space σY where $Y = \{x_0\} \cup \{x_i|\ i \in N\} \cup \{x_{ij}|\ i, j \in N\}$ is easily seen to be homeomorphic to M.

Note that Ψ^* of example 7.1 contains no subspace homeomorphic to M.

Example 1.11, [6], shows that the product of Fréchet spaces need not be Fréchet. This is also an immediate consequence of 6.2 (see [3], page 7 or [7], footnote (3)). In each of these cases, the product is not even sequential. The next example shows that this need not always be the case.

7.4. EXAMPLE. *The product of two Hausdorff Fréchet spaces can be sequential without being Fréchet.*

Proof. Let $X = R/Z$, the real line with the integers identified, and $I = [0, 1]$ the closed unit interval. I is first countable and hence Fréchet. The quotient map φ: $R \to X$ is pseudo-open and hence by 2.3, [6], X is Fréchet. Since I is compact, by Boehme's Theorem 1, [3], $X \times I$ is sequential. For each $n \in N$ let $A_n = \{(n-1/k, 1/n)|\ k \in N\}$ and let $A = \bigcup \{A_n|\ n \in N\}$. Then $(0, 0) \in \operatorname{cl} A$ but no sequence in A converges to $(0, 0)$. Hence $X \times I$ is not Fréchet.

7.5. EXAMPLE. *The product of two hereditarily quotient (pseudo-open) maps may be a quotient map without being hereditarily quotient (pseudo-open).*

Proof. The natural identifications φ_X: $X^* \to X$ and φ_I: $I^* \to I$ (see 5.2) are pseudo-open by 2.3, [6] but $\varphi_X \times \varphi_I$ is not, since $X^* \times I^*$ is a Fréchet space. However by 5.8 $\varphi_X \times \varphi_I$ is a quotient map.

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On bundles over a sphere with fibre Euclidean space

by

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The origin for this work is a paper of S. P. Novikov [17] on the topological invariance of rational Pontrjagin classes. His paper gives the first method (beyond mere homotopy theory) for proving topological invariance of certain properties. The object of this paper is to consider the special case of (topological) bundles over a sphere with fibre Euclidean space, and to compare the piecewise-linear (hereafter written as PL) and topological classifications. Perhaps the most interesting of the results obtained is that topological equivalence of two such bundles implies (stable) piecewise linear equivalence; however, we go on to extract all the information we can from the method.

I am indebted to Steve Gersten and Larry Siebenmann for pointing out that results from the latter's thesis can be used to fill an apparent gap in the argument of [17]: Novikov's recently published detailed proof [28] appears to use the same reasoning.

Our main result is the following

THEOREM. *The natural homomorphism*

$$j\colon \pi_i(G, \mathrm{PL}) \to \pi_i(G, \mathrm{Top})$$

has a left inverse, for all $i > 0$, except possibly for $i = 2$ or 4. Even in these cases, j is injective.

In the first paragraph we establish our notation. The next is devoted to the lemmas which are needed at the key place in the argument. We then prove the main theorems. A final section is devoted to discussion of special features of low dimensional cases, to which the proofs do not apply without modification.

§ 1. Structure groups and classifying spaces. First it will be convenient to establish our notation and recall some known results.

By O_n we denote the usual *orthogonal group acting on R^n*. PL_n is the *group of piecewise linear homeomorphisms of R^n onto itself*, leaving the origin fixed. It is necessary to define PL_n as a *semi-simplicial group* [14].

Top_n will denote the *group of all homeomorphisms of (R^n, O) onto itself*, with the compact-open topology.

G_n is the *monoid of homotopy self-equivalences* of S^{n-1}, again with the compact-open topology.

We affix a $^+$ if we wish to consider only *orientation-preserving maps*.

The above are selected from several possible definitions which differ only up to homotopy type [12]. In each case we clearly have an associative multiplication with unit, so there exist classifying spaces, defined up to homotopy: cf. [3], [14]. We also have maps, representing natural transformations of bundle functors,

$$\mathrm{B}O_n \to \mathrm{BPL}_n \to \mathrm{BTop}_n \to \mathrm{B}G_n\,.$$

For the first transformation, see [13]; the second is defined by ignoring the PL structure, and the third by deleting the zero cross-section and proceeding to fibre homotopy equivalence. The product structure $\boldsymbol{R}^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}$ induces inclusion homomorphisms $O_n \to O_{n+1}$ and maps $\mathrm{B}O_n \to \mathrm{B}O_{n+1}$, and similarly for PL_n, Top_n, and G_n. We write BO, BPL, BTop and BG for the limit spaces as $n \to \infty$ (which can be defined as 'telescopes' using a sequence of mapping cylinders). We thus obtain spaces and maps, defined up to homotopy,

$$\mathrm{B}O \to \mathrm{BPL} \to \mathrm{BTop} \to \mathrm{B}G;$$

BO, BPL and BG may be taken to be CW complexes.

We next show that these spaces may all be regarded as weakly homotopy associative and weakly homotopy-commutative H-spaces with unit, and the maps as maps of H-spaces. For example, the natural products $\mathrm{PL}_m \times \mathrm{PL}_n \to \mathrm{PL}_{m+n}$ are homomorphisms and are associative, with PL_0 as unit. Up to permutation of coordinates, they are also compatible with the inclusion maps $\mathrm{PL}_m \to \mathrm{PL}_{m+1}$, and commutative. We have induced maps $\mathrm{BPL}_m \times \mathrm{BPL}_n \to \mathrm{BPL}_{m+n}$ with the corresponding properties. Now make the conventions on order of coordinates which are necessary also in the orthogonal case; then our maps are compatible with inclusions, and so induce a product $\mathrm{BPL} \times \mathrm{BPL} \to \mathrm{BPL}$. If K is a finite CW complex, it is easy to check that the induced product on $[K:\mathrm{PL}]$ is associative, commutative, and with unit. Precisely the same arguments work — as well as for BO — for BTop and essentially the same for BG, where we replace the formula $\boldsymbol{R}^m \times \boldsymbol{R}^n = \boldsymbol{R}^{m+n}$ by $S^{m-1} * S^{n-1} = S^{m+n-1}$. Since these products are all induced by the same construction, our maps are maps of H-spaces. Note that all this is equally valid in the oriented case.

The space BO is fairly well known; its homotopy groups were determined by Bott [2], and for its cohomology see e.g. [1]. BG is less familiar: its homotopy groups coincide with stable homotopy of spheres: $\pi_i(\mathrm{B}G) \cong \pi_{i+n-1}(S^n)$ if $n > i$, and the cohomology of BG has only recently begun to be studied (Milnor [15] and Gitler and Stasheff [5]). Less familiar

still, but now accessible, is BPL. Some information on its cohomol is available (Williamson [26]). We also have a fairly complete knowle of its homotopy, which we will discuss in a moment. The main obj of this paper is to study BTop.

The first statement concerning homotopy of BPL is t $\Gamma_n \cong \pi_n(\mathrm{PL}, O) \cong \pi_{n+1}(\mathrm{BPL}, \mathrm{B}O)$ [10], however, this will be of use to us since the calculation of Γ_n involves the simpler isomorphi below. The simplest interpretation of $\pi_n(\mathrm{BPL}) \cong \pi_{n-1}(\mathrm{PL})$ is by sta framings of the trivial bundle over the combinatorial sphere S^{n-1}. the stable tubular neighbourhood theorem of [13], this can be identif with stable framings of the standard imbedding $S^{n-1} \subset S^{n+N-1}$. Now ap the Cairns-Hirsch theorem to smooth these framed imbeddings: we obt stably framed manifolds, combinatorially equivalent to S^{n-1}. A sligh more complicated argument, using the Thom-Pontrjagin construction [2 shows that $\pi_{n+1}(\mathrm{B}G, \mathrm{BPL}) \cong \pi_n(G, \mathrm{PL})$ can be identified with the cob dism group P_n of framed smooth n-manifolds M^n, with boundary co binatorially equivalent to S^{n-1}. Now this group has been computed Kervaire and Milnor, and was to have appeared in the sequel to [1 it was published by Levine [29], also an exposition has been given Haefliger [7]. The result is as follows:

LEMMA 1. $\pi_{n+1}(\mathrm{B}G, \mathrm{BPL}) = P_n$ *is zero if n is odd, is cyclic of orde if* $n \equiv 2\ (\mathrm{mod}\,4)$, *and is infinite cyclic if* $n \equiv 0\ (\mathrm{mod}\,4)$. *If* $n \equiv 2$ (mod *the isomorphism on* $\boldsymbol{Z}/2\boldsymbol{Z}$ *is given by the Kervaire-Arf invariant of if* $n \equiv 0\ (\mathrm{mod}\,4)$, *we use the signature divided by* 8 (*or by* 16 *if* $n = 4$) *give an isomorphism* $P_n \to \boldsymbol{Z}$.

§ 2. Technical preparation.

The main preparation necess is to quote the relevant results from the thesis of L. C. Siebenma (Princeton, 1965). We will do this in our own terminology.

Let W^w be an open manifold — or indeed any locally compa locally path-connected space, and let ε be an end of W (in the sense Freudenthal [4]). We can regard ε as determined by a sequence $P_1 \supset P_2 \supset$ of connected open noncompact subsets of W, with compact frontie and such that any compact subset of W meets only a finite number the P_i. A subset of W is a *neighbourhood* of ε if it contains some P_i. Si the P_i are connected, hence path-connected, we can choose points $x_i \in$ and paths a_i in P_i joining x_i to x_{i+1}, and thus inducing an isomorphi

$$\pi_1(P_i, x_i) \cong \pi_1(P_i, x_{i+1});$$

note that we have an inclusion map of $\pi_1(P_{i+1}, x_{i+1})$ to the latter.

We say that ε is *tame* if

(a) *The sequence*

$$\pi_1(P_1, x_1) \leftarrow \pi_1(P_2, x_2) \leftarrow \pi_1(P_3, x_3) \leftarrow$$

has a subsequence, say

$$G_1 \overset{f_1}{\leftarrow} G_2 \overset{f_2}{\leftarrow} G_3 \overset{f_3}{\leftarrow} \dots$$

such that if $I_i = f_i(G_{i+1})$, f_i *induces an isomorphism* $I_{i+1} \to I_i$, *for each* i.

We write $\pi_1(\varepsilon)$ for the common value. When (a) holds, this is independent of all the choices made.

(b) *Any neighbourhood* U *of* ε *contains a neighbourhood* V *such that there exists a finite* CW *complex* K *and a homotopy commutative diagram*

$$\begin{array}{ccc} V & \subset & U \\ & \searrow \quad \nearrow & \\ & K & \end{array}$$

Our form of condition (b) is slightly more general than Siebenmann's: it is so framed as to permit us to observe that, almost trivially,

Tameness is a property which is invariant under proper homotopy equivalence.

We recall that a map is proper if the preimage of each compact set is compact: the same notion applies to homotopies, so our terms are defined. Siebenmann's arguments are equally valid using our definition (b). We now give his main theorem, (5.7).

Theorem A. *Let* W^w *be an open* PL-*manifold, with a tame end* ε. *Then there exists an obstruction* $\sigma(\varepsilon)$ *in the projective class group* $\widetilde{K}_0(\pi_1(\varepsilon))$. *If* $w \geqslant 6$, $\sigma(\varepsilon)$ *vanishes if and only if* ε *has a neighbourhood* M *which is a* PL-*manifold with boundary, closed in* W, *with compact frontier* ∂M, *and such that the inclusion* $\partial M \subset M$ *is a homotopy equivalence, and inclusion induces an isomorphism of* $\pi_1(\varepsilon)$ *on* $\pi_1(M)$.

Corollary. M *is* PL-*homeomorphic to* $\partial M \times \mathbf{R}$. ($\mathbb{R}_+$?)

This was shown by Siebenmann (loc. cit., (5.2)), using a theorem of Stallings [20]. We now give the special case of this result which we need.

Theorem 1. *Let* V^v *be a closed manifold,* W^{v+1} *a* PL-*manifold,* $v \geqslant 5$, *and* f: $V \times \mathbf{R} \to W$ *a proper homotopy equivalence. Then there is an obstruction* $\sigma(\varepsilon)$ *in the projective class group* $\widetilde{K}_0(\pi_1(V))$; $\sigma(\varepsilon)$ *vanishes if and only if there exist a closed* PL-*manifold* V' *and a* PL-*homeomorphism* h: $V' \times \mathbf{R} \to W$.

Proof. Clearly both ends of $V \times \mathbf{R}$ are tame; by the remark above, the same applies to W. The fundamental group of an end of W is that of the corresponding end of $V \times \mathbf{R}$, which coincides with $\pi_1(V)$. We can now apply Theorem A to define $\sigma(\varepsilon)$ and, if it vanishes, to construct M and $\partial M = V'$. A standard argument using the van Kampen theorem and the Mayer–Vietoris sequence on universal covers now shows that $M' = W - (M - \partial M)$ has $V' = \partial M'$ as deformation retract. Stallings' result then implies that M', too, is PL-homeomorphic to $V' \times \mathbf{R}_+$.

In the application below we will need an extra conclusion to the theorem. This is provided by

Lemma 1. *Suppose* V *and* V' *closed manifolds (or more generally, compact Hausdorff spaces),* G: $V \times \mathbf{R} \to V' \times \mathbf{R}$ *a proper map which does not interchange the ends of* $\mathbf{R}$. *Then there exist a map* g: $V \to V'$ *and a proper homotopy of* G *to* $g \times 1_{\mathbf{R}}$.

Proof. Denote by G_1: $V \times \mathbf{R} \to V'$ and G_2: $V \times \mathbf{R} \to \mathbf{R}$ the component maps of G. Note that since V' is compact, G is proper if and only if G_2 is. We define g by $g(v) = G_1(v, 0)$ for $v \in V$, and the first component of the homotopy by

$$H_t(v, u) = G_1(v, ut) .$$

The second component is provided by

Lemma 2. *Let* V *be a compact Hausdorff space. Then the space* X *of proper maps* $V \times \mathbf{R} \to \mathbf{R}$ *which do not interchange the ends of* $\mathbf{R}$ *is contractible.*

Proof. We define a contraction H: $X \times I \to X$ by

$$H(f, t)(v, u) = (1-t) f(v, u) + ut .$$

Continuity of H follows by standard arguments; that $H(f, t)$ is proper since if $f^{-1}[-n, n] \subset V \times [-m, m]$, with $m \geqslant n$ then $H(f, t)^{-1}[-n, n] \subset V \times [-m, m]$.

We shall also need the following, which is due to A. Grothendieck (see [27]).

Proposition. *If* π *is a free abelian group,* $\widetilde{K}_0(\pi)$ *vanishes.*

This will enable us to get the homeomorphisms we want.

Finally we shall need the relative versions of all these theorems From (10.1) of Siebenmann's thesis we find

Theorem A rel. *Let* W *be a* PL-*manifold with a tame end* ε. *Assum that* ε *defines an end* $\partial\varepsilon$ *of* ∂W *which has a neighbourhood* N *such that* $\partial N \subset N$ *is a homotopy equivalence and* $\pi_1(\partial\varepsilon) = \pi_1(N)$. *Then* $\sigma(\varepsilon)$ *is defined as before If* $w \geqslant 6$, *it vanishes if and only if* ε *has a neighbourhood* M, *wit* $N = M \cap \partial W$ *and the relative boundary* V' *of* M *compact, so that* $V' \subset M$ *is a homotopy equivalence, and* $\pi_1(\varepsilon) = \pi_1(M)$.

We now deduce

Theorem 1 rel. *Let* V^v *be a compact manifold,* W^{v+1} *a* PL-*manifol* $v \geqslant 5$, *and* f: $V \times \mathbf{R} \to W$ *a proper homotopy equivalence with* $f(\partial V \times \mathbf{R}) \subset \partial W$ *Suppose we are given a closed* PL-*manifold* P, *a* PL-*homeomorphis* h: $P \times \mathbf{R} \to \partial W$, *a map* g: $\partial V \to P$, *and a proper homotopy of* $h^{-1} \circ f | \partial V \times \mathbf{R}$ *to* $g \times 1_{\mathbf{R}}$. *Then, if an obstruction in* $\widetilde{K}_0(\pi_1(V))$ *vanishes, there exist a compa* PL-*manifold* V' *with boundary* P, *an extension of* h *to a* PL-*homeomorphis* h: $V' \times \mathbf{R} \to W$, *an extension of* g *to* g: $V \to V'$, *and an extension of th proper homotopy to one of* $h^{-1} \circ f$ *to* $g \times 1_{\mathbf{R}}$.

Proof. Assuming the obstruction to vanish, Theorem A rel provides a PL-submanifold V' of $W \times \boldsymbol{R}$ with boundary $h(P \times 0)$, and Stallings' theorem an extension of h to a PL-homeomorphism. We are given a homotopy of g to the map g' defined by $h^{-1} \circ f$ on $\partial V \times 0$, which extends over V: the homotopy extension theorem gives us an extension of g, and a homotopy of it to g'. Since $V \times 0$ is a deformation retract of $V \times \boldsymbol{R}$, we now obtain (as in Lemma 1) the first component of our proper homotopy. The second is found by using Lemma 2.

§ 3. Novikov's lemma. We now give our extension of Novikov's lemma.

THEOREM 2. *Let M^m be a closed manifold with free abelian fundamental group, $m \geqslant 5$. Let N^{m+q} be a* PL-*manifold, h: $M \times \boldsymbol{R}^q \to N$ a homeomorphism. Then we can construct a* PL-*manifold M_0^m, a* PL-*homeomorphism g_0: $M_0 \times \boldsymbol{R}^q \to N$, a map f_0: $M \to M_0$, and a proper homotopy k_t of $g_0^{-1} \circ h$ to $f_0 \times 1_{\boldsymbol{R}^q}$.*

Suppose also that M bounds the compact manifold V with free abelian fundamental group, that N bounds the PL-*manifold W, and that h extends to a homeomorphism H: $V \times \boldsymbol{R}^q \to W$. Then M_0 bounds a* PL-*manifold V_0, g_0 extends to a* PL-*homeomorphism G_0: $V_0 \times \boldsymbol{R}^q \to W$, f_0 to F_0: $V \to V_0$, and k_t to a proper homotopy K_t of $G_0^{-1} \circ H$ to $F_0 \times 1_{\boldsymbol{R}^q}$.*

Proof. We will write down the details of the first part only, since the second follows essentially the same argument, using Theorem 1 rel in place of Theorem 1. The proof proceeds by induction, essentially on q.

First let T be the boundary of the standard 2-simplex, T^i the product of i copies of T. Choose a PL-embedding $T^{q-1} \times \boldsymbol{R} \subset \boldsymbol{R}^q$. Now the universal covering $\widetilde{T}$ of T is PL-homeomorphic to $\boldsymbol{R}$. We fix such a PL-homeomorphism and choose the embedding so that the composite

$$k:\ \boldsymbol{R}^q \to (T^{q-1} \times \boldsymbol{R})^\sim \to T^{q-1} \times \boldsymbol{R} \to \boldsymbol{R}^q$$

is the identity in a neighbourhood of the origin.

Let $N^{(1)} = h(M \times T^{q-1} \times \boldsymbol{R})$, and let $h^{(1)}$ denote the induced homeomorphism of $M \times T^{q-1} \times \boldsymbol{R}$ on $N^{(1)}$. Let $N^{(i)}$ be the covering space of $N^{(1)}$ induced by the projection

$$N^{(1)} \to M \times T^{q-1} \times \boldsymbol{R} \to T^{q-1} \to T^{i-1}$$

(where T^{i-1} is the product of the last $(i-1)$ factors of T^{q-1}) from the universal cover of T^{i-1}, and denote the homeomorphism lifting $h^{(1)}$ by

$$h^{(i)}:\ M \times T^{q-i} \times \boldsymbol{R}^i \to N^{(i)}.$$

Induction hypothesis. *We have a closed* PL-*manifold M_{q-i}, a* PL-*homeomorphism g_i: $M_{q-i} \times \boldsymbol{R}^i \to N^{(i)}$, a map f_i: $M \times T^{q-i} \to M_{q-i}$, and a proper homotopy of $g_i^{-1} \circ h^{(i)}$ to $f_i \times 1_{\boldsymbol{R}^i}$.*

Induction basis $i = 1$. We apply Theorem 1, taking $h^{(1)}$ for f. By the Proposition the projective class group, hence also the obstruction vanishes. The Theorem then provides M_{q-1} and g_1; Lemma 1 gives us f_1 and the required proper homotopy.

Induction step $i \to i+1$. The covering space $N^{(i+1)}$ of $N^{(i)}$ induces a covering $\widetilde{M}_{q-i}$ of M_{q-i}, a PL-homeomorphism $\widetilde{g}_i$: $\widetilde{M}_{q-i} \times \boldsymbol{R}^i \to N^{(i+1)}$, and a map $\widetilde{f}_i$: $M \times T^{q-i-1} \times \boldsymbol{R} \to \widetilde{M}_{q-i}$, which is a proper homotopy equivalence since f_i is a homotopy equivalence. Also, the proper homotopy lifts to one of $\widetilde{g}_i^{-1} \circ h^{(i+1)}$ to $\widetilde{f}_i \times 1_{\boldsymbol{R}^i}$.

We now apply Theorem 1 and Lemma 1 to

$$\widetilde{f}_i:\ M \times T^{q-i-1} \times \boldsymbol{R} \to \widetilde{M}_{q-i}.$$

We obtain a closed PL-manifold M_{q-i-1}, a PL-homeomorphism e_i: $M_{q-i-1} \times \boldsymbol{R} \to \widetilde{M}_{q-i}$, a map f_{i+1}: $M \times T^{q-i-1} \to M_{q-i-1}$, and a proper homotopy of $e_i^{-1} \circ \widetilde{f}_i$ to $f_{i+1} \times 1_{\boldsymbol{R}}$.

Define $g_{i+1} = \widetilde{g}_i \circ (e_i \times 1_{\boldsymbol{R}^i})$. Multiplying the proper homotopy above by $1_{\boldsymbol{R}^i}$ gives a proper homotopy of $(e_i^{-1} \circ \widetilde{f}_i) \times 1_{\boldsymbol{R}^i}$ to $f_{i+1} \times 1_{\boldsymbol{R}^{i+1}}$. Taking the lift of the proper homotopy of the induction hypothesis, and composing with $e_i^{-1} \times 1_{\boldsymbol{R}^i}$ gives a proper homotopy of

$$(e_i^{-1} \times 1_{\boldsymbol{R}^i}) \circ \widetilde{g}_i^{-1} \circ h^{(i+1)} = g_{i+1}^{-1} \circ h^{(i+1)} \quad \text{to} \quad (e_i^{-1} \circ \widetilde{f}_i) \times 1_{\boldsymbol{R}^i}.$$

The desired proper homotopy is obtained by performing first this, then the proper homotopy above. This completes the induction step.

Conclusion of proof. When $i = q$, the induction gives a closed PL-manifold M_0, a PL-homeomorphism g_q: $M_0 \times \boldsymbol{R}^q \to N^{(q)}$, a map f_q: $M \to M_0$, and a proper homotopy of $g_q^{-1} \circ h^{(q)}$ to $f_q \times 1_{\boldsymbol{R}^q}$. We set $f_0 = f_q$: note that if we had N, h in place of $N^{(q)}$, $h^{(q)}$ we could also set $g_0 = g_q$ and the theorem would be established.

Consider the commutative diagram

$$\begin{array}{ccc} M \times \boldsymbol{R}^q & \xrightarrow{h^{(q)}} & N^{(q)} \\ \downarrow{\scriptstyle 1_M \times k} & & \downarrow{\scriptstyle \lambda} \\ M \times \boldsymbol{R}^q & \xrightarrow{h} & N. \end{array}$$

We will construct a PL-homeomorphism μ: $N^{(q)} \to N$ which agrees with λ on a neighbourhood of $h^{(q)}(M \times 0)$, and a proper homotopy of h to $\mu \circ h^{(q)}$ which is constant near $h^{(q)}(M \times 0)$. Then take $g_0 = \mu \circ g_q$. Compose the proper homotopy with g_0^{-1} to obtain one of $g_0^{-1} \circ h$ to $g_0^{-1} \circ \mu \circ h^{(q)} = g_q^{-1} \circ h^{(q)}$, and follow by the proper homotopy given by the induction. This defines k_t, and establishes the conclusion of the Theorem.

By hypothesis, k agrees with the identity near the origin: choose an open disc E on which it does. E has one end (two if $q = 1$) which is tame and has fundamental group trivial (free cyclic if $q=2$). Corresponding statements are valid for $M\times E$ and for $h^{(q)}(M\times E)$ (which we can identify with $h(M\times E)$). By Theorem A and the Proposition, we can find a closed PL-submanifold W of $h(M\times E)$, a neighbourhood of the end (or disjoint union of neighbourhoods of both, if $q=1$) with compact frontier ∂W, and the inclusion $\partial W\subset W$ a homotopy equivalence with the usual condition on the fundamental group. Now $M\times \mathrm{Cl}E$ is a deformation retract of $M\times \boldsymbol{R}^q$, and ∂E has a product neighbourhood so we can deform slightly inside E too. Combining this with the above we deduce that if $V=h(M\times E)-\mathrm{Int}\,W$, then $N-\mathrm{Int}V$ has $\partial V=\partial W$ as deformation retract. It follows, as usual, by Stallings' theorem that there is a PL-homeomorphism $e\colon \partial V\times \boldsymbol{R}_+\to N-\mathrm{Int}V$. Similarly we have $m\colon \partial V\times\boldsymbol{R}_+\to N^{(q)}-\mathrm{Int}V$. Now define μ equal to the identity on V and to $e\circ m^{-1}$ outside it.

Finally, to construct a proper homotopy of h to $\mu\circ h^{(q)}$ it suffices to construct one of the identity to $h^{-1}\circ\mu\circ h^{(q)}=\nu$, say. In fact we will construct an isotopy. For $0<t\leqslant 1$, $P\in M$, $x\in\boldsymbol{R}^q$ we set

$$\nu_t(P,x)=(Q,y)\qquad\text{where}\qquad \nu(P,tx)=(Q,ty)\,.$$

For any x we have $tx\in E$ for t small enough, and then $\nu_t(P,x)=(P,x)$, so the homotopy remains continuous at $t=0$ if we define ν_0 as the identity. This completes the proof of the theorem.

§ 4. The main theorems.

The following is (except in low dimensions, which will be discussed later) the most precise result we have been able to deduce from the methods of § 3.

THEOREM 3. *For $i\geqslant 6$, the natural homomorphism*

$$j\colon\ \pi_i(G,\mathrm{PL})\to\pi_i(G,\mathrm{Top})$$

has a left inverse.

Remark. If i is odd, $\pi_i(G,\mathrm{PL})$ vanishes and the result is trivial. The cases $i=2,4$ will be dealt with below.

Proof. The first step is the observation that a PL (resp. topological) automorphism of the trivial bundle $S^{i-1}\times\boldsymbol{R}^q\to S^{i-1}$, together with a proper homotopy of it to the identity, represents an element of $\pi_i(G,\mathrm{PL})$ (resp. $\pi_i(G,\mathrm{Top})$); and that conversely, any such element can be so represented, for suitable q. Rather than prove this in detail we prove a result which gives the same argument in somewhat simpler form, viz. the assertion that elements of $\pi_i(G_q)$ are represented by proper homotopy equivalences $S^i\times\boldsymbol{R}^q\to S^i\times\boldsymbol{R}^q$.

In one direction this is clear: G_q is a space of maps $S^{q-1}\to S^{q-1}$, and by taking open cones with vertex the origin, and extending maps conewise, these can be identified with proper maps $\boldsymbol{R}^q\to\boldsymbol{R}^q$. Thus a map $S^i\to G_q$ has as adjoint a proper (fibrewise) map $S^i\times\boldsymbol{R}^q\to S^i\times\boldsymbol{R}^q$, which is a proper

homotopy equivalence as G_q has a homotopy inverse. Conversely, let $F\colon S^i\times\boldsymbol{R}^q\to S^i\times\boldsymbol{R}^q$ be proper. Then if D^q is the unit disc in $\boldsymbol{R}^q$, $F^{-1}(S^i\times D^q)$ is compact, hence contained in $S^i\times\Delta$ for a large enough disc Δ in $\boldsymbol{R}^q$. Now the adjoint of

$$S^i\times\partial\Delta\xrightarrow{F}S^i\times(\boldsymbol{R}^q-\mathrm{Int}\,D^q)\xrightarrow{p_2}\boldsymbol{R}^q-\mathrm{Int}\,D^q\to S^{q-1}$$

(the last map is radial projection) defines a map $S^i\to G_q$, whose homotopy class is clearly not altered by taking Δ larger. It is easy to check that these two constructions induce inverse maps of equivalence classes.

Now suppose given an element of $\pi_i(G,\mathrm{Top})$: represent it by a topological automorphism h of $S^{i-1}\times\boldsymbol{R}^q\xrightarrow{p_1}S^{i-1}$ (for q large enough) with a proper homotopy to the identity. By Theorem 2 with $M=S^{i-1}$, there exist a PL-manifold M_0, a PL-homeomorphism $g_0\colon M_0\times\boldsymbol{R}^q\to S^{i-1}\times\boldsymbol{R}^q$, a map $f_0\colon S^{i-1}\to M_0$, and a proper homotopy k_t of $g_0^{-1}\circ h$ to $f_0\times 1_{\boldsymbol{R}^q}$. Now f_0 is a homotopy equivalence; by Smale's solution of the Poincaré conjecture [18], f_0 is homotopic to a PL-homeomorphism. (Note. This is the only point where the argument breaks down in the C^∞-case.) We may thus replace M_0 by S^{i-1} and f_0 by the identity. So we have a PL-homeomorphism $g_0\colon S^{i-1}\times\boldsymbol{R}^q\to S^{i-1}\times\boldsymbol{R}^q$ and a proper homotopy of $g_0^{-1}\circ h$ to the identity. Unfortunately, g_0 is not fibre-preserving. However, it follows from [9] that if $q>i$, g_0 is PL-isotopic to a fibre-preserving map: the precise deduction goes as follows. The map g_0 determines an element of $\pi_{i-1}(\mathrm{PL}_{q+i-1,i-1})$. By [9], injection gives an isomorphism of $\pi_{i-1}(\mathrm{PL}_q)$ on this group. By the Haefliger–Poenaru theorem [8] it follows that g_0 is PL-regularly homotopic to a bundle automorphism g, with $S^{i-1}\times 0$ fixed. Hence we have an isotopy of a neighbourhood of $S^{i-1}\times 0$; this extends to an isotopy of g_0 to a PL-homeomorphism which agrees with g (hence is fibre preserving) near $S^{i-1}\times 0$. A further PL-isotopy (cf. end of proof of Theorem 2) now takes this map to g. The map g and sequence of proper homotopies determine an element of $\pi_i(G,\mathrm{PL})$.

A slightly more complicated argument which, however, introduces no new idea, applying the second clause of Theorem 2 to the case where $V=S^{i-1}\times I$ and $M=S^{i-1}\times\partial I$, shows that we obtain a well-defined map

$$r\colon\ \pi_i(G,\mathrm{Top})\to\pi_i(G,\mathrm{PL})\,.$$

(For this argument, Smale's theorem is replaced by the result of Gugenheim [6], that homotopic PL-homeomorphisms of S^{i-1} are PL-isotopic.)

In order to prove that $r\circ j=1$, it will suffice to show that if $h\colon S^{i-1}\times\boldsymbol{R}^q\to S^{i-1}\times\boldsymbol{R}^q$ above is a PL-map, then the constructed proper homotopy of $g_0^{-1}\circ h$ to the identity is properly homotopic to a PL-isotopy. But in the application of Theorem 2 we can now choose each f_i equal to the identity, $g_i=h^{(i)}$, and the proper homotopies constant. Then $g_q=h^{(q)}$,

$g_0 = \mu \circ h^{(q)}$, and the proper homotopy of this to h which was constructed was in fact a PL-isotopy.

Finally, we must argue that r is a homomorphism. In order to add two elements of $\pi_i(G, \text{Top})$, we first normalise them: the first element so that the map $S^{i-1} \to \text{Top}$ is trivial on one hemisphere and the proper homotopy constant there, the second similarly with the complementary hemisphere. The sum is then defined by using one map on each hemisphere. We now claim that if in the application of Theorem 2 h is trivial on a hemisphere, we can choose g_0, f_0 and the proper homotopy to be likewise. This follows by using the relative form of Theorem 2. Additivity of r is now immediate. This completes the proof of the Theorem.

COROLLARY 3.1. *Except (perhaps) in low dimensions, we have split short exact sequences*

$$0 \to \pi_i(G, \text{PL}) \to \pi_i(G, \text{Top}) \to \pi_{i-1}(\text{Top}, \text{PL}) \to 0\,,$$

$$0 \to \pi_{i-1}(\text{PL}) \to \pi_{i-1}(\text{Top}) \to \pi_{i-1}(\text{Top}, \text{PL}) \to 0.$$

Theorem 3 implies that in the homotopy sequence of the triple $(G, \text{Top}, \text{PL})$ the maps $\pi_i(\text{Top}, \text{PL}) \to \pi_i(G, \text{PL})$ are zero. It follows that the sequence breaks up into short exact sequences as above; moreover, the theorem provides a splitting of these. Similarly, the composite of the map $\pi_{i-1}(\text{Top}, \text{PL}) \to \pi_i(G, \text{Top})$ which splits the first sequence with the boundary map $\pi_i(G, \text{Top}) \to \pi_{i-1}(\text{Top})$ gives a homomorphism which splits the homotopy sequence of the pair (Top, PL).

We note in particular that the maps $\pi_i(\text{PL}) \to \pi_i(\text{Top})$ are injective, so that topological equivalence of two bundles over S^{i+1} with euclidean fibres implies (stable) PL-equivalence. This result could indeed have been obtained more simply: it needs no reference to proper homotopy. It seems likely that current work on the lines of this paper will soon prove the same result with the base space replaced by an arbitrary finite CW-complex: the stronger result is more useful in striving for such extensions.

It has now (Aug. 1967) been shown independently by D. Sullivan [31] and A. Casson that, for any finite CW-complex X with $H_3(X; \mathbf{Z})$ free of 2-torsion, the mapping of sets of homotopy classes

$$[X: G/\text{PL} \to [X: G/\text{Top}]$$

is injective. Both proofs use Novikov's lemma.

§ 5. Low dimensions. Siebenmann's Theorem A is not known if $w \leqslant 5$. In the case $w = 5$, the place where the proof breaks down is the absence of an embedding theorem for D^3 in W^5, since we cannot even easily embed S^2 in a 4-manifold. We now show how this difficulty can be circumvented in some cases. Observe that the cases in which we are interested are those in which W is properly homotopy equivalent to the product of $\mathbf{R}$, a sphere S^i (or, in the relative case, a disc D^i) and a number of circles.

Before giving the proof in the general case, we first consider the simplest special case.

PROPOSITION. *Let W^5 be a* PL*-manifold, properly homotopy equivalent to $S^4 \times \mathbf{R}$. Then W is* PL*-homeomorphic to $S^4 \times \mathbf{R}$.*

Proof. Siebenmann's argument is strong enough to show in this case that W^5 has a 1-connected closed PL-submanifold M^4 which separates the ends: let V^5 be the closure of one of the parts into which W is divided. The cohomology of V "at infinity" is that of S^4, and by treating V as a cobordism of M^4 to S^4 we show (see Lemma 3 below) that M has zero signature. In fact,

$$K = \text{Ker}\big(H_2(M) \to H_2(V)\big)$$

is isotropic for the quadratic form of intersection numbers on $H_2(M)$, and if $\hat{K}$ is its integral dual, we can write $H_2(M) = K \oplus \hat{K}$. Now by a result of [23], M bounds a manifold V' which is simply-connected, and has $\text{Ker}\big(H_2(M) \to H_2(V')\big) = \hat{K}$. Attaching V' to V along M gives a contractible manifold which is simply-connected at infinity and hence, by a theorem of Stallings [19], PL-homeomorphic to $\mathbf{R}^5$. Now V' is a compact subspace of this, hence contained in a disc, with boundary Σ^4, say. Then $\Sigma^4 \subset V \subset W$, and separates the ends of W; one closed complementary region is PL-homeomorphic to $\Sigma \times \mathbf{R}_+$, and [20] shows that the other is also.

We point out that the Poincaré conjecture in dimension 4 is still unresolved. The above argument has the merit of by-passing this potential difficulty.

For the case $i=4$ of Theorem 3, the best result we can achieve, using the above ideas, is

THEOREM 3_4. *There is a subgroup A of index 1 or 2 in $\pi_4(G, \text{Top})$, such that j factorises as*

$$\pi_4(G, \text{PL}) \xrightarrow{j'} A \subset \pi_4(G, \text{Top})\,,$$

and j' has a left inverse.

The only gap in the proof of Theorem 3 when $i = 4$ was the appeal to Theorem 2 for, in the absolute case S^3; in the relative case, $S^3 \times I$ and D^3. The only gap in the proof of Theorem 2 for these cases was in applying Siebenmann's theorem A. We first consider the cases where the manifold M to be constructed has dimension 4 — thus we would like to be able to construct $S^3 \times S^1$, $S^3 \times I$, and $D^3 \times S^1$. In each case we will seek a manifold PL-homeomorphic (not merely homotopy equivalent) to the desired manifold. The subsequent construction of S^3 and D^3 will then be trivial, and no reference will need to be made to unsolved cases of the Poincaré conjecture. Note in each case that the boundary is given in advance, so we have a problem of relative surgery; also that the only two fundamental groups which we need to consider are $\{1\}$ and $\mathbf{Z}$.

We now consider the case $v = 4$ of Theorem 1. First suppose that V^4 is a Poincare complex (see [25]: a topological manifold would do, but we will anyway need a stronger hypothesis below), W^5 a PL-manifold, and $f: V \times \mathbf{R} \to W$ a proper homotopy equivalence. If g is a homotopy inverse to f, we may suppose $p_2 \circ g: W \to \mathbf{R}$ regular at $0 \in \mathbf{R}$, with preimage $M^4 \subset W$. Also, surgery on M as for Theorem A shows that we may suppose the inclusion map $i: M \to W$ 2-connected. Thus $\varphi = p_1 \circ g \circ i: M \to V$ is 2-connected and of degree 1. Note also that if ν is the stable normal bundle of W, then $i^*\nu$ gives that of M. As $p_1 \circ g$ is a homotopy equivalence, this shows that there exists a bundle α (in fact $(f^*\nu)|V \times 0$) over V, which induces by φ the stable normal bundle of M. We observe that the induced map of Thom spaces

$$M^{i^*\nu} = M^{\varphi^*\alpha} \to V^\alpha$$

shows that V^α is reducible, and hence the spherical fibration corresponding to α is the 'Spivak normal bundle' [25] [30] of the Poincaré complex V. Observe finally that if β is another spherical fibration over V, with reducible Thom space, and the same (large) fibre dimension r as α, then [25], Theorem 3.5, shows that for some map of fibrations $\alpha \to \beta$, which is unique up to fibre homotopy equivalence, the given element of $\pi_{r+5}(V^\alpha)$ goes into the given element of $\pi_{r+5}(V^\beta)$.

THEOREM 4. *Let V^4 be a compact* PL-*manifold*, W^5 *a* PL-*manifold, and $g: W \to V \times \mathbf{R}$ a proper homotopy equivalence inducing a* PL-*homeomorphism of ∂W on $\partial V \times \mathbf{R}$. Suppose g t-regular on $V \times 0$, with $M = g^{-1}(V \times 0)$; define α as above. Assume that there is an isomorphism of α on the stable normal* PL-*bundle of V which is the identity over ∂V and carries the element of $\pi_{n+5}(V^\alpha, \partial V^\alpha)$ to the normal invariant of V as in* [16].

Then if $\pi_1(V) \cong 1$ or $\mathbf{Z}_2$, g is properly homotopic rel ∂V *to a* PL-*homeomorphism. If $\pi_1(V) \cong \mathbf{Z}$, and V is orientable, there is an obstruction in $\mathbf{Z}_2$ to the validity fo this conclusion.*

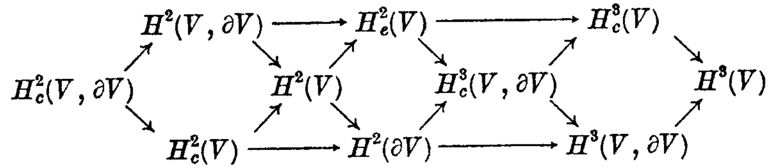
We are not yet in a position to give a result for general $\pi_1(V)$ since the proof depends on non simply-connected surgery. This part of the proof will appear in [24]; the remainder is given below.

Proof of Theorem 3_4. We follow the proof of Theorem 3 up to the point where we wish to construct $S^3 \times I$, $S^3 \times S^1$, or $D^3 \times S^1$; denote this manifold by V^4. We will check below that Theorem 4 applies to the situation; it follows that there is no obstruction to obtaining $S^3 \times I$ (which shows that r, when defined, is well-defined), but we have an obstruction in $\mathbf{Z}_2$ in the other cases. Our uniqueness results show that this depends only on the original element of $\pi_4(G, \mathrm{Top})$. Further, using D^3 instead of S^3 we see as before that we have defined a homomorphism $\pi_4(G, \mathrm{Top}) \to \mathbf{Z}_2$: clearly it vanishes on the image of $\pi_4(G, \mathrm{PL})$. The result thus follows by defining A to be the kernel of this homomorphism.

It remains to check the hypothesis of Theorem 4 in the desired cases. All is clear except for the normal bundle and normal invariant of V. The remarks above show that the obstructions to these being as desired lie in groups $H^i(V, \partial V; \pi_i(G, \mathrm{PL}))$, i.e. $H^2(V, \partial V; \mathbf{Z}_2)$ and $H^4(V, \partial V; \mathbf{Z})$. In our case, the first of these groups vanishes; the second is infinite cyclic. There is thus one obstruction $\in \mathbf{Z}$; it can (cf. [16]) be related to the signature of the manifold M' obtained by glueing M to V by the given PL-homeomorphism of the boundaries. We will now show that under the hypotheses of Theorem 4, $\sigma(M')$ necessarily vanishes. As this was used above also, we give it as a separate lemma.

LEMMA 3. *Let M be a connected finite simplicial complex which is an oriented Poincaré 4-complex, N a* PL *5-manifold properly homotopy equivalent to $M \times \mathbf{R}$, and V a* PL *submanifold of N which is a neighbourhood of one of the ends and has ∂V compact. Then ∂V and M have the same signature*

Proof. Consider the diagram



Here, the suffix c denotes compact cohomology, and two of the sequences are cohomology exact sequences of $(V, \partial V)$ with closed resp. compact supports. The term $H^2_e(V)$ can be defined most conveniently using an (infinite) triangulation of V, and taking the homology groups of the complex of chochains modulo finite cochains: it is then easy to check exactness of the other two sequences. But now $H^2_e(V)$ is invariant under proper homotopy equivalence, and is unaltered by changing a compact subset, so

$$H^2_e(V) \cong H^2_e(M \times \mathbf{R}_+) \cong H^2(M)\,.$$

We now consider V as playing the role of a cobordism of ∂V to M and use the argument of [21] which proves invariance of the signature under ordinary cobordism. Briefly summarised, we consider the Mayer-Vietoris sequence

$$H^2(V) \to H^2_e(V) \oplus H^2(\partial V) \to H^2_c(V, \partial V);$$

observe that the two homomorphisms are dual to each other, and so have the same rank; and then note that cup products vanish on the image of $H^2(V)$. The details offer no difficulty, so the lemma is proved.

Proof of Theorem 4. It follows (cf. [16]) from our hypothesis about the normal invariants that there exist maps f_0, f_1: $(D^{n+5}, \partial D^{n+5}) \to (V^a, \partial V^a)$, homotopic rel ∂D^{n+5}, t-regular on V, with $f_0^{-1}(V) = V$ and $f_1^{-1}(V) = M$. Make the homotopy t-regular on V: then we obtain a cobordism X of V to M which retracts on V, and has $\partial_c X = \partial V \times I$. Now M separates W into two parts, say W_- and W_+. Form Y by glueing X to W_+ along M. Then $\partial Y = V$, and Y retracts on V (the retraction on X was given above; on W_+ it is induced by $p_1 \circ f^{-1}$; these agree on M).

Now perform surgery on Y to make the retraction on V a homotopy equivalence, leaving ∂Y fixed. It will be shown in [24] that under the assumptions of the theorem, there is an integer obstruction to performing surgery, a finite number of times, to obtain a manifold Z with $\partial Z = V$ and $V \subset Z$ a homotopy equivalence. However, we will also see in [24] that we can alter the obstruction by any even integer by choosing a different cobordism X. The conclusion of the proof is now essentially the same as for the proposition above.

First, we will show

LEMMA 4. *Let* V *be a* PL-*manifold with one tame end* ε *and a compact boundary such that the inclusion* $\partial V \subset V$ *is a homotopy equivalence and inclusion induces an isomorphism of* $\pi_1(\varepsilon)$ *on* $\pi_1(V)$. *Then* V *is* PL-*homeomorphic to* $\partial V \times \mathbf{R}_+$.

Proof. We follow the argument of [19], which is in two steps. In step (A), we show that any compact subset C of V is contained in (can be engulfed by) a collar neighbourhood of the boundary. The proof in [19] shows this, provided that C lies in a compact D such that $(V, V-D)$ is 2-connected. But Siebenmann (loc. cit., (3.10)) shows that ε has an arbitrarily small (e.g. not meeting C) "1-neighbourhood" N: using the fact that $\partial N \subset V \to \partial V$ has degree 1, we see easily that we can choose the closure of $V-N$ for D.

To conclude the proof (step (B)), we can now use the results of Stallings in [20].

The rest of the proof of Theorem 4 is immediate.

We last consider the case $i=2$ of Theorem 3. Here there is an argument due to Sullivan [31] which shows that

$$j: \pi_2(G, \mathrm{PL}) \to \pi_1(G, \mathrm{Top})$$

is injective; indeed, the same argument can also be used for $i=4$ to give a much shorter proof of injectivity than the above.

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The hypothesis $2^{\aleph_0} \leqslant \aleph_n$ and ambiguous points of planar functions

by

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Let P be the set of all points in the Euclidean plane provided with a Cartesian coordinate system having a horizontal x-axis and a vertical y-axis. By a line with direction θ we shall mean a straight line in the plane P whose angle of inclination is θ, where $0 \leqslant \theta < \pi$. Suppose that n is a natural number and that $0 \leqslant \theta_1 < \theta_2 < \ldots < \theta_n < \pi$ [1]. We define the relation

$$P = E_1(\theta_1; K_1) \cup E_2(\theta_2; K_2) \cup \ldots \cup E_n(\theta_n; K_n)$$

to mean that P is the union of n sets, $E_1, E_2, \ldots, E_n$, where E_j $(j = 1, 2, \ldots, n)$ intersects every line with direction θ_j in a subset of that line satisfying the condition K_j. In this paper, K_j will take one of the following forms: (i) $< \aleph_\alpha$, (ii) $\leqslant \aleph_\alpha$, (iii) n.d., where $E_j(\theta_j; K_j)$ then means, respectively, that E_j intersects every line with direction θ_j in a set of power less than $\aleph_\alpha$, in a set of power less than or equal to $\aleph_\alpha$, in a linear nowhere dense set of points.

We shall be concerned with the following specific propositions:

(H$_n$) $$2^{\aleph_0} \leqslant \aleph_n ,$$

(Q$_n$) $$P = E_1(\theta_1; < \aleph_0) \cup E_2(\theta_2; < \aleph_0) \cup E_3(\theta_3; < \aleph_0) \cup \ldots \cup E_{n+2}(\theta_{n+2}; < \aleph_0) ,$$

(B$_n$) $$P = E_1(\theta_1; \text{n.d.}) \cup E_2(\theta_2; \leqslant \aleph_0) \cup E_3(\theta_3; \leqslant \aleph_1) \cup \ldots \cup E_{n+2}(\theta_{n+2}; \leqslant \aleph_n) .$$

It is evident that $(Q_n) \Rightarrow (B_n)$. I showed [1] that $(B_1) \Rightarrow (H_1)$, and Davies showed [4] that $(H_1) \Rightarrow (Q_1)$. Subsequently Davies proved [5] that $(H_n) \Rightarrow (Q_n)$ and $(Q_n) \Rightarrow (H_n)$ for every n.

I shall prove that $(B_n) \Rightarrow (H_n)$ for every n, and I shall then apply this result to show that the existence of a function with a certain kind of ambiguous behavior (this term will be defined in the next paragraph) implies (H_n) (whereas the result $(Q_n) \Rightarrow (H_n)$ is insufficient to show this).

Let $\zeta \in P$. By a *segment* Λ at ζ we mean a rectilinear segment extending from a point $\zeta' \in P$, with $\zeta' \neq \zeta$, to the point ζ; Λ is regarded

(1) What is essential here is not that the thetas be in this particular order, but that they be distinct.