Andrew Ranicki

Topology Vol 2, pp. 253–261. Pergamon Press, 1963. Printed in Great Britain

CLASSIFICATION PROBLEMS IN DIFFERENTIAL TOPOLOGY—I

CLASSIFICATION OF HANDLEBODIES

C. T. C. WALL

(*Received* 14 *February* 1963)

IT HAS recently become apparent, from work of Milnor, Smale, Mazur and Novikov among others, that the problem of diffeomorphism classification of manifolds, at least if those manifolds are simply-connected and of dimension at least 5, is capable of a solution which is not merely theoretical, but susceptible of description by fairly easily computed and standard invariants. The obect of this series of papers (about five are planned) is to perform the relevant computations in some comparatively simple cases. Earlier work on these lines can be found in [1], [8] and [11].

The other papers in this series will be:

II: Diffeomorphisms of Handlebodies;
III: Applications to special cases;
Q: Quadratic forms on finite groups and related topics;
IV: Classification of $(s-1)$-connected $(2s+1)$-manifolds.

The titles are fairly self-explanatory. In II we classify diffeomorphisms up to an equivalence relation somewhat stronger than diffeotopy. In III we compute the relevant homotopy groups. We shall also consider the relation of a manifold $N \in \mathscr{H}(2s+1,k,s)$ to a manifold M obtained from ∂N by deleting the interior of a $2s$-disc; we find $M \in \mathscr{H}(2s,2k,s)$, and compare diffeomorphisms of M with those of N. The paper Q will contain a number of preliminary results and notations, and in IV we attain our main objective. As in [11], complete success is obtained only for the problem of classifying almost-closed manifolds.

In this paper we introduce certain diffeomorphism invariants α, λ (which are not by any means new) and examine exhaustively the relations they satisfy in favourable cases (the so-called stable range). It then turns out that the classification of handlebodies in the said cases (due essentially to Haefliger and Smale) can be conveniently expressed by our invariants. Applications of the results are postponed till the third paper of the series.

§1. THE DIFFEOMORPHISM INVARIANTS

We shall use 'manifold' for compact, oriented, C^∞-differential manifold, which may have boundary. Let M^m be an m-manifold. We write $i_s(M)$ for the set of equivalence classes under diffeotopy of imbeddings $f: S^s \to M^m$ (where S^s is the s-sphere). Of course, this is itself a diffeomorphism invariant. If we suppose M simply-connected, then a diffeotopy class of imbeddings determines a homotopy class of maps. We write $\eta_s : i_s(M) \to \pi_s(M)$ for the projection of diffeotopy classes to homotopy classes.

PROPOSITION 1. *If $2m \geqslant 3s+3$, and M^m is $(2s-m+1)$-connected, η_s is onto.*

If $2m \geqslant 3s+4$, and M^m is $(2s-m+2)$-connected, η_s is $(1-1)$.

These results are due to Haefliger [2]. We shall need a mild extension (whose proof we defer for a few pages).

LEMMA 1. *If $2m \geqslant 3s+2$, and M^m is $(2s-m+2)$-connected, two homotopic imbeddings of S^s are regularly homotopic* $(m \geqslant s+3)$.

Thus when $2m \geqslant 3s+4$, M^m $(2s-m+2)$-connected, we can identify $i_s(M)$ with $\pi_s(M)$ and so, in particular, give it a group structure. The range of validity of this result is called the stable range. We observe that in the stable range, the invariants $i_s(M)$ admit homotopy operations. For example, if $\xi \in \pi_r(S^s)$, then composition with ξ gives a map (not in general a homomorphism, unless ξ is a suspension) $\circ\xi : \pi_s(M) \to \pi_r(M)$. Using the identifications of Proposition 1, we deduce a map $\circ\xi : i_s(M) \to i_r(M)$, defined in the stable range.

Now each imbedded (or immersed) sphere has a normal bundle, whose equivalence class is determined by the class of the sphere under diffeotopy (or regular homotopy). We recall the clasification (Steenrod [10]) of bundles over S^s. Let D^s_+, D^s_- denote two hemispheres; then any bundle over S^s has trivial restrictions to these, and so is derived from trivial bundles over D^s_+, D^s_- by identifying along the equator. In particular, for an $(m-s)$-vector bundle over S^s, we have the trivial bundles $D^s_+ \times R^{m-s}$, $D^s_- \times R^{m-s}$ and a characteristic map $\chi : S^{s-1} \to SO_{m-s}$; and a point (P,x) of $S^{s-1} \times R^{m-s} \subset D^s_+ \times R^{m-s}$ is identified with $(P, \chi(P)\cdot x)$ on the lower half. Equivalence classes of bundles are in $(1-1)$ correspondence with homotopy classes of maps χ.

We denote the map which associates to each sphere its normal bundle by $\alpha : i_s(M) \to \pi_{s-1}(SO_{m-s})$. This is our second diffeomorphism invariant. Its sth suspension $S^s\alpha : i_s(M) \to \pi_{s-1}(SO_m)$ associates to each sphere the bundle induced from the tangent bundle of M (we assume $s < m$), which depends only on the homotopy class of the sphere, and in fact defines the homomorphism of $\pi_s(M)$ to $\pi_{s-1}(SO_m) \cong \pi_s\ (B(SO_m))$ induced by a classifying map $M \to B(SO_m)$ for the tangent bundle of M.

Now consider the behaviour of α under the operation $\circ\xi$ defined above in the stable range. In fact for these purposes, the stable range may be extended by a dimension, for by Proposition 1, each homotopy class is represented by imbedded spheres, and as a corollary of Lemma 1, these all have the same normal bundle, so that if $2m \geqslant 3s+3$ and M^m is $(2s-m+2)$-connected, α induces a map from $\pi_s\ (M)$ to $\pi_{s-1}(SO_{m-s})$. Let S^s be a sphere in M, representing $x \in i_s(M)$, then $x \circ \xi$ is represented by an imbedding of S^r homotopic in M to the map defined by ξ of S^r into S^s. Now a tubular neighbourhood of S^s, is certainly $(s-1)$-connected, so by Proposition 1, provided $s-1 \geqslant 2r-m+1$, S^r can be imbedded already in this neighbourhood. If in addition $s-1 \geqslant 2r-m+2$ and $2m \geqslant 3r+3$, the imbedding is unique up to regular homotopy, and so has a well-defined normal bundle, $\alpha(x \circ \xi)$.

LEMMA 2. *Suppose $2m \geqslant 3r + 3$, $m \geqslant 2r - s + 3$, and $r \geqslant s$. Then the above construction determines a map*

$$F : \pi_{s-1}(SO_{m-s}) \times \pi_r(S^s) \to \pi_{r-1}(SO_{m-r}).$$

We pose the problem, to give a homotopy-theoretic interpretation of this map. We present below a number of properties which may shed some light on the problem (in Lemma 5).

We use the Thom construction to define our next invariant. First suppose $x \in i_s(M)$ has $\alpha(x) = 0$, and is represented by a sphere $S^s \subset M$ with normal bundle trivialised. Then it has a tubular neighbourhood $S^s \times D^{m-s}$: project on the factor D^{m-s}, and shrink the boundary to a point (∞), giving a sphere S^{m-s}, and finally extend the map of the tubular neighbourhood to M by mapping the rest of M to ∞. We have defined a map $M \to S^{m-s}$; this induces homomorphisms of homotopy groups $\pi_r(M) \to \pi_r(S^{m-s})$.

The hypothesis $\alpha(x) = 0$ may be dispensed with. For express an imbedded sphere S^s as the union of two hemispheres D^s_+ and D^s_-. A tubular neighbourhood may be derived from $D^s_+ \times D^{m-s}$ and $D^s_- \times D^{m-s}$ by identifying along the equator as above. Now write E for the interior of $D^s_+ \times D^{m-s}$. Then E is an open m-cell and if we remove E from M, $\pi_r(M)$ and $i_r(M)$ are unaltered, for $r \leqslant m - 2$. However, on the remainder of M, the Thom construction may be performed just as before, and now, since D^s_- is contractible, the trivialisation of the bundle is essentially unique. Hence the map is uniquely determined by x. Thus for each $x \in i_s(M)$ and $r \leqslant m - 2$, we have a homomorphism of $\pi_r(M)$ to $\pi_r(S^{m-s})$. This defines maps $\lambda_{sr} : i_s(M) \times \pi_r(M) \to \pi_r(S^{m-s})$, linear in the second variable. It is clear that $\lambda_{st}(x, y \circ \xi) = \lambda_{sr}(x,y) \circ \xi$ for $\xi \in \pi_t(S^r)$, since the map of homotopy groups is induced by a map of spaces. We sometimes use λ_{sr} also for the map of $i_s(M) \times i_r(M)$ defined by first performing η_r on the second variable, and then λ_{sr}.

§2. RELATIONS BETWEEN THE INVARIANTS

We next find all the formal properties of α and λ.

LEMMA 3. *For $x \in i_s(M)$, $\lambda_{ss}(x,x) = S\pi\alpha(x)$.*

Here, π is induced by projection of SO_{m-s} on S^{m-s-1}, and S is the Freudenthal suspension.

Proof. Let $S^s \subset M$ be a sphere representing x, and form a tubular neighbourhood as in the definition of λ. We must find a second sphere representing x, but avoiding E: this we do as follows. Let $*$ be a base point in ∂D^{m-s}. Then, regarding the tubular neighbourhood as a bundle, we choose a cross-section given by $D^s_+ \times *$ over D^s_+. Using the identification, over $P \in S^{s-1} \subset D^s_-$, this gives $(P, \chi(P), *)$. The map $P \to \chi(P) \cdot *$ represents (by definition) the homotopy class $\pi\alpha(x)$. This must be extended to a map of D^s_- to give a map of S^s to S^{m-s} representing $\lambda_{ss}(x,x)$; but extension by hemispheres defines precisely the Freudenthal suspension $S\pi\alpha(x)$.

LEMMA 4. *Suppose M^m $(r+s-m+1)$-parallelisable. Then for $x \in i_r(M)$, $y \in i_s(M)$,*

$$S^r\lambda_{rs}(x,y) = (-1)^{rs}S^s\lambda_{sr}(y,x) \qquad (r,s \leqslant m-2).$$

Proof. Let $f: S^r \to M$, $g: S^s \to M$ represent x,y; w.l.o.g. we suppose these transversal to each other. Then the images meet in a submanifold V of both, of dimension $r+s-m$. Now λ_{rs} is represented by the map of S^s to S^{m-r} induced by trivialising the normal bundle of a hemisphere of S^r in M—or equivalently, by the restriction to V of that trivialisation (now of the normal bundle of V in S^s). Now since $r,s \leqslant m-2$, we can represent $\pi_{r+s}(S^m)$ by maps of $S^r \times S^s$ into S^m. Hence the rth suspension of λ_{rs} is represented by the map of $S^r \times S^s$ into S^m given by the submanifold $1 \times V$, whose normal bundle is trivialised first in S^s, then by adding the trivialisation deduced from a base of the tangent space of S^r at 1. Varying by a homotopy, we may suppose V imbedded diagonally in $S^r \times S^s$ (for we have given imbeddings in either factor), provided the above trivialisation is still used.

Similarly for $S^s\lambda_{sr}(y,x)$, except that reversal of the factors induces a sign $(-1)^{rs}$, and we have a different trivialisation so that if the trivialisations agree (up to homotopy) the result will follow. Now the normal bundle of V in $S^r \times S^s$ is canonically isomorphic to the restriction to V of the tangent bundle of M. For, if $P \in V$ and $w = w_1 + w_2$ is a vector tangent to $S^r \times S^s$ at P, where w_1 is tangent to S^r, w_2 to S^s, we may map w_1 to the corresponding tangent vector to S^r at P in M; similarly w_2, and subtract. This map takes tangent vectors w to $S^r \times S^s$ at $P \times P$ into tangent vectors to M at P; and if the image of w is 0, w_1 is tangent to V in S^r, and w_2 represents the same vector tangent to V in S^s, so $w = w_1 + w_2$ is tangent to V in $S^r \times S^s$. Hence normal vectors are mapped monomorphically.

Now the two trivialisations of the restriction to V of the tangent bundle of M are defined by taking $V \subset D^r(D^s)$, a hemisphere of S^r (or S^s) and trivialising over the contractible space $D^r(D^s)$. Hence they agree if and only if the tangent bundle of M is trivial over $D^r \cup D^s$. But, up to homotopy,

$$D^r \cup_V D^s \simeq \frac{D^r \cup_V D^s}{D^s} = \frac{D^r}{V} \simeq D^r \cup CV \simeq \frac{D^r \cup CV}{D^r} = SV,$$

so $D^r \cup D^s$ has the homotopy type of the suspension of V, which is $(r+s-m+1)$-dimensional, and the result now follows from the hypothesis that M is $(r+s-m+1)$-parallelisable.

COROLLARY. *If $2m \geqslant 3s+3$ and M^m is $(2s-m+2)$-connected, λ_{ss} induces a $(-1)^s$—symmetric bilinear map*

$$\lambda: \pi_s(M) \times \pi_s(M) \to \pi_s(S^{m-s}).$$

Proof. Under the hypothesis, $\pi_s(S^{m-s})$ is a stable group, and so s-fold suspension induces an isomorphism of it. Thus if $2m \geqslant 3s+4$ and M is $(2s-m+2)$-connected, we can identify $i_s(M)$ and $\pi_s(M)$; the symmetry of λ follows from that (proved above) of its s-fold suspension, and λ is linear in the first variable since it is symmetric, and linear in the second.

If $2m = 3s+3$ we have only to observe in addition that if $\eta_s(x) = \eta_s(x^1)$, then $\lambda_{ss}(x,y) = \lambda_{ss}(x^1,y)$ for any y: this follows again from the symmetry and the fact that λ depends only on the homotopy class of the second argument.

We think of λ as a generalised intersection number: for $m = 2s$ it is precisely the usual intersection number, as we see at once.

THEOREM 1. *Let $2m \geqslant 3s + 3$, $s \geqslant 2$, and suppose M^m $(2s - m + 2)$-connected. Then we have maps $\alpha: \pi_s(M) \to \pi_{s-1}(SO_{m-s})$; $\lambda: \pi_s(M) \times \pi_s(M) \to \pi_s(S^{m-s})$ such that λ is bilinear, $\lambda(y,x) = (-1)^s\lambda(x,y)$, $\lambda(x,x) = S\pi\alpha(x)$ and $\alpha(x + y) = \alpha(x) + \alpha(y) + \partial\lambda(x,y)$.*

Here ∂ is the boundary in the homotopy exact sequence of the fibering $SO_{m-s} \to SO_{m-s+1} \to S^{m-s}$. We have already established all the results except for the addition formula for α. The idea of the proof is as follows. We represent x, y by spheres S_1^s, S_2^s transverse to each other. These are joined by a small tube obtained by thickening an arc which joins S_1 to S_2, but is disjoint from them except at the ends. We obtain an immersed sphere representing $x + y$ and with normal bundle $\alpha(x) + \alpha(y)$. We must modify this to be an imbedding, and see how this changes the normal bundle.

HILFSSATZ. *Let $f: S^s \to M^m$ be an immersion which crosses itself in general position along a submanifold V^{2s-m}. Then there is a disc D^m in M^m which meets S^s in a disc containing V^{2s-m}.*

Proof of Hilfssatz. We use results of combinatorial topology. Observe that the conditions $2m \geqslant 3s + 3$, $s \geqslant 2$ imply $m > s + 2$. Moreover S^s is $(2s - m)$-connected and M^m $(2s - m + 1)$-connected. The result—in the combinatorial sense—now follows from a lemma of Zeeman [12]. By a result of Hirsch [5], a small deformation will suffice to make the discs differentiably imbedded.

Proof of Theorem. The desired modification is now simple: we remove the part of $f(S^s)$ within D^m, and replace by an imbedded s-disc spanning $D^m \cap f(S^s)$—that this is possible follows from Proposition 1, in a slightly generalised form [3]. Alternatively, we may describe the old sphere as obtained from the new by taking the connected sum with an immersion of S^s in S^m—and the change in normal bundle will be just the normal bundle of this immersion.

Now the immersion has the property that S^s can be divided into two hemispheres (which correspond to the original spheres S_1 and S_2) such that each is imbedded, and their intersection invariant is $\lambda(x,y)$. The imbedding of one hemisphere may be regarded as standard, and we have to describe the second. In fact, we suppose a neighbourhood of D_+^s to be imbedded flat; then we can ignore a neighbourhood $S^{s-1} \times D^{m-s+1}$ of its boundary, and concentrate on the complementary $D^s \times S^{m-s}$. In this, D_+^s is mapped by

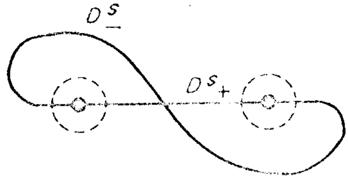

FIG. 1

$D^s \times 1$, and near the boundary, D^s_- by $D^s \times -1$. Moreover, the homotopy class (relative to the boundary) of D^s_- is just $\lambda(x,y) \in \pi_s(S^{m-s})$. Now using Proposition 1 or Lemma 1, we may replace by any homotopic imbedding: we choose a cross-section of the projection of $D^s \times S^{m-s}$ on D^s. Then tangent vectors project isomorphically into D^s; hence normal vectors project isomorphically into S^{m-s}. We trivialise the normal bundle of D^s_- by lifting the map $D^s_- \to S^{m-s}$ (which represents α) to a map $D^s_- \to SO_{m-s+1}$, and the characteristic class of the normal bundle of S^s is now obtained by looking at the restriction of this to the boundary $S^{s-1} \to SO_{m-s}$. But this is just the process used to define the boundary operator in the homotopy exact sequence, which proves our result.

We remark that most of the above considerations for spheres can be parallelled for discs—at least if α is defined relative to a given trivialisation on the boundary. We shall have use for such extensions in the sequel.

We can now give the proof of Lemma 1 (we do *not* use the result of Theorem 1, only the method).

Proof of Lemma 1. Suppose given two homotopic imbeddings of S^s in M^m, and a homotopy, i.e. a map of $S^s \times I$ into $M^m \times I$. The singularity locus of this map (supposed in "general position") has dimension $2(s+1)-(m+1) = 2s-m+1$ (since $2(m+1) > 3(s+1)$—see Haefliger [2]). Again apply Zeeman's lemma to enclose this in a disc. Now on the boundary of this disc we have an imbedding of S^s in S^m; by a result of Kervaire [6], for $2m \geqslant 3s+2$ this is regularly homotopic to the standard imbedding, i.e. spans an immersed disc D^{s+1} in D^{m+1}. We have not actually obtained a regular homotopy, but the existence of an immersed $S^s \times I$ shows (using Hirsch's obstruction theory [4]) that the given spheres are in fact regularly homotopic.

It has been pointed out to me by the referee that the combinatorial argument in the above Hilfssatz can be by-passed by using recent (in part unpublished) imbedding theorems of Haefliger. For Theorem 1, we choose an imbedding of S^s in S^m with each hemisphere imbedded, with intersection number $-\lambda(x,y)$ (this is constructed as above), and take the connected sum with (M^m, S^s). In the new pair (M^m, S^s), the two hemispheres are imbedded, with zero intersection invariant; now it can be deduced from the main theorem of [3] that such an immersion is regularly homotopic to an imbedding. The result now follows as above, by calculating the normal bundle of (S^m, S^s). Lemma 1 follows from Theorem (4.2 (b)) of Haefliger and Hirsch [Immersions in the stable range, *Ann. Math., Princeton* 75 (1962), 231–241] and from the version of Prop. (2.2(b)) of Haefliger's Bourbaki Seminar (December 1962) for homotopy.

We now apply Theorem 1 to deduce some properties of the map F of Lemma 2.

LEMMA 5. *The map* $F: \pi_{s-1}(SO_{m-s}) \times \pi_r(S^s) \to \pi_{r-1}(SO_{m-r})$, *defined for* $2m \geqslant 3r+3$, $m \geqslant 2r-s+3$, *and* $r \geqslant s$, *satisfies*

(1) *F is linear is the first variable, if the second is a suspension.*

(2) *If the first variable is a suspension, it is linear in the second.*

(3) $F(S\alpha.\xi) = SF(\alpha,\xi)$ *when the right hand side is defined.*

(4) *$\pi F(\alpha,\xi)$ is the product of $\pi\alpha,\xi$ and ξ in the stable homotopy ring.*

(5) $F(\alpha, \xi_1 + \xi_2) = F(\alpha, \xi_1) + F(\alpha, \xi_2) + \partial(\pi\alpha . \xi_1 . \xi_2)$

(6) $F(\alpha, k) = k\alpha + \binom{k}{2} \partial S\pi(\alpha)$ *where* $k \in \pi_s(S^s) \cong \mathbf{Z}$.

Proof. For (1), take D^{m-s}-bundles over S^s representing α_1, α_2, and form the sum of the corresponding manifolds. Take imbeddings of S^r in the two parts, defining F and join by a tube in the sum. The homotopy class of the result is $i_1 \circ \xi + i_2 \circ \xi$, which equals $(i_1 + i_2) \circ \xi$ when ξ is a suspension (i_1, i_2 denote the central cross-sections of the two bundles), and the result follows.

For (5) we apply Theorem 1: we have to calculate the intersection invariant of two spheres S_1^r and S_2^r representing ξ_1 and ξ_2. We shall use the fact that our homotopy group is stable to ignore suspensions; then using Lemma 4 and the last remark in §1 we have

$$\lambda(S_1^r, S_2^r) = \lambda(S_1^r, S^s \circ \xi_2) = \lambda(S_1^r, S^s) \circ \xi_2 = (-1)^{rs}\lambda(S^s, S_1^r) \circ \xi_2$$
$$= (-1)^{rs}\lambda(S^s, S^s \circ \xi_1) \circ \xi_2 = (-1)^{rs}\lambda(S^s, S^s) \circ \xi_1 \circ \xi_2 = \pi\alpha \circ \xi_1 \circ \xi_2$$

(the sign may be ignored, as if s is odd, $\pi\alpha$ must have order 2). Now (5) follows from the Theorem; (2) is an immediate corollary, and (4) follows also from the Theorem and the above calculation (with $\xi_1 = \xi_2$). We deduce (6) by induction.

Finally (3) is trivial, since if the containing manifold is multiplied by I, the normal bundle of S^r is suspended once.

It must be confessed that these results are complicated and incomplete; they will, however, suffice for our purposes.

§3. CLASSIFICATION OF HANDLEBODIES

According to Smale [7], a handlebody $M \in \mathscr{H}(m, k, s)$ is a manifold which can be obtained by gluing k s-handles to a disc. More precisely, there is an imbedding

$$f: \bigcup_{i=1}^k (\partial D_i^s \times D_i^{m-s}) \to \partial D^m$$

and M is formed from $D^m \cup \bigcup_{i=1}^k (D_i^s \times D_i^{m-s})$ by identifying corresponding points under f and rounding the corners. The map f is called a *presentation* of M.

We seek to classify handlebodies M up to diffeomorphism. We shall classify presentations up to diffeotopy; to deduce a classification of manifolds from this, we must know how many presentations (in some sense) M has. Clearly, M has the homotopy type of a bouquet of k s-spheres. The reduced homology groups all vanish except the sth, which is free abelian of rank k. Write $H = H_s(M) \cong \pi_s(M)$ (by the Hurewicz theorem). Then a presentation determines a basis $\{e_i\}$ of H consisting of the images in H of

$$(D_i^s, \partial D_i^s) \to (D_i^s \times 0, \partial D_i^s \times 0) \to (M, D^m) \leftarrow (M, \phi).$$

PROPOSITION 2. *M has a presentation corresponding to any basis of H.*

This result is due to Smale [9].

Next we will classify imbeddings f. Let $\bar{f}$ be the restriction of f to $\bigcup_{i=1}^k \partial D_i^s \times 0$. Thus $\bar{f}$ is an imbedding of a disjoint union of k $(s-1)$-spheres in S^{m-1}, a *link* in the sense of Haefliger [3A]. Now (if $m \geqslant s+3$) the complement of the ith S^{s-1} has the homotopy type of S^{m-s-1}, hence the map of the jth determines an element $\lambda_{ij} \in \pi_{s-1}(S^{m-s-1})$. Moreover, as is easily seen, $\lambda_{ji} = (-1)^s \lambda_{ij}$. The λ_{ij} are linking invariants, or (if $m = 2s$) linking numbers in the classical sense, and have been used by Kervaire.

PROPOSITION 3. *If $2m \geqslant 3s+3$, $s \geqslant 2$, diffeotopy classes of imbeddings $\bar{f}$ are in* (1 – 1) *correspondence with sets of linking invariants $\lambda_{ij} \in \pi_{s-1}(S^{m-s-1})$ $(i<j)$.*

The proof is given in [3A].

Now, given $\bar{f}$, to obtain f we merely have to extend imbeddings from spheres to tubular neighbourhoods. In fact, the normal bundle of each S^{s-1} admits a canonical trivialisation, since by Proposition 1 (and Lemma 1), S^{s-1} spans an imbedded disc D^s in D^m, unique up to diffeotopy (regular homotopy), and we trivialise the normal bundle of that. By the tubular neighbourhood theorem, given an imbedded S^{s-1} in S^{m-1} with trivial normal bundle, diffeotopy classes of extensions to an imbedding of $S^{s-1} \times D^{m-s}$ are in (1 – 1) correspondence with trivialisations of the normal bundle, i.e., with elements of $\pi_{s-1}(SO_{m-s})$. This proves

LEMMA 6. *Diffeotopy classes of imbeddings f are in* (1 – 1) *correspondence with sets of invariants*

$$\lambda_{ij} \in \pi_{s-1}(S^{m-s-1}) \ (1 \leqslant i < j \leqslant k), \quad \alpha_i \in \pi_{s-1}(S0_{m-s}) \ (1 \leqslant i \leqslant k).$$

(The hypotheses of the preceding Proposition are preserved here, as below).

Now we take an f with the given invariants and form a manifold M. Let $\{e_i\}$ be the basis of $\pi_s(M)$ defined above.

LEMMA 7. $\lambda(e_i, e_j) = S\lambda_{ij}$, $\alpha(e_i) = \alpha_i$.

Proof. Observe that since M is $(s-1)$-connected, and $(s-1) \geqslant (2s-m+2)$ follows from our hypotheses, we are indeed in a situation where Theorem 1 is applicable and λ, α are defined on the homotopy group.

Represent e_i by the sphere S_i with $D_i^s \times 0$ as one hemisphere and a disc D_i' in D^m as the other. Then S_i, S_j meet only in D^m. To compute their intersection, we deform D_i' to lie on ∂D^m, which is possible by Proposition 1. Thus it only meets S_j on $\partial D_j^s \times 0$. If we now perform the Thom construction, ∂D^m is mapped to S^{m-s-1}, and D^m to one complementary hemisphere. The induced map on $\partial D_j^s \times 0$ is just that used to define the linking invariant λ_{ij}; it has to be extended to map the hemispheres of S_j into those of S^{m-s}, and thus yields the suspension $S\lambda_{ij}$.

To compute $\alpha(e_i)$ we again use the hemispheres D_i', $D_i \times 0$, and trivialise the normal bundles; that of $D_i \times 0$ is indeed already trivialised. The fitting together on the boundary is given by f, and we defined α_i above as the element of $\pi_{s-1}(SO_{m-s})$ corresponding to the identification.

Since λ_{ij} belongs to a stable group, so is determined by its suspension, it follows that α and λ suffice for the diffeomorphism classification of f. Thus their values on the generators e_i are independent, and determine the presentation.

By Theorem 1, these values determine α and λ uniquely on the whole of H. Thus there is a $(1-1)$ correspondence between handlebodies with a presentation, and structures (H, α, λ) with a chosen basis for H. Using Proposition 2, we now deduce

THEOREM 2. *If $s \geqslant 2, 2m \geqslant 3s+3$, diffeomorphism classes of handlebodies $M \in \mathscr{H}(m, k, s)$ are in $(1-1)$ correspondence with isomorphism classes of structures of the following type on free abelian groups H of rank k: maps $\alpha : H \to \pi_{s-1}(SO_{m-s})$, $\lambda : H \times H \to \pi_s(S^{m-s})$ satisfying the conditions of Theorem 1.*

The case $m = 2s$ of this theorem was proved and exploited in our paper [11].

Now let $(H_1, \alpha_1, \lambda_1)$ and $(H_2, \alpha_2, \lambda_2)$ be the systems of invariants for handlebodies M_1 and M_2. Define H as the direct sum $H_1 \oplus H_2$, and $\alpha : H \to \pi_{s-1}(SO_{m-s})$, $\lambda : H \times H \to \pi_s(S^{m-s})$ by components:

$$\alpha(x_1, x_2) = \alpha_1(x_1) + \alpha_2(x_2),$$
$$\lambda((x_1, x_2), (y_1, y_2)) = \lambda_1(x_1, y_1) + \lambda_2(x_2, y_2).$$

It is immediate that (H, α, λ) satisfies the conditions of Theorem 1. We remind the reader that the sum of two bounded manifolds is defined by identifying discs imbedded in the boundaries of each, and rounding corners; this is well-defined if the boundaries are connected and oriented and the discs have opposite orientations.

COROLLARY. *The invariants of $M_1 + M_2$ are (H, α, λ).*

Clearly $H_s(M_1 + M_2) = H_s(M_1) \oplus H_s(M_2)$. The normal bundle of a sphere in M_1, and the intersection of two such spheres, are clearly unaltered by regarding them as spheres in $M_1 + M_2$. The result follows.

REFERENCES

1. J. EELLS and N. KUIPER: Manifolds which are like projective planes, *Publ. Math. Inst. Hautes Etudes Sci.*, No. 14.
2. A. HAEFLIGER: Plongements différentiables de variétés dans variétés, *Comment Math. Helvet.* 36 (1961), 47–82.
3. A. HAEFLIGER: Plongements différentiables dans le domaine stable, *Comment Math. Helvet.* 37 (1962), 155–176.

3A. A. HAEFLIGER: Differentiable links, *Topology* 1 (1962), 241–244.

4. M. W. HIRSCH: Immersions of manifolds, *Trans. Amer. Math. Soc.* 93 (1959), 242–276.
5. M. W. HIRSCH: On combinatorial submanifolds of differentiable manifolds. *Comment Math. Helvet.* 36 (1961), 103–111.
6. M. KERVAIRE: Sur l'invariant de Smale d'un plongement, *Comment Math. Helvet.* 34 (1960), 127–139.
7. S. SMALE: Generalized Poincaré's Conjecture in dimensions greater than four, *Ann. Math., Princeton* 74 (1961), 391–406.
8. S. SMALE: On the Structure of 5-manifolds, *Ann. Math., Princeton* 75 (1962), 38–46.
9. S. SMALE: On the structure of manifolds, *Amer. J. Math.* 84 (1962), 387–399.
10. N. E. STEENROD: *The Topology of Fibre Bundles*, Princeton University Press, 1951.
11. C. T. C. WALL: Classification of $(n-1)$-connected $2n$-manifolds, *Ann. Math., Princeton* 75 (1962), 163–189.
12. E. C. ZEEMAN: The Poincaré conjecture for $n \geqslant 5$, *Topology of 3-Manifolds and Related Topics*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

Trinity College,
Cambridge.