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## FIXED SETS OF FRAMED G-MANIFOLDS

## STEFAN WANER

ABSTRACT. This note describes restrictions on the framed bordism class of a framed manifold Y in order that it be the fixed set of some framed Gmanifold M with G a finite group. These results follow from a recently proved generalization of the Segal conjecture, and imply, in particular, that if M is a framed G-manifold of sufficiently high dimension, and if G is a p-group, then the number of "noncancelling" fixed points is either zero or approaches infinity as the dimension of M goes to infinity. Conversely, we give sufficient conditions on the framed bordism class of a manifold Y that it be the fixed set of some framed G-manifold M of arbitrarily high dimension.

Introduction and statement of results. In this note, we show how the recently proved Segal conjecture on the stable cohomotopy of the classifying space BG of a finite group G turns out to place severe restrictions on the fixed-sets of framed G-manifolds of large dimension.

Conner and Floyd proved the following result in  $[\mathbf{CF}, 40.1]$ . Let  $G = \mathbf{Z}/p$  (p an odd prime), and let M be a smooth compact oriented G-manifold with fixed set Y of codimension n and framed in M. (That is, the normal bundle of Y in M is equivariantly framed.) Assume also that the local representation normal to Y is the same for all components of Y. Then, denoting oriented bordism by  $\Omega_*$ , one has  $[Y] \in p^{s(n)}\Omega_8$ , where  $s(n) \to \infty$  as  $n \to \infty$ . When Y is discrete, this means that the number of "noncancelling" fixed points is either zero or becomes large as the dimension of M increases.

Here, we examine this phenomenon in the context of framed G-manifolds, and give a direct generalization for arbitrary finite groups G. As alluded to above, our proof makes extensive use of the Segal conjecture proved by Carlsson [C1], or, more precisely, its generalization due to Adams, Haeberly, Jackowski, and May [A1]. This suggests that even the "stable" (high-dimensional) properties of fixed sets of G-manifolds are subtle, and that a generalization of the Conner-Floyd result to oriented G-manifold for arbitrary G might require some form of completion result for oriented bordism analogous to the Segal conjecture.

If M is a (smooth) framed G-manifold, then there exists an orthogonal G-module V such that M is "modelled locally on V" in the sense of Pulikowski [**P1**] and Kosniowski [**K1**]. This means that if  $x \in M$ , then there is a neighborhood U of x which is  $G_x$ -diffeomorphic with  $V|G_x$ .

Our result is the following.

THEOREM A. Let G be a finite group, let V be an orthogonal G-module with  $V^G = \{0\}$ , and let  $k \ge 0$ . Then there exists an integer j as well as a sequence  $(s_n)$ 

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with  $s_n \to \infty$  as  $n \to \infty$ , such that, if M is any framed G-manifold modelled locally on the representation  $V^n \oplus \mathbf{R}^k$  with G-fixed set Y, one has  $[Y] \in j^{s_n} \Omega_k^{\mathrm{fr}}$ , where  $\Omega_*^{\mathrm{fr}}$ denotes nonequivariant framed bordism.

The integer j is determined by the isotropy subgroups of points in V and the algebra of the Burnside ring of G, and will be described fully in §1. When j > 1, the theorem implies that one cannot have a framed G-manifold modelled on V possessing a single fixed point (see §2, Corollary 2). If G is a p-group, it will turn out that j is always a power of p. When G has odd order and V is so large as to contain arbitrary G-orbits, then j > 1. On the other hand, if, for example,  $G = \mathbf{Z}/p \times \mathbf{Z}/q$  with p and q distinct primes, then there exist V's such that j = 1. Theorem A has the following converse.

THEOREM B. Let G be a finite group, let V be an orthogonal G-module with  $V^G = \{0\}$ , and let  $m, k \ge 0$ . Then, with j as in Theorem A and Y an arbitrary framed manifold of dimension k, there exists an integer n and a framed G-manifold M modelled locally on  $V^m \oplus \mathbf{R}^k$  with fixed set framed cobordant with  $j^n Y$ .

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1. A consequence of the Segal conjecture. Let G be a finite group and let  $\mathcal{U} = R^{\infty}$ , where R denotes the real regular representation of G, endowed with its natural inner product. We shall write  $V < \mathcal{U}$  to indicate that V is a finitedimensional G-invariant subspace of  $\mathcal{U}$ . The one-point compactification of  $V < \mathcal{U}$ will be denoted by  $S^V$  and, if X is a based G-space, the smash product  $X \wedge S^V$ will be denoted by  $\Sigma^V X$ . The stable equivariant cohomotopy of X is given by

$$\omega_G^{\gamma}(X) = \operatornamewithlimits{colim}_{U < \mathcal{U}} [\Sigma^{W \oplus U} X, S^{V \oplus U}]_G,$$

where  $\gamma = [V - W] \in RO(G)$  and where  $[-, -]_G$  denotes G-homotopy classes of based G-maps. Dually, the  $-\gamma$ th stable equivariant homotopy group,  $\omega_{-\gamma}^G(X)$ , is given by

$$\omega_{-\gamma}^G(X) = \underset{U < \mathcal{U}}{\operatorname{colim}} [\Sigma^{W \oplus U}, \omega^{V \oplus U} X]_G.$$

We shall require the following result.

LEMMA 1.1. Let n > 0,  $m \ge 0$ , and  $V < \mathcal{U}$  with  $V^G = \{0\}$ . Then  $\omega_{mV+n}^G$  is finite.

**PROOF.** Consider first the case m = 0. One has, by a result of Hauschild [H1],

$$\omega_n^G \cong \Sigma_{(H)} \pi_n^s (B(NH/H)_+)$$

for  $n \ge 0$ , where the sum is taken over a complete set of conjugacy classes (H) of subgroups of G. The subscript + denotes addition of a disjoint basepoint. If n > 0, then  $\pi_n^s(B(NH/H)_+)$  is finite. Now let  $m \ge 0$ . Then

$$\omega_{mV+n}^G \cong [S^{mV}, \Omega^n Q_G S^0]_G,$$

where  $Q_G S^0$  is the equivariant loop space  $\operatorname{colim}_{W < \mathcal{U}} \Omega^W S^W$ ,  $\Omega^W S^W$  denoting the *G*-space of self-maps of  $S^W$  (see, for example, [**H1** or **CW**]). Since  $n \ge 0$ , all the homotopy groups of all fixed sets of  $\Omega^n Q_G S^0$  are finite by the case n = 0 applied to the subgroups  $H \subset G$ . It now follows by induction over the skeleta of  $S^{mV}$  that  $\omega_{mV+n}^G$  is finite.  $\Box$ 

Let  $V < \mathcal{U}$  be any G-module with  $V^G = \{0\}$ . Define an associated family  $\mathcal{F}(V)$  of subgroups of V by

$$\mathcal{F}(V) = \{ H \subset G \colon V^H \neq 0 \}.$$

One has a universal G-space  $E\mathcal{F}(V)$  associated with  $\mathcal{F}(V)$ ;  $E\mathcal{F}(V)$  is the unique (up to G-homotopy) G-CW complex with  $E\mathcal{F}(V)^H$  contractible for each  $H \in \mathcal{F}(V)$  and empty otherwise. There is then a G-cofiber sequence

$$E\mathcal{F}(V)_+ \to S^0 \to \underline{E}\mathcal{F}(V) \to \cdots$$

associated with the projection of  $E\mathcal{F}(V)$  onto a point. Note that, with S(U) denoting the unit sphere in  $U < \mathcal{U}$ , one has

$$E\mathcal{F}(V) \simeq S(\infty V) = \operatorname{colim}_{n} S(nV),$$

while

$$\underline{E}\mathcal{F}(V) \simeq S^{\infty V} = \operatorname{colim}_n S^{nV},$$

both colimits being taken with respect to the natural inclusions. Passing to stable equivariant cohomotopy gives an exact sequence

(1) 
$$\cdots \to \omega_G^{\gamma}(\underline{E}\mathcal{F}(V)) \xrightarrow{\beta} \omega_G^{\gamma}(S^0) \xrightarrow{\alpha} \omega_G^{\gamma}(E\mathcal{F}(V)_+) \to \cdots$$

in which  $\alpha$  is the Segal map in the generalized context of [A1]. In this setting, the Segal conjecture takes the following form. Let A(G) denote the Burnside ring of G, and let, for  $H \subset G$ ,

$$d_H \colon A(G) \to \mathbf{Z}$$

be the homomorphism assigning to the virtual G-set s - t the integer  $|s^{H}| - |t^{H}|$ . Denote the ideal  $\bigcap_{(H) \in \mathcal{F}(V)} \ker d_{H}$  by I(V), and I(V)-adic completion of the A(G)-module M by M. The conjecture as proved in [A1] then states that  $\alpha$  induces an isomorphism

$$\alpha \widehat{:} (\omega_G^{\gamma}(S^0)) \widehat{\to} \omega_G^{\gamma}(E\mathcal{F}(V)_+)$$

for each  $\gamma \in RO(G)$ . (In particular,  $\omega_G^{\gamma}(E\mathcal{F}(V)_+)$  is I(V)-adically complete.)

Let  $k \in \mathbb{Z}$ . The exact sequence (1) is closely related to the exact sequence

(2) 
$$\cdots \to \omega_{nV+k}^G(S^0) \xrightarrow{(\beta_n)} \omega_k^G(S^0) \xrightarrow{(\alpha_n)} \omega_{nV+k-1}^G(S(nV)_+) \to \cdots$$

is stable G-homotopy induced by the cofiber sequence

$$S(nV)_+ \to D(nV)_+ \to S^{nV} \to \Sigma S(nV)_+ \to \cdots$$

The sequence (2) gives rise to short exact sequences

(3) 
$$0 \to \omega_k^G / \operatorname{Im} \beta_n \xrightarrow{\alpha_n} \omega_{nV+k-1}^G (S(nV)_+) \to \operatorname{coker} \alpha_n \to 0,$$

where  $\omega^G_* = \omega^G_*(S^0)$ . One has natural homomorphisms

$$\gamma_* : \omega_{(n+1)V+k-1}^G(S(n+1)V_+) \to \omega_{nV+k-1}^G(SnV_+)$$

(omitting some parentheses), given as follows. Let  $\nu: S(n+1)V_+ \to \Sigma^V SnV_+$ denote the natural quotient, obtained by collapsing about a tubular neighborhood of S(nV) in S((n+1)V), and define  $\gamma_*$  as the composite

$$\omega_{(n+1)V+k-1}^G(S(n+1)V_+) \xrightarrow{\nu_\star} \omega_{(n+1)V+k-1}^G(\Sigma^V SnV_+) \cong \omega_{nV+k-1}^G(SnV_+).$$

It may be checked that, under Spanier Whitehead duality, the maps  $\gamma_*$  agree with the inverse system maps

$$\gamma^* \colon \omega_G^{-k}(S(n+1)V_+) \to \omega_G^{-k}(SnV_+)$$

induced by inclusion. One also has natural homomorphisms

$$\mu_* \colon \omega^G_{(n+1)V+k}(S^0) \to \omega^G_{nV+k}(S^0),$$

given by the composites

$$\omega_{(n+1)V+k}^G(S^0) \to \omega_{(n+1)V+k}^G(\Sigma^V S^0) \cong \omega_{nV+k}^G(S^0),$$

where the first map is induced by inclusion  $S^0 \to S^V \cong \Sigma^V S^0$ . The maps  $\gamma_*$  and  $\mu_*$  commute the maps in the sequence (2), giving commutative diagrams:

$$\cdots \to \begin{array}{ccc} \omega_{(n+1)V+k}^{G}(S^{0}) & \stackrel{(\beta_{n})}{\to} & \omega_{k}^{G}(S^{0}) & \stackrel{(\alpha_{n})}{\to} & \omega_{(n+1)V+k-1}^{G}(S(n+1)V_{+}) & \to \cdots \\ & \downarrow \mu_{*} & \parallel & & \downarrow \gamma_{*} \\ \cdots \to & \omega_{nV+k}^{G}(S^{0}) & \stackrel{(\beta_{n})}{\to} & \omega_{k}^{G}(S^{0}) & \stackrel{(\alpha_{n})}{\to} & \omega_{nV+k-1}^{G}(SnV_{+}) & \to \cdots \end{array}$$

Passing the sequences (3) to (inverse) limits gives an exact sequence

(4) 
$$0 \to \lim_{n} \omega_{k}^{G} / \operatorname{Im} \beta_{n} \xrightarrow{\alpha} \lim_{n} \omega_{nV+k-1}^{G} (S(nV)_{+}) \to \lim_{n} \operatorname{coker} \alpha_{n} \to 0$$

since  $\lim^{1} \omega_{k}^{G} / \operatorname{Im} \beta_{n} = 0$ , the bonding maps being surjections. The map  $\alpha = \lim_{n \to \infty} \alpha_{n}$  is reminiscent of the Segal map  $\alpha$ . Write the latter (dually) as

$$\alpha : \lim_{n} \omega_{k}^{G} / I(V)^{n} \omega_{k}^{G} \to \lim_{n} \omega_{nV+k-1}^{G} (S(nV)_{+}).$$

(The target is  $\omega_G^{-k}(E\mathcal{F}(V)_+)$  by vanishing of the lim<sup>1</sup> terms [A1].) Abbreviate  $\lim_n \omega_k^G / \operatorname{Im} \beta_n$  as  $(\omega_k^G)_{\beta}$ . One then has

**PROPOSITION 1.2.** There exists a natural homomorphism

$$\psi \colon (\omega_k^G) \widehat{\to} (\omega_k^G)_\beta^{\widehat{}}$$

making the diagram

$$\begin{array}{ccc} (\omega_k^G) & \stackrel{\alpha}{\longrightarrow} & \omega_G^{-k}(E\mathcal{F}(V)_+) \\ \psi_{\downarrow} & & \\ (\omega_k^G)_{\beta} & & \\ \end{array}$$

commute. It now follows from injectivity of  $\alpha$  (in (4)) that both  $\alpha$  and  $\psi$  are isomorphisms.

PROOF. If k < 0, the conclusion is immediate since  $\omega_k^G = 0$ . Thus assume  $k \ge 0$ . It suffices to show that, for each  $n \ge 0$ , there exists an integer r(n) with

$$I(V)^{r(n)} \subset \operatorname{Im} \beta_n.$$

(This will then technically define a pro-map from the one inverse system to the other.)

Let  $x \in \omega_k^G$ . Then x is represented by a G-map  $S^{W+k} \to S^W$  for some  $W < \mathcal{U}$ . Our object is now to extend a representative of  $\rho x$  over  $S^{W+k+nV}$  (stably) for arbitrary  $\rho \in I(V)^{r(n)}$  with r(n) independent of x. Regard the pair  $(S^{nV}, S^0)$  as a relative G-CW complex with relative G-cells of the form  $G/H \times D^i$  for  $H \in \mathcal{F}(V)$ (which one may assume by the orbit structure of  $S^{nV}$ ).

We define r(n) as the number of relative G-cells in  $(S^{nV}, S^0)$ . Assume, inductively over the skeleta of the pair, that for each  $\rho \in I(V)^{s(p)}$ , with s(p) the number of relative G-cells in the p-skeleton  $((S^{nV})^p, S^0)$ , one has a stably G-homotopy commutative diagram:

$(S^{nV})^p \wedge S^{W+k}$	$\xrightarrow{g_p}$	$S^W$
j↑		
$S^{W+k}$	$\xrightarrow{f_p}$	$S^W$

Here,  $f_p$  represents  $\rho x$  and j is inclusion. The obstruction to extending  $g_p$  stably over a typical (p + 1)-cell of the form  $G/H \times D^{p+1}$  defines a stable *H*-equivariant map

$$\theta \colon S^p \wedge S^{W+k} \xrightarrow{c} (S^{nV})^p \wedge S^{W+k} \xrightarrow{g_p} S^W,$$

where c is adjoint to the attaching map for that cell. If  $k \in I(V)$ , one may represent k by a stable G-map  $\underline{k}: S^X \to S^X$  for suitable  $X < \mathcal{U}$ . Consider the diagram:

$$\begin{array}{cccc} (S^{nV})^p \wedge S^{W+k} \wedge S^X & \stackrel{g_p \wedge k}{\to} & S^W \wedge S^X \\ j^{\dagger} & & || \\ S^{W+k} \wedge S^X & \stackrel{f_p \wedge k}{\to} & S^W \wedge S^X \end{array}$$

The obstruction to extending  $g_p \wedge \underline{k}$  stably over this cell is now represented by  $\theta \wedge \underline{k}$ , regarded as an *H*-equivariant map. Since  $k \in I(V)$  and  $H \in \mathcal{F}(V)$ , this is *H*-homotopy trivial. Thus one may extend  $g_p \wedge \underline{k}$  stably over this cell. Note that  $f_p \wedge \underline{k}$  represents  $k\rho x$ , so that one may continue this process over the relative (p+1)-cells and obtain the inductive step, and hence the result.  $\Box$ 

One has the following converse to Proposition 1.2.

PROPOSITION 1.3. Let  $k \in \mathbb{Z}$ . Then there exists a sequence  $s(n) \to \infty$  as  $n \to \infty$  such that  $\operatorname{Im} \beta_n \subset I(V)^{s(n)} \omega_k^G$  for each n sufficiently large.

**PROOF.** Define a preliminary sequence r(n) by

$$r(n) = \min\{n, \max\{j \in \mathbf{N} \colon \operatorname{Im} \beta_n \subset I(V)^j \omega_k^G\}\}.$$

(Note that one must allow  $\max\{j \in \mathbf{N} \colon \operatorname{Im} \beta_n \subset I(V)^j \omega_k^G\} = \infty$ .) Then, by definition,  $\operatorname{Im} \beta_n \subset I(V)^{r(n)} \omega_k^G$ . To prove the proposition, it suffices to show that there exists a subsequence q(n) of r(n) with  $q(n) \to \infty$  as  $n \to \infty$ . Assume that no such subsequence exists. Then there exists an integer  $j \in \mathbf{N}$  and a subsequence t(n) of the natural numbers with

$$\operatorname{Im} \beta_{t(n)} \subset I(V)^{j} \omega_{k}^{G} \quad ext{and} \quad \operatorname{Im} \beta_{t(n)} \not\subset I(V)^{j+1} \omega_{k}^{G}.$$

It follows that there is a sequence of stable G-maps

$$x_{t(n)} \colon S^{nV+W+k} \to S^W$$

with the composite

$$y_{t(n)} \colon S^{W+k} \to S^{nV+W+k} \to S^W$$

defining a class  $[y_{t(n)}] \in I(V)^{j} \omega_{k}^{G} - I(V)^{j+1} \omega_{k}^{G}$  for each n.

If k > 0, then, by Lemma 1.1,  $\omega_k^G$  and  $\omega_{k+nV}^G$  are finite. Since the maps  $\beta_n$  define a map  $\beta$  into the constant system  $\{\omega_k^G\}$ , it now follows that there exists an element  $z = ([z_n]) \in \lim_n \omega_{k+nV}^G$ , obtained from the  $[x_{t(n)}]$  by application of the bonding homomorphisms, with

$$\beta_n([z_n]) \in I(V)^j \omega_k^G - I(V)^{j+1} \omega_k^G$$

for each  $n \ge 0$ . However,  $\beta_n([z_n]) = \beta(z)$  is now independent of n, since it is in a constant system, and  $\beta(z) \in \bigcap_n \operatorname{Im} \beta_n$ , by construction. Thus the completion

$$\omega_k^G \to (\omega_k^G)_\beta^\frown$$

maps  $\beta(z)$  to zero. Thus, by Proposition 1.2, I(V)-adic completion  $\omega_k^G \to (\omega_k^G)^{\widehat{}}$ maps  $\beta(z)$  to zero as well. It now follows that  $\beta(z) \in \bigcap_n I(V)^n \omega_k^G$ , by definition of I(V)-adic completion. But  $\beta(z) \in I(V)^j \omega_k^G - I(V)^{j+1} \omega_k^G$ , a contradiction. We now consider the case k = 0. Here, by definition of the  $x_n$ , one has  $\beta_n(x_n) \in I(V)^{-1} \otimes I(V$ 

We now consider the case k = 0. Here, by definition of the  $x_n$ , one has  $\beta_n(x_n) \in I(V)\omega_0^G = I(V)$ , since  $\omega_0^G \cong A(G)$ . However,  $I(V)/I(V)^m$  is finite for each  $m \ge 1$ , so that there exists a sequence  $([z_n])$  with

$$z_n \colon S^{nV+W} \to S^W$$

such that  $\beta_n[z_n] \in I(V)^j - I(V)^{j+1}$  and such that  $([z_n])$  maps under the natural quotient

$$\prod_{n} \omega_{nV}^{G} \to \prod_{n} A(G)/I(V)^{n}$$

to an element  $\underline{a} = ([a_n])$  of  $\lim_n A(G)/I(V)^n = A(G)$ . Thus if  $a_n \in A(G)$  represents  $[a_n]$ , one has  $a_n - \beta_n[z_n] \in I(V)^n$ . Consider  $\psi(\underline{a}) \in A(G)_{\beta}$ . By the construction of  $\psi$ , there is a sequence q(n) with  $q(n) \leq n$  and  $q(n) \to \infty$  such that

$$a_n - \beta_n[z_n] \in \operatorname{Im} \beta_{q(n)}$$

It now follows that  $\psi(\underline{a}) = 0$ , whence  $\underline{a} = 0$ . But  $a_n = \beta_n[z_n] \in I(V)^j - I(V)^{j+1}$ , which is again a contradiction.

When k < 0,  $\omega_k^G = 0$ , so the conclusion is automatic in this case.  $\Box$ 

2. Application to framed G-manifolds. Fix  $V < \mathcal{U}$ , and let M be a smooth G-manifold. Then M is said to have equivariant dimension V (or to be a V-manifold) if, for each  $x \in \operatorname{Int} M$ , there is a smooth  $G_x$ -equivariant diffeomorphism  $i: V \to M$ , taking 0 to x. More generally, M is a (V - W)-manifold for V and  $W < \mathcal{U}$  if  $M \times D(W)$  is a V-manifold. This notion is due originally to Pulikowski [P1] and Kosniowski [K1], but we shall not be requiring such generalizations here. We shall refer to a G-manifold of dimension  $V^n \pm \mathbf{R}^k$  (where  $\mathbf{R}^k$  is given the trivial G-action) as an  $(nV \pm k)$ -manifold, and all G-manifolds considered will be assumed compact.

The normal bundle of a G-manifold with equivariant dimension V has fibers similarly modelled on a fixed representation W in the sense that the fiber over a typical point x is  $G_x$ -isomorphic with W. Such G-bundles are discussed in [W1 and W2]. M is equivariantly framed if its normal bundle  $\mu_M$  with respect to a smooth embedding in some (large) finite-dimensional G-module U is a product,  $\mu_M \cong \mathcal{E}_M(W)$ , where  $\mathcal{E}_M(W)$  is the product G-bundle  $M \times W \to M$ , and where  $V \oplus W \cong U$ , as a G-module.

REMARK 2.1. This last condition, that  $V \oplus W \cong U$ , is necessary to obtain a welldefined homomorphism from framed *G*-bordism into equivariant stable homotopy. For example, if  $G = \mathbb{Z}/p$  (*p* prime) and *V* is any nontrivial irreducible *G*-module, then the unit sphere S(V) is equivariantly framed, and may be viewed as either a (V-1)-manifold or a (v-1)-manifold, where  $v = \dim V$ . However, it is not equivariantly framed, in the above sense, as a (v-1)-manifold.

LEMMA 2.2. Let V be such that  $V^G = \{0\}$ , and let n be a nonnegative integer. Then there exists a nonnegative integer N = N(n, V) such that, if M is any framed (nV+k)-manifold with n > N, then the normal bundle  $\gamma_G$  of  $M^G$  in M is a product G-bundle.

**PROOF.** Embed M equivariantly in the (large) G-module U and choose a trivialization,  $\mu_M \cong \mathcal{E}_M(W)$ , of the normal bundle of M. Write

$$U = W \oplus V^n \oplus \mathbf{R}^n \cong U_0 \oplus V^r \oplus V^n \oplus \mathbf{R}^k,$$

where  $U_0$  has no summands isomorphic with a summand of V. Then  $\gamma_G \oplus \mathcal{E}(V^r)$  has fiber dimension (n+r)V, and is canonically a product G-bundle. The G-bundle  $\gamma_G$  is classified by the space  $BO_G(nV)$ , where  $O_G(jV)$  is the group of equivariant orthogonal isomorphisms of  $jV = V^j$ . The composite

$$M^G \to BO_G(nV) \to \operatorname{colim}_j BO_G(jV)$$

of the natural inclusion with a classifying map is therefore null-homotopic. Since the second arrow is an *n*-equivalence for sufficiently large m (depending only on n and V), the result now follows.  $\Box$ 

It follows from the lemma that the fixed-sets of framed G-manifolds admit stable framings, given sufficiently large "codimension" n. The above argument may easily be elaborated to show that, for each  $H \subset G$ ,  $M^H$  is equivariantly framed as an NH/H-manifold.

Denote by  $\Omega_{\mathrm{fr}}$  nonequivariant framed bordism (stable homotopy). If  $H \subset G$ , then let  $J(H) \subset \mathbb{Z}$  be the ideal

$$J(H) = \operatorname{Im} d_H \colon I(V) \to \mathbf{Z}.$$

We reformulate Theorem A, including a description of the integer j.

THEOREM A. Let G be a finite group, let V be any orthogonal G-module with  $V^G = \{0\}$ , and let  $k \ge 0$ . Let  $H \subset G$  be such that  $V^H = 0$ . Then there exists a sequence  $(s_n)$  with  $s_n \to \infty$  as  $n \to \infty$  such that, if M is any framed (nV + k)-manifold with H-fixed set  $Y^k$ , one has  $[Y] \in J(H)^{s_n} \Omega_k^{\text{fr}}$ .

COROLLARY 1. Let G be a p-group, let V be any orthogonal G-module with  $V^G = \{0\}$ , and let  $k \ge 0$ . Let  $H \subset G$  be such that  $V^H = \{0\}$ . Then there exists a sequence  $(s_n)$  with  $s_n \to \infty$  as  $n \to \infty$  such that, if M is any framed (nV + k)-manifold with H-fixed set  $Y^k$ , one has  $[Y] \in p^{s_n} \Omega_k^{f_n}$ .

PROOF. This is now an immediate consequence of the fact that, for a p-group,  $J(H) \subset p\mathbb{Z}$ .  $\Box$ 

COROLLARY 2. If G is any p-group, there does not exist any framed V-manifold possessing a single fixed point.

PROOF. If M were a framed V-manifold with a single fixed point, then the sequence  $M_n = (M \times M \times \cdots \times M)$  (*n* times) is a sequence of framed *nV*-manifolds each possessing a single fixed point, contradicting Corollary 1.  $\Box$ 

REMARK 2.3. Corollary 2 fails if G is not a p-group. For example, let  $G = \mathbf{Z}/p \times \mathbf{Z}/q$ , with p and q distinct primes. Choose integers m and n with mp+nq = 1, and let  $V = \rho$ , any one-dimensional semifree irreducible complex  $\mathbf{Z}/pq$ -module. The element  $a = [1 - m\mathbf{Z}/p - n\mathbf{Z}/q] \in A(\mathbf{Z}/pq)$  lies in I(V), since  $\mathcal{F}(V) = \{1\}$  (where 1 is the trivial subgroup). By the proof of Proposition 1.1, there exists an integer r(n) with  $I(V)^{r(n)} \subset \text{Im } \beta_n$  for any  $n \geq 0$ . Choose any such n, and let  $f_n: S^{nV+W} \to S^W$  be such that  $\beta_n[f_n] = a^n$ . One may G-homotope f to a G-map transverse to  $0 \in S^W$ , so that  $f_n^{-1}(0)$  is a framed nV-manifold, M. The fixed-set of M corresponds to the class of  $f_n^G: (S^W)^G \to (S^W)^G \in \pi_0^s \cong \mathbf{Z}$ . By definition of  $f_n$ , however, one has

$$\log f_n^G = d_G(a^n) = 1,$$

so that M possesses only a single "essential" (noncancelling) fixed point in the sense of [**K2**]. One can thus attach copies of  $S(V) \times I$  to M to obtain a framed G-manifold of dimension nV possessing a single fixed point.

3. Proof of Theorems A and B. We first prove Theorem A. If  $M_n$  is a framed G-manifold of dimension nV + k, then the Pontryagin-Thom construction defines a G-map

$$f_n \colon S^{nV+k+W} \to S^W$$

for some W. Let s(n) be the sequence obtained in Proposition 1.2. Then the composite

$$S^{k+W} \to S^{nV+k+W} \to S^W$$

of  $f_n$  with inclusion defines a class  $x \in I(V)^{s(n)}\omega_k^G$ . Let H be such that  $V^H = \{0\}$ . Then restriction of a G-map to the H-fixed subset defines a homomorphism  $\omega_k^G \to \pi_k^s$  such that, if  $a \in A(G)$ , then  $\varphi(ay) = d_H(a)\varphi(y)$ , where  $\varphi: A(G) \to A(H)$  is the forgetful homomorphism. This may be seen directly from the definition of the A(G)-action on  $\omega_k^G$ . Thus

$$\varphi(x) = \varphi(ay) = d_H(a)\varphi(y)$$

for some  $a \in I(V)^n$ , where  $d_H(I(V)^n) \subset J(H)^n$ . Since the *H*-fixed set of *M* corresponds to the class  $\varphi(x)$ , the result now follows.  $\Box$ 

REMARKS 3.1. If  $H \in \mathcal{F}(V)$ , then all information on the *H*-fixed set is lost upon application of  $\beta_n$ , so no analogous result can be drawn.

Turning to the proof of Theorem B, and with j a generator of the ideal J(G), let  $[Y] \in \pi_k^s$  be the stable homotopy class determined by the framed manifold Y. Then, under the natural map  $\pi_k^s \to \omega_k^G$ , [Y] determines a stable homotopy class of G-maps  $\varsigma: S^{U+k} \to S^U$  with  $U < \mathcal{U}$ . Following the proof of Proposition 1.2, one extends  $\rho^n \varsigma$  (stably) to a G-map

$$\varsigma' \colon S^{U+mV+k} \to S^U$$

for suitable n and arbitrary  $\rho \in I(V)$ . Now G-homotope  $\zeta'$  to a G-map  $\underline{\zeta}$  transverse to  $0 \in S^U$ , and let M be the framed G-manifold  $\underline{\zeta}^{-1}(0)$ . Then M has dimension

mV + k, and its *G*-fixed set *Z* is the preimage of  $(S^U)^G$  under the restriction  $\underline{\varsigma}|(S^{U+mV+k})^G = \underline{\varsigma}|(S^{U+k})^G$ , since  $V^G = \{0\}$ . Since  $\underline{\varsigma}$  is stably *G*-homotopic to an extension of  $\rho^n \varsigma$ , restricting to the *G*-fixed set gives a framed cobordism of *Z* with  $d_G(\rho)^n Y$ . The theorem now follows by choosing  $\rho \in d_G^{-1}(j)$ .  $\Box$ 

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DEPARTMENT OF MATHEMATICS, HOFSTRA UNIVERSITY, HEMPSTEAD, NEW YORK 11550