if we break up (31) into short exact sequences and take its graded cohomology, we can deduce that $H^i_*(\Lambda^2\mathcal{E})) \cong \mathcal{H}^{\rangle-\in}_*(\mathcal{S}_\in\mathcal{F})$ for $r+1 < i < N$. Since $r+1 < 2r$, this gives parts (a) and (b) of the lemma.

Part (c) of the lemma follows from part (b) by the same argument as in Lemma 4.2. □

We have the following corollary in analogy with Corollary 4.3.

Corollary 5.4. *Let n, l, and $X \subset \mathbb{P}^{\ltimes+\not\models}_\urcorner$ be as in Theorem 0.1 with k a field of characteristic 2. Suppose that $U \subset H^{n/2}(\mathcal{O}_\mathcal{X}(\updownarrow/\in))$ is the maximal isotropic subspace defined in* (16), *and that $\mathcal{F}_\infty$ is the locally free sheaf defined in* (18). *Then*

$$H^{n+2}(\Lambda^2\mathcal{F}_\infty(\updownarrow)) \cong \begin{cases} 0 & \textit{if } n \textit{ or } l \textit{ is odd,} \\ S_2U & \textit{if } n \textit{ and } l \textit{ are even.} \end{cases}$$

The proof of Lemma 4.4 in characteristic 2 is essentially the same as in the previous section, only with Lemma 5.3 and Corollary 5.4 replacing their analogues, and with $T_2 = S_2$ always. Hence Theorem 0.1 also holds in characteristic 2 as long as one treats all even n the same.

6. The Local Version of the Main Theorem

In this section we consider Theorem 0.3, the local version of our main result. We state a variant version which is clearly a local analogue of Theorem 0.1 with an identical proof, and then show that this variant version is equivalent to Theorem 0.3.

Let $(R, \mathfrak{m}, \mathfrak{k})$ be a regular local ring, and let $U = \operatorname{Spec}(R) - \{\mathfrak{m}\}$ be the punctured spectrum of R. We say that a closed subscheme $Y \subset U$ of pure codimension 3 is *Pfaffian* if O_X has a locally free resolution on U

$$0 \to \mathcal{O}_\mathcal{U} \xrightarrow{\langle} \mathcal{E}^\vee \xrightarrow{\{} \mathcal{E} \xrightarrow{\}} \mathcal{O}_\mathcal{U} \to \mathcal{O}_\mathcal{X}$$

where $\mathcal{E}$ is a locally free $\mathcal{O}_{\mathcal{U}}$-module of odd rank $2p+1$, f is skew-symmetric, and g and $h = g^\vee$ are given locally by the Pfaffians of order $2p$ of f. The following theorem is the obvious local analogue of Theorem 0.1.

Theorem 6.1. *Let $(R, \mathfrak{m}, \mathfrak{k})$ be a regular local ring of dimension $n+4>4$ with residue field not of characteristic 2, and let $U = \operatorname{Spec}(R) - \{\mathfrak{m}\}$. Let $X \subset U$ be a closed subscheme of pure codimension 3. Then X is Pfaffian if and only if the following three conditions hold:*

(*a*) *X is locally Gorenstein,*

(*b*) *$\omega_X \cong \mathcal{O}_{\mathcal{X}}$, and*

(*c*) *if $n \equiv 0 \pmod 4$, then $H^{n/2}(\mathcal{O}_{\mathcal{X}})$ is of even length.*

Theorem 6.1 may be proven in exactly the same manner as Theorem 0.1. All results concerning the Buchsbaum-Eisenbud proof, the Horrocks correspondence, Serre/local duality, the cohomology of $H^{n+2}(\Lambda^2\mathcal{F}_\infty)$ work identically for graded modules over polynomial rings over k and for modules over regular local k-algebras. There is only one point which is in any way more subtle in the local case. Namely, if n is even, then one has a Matlis duality pairing of R-modules of even finite length

$$H^{n/2}(\mathcal{O}_{\mathcal{X}}) \times \mathcal{H}^{\setminus/\in}(\mathcal{O}_{\mathcal{X}}) \to \|.$$

This pairing is perfect in the sense that for any submodule $M \subset H^{n/2}(\mathcal{O}_{\mathcal{X}})$ one has

$$\operatorname{length}(M) + \operatorname{length}(M^\perp) = \operatorname{length}(H^{n/2}(\mathcal{O}_{\mathcal{X}})).$$

In order to be able to define C_X^* and $\mathcal{F}_\infty$ as in (18) one must choose an isotropic submodule W of length equal to half that of $H^{n/2}(\mathcal{O}_{\mathcal{X}})$. But it is not difficult to show that this is possible.

We now compare Theorems 0.3 and 6.1. First of all, $E = \Gamma(\mathcal{E})$ gives a bijective correspondence between locally free sheaves $\mathcal{E}$ on U and reflexive R-modules E such that $E_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$. There is also bijective correspondence betweenclosed subschemes $X \subset U$ of pure codimension 3 and unmixed ideals $I \subset R$ of height 3 given by $I = \Gamma(\mathcal{I}_{\mathcal{X}})$. Hence an ideal I is Pfaffian in the sense of Theorem 0.3 if and only if the corresponding subscheme $X \subset U$ is Pfaffian in the sense of Theorem 6.1.

The three conditions (a), (b), and (c) of the two theorems also correspond. In the case of (a) this is obvious. For (b) note that $\omega_{R/I} \cong \Gamma(\omega_X)$ since for all $\mathfrak{p} \in \mathfrak{U}$ one has $\omega_{R/I,\mathfrak{p}} = \operatorname{Ext}^3_{R_\mathfrak{p}}((R/I)_\mathfrak{p}, R_\mathfrak{p}) = \omega_{X,\mathfrak{p}}$, and $\omega_{R/I}$ is saturated. Similarly $(R/I)^{\text{sat}} \cong \Gamma(\mathcal{O}_{\mathcal{X}})$. This gives the equivalence of the two conditions (b).

As for the conditions (c), first note that the dimension n in Theorem 0.3 corresponds to $n+4$ in Theorem 6.1. But if one uses n as in the latter theorem, one has

$$H^{n/2}(U, \mathcal{O}_{\mathcal{X}}) \cong \mathcal{H}^{(\setminus+\in)/\in}(\mathcal{U}, \mathcal{I}_{\mathcal{X}}) \cong \mathcal{H}_\mathfrak{m}^{(\setminus+\triangle)/\in}(\mathcal{I}).$$

Hence the two conditions (c) correspond.

Therefore the two theorems 0.3 and 6.1 are equivalent, as claimed.

In equicharacteristic 2 the computations of Section 5 remain true in the local case. So Theorems 0.3 and 6.1 are true in equicharacteristic 2 provided one changes the phrase "$n \equiv 0 \pmod 4$" in the parity condition to "n is even." If R is a regular local ring with residue field of characteristic 2 and quotient field of characteristic 0, a different set of calculations is needed. These are unfortunately somewhat involved, and we do not reproduce them here.

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