# A PROBLEM OF ANALYSIS SITUS

*By* G. N. WATSON.

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1. The theorem, which is due to Jordan,* that a regular closed curve possesses an interior and an exterior, is well known and has far reaching consequences in the theory of functions of a complex variable as developed by mathematicians on the lines first indicated by Cauchy.

It will be recalled that, when a regular closed curve is given, if $Q$ is a fixed point not on the curve, while $P$ is a variable point of the curve, then, if $\theta$ be the angle between $QP$ and the $x$-axis, Jordan's theorem is, effectively, that $\theta$ changes by $2\pi$ or by zero as $P$ describes the curve, and that in the former case $Q$ is called an interior point of the curve, while in the latter case $Q$ is called an exterior point of the curve.

In this paper, I propose to investigate a theorem somewhat similar to Jordan's theorem, namely that, when a simple closed curve with a *continuously turning tangent*† is given, if $\psi$ be the angle between the tangent at a variable point $P$ of the curve and the $x$-axis, *then* $\psi$ *changes by* $2\pi$ *as* $P$ *describes the curve.*

This theorem can be employed‡ in proving Macdonald's theorem concerning a relation between the number of zeros of an analytic function $f(z)$, and the number of zeros of its derivate $f'(z)$ in a prescribed region of the plane of the complex variable $z$.

2. As the terminology of *Analysis Situs* is not fixed, and some of its

* *Cours d'Analyse*, t. I, pp. 96–103. The substance of a fairly simple proof (due to Ames, *American Journal*, Vol. XXVII, pp. 343–380) is given in my tract, "Complex Integration and Cauchy's Theorem" (*Camb. Math. Tracts*, No. 15).

† The precise meaning to be attributed to this phrase will be found below in § 3; the case in which the tangent has a finite number of abrupt changes of direction is discussed in §§ 8, 9.

‡ Whittaker and Watson, *Modern Analysis*, p. 121. Macdonald stated and proved his theorem in the *Proceedings*, Vol. XXIX, p. 576. His proof is of a different nature from that given in the *Modern Analysis*, and hence it does not raise the question discussed in this paper.

theorems are little known, it will be convenient to give a summary of the definitions and theorems* which are preliminary to the analysis employed in this paper.

Let
$$x = x(t), \quad y = y(t),$$

where $x(t)$, $y(t)$ are real continuous one-valued functions of the real parameter $t$ for all values of $t$ such that $t_0 \leqslant t \leqslant T$. The functions are to be such that they do not assume the same pair of values† for *different* values of $t$ in the range $t_0 < t < T$. Then the set of points $(x, y)$ determined by the values of $t$ in the range $t_0 \leqslant t \leqslant T$ is called a *simple curve* joining the end points $(x_0, y_0)$ and $(X, Y)$, where $x_0, \ldots$ have been written for $x(t_0), \ldots$. If the end points coincide, the curve is said to be *closed*.

A curve defined *either* by the equation $y = f(x)$, $(x_0 \leqslant x \leqslant x_1)$, *or else* by the equation $x = \phi(y)$, $(y_0 \leqslant y \leqslant y_1)$, where $f$ (or $\phi$) is a one-valued continuous function of its argument is called an *elementary curve*.

A simple curve is said to be *regular*‡ if it can be divided into a *finite* number of parts, each of which is an elementary curve.

If a simple closed curve is given, and if $T - t_0 = \omega$, then $\omega$ is called the *primitive period* associated with the pair of functions $x(t)$, $y(t)$; these functions are then defined for *all* real values of $t$ by the equations

$$x(t+n\omega) = x(t), \quad y(t+n\omega) = y(t),$$

where $n$ is any integer.

If $Q(x_1, y_1)$ is a point not on a given simple closed curve, it can be shewn that it is possible to define a continuous function of $t$, $\theta(t)$ say, such that
$$\cos\theta(t) = \{x(t) - x_1\}/r, \quad \sin\theta(t) = \{y(t) - y_1\}/r,$$

where
$$r = +\sqrt{[\{x(t)-x_1\}^2 + \{y(t)-y_1\}^2]}.$$

It can then be shewn that, if the curve is *regular*,

$$\theta(t+\omega) - \theta(t) = 2n\pi,$$

where $n$ is an integer depending only on $(x_1, y_1)$; this integer is called the *order* of $Q$ with respect to the regular closed curve.

Jordan's theorem obviously is that $n = 0$ or $\pm 1$; if $n = 0$, $Q$ is said to be a point exterior to the given curve, while if $n = \pm 1$, $Q$ is said to be a point of the interior of the curve.

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* The reader is referred to Ames' memoir, or to my tract, for a complete account of the theorems now quoted.

† This condition ensures that the curve has no double points.

‡ Ames uses this term in a more restricted sense.

Since the values of $t$ form an ordered set,* the points of a simple curve can be regarded as ordered, the order of points on the curve being the order of the corresponding values of $t$. Such an ordered set of points is called an *oriented* curve; two oriented curves $C_1$, $C_2$ having a common arc $\sigma$ are said to have the *same orientation* if the points of $\sigma$ are in the same order, whether $\sigma$ is regarded as belonging to $C_1$ or to $C_2$. If the points are not in the same order, $C_1$ and $C_2$ are said to have *opposite orientations.*

If an oriented curve $C_1$ is given by the equations

$$x = x(t), \quad y = y(t), \quad (t_0 \leqslant t \leqslant T),$$

and if we put $t = -t'$, $x(t) \equiv x'(t')$, $y(t) \equiv y'(t')$, then the point $(x', y')$ traces out an oriented curve $C_2$ as $t'$ increases from $-T$ to $-t_0$; the curves $C_1$, $C_2$ consist of the same set of points but have opposite orientations. It is consequently easily seen that when a regular closed curve is given it is possible to orient it (by suitable choice of parameter) in such a way that the order of interior points is $+1$ (not $-1$); such an oriented curve is said to be described *counter clockwise* as its parameter increases by the primitive period. It is invariably supposed that this choice of parameter is made.

The following theorem, due to Ames, is of importance in our investigation:—

AMES' THEOREM.†—Let $R_1$, $R_2$ be the sets of points formed by the interiors of two regular closed curves $C_1$, $C_2$, respectively; and let an arc $\sigma_1$ of $C_1$ coincide in position with an arc $\sigma_2$ of $C_2$. Then (i) if $R_1$, $R_2$ have no common point, the orientations of $\sigma_1$, $\sigma_2$ are opposite; but (ii) if every point of $R_1$ be a point of $R_2$, the orientations of $\sigma_1$, $\sigma_2$ are the same.

3. It is now possible to enunciate the theorem to be proved in this paper.

*Let a simple closed curve* $\Gamma$ *be defined by the equations*

$$x = x(t), \quad y = y(t), \quad (t_0 \leqslant t \leqslant T),$$

*and let the continuous functions* $x(t)$, $y(t)$ *be defined for all real values of* $t$ *outside the range* $(t_0, T)$ *in the customary manner, so that they form a pair of periodic functions which has primitive period* $T - t_0 = \omega$.

* They may be ordered in ascending order of magnitude.

† *Loc. cit.*, p. 362.

*Further, let the functions $x(t)$, $y(t)$, thus defined for all real values of $t$, have continuous differential coefficients* with regard to $t$ for all such values of $t$; it is to be assumed that these differential coefficients do not simultaneously vanish† for any value of $t$.*

*Then* (I) *the curve is regular,‡ so that Jordan's theorem is applicable to it:§ and* (II) *it is possible to define a function of $t$, say $\psi(t)$, by the equations*

$$\cos\psi(t) = \frac{\dot{x}}{+\sqrt{(\dot{x}^2+\dot{y}^2)}}, \qquad \sin\psi(t) = \frac{\dot{y}}{+\sqrt{(\dot{x}^2+\dot{y}^2)}},$$

*which is continuous for all real values of $t$, and is such that*

$$\psi(t+\omega) - \psi(t) = 2\pi.$$

4. The method of investigation which will be adopted is substantially "the method of subdividing regions into *suitable* regions" employed in a well known proof of the Heine-Borel theorem and kindred theorems.

We first observe that, since $\dot{x}$ and $\dot{y}$ never simultaneously vanish, and since $\dot{x}$ and $\dot{y}$ are continuous functions of $t$ for all real values of $t$, there exists‖ a function $\psi(t)$, which may be regarded as a real, one-valued, continuous¶ function of $t$ for all real values of $t$, and which satisfies the equations

$$\cos\psi(t) = \dot{x}/\sqrt{(\dot{x}^2+\dot{y}^2)}, \quad \sin\psi(t) = \dot{y}/\sqrt{(\dot{x}^2+\dot{y}^2)}.$$

Further, since $x$ and $y$, *qua* functions of $t$, have period $\omega$, it follows that $\dot{x}$ and $\dot{y}$ also have period $\omega$, and so

$$\cos\psi(t+\omega) = \cos\psi(t), \quad \sin\psi(t+\omega) = \sin\psi(t):$$

* This is the analytical interpretation of the phrase "continuously turning tangent" employed in § 1; Ames finds it convenient to describe the curve as "smooth" at a point at which $x(t)$ and $y(t)$ have continuous differential coefficients. In future we shall denote differential coefficients with regard to $t$ by dots.

† The removal of this restriction is discussed in § 8.

‡ This result is stated by Ames, who considered that the methods to be employed in a formal proof were so obvious that a proof might be left to the reader; the proof is given here only because it follows in a very few lines from part of the analysis employed in proving (II).

§ Since the results stated in § 2 are true for the curve under discussion, it will be tacitly assumed after the end of § 4 (except in § 10), that the requisite change of parameter is made (if necessary) to ensure that the curve $\Gamma$ has the conventional orientation; this change will not affect the continuity of the differential coefficients of the coordinates with respect to the parameter; in fact, it will merely change their signs.

‖ Cf. *Modern Analysis*, pp. 538, 539.

¶ The function $\psi(t)$ may be rendered *definite* in any convenient manner, *e.g.*, by choosing $\psi(0)$ to satisfy the inequalities $-\pi < \psi(0) \leqslant \pi$.

and therefore $$\psi(t+\omega)-\psi(t) = 2n\pi,$$

where $n$ is an integer.

Moreover $n$ is constant, *i.e.*, independent of $t$. For, as $t$ varies continuously, so do $\psi(t)$ and $\psi(t+\omega)$, while $n$ can vary only *per saltus*; and therefore $n$ is constant.*

Let $\Gamma'$ be any arc of $\Gamma$ and let $\tau$ be the parameter of any point of $\Gamma'$.

Then $\Gamma'$ belongs to one of the four following classes† according to the values of $\psi(\tau)$ on $\Gamma'$:

(A) For *one or more* values of $\tau$, $|\cos\psi(\tau)| \geqslant \cos\frac{1}{10}\pi$, and, for *all* values of $\tau$, $|\cos\psi(\tau)| > \sin\frac{1}{10}\pi$.

(B) For *one or more* values of $\tau$, $|\cos\psi(\tau)| \leqslant \sin\frac{1}{10}\pi$, and, for *all* values of $\tau$, $|\cos\psi(\tau)| < \cos\frac{1}{10}\pi$.

(C) For *all* values of $\tau$, $\sin\frac{1}{10}\pi < |\cos\psi(\tau)| < \cos\frac{1}{10}\pi$.

(D) For *one or more* values of $\tau$, $|\cos\psi(\tau)| \geqslant \cos\frac{1}{10}\pi$, and for *one or more* other values of $\tau$, $|\cos\psi(\tau)| \leqslant \sin\frac{1}{10}\pi$.

Arcs of these types will be denoted by the symbols $\Gamma_A$, $\Gamma_B$, $\Gamma_C$, $\Gamma_D$ respectively.

It is now easy to prove that $\Gamma$ is a *regular* curve.

For $\psi(t)$ is a continuous function of $t$, and, since continuity involves *uniformity* of continuity (Heine's theorem), we can divide the range $(t_0, T)$ into a *finite* number of (closed) subdivisions, such that if $t$, $t'$ are *any* two values of the parameter which are both in any one subdivision, then $|\psi(t')-\psi(t)|$ is less than, say, $\frac{1}{10}\pi$.

Corresponding to this subdivision of the range $(t_0, T)$, $\Gamma$ is divided into a finite number of arcs, *which are all of the types* $\Gamma_A$, $\Gamma_B$, $\Gamma_C$ [for, on an arc of the type $\Gamma_D$, $\psi(t)$ must change by at least as much as $\frac{3}{10}\pi > \frac{1}{10}\pi$].

It will now be shewn that each of the arcs $\Gamma_A$ and $\Gamma_C$ is an elementary curve of the type $y=f(x)$; an exactly similar proof (which is left to the reader) will then shew that each of the curves $\Gamma_B$ and $\Gamma_C$ is an elementary curve of the type $x=\phi(y)$: where $f$ and $\phi$ denote continuous functions of their respective arguments.

Take any one of the arcs $\Gamma_A$ or $\Gamma_C$, and let the parameters of its end points be $t_1$, $t_2$ $(0 < t_2-t_1 < \omega)$.

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* Cf. *Tract*, p. 6, footnote.

† The angle $\frac{1}{10}\pi$ is introduced in the following statements for the sake of definiteness. It could be replaced by any sufficiently small angle.

Then, when $t_1 \leqslant t \leqslant t_2$, we have

$$|\cos\psi(t)| > \sin\tfrac{1}{10}\pi,$$

and so $\dot{x}(t)$ is never zero and is therefore one-signed in the range $(t_1, t_2)$; therefore $x(t)$ is monotonic (in the strict sense) in this range, and it is also continuous. Hence, writing $x(t_1) \equiv x_1$, $x(t_2) \equiv x_2$, we see that, in the range $(x_1, x_2)$, $t$ is a continuous function* of $x$; and so, since $y$ is a continuous function of $t$, $y$ is also a continuous function of $x$; indeed the differential coefficient of $y$ with regard to $x$ exists and is equal to $\dot{y}/\dot{x}$.

Consequently each of the arcs $\Gamma_A$ and $\Gamma_C$ is an elementary curve of the type $y = f(x)$, and, similarly, each of the arcs $\Gamma_B$ and $\Gamma_C$ is an elementary curve of the type $x = \phi(y)$. Since the number of arcs $\Gamma_A$, $\Gamma_B$, $\Gamma_C$ is finite, the curve $\Gamma$ is regular; so theorem (I) is completely proved.

5. It is convenient to base the proof of theorem (II), namely that

$$\psi(t+\omega)-\psi(t) = 2\pi,$$

on the following lemma :—

LEMMA.—Let $P_0$ be a point (which is not an end point) of the elementary curve $y = f(x)$, which forms part of the given regular closed curve $\Gamma$; let $P_0T$ and $P_0U$ be the progressive and regressive tangents† at $P_0$ to $\Gamma$; also let $P_0Q$ be drawn parallel to that one of the directions of the $y$-axis which makes all points of the line $P_0Q$ which are sufficiently near to $P_0$ (except the end point $P_0$) lie *inside* $\Gamma$.

*Then the principal values of the angles* $T\hat{P}_0Q$, $Q\hat{P}_0U$ *are both* positive *and definitely* less than $\pi$.

[The corresponding form of the lemma for the elementary curve $x = \phi(y)$ merely involves the alteration that $P_0Q$ is to be drawn parallel to one of the directions of the $x$-axis.]

Let $P_0$ have parameter $t_0$ and coordinates $(x_0, y_0)$.

Then it can be shewn that it is possible to choose‡ a positive number $r$ so small that no points of $\Gamma$ except those on $y = f(x)$ lie inside or on the circle with centre $P_0$ and radius $r$.

With such a choice of $r$, let the point $(x_0, y_0+r)$ be called $B$, while $(x_0, y_0-r)$ is called $B_1$.

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* See, *e.g.*, Hardy, *A Course of Pure Mathematics*, § 109.

† So that $UP_0T$ is a straight line inclined $\psi(t) + 2r\pi$ [not $\psi(t) + (2r+1)\pi$] to the $x$-axis.

‡ Ames' memoir, p. 356; or *Tract*, p. 10.

Let $P$ be any point of $\Gamma$, with parameter $t$, and let the angles which $BP$ and $B_1P$ make with the $x$-axis be* $\theta(t)$ and $\theta_1(t)$; and let

$$\phi(t) = \theta(t) - \theta_1(t).$$

Now it can be proved that,† *if* $x(t_0+\delta) > x(t_0)$ *for sufficiently small positive values of* $\delta$, then integers $n_1$, $n_2$ exist such that

$$\phi(t_0) = (2n_1+1)\pi, \quad \phi(t_0+\delta) > (2n_1+1)\pi,$$

$$\phi(t_0+\omega-\delta) < (2n_2+1)\pi, \quad \phi(t_0+\omega) = (2n_2+1)\pi,$$

*and that* $\phi(t)$ *is not an odd multiple of* $\pi$ *when* $t_0+\delta < t < t_0+\omega-\delta$.

Hence $n_2 > n_1$, and so the order of $B$ exceeds the order of $B_1$; and therefore $B$ is *inside the curve and above* $P_0$; therefore $Q$ is above $P_0$ when $\dot{x}(t_0)$ is positive; similarly $Q$ is below $P_0$ when $\dot{x}(t_0)$ is negative.

The principal value of $T\hat{P_0}Q$ is therefore the principal value of $\frac{1}{2}\pi - \psi(t_0)$ or $\frac{3}{2}\pi - \psi(t_0)$, according as $\dot{x}(t_0) > 0$ or $< 0$.

Also, according as $\dot{x}(t_0) \gtrless 0$, we have

$$\sin T\hat{P_0}Q = \sin\{\pi \mp \tfrac{1}{2}\pi - \psi(t_0)\}$$

$$= \pm \cos\psi(t_0)$$

$$= \pm \dot{x}(t_0) \div \surd[\{\dot{x}(t_0)\}^2 + \{\dot{y}(t_0)\}^2],$$

and consequently $\sin T\hat{P_0}Q$ is essentially positive. Since the principal value of $T\hat{P_0}Q$ does not exceed $\pi$ numerically, *it follows* that the principal value of $T\hat{P_0}Q$ is *positive and less than* $\pi$.

Similarly the principal value of $Q\hat{P_0}U$ is the principal value of

$$\{\psi(t_0)+\pi\} - \tfrac{1}{2}\pi \quad \text{or of} \quad \{\psi(t_0)+\pi\} - \tfrac{3}{2}\pi,$$

according to the sign of $\dot{x}(t_0)$, and hence the principal value of $Q\hat{P_0}U$ is also positive and less than $\pi$; and the lemma is proved.

[The proof for an elementary curve of the type $x = \phi(y)$ is left to the reader.]

6. We shall now investigate the total change in $\psi$ as the point of contact of a tangent of a curve traverses the curve when the curve is of a special type and has a *discontinuously turning tangent*.

* The functional symbols $\theta$ and $\phi$ are used in this sense only in this section of the paper.

† See note ‡ on opposite page.

Suppose that the curve $\Gamma$ of the type specified in § 3 is given, and that a rectangle $R$ is given* which meets $\Gamma$ in two points $E$, $F$ only, the arc† $EF$ of $\Gamma$ which lies inside $R$ being part (or the whole) of an arc of the type‡ $\Gamma_A$; it is supposed that, if $t$, $t'$ be the parameters of *any* two points of the arc $EF$, then $|\psi(t)-\psi(t')| < \frac{1}{10}\pi$, that the sides of the rectangle are parallel to the axes, and that $E$, $F$ lie on the sides $m_1$, $m_2$ of the rectangle parallel to the $y$-axis; let $G$ be the end of $m_2$ which is inside $\Gamma$, and let $H$ be the end of $m_1$ which is inside $\Gamma$.

Then Jordan's theorem is applicable to the curve $EFGH$, and by Ames' theorem (see § 2) the curves $EFGH$ and $\Gamma$ have the same orientations.§

Let $t_1$ and $t_2$ be the parameters of $E$ and $F$ regarded as points of $\Gamma$.

Let $t'$ be a parameter for the curve $EFGH$ and $\omega'$ the associated period; and let $\psi'(t')$ be the angle between the tangent and the $x$-axis. It is obviously legitimate to take $t' \equiv t$ when $t_1 \leqslant t' \leqslant t_2$, and also

$$\psi'(t') = \psi(t)$$

in this range.

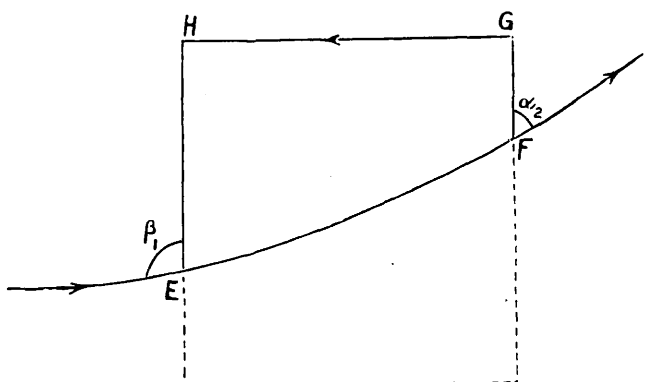


FIG. 1.

Now *define* the changes of $\psi'$ at $E$, $F$, $G$, $H$ to be the *principal values* of the respective angles. Let $\alpha_2$, $\beta_1$ be the changes of $\psi'$ at $F$ and $E$; then $\alpha_2$ and $\beta_1$ are positive and less than $\pi$, by § 5; also the changes of $\psi'$ at $G$ and $H$ are $+\frac{1}{2}\pi$ (not $-\frac{1}{2}\pi$) by a proof similar to that of § 5.

* Methods of constructing such rectangles are given in § 7.

† It is supposed that this arc $EF$ is given the same orientation as $\Gamma$.

‡ The investigation for an arc of the type $\Gamma_B$ is similar to that for $\Gamma_A$, and is consequently left to the reader.

§ The proof that every point inside $EFGH$ is also inside $\Gamma$ is left to the reader.

Now, as in § 4, $\psi'(t'+\omega')-\psi'(t') = 2n'\pi,$

where $n'$ is constant for values of $t'$ in the range $(t_1, t_2)$.

Now, if $t'_3$, $t'_4$ be the parameter of $G$ and $H$, we have

$$\begin{aligned}\psi'(t'+\omega')-\psi'(t') = & \{\psi'(t'+\omega')-\psi'(t'_4+0)\}+\{\psi'(t'_4+0)-\psi'(t'_4-0)\} \\ & +\{\psi'(t'_4-0)-\psi'(t'_3+0)\}+\{\psi'(t'_3+0)-\psi'(t'_3-0)\} \\ & +\{\psi'(t'_3-0)-\psi'(t'_2+0)\}+\{\psi'(t'_2+0)-\psi'(t'_2-0)\} \\ & +\psi'(t'_2-0)-\psi'(t').\end{aligned}$$

Make $t' \to t_1+0$, and we get*

$$\psi'(t'+\omega')-\psi'(t') = \beta_1+\tfrac{1}{2}\pi+\tfrac{1}{2}\pi+\alpha_2-\{\psi'(t_2-0)-\psi(t_1+0)\}.$$

The expression on the right lies between $\pi-\frac{1}{10}\pi$ and $3\pi+\frac{1}{10}\pi$; and so $2n'\pi = 2\pi$, *i.e.*, $n' = 1$.

Therefore $\qquad 2\pi = \beta_1+\pi+\alpha_2+\psi(t_2)-\psi(t_1).$

That is to say $\qquad \psi(t_2)-\psi(t_1) = 2\pi-(\beta_1+\alpha_2+\pi).$

6*a*. Next suppose that the rectangle $R$ (with sides parallel to the arcs) meets $\Gamma$ in two points $E$, $F$, only, as before, but that the arc $EF$ of $\Gamma$ which lies inside $R$ is part (or the whole) of one of the arcs $\Gamma_C$. If $E$ and $F$ are on opposite sides of $R$, we get precisely the result of § 6. Accordingly, we suppose that $E$ and $F$ are on adjacent sides of $R$; there are two cases to be considered, according as one or three vertices of the rectangle inside $\Gamma$. We shall investigate the former case (Fig. 2) in detail and leave the latter (Fig. 3) to the reader.

Let $G$ be the vertex of $R$ which is inside $\Gamma$; then since an arc $\Gamma_C$ may be represented by an equation of the form $y = f(x)$, *and also* by an equation of the form $x = \phi(y)$, it follows that if $\alpha_2$ be the principal value of the change of $\psi'$ at $F$ while $\beta_1$ is the principal value of the change of $\psi'$ at $E$, where $\psi'$ is the angle between the tangent at a point of $EFG$ and the $x$-axis, then $\alpha_2$ and $\beta_1$ both lie between 0 and $\pi$. Hence, as in § 6, we get†

$$\psi(t_2)-\psi(t_1) = 2\pi-(\beta_1+\alpha_2+\tfrac{1}{2}\pi),$$

where $t_1$, $t_2$ are the parameters of $E$ and $F$ on $\Gamma$.

* $\psi'(t')$ is continuous in each of the ranges $(t'_2+0,\ t'_3-0)$, $(t'_3+0,\ t'_4-0)$, $(t'_4+0,\ t'_1+\omega'-0)$, and is therefore constant in each of these ranges.

† Each corner of the rectangle inside $\Gamma$ contributes $\frac{1}{2}\pi$ in the analysis of § 6; hence the difference between this result and that of § 6.

In like manner, if there were three vertices of the rectangle inside $\Gamma$, we should get

$$\psi(t_2)-\psi(t_1) = 2\pi-(\beta_1+\alpha_2+\tfrac{3}{2}\pi).$$

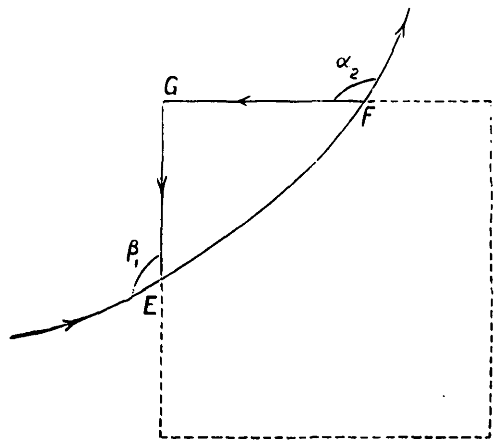


FIG. 2.

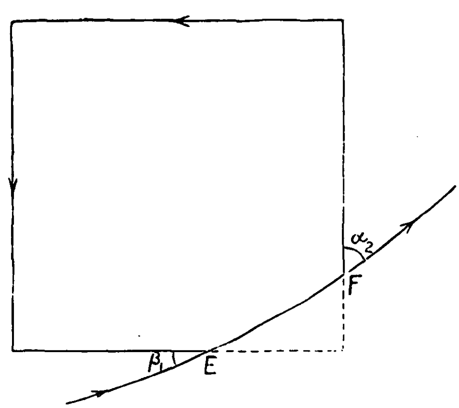


FIG. 3.

To sum up, we see that, if a rectangle $R$ with sides parallel to the axes meets $\Gamma$ in two points $E$ and $F$ only, so that the oriented arc $EF$ of $\Gamma$ is inside $R$, where all points of $EF$ belong to not more than one of the arcs $\Gamma_A$, $\Gamma_B$, $\Gamma_C$, and $E$, $F$ are on opposite sides of $R$ (unless $EF$ is part of an arc $\Gamma_C$, in which case they may be on adjacent sides), and if $t_1$, $t_2$ $(0 < t_2-t_1 < \omega)$ be the parameters of $E$ and $F$, then

$$\psi(t_2)-\psi(t_1) = 2\pi-(\beta_1+\alpha_2+\tfrac{1}{2}r\pi).$$

In this equation $r$ is the number of vertices of the rectangle which are inside $\Gamma$, $a_2$ is the principal value of $\lambda(t_2)-\psi(t_2)$, and $\beta_1$ is the principal value of $\psi(t_1)-\lambda(t_1)+\pi$, where $\lambda(t)$ denotes a value of the angle between the side of the rectangle (drawn *towards the interior* of $\Gamma$) through the point with parameter $t$ and the $x$-axis.

7. We shall now divide the interior of $\Gamma$ into a number of smaller regions. By the arguments of § 4, we can divide $\Gamma$ into a *finite* number of arcs, such that if $t$, $t'$ be the parameters of any two points on the same arc, then $|\psi(t)-\psi(t')| < \frac{1}{20}\pi$. Let these arcs (in order) be called $\sigma_1, \sigma_2, \ldots, \sigma_s$.

Let $P$ be any point on $\Gamma$, and let $Q$ be any other point on $\Gamma$, such that $P$, $Q$ are not on the same arc $\sigma$ nor on consecutive arcs. Then the distance $PQ$ has a positive lower bound, say $4\delta$. Cover the plane with a network of squares with sides parallel to the axes, the side of each square being of length $\delta$.

[If a vertex of any square lies on $\Gamma$, shift the network slightly until this is no longer the case.]

The arcs $\sigma$ are of the types (A), (B) and (C); not of type (D), also portions of $\Gamma$ which are in the same square, or in two consecutive squares, must belong to the same arc $\sigma$ or to consecutive arcs; also arcs $\sigma$ of types (A) and (B) cannot be consecutive, for they must be separated by at least five arcs of type (C).

Now the arcs $\sigma$ (taken in order) may be arranged in groups, each group consisting of one or other of three types:

(1) The first type consists of a number of consecutive arcs of types (A) and (C), each arc of type (C) being consecutive to at least one arc of type (A); the number of arcs in the group should be the largest possible. It therefore consists of arcs of types (A) and (C), those of the former type which are not consecutive being separated by not more than two arcs of type (C); and the group begins and ends with an arc of type (C).

(2) The second type consists of a number of consecutive arcs of types (B) and (C), arranged according to same laws of arrangement as were postulated for types (A) and (C) in (1).

(3) Each group of this type consists of a number of consecutive arcs of type (C) which do not adjoin an arc of type (A) or type (B).

Any group of arcs of types (1), (2), (3) respectively, will be called a curve of type (A′), (B′), (C′) respectively.

Then each arc of type (A′) or type (C′) is an elementary curve of the form $y = f(x)$; and each arc of type (B′) or type (C′) is an elementary curve of the form $x = \phi(y)$.

Take any curve of type (A′), and let the lines of the network (parallel to the $y$-axis) which it crosses be $l_1, l_2, \ldots, l_p$; the points where it crosses $l_1$ and $l_p$ must be on the extreme arcs of type (C), because on these extreme arcs $|\cos\psi(t)| \geqslant \cos(\frac{1}{10}\pi + \frac{1}{20}\pi + \frac{1}{20}\pi)$, and so the difference of the abscissæ of the ends of one of these arcs $\geqslant 4\delta\cos\frac{1}{5}\pi > \delta$; and so also the difference of the abscissæ of the ends of the other of the arcs of type (C) exceeds $\delta$.

Between $l_1$ and $l_2$ the curve cannot cross more than one side of a square parallel to the $x$-axis; for the difference of the ordinates at the points where the curve crosses $l_1$ and $l_2$ is less than $\delta\tan\frac{1}{5}\pi < \delta$. Similar reasoning applies to the sides of squares between $l_2$ and $l_3$, $l_3$ and $l_4$, and so on.

Obliterate all* the sides of squares which are parallel to the axis of $x$, and which meet the curve between $l_1$ and $l_p$.

Carry out this process of obliteration for all the curves (A′), and a similar process of obliteration† of sides of squares parallel to the $y$-axis for all the curves (B′).

We then have the plane divided into a number of squares and rectangles, such that the curve $\Gamma$ meets the perimeter of any square or rectangle either in no points or in two points (but not more).

Now obliterate the whole of the network outside $\Gamma$, and the surviving part of the network gives (i) a number of squares inside $\Gamma$, and (ii) a number of incomplete squares and rectangles of which the curvilinear boundaries make up $\Gamma$; each of these latter curves consists of an arc of $\Gamma$ and part of a rectangle inside $\Gamma$, making up a curve with discontinuously turning tangent of the type considered in §§ 6, 6*a*.

Let $\nu$ be the number of squares, and $\mu$ the number of other regions, so that the number of points where surviving lines of the network meet $\Gamma$ is $\mu$; let $t_1, t_2, \ldots, t_\mu$ be the parameters of these points, in order, and let

$$t_{\mu+1} = t_1 + \omega.$$

Add up the equations of the type

$$\psi(t_2) - \psi(t_1) = 2\pi - (\beta_1 + \alpha_2 + \tfrac{1}{2}r_1\pi),$$

* The actual side of the square only is to be obliterated; not the whole of the line of the network on which it lies.

† This process is not carried out for the curves (C′), since they do not intersect the perimeter of any square in more than two points.

and we get

$$\psi(t_{\mu+1})-\psi(t_1) = 2\mu\pi - \sum_{m=1}^{\mu} (\alpha_m+\beta_m+\tfrac{1}{2}r_m\pi)$$

$$= 2(\mu+\nu)\pi - \left\{ \sum_{m=1}^{\mu} (\alpha_m+\beta_m+\tfrac{1}{2}r_m\pi)+2\nu\pi \right\}.$$

Now $\alpha_m$, $\beta_m$ are positive angles less than $\pi$ whose cosines are equal but opposite in sign; and so $\alpha_m+\beta_m = \pi$.

Also $4\nu+\Sigma r_m$ is the total number of *corners** inside $\Gamma$, while $2\mu$ is the number of corners on $\Gamma$. Consequently

$$\psi(t_1+\omega)-\psi(t_1) = 2N\pi-\tfrac{1}{2}M\pi,$$

where $N$ is the total number of regions into which the interior of $\Gamma$ is divided, and $M$ is the total number of corners inside and on $\Gamma$.

Now some lines of the network terminate on other lines of the network and not on the curve. Produce these lines until they meet the curve but no further.† Then the process of drawing each segment of these lines increases‡ $N$ by 1, and $M$ by 4, since two additional corners at each end of the segment are inserted.

Consequently $2N-\frac{1}{2}M$ is unaffected by drawing these lines. Now obliterate in turn each segment of the lines parallel to the $x$-axis; each obliteration diminishes $N$ by 1 and $M$ by 4, and at the end of these obliterations we are left with $\Gamma$ and a number of lines parallel to the $y$-axis and terminated by $\Gamma$. Obliterate these in turn, and (as before) each obliteration diminishes $N$ by 1 and $M$ by 4, and so does not affect $2N-\frac{1}{2}M$. Therefore

$$2N\pi-\tfrac{1}{2}M\pi = 2N_0\pi-\tfrac{1}{2}M_0\pi,$$

where $N_0$ is the number of regions inside $\Gamma$ and $M_0$ the number of corners when the whole network is obliterated.

But, by Jordan's theorem, $N_0 = 1$, and $M_0$ is obviously zero. Therefore

$$\psi(t_1+\omega)-\psi(t_1) = 2\pi,$$

which is the result stated in Theorem II.

8. We now consider what happens when $\dot{x}$ and $\dot{y}$ both vanish for a finite number of values of $t$ in the range $(t_0, T)$.

* A corner is an angle less than or equal to $\pi$ formed by two intersecting arcs $AP$, $BP$ with tangents at $P$.

† Cf. *Tract*, note at end of p. 11.

‡ This is a slight extension of a theorem due to Ames, *loc. cit.*, p. 356.

Let $\tau$ be such a value of $t$ and let $Q$ be the point with parameter $\tau$. Then, if both $\dot{y}/\dot{x}$ and $\dot{x}/\dot{y}$ are discontinuous at $\tau$, $\psi(t)$ is discontinuous: in this case we shall agree to define $\psi(\tau+0)$ by the convention that $\psi(\tau+0)-\psi(\tau-0)$ is to have its principal value (which is *not* equal to $\pi$).

If either $\dot{y}/\dot{x}$ or $\dot{x}/\dot{y}$ (or both) is continuous, and if neither $\dot{x}$ nor $\dot{y}$ changes sign at $\tau$, $\psi(t)$ may be defined so as to be continuous at $\tau$, and the proof of § 7 is valid.

If either $\dot{y}/\dot{x}$ or $\dot{x}/\dot{y}$ (or both) is continuous, and* $\dot{x}$ or $\dot{y}$ changes sign, $\psi(\tau+0)-\psi(\tau-0)$ must be an odd multiple of $\pi$.

In this case we define $\psi(\tau+0)$ to be $\psi(\tau-0)+\pi$ or $\psi(\tau-0)-\pi$, according as a line drawn through $Q$ parallel to the $y$-axis† has all points on it (which are sufficiently near to $Q$) *outside* or *inside* the curve.

It seems hardly necessary to write out all possible cases; that shewn in Fig. 4, which is one of the most difficult, will be sufficient; $Q$ is a cusp, and $\psi(\tau+0)-\psi(\tau-0) = +\pi$.

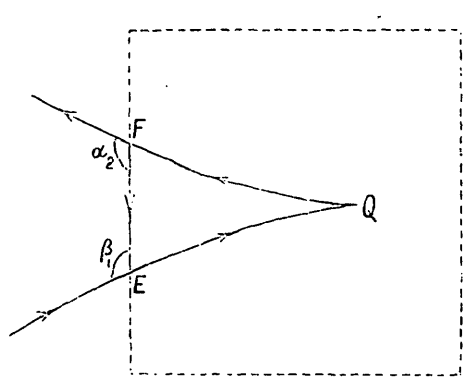


FIG. 4.

Choose the points $\tau$ to be end points of the arcs $\sigma$ of § 7. Draw the network as before, with the additional proviso that none of the points $Q$ are to be on the sides of the network. Obliterate temporarily the interiors of all the squares containing the points $Q$ and let $4\delta_1$ be the lower bound of distances between surviving portions of consecutive arcs $\sigma$ which have portions obliterated. Cut up the squares of the network into smaller squares whose sides are less than $\delta_1$.

* In this case $Q$ is a cusp.

† If $\psi(\tau-0)$ is an odd multiple of $\frac{1}{2}\pi$, we draw it parallel to the $x$-axis.

Consider a square containing one of the points $Q$; treating the part of the curve inside this square after the manner of § 6, we find that there is an integer $n$, such that

$$\begin{aligned} 2n_1\pi &= \psi'(t_1+\omega'+0)-\psi'(t_1+0) \\ &= \alpha_2+\beta_1+\psi(t_2)-\psi(t_1) \\ &= \alpha_2+\beta_1+\{\psi(\tau+0)-\psi(\tau-0)\}+\{\psi(t_2)-\psi(\tau+0)\} \\ &\qquad +\{\psi(\tau-0)-\psi(t_1)\}. \end{aligned}$$

Now $\alpha_2$ and $\beta_1$ lie between $\frac{1}{2}\pi \pm \frac{1}{20}\pi$, and so $2n_1\pi$ lies between

$$2\pi+\tfrac{1}{10}\pi+\tfrac{1}{20}\pi+\tfrac{1}{20}\pi \quad \text{and} \quad 2\pi-\tfrac{1}{10}\pi-\tfrac{1}{20}\pi-\tfrac{1}{20}\pi,$$

and therefore $n_1 = 1$.

Consequently $\psi(t_2)-\psi(t_1) = 2\pi-(\alpha_2+\beta_1)$,

an equation of the same form as those in §§ 6, 6$a$ with $r = 0$.

The portions of the curve inside the smaller squares are now treated after the manner of § 7, and the required result follows.

9. The case in which $\dot{x}$ and $\dot{y}$ have a finite number of finite discontinuities is worth mention; at such a point (with parameter $\tau$), where $\dot{x}$ or $\dot{y}$ is discontinuous, $\psi(t)$ is (in general) discontinuous, and the curve has an abrupt change of direction. To define $\psi(t)$ in such circumstances, we suppose that $\psi(t)$ varies continuously (except at the points $\tau$) as $t$ increases from $t_0$ to $t_0+\omega$, and that the change in $\psi(t)$ as $t$ increases through the value $\tau$ is *numerically less than* $\pi$, so that $\psi(\tau+0)-\psi(\tau-0)$ has its principal value.* By the methods of § 8, it is now easy to shew that, as usual

$$\psi(t_0+\omega)-\psi(t_0) = 2\pi,$$

and it seems unnecessary to write out the details of a proof. For instance, if the cusp in Fig. 4 be replaced by an angle for which the discontinuity in $\psi$ is positive, the arguments of the case worked out in § 8 hold almost *verbatim*.

A verbal statement in the case of a curve with a finite number of discontinuities in the direction of its tangent is as follows.

The sum of the changes in $\psi$ along the continuously turning portions of the curve plus the sum of the exterior angles (taken numerically less than $\pi$) at the discontinuities is equal to $2\pi$.

* These considerations shew that the case under discussion is, in reality, no more general than the case worked out in § 8.

10. In the case when $\ddot{x}$, $\ddot{y}$ are continuous for all values of $t$, the result of Theorem II can be put into a definitely arithmetical form, for we have

$$\psi(t+\omega)-\psi(t) = \int_t^{t+\omega} \frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2}\,dt,$$

and hence the result:

*Let $x(t)$, $y(t)$ be real functions with continuous second differential coefficients, such that they do not assume the same pair of values for different values of $t$ in the range $t_0 < t < T$, such that $\dot{x}$ and $\dot{y}$ do not simultaneously vanish, so that $x$ and $y$ have period $\omega = T-t_0$. Then it is possible to find values of $h$ and $k$, such that*

$$(x-h)^2+(y-k)^2$$

*does not vanish, and such that*

$$\int_t^{t+\omega} \frac{(x-h)\,\dot{y}-(y-k)\,\dot{x}}{(x-h)^2+(y-k)^2}\,dt$$

*is not zero; the value of this integral can then only be $+2\pi$ or $-2\pi$. And*

$$\int_t^{t+\omega} \frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2}\,dt$$

*is equal to $+2\pi$ or $-2\pi$ according as the former integral was equal to $+2\pi$ or $-2\pi$.*

It does not seem to be possible to prove that the value of the last integral is $\pm 2\pi$ by any obvious method more direct than that employed in this paper.