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CLASSIFICATION PROBLEMS IN DIFFERENTIAL TOPOLOGY—II

DIFFEOMORPHISMS OF HANDLEBODIES

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THE OBJECT of this paper is to apply the methods and results of paper I to the next harder prolem—*viz.*, classifying up to diffeotopy diffeomorphisms of handlebodies. (Of course, other classifications, particularly up to conjugacy, are of interest, but this seems to belong to another realm of ideas). It turns out, however, that the problem as proposed is not quite practicable, and we have to work with a stronger equivalence relation. We obtain essentially complete results for this simplified problem, which are best stated as an exact sequence. Our numeration follows on from that of paper I.

§4. QUASI-DIFFEOTOPY

Recall that two diffeomorphisms h_0, h_1 of a manifold M are said to be diffeotopic if there is a diffeomorphism H of $M \times I$ of the form $H(P, t) = (h_t(P), t)$ for $P \in M$, $t \in I$.

DEFINITION. *A quasi-diffeotopy Q of h_0 to h_1 is a diffeomorphism of $M \times I$ such that $Q(P, O) = (h_0(P), O)$ and $Q(P, 1) = (h_1(P), 1)$ for $P \in M$.*

This concept is not essentially new (see, e.g., Brown and Gluck [15]; the term 'weak diffeotopy' would however be unacceptable, since it has quite a different meaning already).

We write Diff (M) for the group of orientation-preserving diffeomorphisms of M; this is to be thought of as a topological group with the C^∞-topology, but we shall not need this. Thus $\pi_0(\text{Diff}\,(M))$ is the group of diffeotopy classes of diffeomorphisms. We shall write $\tilde{\pi}_0(\text{Diff}\,(M))$ for the set of quasi-diffeotopy classes.

Remark. One can also define $\tilde{\pi}_r(\text{Diff}\,(M))$ as the group of diffeomorphisms of $S^r \times M$ (say, keeping the axes fixed) modulo those which extend to a diffeomorphism of $D^{r+1} \times M$. It is then possible to obtain a classification similar to that below, in a similarly defined stable range. We refrain from this since it is not clear that $\tilde{\pi}_r(\text{Diff}\,(M))$ has any direct relation to $\pi_r(\text{Diff}\,(M))$; however, it will be useful to bear these considerations in mind.

We also use relative groups; Diff (M, V) and Diff $(M$ rel $V)$ will be the subgroups of Diff (M) which keep V invariant, resp. pointwise fixed; and [illegible]

$\tilde{\pi}_0(\text{Diff}(M \text{ rel } V))$ their quotients by the corresponding equivalence relations that Q (as above) must keep $V \times I$ invariant resp. pointwise fixed.

We first list the trivial properties of the relation.

LEMMA 8. *Let h_0, h_1, h_0', h_1', be diffeomorphisms of M.*

(i) *If h_0, h_1 are diffeotopic, they are quasi-diffeotopic.*

(ii) *If h_0, h_1 are quasi-diffeotopic, so are their restrictions to ∂M.*

(iii) *If h_0, h_1 and h_0', h_1' are quasi-diffeotopic, so are $h_0 \circ h_0'$, $h_1 \circ h_1'$.*

(iv) *Let h be a diffeomorphism of $M \times I$ agreeing with 1 on $M \times 1$. Then h is quasi-diffeotopic to* 1.

Proof. (i) Any diffeotopy is, by definition, a quasi-diffeotopy.

(ii) If Q is a quasi-diffeotopy from h_0 to h_1, we may take $Q|\partial M \times I$.

(iii) Let Q, Q' be quasi-diffeotopies; then we take $Q_\circ Q'$.

(iv) is less trivial, and is indeed the property which distinguishes quasi-diffeotopy from diffeotopy. Note that first by the usual 'normalisation' process—a transformation of I—we replace h by a diffeomorphism 'constant' near $M \times 0$ and near $M \times 1$—i.e. having there the form $h(P, u) = (k(P), u)$. This only alters h by a diffeotopy, hence by (i) by a quasi-diffeotopy; and now we may suppose that h agrees with 1 near $M \times 1$.

Write $h(P, r) = (P', \lambda r)$, then define

$$\begin{aligned} Q(P, u; t) &= (P', \lambda u; \lambda t) \qquad & r^2 = t^2 + u^2 \leqslant 1 \\ &= (P, u; t) \qquad & r^2 = t^2 + u^2 \geqslant 1. \end{aligned}$$

Then $Q(P, u; 0) = (h(P, u); 0)$ and $Q(P, u; 1) = (P, u; 1)$. Moreover our conditions on h ensure that Q is indeed a diffeomorphism.

Note that (iii) shows that $\tilde{\pi}_0(\text{Diff}(M))$ is in fact a group. We shall now apply (iv); the pattern of the application is as follows. For any manifold M', we know that the boundary admits a product neighbourhood. It follows that if a 'collar' $\partial M' \times I$ is attached along $\partial M' \times 0$, the resulting manifold M is diffeomorphic to M'.

LEMMA 9. (i) *If h_0, h_1 are quasi-diffeotopic diffeomorphisms of ∂M, and h_0 extends to a diffeomorphism of M, then so does h_1;*

(ii) *If h_0, h_1 are diffeomorphisms of M whose restrictions to M' are quasi-diffeotopic, then so are h_0, h_1;*

(iii) $\tilde{\pi}_0(\text{Diff } D^n) = 1$;

(iv) $\tilde{\pi}_0(\text{Diff } S^{n-1}) = \text{Diff } S^{n-1}/i^*(\text{Diff }(D^n))$.

Proof. (i) We know M' is diffeomorphic to M; now let H_0 be the diffeomorphism of M' extending h_0 on $\partial M'$, and Q (on $\partial M \times I$) the quasi-diffeotopy of h_0 to h_1; these fit together on the common boundary $\partial M'$, and we may smooth off there (e.g. use product neighbourhoods and let H_0, Q be 'constant' near $\partial M'$).

(ii) If Q' is a quasi-diffeotopy on M', extend to some diffeomorphism Q_0 of $M \times I$ (e.g. constant on $\partial M' \times I$); this gives a quasi-diffeotopy of h_0 to a map agreeing on M' with h_1; by Lemma 8 (iii) we can assume h_0, h_1 agree on M'. They induce diffeomorphisms of $\partial M' \times I$ agreeing on $\partial M' \times 0$; by Lemma 8 (iv) these are quasi-diffeotopic, and the quasi-diffeotopy is the identity on $\partial M'$, so extends by the identity to M'.

(iii) Let $f: D^n \to D^n$ imbed a concentric disc (e.g. multiplication by $\frac{1}{2}$). If h is a diffeomorphism of D^n, then by the Disc Theorem f and $h \circ f$ are weakly, hence strongly diffeotopic; i.e. h is diffeotopic to a map g which keeps $f(D^n)$ fixed. But g is quasi-diffeotopic to the identity by (ii). Hence, by Lemma 8 (i) and (iii), so is h.

(iv) By (ii) and Lemma 8 (ii), any diffeomorphism of S^{n-1} which extends to one of D^n is quasi-diffeotopic to the identity. The converse follows from (i). The result in general follows (by Lemma 8 (iii)).

The group on the right hand side of equation (iv) was defined by Thom [14] as Γ_n, and is well studied (e.g. it is known that Γ_n is zero for $n \leqslant 6$ and finite for all n). We shall now generalize one of Thom's results. We first need

HILFSSATZ. *Let M^m be simply connected, D^m be a disc in the interior of M^m. Then* $\tilde{\pi}_0(\text{Diff}\,(M^m, D^m)) \xrightarrow{\sim} \tilde{\pi}_0(\text{Diff}\,(M^m))$. *Moreover, if M is a sphere,*

$$\tilde{\pi}_0(\text{Diff}\,(D^m \text{ rel } S^{m-1})) \xrightarrow{\sim} \tilde{\pi}_0(\text{Diff}\,(S^m \text{ rel } D^m)) \xrightarrow{\sim} \tilde{\pi}_0(\text{Diff}\,(S^m, D^m)).$$

Proof. There is certainly a natural map between the first two groups. Since, by the disc theorem, any diffeomorphism of M is diffeotopic to one keeping D^m fixed, it is onto, indeed, so is the natural map of $\tilde{\pi}_0(\text{Diff}\,(M^m \text{ rel } D^m)$, as we need for the second part. Now let $h_0, h_1 \in \text{Diff}\,(M^m, D^m)$ be quasi-diffeotopic in M, with quasi-diffeotopy Q. Then for $m \geqslant 2$ (the result is trivial for $m = 1$), $Q\,(0 \times I)$ is diffeotopic in $M \times I$ to $0 \times I$ (we use 0 for the centre of the disc D^m); modifying Q by a diffeotopy, we may suppose it the identity on $0 \times I$. But then $D^m \times I$ and $Q(D^m \times I)$ are tubular neighbourhoods of $0 \times I$, and modifying Q by a further diffeotopy (fixed at the ends), we can make these the same point-set, as required.

For $\tilde{\pi}_0(\text{Diff}\,(M^m \text{ rel } D^m))$ we proceed as above, but at the last stage, applying the tubular neighbourhood theorem, we may suppose that Q induces a bundle map of $D^m \times I$ (over I) on itself. The obstruction to making this the identity thus lies in $\pi_1(SO_m)$, and is represented by a map of I to SO_m. Now if, for example, M^m is a sphere S^m, then SO_m acts on S^m, so rotating the whole sphere (for each $t \in I$) by the corresponding element of SO_m, we reduce Q to the identity on $D^m \times I$. The remaining assertions follow easily.

Remark. Observe that this discussion can be informally stated as follows. There is an exact sequence.

$$\tilde{\pi}_1(\text{Diff}(M^m, D^m)) \to \tilde{\pi}_1(\text{Diff}(D^m)) \to \tilde{\pi}_0(\text{Diff}(M^m \text{ rel } D^m)) \to \tilde{\pi}_0(\text{Diff}(M^m, D^m)) \to \tilde{\pi}_0(\text{Diff}(D^m)).$$

The last term is zero, the second isomorphic to $\pi_1(SO_m)$, and we have shown that if M^m is S^m the first map is onto. In fact even if $\tilde{\pi}_i$ is replaced by π_i, it is not hard to justify this sequence.

Now let M^m be a closed oriented simply-connected manifold, D^m an imbedded disc, N^m the closure of $M^m - D^m$. Write $\{M/N\}$ for the set of diffeomorphism classes of closed manifolds M_g formed from N by attaching a disc to the boundary by some diffeomorphism g of S^{m-1}; thus $M_1 = M$.

THEOREM 3. *There is an exact sequence*

$$\Gamma_{m+1} \to \tilde{\pi}_0(\mathrm{Diff}(M)) \to \tilde{\pi}_0(\mathrm{Diff}(N)) \to \Gamma_m \to \{M/N\} \to 0.$$

Proof. By the last lemma and Hilfssatz the sequence may be written as

$$\tilde{\pi}_0(\mathrm{Diff}(D^m \text{ rel } S^{m-1})) \xrightarrow{\alpha} \tilde{\pi}_0(\mathrm{Diff}(M^m, D^m)) \xrightarrow{\beta} \tilde{\pi}_0(\mathrm{Diff}(N^m)) \xrightarrow{\gamma} \tilde{\pi}_0(\mathrm{Diff}(S^{m-1})) \xrightarrow{\delta} \{M/N\} \to 0.$$

The maps are now easy to define: α is induced by inclusion—if f is a diffeomorphism of D^m rel S^{m-1}, $\alpha(f)$ is defined as the identity on N^m and f on D^m; for $g \in \mathrm{Diff}(M^m, D^m)$ we let $\beta(g)$ be the restriction of g to N^m. Similarly γ is defined by restricting to $\partial N = S^{m-1}$. Finally, $\delta(g) = M_g$. In each case the definition is compatible with quasi-diffeotopies: this is clear for α, β, γ and follows for δ from Lemma 9 (i).

Next, $\beta\alpha(f)$ is the identity map, hence correspondingly $\beta\alpha$ is zero for equivalence classes. Also $\gamma\beta$ is zero, for if $\gamma\beta(g) = g'$, then g' can be extended to D^m. And $\delta\gamma = 0$, for if g can be extended to a diffeomorphism of N, that diffeomorphism (with the identity on D^m) induces a diffeomorphism of M_g on M_1, and also of M_{gf} on M_f.

If now g represents an element of Ker β, there is a quasi-diffeotopy Q of $g|N$ to 1. Extending Q, we find that g is quasi-diffeotopic to some g', which agrees with 1 on N, so is in the image of α. Extendability follows from Lemma 9 (i) essentially. If the class of h is in Ker γ, then $h|S^{m-1}$ is quasi-diffeotopic to 1, so by Lemma 9 (iv), extends to D^m, thus h is in Imβ. Finally suppose g in Ker δ, so M_g diffeomorphic to M_1. Using the disc theorem, we modify the diffeomorphism of M_g on M_1 to induce the identity on D^m; it then gives a diffeomorphism of N which agrees with g on the boundary, so that g is in Imγ.

COROLLARY. *For* $M = S^m$, $N = D^m$, $\delta: \Gamma_m \cong \{M/D^m\}$.

This follows from the Theorem and Lemma 9 (iii). This result—that Γ_m may be defined by diffeomorphism classes of manifolds obtainable by attaching 2 discs—is due to Thom [14]. The other case we have in mind is for M an $(s-1)$-connected $2s$-manifold; we shall apply the theorem in a later paper of this series to obtain information on $\{M/N\}$.

§5. OBSTRUCTIONS TO DIFFEOTOPY

We now let $M \in \mathscr{H}(m, k, s)$ be a handlebody, of the type classified in the preceding paper—i.e. $s \geq 2$, $2m \geq 3s+3$—and seek to classify diffeomorphisms of M up to quasi-diffeotopy. We first classify them up to homotopy. Now M has the homotopy type of a bouquet of k s-spheres, and $s \geq 2$, so that $\pi_s(M) \cong H_s(M)$. It follows that the homotopy class of a map h of M in itself is determined by the induced map h_* of $H = H_s(M)$. Now we have invariantly defined functions $\alpha: H \to \pi_{s-1}(SO_{m-s})$ and $\lambda: H \times H \to \pi_s(S^{m-s})$; which must be preserved by any diffeomorphism, i.e. h_* satisfies

$$\alpha(h_*(x)) = \alpha(x), \quad \lambda(h_*(x), h_*(y)) = \lambda(x, y) \quad \text{for } x, y \in H.$$

Conversely, let h_* be an automorphism of H with this property. Let $\{e_i\}$ be a basis of H; by Proposition 2, M has presentations corresponding to the bases $\{e_i\}$ and $\{h_*(e_i)\}$. But the formulae above, with Lemma 7, show that these presentations have the same invariants α_i and λ_{ij}. Since, by Lemma 6, these gave a complete set of invariants, there is a diffeomorphism of M which carries one presentation into the other, and so induces h_*. We write Aut H for the group of automorphisms of H which preserve (in the sense above) α and λ.

LEMMA 10. *The homotopy classes of diffeomorphisms of M are in* (1 — 1) *correspondence with elements of* Aut H.

Next we must study when two homotopic diffeomorphisms are quasi-diffeotopic: this problem lies somewhat deeper. We attempt to build up a quasi-diffeotopy in steps, using a filtration of M by subcomplexes. We first write $M = M' \cup (\partial M' \times I)$, as for Lemma 9, and use a presentation $M = D^m \cup_f \bigcup_{i=1}^k (D_i^s \times D_i^{m-s})$ of M as a handlebody. Our filtration starts with D^m; then adds $D_i^s \times 0$; then $D_i^s \times D_i^{m-s}$, and finally $\partial M' \times I$. The idea is to use the disc theorem to deal with D^m, use Haefliger's imbedding theorem (Proposition 1) for discs $D_i^s \times 0$, extend to $D_i^s \times D_i^{m-s}$ by the tubular neighbourhood theorem, and then out to the boundary using a result of Smale.

Suppose that $h: M \to M$ is a diffeomorphism homotopic to 1. We regard h as a diffeomorphism of $M \times 1$, and seek to extend h on $M \times 1$ and 1 on $M \times 0$ to a diffeomorphism Q of $M \times I$.

Step 1. By the disc theorem, $h|D^m$ and $1|D^m$ are diffeotopic, and we choose a diffeotopy and define it as $Q|D^m \times I$.

Step 2. We next define Q on $D_i^s \times I$: now $Q|\partial(D_i^s \times I)$ is already given: on $\partial D_i^s \times I$ by Step 1, on $D_i^s \times 0$ by 1, and on $D_i^s \times 1$ by h. We can extend to some map of $D_i^s \times I$—this follows since h is homotopic to 1. By Haefliger's theorem [3], this map may be replaced by an imbedding, provided that $2(m+1) \geqslant 3(s+1)+3$, i.e. $2m \geqslant 3s+4$ and $M \times I$ is $\{2(s+1)-(m+1)+1\}$-connected. The second condition follows from the first, since M is certainly exactly $(s-1)$-connected, and $s \leqslant m-3$. From now on, we shall assume $2m \geqslant 3s+4$.

Step 3. We have now defined imbeddings $Q(D_i^s \times I)$; however, we have no reason to suppose that these will have disjoint images. Since their boundaries were fixed in advance and disjoint, we can appeal to §1 for a measure (λ) of their intersection, defined using the Thom construction. We obtain elements $\mu_{ij} \in \pi_{s+1}(S^{m-s})$, for $i \neq j$, which are $(-1)^{s+1}$-symmetric. Provided that these elements vanish, Haefliger's theorem [3] assures us of the possibility of separating the images $Q(D_i^s \times I)$, and so getting an imbedding at this step also.

Step 4. By Step 2, Q is defined on $D^m \times I$ and on $D_i^s \times I$. We now wish to extend to $D_i^s \times D_i^{m-s} \times I$—i.e., to a tubular neighbourhood of $D_i^s \times I$. By the tubular neighbourhood theorem, it is sufficient to find a trivialisation of the normal bundle with the desired properties. As in Step 2, these turn out to be that a trivialisation is already given on the boundary $\partial(D_i^s \times I)$ (the corners clearly present no special features). The obstruction to extending this over $D_i^s \times I$ (since this is contractible, the bundle certainly is trivial) is an element β_i of $\pi_s(SO_{m-s})$. If $\beta_i = 0$, the extension of the framing, and hence of Q, is possible.

Step 5. We now assume that Steps 1–4 have been successfully performed; we assert that Q can then be defined on the whole of $M \times I$, so that h and 1 are indeed quasi-diffeotopic. Observe that since M', and M, and $M \times I$ have the same homotopy type, and $Q: M' \times I \to M \times I$ agrees with the identity on $M' \times 0$, the inclusion of $Q(M' \times I)$ is a homotopy equivalence. Let W be the closure of the complement. Clearly W is simply connected (recall $s \leqslant m-3$), so that by a theorem of Smale [9], W gives an 'h-cobordism' of $\partial M' \times I$ to $\partial M \times I$, and is diffeomorphic to $\partial M \times I \times I$. We can then extend Q using such a diffeomorphism.

We may summarise the above discussion as follows.

LEMMA 11. *Let $s \geqslant 2$, $2m \geqslant 3s+4$, $M \in \mathscr{H}(m, k, s)$; then two homotopic diffeomorphisms of M are also quasi-diffeotopic, provided certain obstructions $\beta_i \in \pi_s(SO_{m-s})$, $\mu_{ij} \in \pi_{s+1}(S^{m-s})$ $(1 \leqslant i < j \leqslant k)$ vanish.*

We warn the reader that this result is far from conclusive (which misled the author for many months), and we shall make a more complete study in the next section. We also observe that by taking more care (and using deeper results of Haefliger), we could have obtained actual, rather than quasi-diffeotopy, in each of the first four of the above steps. For Step 5, however, there does not seem any method to obtain actual diffeotopy.

The next question is, which values of the obstructions β_i and μ_{ij} are possible. We first formulate

LEMMA 12. *If the diffeomorphisms h, h' of M, homotopic to 1, admit obstructions β_i, μ_{ij} and β'_i, μ'_{ij} as above, then $h' \circ h$ admits $\beta_i + \beta'_i$, $\mu_{ij} + \mu'_{ij}$.*

Proof. Given h, h' we construct maps Q, Q' as in Step 2 above, defining the given obstructions. Then for $h' \circ h$, we can define Q'' by

for $0 \leqslant t \leqslant \frac{1}{2}$, $Q''(P, t) = (P_1, t_1)$ where $Q(P, 2t) = (P_1, 2t_1)$;
for $\frac{1}{2} \leqslant t \leqslant 1$, $Q''(P, t) = (h(P_2), t_2)$ where $Q'(P, 2t-1) = (P_2, 2t_2 - 1)$.

Now recalling the definitions of μ, β by maps of discs, we see that their additivity follows at once from the definition of addition in homotopy groups.

Now if arbitrary β_i are assigned, and $g_i: (D^s, S^{s-1}) \to (SO_{m-s}, 1)$ are maps representing them, we can define a diffeomorphism h of $D^m \bigcup_{i=1}^k (D_i^s \times D_i^{m-s})$ as the identity on D^m, and by the formula

$$h(P, x) = (P, g_i(P).x)$$

for $P \in D_i^s$, $x \in D_i^{m-s}$. (For a diffeomorphism, we ought, strictly, to insist that g_i be smooth, and map a neighbourhood of S^{s-1} to 1). Clearly, h is homotopic to 1, all the μ_{ij} are zero (we can take Q as the identity on $D_i^s \times 0 \times I$) and the β_i are as chosen. It remains to consider the μ_{ij}; this is rather more complicated, but we shall prove

LEMMA 13. *Given arbitrary $\beta_i \in \pi_s(SO_{m-s})(1 \leqslant i \leqslant k)$ and $\mu_{ij} \in \pi_{s+1}(S^{m-s})(1 \leqslant i < j \leqslant k)$, there is a diffeomorphism h of M, homotopic to 1, such that β_i, μ_{ij}, arise as obstructions to a diffeotopy to 1.*

Proof. By Lemma 12, the obstructions are additive for compositions, so it is sufficient to show that any given one can be arbitrary, the rest being zero, as the result follows on adding up. We have just shown this for the β_i; we now do it for μ_{12} (typical for μ_{ij})—it is sufficient to obtain arbitrary μ_{12}, the other μ_{ij} being zero, and ignore β_i.

We take $Q = 1$ on $(D^m \bigcup_{i \geqslant 2} D_i^s \times 0) \times I$; we must define $Q(D_1^s \times 0 \times I)$ to intersect $D_2^s \times 0 \times I$ in V to give the chosen μ_{12} and disjoint from the other $D_i^s \times 0 \times I$. The following argument was suggested by the referee.

Let Δ^m be a disc in M, disjoint from $D^m \bigcup_{i \geqslant 3}(D_i^s \times 0)$, and meeting $D_1^s \times 0$, $D_2^s \times 0$ in discs Δ_1^s, Δ_2^s in their interiors, and naturally imbedded in Δ^m. (For example, first choose Δ_i' concentric with $D_i^s \times 0$, join Δ_1' and Δ_2' by an arc avoiding D^m and the $D_i^s \times 0$, and thicken). Now take $Q = 1$ on $(D_1^s \times 0 - \Delta_1^s) \times I$. Then imbed $\Delta_1^s \times 1$ in $(\Delta^m - \Delta_2^s) \times 1$, to satisfy the following condition. It completes disjoint imbeddings of $\partial(\Delta_1^s \times I)$, $\partial(\Delta_2^s \times I)$ in $\partial(\Delta^m \times I)$, their linking number $\mu' \in \pi_s(S^{m-s-1})$ is to satisfy $S\mu' = \mu_{12}$ (this is possible since μ_{12} is in a stable group, so is a suspension element). Finally, we 'fill in' $Q(\partial(\Delta_1^s \times I))$ by an $(s+1)$-disc imbedded in $\Delta^m \times I$. This completes the definition of Q on $(D^m \bigcup_i D_i^s \times 0) \times I$; it is clear that it has the required intersection invariants.

We extend this to a product neighbourhood $D_1^s \times D_1^{m-s} \times I$ using a trivialisation of the normal bundle, which agrees with the given maps of $M \times 0$ and $D^m \times I$, and is disjoint from the parts already mapped except in a neighbourhood of V. If we can now extend $Q|M' \times 1$ to a diffeomorphism h of $M \times 1$ the above discussion shows that the obstructions to a quasi-diffeotopy of h and 1 are as stated. But this follows, just as in Step 5, by a result of Smale.

We shall now summarise the results so far obtained. Now what we have shown (Lemma 11) does not imply that the diffeotopy class of f determines uniquely the obstructions β_i and μ_{ij}, but does imply the converse: a remark reinforced by Lemma 13. We write L for the direct sum of $\binom{k}{2}$ copies of $\pi_{s+1}(S^{m-s})$ and k copies of $\pi_s(SO_{m-s})$: L denotes the total range of values of μ_{ij} and β_i, varying independently. As we just said, any element of L determines uniquely a quasi-diffeotopy class of diffeomorphisms; by Lemma 12 this defines a homomorphism of L to $\tilde{\pi}_0(\text{Diff}(M))$ and a class arises if and only if it is homotopic to the identity, by Lemma 11. If we appeal also to Lemma 10, we deduce

Summary. There is an exact sequence

$$L \to \tilde{\pi}_0(\text{Diff}(M)) \to \text{Aut } M \to 0.$$

We do not state this formally, since we shall obtain a more complete result below. Indeed, our next task is to investigate the kernel of the first of the above homomorphisms.

§6. VARIABILITY OF THE OBSTRUCTIONS

It is now necessary to observe that the obstructions μ, β of the preceding paragraph are indeed not well-determined by h. In fact they remain invariant certainly for as long as $Q((D^m \bigcup_{i=1}^k D^s) \times I)$ is only varied by a diffeotopy, and, for $2m \geqslant 3s + 5$, we can apply Haefliger's theorem to obtain diffeotopies of $D^s \times I$ in $M \times I$. Since the boundary of $D^s \times I$ will be fixed, we will find that the absolute $(s+1)$st homotopy group of M plays a rôle.

In the four steps of §5, we twice had to make a choice, at Steps 1 and 2. In Step 1 we chose a diffeotopy of D^m to $h(D^m)$. Now the proof of the Hilfssatz to Theorem 3 showed that this diffeotopy is unique up to a diffeotopy, and operation of $\pi_1(SO_m)$. But varying $Q(D^m \times I)$ by a diffeotopy is of no concern to us, and we now see that we have the distinction between $\tilde{\pi}_0(\mathrm{Diff}\,(M)) \cong \tilde{\pi}_0(\mathrm{Diff}\,(M,\ D^m))$, which we wish to calculate, and $\tilde{\pi}_0(\mathrm{Diff}\,(M \text{ rel } D^m))$, which is what we have nearly calculated (in Steps 2–5).

We shall concentrate on $\tilde{\pi}_0(\mathrm{Diff}\,(M \text{ rel } D^m))$. At Step 2, we had $Q|\partial(D_i^s \times 0 \times I)$ given, and chose an extension to a map of $D_i^s \times 0 \times I$. For any other such, we have a difference map of S^{s+1} into $M \times I$. Conversely we can alter Q on $D_i^s \times 0 \times I$ by any element of $\pi_{s+1}(M)$, and by Proposition 1 still take Q an imbedding which, indeed, is unique up to diffeotopy if $2m \geqslant 3s+5$ and up to regular homotopy (by Lemma 1) if $2m = 3s+4$.

Now suppose Q changed, by altering Q on $D_1^s \times 0 \times I$ by the element ξ_1 of $\pi_{s+1}(M)$; then we must calculate the change of the β_i and μ_{ij}. These are clearly unaltered if $i, j \neq 1$. Now for μ_{i1}, note that Q is unaltered on $D_i^s \times 0 \times I$, so the Thom construction defines the same map f of $(M - D^m) \times I$ to S^{m-s}. But the homotopy class of $D_1^s \times 0 \times I$ is changed by ξ_1, so μ_{i1} is changed by $[f] \circ \xi_1$.

We can express this more explicitly. For M has the homotopy type of a bouquet of k s-spheres, and since $2m \geqslant 3s+3$, $\pi_{s+1}(S^{m-s})$ is a stable group. Thus $[f]$ is determined by k homotopy classes of maps $S^s \to S^{m-s}$, and these maps are precisely the λ_{ij} (by definition). Likewise, $\xi_1 \in \pi_{s+1}(M)$, which is a direct sum of k copies of $\pi_{s+1}(S^s)$, provided $s \geqslant 3$. We write ξ_{j1} for the components of ξ_1. Then we have $[f] \circ \xi_1 = \Sigma_j \lambda_{ij} \circ \xi_{j1}$ since the ξ_{j1} are suspension elements.

Assembling these results we have

LEMMA 14. *Let Q be altered so that the components of the change of $Q(D_i^s \times 0 \times 1)$ are $\xi_{ri} \in \pi_{s+1}(S^s)$. Then*

$$\mu'_{ij} = \mu_{ij} + \sum_r \lambda_{jr} \circ \xi_{ri} + (-1)^{s+1} \sum_r \lambda_{ir} \circ \xi_{rj}.$$

The second term is necessary since the variation considered above must alter not only μ_{i1} but also μ_{1i}. Also, since $s > 2$, the ξ_{ri} have order 2, so the sign can be omitted.

To compute the change in β_i, we first again suppose Q only changed on $D_1^s \times 0 \times I$. Then as remarked after Theorem 1, we can apply that result to calculate the change. The result is

$$\beta'_1 = \beta_1 + \alpha(\xi_1) + \partial\lambda(Q(D_1^s \times 0 \times I), \xi_1).$$

But since ξ_1 has components ξ_{j1}, we have by definition

$$\alpha(\xi_1) = \sum_j F(S\alpha_j, \xi_{j1})$$

since the normal bundle in M must be suspended to find that in $M \times I$ and the ξ_{j1} may be represented by disjoint maps in $M \times I$. Moreover, since λ depends on the second argument only up to homotopy, we have

$$\lambda(Q(D_1^s \times 0 \times I), \xi_1) = \sum_j \lambda_{1j} \circ \xi_{j1}.$$

Assembling these results, and again passing to the general case, we have

LEMMA 15. *Under the hypotheses of Lemma* 14,

$$\beta_i' = \beta_i + \sum_j F(S\alpha_j, \xi_{ji}) + \sum_j \hat{c}(\lambda_{ij} \circ \xi_{ji}).$$

Hence the change in both μ_{ij} and β_i depends only on the ξ_{ji}, and depends additively on these—by right-distributivity of composition in homotopy groups, and by (2) of Lemma 5.

We write K for the direct sum of k^2 copies of $\pi_{s+1}(S^s)$, regarded as the range of possible values of the ξ_{ij}. We may regard the formulae of Lemmas 14 and 15 as defining a homomorphism from K to L.

THEOREM 4. *Let* $M \in \mathscr{H}(m, k, s)$, $s \geqslant 3$, $2m \geqslant 3s + 4$. *Then there is an exact sequence*

$$K \to L \to \tilde{\pi}_0(\mathrm{Diff}\,(M \text{ rel } D^m)) \to \mathrm{Aut}\, H \to 0.$$

Proof. In view of the summary in §5, it remains only to check exactness at L. Clearly an element of L determines a diffeomorphism quasi-diffeotopic to the identity if and only if Q can be changed so that the β_i and μ_{ij} become zero. But in the last three lemmas we have shown that the element of L can be changed precisely by some element in the image of K.

Remark. In this sequence, Aut H determines the allowable homotopy equivalences, and so a part of $\tilde{\pi}_0(\mathrm{Map}\,(M, M))$. Also we have $K = \pi_1(\mathrm{Map}\,(M, M))$. The group L involves also tangential structure of M. It seems that the sequence can be prolonged further to the left, provided sufficient dimensional restrictions hold; for example, the next term would be $\tilde{\pi}_1(\mathrm{Diff}\,(M))$. Rather than a piecemeal proof on the lines of this paper, we feel that the sequence should be obtained abstractly, and then interpreted.

We do not yet possess an expression of the relation of $\tilde{\pi}_0(\mathrm{Diff}\,(M \text{ rel } D^m))$ to $\tilde{\pi}_0(\mathrm{Diff}\,(M, D^m)) \cong \tilde{\pi}_0(\mathrm{Diff}\,(M))$ for general handlebodies of the type considered in Theorem 4.

There is another respect in which the result of Theorem 4 is incomplete. The group $\tilde{\pi}_0(\mathrm{Diff}\,(M \text{ rel } D^m))$ is determined as an extension of $L/\mathrm{Im}(K)$ by Aut H, but gives no information as to which extension we have. Now the operation of Aut H on L and K (and hence on the quotient) is natural, since L and K can be regarded (using the formulae of Theorem 1 for the extensions) as groups of maps of H or $H \times H$, and such maps may be simply composed with automorphisms of H.

To determine the extension, however, we also need a factor set, and we have only succeeded in finding this in one special case.

Suppose that the M of Theorems 2 and 4 can be imbedded in $\mathbf{R}^m$. Then any s-sphere has (by Proposition 1) a trivial normal bundle in $\mathbf{R}^m$, hence in M, so that the function α is identically zero. Similarly, λ must be the same whether reckoned in M or $\mathbf{R}^m$, and since it depends only on homotopy of the arguments, it too vanishes. Conversely, the conditions $\alpha = \lambda = 0$ entirely determine $M \in \mathscr{H}(m, k, s)$—in fact, M is a sum of copies of $S^s \times D^{m-s}$, and so certainly imbeds in $\mathbf{R}^m$.

Write S for the subgroup of $\tilde{\pi}_0(\mathrm{Diff}\,(M))$ of classes of diffeomorphisms which extend to $\mathbf{R}^m$.

LEMMA 17. *When* $2m \geqslant 3s+4$, $s \geqslant 3$, *and for* $M \in \mathscr{H}(m, k, s)$, $\alpha = \lambda = 0$, *the map from* S *to* Aut H *is an isomorphism, and so* $\tilde{\pi}_0(\text{Diff}(M))$ *is a* split *extension of* L *by* Aut H.

Proof. Observe that since $\alpha = \lambda = 0$, by Lemmas 14 and 15, the image of K in L is zero.

Now just as the imbeddability of M in R^m proved that α and λ were zero, it follows that for extendability of a diffeomorphism h, β and μ must vanish. [Indeed, they must vanish for any choices of the indeterminates at steps 1 and 2, and since a representative of the kernel of $\tilde{\pi}_0(\text{Diff}(M \text{ rel } D^m)) \to \tilde{\pi}_0(\text{Diff}(M))$ certainly extends, it must have β and μ zero, so the kernel is zero, and this map is always an isomorphism.] Hence the map from S to Aut H is a monomorphism.

Since $\alpha = \lambda = 0$, Aut H contains all automorphisms of the abelian group H, and so, by [13], is generated by

(i) Permutations of the e_i;

(ii) R, where $Re_1 = -e_1$ and $Re_i = e_i$ for $i > 1$;

(iii) T, where $Te_1 = e_2 - e_1$, $Te_2 = -e_1$, and $Te_i = e_i$ for $i > 2$.

We prove each generator in turn in the image of S. Recall that M is the sum of k copies of $S^s \times D^{m-s}$.

If $k = 2$, we can interchange 2 copies by a rotation. Using the disc theorem to modify this to keep a disc fixed, we now see that we can interchange any two of the summands, and so obtain (i). Similarly, using the disc theorem, we reduce the proof of (ii) and (iii) to the cases $k = 1$, $k = 2$ respectively.

If $S^s \times D^{m-s} \subset R^m$ as the standard tubular neighbourhood of S^s in $R^{s+1} \subset R^m$, we can represent R by the rotation which changes the signs of first and last co-ordinates.

Finally, for $k = 2$, note that M is a 'thickening' of $S^s \cup S^s$—or, equivalently, of the join of S^{s-1} to 3 points. We may imbed M by putting a standard S^{s-1} in R^s, the vertices of an equilateral triangle in R^2, taking the join, and a smooth neighbourhood of it in R^m. For homology, we can take the thickened join to one of the points as the basic D^m, and the other two as the handles. The required diffeomorphism is now given by the rotation in R^2 through an angle $2\pi/3$.

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